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Small Cycle Double Covers and Line Graphs

by

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Abstract

A cycle double cover (CDC) of a graph is a collection of cycles of the graph with the property that every edge of the graph is included in exactly two cycles. The Cycle Double Cover Conjecture, proposed by both Seymour and Szekeres, independently, states that every bridgeless graph has a CDC.

A small cycle double cover (SCDC) of a simple graph on n vertices is a CDC with at most $n-1$ cycles. The Small Cycle Double Cover Conjecture, due to Bondy, states that every simple, bridgeless graph has an SCDC.

If a graph has a CDC with certain properties, then its line graph (a simple graph) has an SCDC. By showing that all complete multipartite graphs have CDCs with appropriate properties, it is thus proved that the line graphs of all complete multipartite graphs except $K_{1,2}$ (whose line graph has a bridge) have SCDCs.

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Chapter 1

Introduction

1.1 Overview

There are numerous problems in graph theory concerning the covering of the edges of a graph with paths or cycles. One of the best known problems of this type requires that each edge of a graph be covered by exactly two cycles; the resulting collection of cycles is called a *cycle double cover (CDC)* of the graph. A bridge, or cut-edge, of a graph cannot be contained in a cycle, and thus for a graph to have a CDC, it must necessarily be bridgeless. Seymour [29] and Szekeres [30] independently theorized that this condition is also sufficient, and conjectured the following.

Conjecture 1 (Cycle Double Cover Conjecture) *Every bridgeless graph has a cycle double cover.*

A natural extension of the CDC Conjecture arises from attempts to find CDCs with the smallest possible number of cycles. The number of cycles in a CDC of a graph with multiple edges or loops cannot be related to the number of vertices in that graph because the number of edges of that graph cannot be given in terms of the number of vertices. For example, a graph on two vertices with $m \geq 2$ edges between them has a CDC with no fewer than m cycles. However, because a vertex of degree d requires d cycles to cover all of its incident edges twice, and since a simple graph on n vertices has maximum degree at most $n-1$, a general lower bound on the number of cycles in a CDC of a simple graph is $n-1$. Whether this value of $n-1$ is also an upper bound on the minimum number of cycles required to cover a simple graph is yet to be determined. This counting of cycles in a CDC gives rise to the notion of a *small cycle double cover (SCDC)*, a CDC of a simple graph on n vertices which contains at most $n-1$ cycles (see [4]). As with CDCs, any graph with an SCDC must be bridgeless. Bondy [4] thus proposes the following.

Conjecture 2 (Small Cycle Double Cover Conjecture) *Every simple, bridgeless graph has a small cycle double cover.*

Various types of graphs which satisfy the SCDC Conjecture are mentioned in the following chapter, while the remainder of this thesis is devoted to proving the SCDC Conjecture for line graphs of complete multipartite graphs. In Chapter 3, lemmas developed in [24] which prove the existence of an SCDC of a line graph, $L(G)$, when a CDC of the graph, G, has certain properties, are proved. Subsequent chapters then show that there exist CDCs of complete multipartite graphs possessing these properties. A number of technical results are presented in Chapter 4, including proofs of the existence of CDC's with the required properties for specific classes of graphs. Finally, the main result is proved in Chapter 5. This result shows that the line graph of any complete multipartite graph (excluding $K_{1,2}$) has an SCDC.

1.2 Notation and Terminology

A graph, G, consists of a set of vertices, $V(G)$, and a collection of edges, $E(G)$, where each edge is an unordered pair of vertices. For an edge $e = \{u, v\}$, $u, v \in G$, we write $e = uv$ and say that u and v are the *endpoints* of e. If $e \in E(G)$ and $e = uv$, then u and v are *adjacent*, and are *joined* by edge e. As well, e is *incident* with u and v. Similarly, if $e, f \in E(G)$ and e and f have a common endpoint, then e and f are *adjacent*. The *neighbour set* of a vertex $v \in V(G)$, denoted $N(v)$, consists of all vertices in $V(G)$ which are adjacent to v.

For a vertex $v \in V(G)$, the *degree* of v, denoted $d(v)$, is simply the number of edges incident with v. A graph is *even* if each of its vertices has even degree, and is *odd* if each of its vertices has odd degree. A graph is *k-regular* if each of its vertices has degree k. If a graph is 3-regular, we also say it is *cubic*.

A *loop* is an edge whose two endpoints coincide, and a *multiple edge* in a graph occurs when two vertices are joined by more than one edge. A *simple graph* is a graph that has no loops and no multiple edges, whereas a *multigraph* is a graph that may contain loops or multiple edges. From now on, the term graph will always refer to a simple graph.

Let G be a graph with edge set $E(G) = \{e_1, e_2, \ldots, e_n\}$. The *line graph* of G, denoted $L(G)$, is a graph with vertices $V(G) = \{e_1, e_2, \ldots, e_n\}$, where $e_ie_j \in E(L(G))$ if and only if e_i and e_j are adjacent in G.

A *subgraph*, H, of a graph, G, is a graph with a vertex set $V(H) \subseteq V(G)$, and an edge set $E(H) \subseteq E(G)$, such that if an edge, e, is in $E(H)$, then both endpoints of e are in $V(H)$. For a set $S \subseteq V(G)$, $G[S]$ is the subgraph *induced* by S, and consists of the vertices in S and every edge of G whose two endpoints are vertices in S. Subgraphs $H_1, H_2, \ldots, H_n$ of G *partition* the edge set of G if every edge of G is in exactly one of the subgraphs.

For a finite set, X, we use $|X|$ to denote the number of elements in X, and call $|X|$ the *cardinality*, or *size*, of X.

The *complete graph* of n vertices, K_n, is a graph where every pair of vertices is joined by an edge. A *clique* of a graph, G, is a subgraph of G that is a complete graph.

In a graph, G, a subset $S \subseteq V(G)$ is *independent* if and only if $G[S]$ has no edges. For integers $m, n \geq 1$, $K_{m,n}$ denotes the *complete bipartite graph*, a graph whose vertex set is partitioned into two nonempty, independent sets, X and Y, called the *parts* of $K_{m,n}$, with $|X| = m$ and $|Y| = n$, where every vertex in X is adjacent to every vertex in Y. Similarly, a *complete multipartite*, or *p-partite graph*, $p \geq 2$, has a vertex set that is partitioned into p nonempty, independent sets, or parts, such that any two vertices from different parts are joined by an edge.

A *path* in a simple graph is a sequence of distinct vertices, $v_0 v_1 \ldots v_k$, where vertices v_{i-1} and v_i are joined by an edge. A *cycle* is similar to a path, except that its first vertex is the same as its last vertex. It should be noted that paths and cycles can be considered to be sequences of vertices and edges within a graph, or to be graphs, themselves. The *length* of a path or of a cycle is the number of edges it contains. The term *k-cycle* refers to a cycle of length k. A 2-cycle (possible only in multigraphs) is also called a *digon*.

An edge of a graph is *covered* by a path or by a cycle if the path or cycle contains that edge.

A *path double cover (PDC)* is a collection of paths of a graph such that each edge of the graph is covered exactly twice. A *perfect path double cover (PPDC)* is a PDC where every vertex occurs twice as an endpoint of a path (and hence, the number of paths is equal to the number of vertices).

A *cycle decomposition* of a graph, G, is a collection of cycles in G such that every edge of G lies in exactly one of the cycles.

Let $c = v_0 v_1 \ldots v_i \ldots v_j \ldots v_k v_0$ be a cycle in a graph, G. The edge $e = v_i v_j$ is called a *chord* of c if and only if v_i and v_j are not adjacent in c but are adjacent in G. The cycle is said to be *chordless* if it has no chords. Let $\mathcal{C}$ be a collection of cycles in a graph, G. Then we define β_n to be the number of cycles in $\mathcal{C}$ that contain exactly n chords. Let H be a subgraph of G and suppose c is a cycle in H. Then c is also a cycle in G, and we define $ch(c)$ to be the number of chords of c when c is considered as a cycle of G. If $\mathcal{C}_H$ is a collection of cycles in H, we denote by $ch(\mathcal{C}_H)$, the total number of chords of the cycles of $\mathcal{C}_H$ when these cycles are considered on the subgraph $G[V(H)]$. We define the function $\gamma(\mathcal{C}_H)$ to be $\gamma(\mathcal{C}_H) = |\mathcal{C}_H| + ch(\mathcal{C}_H)$.

A graph is *connected* if there is a path between any two of its vertices. A *cut-edge* or *cut-vertex* of a connected graph, G, is an edge or vertex, respectively, whose deletion results in a graph that is no longer connected. An *edge cut* or *vertex cut* of G is a subset $E' \subseteq E(G)$ or $V' \subseteq V(G)$, respectively, such that $G \backslash E'$, or $G \backslash V'$ is not connected. A graph is thus *k-edge-connected* or *k-connected*, respectively, if the size of its smallest edge cut or vertex cut is at least $k+1$. A *bridge* in a graph is a cut-edge and so a graph is *bridgeless* if it has no cut-edges.

For any terms not defined here, see [5].

Chapter 2

CDCs and SCDCs

In 1736, in what is considered to be the first paper in graph theory, Euler proved that if each edge of a connected multigraph can be covered exactly once by a sequence of vertices and edges, starting and ending on the same vertex, and passing through adjacent edges (a *closed tour*, or *Euler tour*), then the multigraph is necessarily even [8]. Although Euler also stated the converse, it was not until over a century later, in 1873, that Hierholzer [15] published a proof of the fact that any even, connected multigraph has an Euler tour. Further properties of multigraphs with Euler tours, called eulerian multigraphs, were investigated by Veblen [33], who characterized these multigraphs in terms of cycle decompositions. It thus follows that for a connected multigraph, M, the following are equivalent:

1. M is eulerian.
2. M is even.
3. M has a cycle decomposition.

A proof of this equivalence can be found in [11].

Because every cycle through a vertex requires two edges incident with that vertex, no graph with vertices of odd degree can be decomposed into cycles. However, if every edge of a graph, G, is replaced by a digon, then the resulting multigraph, G', has only vertices of even degree. Thus G' has a cycle decomposition, where some cycles may have length two. A cycle of length two in G', a digon, corresponds to an edge rather than to a cycle in G. However, other cycles in G' correspond to cycles in G, and so if G' has a cycle decomposition with no cycles of length two, then this same collection of cycles, considered as cycles of G, forms a cycle double cover (CDC) of G. Therefore, a CDC generalizes the concept of cycle decomposition to graphs containing vertices of odd degree.

There is also a connection between CDCs of graphs and graph embeddings. A multigraph is *embeddable* on a surface if it can be drawn on that surface so that none of its edges intersect except at their ends. A *strong embedding* of a multigraph on a surface is an embedding where the boundary of each face is a cycle. If a multigraph has a strong embedding, then it also has a CDC composed of cycles corresponding to its face boundaries. Therefore, the truth of the CDC Conjecture could be deduced if the following were true.

Conjecture 3 (Strong Embedding Conjecture [16]) *Every 2-connected multigraph has a strong embedding (on some surface).*

The CDC Conjecture has been studied extensively by numerous authors. One approach to solving the problem has been an attempt to describe the characteristics of a minimum counterexample to the conjecture (a counterexample with the smallest number of vertices, and subject to this condition, the smallest number of edges), with the hope that it will then be possible to show that no such counterexample exists. Let the multigraph, G, be a minimum counterexample to the CDC Conjecture. By minimality, G is connected and thus has no vertices of degree zero, and since G is bridgeless, it has no vertices of degree one. Suppose G has a vertex, v, of degree two, and let its two adjacent vertices be u and w. Let G' be the multigraph obtained from G by deleting v (thus eliminating edges uv and vw) and adding an edge uw (there may already be one or more of these edges). By the minimality of G, the multigraph G' has a CDC with the new edge uw contained in two cycles. A CDC of G can be obtained from the CDC of G' simply by modifying the cycles passing through the new edge, so that instead, they pass through edges uv and vw. Consequently, if G is a minimum counterexample to the CDC Conjecture, it must have minimum degree at least three.

Suppose now that G has a vertex, v, of degree greater than three. Fleischner [10] has proved that there are two edges, uv and vw, incident to v such that the multigraph, G', obtained by the deletion of these two edges and the addition of an edge, uw, is still bridgeless. By minimality, G' has a CDC, and thus the new edge uw is included in two cycles. Therefore, a CDC of G exists and is obtained from the CDC of G' by altering the cycles passing through uw so that they pass through uv and vw instead. The minimum counterexample to the CDC Conjecture must therefore be 3-regular.

Finally, G has no edge cut of size two. To see this, suppose that G does have an edge cut of size two. Contracting one of the edges in this edge cut results in a bridgeless multigraph, G', with fewer edges than G. By the minimality of G, the multigraph G' has a CDC which can be modified to produce a CDC of G [16].

We can also see that G is not 3-edge-colourable. If it were, then each colour class would be a perfect matching, and each of the three pairs of perfect matchings would form a 2-regular, spanning subgraph of G. Each 2-regular subgraph is a collection of cycles, so the union of these subgraphs would be a CDC of the graph. Another property of the minimum counterexample, G, is that it must be *cyclically-4-edge-connected* [16], meaning that if $V(G)$ is partitioned into sets S and $V(G)\backslash S$ so that

$G[S]$ and $G[V(G)\backslash S]$ each contain a cycle, then there are at least four edges in G which have one endpoint in S and one endpoint in $V(G)\backslash S$.

A minimum counterexample to the CDC Conjecture, a multigraph which is 3-regular, cyclically-4-edge-connected, and is not 3-edge-colourable, is called a *snark* (see [12]). Goddyn [13] shows that a minimum counterexample to the CDC Conjecture has *girth* (length of the shortest cycle in the graph) at least ten, and it has been been conjectured that no snarks of this girth exist [18].

The CDC Conjecture has also been proved for various classes of multigraphs. It has been shown that it holds for multigraphs with Hamilton paths [32] and for 4-edge-connected graphs [17].

Alspach, Goddyn and Zhang [1] have addressed the more general problem of covering the edges of a weighted multigraph with cycles, where the weight of an edge corresponds to the number of times that the edge must be covered. When all edges are assigned weight two, the problem is reduced to that of finding a CDC of the multigraph. These authors show that a 2-edge-connected multigraph has a CDC provided that it has no subgraph homeomorphic with the Petersen graph. In other words, a 2-edge-connected multigraph has a CDC if none of its subgraphs can be transformed into the Petersen graph through a series of edge contractions.

As previously observed, the Small Cycle Double Cover (SCDC) Conjecture can be viewed as a strengthening of the CDC Conjecture that restricts the number of cycles. However, it can also be considered to be a generalization of the following conjecture due to Hajós (see Lovász [23]).

Conjecture 4 (Hajós) *If G is a simple, even graph on n vertices, then G can be decomposed into $\lfloor (n-1)/2 \rfloor$ cycles.*

Originally, Hajós conjectured that the number of cycles should be at most $n/2$, but Dean [7] shows that the value $\lfloor (n-1)/2 \rfloor$ is equivalent.

If Hajós' Conjecture holds, then every eulerian graph has an SCDC obtained by taking two copies of the cycles used in its decomposition. As Hajós' Conjecture is true for even graphs with maximum degree four ([9], [14]) and for planar graphs ([26], [31]), these graphs have SCDCs. Using the fact that Hajós' Conjecture holds for planar graphs, Seyffarth [28] proves that every 4-connected planar graph has an SCDC.

Other classes of graphs have also been shown to satisfy the SCDC Conjecture. Bondy [4] describes how to construct SCDCs of complete graphs, K_n, $n \geq 3$, and complete bipartite graphs, $K_{m,n}$, $m, n \geq 2$. The SCDCs of the complete graphs consist of $n-1$ Hamilton cycles, while those of the complete bipartite graphs consist of $\max\{m, n\}$ cycles, each of length $\min\{2m, 2n\}$ [4].

Bondy verifies that the SCDC Conjecture holds for squares of trees [3], and also studies the conjecture as it applies to trigraphs (see [3]). A *trigraph* is defined to be a connected graph, G, with a spanning tree, T, such that the addition to T of any edge of G not already in T results in a triangle. The tree T is called a tritree of G. Bondy shows that every simple trigraph has an SCDC if and only if every simple, bridgeless trigraph also has a perfect path double cover (PPDC) [3]. Li [22] has proved that every simple graph admits a PPDC, and so it follows that every simple, bridgeless trigraph has an SCDC.

Bondy and Seyffarth [27] show that a simple triangulation of any surface has an SCDC.

The verification of the SCDC Conjecture has been approached from the point of view of finding a minimum counterexample, as has been done with the CDC Conjecture. However, unlike the CDC Conjecture, it is not sufficient to prove the SCDC Conjecture for 3-regular graphs, because the reductions used to obtain these 3-regular graphs may change the simple graphs into graphs with multiple edges. Bondy, however, has shown [4] that a minimum counterexample to the SCDC Conjecture is 3-connected and cyclically-4-edge-connected. For 3-regular graphs, Bondy [4] proposes the following stronger version of the SCDC Conjecture.

Conjecture 5 (Bondy) *Let G be a simple, 2-connected, 3-regular graph on n vertices, $n \geq 6$. Then G has a cycle double cover, $\mathcal{C}$, such that $|\mathcal{C}| \leq n/2$.*

Lai, Yu, and Zhang [21] prove that if a 2-connected, 3-regular graph on n vertices has a CDC, it has a CDC with at most $n/2$ cycles, showing that Conjecture 5 is equivalent to the CDC Conjecture for 2-connected, 3-regular graphs.

Because of the need to count cycles, it seems that generally, more information about the structure of a graph is required to prove the existence of an SCDC than to prove the existence of a CDC. The well-defined structures of complete graphs, complete bipartite graphs, and simple triangulations of surfaces make the construction of their SCDCs straightforward. Line graphs also have well-defined structures, and so they are an ideal class of graphs to study with respect to the CDC Conjecture and the SCDC Conjecture.

Cai and Corneil prove that if a graph has a CDC, its line graph also has a CDC [6]. The authors note that any graph can be obtained by contracting cliques in the line graph of its subdivided graph. This relationship between the graph and its line graph means that if a bridgeless line graph has a CDC, then every bridgeless graph has a CDC, and so proving the CDC Conjecture for line graphs is equivalent to proving the CDC Conjecture [6].

Klimmek [20] proves that every eulerian line graph with n vertices has a cycle decomposition into $\lfloor (n-1)/2 \rfloor$ cycles, thus verifying Hajós' Conjecture for such graphs. By taking two copies of the cycle decomposition, the SCDC Conjecture is also verified for these graphs. Since a line graph, $L(G)$, is eulerian if and only if the graph, G, is connected and either even or odd, the line graph of every even graph and of every odd graph has an SCDC. Additionally, MacGillivray and Seyffarth [24] prove that if G is a complete graph, a complete bipartite graph, or a planar graph, and $L(G)$ is bridgeless, then $L(G)$ has an SCDC. We note that of all line graphs of complete bipartite graphs, only $L(K_{1,2})$ has a bridge.

Chapter 3

Theory Relating CDCs of Graphs to SCDCs of Line Graphs

In their paper [24], MacGillivray and Seyffarth prove that line graphs of complete graphs and of complete bipartite graphs (except $K_{1,2}$) have SCDCs. Our proof that line graphs of all complete multipartite graphs (except $K_{1,2}$) have SCDCs uses the techniques that they developed, and so we begin with a review of some of their results. Lemmas 6, 7, 8, and 9 can all be found in [24].

A *transition* at a vertex, x, in a graph, G, is a pair of distinct edges $\{ax, xb\}$ incident with that vertex. Therefore, any cycle which passes through vertex x defines a transition at x. If $\mathcal{C}$ is a CDC of G, then the collection of cycles in $\mathcal{C}$ containing vertex x defines a *system of transitions*, denoted $T(x)$, consisting of all the transitions at x defined by the cycles in the collection. It is clear that for all $x \in V(G)$, $|T(x)| = d(x)$.

A vertex, x, in a graph, G, corresponds to a clique, $K(x)$, on $d(x)$ vertices, in the line graph, $L(G)$. If $xy \in E(G)$, then in $L(G)$, the cliques $K(x)$ and $K(y)$ intersect at the vertex xy. Since every complete graph, K_n, $n \geq 1$, has a perfect path double cover (PPDC) (see [2]), every clique $K(x)$ in $L(G)$ also has a PPDC, $\mathcal{P}(x)$. Thus by taking all of these PPDCs, we see that the line graph $L(G)$ has a PDC, $\mathbf{P}$.

Let G be a bridgeless graph, $\mathcal{C}$ a CDC of G, and $\mathbf{P}$ a PDC of the line graph, $L(G)$. The PDC, $\mathbf{P}$, of $L(G)$ and the CDC, $\mathcal{C}$, of G, are *compatible* if and only if, for every vertex $x \in V(G)$, there is a bijection between the transitions in the set $T(x)$ and the endpoints of the paths in $\mathcal{P}(x)$, requiring that every transition, $\{ax, xb\} \in T(x)$, corresponds to a path in $\mathcal{P}(x)$ with endpoints ax and xb. Let f_x be such a bijection. We will call

$$f_x : T(x) \to \mathcal{P}(x)$$

a *compatibility function.*

A simple test for compatibility between a CDC of a graph and a PDC of a line graph is desirable, and with this in mind, we introduce transition multigraphs and associated multigraphs.

The *transition multigraph,* $M_T(x)$, of a vertex $x \in V(G)$, is the graph on the vertices of the neighbour set, $N(x)$, with edges $\{ab : \{ax, xb\} \in T(x)\}$. Because the cycles defining the transitions in $T(x)$ are part of a CDC, each edge incident to x occurs in two transitions in $T(x)$. Therefore, $M_T(x)$ is 2-regular.

The *associated multigraph,* $M_{\mathcal{P}}(x)$, of a vertex $x \in V(G)$, is also a graph on the vertices of the set $N(x)$; endpoints vx and ux of a path in $\mathcal{P}(x)$ give rise to the edge vu in the associated multigraph $M_{\mathcal{P}}(x)$. Because $\mathcal{P}(x)$ is a PPDC of the clique $K(x)$, each vertex in the clique is an endpoint of two paths, and thus $M_{\mathcal{P}}(x)$ is 2-regular.

The concepts of transition, compatibility, transition multigraphs, and associated multigraphs are illustrated using the graph, G, shown in Figure 3.1. The graph has vertex set $V(G) = \{u, v, x, y, z\}$, and edge set $E(G) = \{uv, ux, uy, uz, vx, xy, xz, yz\}$.

