THE UNIVERSITY OF CALGARY

FINITE GRAPH RAMSEY THEORY

BY

David S. Gunderson

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

November, 1990

ISBN 0-315-66995-0

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Finite Graph Ramsey Theory," submitted by David S. Gunderson in partial fulfillment of the requirements for the degree of Master of Science in Pure Mathematics.

Chairman, Dr. Norbert W. Sauer
Department of Mathematics and Statistics

Dr. Eric C. Milner
Department of Mathematics and Statistics

Dr. Robert E. Woodrow
Department of Mathematics and Statistics

Dr. G. W. Sands
Department of Mathematics and Statistics

Dr. Ian H. Witten
Department of Computer Science

Date Jan. 9/91

Abstract

For relational structures F, G, H, and a positive integer r, the Ramsey arrow notation $F \longrightarrow (G)^H_r$ means that for any partition of the H-substructures induced in F into at most r classes, there is a G-substructure induced in F having all its induced H-substructures in one partition class. The central aim in this thesis is to determine for which pairs of finite graphs G, H, an F can be found satisfying the Ramsey arrow. This pursuit extends to hypergraphs, both ordered and unordered.

Some background results are surveyed. Both the finite and infinite form of Ramsey's theorem are proved, together with results on arithmetic progressions. A theorem by Hales and Jewett concerning partitions of combinatorial spaces is proved. Van der Waerden's theorem is then derived from this. Shelah's proof that the Hales-Jewett function is primitive recursive is given.

Well known vertex and edge partition Ramsey theorems for graphs are proved. The edge partition theorem is proved using 'partite amalgamation', a technique developed by Nešetřil and Rödl. The existence of a k-uniform hypergraph which has arbitrarily large girth and chromatic number is shown by means of a constructive proof by Nešetřil and Rödl.

The Ramsey theorem for ordered hypergraphs is given in its utmost generality. A proof by Nešetřil and Rödl is given in which both the Hales-Jewett theorem and partite amalgamation is used. The arguments used for this powerful theorem are in every way complete, and until now, have been inaccessible due to the complexity of the proof and technical difficulties in the literature. This theorem completely answers the main question of this thesis for ordered hypergraphs.

Applications of the Ramsey theorem for ordered hypergraphs were developed by V. Rödl, N. W. Sauer, and the present author. These include a complete characterization of those pairs of finite *unordered* graphs and hypergraphs G,H and integers r for which there exists a 'Ramsey graph' F satisfying $F \longrightarrow (G)^H_r$. This characterization settles many questions, but since it is in terms of chromatic numbers, it is difficult to employ in the actual production of the pairs G,H. For P_2, a path on three vertices, it is found that those graphs G for which there exists an F satisfying $F \longrightarrow (G)^{P_2}_2$ are precisely those which are both chordal and comparability graphs or those satisfying an easily stated ordering condition. This is also a new significant step by the three authors in the direction of a complete answer to the main question of this thesis.

Theorems and conjectures regarding minimal Ramsey graphs are also surveyed. The author presents a new minimal ordered graph and a conjecture is extended.

For the most part, Chapters 5 and 6 represent new results jointly discovered by V. Rödl, N. W. Sauer and the present author. Chapter 7 contains musings regarding some independent research of the author.

Acknowledgements

My deepest gratitude extends to my supervisor, Dr. Norbert W. Sauer, for asking the right questions, then seriously considering my answers. For keeping me on the right path while always allowing me to learn for myself, I thank you Norbert; I can only hope to acquire such patience. Allow me to thank you again for your support in sending me to an international conference. Both your financial assistance and your confidence in me are appreciated more than you can ever realize.

I would like to thank the Department of Mathematics and Statistics for both the financial and emotional support which enabled me to thrive as a graduate student. I thank the many people in the department who were so generous with a smile, an ear, or a hand; among these many are L. Bates, T. Bisztriczky, H. K. Farahat, R. K. Guy, E. C. Milner, R. A. Mollin, G. W. Sands, J. Schaer, M. G. Stone, R. E. Woodrow, P. Zvengrowski, the administration, the secretaries, and fellow graduate students. A debt is also owed to V. Rödl, whose understanding went a long way.