Let $\mathcal{C}$ consist of the following cycles.

$$\begin{aligned} \text{C1} &= uvxu \\ \text{C2} &= uxyu \\ \text{C3} &= uyzu \\ \text{C4} &= xyzx \\ \text{C5} &= uvxzu \end{aligned}$$

It can easily be verified that $\mathcal{C}$ is a CDC of G. Vertex v in G has degree two, and corresponds to clique $K(v)$ in $L(G)$, a graph with vertices uv and vx connected by an edge which we write as $(uv)(vx)$.

In G, cycle C1 passes through v and defines the transition $\{uv, vx\}$ at v. The only other cycle in $\mathcal{C}$ passing through v is cycle C5, which also defines the transition $\{uv, vx\}$ at v. The paths P1 $= (uv)(vx)$ and P2 $= (uv)(vx)$ form a PPDC, $\mathcal{P}(v)$, of $K(v)$ and their endpoints, uv and vx for P1, and uv and vx for P2 correspond to the transitions at v defined by the cycles C1 and C5.

The transition multigraph of v, $M_T(v)$, consists of two copies of edge ux, one copy contributed by cycle C1 and the other by cycle C5, and so is a digon. The associated multigraph of v, $M_{\mathcal{P}}(v)$, also consists of two copies of edge ux, one copy contributed by path P1, and the other by path P2.

Cycles C1, C2, C3, and C5 pass through vertex u and define the system of transitions

$$T(u) = \{\{xu, uv\}, \{yu, ux\}, \{zu, uy\}, \{zu, uv\}\}.$$

The transition multigraph of u, $M_T(u)$, consists of edges xv, yx, zy, and zv, and is thus a 4-cycle through vertices x, y, z, and v.

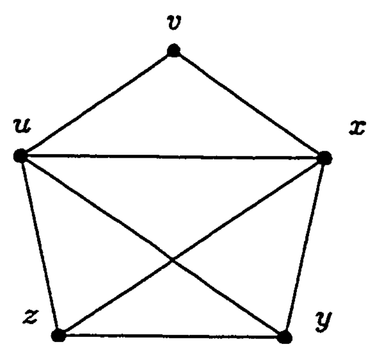

Figure 3.1: Graph G.

Paths

$$\begin{aligned}
P1 &= (uv)(uz)(uy)(ux) \\
P2 &= (uz)(ux)(uv)(uy) \\
P3 &= (ux)(uv)(uz)(uy) \\
P4 &= (uv)(uy)(ux)(uz)
\end{aligned}$$

form a PPDC, $\mathcal{P}(u)$, of $K(u)$, and their endpoints uv and ux, uz and uy, ux and uy, and uv and uz correspond to the transitions in $T(u)$. Like the transition multigraph $M_T(u)$, the associated multigraph $M_{\mathcal{P}}(u)$ consists of the 4-cycle through the vertices x, y, z, and v.

For $K(x)$, $K(y)$, and $K(z)$, the PPDCs $\mathcal{P}(x)$, $\mathcal{P}(y)$, and $\mathcal{P}(z)$ that follow complete a PDC of $L(G)$ that is compatible with the CDC, $\mathcal{C}$, of G.

$$\begin{aligned}
\mathcal{P}(x) = \; &\{(xv)(xu)(xy)(xz), \;\; (xu)(xz)(xv)(xy), \\
&(xv)(xz)(xy)(xu), \;\; (xy)(xv)(xu)(xz)\}
\end{aligned}$$

$$\mathcal{P}(y) = \{(yx)(yu)(yz), (yu)(yz)(yx), (yz)(yx)(yu)\}$$

$$\mathcal{P}(z) = \{(zu)(zx)(zy), (zx)(zy)(zu), (zy)(zu)(zx)\}$$

The following lemma shows that we may test for compatibility by examining the transition multigraphs and associated multigraphs of vertices of a graph.

Lemma 6 *Let G be a bridgeless graph, $\mathcal{C}$ a cycle double cover of G, and $\mathbf{P}$ a path double cover of the line graph, $L(G)$, consisting of perfect path double covers, $\mathcal{P}(x)$, of the vertex cliques $K(x)$, $x \in V(G)$. Then $\mathcal{C}$ and $\mathbf{P}$ are compatible if and only if, for each vertex $x \in V(G)$, the transition multigraph, $M_T(x)$, is isomorphic to the associated multigraph, $M_{\mathcal{P}}(x)$.*

Proof: Suppose the CDC, $\mathcal{C}$, and the PDC, $\mathbf{P}$, are compatible. Then for every vertex $v \in V(G)$, there exists a compatibility function $f_v : T(v) \to \mathcal{P}(v)$. For a particular vertex $x \in V(G)$, with a neighbour set $N(x) = \{y_1, y_2, \ldots, y_n\}$, the compatibility function induces a bijection, g_x, on the vertices in $N(x)$, where

$$g_x : V(M_T(x)) \to V(M_{\mathcal{P}}(x)),$$

and $g(y_i) = y_i$. If the edge $y_i y_j \in E(M_T(x))$, then $\{y_i x, x y_j\} \in T(x)$, and so $y_i x$ and $x y_j$ are endpoints of a path in $\mathcal{P}(x)$. Therefore, $y_i y_j \in E(M_{\mathcal{P}}(x))$. Similarly, if $y_i y_j \in E(M_{\mathcal{P}}(x))$, then $y_i y_j \in E(M_T(x))$, and so the function g_x is an isomorphism.

To prove the converse, again let $x \in V(G)$, and let x have the neighbour set $N(x) = \{y_1, y_2, \ldots, y_n\}$. Suppose g_x is an isomorphism between the multigraphs $M_T(x)$ and $M_{\mathcal{P}}(x)$. Then the edge $y_i y_j \in E(M_T(x))$ if and only if the edge

$$g_x(y_i) g_x(y_j) \in E(M_{\mathcal{P}}(x)).$$

Because $\mathcal{P}(x)$ is a PPDC of the clique $K(x)$, any permutation in the labelling of the vertices of $K(x)$ results in a PPDC. Therefore, we can choose labels of the vertices and thus also determine the labelling in the paths, so that g_x is the identity function. If, for every x, g_x is the identity function, then for all $x \in V(G)$, $y_iy_j \in E(M_T(x))$ if and only if $y_iy_j \in E(M_{\mathcal{P}}(x))$, and thus $\mathcal{C}$ and $\mathbf{P}$ are compatible.

■

Now that we have a method for determining the compatibility of a CDC of a graph and the PDC of its line graph, we will develop a technique for finding an SCDC of the line graph using our knowledge about compatibility. The following lemma is not directly applied to this problem but will be used in the proof of the subsequent lemma.

Lemma 7 *If H is a simple, eulerian graph with k vertices of degree four, and all other vertices of degree two, then H has a cycle decomposition with at most $k+1$ cycles.*

Proof: We proceed by induction on k. When $k = 0$, H is a cycle, and so the statement is clearly true. Now suppose the result holds for all simple, eulerian graphs with l vertices of degree four, $0 \leq l < k$, $k \geq 1$, whose remaining vertices all have degree two, and let H be a simple, eulerian graph with k vertices of degree four, and all other vertices of degree two. Construct an Euler tour of H. The tour passes through all vertices of degree two once, and all vertices of degree four twice. Because H is simple, this tour can be described by the sequence of vertices through which it passes. In such a sequence, there is at least one subsequence, starting and ending on the same vertex of degree four, but otherwise having no repetition of vertices.

This subsequence corresponds to a cycle, c, in the graph, H, and the removal of its edges and any isolated vertices results in a simple, eulerian graph, H', with at most $k-1$ vertices of degree four, and all other vertices of degree two. By the induction hypothesis, H' has a cycle decomposition with at most $(k-1)+1=k$ cycles; these cycles, along with the cycle, c, give us a cycle decomposition of H with at most $k+1$ cycles.

■

The next lemma gives us the desired technique for finding an SCDC of a line graph using compatible CDCs and PDCs.

Lemma 8 *Let G be a bridgeless graph, $\mathcal{C}$ a cycle double cover of G, and $\mathbf{P}$ a path double cover of the line graph, $L(G)$, consisting of the perfect path double covers, $\mathcal{P}(x)$, of the vertex cliques, $K(x), x \in V(G)$. Assume that $\mathcal{C}$ and $\mathbf{P}$ are compatible. For each vertex $u \in V(G)$, fix a compatibility function f_u from $T(u)$ to $\mathcal{P}(u)$. Let $c = v_0v_1 \ldots v_{q-1}v_0$ be a cycle in $\mathcal{C}$, and for each i, $0 \le i \le q-1$, let $f_i = f_{v_i}$. By the definition of a compatibility function, $f_i(\{v_{i-1}v_i, v_iv_{i+1}\}) = P_i$, where P_i is a path in $\mathcal{P}(v_i)$ with endpoints $v_{i-1}v_i$ and v_iv_{i+1} (subscripts taken modulo q). Then,*

(i) $E_c = \bigcup_{i=0}^{q-1} P_i$ is an eulerian subgraph of $L(G)$ with maximum degree at most four. Also,

(ii) if for each $c \in \mathcal{C}$, $D(c)$ is the set of cycles in a cycle decomposition of E_c, then $\mathbf{C} = \bigcup_{c \in \mathcal{C}} D(c)$ is a cycle double cover of $L(G)$, and $|\mathbf{C}| \le |\mathcal{C}| + ch(\mathcal{C}) = \gamma(\mathcal{C})$.

Proof:

(i) Because P_i is a path with endpoints $v_{i-1}v_i$ and v_iv_{i+1} (subscripts taken modulo q), an Euler tour of E_c can be constructed by joining, successively, all paths P_i,

$0 \leq i \leq q-1$. Therefore E_c is eulerian.

Notice that paths P_i and P_{i+1} both have the endpoint v_iv_{i+1}. Suppose a path P_j, $j \neq i$, $j \neq i+1$, also contains vertex v_iv_{i+1}. Then v_iv_{i+1} is a vertex in $K(v_j)$ and so in G, edge v_iv_{i+1} must be incident with vertex v_j. This is impossible as the edge v_iv_{i+1} has only two endpoints, v_i and v_{i+1}. Therefore, v_iv_{i+1} is an endpoint of two paths, P_i and P_{i+1}, and is not contained in any other path, and so, in E_c, we conclude that for all i, $0 \leq i \leq q-1$, vertex v_iv_{i+1} has degree two.

Let xy be a vertex in E_c that is not an endpoint of a path P_i, $0 \leq i \leq q-1$. Then $xy \notin \{v_0v_1, v_1v_2, \ldots, v_{q-1}v_0\}$, the set of endpoints of all paths P_i. We claim that xy lies in at most two paths P_i, $0 \leq i \leq q-1$. To see this, suppose that xy lies in three paths, P_i, P_j, and P_k, $0 \leq i, j, k \leq q-1$, i, j, k distinct. Then xy is a vertex in the cliques $K(v_i)$, $K(v_j)$, and $K(v_k)$, and so, in G, the three distinct vertices v_i, v_j, and v_k must all be incident to edge xy. This is impossible, hence xy lies in at most two paths P_i, $0 \leq i \leq q-1$.

If xy lies in one path P_i, $0 \leq i \leq q-1$, then it has degree two in E_c, whereas if it lies in two such paths, it has degree four in E_c. Suppose a vertex xy has degree four in E_c and lies in paths P_i and P_j, $0 \leq i, j \leq q-1$, $i \neq j$. Then $x = v_i$ and $y = v_j$, or $x = v_j$ and $y = v_i$. However, $xy \notin \{v_0v_1, v_1v_2, \ldots, v_{q-1}v_0\}$ so $i \neq j \pm 1$. Therefore, v_i and v_j are not adjacent in the cycle c and so the vertex xy of degree four in E_c corresponds to a chord in c.

(ii) Because $\mathcal{C}$ and $\mathbf{P}$ are compatible, there is a one-to-one correspondence between transitions of the cycles in $\mathcal{C}$ and paths in $\mathbf{P}$. Thus, the transitions of a cycle $c \in \mathcal{C}$ have a one-to-one correspondence with the paths in E_c, and thus, every path in $\mathbf{P}$ lies in exactly one E_c. Therefore, if $D(c)$ is the set of cycles in a cycle decomposition of E_c, $c \in \mathcal{C}$, then $\mathbf{C} = \bigcup_{c\in\mathcal{C}} D(c)$ is a CDC of $L(G)$.

Since each E_c is an eulerian graph with maximum degree four, where each vertex of degree four corresponds to a chord in the cycle, c, then by applying Lemma 7, we see that E_c has a decomposition into at most $1 + ch(c)$ cycles. There are $|\mathcal{C}|$ cycles in the CDC, and so

$$\begin{aligned} |\mathbf{C}| &\leq \sum_{c\in\mathcal{C}}(1 + ch(c)) \\ &= |\mathcal{C}| + ch(\mathcal{C}) \\ &= \gamma(\mathcal{C}). \end{aligned}$$

■

We have now determined that a graph, G, has a CDC, $\mathcal{C}$, and its line graph, $L(G)$, has a PDC, $\mathbf{P}$, which are compatible if and only if the associated multigraph and transition multigraph of each vertex are isomorphic (Lemma 6). We have also shown that if there is a CDC, $\mathcal{C}$, compatible with a PDC, $\mathbf{P}$, then there is a CDC, $\mathbf{C}$, of the line graph $L(G)$, with $|\mathbf{C}| \leq |\mathcal{C}| + ch(\mathcal{C})$ (Lemma 8).

Because a graph usually has fewer vertices and edges than its line graph, it is generally easier to directly find results concerning the graph, rather than the line graph. It would therefore be advantageous if we could describe a few associated multigraphs that arise naturally, and then attempt to find CDCs whose transition multigraphs are compatible. The following lemma describes associated multigraphs arising from cliques.

Lemma 9 *(i) The complete graph, K_{2m}, $m \geq 1$, has a perfect path double cover whose associated multigraph consists of m digons.*

(ii) The complete graph, K_{2m+1}, $m \geq 1$, has a perfect path double cover whose associated multigraph consists of one triangle and $m-1$ digons.

Proof: Let $m \geq 1$ and let $V = \{v_0, v_1, \ldots, v_{2m-1}\}$ be the set of vertices of K_{2m}.

(i) For $0 \leq i \leq 2m-1$, define the path P_i as $P_i = v_i v_{i+1} v_{i-1} v_{i+2} v_{i-2} \ldots v_{i+m+1} v_{i+m}$ (subscripts modulo $2m$). Each path P_i is a Hamilton path of K_{2m}, and each edge $v_j v_{j+k}$, $0 \leq j, k \leq 2m-1$, is in exactly one path P_i, $0 \leq i \leq m-1$, and one path P_l, $m \leq l \leq 2m-1$. Let $\mathcal{P} = \{P_i : 0 \leq i \leq 2m-1\}$. Because $P_i = P_{i+m}$ for $0 \leq i \leq m-1$, $\mathcal{P}$ is a PPDC of K_{2m} consisting of two copies of a decomposition into m Hamilton paths. Therefore, the associated multigraph, $M_{\mathcal{P}}(K_{2m})$, consists of m digons.

(ii) To the graph K_{2m}, add a new vertex, v_{2m}, and edges connecting this vertex to all vertices of K_{2m} to form the graph K_{2m+1}. In order to construct a PPDC, $\mathcal{Q}$, of this new graph, the paths P_i, $1 \leq i \leq 2m-1$ as described in part (i) can be modified by replacing the edges $v_i v_{i+1}$ by the edges $v_i v_{2m}$ and $v_{2m} v_{i+1}$. Note that the endpoints of these paths remain the same. Path P_0, described in part (i), is changed by adding the edge $v_{2m} v_0$ to the beginning of the path. Finally, a new path $P_* = v_{2m} v_1 v_2 \ldots v_{2m-1} v_0$ is created, and consists of edge $v_{2m} v_1$, which is only covered by P_1, and also, those edges lost when paths P_i, $1 \leq i \leq 2m-1$, were modified. For $1 \leq i \leq m-1$ and for $m+1 \leq i \leq 2m-1$, the paths P_i contribute a total of $m-1$ digons to the associated multigraph, $M_{\mathcal{Q}}(K_{2m+1})$. Path P_0 has endpoints v_{2m} and v_m, path P_* has endpoints v_{2m} and v_0, and path P_m has endpoints v_m and v_0. Therefore, these paths contribute the triangle $v_{2m} v_m v_0 v_{2m}$ to the associated multigraph, $M_{\mathcal{Q}}(K_{2m+1})$.

■

Chapter 4

Preliminary Results

In Lemma 8 in Chapter 3, we saw that if a graph has a CDC, $\mathcal{C}$, which is compatible with a PDC of its line graph, then the line graph has an SCDC, $\mathbf{C}$, where $|\mathbf{C}| \leq |\mathcal{C}| + ch(\mathcal{C}) = \gamma(\mathcal{C})$. Lemma 9 describes two typical associated multigraphs of the vertex cliques of the line graph, one occurring when the number of vertices in the clique is even, the other occurring when the number is odd. These associated multigraphs are composed of a collection of digons, and a triangle and a collection of digons, respectively. In Lemma 6, also in Chapter 3, it was proved that a CDC of a graph and a PDC of its line graph are compatible if and only if, for each vertex, the associated multigraph and transition multigraph are isomorphic. In order to prove that a line graph has an SCDC, all that remains to be shown is that its graph has an appropriate CDC. This chapter and the one that follows are devoted to proving the result that line graphs of all complete multipartite graphs except $K_{1,2}$ have SCDCs. This result is obtained by showing that, for a complete multipartite graph, G (G is not $K_{1,2}$), G has a CDC, $\mathcal{C}_G$, such that:

1. For each vertex of G, the transition multigraph defined by $\mathcal{C}_G$ consists of a collection of digons or of a triangle and a collection of digons, and

2. $\gamma(\mathcal{C}_G) = |\mathcal{C}_G| + ch(\mathcal{C}_G) < |E(G)|$, where $|E(G)|$ is the number of vertices in $L(G)$.

4.1 Graph Decomposition

In order to prove that a complete multipartite graph, G, has a CDC, $\mathcal{C}_G$, with appropriate properties, we will often analyze parts of the graph separately. The following lemma shows the validity of this method, and also shows how the value of $\gamma(\mathcal{C}_G)$ is obtained.

Lemma 10 *Let G be a graph with subgraphs $H_1, \ldots, H_n$ that partition the edge set $E(G)$. For each i, $1 \leq i \leq n$, suppose H_i has a cycle double cover, $\mathcal{C}_{H_i}$. Then $\mathcal{C}_G = \bigcup_{i=1}^{n} \mathcal{C}_{H_i}$ is a cycle double cover of G. Furthermore, if $\gamma(\mathcal{C}_{H_i}) \leq |E(H_i)|$ for all i, $1 \leq i \leq n$, with strict inequality for at least one i, then $\gamma(\mathcal{C}_G) < |E(G)|$.*

Proof: Since the subgraphs $H_1, \ldots, H_n$ partition $E(G)$,

$$|E(G)| = \sum_{i=0}^{n} |E(H_i)|.$$

Also, because these same subgraphs partition $E(G)$, and because $\mathcal{C}_{H_i}$ is the CDC of H_i, $1 \leq i \leq n$, it follows that $\mathcal{C}_{H_i} \cap \mathcal{C}_{H_j} = \emptyset$ for all $i \neq j$. Therefore, $\mathcal{C}_G = \bigcup_{i=1}^{n} \mathcal{C}_{H_i}$ is a CDC of G, and $|\mathcal{C}_G| = \sum_{i=1}^{n} |\mathcal{C}_{H_i}|$.

We claim that $ch(\mathcal{C}_G) = \sum_{i=1}^{n} ch(\mathcal{C}_{H_i})$. Let c be a cycle in $\mathcal{C}_{H_i}$, and let e be a chord of c. The cycle c is also a cycle in $\mathcal{C}_G$, and still has chord e. Conversely, suppose that a cycle, c, of $\mathcal{C}_G$, through vertices u and v of G, has a chord uv. There is a unique i, $1 \leq i \leq n$, such that $c \in \mathcal{C}_{H_i}$, and so u and v must be vertices in the subgraph H_i. Therefore, uv is a chord in $G[V(H_i)]$, the subgraph induced by the vertices of H_i, and so by the definition of the ch function, uv is counted as a chord in $ch(H_i)$. Thus

$$\gamma(\mathcal{C}_G) \quad = \quad |\mathcal{C}_G| + ch(\mathcal{C}_G)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |\mathcal{C}_{H_i}| + \sum_{i=1}^{n} ch(\mathcal{C}_{H_i}) \\
&= \sum_{i=1}^{n} [|\mathcal{C}_{H_i}| + ch(\mathcal{C}_{H_i})] \\
&= \sum_{i=1}^{n} \gamma(\mathcal{C}_{H_i}) \\
&\leq \sum_{i=1}^{n} |E(H_i)| = |E(G)|.
\end{aligned}
$$

Therefore, $\gamma(\mathcal{C}_G) \leq |E(G)|$, and if $\gamma(\mathcal{C}_{H_i}) < |E(H_i)|$ for at least one i, $1 \leq i \leq n$, then $\gamma(\mathcal{C}_G) < |E(G)|$.

■

4.2 Cycle Decomposition

As previously discussed, two copies of a cycle decomposition of a graph form a cycle double cover of that same graph. The following lemma confirms this property, and describes the resulting transition multigraphs.

Lemma 11 *Let G be an eulerian graph and let $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ be a cycle decomposition of G. Then G has a cycle double cover, $\mathcal{C}'$, that consists of two copies of $\mathcal{C}$. Furthermore, the transition multigraph $M_T(v)$ of each vertex v in the vertex set $V(G)$ consists of digons.*

Proof: Since $\mathcal{C}$ is a cycle decomposition, every edge in G is covered exactly once by a cycle c_i, $1 \leq i \leq n$, in $\mathcal{C}$. Therefore, two copies of $\mathcal{C}$, denoted $\mathcal{C}'$, cover the edges of G twice, so $\mathcal{C}'$ is a CDC of G.

Suppose a cycle $c \in \mathcal{C}$ passes through the vertex v. Then, c induces a transition $\{av, vb\}$, and so ab is an edge in the transition multigraph, $M_T(v)$. Since $\mathcal{C}'$ has two copies of c, the transition multigraph, $M_T(v)$, has a second copy of ab, and thus, $M_T(v)$ has a digon on the vertices a and b. This fact holds for all cycles in C passing through v, and thus $M_T(v)$ consists of digons.

∎

4.3 CDCs of Some Classes of Graphs

The remainder of this chapter is devoted to showing that complete bipartite and complete 3-partite graphs, as well as certain complete 4-partite graphs have CDCs with suitable numbers of cycles and chords and appropriate transition multigraphs. In showing this, not only do we prove that their line graphs have SCDCs, but also in conjunction with Lemma 10, we can use these graphs in the following chapter to prove that the line graph of any complete multipartite graph except $K_{1,2}$ has an SCDC.

4.3.1 Bipartite graphs

We start by finding CDCs of complete bipartite graphs with at least two vertices in each part. Three separate treatments are required and are determined by the parity of the parts.

Lemma 12 *If G is a complete bipartite graph with parts X and Y, where both $|X|$ and $|Y|$ are even, then G has a decomposition, $\mathcal{C}_G$, consisting of chordless cycles of length four. Furthermore, $\gamma(\mathcal{C}_G) = |E(G)|/4$.*

Proof: Let $X = \{x_0, x_1, \ldots, x_{m-1}\}$ and let $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. For $0 \leq i \leq (m-2)/2$ and $0 \leq j \leq (n-2)/2$, let

$$C_{ij} = x_{2i}y_{2j}x_{2i+1}y_{2j+1}x_{2i}.$$

We claim that

$$\mathcal{C} = \{C_{ij} \mid 0 \leq i \leq (m-2)/2, 0 \leq j \leq (n-2)/2\}$$

is a cycle decomposition of G.

Let $x_a y_b \in E(G)$. Then for some integers i and j, where $a = 2i$ or $a = 2i+1$, and $b = 2j$ or $b = 2j+1$, the edge, $x_a y_b$, is in the cycle $x_{2i}y_{2j}x_{2i+1}y_{2j+1}x_{2i}$. Therefore, every edge $x_a y_b$ is in at least one cycle. The number of cycles in $\mathcal{C}_G$ is $((m-2)/2+1)((n-2)/2+1) = mn/4$, and thus the number of edges covered by the 4-cycles is $4(mn/4) = mn = |E(G)|$. Therefore, as each edge of G is in at least one cycle, and as the number of edges in the graph is equal to the number of edges in the 4-cycles, each edge is in exactly one 4-cycle.

The vertices in each 4-cycle of $\mathcal{C}_G$ alternate between the two parts of G. Therefore, non-consecutive vertices in these cycles are in the same part, and so are non-adjacent in G. Thus, the 4-cycles in $\mathcal{C}_G$ are chordless. Consequently, $\gamma(\mathcal{C}_G) = |\mathcal{C}_G| = |E(G)|/4$.

■

Notice that in the construction just described, the vertices in each part can easily be paired, with X having pairs $\{x_0, x_1\}, \{x_2, x_3\}, \ldots, \{x_{m-2}, x_{m-1}\}$ and Y having pairs $\{y_0, y_1\}, \{y_2, y_3\}, \ldots, \{y_{n-2}, y_{n-1}\}$. These pairs now define the cycles of the cycle decomposition: cycle C_{ij} has, as its vertex set, the i^{th} pair of X and the j^{th} pair of Y. This pairing will be relevant in later applications of this lemma.

By applying Lemma 11 to this result, we immediately get:

Corollary 13 *If G is a complete bipartite graph with parts X and Y, where both $|X|$ and $|Y|$ are even, then G has a cycle double cover, C_G, composed of chordless 4-cycles, such that $\gamma(C_G) < |E(G)|$. Furthermore, for each $v \in V(G)$, the transition multigraph $M_T(v)$ consists of $d(v)/2$ digons.*

Now, by applying Lemmas 6, 8, and 9, we get:

Corollary 14 *If G is a complete bipartite graph with parts X and Y, where both $|X|$ and $|Y|$ are even, then the line graph, $L(G)$, has a small cycle double cover.*

The following lemma uses a construction similar to that of Lemma 12, but with a slight modification necessary to compensate for the part of odd size.

Lemma 15 *If G is a complete bipartite graph with partitions X and Y, where $|X|$ is odd and at least three, and $|Y|$ is even, then G has a cycle double cover, C_G, consisting of chordless cycles of length four, with $\gamma(C_G) < |E(G)|$. Furthermore, for each $x \in X$, $M_T(x)$ consists of $|Y|/2$ digons, and for each $y \in Y$, $M_T(y)$ consists of one triangle and $(|X| - 3)/2$ digons.*

Proof: Let $X = \{x_0, x_1, \ldots, x_{m-1}\}$, let $X' = X \backslash \{x_{m-1}, x_{m-2}, x_{m-3}\}$, and let $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. Let C_G consist of the following cycles.

Cycle Type	Cycles		Number of Cycles
C1	$y_{2j}x_{m-1}y_{2j+1}x_{m-2}y_{2j}$,	$0 \leq j \leq \frac{n-2}{2}$	$\frac{n}{2}$
C1	$y_{2j}x_{m-2}y_{2j+1}x_{m-3}y_{2j}$,	$0 \leq j \leq \frac{n-2}{2}$	$\frac{n}{2}$
C1	$y_{2j}x_{m-3}y_{2j+1}x_{m-1}y_{2j}$,	$0 \leq j \leq \frac{n-2}{2}$	$\frac{n}{2}$
C2 (2 copies)	$x_{2i}y_{2j}x_{2i+1}y_{2j+1}x_{2i}$,	$\left\{\begin{array}{l} 0 \leq i \leq \frac{m-5}{2} \\ 0 \leq j \leq \frac{n-2}{2} \end{array}\right\}$	$2(\frac{m-3}{2})(\frac{n}{2})$

We see that for each j, $0 \leq j \leq (n-2)/2$, the edges $x_{m-1}y_{2j}$ and $x_{m-1}y_{2j+1}$ lie in exactly two of the C1 cycles. Similarly, for each j, $0 \leq j \leq (n-2)/2$, the edges $x_{m-2}y_{2j}$, $x_{m-2}y_{2j+1}$, $x_{m-3}y_{2j}$, and $x_{m-3}y_{2j+1}$ each lie in exactly two of the C1 cycles. Therefore, all edges incident with x_{m-1}, x_{m-2}, and x_{m-3} are covered twice. The C2 cycles form a cycle decomposition of the complete bipartite graph with parts X' and Y (see Lemma 12), and thus all edges incident with vertices of X' are covered twice by the copies of the C2 cycles. Because G is a bipartite graph, every edge of G is incident with some vertex in X, and so, $\mathcal{C}_G$ covers all edges of G twice.

The CDC $\mathcal{C}_G$ consists only of 4-cycles; since G is bipartite, these 4-cycles must be chordless. Therefore,

$$\gamma(\mathcal{C}_G) = 2(mn/4) < mn = |E(G)|.$$

The transition multigraph for each vertex in X consists of two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-2)/2$, and thus consists of $n/2 = |Y|/2$ digons. The transition multigraph of each vertex in the set Y has edges $x_{m-1}x_{m-2}$, $x_{m-2}x_{m-3}$, and $x_{m-3}x_{m-1}$, resulting from the C1 cycles, and two of each edge $x_{2i}x_{2i+1}$, $0 \leq i \leq (m-5)/2$, resulting from the two copies of the C2 cycles. These transition multigraphs, therefore, consist of the triangle $x_{m-1}x_{m-2}x_{m-3}x_{m-1}$ and $(m-3)/2 = (|X|-3)/2$ digons.

■

The next corollary follows immediately from the previous lemma and Lemmas 6, 8, and 9.

Corollary 16 *If G is a complete bipartite graph with parts X and Y, where $|X|$ is odd and at least three, and $|Y|$ is even, then the line graph $L(G)$ has a small cycle double cover.*

As part of Theorem 10 in MacGillivray and Seyffarth [24], the following is proved.

Lemma 17 *If G is a complete bipartite graph with parts X and Y, where both $|X|$ and $|Y|$ are odd and at least three, then G has a cycle double cover, C_G, consisting of one cycle of length six and $(mn-3)/2$ cycles of length four, with $C_G < |E(G)|$. Furthermore, for each vertex $v \in V(G)$, the transition multigraph $M_T(v)$ consists of one triangle and $(d(v)-3)/2$ digons.*

Proof: Let $X = \{x_0, x_1, \ldots, x_{m-1}\}$, $X' = \{x_{m-3}, x_{m-2}, x_{m-1}\}$, and $X'' = X \backslash X'$. Similarly, let $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, $Y' = \{y_{n-3}, y_{n-2}, y_{n-1}\}$, and $Y'' = Y \backslash Y'$. We will define four subgraphs, H, J_1, J_2, and L, as follows. Let H be the subgraph induced by the vertices of X' and Y', let J_1 be the subgraph induced by the vertices in X' and Y'', let J_2 be the subgraph induced by the vertices in X'' and Y', and let L be the subgraph induced by the vertices in X'' and Y''. The subgraphs are all complete bipartite graphs, and together, they partition the edge set of G.

The subgraph H is simply $K_{3,3}$. Let $\mathcal{C}_H$ be composed of the following cycles.

Cycle Type	Cycles
C1	$x_{m-3}y_{n-3}x_{m-2}y_{n-2}x_{m-3}$
C1	$x_{m-2}y_{n-2}x_{m-1}y_{n-1}x_{m-2}$
C1	$x_{m-1}y_{n-1}x_{m-3}y_{n-3}x_{m-1}$
C2	$x_{m-3}y_{n-2}x_{m-1}y_{n-3}x_{m-2}y_{n-1}x_{m-3}$

Each edge $x_{m-i}y_{n-i}$, $i = 1, 2, 3$, is in two of the C1 cycles, while each edge $x_{m-i}y_{n-j}$, $i, j = 1, 2, 3$, $i \neq j$, is in one C1 cycle and the C2 cycle. Therefore, $\mathcal{C}_H$ is a CDC of H.

There are a total of four cycles in $\mathcal{C}_H$. None of the 4-cycles have chords, and the 6-cycle has three chords, so

$$\gamma(\mathcal{C}_H) = |\mathcal{C}_H| + ch(\mathcal{C}_H) = 4 + 3 = 7 < 9 = |E(H)|.$$

For each vertex in the sets X' and Y', $\mathcal{C}_H$ contributes a triangle to the transition multigraph. If $|X| = 3$ and $|Y| = 3$, then $\mathcal{C}_H$ is the CDC of G and we are done. Henceforth, we can assume $|X| > 3$ or $|Y| > 3$.

The two subgraphs, J_1 and J_2, are complete bipartite graphs with one part of even size, and one part of odd size. Therefore, by Lemma 15, we know that they have CDCs, $\mathcal{C}_{J_1}$ and $\mathcal{C}_{J_2}$, where $\gamma(\mathcal{C}_{J_i}) < |E(J_i)|$, $i = 1, 2$. These CDCs contribute $|Y''|/2$ and $|X''|/2$ digons, respectively, to the transition multigraph of each vertex in the sets X' and Y', and a triangle to the transition multigraph of each vertex in the sets X'' and Y''.

The subgraph L is a complete bipartite graph with two parts of even size, and so by Corollary 13, we know L has a CDC, $\mathcal{C}_L$, with $\gamma(\mathcal{C}_L) < |E(L)|$, that contributes $|Y''|/2$ and $|X''|/2$ digons, respectively, to the transition multigraph of each vertex in X'' and Y''.

Because the subgraphs H, J_1, J_2, and L partition the edge set of G, and because $\gamma(\mathcal{C}_H) < |E(H)|$, $\gamma(\mathcal{C}_{J_i}) < |E(J_i)|$, $i = 1, 2$, and $\gamma(\mathcal{C}_L) < |E(L)|$, then by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The CDC of H contributes a triangle to the transition multigraph of each vertex in X' and Y'. The CDCs of J_1 and J_2 contribute a triangle to the transition multigraph of each vertex in X'' and Y'', and $|Y''|/2$ and $|X''|/2$ digons, respectively, to the transition multigraph of each vertex in X' and Y'. Finally, the CDC of L contributes $|Y''|/2$ and $|X''|/2$ digons, respectively, to the transition multigraph of each vertex in X'' and Y''. Therefore, the transition multigraph of each vertex $x \in X$ consists of a triangle and $|Y''|/2 = (d(x) - 3)/2$ digons, and the transition multigraph of each vertex $y \in Y$ consists of a triangle and $|X''|/2 = (d(y) - 3)/2$ digons.

■

By applying the results of the previous lemma and Lemmas 6, 8, and 9, we obtain the following:

Corollary 18 *If G is a complete bipartite graph with parts X and Y, where both $|X|$ and $|Y|$ are odd and at least three, then the line graph $L(G)$ has a small cycle double cover.*

4.3.2 Complete Graphs

In Theorem 9 of MacGillivray and Seyffarth [24], it is shown that for all $n \geq 2$, $L(K_n)$ has a small cycle double cover. The proof uses the following, which we state without proof.

Lemma 19 *For all integers $n \geq 3$, K_n has a cycle decomposition into triangles if and only if $n \equiv 1, 3 \pmod 6$.*

The truth of this result can be verified as follows. In the graph K_n, every pair of vertices is joined by an edge. Therefore, in order to decompose the graph into triangles, we want each pair of vertices to appear together once, in a grouping of three vertices. The required groupings of vertices can be thought of as blocks in a Steiner Triple System, which exists if and only if the number of varieties (in this case, vertices) is $n \equiv 1, 3 \pmod 6$ (see [19]).

As an immediate consequence of this Lemma 19 and of Lemma 11, we have

Corollary 20 *For all integers $n \geq 3$, $n \equiv 1, 3 \pmod 6$, K_n has a cycle double cover such that for each $v \in V(G)$, the transition multigraph $M_T(v)$ consists of $d(v)/2$ digons.*

4.3.3 3-partite Graphs

In the following theorem, we use results obtained in Corollary 13, Lemma 15, and Lemma 17 (the results concerning complete bipartite graphs) to find CDCs with appropriate properties of all complete 3-partite graphs.

Theorem 21 *If G is a complete 3-partite graph, then G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) \leq |E(G)|$. Except for the graph with parts of size one, two, and two, this inequality is strict. Furthermore, depending on the parity of the parts, the transition multigraph of each vertex in G consists of either a collection of digons or a triangle and a collection of digons.*

Proof: Let G be a complete 3-partite graph, with parts $X = \{x_0, x_1, \ldots, x_{m-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, and $Z = \{z_0, z_1, \ldots, z_{p-1}\}$.

In developing a technique for the construction of CDCs of complete 3-partite graphs, the method used to cover the edges of a complete bipartite graph with two parts of even size (see Lemma 12) is used as a model. Variations in the descriptions of the CDCs occur because some 3-partite graphs contain parts of odd size.

Case 1. m, n, and p even.

Since the three parts each have an even number of vertices, the subgraphs induced by the vertices in X and Y, the vertices in X and Z, and the vertices in Y and Z, are each complete bipartite graphs with parts of even size. Therefore, Lemma 12 can be applied to each subgraph, in turn, and thus G has a decomposition into chordless 4-cycles. Two copies of this decomposition form a CDC, $\mathcal{C}_G$, of G, and consequently, the transition multigraph of each vertex $v \in V(G)$ consists of $d(v)/2$ digons. Every

4-cycle in $\mathcal{C}_G$ contains vertices from two parts, the vertices from each part appearing alternately along the cycle. Therefore, all 4-cycles in $\mathcal{C}_G$ are chordless. Since

$$|E(G)| = mn + mp + np,$$

we get

$$\gamma(\mathcal{C}_G) = 2(mn/4 + mp/4 + np/4) = (mn + mp + np)/2 < |E(G)|.$$

Case 2. m and n even, p odd.

Let $X' = X \backslash \{x_{m-1}, x_{m-2}\}$, $Y' = Y \backslash \{y_{n-1}, y_{n-2}\}$, and $Z' = Z \backslash \{z_{p-1}\}$. Let C_G consist of the following cycles.

Cycle Type	Cycles		Number of Cycles
C1	$y_{2j}z_{p-1}y_{2j+1}x_{m-1}y_{2j}$,	$0 \leq j \leq \frac{n-2}{2}$	$\frac{n}{2}$
C1	$y_{2j}z_{p-1}y_{2j+1}x_{m-2}y_{2j}$,	$0 \leq j \leq \frac{n-2}{2}$	$\frac{n}{2}$
C1	$x_{2i}z_{p-1}x_{2i+1}y_{n-1}x_{2i}$,	$0 \leq i \leq \frac{m-2}{2}$	$\frac{m}{2}$
C1	$x_{2i}z_{p-1}x_{2i+1}y_{n-2}x_{2i}$,	$0 \leq i \leq \frac{m-2}{2}$	$\frac{m}{2}$
C2	$x_{m-1}y_{2j}x_{m-2}y_{2j+1}x_{m-1}$,	$0 \leq j \leq \frac{n-4}{2}$	$\frac{n-2}{2}$
C2	$y_{n-1}x_{2i}y_{n-2}x_{2i+1}y_{n-1}$,	$0 \leq i \leq \frac{m-4}{2}$	$\frac{m-2}{2}$
C3 (2 copies)	$x_{2i}y_{2j}x_{2i+1}y_{2j+1}x_{2i}$,	$\left\{ \begin{array}{l} 0 \leq i \leq \frac{m-4}{2} \\ 0 \leq j \leq \frac{n-4}{2} \end{array} \right\}$	$2(\frac{m-2}{2})(\frac{n-2}{2})$
C4 (2 copies)	$x_{2i}z_{2l}x_{2i+1}z_{2l+1}x_{2i}$,	$\left\{ \begin{array}{l} 0 \leq i \leq \frac{m-2}{2} \\ 0 \leq l \leq \frac{p-3}{2} \end{array} \right\}$	$2(\frac{m}{2})(\frac{p-1}{2})$
C5 (2 copies)	$y_{2j}z_{2l}y_{2j+1}z_{2l+1}y_{2j}$,	$\left\{ \begin{array}{l} 0 \leq j \leq \frac{n-2}{2} \\ 0 \leq l \leq \frac{p-3}{2} \end{array} \right\}$	$2(\frac{n}{2})(\frac{p-1}{2})$

For each j, $0 \leq j \leq (n-2)/2$, the edges $z_{p-1}y_{2j}$ and $z_{p-1}y_{2j+1}$ lie in two of the C1 cycles. Similarly, for each i, $0 \leq i \leq (m-2)/2$, the edges $z_{p-1}x_{2i}$ and $z_{p-1}x_{2i+1}$ lie in two of the C1 cycles. As z_{p-1} lies in no other cycles of $\mathcal{C}_G$, every edge incident with this vertex is covered twice.

The C4 cycles form a cycle decomposition of the complete bipartite graph with parts Z' and X, and the C5 cycles form a cycle decomposition of the complete bipartite graph with parts Z' and Y. Vertices of Z' do not lie in any other cycles of $\mathcal{C}_G$, and thus because two copies of the C4 cycles and the C5 cycles are used, all edges incident with vertices of Z' are covered twice.

For each j, $0 \leq j \leq (n-2)/2$, the edges $x_{m-1}y_{2j}$, $x_{m-1}y_{2j+1}$, $x_{m-2}y_{2j}$, and $x_{m-2}y_{2j+1}$ are each covered exactly twice, once by the C1 cycles and once by the C2 cycles. Similarly, for each i, $0 \leq i \leq (m-2)/2$, the edges $y_{n-1}x_{2i}$, $y_{n-1}x_{2i+1}$, $y_{n-2}x_{2i}$, and $y_{n-2}x_{2i+1}$ are each covered exactly twice, once by the C1 cycles and once by the C2 cycles. Thus all edges incident with vertices x_{m-1}, x_{m-2}, y_{n-1}, and y_{n-2} are covered twice.

Finally, the C3 cycles form a cycle decomposition of the complete bipartite graph with parts X' and Y'. Edges between vertices in these two parts do not lie in any other cycles of $\mathcal{C}_G$, and so because two copies of the C3 cycles are used, all edges incident with vertices in X' and Y' are covered twice. Combining these results, we see that all edges in G are covered exactly twice.

Because the CDC of G consists only of 4-cycles, $|\mathcal{C}_G| = 2|E(G)|/4$, and thus,

$$\begin{aligned}\gamma(\mathcal{C}_G) &= 2|E(G)|/4 + ch(\mathcal{C}_G)\\ &= |E(G)|/2 + ch(\mathcal{C}_G).\end{aligned}$$

No 4-cycle in $\mathcal{C}_G$ has vertices from more than three different parts, and so no 4-cycle in $\mathcal{C}_G$ has more than one chord. Recalling that β_0 denotes the number of chordless cycles in $\mathcal{C}_G$, $ch(\mathcal{C}_G) = |\mathcal{C}_G| - \beta_0$. Therefore,

$$\begin{aligned}\gamma(\mathcal{C}_G) &= |E(G)|/2 + (|E(G)|/2 - \beta_0) \\ &= |E(G)| - \beta_0 \\ &< |E(G)|\end{aligned}$$

if there is at least one cycle with no chord.

By examining the CDC, we can see that the C2 cycles, the C3 cycles, the C4 cycles, and the C5 cycles are all chordless. The C2 cycles and the C3 cycles exist if $m > 2$ and $n > 2$, the C4 cycles exist if $m > 2$ and $p > 1$, and the C5 cycles exist if $n > 2$ and $p > 1$. Therefore, unless $m = 2$, $n = 2$, and $p = 1$, there is at least one 4-cycle with no chords in $\mathcal{C}_G$.

The transition multigraph of vertex z_{p-1} consists of two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-2)/2$, and $x_{2i}x_{2i+1}$, $0 \leq i \leq (m-2)/2$, resulting from the C1 cycles, and thus $M_T(z_{p-1})$ consists of $n/2 + m/2$ digons. The transition multigraphs of vertices in Z' each have two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-2)/2$, and two copies of edges $x_{2i}x_{2i+1}$, $0 \leq i \leq (m-2)/2$, resulting from the C4 and C5 cycles, respectively, and so, as with the transition multigraph of vertex z_{p-1}, each consists of $n/2 + m/2$ digons.

The transition multigraphs of vertices x_{m-1} and x_{m-2} have one copy each of edges $z_{p-1}y_{n-1}$, $z_{p-1}y_{n-2}$, and $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-2)/2$, resulting from the C1 cycles. As well, $M_T(x_{m-1})$ and $M_T(x_{m-2})$ each have edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-4)/2$, resulting from the C2 cycle in which the vertices both appear. Both have two copies of edges

$z_{2l}z_{2l+1}$, $0 \leq l \leq (p-3)/2$, resulting from the C4 cycles. Thus both $M_T(x_{m-1})$ and $M_T(x_{m-2})$ consist of the triangle $z_{p-1}y_{n-2}y_{n-1}z_{p-1}$ and $(n-2)/2+(p-1)/2$ digons. The transition multigraph of each vertex in X' has edges $z_{p-1}y_{n-1}$ and $z_{p-1}y_{n-2}$, resulting from the C1 cycles, edge $y_{n-1}y_{n-2}$, from the C2 cycles, two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-4)/2$, from the C3 cycles and two copies of edges $z_{2l}z_{2l+1}$, $0 \leq l \leq (p-3)/2$, from the C5 cycles. Thus the transition multigraph of each vertex in X' consists of the triangle $z_{p-1}y_{n-2}y_{n-1}z_{p-1}$ and $(n-2)/2+(p-1)/2$ digons.