And Ute, only I know how vital you were to my progress. I thank you.

Contents

List of Figures

Chapter 0

Introduction

0.1 The Party

There seems to be a standard way to introduce Ramsey theory, and that is by means of the 'party puzzle'. I was first introduced to it by Prof. Ron Aharoni in an undergraduate set theory course. I was told a story similar to the following one.

Suppose there was a party, a very crowded party, and six people were forced into a room away from the main crowd. The door was closed behind the sixth person and the introductions were about to begin. But before we let these six people talk to each other, let us just freeze the action for a moment. If three of these six people were selected, what is the chance that these three people already knew each other, that is, that each of the three knew the other two? If the likelihood of any one person at the party knowing another is one half, then there are 7 to 1 odds against the three selected people all knowing each other. (Here we assume that if A knows B, then B knows A.) This works for any three people at the party, not just those frozen in time in the side room. Similarly, based on the same probabilities, there are also 7 to 1 odds against these three people being complete strangers to one another. What is the likelihood that of these six people, either three people are mutual acquaintances, or three people are mutual strangers, (or both)? The claim is, that no matter what select group of six people are ushered into the room, at least one of the two scenarios holds.

For the sake of discussion, let A, B, C, D, E, and F be the names of the six people. The possibility exists that three of the six all know each other *and* the other three are mutual strangers, however the claim is only that at least one of these situations hold. We will now prove our claim.

Take a look at A. She faces five remaining people in the room. Classifying the five, she divides them into two groups, those she knows and those she does not. Of these two groups, one must have at least three people in it. As we will soon see, it matters not which, so suppose that the group of strangers is largest and that, say, C, D, and E are three she hasn't met. If C has never met D then the triple A,C,D is a group of three mutual strangers and we are done. So suppose C knows D. Similarly, if C has never achieved enough courage to be introduced to E, the triple A,C,E would suffice as the group of strangers, so assume C and E know each other, too. Lastly, if D and E did not know each other, then the triple A,D,E would be the trio of strangers, so suppose D and E have met. Then C,D,E must be a triple of acquaintances. In each possible case we have given, one of the required situations arises. Now apply the same argument when the group of friends is the larger lot and obtain the analogous result. So the claim has been shown.

If only five people were in the room, one could suggest a scenario which shows that the 'three-three' result does not occur. Namely, if A knows B, B knows C, C knows D, D knows E, E knows A, and no one knows anyone else in the room, then no such result holds. This shows that six is the minimum number of people required in the room to guarantee this 'three-three' phenomenon. Naturally, if more people were allowed into the room, the same bizarre claim would hold.

If we required a foursome to exist which either all knew each other or were total strangers to each other, then eighteen people in the room would suffice. The same claim for fivesomes holds but the minimum number of people has not been calculated yet.

0.2 Ramsey Theory

The fact that six people suffices for either a 'stranger trio' or 'friendly trio' is abbreviated by $6 \longrightarrow (3,3)$. Similarly, $18 \longrightarrow (4,4)$. We could have asked how many people are required so that, say, either 3 are friends or 4 are strangers. This number is 9, denoted by $9 \longrightarrow (3,4)$. For any such party, one could draw a chart with dots representing people, and lines between pairs of dots representing the relationship between these people. Every pair would have a colored line joining them, say, red indicating friendship, and blue indicating those pairs who have not yet met. Then such a chart, or graph, on 6 points would contain either a red triangle or a blue triangle by thc carlier claim. Notice that a triangle contains all the edges possible on three points, hence we call it 'complete'. So the statement $6 \longrightarrow (3,3)$ is the same as saying that if we are given a complete graph on six vertices with edges colored red and blue, we could find in it a red triangle or a blue triangle (or possibly both). For any number k, does there always exist a number $\mathrm{R}(k,k)$ so that $\mathrm{R}(k,k) \longrightarrow (k,k)$ holds?