Because of the symmetry of X and Y in the description of the cycles, the transition multigraph of each vertex in Y also consists of a triangle and a collection of digons. In this case, $z_{p-1}x_{m-2}x_{m-1}z_{p-1}$ is the triangle, and the number of digons is $(m-2)/2+(p-1)/2$.

Case 3. m, n, and p odd.

Let $X' = X\backslash\{x_{m-1}\}$, $Y' = Y\backslash\{y_{n-1}\}$, and $Z' = Z\backslash\{z_{p-1}\}$. Let $\mathcal{C}_G$ consist of two copies of the following cycle decomposition.

Cycle Type	Cycles		Number of Cycles
Triangle	$x_{m-1}y_{n-1}z_{p-1}x_{m-1}$		1
C1	$y_{2j}z_{p-1}y_{2j+1}x_{m-1}y_{2j}$,	$0 \le j \le \frac{n-3}{2}$	$(n-1)/2$
C1	$x_{2i}z_{p-1}x_{2i+1}y_{n-1}x_{2i}$,	$0 \le i \le \frac{m-3}{2}$	$(m-1)/2$
C1	$z_{2l}x_{m-1}z_{2l+1}y_{n-1}z_{2l}$,	$0 \le l \le \frac{p-3}{2}$	$(p-1)/2$
C2	$x_{2i}y_{2j}x_{2i+1}y_{2j+1}x_{2i}$,	$\left\{\begin{array}{c} 0 \le i \le \frac{m-3}{2} \\ 0 \le j \le \frac{n-3}{2} \end{array}\right\}$	$((m-1)/2)((n-1)/2)$
C2	$x_{2i}z_{2l}x_{2i+1}z_{2l+1}x_{2i}$,	$\left\{\begin{array}{c} 0 \le i \le \frac{m-3}{2} \\ 0 \le l \le \frac{p-3}{2} \end{array}\right\}$	$((m-1)/2)((p-1)/2)$
C2	$y_{2j}z_{2l}y_{2j+1}z_{2l+1}y_{2j}$,	$\left\{\begin{array}{c} 0 \le j \le \frac{n-3}{2} \\ 0 \le l \le \frac{p-3}{2} \end{array}\right\}$	$((n-1)/2)((p-1)/2)$

Edges $x_{m-1}y_{n-1}$ and $x_{m-1}z_{p-1}$ are covered exactly once by the triangle, while edges $x_{m-1}y_{2j}$ and $x_{m-1}y_{2j+1}$, $0 \le j \le (n-3)/2$, and edges $x_{m-1}z_{2l}$ and $x_{m-1}z_{2l+1}$, $0 \le l \le (p-3)/2$, are covered exactly once by the C1 cycles.

For the remaining edges incident with the vertices of X, notice that for any i, $0 \le i \le (m-3)/2$, the edges $x_{2i}y_{2j}$, $x_{2i}y_{2j+1}$, $x_{2i+1}y_{2j}$, and $x_{2i+1}y_{2j+1}$, $0 \le j \le (n-3)/2$, and also the edges $x_{2i}z_{2l}, x_{2i}z_{2l+1}$, $x_{2i+1}z_{2l}$, and $x_{2i+1}z_{2l+1}$, $0 \leq l \leq (p-3)/2$, are covered exactly once, in this case by the C2 cycles. Thus all edges incident with vertices of X are covered exactly once.

Because of the symmetry of X, Y, and Z in the cycles described in this case, all edges incident with vertices of Y and Z are also covered exactly once.

The cycle decomposition consists of one triangle and $(|E(G)| - 3)/4$ cycles of length four. As two copies of this decomposition are used for the CDC, $\mathcal{C}_G$,

$$\begin{aligned} \gamma(\mathcal{C}_G) &= |\mathcal{C}_G| + ch(\mathcal{C}_G) \\ &= 2\left[1 + \frac{|E(G)| - 3}{4}\right] + ch(\mathcal{C}_G) \\ &= (|E(G)| + 1)/2 + ch(\mathcal{C}_G). \end{aligned}$$

A triangle has no chords since it is the complete graph on three vertices. As in Case 2, no 4-cycle in $\mathcal{C}_G$ has vertices from more than three parts, and so no 4-cycle has more than one chord. Since $\mathcal{C}_G$ has $(|E(G)| - 3)/2$ cycles of length four, $ch(\mathcal{C}_G) \leq (|E(G)| - 3)/2$. Therefore,

$$\begin{aligned} \gamma(\mathcal{C}_G) &= (|E(G)| + 1)/2 + ch(\mathcal{C}_G) \\ &\leq (|E(G)| + 1)/2 + (|E(G)| - 3)/2 \\ &= |E(G)| - 1 < |E(G)|. \end{aligned}$$

Because two copies of a cycle decomposition of G are used for the CDC, the transition multigraph of each vertex $v \in V(G)$ consists of $d(v)/2$ digons.

Case 4. m and n odd, p even.

Let $X' = X \backslash \{x_{m-1}\}$, $Y' = Y \backslash \{y_{n-1}\}$, and $Z' = Z \backslash \{z_{p-1}, z_{p-2}\}$. Let $\mathcal{C}_G$ consist of the following cycles.

Cycle Type	Cycles		Number of Cycles
Triangle	$x_{m-1}z_{p-1}y_{n-1}x_{m-1}$		1
Triangle	$x_{m-1}z_{p-2}y_{n-1}x_{m-1}$		1
C1	$x_{m-1}z_{p-1}y_{n-1}z_{p-2}x_{m-1}$		1
C2	$y_{2j}z_{p-1}y_{2j+1}x_{m-1}y_{2j},$	$0 \le j \le \frac{n-3}{2}$	$(n-1)/2$
C2	$y_{2j}z_{p-2}y_{2j+1}x_{m-1}y_{2j},$	$0 \le j \le \frac{n-3}{2}$	$(n-1)/2$
C2	$x_{2i}z_{p-1}x_{2i+1}y_{n-1}x_{2i},$	$0 \le i \le \frac{m-3}{2}$	$(m-1)/2$
C2	$x_{2i}z_{p-2}x_{2i+1}y_{n-1}x_{2i},$	$0 \le i \le \frac{m-3}{2}$	$(m-1)/2$
C3 (2 copies)	$z_{2l}x_{m-1}z_{2l+1}y_{n-1}z_{2l},$	$0 \le l \le \frac{p-4}{2}$	$2(p-2)/2$
C4	$x_{2i}z_{p-1}x_{2i+1}z_{p-2}x_{2i},$	$0 \le i \le \frac{m-3}{2}$	$(m-1)/2$
C4	$y_{2j}z_{p-1}y_{2j+1}z_{p-2}y_{2j},$	$0 \le j \le \frac{n-3}{2}$	$(n-1)/2$
C5 (2 copies)	$x_{2i}y_{2j}x_{2i+1}y_{2j+1}x_{2i},$	$\left\{\begin{array}{l} 0 \le i \le \frac{m-3}{2} \\ 0 \le j \le \frac{n-3}{2} \end{array}\right\}$	$2((m-1)/2)((n-1)/2)$
C6 (2 copies)	$x_{2i}z_{2l}x_{2i+1}z_{2l+1}x_{2i},$	$\left\{\begin{array}{l} 0 \le i \le \frac{m-3}{2} \\ 0 \le l \le \frac{p-4}{2} \end{array}\right\}$	$2((m-1)/2)((p-2)/2)$
C7 (2 copies)	$y_{2j}z_{2l}y_{2j+1}z_{2l+1}y_{2j},$	$\left\{\begin{array}{l} 0 \le j \le \frac{n-3}{2} \\ 0 \le l \le \frac{p-4}{2} \end{array}\right\}$	$2((n-1)/2)((p-2)/2)$

Edges $z_{p-1}x_{m-1}$, $z_{p-1}y_{n-1}$, $z_{p-2}x_{m-1}$, and $z_{p-2}y_{n-1}$ are each covered exactly twice, once by a triangle and once by the C1 cycle. For each j, $0 \le j \le (n-3)/2$, edges $z_{p-1}y_{2j}$, $z_{p-1}y_{2j+1}$, $z_{p-2}y_{2j}$, and $z_{p-2}y_{2j+1}$ lie in both the C2 cycles and the C4 cycles. Similarly, for each i, $0 \le i \le (m-3)/2$, edges $z_{p-1}x_{2i}$, $z_{p-1}x_{2i+1}$, $z_{p-2}x_{2i}$, and $z_{p-2}x_{2i+1}$ also lie in both the C2 cycles and the C4 cycles. Thus, all edges incident with the vertices z_{p-1} and z_{p-2} are covered twice.

For each l, $0 \le l \le (p-4)/2$, edges $z_{2l}x_{m-1}$, $z_{2l+1}x_{m-1}$, $z_{2l}y_{y-1}$, and $z_{2l+1}y_{n-1}$ are each covered twice, by two copies of a C3 cycle. The C6 cycles and the C7 cycles form cycle decompositions of the complete bipartite graphs with parts Z' and X', and with parts Z' and Y'. Thus edges incident with vertices in Z' are all covered twice.

Edge $x_{m-1}y_{n-1}$ is covered twice, once by each triangle. For each j, $0 \le j \le (n-3)/2$, edges $x_{m-1}y_{2j}$ and $x_{m-1}y_{2j+1}$ lie in two C2 cycles, and so are covered twice. Similarly, for each i, $0 \le i \le (m-3)/2$, edges $y_{n-1}x_{2i}$ and $y_{n-1}x_{2i+1}$ also lie in two C2 cycles, and so are covered twice. Finally, because the C5 cycles form a cycle decomposition of the complete bipartite graph with parts X' and Y', edges incident with vertices in both X' and Y' are covered twice.

The CDC, $\mathcal{C}_G$, of G consists of two triangles and $(2|E(G)|-6)/4$ cycles of length four. Therefore,

$$\begin{aligned}\gamma(\mathcal{C}_G) &= 2 + \frac{2|E(G)|-6}{4} + ch(\mathcal{C}_G) \\ &= (|E(G)|+1)/2 + ch(\mathcal{C}_G).\end{aligned}$$

Triangles have no chords, and as in Cases 2 and 3, no 4-cycle in $\mathcal{C}_G$ has vertices from more than three parts, so no 4-cycle has more than one chord. Thus, since $\mathcal{C}_G$ has $(2|E(G)|-6)/4 = (|E(G)|-3)/2$ cycles of length four, $ch(\mathcal{C}_G) \leq (|E(G)|-3)/2$. Therefore,

$$\begin{aligned}\gamma(\mathcal{C}_G) &= (|E(G)|+1)/2 + ch(\mathcal{C}_G)\\ &\leq (|E(G)|+1)/2 + (|E(G)|-3)/2\\ &= |E(G)|-1\\ &< |E(G)|.\end{aligned}$$

The transition multigraph of vertex z_{p-1} has two copies of edge $x_{m-1}y_{n-1}$, one resulting from a triangle, and one from the C1 cycle. It has two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-3)/2$, and two copies of edges $x_{2i}x_{2i+1}$, $0 \leq i \leq (m-3)/2$, resulting from the C2 cycles, and thus it consists of $1+(n-1)/2+(m-1)/2$ digons. The transition multigraph of vertex z_{p-2} is the same as that of vertex z_{p-1}, because of the symmetry of X and Y in the description of $\mathcal{C}_G$. The transition multigraph of each vertex in Z' has two copies of edges $x_{m-1}y_{n-1}$, resulting from the two copies of the C3 cycle, and two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-1)/2$, and $x_{2i}x_{2i+1}$, $0 \leq i \leq (m-1)/2$, resulting from the C6 cycles and the C7 cycles. These transition multigraphs, therefore, also consist of $1+(n-1)/2+(m-1)/2$ digons.

The transition multigraph of vertex x_{m-1} has edges $z_{p-1}y_{n-1}$, $z_{p-2}y_{n-1}$, and $z_{p-1}z_{p-2}$, resulting from the triangles and the C1 cycle. It has two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-3)/2$, from the C2 cycles, and two copies of edges $z_{2l}z_{2l+1}$, $0 \leq l \leq (p-4)/2$, one from each C3 cycle. As x_{m-1} appears in no other cycles, we can see that $M_T(x_{m-1})$ consists of the triangle, $z_{p-1}y_{n-1}z_{p-2}z_{p-1}$, and $(n-1)/2+(p-2)/2$

digons. The transition multigraph of each vertex in X' has one copy of edges $z_{p-1}y_{n-1}$ and $z_{p-2}y_{n-1}$, resulting from the C2 cycles, and one copy of edge $z_{p-1}z_{p-2}$ from the C4 cycles. Each also has two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-3)/2$, and $z_{2l}z_{2l+1}$, $0 \leq l \leq (p-4)/2$, from the C5 cycles and the C6 cycles. Therefore, as with $M_T(x_{m-1})$, the transition multigraph of each vertex in X' consists of the triangle, $z_{p-1}y_{n-1}z_{p-2}z_{p-1}$, and $(n-1)/2 + (p-2)/2$ digons.

Because of the symmetry of X and Y in the description of the cycles, it is easy to see that the transition multigraph of vertex y_{n-1} and vertices in Y' will consist of triangle $z_{p-1}x_{m-1}z_{p-2}z_{p-1}$, and $(m-1)/2 + (p-2)/2$ digons.

In summary, we have found that G has a CDC, $\mathcal{C}_G$, where $\gamma(\mathcal{C}_G) \leq |E(G)|$, equality occurring only when $m = 2$, $n = 2$, and $p = 1$. We have also found that the CDCs contribute either a collection of digons, or a triangle and a collection of digons, to the transition multigraph of each vertex in G.

■

Corollary 22 *If G is a complete 3-partite graph, then the line graph $L(G)$ has a small cycle double cover.*

Proof: Unless G is the complete 3-partite graph $K_{1,2,2}$, the result follows directly from Theorem 21 and Lemmas 6, 8, and 9.

For $G = K_{1,2,2}$, let the parts of G be $X = \{x_0, x_1\}$, $Y = \{y_0, y_1\}$, and $Z = \{z_0\}$. It follows that the vertices of $L(G)$ will be

$$V(L(G)) = \{x_0z_0, x_1z_0, y_0z_0, y_1z_0, x_0y_0, x_0y_1, x_1y_0, x_1y_1\}.$$

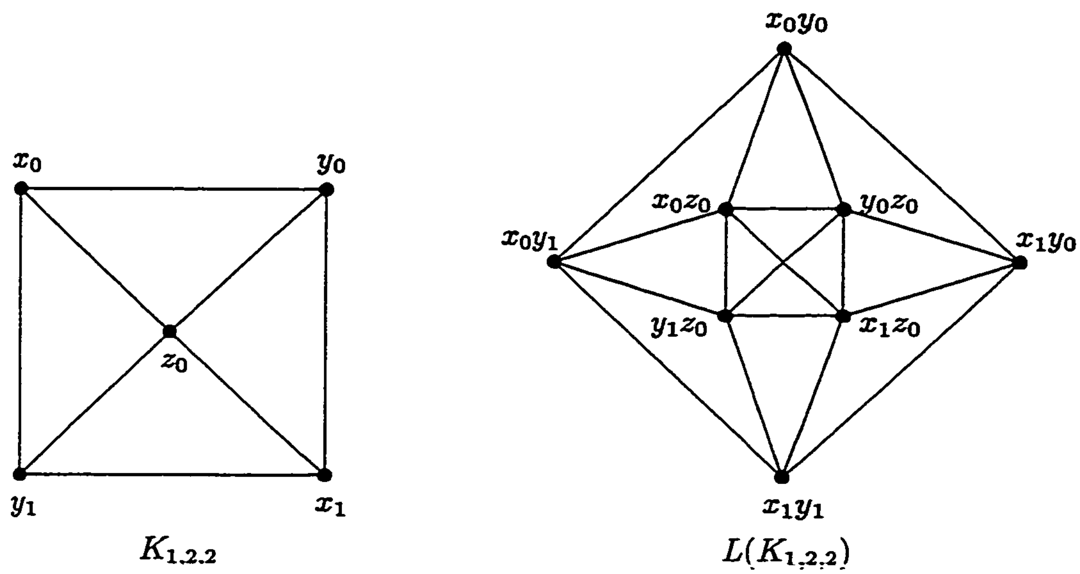

Figure 4.1: $K_{1,2,2}$ and $L(K_{1,2,2})$.

As consistent with the notation previously used, an edge in $L(G)$ will be identified by the vertices at its endpoints, and a cycle in $L(G)$ will be described by the vertices through which it passes. However, for ease of reading, parentheses will be used in the description of an edge or cycle to separate the vertex names. For example, an edge between vertices x_0y_0 and x_0y_1 of $L(G)$ is written $(x_0y_0)(x_0y_1)$.

Let $\mathcal{C}_G$ consist of the following cycles:

Cycle Type	Cycles
C1	$(x_0y_0)(x_0y_1)(x_1y_1)(y_1z_0)(x_1z_0)(y_0z_0)(x_0z_0)(x_0y_0)$
C1	$(x_0y_0)(x_0y_1)(x_0z_0)(x_1z_0)(y_1z_0)(y_0z_0)(x_1y_0)(x_0y_0)$
C1	$(x_1y_1)(x_1y_0)(x_1z_0)(y_0z_0)(x_0z_0)(y_1z_0)(x_0y_1)(x_1y_1)$
C1	$(x_1y_1)(x_1y_0)(x_0y_0)(y_0z_0)(y_1z_0)(x_0z_0)(x_1z_0)(x_1y_1)$
C2	$(x_1y_1)(x_1z_0)(x_1y_0)(y_0z_0)(x_0y_0)(x_0z_0)(x_0y_1)(y_1z_0)(x_1y_1)$.

The preceding cycles form a CDC, $\mathcal{C}_G$ of G. Each of the ten edges $(x_0y_0)(x_0y_1)$, $(x_0y_0)(x_1y_0)$, $(x_1y_1)(x_1y_0)$, $(x_1y_1)(x_0y_1)$, $(x_0z_0)(x_1z_0)$, $(y_0z_0)(y_1z_0)$, $(x_0z_0)(y_0z_0)$, $(x_0z_0)(y_1z_0)$, $(x_1z_0)(y_0z_0)$, and $(x_1z_0)(y_1z_0)$ is in exactly two C1 cycles, while each edge of the form $(x_iz_0)(x_iy_j)$ or $(y_iz_0)(x_jy_i)$, $0 \leq i, j \leq 1$, is in one C1 cycle and in the C2 cycle. Because $|\mathcal{C}_G| = 5$ and the number of vertices in $L(G)$ is eight, $|\mathcal{C}_G| < |V(L(G))|$, and the CDC, $\mathcal{C}_G$, is, in fact, an SCDC.

■

4.3.4 4-partite Graphs

Before we can prove the general result that the line graph of any complete multipartite graph has an SCDC, we need one final result: that the 4-partite graph with all odd parts has a CDC with the properties required to use Lemma 8.

Lemma 23 *If G is a complete 4-partite graph with parts X, Y, Z, and W, where each part has an odd number of vertices, then G has a cycle double cover, $\mathcal{C}_G$, with*

$\gamma(\mathcal{C}_G) < |E(G)|$. *Furthermore, the transition multigraph of each vertex in G consists of one triangle and $(d(v)-3)/2$ digons.*

Proof: Let

$$
\begin{aligned}
X &= \{x_0, x_1, \ldots, x_{m-2}, x_{m-1}\}, \\
Y &= \{y_0, y_1, \ldots, y_{n-2}, y_{n-1}\}, \\
Z &= \{z_0, z_1, \ldots, z_{p-2}, z_{p-1}\}, \\
W &= \{w_0, w_1, \ldots, w_{q-2}, w_{q-1}\}, \\
X' &= X \backslash \{x_{m-1}\}, \\
Y' &= Y \backslash \{y_{n-1}\}, \\
Z' &= Z \backslash \{z_{p-1}\}, \\
W' &= W \backslash \{w_{q-1}\}.
\end{aligned}
$$

Let $T_1 = G[X \cup Y \cup Z]$, the subgraph induced by the vertices of $X \cup Y \cup Z$, $T_2 = G[X \cup Y \cup W]$, $T_3 = G[X \cup Z \cup W]$, and $T_4 = G[Y \cup Z \cup W]$. Each of these subgraphs is a complete 3-partite graph with three odd parts. Therefore, by Case 3 of Theorem 21, each of these subgraphs has a cycle decomposition into one triangle and a collection of 4-cycles. Every edge of G lies in two of T_1, T_2, T_3, and T_4: edges incident with vertices in X and Y lie in T_1 and T_2, edges incident with vertices in X and Z lie in T_1 and T_3, and edges incident with vertices in X and W lie in T_2 and T_3. Similarly, edges incident with vertices in Y and Z lie in T_1 and T_4, edges incident with Y and W lie in T_2 and T_4, and edges incident with vertices in Z and W lie in T_3 and T_4. Therefore, the union of these cycle decompositions is a CDC, $\mathcal{C}_G$, consisting of four triangles and a collection of 4-cycles. The triangles in $\mathcal{C}_G$ have no chords, and like the 4-cycles in the CDC described in Case 3 of Theorem 21, no 4-cycle has more than one chord. The number of 4-cycles in $\mathcal{C}_G$ is

$$(2|E(G)| - 12)/4 = |E(G)|/2 - 3$$

and so, $ch(\mathcal{C}_G) \leq |E(G)|/2 - 3$. It now follows that

$$\begin{aligned}\gamma(\mathcal{C}_G) &= |\mathcal{C}_G| + ch(\mathcal{C}_G)\\ &\leq (4 + |E(G)|/2 - 3) + (|E(G)|/2 - 3)\\ &= 4 + |E(G)| - 6\\ &= |E(G)| - 2\\ &< |E(G)|.\end{aligned}$$

Vertex x_{m-1} is a vertex in subgraphs T_1, T_2, and T_3, and hence, edges in its transition multigraph are contributed by these three subgraphs. Edges $y_{n-1}z_{p-1}$, $y_{n-1}w_{q-1}$, and $z_{p-1}w_{q-1}$ are contributed to $M_T(x_{m-1})$ by the triangles of these three subgraphs. Two copies of edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-3)/2$, one from the decomposition of subgraph T_1, and one from the decomposition of T_2 are also contributed to $M_T(x_{m-1})$. Similarly, $M_T(x_{m-1})$ contains two copies of edges $z_{2l}z_{2l+1}$, $0 \leq l \leq (p-3)/2$, and edges $w_{2t}w_{2t+1}$, $0 \leq t \leq (q-3)/2$, from the decomposition of the T_1 and T_3 subgraphs, and the T_2 and T_3 subgraphs, respectively. Therefore, the transition multigraph of vertex x_{m-1} consists of the triangle $y_{n-1}z_{p-1}w_{q-1}y_{n-1}$, and $[(n-1)+(p-1)+(q-1)]/2 = (d(x_{m-1})-3)/2$ digons.

Vertices in X' are also included in the T_1, T_2, and T_3 subgraphs. The decompositions of these subgraphs contribute edges $y_{n-1}z_{p-1}$, $y_{n-1}w_{q-1}$, and $z_{p-1}w_{q-1}$ to the transition multigraphs of each vertex of X', while they also contribute a total of two copies of the edges $y_{2j}y_{2j+1}$, $0 \leq j \leq (n-3)/2$, $z_{2l}z_{2l+1}$, $0 \leq l \leq (p-3)/2$, and $w_{2t}w_{2t+1}$, $0 \leq t \leq (q-3)/2$. Therefore, like the transition multigraph of vertex x_{m-1}, the transition multigraph of each vertex in X' consists of the triangle $y_{n-1}z_{p-1}w_{q-1}y_{n-1}$ and $[(n-1)+(p-1)+(q-1)]/2 = (d(x_i)-3)/2$ digons, $0 \leq i \leq m-2$.

Because of the symmetry in the description of the cycles in C_G, the transition multigraph of each vertex in Y, Z, and W also consists of one triangle and a collection of digons.

■

As an immediate consequence of this result and of Lemmas 6, 8, and 9, we get

Corollary 24 *If G is a complete 4-partite graph, with four parts of odd size, then the line graph, $L(G)$, has a small cycle double cover.*

Chapter 5

Complete Multipartite Graphs

In Chapter 3, we saw that in order to prove that the line graph, $L(G)$, of a graph, G, has an SCDC, it suffices to show that G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. Furthermore, $\mathcal{C}_G$ must have the property that it contributes a collection of digons, or a triangle and a collection of digons, to the transition multigraph of each vertex in G. In Chapter 4, we saw that complete bipartite graphs with at least two vertices in each part, complete 3-partite graphs, and complete 4-partite graphs with four odd parts all have CDCs with the properties mentioned above, and thus their line graphs have SCDCs. In Chapter 4, we also found that if G has subgraphs that partition its edge set, and if for each such subgraph H, we have $\gamma(\mathcal{C}_H) < |E(H)|$, then G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$.

In this chapter, by using the results of the previous two chapters, we prove that line graphs of all complete multipartite graphs (except $K_{1,2}$) have SCDCs.

Theorem 25 *If G is a complete multipartite graph other than $K_{1,2}$, then the line graph, $L(G)$, has a small cycle double cover.*

The proof of this theorem is divided into cases, depending on the parity and size of the parts of the graph. In most cases, the results of Lemma 10 are used, and a CDC of the graph, G, is found by combining the CDCs of subgraphs which partition the edge set of G. In order to obtain subgraphs with parts of size and parity with which we can work, the vertices in the original parts of the graph are often rearranged. A part in the original graph may be considered as two smaller sets or a part in the original graph may be enlarged, by grouping with it, vertices from other parts.

Proof of Theorem 25: Let G be a complete multipartite graph.

In each of the following cases, we will find a CDC, $\mathcal{C}_G$, ensuring that $\gamma(\mathcal{C}_G) < |E(G)|$, and ensuring that the transition multigraph of each vertex consists of digons or a triangle and a collection of digons. Once these facts have been established, the results of [24], restated as Lemmas 6, 8, and 9, can be applied to show that the CDC, $\mathcal{C}_G$, results in an SCDC of the line graph, $L(G)$. The cases in this proof are grouped as follows.

Case 1		G consists of $k \geq 2$ parts, each of even size
Case 2		G consists of $k \geq 2$ parts, each of odd size greater than one
	2a	k is even
	2b	k is odd
Case 3		G consists of $k \geq 2$ parts, each of size one
Case 4		G consists of $r \geq 1$ parts of size one and $n \geq 1$ parts, each of odd size greater than one
	4a	$r = 1$, n even
	4b	$r = 1$, n odd
	4c	$r = 2$, n even
	4d	$r = 2$, n odd
	4e	$r \equiv 1, 3, 5 \pmod 6$, $r \geq 3$, n even $r \equiv 0, 2, 4 \pmod 6$, $r \geq 4$, n odd
	4f	$r \equiv 1, 3, 5 \pmod 6$, $r \geq 3$, n odd $r \equiv 0, 2, 4 \pmod 6$, $r \geq 4$, n even
Case 5		G consists of $s \geq 1$ parts of odd size and $t \geq 1$ parts of even size
	5a	$s = 1$, t even
	5b	$s = 1$, t odd
	5c	$s = 2$
	5d	$s \geq 3$

Case 1. G has $k \geq 2$ parts, each of even size.

The graph, G, is the union of the bipartite subgraphs induced by all possible pairs of parts ($\binom{k}{2}$ subgraphs, in total). Because different pairs of parts are used in each subgraph, the edge sets of the induced subgraphs are all disjoint, and so the edge set of G is partitioned by these subgraphs. As each subgraph is a complete bipartite graph with two even parts, we know from Lemma 12 that each subgraph can be decomposed into chordless 4-cycles. Hence, by Corollary 13, each subgraph has a CDC whose cycles number half the number of edges of the subgraph. Each of the subgraphs of G thus meets the requirements of Lemma 10, and so G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$.

The CDC of any of the subgraphs of G is formed from two copies of a cycle decomposition, and therefore the CDC of G is also composed of two copies of a cycle decomposition. In applying Lemma 11, we find that the transition multigraph of each vertex of G consists only of digons.

Case 2. G consists of $k \geq 2$ parts $X_1, \ldots, X_k$, each of odd size greater than one.

2a. k even.

If $k = 2$, G is a bipartite graph with two odd parts, and thus by Lemma 17, has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$, which contributes a triangle and a collection of digons to the transition multigraph of each vertex in G. Also, its line graph, $L(G)$ has an SCDC, as shown in Corollary 18.

If $k \geq 4$, let H_i be the subgraph $G[X_{2i-1} \cup X_{2i}]$, $1 \leq i \leq k/2$, and let H be the subgraph of G, with $H = \bigcup_{i=1}^{k/2} H_i$. Note that the vertex sets of H and G are the same. Each subgraph H_i, $1 \leq i \leq k/2$, is a complete bipartite graph with parts of

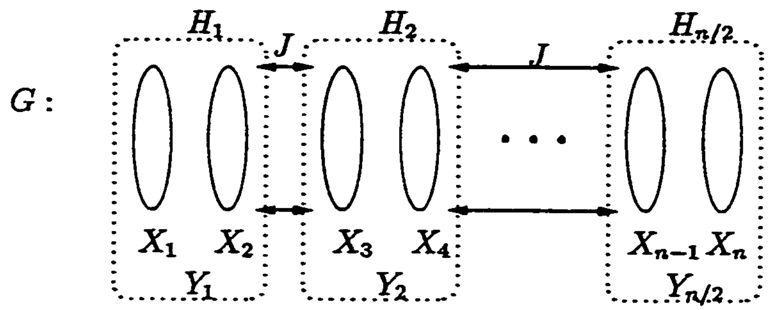

Figure 5.1: Case 2a.

odd size at least three, and so we can apply Lemma 17 to each H_i to find a CDC, $\mathcal{C}_{H_i}$, with $\gamma(\mathcal{C}_{H_i}) < |E(H_i)|$. Because the subgraphs H_i, $1 \le i \le k/2$, partition the edge set of the subgraph H, we can apply Lemma 10 to show that H has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) < |E(H)|$. Each CDC $\mathcal{C}_{H_i}$ contributes a triangle and a collection of digons to the transition multigraph of each vertex in its respective subgraph H_i. Therefore, $\mathcal{C}_H$ contributes one triangle and a collection of digons to the transition multigraph of each vertex in G.

We now have only to double cover the edges between all pairs $Y_i = X_{2i-1} \cup X_{2i}$ and $Y_j = X_{2j-1} \cup X_{2j}$, $1 \le i, j \le k/2$, $i \ne j$. Let J be the graph on $V(G)$ consisting of these edges. Because each of the parts X_l, $1 \le l \le k$, has an odd number of vertices and has at least three vertices, each of the sets Y_i, $1 \le i \le k/2$, has an even number of vertices (at least six), and so J is a complete multipartite graph with even parts. From Case 1, we know that J has a CDC, $\mathcal{C}_J$, consisting of two copies of a cycle decomposition into 4-cycles, which contributes digons to the transition multigraph of each vertex of $V(G)$.

Any 4-cycle in $\mathcal{C}_J$ passing through vertices in Y_i and Y_j, $1 \leq i, j \leq k/2$, $i \neq j$, contains a pair of vertices from Y_i and a pair of vertices from Y_j. Therefore, the 4-cycles that double cover the edges between sets Y_i and Y_j could include vertices from up to four parts (one vertex each from X_{2i-1}, X_{2i}, X_{2j-1}, and X_{2j}). To ensure a minimum number of chords in these 4-cycles, vertices within each set Y_i, $1 \leq i \leq k/2$, are paired as follows. Let $X_{2i-1} = \{x_0, x_1, \ldots, x_{m-1}\}$ and $X_{2i} = \{v_0, v_1, \ldots, v_{n-1}\}$, and let $\{x_0, x_1\}, \ldots, \{x_{m-3}, x_{m-2}\}, \{v_0, v_1\}, \ldots, \{v_{n-3}, v_{n-2}\}, \{x_{m-1}, v_{n-1}\}$ be vertex pairings. These pairs are to be used in defining the cycles in the cycle decomposition. Except for vertices x_{m-1} and v_{n-1}, every vertex in Y_i is paired with another vertex from the same part. Therefore, except for the pair $\{x_{m-1}, v_{n-1}\}$, vertices in each pair are non-adjacent in G, and so contribute no chords to any cycle which passes through them. The vertices x_{m-1} and v_{n-1} are adjacent in G, and consequently contribute a chord to any cycle in which they appear as non-consecutive vertices. With the vertices in the set Y_j paired in an analogous fashion to those vertices in set Y_i, it turns out that for any pair Y_i and Y_j, there are two cycles passing through the two pairs containing adjacent vertices ($\{x_{m-1}, v_{n-1}\}$ in Y_i, and its counterpart in Y_j), and thus, there are two cycles each with two chords. Because each part X_l, $0 \leq l \leq k$, has at least three vertices, we are guaranteed that at least one pair of vertices in each part X_l, or at least two pairs in each set Y_i, $1 \leq i \leq k/2$, is comprised of non-adjacent vertices. Therefore, there are at least eight chordless cycles between pairs of sets in Y_i and Y_j. Since there are $k/2$ sets, Y_i, there are $\binom{k/2}{2}$ pairs Y_i and

Y_j, $i \neq j$, and thus there are

$$\beta_2 = 2\binom{k/2}{2} = (k^2 - 2k)/4$$

4-cycles with two chords, and

$$\beta_0 \geq 8\binom{k/2}{2} = (k^2 - 2k)$$

chordless 4-cycles. Therefore,

$$\begin{aligned}
\gamma(\mathcal{C}_J) &= |\mathcal{C}_J| + ch(\mathcal{C}_J) \\
&= |\mathcal{C}_J| + (|\mathcal{C}_J| - \beta_0 + \beta_2) \\
&\leq 2(2|E(J)|/4) - (k^2 - 2k) + (k^2 - 2k)/4 \\
&= |E(J)| - 3(k^2 - 2k)/4 \\
&< |E(J)|.
\end{aligned}$$

Since H and J partition the edge set of G, and have CDCs, $\mathcal{C}_H$ and $\mathcal{C}_J$, with $\gamma(\mathcal{C}_H) < |E(H)|$, and $\gamma(\mathcal{C}_J) < |E(J)|$, then by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The transition multigraph of each vertex in G consists of a triangle, contributed by the cycles in $\mathcal{C}_H$, and a collection of digons, resulting from the cycles in $\mathcal{C}_H$ and $\mathcal{C}_J$.

2b. k odd.

From Case 3 of Theorem 21, we know there is a cycle decomposition, and hence a CDC, of the edges among any three odd sets. Therefore, the subgraphs

$$H_i = G[X_1 \cup X_{2i} \cup X_{2i+1}], 1 \leq i \leq (k-1)/2$$

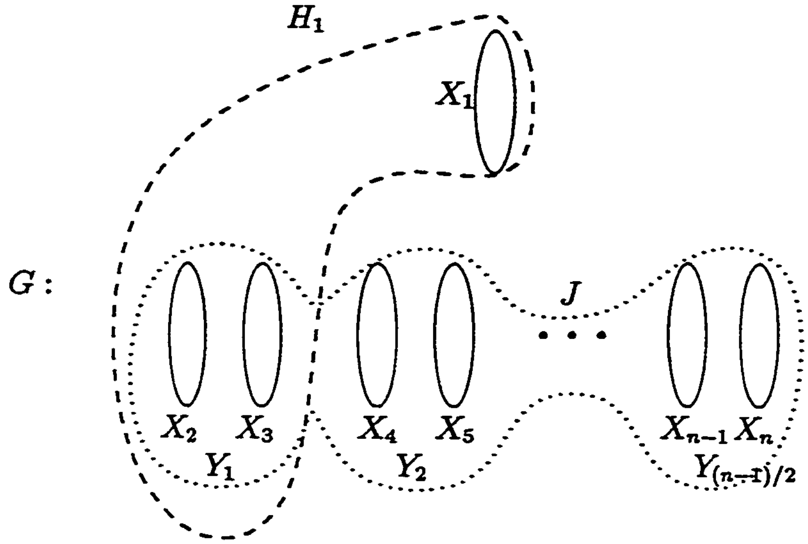

Figure 5.2: Case 2b.

each have a CDC, $\mathcal{C}_{H_i}$, with $\gamma(\mathcal{C}_{H_i}) < |E(H_i)|$. Let $H = \bigcup_{i=1}^{(k-1)/2} H_i$. Applying Lemma 10 shows that H has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) < |E(H)|$. Because each $\mathcal{C}_{H_i}$ is two copies of a cycle decomposition, $\mathcal{C}_H$ is also two copies of a cycle decomposition, and so contributes only digons to the transition multigraph of each vertex in G.

If $k = 3$, then all edges of G are covered twice by the cycles of $\mathcal{C}_H$, and we are done. However, if $k \geq 5$, consider the $(k-1)/2$ supersets $Y_i = X_{2i} \cup X_{2i+1}$, $1 \leq i \leq (k-1)/2$. We have double covered all edges within each of these supersets, and also double covered all edges incident to X_1, so all that remains to be double covered are the edges between pairs of supersets. Let J be the subgraph of G consisting of vertices $V(G) \backslash X_1$, and these uncovered edges. The subgraph J is a complete multipartite graph with parts Y_i, $1 \leq i \leq (k-1)/2$. Because each of the original parts X_l, $2 \leq l \leq k$, has an odd number of vertices greater than one, each Y_i, $1 \leq i \leq (k-1)/2$, has an even number of vertices greater than four. From Case 1 of this proof, we know, therefore, that J has a CDC, $\mathcal{C}_J$, composed of two copies of a decomposition into 4-cycles. As in Case 2a, the number of chords contributed by the 4-cycles can be minimized by pairing the vertices within each superset Y_i, $1 \leq i \leq (k-1)/2$, so that in the collection of pairs of these vertices, only one pair of vertices is adjacent in G. Therefore, the double covering between any two supersets Y_i and Y_j, $1 \leq i, j \leq (k-1)/2$, $i \neq j$ has two 4-cycles with two chords, and at least eight chordless 4-cycles. Since there are $(k-1)/2$ supersets, and hence $\binom{(k-1)/2}{2}$ edge sets between pairs of supersets, there are

$$\beta_2 = 2\binom{(k-1)/2}{2} = (k^2 - 4k + 3)/4$$

4-cycles with two chords, and

$$\beta_0 \geq 8\binom{(k-1)/2}{2} = (k^2 - 4k + 3)$$

chordless 4-cycles contributed by CDC of J, $\mathcal{C}_J$. Therefore,

$$\begin{aligned}
\gamma(\mathcal{C}_G) &= |\mathcal{C}_J| + ch(\mathcal{C}_J) \\
&= |\mathcal{C}_J| + (|\mathcal{C}_J| - \beta_0 + \beta_2) \\
&\leq 2(2|E(J)|/4) - (k^2 - 4k + 3) + (k^2 - 4k + 3)/4 \\
&= |E(J)| - 3(k^2 - 4k + 3)/4 \\
&< |E(J)|.
\end{aligned}$$

The CDC, $\mathcal{C}_G$, of G is the union of the CDCs $\mathcal{C}_H$ and $\mathcal{C}_J$ of the subgraphs H and J. As the subgraphs partition the edge set of G, and as $\gamma(\mathcal{C}_H) < |E(H)|$ and

$\gamma(\mathcal{C}_J) < |E(J)|$, we know from Lemma 10 that $\gamma(\mathcal{C}_G) < |E(G)|$. All edges in G are covered by two copies of cycle decompositions, and so by applying the result of Lemma 11, we know that the transition multigraph of each vertex in G consists of digons.

Case 3. G consists of $k \geq 2$ parts, each of size one.

A multipartite graph with $k \geq 2$ parts of size one is a complete graph on k vertices. In their paper [24], MacGillivray and Seyffarth prove that K_k has a CDC, $\mathcal{C}$, such that $\gamma(\mathcal{C}) = |\mathcal{C}| + ch(\mathcal{C}) < |E(K_k)|$ where, for all graphs but K_6, the cycles in $\mathcal{C}$ contribute a collection of digons, or a triangle and a collection of digons to the transition multigraph of each vertex in the graph. Furthermore, they prove that if $k \geq 3$, and trivially if $k = 2$, then the line graph $L(K_k)$ has an SCDC.

Case 4. G consists of $r \geq 1$ parts of size one and $n \geq 1$ parts, each of odd size greater than one.

We will first look at methods for finding an appropriate CDC of G when $r = 1$ or $r = 2$. Various subgraphs which partition the edge set of G will be defined, and CDCs of these subgraphs will be described. Lemma 10 will then be applied to show that $\gamma(\mathcal{C}_G) < |E(G)|$.

4a. $r = 1$, n even.

Let v be the vertex in the part of size one, and let $X_1, \ldots, X_n$ be the remaining parts of G, each containing at least three vertices. Let $H_i = G[\{v\} \cup X_{2i-1} \cup X_{2i}]$, $1 \leq i \leq n/2$. By Case 3 of Theorem 21, H_i has a CDC, $\mathcal{C}_{H_i}$, consisting of two copies of a cycle decomposition, with $\gamma(\mathcal{C}_{H_i}) < |E(H_i)|$. Let $H = \bigcup_{i=1}^{n/2} H_i$, and $\mathcal{C}_H = \bigcup_{i=1}^{n/2} \mathcal{C}_{H_i}$. By Lemma 10, $\gamma(\mathcal{C}_H) < |E(H)|$, and because each CDC, $\mathcal{C}_{H_i}$, $1 \leq i \leq n/2$, is created by taking two copies of a cycle decomposition, the transition multigraph of each vertex in G, attributed to the CDC $\mathcal{C}_H$, consists of digons.

If $n = 2$, $\mathcal{C}_H$ is a CDC of G and we are done. If $n \geq 4$, we notice that the CDC of the subgraph, H, double covers all edges incident to vertex v, and also, all edges between vertices in the pairs X_{2i-1} and X_{2i}, $1 \leq i \leq n/2$. Therefore, the only edges that remain to be double covered are those edges that connect any two pairs of sets $Y_i = X_{2i-1} \cup X_{2i}$, and $Y_j = X_{2j-1} \cup X_{2j}$, $1 \leq i,j \leq n/2$, $i \neq j$. Let J be the subgraph on vertices $V(G)\backslash\{v\}$, with edges $E(G)\backslash E(H)$. This subgraph is a complete multipartite graph with parts $Y_1, \ldots, Y_{n/2}$, all even, and so by Case 1 of this proof, J has a CDC, $\mathcal{C}_J$, consisting of two copies of a decomposition into 4-cycles whose cycles contribute digons to the transition multigraph of each vertex in $V(J)$.

The number of chords associated with the 4-cycles of $\mathcal{C}_J$ can be minimized. As in Case 2a of this proof, it is possible to define cycles in $\mathcal{C}_J$ such that for each double covering of edges between two sets Y_i and Y_j, $1 \leq i,j \leq n/2$, $i \neq j$, there are two 4-cycles with two chords and at least eight chordless cycles. Since there are $n/2$ sets Y_i, $1 \leq i \leq n/2$, and hence $\binom{n/2}{2}$ edge sets between pairs Y_i and Y_j, $1 \leq i,j \leq n/2$, $i \neq j$, there are

$$\beta_2 = 2\binom{n/2}{2} = (n^2 - 2n)/4$$

4-cycles with two chords, and

$$\beta_0 \geq 8\binom{n/2}{2} = n^2 - 2n$$

chordless 4-cycles. Therefore,

$$\gamma(\mathcal{C}_J) \quad = \quad |\mathcal{C}_J| + ch(\mathcal{C}_J)$$

$$
\begin{aligned}
&= |\mathcal{C}_J| + (|\mathcal{C}_J| - \beta_0 + \beta_2) \\
&\leq 2(2|E(J)|/4) - (n^2 - 2n) + (n^2 - 2n)/4 \\
&= |E(J)| - 3(n^2 - 2n)/4 \\
&< |E(J)|.
\end{aligned}
$$

By Lemma 10, we know that since the subgraphs H and J partition the edge set of G, and have CDCs $\mathcal{C}_H$ and $\mathcal{C}_J$, with $\gamma(\mathcal{C}_H) < |E(H)|$ and $\gamma(\mathcal{C}_J) < |E(J)|$, the union of the cycles of $\mathcal{C}_H$ and $\mathcal{C}_J$ forms a CDC, $\mathcal{C}_G$, of G with $\gamma(\mathcal{C}_G) < |E(G)|$. Because the CDCs $\mathcal{C}_H$ and $\mathcal{C}_J$ contribute only digons to the transition multigraphs of each vertex in the graph G, we know that the transition multigraph of each vertex, attributed to $\mathcal{C}_G$, also consists only of digons.

4b. $r = 1$, n odd.

If $n = 1$, then G is a *star* (a graph with edges radiating out from one vertex to all other vertices) and $L(G)$ is a complete graph on at least three vertices. As discussed in Chapter 2, all complete graphs except K_2 have SCDCs, and therefore, $L(G)$ has an SCDC.

If $n \geq 3$, let v be the vertex in the part of size one, and let $X_1, \ldots, X_n$ be the remaining parts of G, each containing at least three vertices. Let H be the subgraph $G[\{v\} \cup X_{n-2} \cup X_{n-1} \cup X_n]$. The subgraph H is a complete 4-partite graph, with all parts of odd size, and thus, by Lemma 23, has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) < |E(H)|$, which contributes a triangle and a collection of digons to the transition multigraph of each vertex of H. If $n = 3$, $\mathcal{C}_H$ is a CDC of G, and we are done. Therefore, from now on, we assume that $n \geq 5$.

Because n is odd, $n - 3$ is even, and so by Case 4a of this proof, we know that

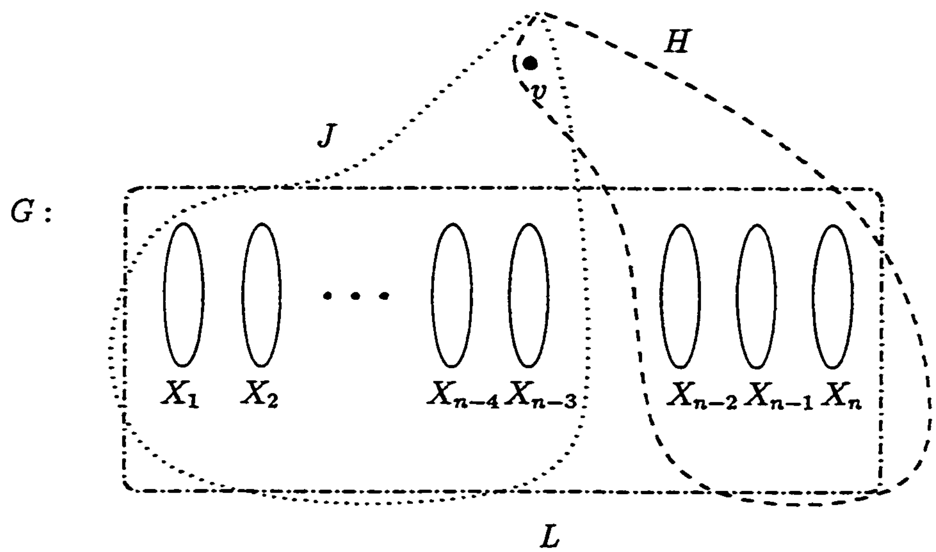

Figure 5.3: Case 4b.

the subgraph $J = G[\{v\} \cup X_1 \cup \ldots \cup X_{n-3}]$ has a CDC, $\mathcal{C}_J$, with $\gamma(\mathcal{C}_J) < |E(G)|$, which contributes only digons to the transition multigraphs of the vertices of J.

All edges incident to vertex v have been covered twice, and in fact, the only uncovered edges in G are those between vertices in $A = V(H)\backslash\{v\}$ and vertices in $B = V(J)\backslash\{v\}$. Let L be the graph on the vertices in the sets A and B containing these yet uncovered edges. The subgraph L is a complete bipartite graph with parts A and B. By definition of the subgraph H, we know that $A = V(H)\backslash\{v\} = X_{n-2} \cup X_{n-1} \cup X_n$, and since each of these parts has odd size, $|A|$ is odd. Because $B = V(J)\backslash\{v\} = X_1 \cup \ldots \cup X_{n-3}$, where $n-3$ is even, and each set X_i, $1 \leq i \leq n-3$, has odd size, $|B|$ is even. Therefore, the edges in L can be double covered by 4-cycles, using the method of Lemma 15. Let $\mathcal{C}_L$ be the CDC of L. This CDC contributes a collection of digons to the transition multigraph of each vertex in A, and a triangle and a collection of digons to the transition multigraph of each vertex in B.

The CDC $\mathcal{C}_L$ consists of cycles between pairs of vertices in $A = X_{n-2} \cup X_{n-1} \cup X_n$ and pairs of vertices in $B = X_1 \cup \ldots \cup X_{n-3}$. In order to minimize the number of chords in these cycles, we must choose the pairs of vertices so that, as often as possible, they are from the same part, X_i, $1 \leq i \leq n$. We can pair all but one of the vertices from each set X_{n-2}, X_{n-1}, and X_n, with other vertices from the same set and then use these pairs in the C2 cycles of $\mathcal{C}_L$, as described in Lemma 15. Let the unpaired vertices from the sets X_{n-2}, X_{n-1}, and X_n be x^*_{n-2}, x^*_{n-1}, and x^*_n, respectively, and let $\{x^*_{n-2}, x^*_{n-1}\}$, $\{x^*_{n-2}, x^*_n\}$, and $\{x^*_{n-1}, x^*_n\}$ be the pairs obtained from these vertices, and used in the C1 cycles. Each such cycle has a chord resulting from these pairs of vertices. As in Case 2a of this proof, the vertices in the even-sized set, B, can be paired so that all but one of the vertices from each set $X_1, \ldots, X_{n-3}$ is paired with other vertices from the same set. Because $n-3$ is even, the remaining vertices can be paired, and each of these pairs contributes a chord to the cycles in which they are included.

There are a total of $\beta_2 = 3(n-3)/2$ cycles of length four with two chords, and because every set X_i, $1 \leq i \leq n$, has at least three vertices, there are $\beta_0 \geq 6(n-3)$ cycles of length four with no chords. Therefore,

$$\begin{aligned}
\gamma(\mathcal{C}_L) &= |\mathcal{C}_L| + ch(\mathcal{C}_L) \\
&= |\mathcal{C}_L| + (|\mathcal{C}_L| - \beta_0 + \beta_2) \\
&\leq 2(2|E(L)|/4) - 6(n-3) + 3(n-3)/2 \\
&= |E(L)| - 9(n-3)/2
\end{aligned}$$

$$< \ |E(L)|.$$

The graph, G, has subgraphs H, J, and L, whose corresponding CDCs, $\mathcal{C}_H$, $\mathcal{C}_J$, and $\mathcal{C}_L$, satisfy $\gamma(\mathcal{C}_H) < |E(H)|$, $\gamma(\mathcal{C}_J) < |E(J)|$, and $\gamma(\mathcal{C}_L) < |E(L)|$. Thus by Lemma 10, we know that G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The transition multigraph of each vertex in $V(G)$ has a triangle and a collection of digons; for vertex v and for vertices in the sets X_{n-2}, X_{n-1}, and X_n, the triangle is contributed by the CDC $\mathcal{C}_H$, while for vertices in the sets $X_1, \ldots, X_{n-3}$, the triangle is contributed by the CDC $\mathcal{C}_L$.

4c. $r = 2$, n even.

Let u and v be the vertices in the two parts of size one, and let $X_1, \ldots, X_n$ be the remaining parts of G, each containing at least three vertices. The subgraph H, induced by the vertices u and v and the vertices from the sets X_{n-1} and X_n, is a complete 4-partite graph with parts of odd size, and so, by Lemma 23, it has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) < |E(H)|$. The transition multigraph of each vertex in $V(H)$ consists of a triangle and a collection of digons. If $n = 2$, then $\mathcal{C}_H$ is a CDC of G and we are done. Henceforth, we assume that $n \geq 4$.

Let J be the subgraph induced by the sets $X_1, \ldots, X_{n-2}$. As each set has odd size at least three, we can apply Case 2a of this proof, and consequently construct a CDC, $\mathcal{C}_J$, of J, with $\gamma(\mathcal{C}_J) < |E(J)|$. The CDC $\mathcal{C}_J$ contributes a triangle and a collection of digons to the transition multigraph of each vertex in the subgraph J.

The only edges of G not included in either H or J are those that connect vertices in the subgraph H to vertices in the subgraph J. Let L be the graph on vertices $V(G)$ that consists of these edges. The subgraph L is a complete bipartite graph with

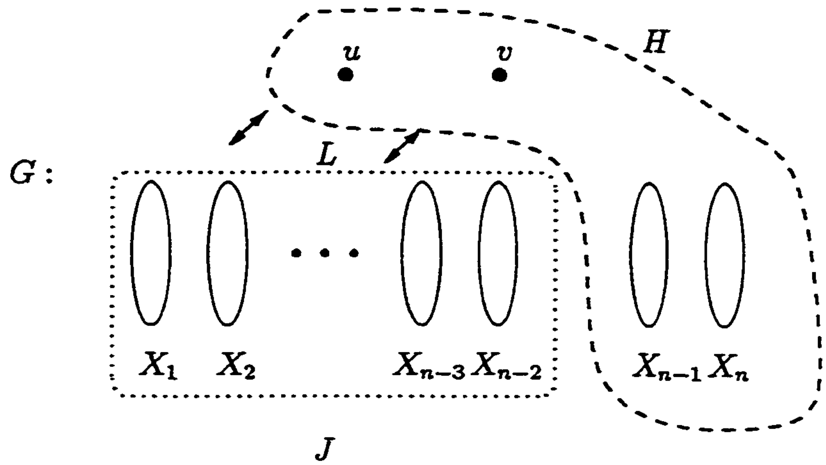

Figure 5.4: Case 4c.

parts $V(H)$ and $V(J)$. Because X_{n-1} and X_n each have an odd number of vertices, the set $V(H) = \{u\} \cup \{v\} \cup X_{n-1} \cup X_n$ has even size. Similarly, because $n-2$ is even and each set X_i, $1 \leq i \leq n-2$, has an odd number of vertices, $V(J) = \bigcup_{i=1}^{n-2} X_i$ has even size. Therefore, by Lemma 12 and Corollary 13, L has a CDC, $\mathcal{C}_L$, consisting of two copies of a 4-cycle decomposition. The CDC, $\mathcal{C}_L$, contributes digons to the transition multigraph of each vertex in L.

As in the preceding cases of this proof, the vertices in the sets $V(H)$ and $V(J)$ can be paired so as to minimize the number of chords in the 4-cycles of L. If vertices $x^*_{n-1} \in X_{n-1}$ and $x^*_n \in X_n$, and vertices u and v are paired, then all remaining vertices in $V(H)$ can be paired with vertices from the same part. Thus, only pairs $\{x^*_{n-1}, x^*_n\}$ and $\{u, v\}$ contribute chords to the 4-cycles in which they appear. In the set $V(J)$, all but one vertex from each set $X_1, \ldots, X_{n-2}$ can be paired with vertices from the same part. These remaining vertices can be paired, and each of these $(n-2)/2$ pairs contributes a chord to any 4-cycle in which it appears. Therefore, the number of cycles in $\mathcal{C}_L$ with two chords is $\beta_2 = 2(2((n-2)/2)) = 2(n-2)$. Because each set X_i, $1 \leq i \leq n$, has at least three vertices, and hence, at least one non-adjacent pair of vertices, there are $\beta_0 \geq 2(2(n-2)) = 4(n-2)$ chordless cycles. Therefore,

$$\begin{aligned}
\gamma(\mathcal{C}_L) &= |\mathcal{C}_L| + ch(\mathcal{C}_L) \\
&= |\mathcal{C}_L| + (|\mathcal{C}_L| - \beta_0 + \beta_2) \\
&\leq 2(2|E(L)|/4) - 4(n-2) + 2(n-2) \\
&= |E(L)| - 2(n-2) \\
&< |E(L)|.
\end{aligned}$$

Subgraphs H, J, and L partition $E(G)$ and have CDCs $\mathcal{C}_H$, $\mathcal{C}_J$ and $\mathcal{C}_L$, respectively, such that $\gamma(\mathcal{C}_H) < |E(H)|$, $\gamma(\mathcal{C}_J) < |E(J)|$, and $\gamma(\mathcal{C}_L) < |E(L)|$. Therefore, by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The transition multigraph of each vertex in G consists of a triangle and a collection of digons. For the vertices u and v, and the vertices in the sets X_{n-1} and X_n, the triangle is contributed by $\mathcal{C}_H$, while for the vertices in the sets $X_1, \ldots, X_{n-2}$, the triangle is contributed by $\mathcal{C}_J$.

4d. $r = 2$, n odd.

Let u and v be the vertices in the two parts of size one, and let $X_1, \ldots, X_n$ be the remaining parts of G, each containing at least three vertices. The subgraph H, induced by the vertices u and v and the vertices of X_n, is a complete 3-partite graph with three odd parts. Therefore, by applying Case 3 of Theorem 21, H has a CDC, $\mathcal{C}_H$, composed of two copies of a cycle decomposition, with $\gamma(\mathcal{C}_H) < |E(H)|$. Because the CDC is composed of two copies of a cycle decomposition, $\mathcal{C}_H$ contributes digons

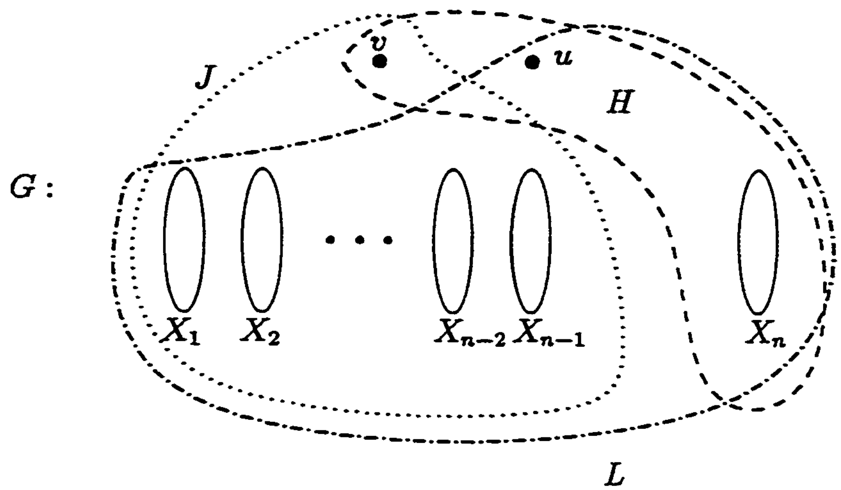

Figure 5.5: Case 4d.

to the transition multigraph of each vertex in H. If $n = 1$, G is 3-partite, and thus $\mathcal{C}_H$ is a CDC of G. Thus, we assume from now on that $n \geq 3$.

Let J be the subgraph induced by the vertex v and the vertices in the sets $X_1, \ldots, X_{n-1}$. Then J is a complete multipartite graph, with one part of size one and the other parts of odd size at least three. Case 4a of this proof shows that J has a CDC, $\mathcal{C}_J$, with $\gamma(\mathcal{C}_J) < |E(J)|$, which contributes digons to the transition multigraph of each vertex in $V(J)$.

The only edges of G not included in the subgraphs H and J are those between the vertices in the set $A = X_n \cup \{u\}$ and the vertices in the set $B = \bigcup_{i=1}^{n-1} X_i$. Let L be the subgraph on the vertices in the set $A \cup B$ containing these edges. Because the set X_n has odd size, the set A has even size; because each of the sets $X_1, \ldots, X_{n-1}$ also has odd size, and because $n-1$ is even, the set B has an even number of vertices.

Therefore, L is a complete bipartite graph with two even parts, and so by Lemma 12, has a CDC, $\mathcal{C}_L$, consisting of two copies of a 4-cycle decomposition. The CDC, $\mathcal{C}_L$, contributes only digons to the transition multigraphs of the vertices in L.

The vertices in A and B can be grouped so as to minimize the number of chords in $\mathcal{C}_L$. In considering the set A, we see that all but one vertex from X_n can be paired with another vertex from this part; the unpaired vertex can be paired with vertex u, and this pair contributes a chord to every 4-cycle in which it appears. Similarly, for the set B, all but one vertex from each set $X_1, \ldots, X_{n-1}$ can be paired with another vertex from the same part, while the remaining $n-1$ vertices can be paired with each other. These latter $(n-1)/2$ pairs each contribute a chord to every 4-cycle in which they appear. Therefore, the number of 4-cycles in $\mathcal{C}_L$ with two chords is $\beta_2 = 2(n-1)/2 = (n-1)$. Because each set $X_1, \ldots, X_n$ has at least three vertices, and thus, at least one non-adjacent pair of vertices, the number of chordless 4-cycles in $\mathcal{C}_L$ is $\beta_0 \geq 2(n-1)$. Therefore,

$$\begin{aligned}
\gamma(\mathcal{C}_L) &= |\mathcal{C}_L| + ch(\mathcal{C}_L) \\
&= |\mathcal{C}_L| + (|\mathcal{C}_L| - \beta_0 + \beta_2) \\
&\leq 2(2|E(L)|/4) - 2(n-1) + (n-1) \\
&= |E(L)| - (n-1) \\
&< |E(L)|.
\end{aligned}$$

Because the subgraphs H, J, and L partition the edge set of G, and because the CDCs, $\mathcal{C}_H$, $\mathcal{C}_J$, and $\mathcal{C}_L$ satisfy $\gamma(\mathcal{C}_H) < |E(H)|$, $\gamma(\mathcal{C}_J) < |E(J)|$, and $\gamma(\mathcal{C}_L) \leq |E(L)|$, then by Lemma 10, the graph G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The CDCs of the three subgraphs of G each contribute only digons to the transition multigraphs of their vertices. Therefore, the transition multigraph of each vertex of G consists only of digons.

4e. $r \equiv 1,3,5 \pmod 6$, $r \geq 3$ and n even; $r \equiv 0,2,4 \pmod 6$, $r \geq 4$ and n odd. As in the previous cases, we proceed by partitioning the edges of G into three subgraphs, H, J, and L. The way these subgraphs are determined depends on the value of r. We thus define two types of graphs, as follows.

Type 1 graph has	$r \equiv 1,3 \pmod 6$,	$r \geq 3$, n even, or
	$r \equiv 0,4 \pmod 6$,	$r \geq 4$, n odd
Type 2 graph has	$r \equiv 5 \pmod 6$,	$r \geq 5$, n even, or
	$r \equiv 2 \pmod 6$,	$r \geq 8$, n odd.

Let Y be the set containing the r vertices from the parts of size one and let $X_1, \ldots, X_n$ be the parts with at least three vertices. For Type 1 graphs, let X_i^{odd}, $1 \leq i \leq n$, be a subset of X_i containing three vertices. For Type 2 graphs, let X_i^{odd}, $1 \leq i \leq n-1$, be a subset of X_i containing three vertices and let X_n^{odd} be a subset of X_n containing one vertex. For both types of graph, let $X_i^{even} = X_i \backslash X_i^{odd}$, $1 \leq i \leq n$, and let $\mathbf{X}^{\mathbf{odd}} = \bigcup_{i=1}^{n} X_i^{odd}$.

Subgraph H. Consider the subgraph H, induced by the vertices $Y \cup \mathbf{X}^{\mathbf{odd}}$. Let $Y = \{y_1, y_2, \ldots, y_r\}$, and let $X_i = \{x_{i,j} : 1 \leq j \leq |X_i|\}$, $1 \leq i \leq n$. Then $V(H) = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, \ldots, x_{n,1}, x_{n,2}, x_{n,3}, y_1, y_2, \ldots, y_r\}$ for Type 1 graphs, and $V(H) = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, \ldots, x_{n-1,1}, x_{n-1,2}, x_{n-1,3}, x_{n,1}, y_1, y_2, \ldots, y_r\}$ for Type 2 graphs. For both types of graphs, let $|V(H)| = |Y \cup \mathbf{X}^{\mathbf{odd}}| = h$. When

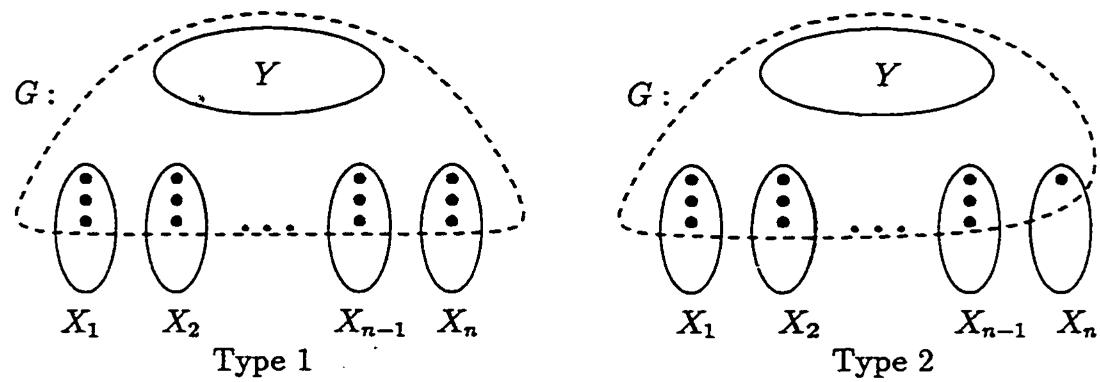

Figure 5.6: Case 4e, Subgraph H.

$r \equiv 1,3 \pmod 6$ and n even, $|\mathbf{X}^{\mathbf{odd}}| = 3n \equiv 0 \pmod 6$,

$r \equiv 0,4 \pmod 6$ and n odd, $|\mathbf{X}^{\mathbf{odd}}| = 3n + 3 \equiv 3 \pmod 6$,

$r \equiv 5 \pmod 6$ and n even, $|\mathbf{X}^{\mathbf{odd}}| = 3(n-1) + 1 \equiv 4 \pmod 6$, and

$r \equiv 2 \pmod 6$ and n odd, $|\mathbf{X}^{\mathbf{odd}}| = 3(n-1) + 1 \equiv 1 \pmod 6$.

Therefore, when

$r \equiv 1,3 \pmod 6$ and n even, $|Y \cup \mathbf{X}^{\mathbf{odd}}| = h \equiv 1,3 \pmod 6$,

$r \equiv 0,4 \pmod 6$ and n odd, $|Y \cup \mathbf{X}^{\mathbf{odd}}| = h \equiv 1,3 \pmod 6$,

$r \equiv 5 \pmod 6$ and n even, $|Y \cup \mathbf{X}^{\mathbf{odd}}| = h \equiv 3 \pmod 6$, and

$r \equiv 2 \pmod 6$ and n odd, $|Y \cup \mathbf{X}^{\mathbf{odd}}| = h \equiv 3 \pmod 6$.

Because $h \equiv 1,3 \pmod 6$ no matter what the value of r, the complete graph, K_h, has a cycle decomposition into triangles (see Lemma 19). To find a cycle decomposition of H, the cycle decomposition of K_h can be modified by removing the edges between the three vertices in each set X_i^{odd}, $1 \leq i \leq n$ (Type 1), and X_i^{odd},

$1 \le i \le n-1$ (Type 2).

For each i, $1 \le i \le n$ (Type 1), and $1 \le i \le n-1$ (Type 2), if there is a triangle through the vertices $x_{i,1}$, $x_{i,2}$, and $x_{i,3}$, then its edges can be deleted. Otherwise, edges $x_{i,1}x_{i,2}$, $x_{i,1}x_{i,3}$, and $x_{i,2}x_{i,3}$ lie in three distinct triangles, $c_1 = x_{i,1}x_{i,2}p_1x_{i,1}$, $c_2 = x_{i,1}x_{i,3}p_2x_{i,1}$, and $c_3 = x_{i,2}x_{i,3}p_3x_{i,2}$, $p_1 \ne p_2 \ne p_3$, $p_1, p_2, p_3 \in V(H)\backslash X_i^{odd}$, of the cycle decomposition of K_h. In this latter case, the removal of the edges results in the 6-cycle $x_{i,1}p_1x_{i,2}p_3x_{i,3}p_2x_{i,1}$ in H. Because the vertices $x_{i,1}$, $x_{i,2}$, and $x_{i,3}$ are all from the same part, this 6-cycle has at most six chords.

With the deletion of edges of K_h to form H, the cycle decomposition of K_h into triangles is also altered, and becomes a cycle decomposition of H composed of triangles and 6-cycles. Each 6-cycle in the cycle decomposition of H indicates that three triangles which existed in the cycle decomposition of K_h are lost. Let g be the number of 6-cycles in the cycle decomposition of H. Then the number of sets X_i^{odd}, whose three vertices were in a triangle in the cycle decomposition of K_h is, for Type 1 graphs, $n-g$, and for Type 2 graphs, $(n-1)-g$. For each X_i^{odd} whose vertices satisfy these conditions, only one triangle that existed in the cycle decomposition of K_h is absent in the cycle decomposition of H. Overall, in removing the edges between the vertices in each X_i, we therefore lose, for Type 1 graphs, $(n-g)+3g$, or $n+2g$, triangles, and for Type 2 graphs, $((n-1)-g)+3g$, or $n-1+2g$, triangles. In either case, we gain g cycles of length six, each with at most six chords. The cycle decomposition of K_h can be constructed so that the vertices in at least one set, X_i^{odd}, form a triangle, and so, for Type 1 graphs, $g \le n-1$, and for Type 2 graphs, $g \le n-2$.

Let $\mathcal{C}_H$ be the CDC of H consisting of two copies of this cycle decomposition into triangles and 6-cycles. Then $\mathcal{C}_H$ contributes digons to the transition multigraph of each vertex in H.

For a Type 1 graph, the number of edges in H is

$$|E(H)| = \binom{r+3n}{2} - 3n \tag{5.1}$$

while for a Type 2 graph, the number of edges is

$$|E(H)| = \binom{r+3(n-1)+1}{2} - 3(n-1) = \binom{r+3n-2}{2} - 3(n-1). \tag{5.2}$$

For a Type 1 graph, the number of cycles in the cycle decomposition of H is

$$\frac{1}{3}\left[\binom{r+3n}{2} - 3n - 6g\right] + g = \frac{1}{3}\left[\binom{r+3n}{2} - 3n\right] - g,$$

and so the number of cycles in the CDC, $\mathcal{C}_H$, is

$$|\mathcal{C}_H| = \frac{2}{3}\left[\binom{r+3n}{2} - 3n\right] - 2g = \frac{2}{3}|E(H)| - 2g.$$

For a Type 2 graph, the number of cycles in the cycle decomposition is

$$\frac{1}{3}\left[\binom{r+3n-2}{2} - 3(n-1) - 6g\right] + g = \frac{1}{3}\left[\binom{r+3n-2}{2} - 3(n-1)\right] - g,$$

and so the number of cycles in the CDC, $\mathcal{C}_H$, is

$$|\mathcal{C}_H| = \frac{2}{3}\left[\binom{r+3n-2}{2} - 3(n-1)\right] - 2g = \frac{2}{3}|E(H)| - 2g,$$

the same number obtained for that of Type 1 graphs. Because each 6-cycle has at most six chords, and each cycle is used twice in the CDC, $\mathcal{C}_H$, then for either type of graph,

$$\begin{aligned}
\gamma(\mathcal{C}_H) &= |\mathcal{C}_H| + ch(\mathcal{C}_H) \\
&\leq 2|E(H)|/3 - 2g + 12g \\
&= 2|E(H)|/3 + 10g.
\end{aligned}$$

Recall that for Type 1 graphs, $g \le n-1$, and so

$$\gamma(\mathcal{C}_H) \le 2|E(H)|/3 + 10(n-1), \tag{5.3}$$

while for Type 2 graphs, $g \le n-2$, and so

$$\gamma(\mathcal{C}_H) \le 2|E(H)|/3 + 10(n-2). \tag{5.4}$$

For Type 1 graphs, because $|E(H)| = \binom{r+3n}{2} - 3n$ and $r \ge 3$, we get

$$\begin{aligned} |E(H)| &= \binom{r+3n}{2} - 3n \\ &\ge \binom{3+3n}{2} - 3n \\ &= (3n+3)(3n+2)/2 - 3n \\ &= [(9n^2+15n+6) - 6n]/2 \\ &= (9n^2+9n+6)/2. \end{aligned}$$

Using this inequality and Equation 5.3, we get

$$\begin{aligned} \gamma(\mathcal{C}_H) &\le 2|E(H)|/3 + 10(n-1) \\ &= |E(H)| - |E(H)|/3 + 10(n-1) \\ &\le |E(H)| - [(9n^2+9n+6)/2]/3 + 10(n-1) \\ &= |E(H)| - (3n^2+3n+2)/2 + 10n - 10 \\ &= |E(H)| - (3n^2+3n+2-20n+20)/2 \\ &= |E(H)| - (3n^2-17n+22)/2 \\ &= |E(H)| - (3n-11)(n-2)/2 \\ &< |E(H)| \end{aligned}$$

when $n > 11/3$. We must therefore consider the cases when $n = 2$ and $n = 3$ separately.

When $n = 3$, we can substitute this value into the calculation of the number of edges in a Type 1 graph (Equation 5.1) to obtain

$$\begin{aligned} |E(H)| &= \binom{r+3n}{2} - 3n \\ &= \binom{r+9}{2} - 9 \\ &= (r+9)(r+8)/2 - 9 \\ &= [(r^2+17r+72) - 18]/2 \\ &= (r^2+17r+54)/2. \end{aligned}$$

Also, $10(n-1) = 20$, so Equation 5.3 becomes

$$\begin{aligned} \gamma(\mathcal{C}_H) &\leq 2|E(H)|/3 + 10(n-1) \\ &= |E(H)| - |E(H)|/3 + 20 \\ &= |E(H)| - [(r^2+17r+54)/2]/3 + 20 \\ &= |E(H)| - (r^2+17r+54-120)/6 \\ &= |E(H)| - (r^2+17r-66)/6 \\ &< |E(H)| \end{aligned}$$

when $r \geq 4$. However, n is odd, G is a Type 1 graph, and $r \equiv 0, 4 \pmod 6$, so $r \geq 4$, and the inequality holds.

When $n = 2$, substituting this value into Equation 5.1 gives

$$|E(H)| = \binom{r+3n}{2} - 3n$$

$$
\begin{aligned}
&= \binom{r+6}{2} - 6 \\
&= (r+6)(r+5)/2 - 6 \\
&= [(r^2 + 11r + 30) - 12]/2 \\
&= (r^2 + 11r + 18)/2.
\end{aligned}
$$

Also, $10(n-1) = 10$, so Equation 5.3 becomes

$$
\begin{aligned}
\gamma(\mathcal{C}_H) &\leq 2|E(H)|/3 + 10(n-1) \\
&= |E(H)| - |E(H)|/3 + 10 \\
&= |E(H)| - [(r^2 + 11r + 18)/2]/3 + 10 \\
&= |E(H)| - (r^2 + 11r + 18 - 60)/6 \\
&= |E(H)| - (r^2 + 11r - 42)/6 \\
&= |E(H)| - (r + 14)(r - 3)/2. \\
&< |E(H)|
\end{aligned}
$$

when $r > 3$.

We have now seen that $\gamma(\mathcal{C}_H) < |E(H)|$ for all Type 1 graphs, except when $r = 3$ and $n = 2$; this graph will be examined in detail at the end of this case.

We must now determine whether the inequality $\gamma(\mathcal{C}_H) < |E(H)|$ holds for Type 2 graphs. Recall that for Type 2 graphs, Equation 5.2 gives

$$
|E(H)| = \binom{r + 3n - 2}{2} - 3(n-1).
$$

Since $r \geq 5$,

$$
|E(H)| = \binom{r + 3n - 2}{2} - 3(n-1)
$$

$$\begin{aligned}
&\geq \binom{3n+3}{2} - 3(n-1) \\
&= (3n+3)(3n+2)/2 - 3n + 3 \\
&= [(9n^2 + 15n + 6) - 6n + 6]/2 \\
&= (9n^2 + 9n + 12)/2.
\end{aligned}$$

Therefore, Equation 5.4 becomes

$$\begin{aligned}
\gamma(\mathcal{C}_H) &\leq 2|E(H)|/3 + 10(n-2) \\
&= |E(H)| - |E(H)|/3 + 10(n-2) \\
&\leq |E(H)| - [(9n^2 + 9n + 12)/2]/3 + 10(n-2) \\
&= |E(H)| - (3n^2 + 3n + 4)/2 + 10n - 20 \\
&= |E(H)| - (3n^2 + 3n + 4 - 20n + 40)/2 \\
&= |E(H)| - (3n^2 - 17n + 44)/2 \\
&< |E(H)|
\end{aligned}$$

for all values of n. Therefore, for Type 2 graphs, $\gamma(\mathcal{C}_H) < |E(H)|$.

Subgraph J. Let J_i, $1 \leq i \leq n$, be the bipartite subgraph of G with parts X_i^{even} and $V(H)$, and all edges incident to vertices in both parts, and let $J = \bigcup_{i=1}^{n} J_i$.

For each integer i, $1 \leq i \leq n$, X_i is an independent set, and so the vertices of X_i^{even} are adjacent to all vertices in $V(H)$ except those in X_i^{odd}. Consequently, we can consider each subgraph J_i, $1 \leq i \leq n$, to be a complete bipartite graph with parts X_i^{even} and $V(H)\backslash X_i^{odd}$. For Type 1 graphs, $|V(H)\backslash X_i^{odd}| = |V(H)| - 3$ for all i, $1 \leq i \leq n$, while for Type 2 graphs $|V(H)\backslash X_i^{odd}| = |V(H)| - 3$ for all i, $1 \leq i \leq n-1$, and $|V(H)\backslash X_n^{odd}| = |V(H)| - 1$ for $i = n$. For both types of graphs, $|V(H)| \equiv 1, 3$

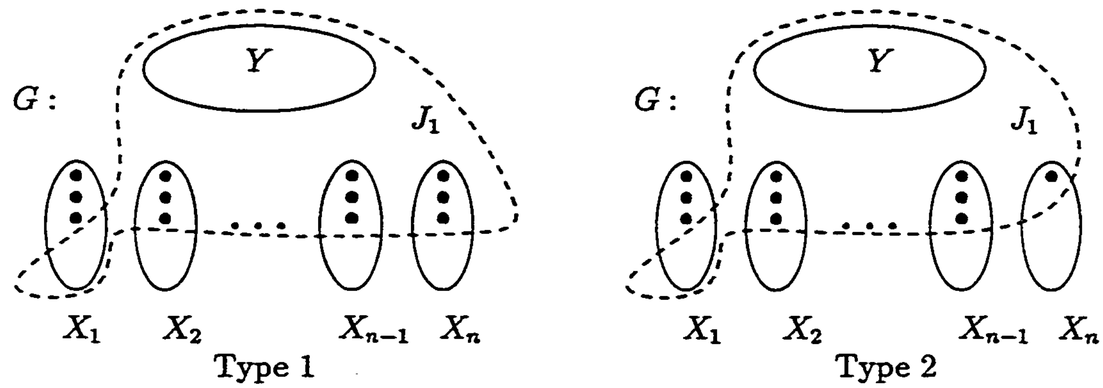

Figure 5.7: Case 4e, Edges of subgraph J_1.

(mod 6), and so $|V(H)\backslash X_i^{odd}|$ is even for all i, $1 \leq i \leq n$. As X_i^{even} also has even size, a CDC, $\mathcal{C}_{J_i}$, of each subgraph J_i, $1 \leq i \leq n$, can be found using the method described in Lemma 12 and Corollary 13 for complete bipartite graphs with two even parts. Each CDC consists of two copies of a decomposition into 4-cycles, and hence each CDC contributes a collection of digons to the transition multigraph of each vertex in the set $V(J_i)$. Because X_i^{even} is a subset of the part X_i, $1 \leq i \leq n$, of G, its vertices are non-adjacent, and so none of the 4-cycles in $\mathcal{C}_{J_i}$, $1 \leq i \leq n$, has more than one chord. Therefore, $ch(\mathcal{C}_{J_i}) \leq |\mathcal{C}_{J_i}|$, and hence

$$\begin{aligned}\gamma(\mathcal{C}_{J_i}) &= |\mathcal{C}_{J_i}| + ch(\mathcal{C}_{J_i}) \\ &\leq 2|\mathcal{C}_{J_i}| \\ &= 2(2|E(J_i)|/4) \\ &= |E(J_i)|.\end{aligned}$$

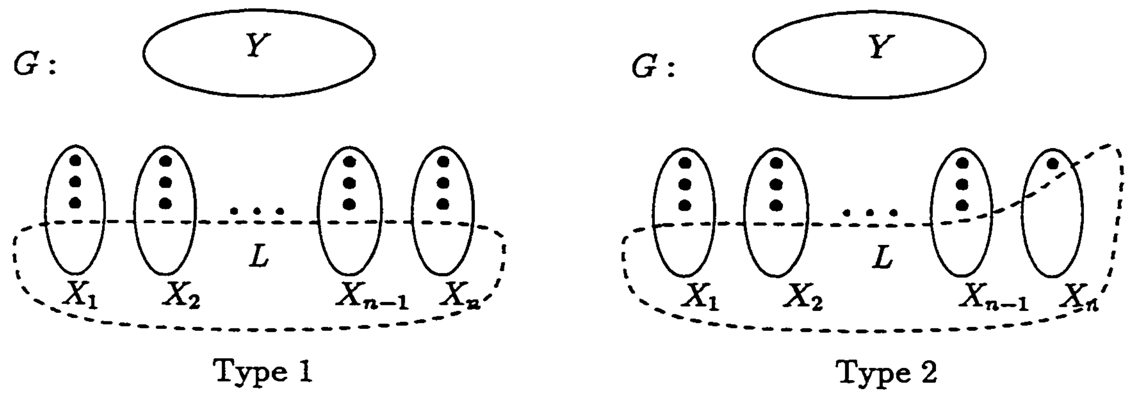

Figure 5.8: Case 4e, Subgraph L.

The subgraphs J_i, $1 \leq i \leq n$, partition the edge set of J, and so, by Lemma 10, $\gamma(\mathcal{C}_J) \leq |E(J)|$.

Subgraph L. Finally, we must find a CDC for the edges between the vertices in each pair of sets X_i^{even}, $1 \leq i \leq n$. Let L be the subgraph induced by the vertices $\bigcup_{i=1}^{n} X_i^{even}$.

This subgraph consists of subsets, X_i^{even}, $1 \leq i \leq n$, of the parts $X_1, \ldots, X_n$ of the graph, G. Because each of these subsets has even size, the graph L is a complete multipartite graph with all even parts. Using the construction in Case 1 of this proof, we get a CDC, $\mathcal{C}_L$, consisting of two copies of a decomposition into 4-cycles. All the cycles are chordless, so $\gamma(\mathcal{C}_L) < |E(L)|$, and $\mathcal{C}_L$ contributes digons to the transition multigraph of each vertex in $V(L)$.

Except for the case $r = 3$ and $n = 2$, the subgraphs H, J, and L partition the edge set of the graph, G. Furthermore, the CDCs, $\mathcal{C}_H$, $\mathcal{C}_J$, and $\mathcal{C}_L$ satisfy $\gamma(\mathcal{C}_H) < |E(H)|$, $\gamma(\mathcal{C}_J) \leq |E(J)|$, and $\gamma(\mathcal{C}_L) < |E(L)|$. By Lemma 10, the union of these CDCs is a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. Because the CDCs $\mathcal{C}_H$, $\mathcal{C}_J$, and $\mathcal{C}_L$ are each two copies of a cycle decomposition of the subgraphs H, J, and L, the transition multigraph of each vertex in the graph G consists of digons.

What remains is to show that when $r = 3$ and $n = 2$, $\gamma(\mathcal{C}_G) < |E(G)|$.

Let G be a complete multipartite graph with $r = 3$ parts of size one, and $n = 2$ parts of odd size, each with at least three vertices. The graph G can be thought of as the union of a triangle (subgraph T), and a 3-partite graph (subgraph W) with one part consisting of the three vertices of the triangle (vertices $V(T)$) and the other two parts of odd size, each with at least three vertices. The subgraph T has a CDC, $\mathcal{C}_T$, consisting of two copies of the triangle, and thus,

$$\gamma(\mathcal{C}_T) = |\mathcal{C}_T| = 2 < 3 = |E(T)|.$$

Denote the parts of W by $X = \{x_0, x_1, x_2\}$, $Y = \{y_0, \ldots, y_{n-1}\}$, and $Z = \{z_0, \ldots, z_{p-1}\}$, where $X = V(T)$.

By Case 3 of Theorem 21, W has a CDC, $\mathcal{C}_W$, consisting of two copies of a cycle decomposition. From the proof of this case, $\mathcal{C}_W$ consists of two triangles and $(|E(W)| - 3)/2$ cycles of length four. The only cycles of length four with two chords are the two copies of the C1 cycles $x_0 z_{p-1} x_1 y_{n-1} x_0$, and so $\beta_2 = 2$. Also, because $|Y| \geq 3$ and $|Z| \geq 3$, at least one of the C2 cycles in the decomposition is chordless, and so in the CDC, $\beta_0 \geq 2$. Therefore,

$$\gamma(\mathcal{C}_W) \;=\; |\mathcal{C}_W| + ch(\mathcal{C}_W)$$

$$
\begin{aligned}
&= 2 + (|E(W)| - 3)/2 + ch(\mathcal{C}_W) \\
&= (|E(W)| + 1)/2 + ((|E(W)| - 3)/2 - \beta_0 + \beta_2) \\
&\leq |E(W)| - 1 - 2 + 2 \\
&= |E(W)| - 1 \\
&< |E(W)|.
\end{aligned}
$$

Subgraphs T and W partition $E(G)$ and have CDCs $\mathcal{C}_T$ and $\mathcal{C}_W$, respectively, such that $\gamma(\mathcal{C}_T) < |E(T)|$ and $\gamma(\mathcal{C}_W) < |E(W)|$. Therefore, by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. Because both $\mathcal{C}_T$ and $\mathcal{C}_W$ are each composed of two copies of a cycle decomposition, $\mathcal{C}_G$ is also composed of two copies of a cycle decomposition, and thus the transition multigraph of each vertex in G consists of a collection of digons.

4f. $r \equiv 1, 3, 5 \pmod 6$, $r \geq 3$, and n odd; $r \equiv 0, 2, 4 \pmod 6$, $r \geq 4$, and n even. Let Y be the set containing the r vertices from the parts of size one, and let $X_1, \ldots, X_n$ be the parts of G containing at least three vertices.

Let H be the subgraph of G induced by $Y \cup X_1 \cup \ldots \cup X_{n-1}$. The subgraph, H, is therefore a complete multipartite graph with $r \equiv 1, 3, 5 \pmod 6$, $r \geq 3$, and n even, or $r \equiv 0, 2, 4 \pmod 6$, $r \geq 4$, and n odd, and so, from the results of Case 4e, we know that H has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) < |E(H)|$, that contributes only digons to the transition multigraph of each vertex in $V(H)$.

The only remaining edges in G to be covered are those that join the vertices in $V(H)$ to the vertices in X_n.

Let J be the complete bipartite subgraph of G with parts $A = (\bigcup_{i=1}^{n-2} X_i) \cup Y$ and X_n. For $r \equiv 1, 3, 5 \pmod 6$, $|Y|$ is odd, and so, since $n - 2$ is also odd, $|A|$ is even.

For $r \equiv 0, 2, 4 \pmod 6$, $|Y|$ is even, and so, since $n-2$ is also even, $|A|$ is again even. Therefore, as $|X_n|$ is odd, we can apply Lemma 15 to find a CDC, $\mathcal{C}_J$, of J. The CDC consists of 4-cycles, each containing a pair of non-adjacent vertices of X_n. Therefore, each cycle in $\mathcal{C}_J$ has at most one chord, and it follows that $ch(\mathcal{C}_J) \leq |\mathcal{C}_J|$. Hence,

$$\begin{aligned}\gamma(\mathcal{C}_J) &= |\mathcal{C}_J| + ch(\mathcal{C}_J)\\ &\leq 2|\mathcal{C}_J|\\ &= 2(2|E(J)|/4)\\ &= |E(J)|.\end{aligned}$$

Because $|A|$ is even and $|X_n|$ is odd, by Lemma 15 the transition multigraph of each vertex in A consists of a triangle and a collection of digons, while the transition multigraph of each vertex in X_n consists only of digons.

Let L be the subgraph of G, whose edges are induced by the vertices in the sets X_{n-1} and X_n. The subgraph is a complete bipartite graph with parts of odd size, each at least three. Lemma 17 can be applied directly, and so L has a CDC, $\mathcal{C}_L$, with $\gamma(\mathcal{C}_L) < |E(L)|$. From Lemma 17, we find that the transition multigraph of each vertex in $V(L)$ consists of a triangle and a collection of digons.

Because the subgraphs H, J, and L partition the edge set of G, and because each of these subgraphs has a CDC, $\mathcal{C}_H$, $\mathcal{C}_J$, and $\mathcal{C}_L$, respectively, satisfying $\gamma(\mathcal{C}_H) < |E(H)|$, $\gamma(\mathcal{C}_J) \leq |E(J)|$, and $\gamma(\mathcal{C}_L) < |E(L)|$, then by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The transition multigraph of each vertex in G consists of a triangle and a collection of digons; for the vertices in the sets $Y, X_1, \ldots, X_{n-2}$, the triangle results from the cycles of $\mathcal{C}_J$, while for the vertices in the sets X_{n-1} and X_n, the triangle results from the cycles of $\mathcal{C}_L$.

Case 5. G consists of s parts of odd size, and t parts of even size.
The proof of this case depends on the results of the previous cases of this theorem, and also on the results presented in Chapter 4. Graphs that have only one or two odd-sized parts, and in which each of these odd-sized parts has only one vertex, must be considered separately from all other graphs. This special consideration is necessary because it is either not possible to find a CDC of the subgraph consisting of the edges connecting a single odd-sized part to the vertices in the even-sized parts if there is only one vertex in the odd-sized part, or to form a CDC of the subgraph induced by two odd-sized parts, when there is only one vertex in each odd-sized part.

5a. $s = 1$, t even.
Let S be the part of odd size and let $X_1, \ldots, X_t$ be the parts of G which contain an even number of vertices.

If $t = 2$, then G is a 3-partite graph with one part of odd size and two parts of even size. We can therefore apply the results of Theorem 21 and Corollary 22 to find that $L(G)$ has an SCDC.

If $t \geq 4$, the even-sized parts can be paired to form supersets $Y_i = X_{2i-1} \cup X_{2i}$, $1 \leq i \leq t/2$. The subgraph H_i, induced by S and the vertices in the superset, Y_i, is a 3-partite graph, with one odd part and two even parts, and hence, as per Case 2 of Theorem 21, has a CDC, $\mathcal{C}_{H_i}$, with $\gamma(\mathcal{C}_{H_i}) \leq |E(H_i)|$ (equality only when H_i has one part of size one and two parts of size two). Therefore, by Lemma 10, the subgraph

$H = \bigcup_{i=1}^{t/2} H_i$ of G has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) \leq |E(H)|$. As each of the CDCs $\mathcal{C}_{H_i}$, $1 \leq i \leq t/2$, contributes digons to the transition multigraph of each vertex in S, and a triangle and a collection of digons to the transition multigraph of each vertex in the superset Y_i, the CDC, $\mathcal{C}_H$, also contributes digons to the transition multigraph of each vertex in S and a triangle and a collection of digons to the transition multigraph of each vertex in the parts $X_1, \ldots, X_t$.

In order to complete the CDC of G, it is necessary to double cover all edges between the pairs of supersets Y_i and Y_j, $1 \leq i, j \leq t/2$, $i \neq j$. Let J be the subgraph on the vertices $\bigcup_{i=1}^{t/2} Y_i$ containing these yet uncovered edges. The subgraph J is a complete multipartite graph with $t/2$ even parts, and so from Case 1 of this proof, J has a CDC, $\mathcal{C}_J$, consisting of two copies of a decomposition into 4-cycles. The CDC $\mathcal{C}_J$ thus contributes digons to the transition multigraph of each of its vertices. Because each of the parts X_l, $1 \leq l \leq t$, has an even number of vertices, the pairs of vertices on which the cycles are defined can be chosen so that in any pair, the vertices are from the same part X_l, $1 \leq l \leq t$. Each cycle in $\mathcal{C}_J$ is thus chordless. Hence

$$\begin{aligned}\gamma(\mathcal{C}_J) &= 2(|E(J)|/4) \\ &< |E(J)|.\end{aligned}$$

Combining the results for the CDCs of the subgraphs H and J, and applying Lemma 10, we find that G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The transition multigraph of each vertex in S consists of digons, while the transition multigraph of any vertex in an even part of G consists of a triangle and a collection of digons.

5b. $s = 1$, t odd; G is not $K_{1,2}$.

The line graph $L(K_{1,2})$ is the complete graph K_2, which trivially has a bridge. Therefore, as discussed in Chapter 2, $L(K_{1,2})$ does not have a CDC or an SCDC, and thus, $K_{1,2}$ must be excluded from this case.

If $t = 1$ and the odd part contains only one vertex, then as in Case 2b, G is a star and $L(G)$ is a complete graph. Therefore, as long as the part of even size has more than two vertices (so G is not $K_{1,2}$), $L(G)$ has an SCDC.

If $t = 1$ and the odd part contains at least three vertices, then G is a bipartite graph with one part of odd size at least three, and one part of even size. Therefore, we can apply Lemma 15 to find that G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$, that contributes a triangle and digons to the transition multigraph of each vertex in the part of even size, and a collection of digons to the transition multigraph of each vertex in the part of odd size.

If $t \geq 3$, let S be the part of odd size, and let $X_1, \ldots, X_t$ be the parts of G containing even numbers of vertices.

Let H be the subgraph induced by the vertices in S and the vertices in the parts $X_1, \ldots, X_{t-1}$. Because $t-1$ is even, we know from Case 5a that H has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) < |E(H)|$, which contributes digons to the transition multigraph of each vertex in S, and a triangle and a collection of digons to the transition multigraph of each vertex in the sets $X_1, \ldots, X_{t-1}$.

Let J be the subgraph on $V(G)$ containing the yet uncovered edges in the graph G. These edges are the edges between vertices in the subgraph, H, and vertices in the set X_t. Because the set $V(H)$ consists of vertices from $t-1$ even-sized parts, as well as the vertices in the odd-sized part, S, the number of vertices in $V(H)$ is odd. The set X_t, however, has an even number of vertices. Therefore, the subgraph J is a complete bipartite graph with one even part and one odd part. By Lemma 15, J has a CDC, $\mathcal{C}_J$, consisting of 4-cycles, which contributes a triangle and a collection of digons to the transition multigraph of each vertex in the set X_t, and a collection of digons to the transition multigraph of each vertex in the set $V(H)$. Because the set X_t is an independent set in G, the pairs of its vertices used in the 4-cycles of $\mathcal{C}_J$ are non-adjacent, and so contribute no chords to the cycles. Therefore, each 4-cycle has at most one chord, so $ch(\mathcal{C}_J) \leq |\mathcal{C}_J|$. Thus,

$$\begin{aligned}\gamma(\mathcal{C}_J) &= |\mathcal{C}_J| + ch(\mathcal{C}_J)\\ &\leq 2|\mathcal{C}_J|\\ &= 2(2|E(J)|/4)\\ &= |E(J)|.\end{aligned}$$

Because the subgraphs H and J partition the edge set of G, and because each of these subgraphs has a CDC, $\mathcal{C}_H$ and $\mathcal{C}_J$ respectively, where $\gamma(\mathcal{C}_H) < |E(H)|$, and $\gamma(\mathcal{C}_J) \leq |E(J)|$, then by Lemma 10, the graph G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. Each vertex in S has a transition multigraph that consists of digons, while each vertex in the sets $X_1, \ldots, X_t$ has a transition multigraph that consists of a triangle and a collection of digons; the CDC $\mathcal{C}_H$ contributes the triangle to the transition multigraph of each vertex in the sets $X_1, \ldots, X_{t-1}$, while the CDC $\mathcal{C}_J$ contributes the triangle to the transition multigraph of each vertex in the set X_t.

5c. $s = 2$.

Let Y_1 and Y_2 be the parts that contain an odd number of vertices, and let $X_1, \ldots, X_t$ be the parts of G that contain an even number of vertices.

Let H be the subgraph induced by the vertices in the sets Y_1, Y_2, and X_t. The subgraph H is a complete 3-partite graph with two odd parts and one even part, and hence has a CDC, $\mathcal{C}_H$, as described in Case 4 of Theorem 21, with $\gamma(\mathcal{C}_H) < |E(H)|$. From that case, we know that with respect to the CDC, $\mathcal{C}_H$, each vertex in Y_1 and Y_2 has a transition multigraph consisting of a triangle and a collection of digons, while each vertex in X_t has a transition multigraph consisting only of digons.

If $t = 1$, then we are done. Otherwise, let the set $A = Y_1 \cup Y_2 \cup X_t$. As Y_1 and Y_2 both have odd size and X_t has even size, $|A|$ is even. The edges in the graph which remain to be covered are those edges between every pair of the sets $A, X_1, \ldots, X_{t-1}$. Let J be the subgraph of G with vertices $V(G)$ containing these uncovered edges. The subgraph J is a complete multipartite graph with all even parts, and so Case 1 of this proof implies that J has a CDC, $\mathcal{C}_J$, which consists of two copies of a decomposition into 4-cycles. The 4-cycles of $\mathcal{C}_J$ connect pairs of vertices from different sets, and so if one vertex, y_1^* from Y_1, is paired with one vertex, y_2^* from Y_2, no other pair of vertices need include adjacent vertices. Thus, any 4-cycle including the pair of vertices $\{y_1^*, y_2^*\}$ has one chord, while all other 4-cycles are chordless. Therefore, $ch(\mathcal{C}_J) \leq |\mathcal{C}_J|$, and so

$$\begin{aligned}
\gamma(\mathcal{C}_J) &= |\mathcal{C}_J| + ch(\mathcal{C}_J) \\
&\leq 2|\mathcal{C}_J| \\
&= 2(2|E(J)|/4) \\
&= |E(J)|.
\end{aligned}$$

Because the CDC $\mathcal{C}_J$ consists of two copies of a cycle decomposition, it contributes only digons to the transition multigraph of each vertex in the subgraph, J.

The subgraphs H and J partition the edge set of G, and since each has a CDC, $\mathcal{C}_H$ and $\mathcal{C}_J$ respectively, where $\gamma(\mathcal{C}_H) < |E(H)|$, and $\gamma(\mathcal{C}_J) \leq |E(J)|$, then by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. Each vertex in Y_1 and each vertex in Y_2 has a transition multigraph that consists of a triangle and a collection of digons, while each vertex in the sets $X_1, \ldots, X_n$ has a transition multigraph that consists only of digons.

5d. $s \geq 3$.

Let H be the subgraph of G induced by the parts of odd size. The subgraph is, therefore, a complete multipartite graph with all odd parts, and so from Cases 2, 3, and 4 of this proof, we know that it has a CDC, $\mathcal{C}_H$, with $\gamma(\mathcal{C}_H) < |E(H)|$. Furthermore, unless $H = K_6$ (G has six parts of odd size, each containing one vertex), the transition multigraph of each vertex in H consists of a collection of digons, or a triangle and a collection of digons.

Let J be the subgraph induced by the parts of even size. If $t = 1$, then J has no edges and so, $\gamma(\mathcal{C}_J) = 0 = |E(J)|$. However, if $t \geq 2$, then from Case 1 of this proof, we know that the subgraph has a CDC, $\mathcal{C}_J$, with $\gamma(\mathcal{C}_J) < |E(J)|$, and that the transition multigraph of each vertex of J consists of digons.

Let L be the subgraph of G with vertices $V(G)$, containing the edges which connect the vertices of the subgraph H to the vertices of the subgraph J. This subgraph is a complete bipartite graph with parts $V(H)$ and $V(J)$ where $|V(H)|$ is even or odd, and $|V(J)|$ is even. By Lemma 12 or Lemma 15, depending on the parity of $|V(H)|$, the subgraph L has a CDC, $\mathcal{C}_L$, consisting only of 4-cycles. If the number of vertices in $V(H)$ is even, then $\mathcal{C}_L$ contributes only digons to the transition multigraph of each vertex in the subgraphs H and J, while if the number of vertices in $V(H)$ is odd, $\mathcal{C}_L$ contributes digons to the transition multigraph of each vertex in $V(H)$, but contributes a triangle and a collection of digons to the transition multigraph of each vertex in $V(J)$. Because the set $V(J)$ consists of even-sized parts, vertices in this set can be paired so that the two vertices in each pair are in the same part of G. Therefore, the pairs of vertices of the set $V(J)$ contribute no chords to the 4-cycles in $\mathcal{C}_L$. Some pairs of vertices in $V(H)$, however, contain vertices which are adjacent in G, but the number of these pairs can be minimized by grouping, whenever possible, two vertices from the same part. Each pair of adjacent vertices in $V(H)$ contributes one chord to each cycle in which it appears, and so each cycle in $\mathcal{C}_L$ has at most one chord. Therefore, $ch(\mathcal{C}_L) \leq |\mathcal{C}_L|$, with equality only when all vertices in $V(H)$ are adjacent (H is a complete graph). Hence,

$$\begin{aligned}\gamma(\mathcal{C}_L) &= |\mathcal{C}_L| + ch(\mathcal{C}_L)\\ &\leq 2|\mathcal{C}_L|\\ &= 2(2|E(L)|/4)\\ &= |E(L)|.\end{aligned}$$

Because the subgraphs H, J, and L partition the edge set of G, and because each of these subgraphs has a CDC, $\mathcal{C}_H$, $\mathcal{C}_J$, and $\mathcal{C}_L$, respectively, where $\gamma(\mathcal{C}_H) < |E(H)|$, $\gamma(\mathcal{C}_J) \leq |E(J)|$ (equality only when $t = 1$), and $\gamma(\mathcal{C}_L) \leq |E(L)|$ (equality only when H is complete), then by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. Depending on the structure of the odd-sized parts in the graph, the transition multigraph of a vertex in G consists of a collection of digons, or of a triangle and a collection of digons, except when G has exactly six parts of odd size, each containing one vertex.

We must now examine the graph, G, with exactly six parts of odd size, each containing one vertex (the subgraph H, as described previously, is K_6). Let $Y = \{y_0, \ldots, y_5\}$ be the set containing the six vertices from the parts of size one, and let $X_1, \ldots, X_t$ be the parts of even size. Let u and v be two vertices in the set X_t.

We define Q as the subgraph $G[Y \cup \{u\} \cup \{v\}]$, and define Q_1 as the subgraph of Q induced by the vertex u and the vertices in the set Y, and Q_2 as the subgraph of Q induced by v and vertices in the set Y. The subgraph Q_1 is a complete graph on seven vertices and since $7 \equiv 1 \pmod 6$, it has a cycle decomposition, $\mathcal{C}_{Q_1}$, into $|E(Q_1)|/3 = 7$ triangles (see Lemma 19). Because of the symmetry of the vertices in Y, we can label the vertices so that the triangles in $\mathcal{C}_{Q_1}$ containing vertex u are the triangles uy_0y_1u, uy_2y_3u, and uy_4y_5u.

The subgraph Q_2 is also a complete graph on seven vertices, and has a cycle decomposition, $\mathcal{C}_{Q_2}$, obtained by replacing the vertex u with the vertex v in the cycles of $\mathcal{C}_{Q_1}$. Therefore, $\mathcal{C}_{Q_2}$ also consists of seven triangles, including vy_0y_1v, vy_2y_3v, and vy_4y_5v, the triangles which pass through vertex v.

In defining the 4-cycles

$$\begin{aligned} \text{C1} &= uy_0vy_1u \\ \text{C2} &= uy_2vy_3u \\ \text{C3} &= uy_4vy_5u, \end{aligned}$$

we find that

$$\mathcal{C}_Q = \mathcal{C}_{Q_1} \cup \mathcal{C}_{Q_2} \cup \{\text{C1, C2, C3}\}$$

is a CDC of Q. Edges incident with two vertices in Y are covered twice, once by a cycle in $\mathcal{C}_{Q_1}$, and once by a cycle in $\mathcal{C}_{Q_2}$. Edges incident with vertex u are covered twice, once by a cycle in $\mathcal{C}_{Q_1}$, and once by a 4-cycle. Similarly, edges incident with vertex v are covered twice, once by a cycle in $\mathcal{C}_{Q_2}$, and once by a 4-cycle.

Triangles have no chords, but the 4-cycles C1, C2, and C3 each contain one chord, contributed by the pair of vertices from Y. Therefore, $ch(\mathcal{C}_Q) = 3$, and so

$$\begin{aligned}\gamma(\mathcal{C}_Q) &= |\mathcal{C}_Q| + ch(\mathcal{C}_Q) \\ &= (7+7+3)+3 \\ &= 20 \\ &< 27 = |E(Q)|.\end{aligned}$$

The transition multigraph of vertex y_0 contains edges uy_1, vy_1 and uv, contributed by the triangles uy_0y_1u and vy_0y_1v and the 4-cycle C1. Because the cycles in $\mathcal{C}_{Q_2}$ were constructed by replacing vertex u with vertex v in the cycles of $\mathcal{C}_{Q_1}$, the CDC $\mathcal{C}_Q$ has two copies of each triangle that passes through three vertices in Y. Therefore, these triangles contribute digons to the transition multigraph, $M_T(y_0)$, and so $M_T(y_0)$ consists of the triangle uy_1vu and $(d(y_0)-3)/2 = 2$ digons. Because of the symmetry in construction of the cycles in $\mathcal{C}_Q$, the transition multigraph of every other vertex in Y also consists of a triangle and two digons.

The transition multigraph, $M_T(u)$, consists of the edges y_0y_1, y_2y_3, and y_4y_5, contributed by the triangles of $\mathcal{C}_{Q_1}$ that pass through u, and one other copy of the edges y_0y_1, y_2y_3, and y_4y_5, contributed by the 4-cycles C1, C2, and C3. Thus $M_T(u)$ consists of three digons. Because of the symmetry between u and v in the cycles of $\mathcal{C}_Q$, $M_T(v)$ also consists of three digons.

Let R be the subgraph of G on vertices Y and $A = X_t \backslash \{u, v\}$, consisting of edges incident with vertices in both sets. The subgraph R is a complete bipartite graph with two even parts, and so by applying the results of Lemma 12 and Corollary 13, R has a CDC, $\mathcal{C}_R$, consisting of two copies of a cycle decomposition of R into 4-cycles. The CDC $\mathcal{C}_R$ thus contributes digons to the transition multigraph of each vertex in $V(R)$.

Because A is a subset of the part X_t, vertices in A are non-adjacent in G. All vertices of Y are adjacent in G, however, hence each 4-cycle in $\mathcal{C}_R$ has one chord. Therefore, $ch(\mathcal{C}_R) = |\mathcal{C}_R|$, and

$$\begin{aligned}\gamma(\mathcal{C}_R) &= |\mathcal{C}_R| + ch(\mathcal{C}_R) \\ &= 2|\mathcal{C}_R| \\ &= 2(2|E(R)|/4) \\ &= |E(R)|.\end{aligned}$$

Let W be the subgraph on $V(G)$ containing the yet uncovered edges of G. Then W is a complete multipartite graph, with parts $Y \cup X_t, X_1, \ldots, X_{t-1}$. Each part X_i, $1 \leq i \leq t$ has even size, and $|Y| = 6$, so the parts of W all have even size. Therefore, by Case 1 of this proof, W has a CDC, $\mathcal{C}_W$, consisting of two copies of a cycle decomposition into 4-cycles, which contributes digons to the transition multigraph of each vertex in G. Because each set X_i, $1 \leq i \leq t-1$ is an independent set, pairs of vertices from these sets contribute no chords to the cycles of $\mathcal{C}_W$. Therefore, each cycle has at most one chord, and so $ch(\mathcal{C}_W) \leq |\mathcal{C}_W|$. Thus

$$\gamma(\mathcal{C}_W) = |\mathcal{C}_W| + ch(\mathcal{C}_W)$$

$$\begin{aligned} &\leq 2|\mathcal{C}_W| \\ &= 2(2|E(W)|/4) \\ &= |E(W)|. \end{aligned}$$

The subgraphs Q, R, and W partition the edge set of G, and since each has a CDC, $\mathcal{C}_Q$, $\mathcal{C}_R$, and $\mathcal{C}_W$, respectively, where $\gamma(\mathcal{C}_Q) < |E(Q)|$, $\gamma(\mathcal{C}_R) = |E(R)|$, and $\gamma(\mathcal{C}_W) \leq |E(W)|$, then by Lemma 10, G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$. The transition multigraph of each vertex in Y consists of a triangle, contributed by $\mathcal{C}_Q$, and a collection of digons, while the transition multigraph of each vertex in the sets $X_1, \ldots, X_t$ consists of a collection of digons.

In each case of this proof, we have shown that either the line graph $L(G)$ of the graph G has an SCDC, or that G has a CDC, $\mathcal{C}_G$, with $\gamma(\mathcal{C}_G) < |E(G)|$, which contributes a collection of digons or a triangle and a collection of digons to the transition multigraph of each vertex in G. In the latter case, by applying Lemmas 6, 8, and 9 to the stated result, we conclude that the line graph $L(G)$ has an SCDC.

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Chapter 6

Conclusion

The Small Cycle Double Cover Conjecture states that every simple, bridgeless graph has a small cycle double cover. No-one has yet found a counterexample to this conjecture, but the conjecture has been verified for various classes of graphs: even graphs with maximum degree four, even planar graphs, 4-connected planar graphs, complete graphs, complete bipartite graphs, squares of trees, trigraphs, simple triangulations, and eulerian line graphs.

MacGillivray and Seyffarth have proved that the SCDC Conjecture holds for bridgeless line graphs of planar graphs, and for bridgeless line graphs of two specific classes of complete multipartite graphs: those with one vertex in each part (complete graphs), and those with only two parts (bipartite graphs). In this thesis, the results of MacGillivray and Seyffarth are extended, and it is proved that bridgeless line graphs of all complete multipartite graphs have SCDCs.

The technique developed by MacGillivray and Seyffarth for proving that the line graph of some graph has an SCDC was used in this thesis. This technique depends on the fact that the edge set of the line graph is partitioned by a collection of cliques corresponding to the vertices of the original graph. Because each of these cliques can be covered by a PPDC, their union forms a PDC of the line graph. The paths in this PDC can be joined together in a certain pattern to form a CDC of the line graph. The method for joining these paths is determined by the cycles of a CDC of the original graph. By counting the number of cycles in the CDC of the line graph, as well as the number of their chords, an upper limit to the number of cycles in the CDC of the line graph can be given, and thus it can easily be verified whether or not the CDC of the line graph is actually an SCDC. Consequently, in using this method, the problem of finding an SCDC of a line graph is reduced to finding a CDC, with appropriate properties, of the original graph.

Since a graph usually has a less complex structure than its line graph, it makes sense to continue to use the methodology developed by MacGillivray and Seyffarth and used in this thesis for proving that line graphs of other classes of graphs have SCDCs. The more structured a graph, the easier it is to find a CDC of that graph with the appropriate properties. Therefore, if one were to use this technique to prove that the line graphs of certain classes of graphs have SCDCs, it would be logical to study other well-structured classes of graphs.

The method developed by MacGillivray and Seyffarth could perhaps be enhanced if one were to find other PPDCs of vertex cliques with easily predictable associated multigraphs, since then the number of appropriate CDCs would be greater.

In verifying that certain classes of graphs fulfill the SCDC Conjecture, the general statement of the conjecture is not proved. However, the techniques developed for proving that various classes of graphs have SCDCs may eventually lead to a proof of the conjecture or to a counterexample to the conjecture.

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