The answer is an astounding "yes"; these numbers are called *Ramsey numbers.* The existence of these numbers was proved by F. P. Ramsey earlier this century while trying to solve a problem in logic. Three Hungarians rediscovered this fact, and one of them, Pal Erdős, generalized and popularized the theory. Extensions were made to the infinite too, however, in this thesis, we restrict ourselves to finite.

As it turns out, Ramsey theory is intimately connected to many areas of mathematics. Geometry, arithmetic progressions, field theory, lattice theory, block designs, and graph theory are just a few of the areas which Ramsey theory has contributed to. Besides graph theoretic results, we include, as a matter of taste, a few results from other areas of Ramsey theory.

0.3 The Objective

Without being too precise, an *induced subgraph* of a graph G is a collection of vertices in G together with all the edges of G found among vertices in the collection. If F, G, and H are graphs and r is a positive integer, the Ramsey arrow notation $F \longrightarrow (G)_r^H$ means that for any labelling of the induced H-subgraphs (subgraphs isomorphic to H) of F with at most r distinct labels, there exists an induced G-subgraph of F, call it G', so that all the induced H-subgraphs of G' have the same label.

The main objective in this thesis *was* to completely 'classify' all those pairs of graphs G,H and integers r for which there existed a Ramsey F, that is, a graph F satisfying $F \longrightarrow (G)_r^H$. This is one of the main problems in the field of 'finite graph Ramsey theory', or more properly, 'finite induced graph Ramsey theory'. The case for ordered graphs (graphs with a given orientation of the vertices) has already been solved—surprisingly, every pair qualifies! The situation for unordered graphs is more complicated.

A complete characterization is given of those graphs and hypergraphs G,H for which there exists a Ramsey F for r colors. This characterization involves orderings and chromatic numbers of hypergraphs, so it does not satisfactorily meet the main objective. It does, however, serve nicely as a tool by which to answer many questions. For example, when $H = P_2$, a path on 3 vertices, all those graphs G for which there exists an F satisfying $F \longrightarrow (G)_2^H$ are explicitly given. Cases for most other non-trivial cases are intractable.

It should be noted that it is not part of the main objective to give a complete overview of the field. There are many areas of Ramsey theory , even topics involving finite graph Ramsey theory which are not discussed here. In 1987, H. J. Prömel [100] produced *Ramsey theory for discrete structures*, a manuscript over 350 pages long. It contains wonderful accounts of many early results together with a good cross section of modern results. See [55] for another excellent compilation of theorems, proofs, history, and anecdotes which describe the field of Ramsey theory more eloquently than we can hope to do here. Both works contain splendid bibliographies. Among other significant works which contain excellent surveys are [51], [54], [64], [84], [86], [108], [109], [112], [113], all vast in their scope.

There have been easy to read, albeit enlightening, articles on Ramsey theory. One very recent one [57] appeared in *Scientific American*. That particular introduction to Ramsey theory should perhaps be required reading for anyone interested in the area—it will only serve to enhance one's enthusiasm for puzzle solving. Other very good introductory reading includes [49].

Of Chapter 1, only one result is required later in the thesis, namely the finite version of Ramsey's original theorem. The remainder of the contents of Chapter 1 are included only to give a very brief historical perspective—hence some results are only stated, not proven. The two works of Prömel and Voigt, [112] and [113] represent the definitive work on two chapters of early Ramsey theory (integrated with modern material).

Similarly, the only result contained in Chaptcr 2 which is critical to latcr chapters is the Hales-Jewett Theorem. The implications of the Hales-Jewett theorem are so numerous and exciting that we include some related results. All other results found in Chapters 1 and 2 are included solely as a matter of preference. The main theorems are contained in Chapters 3 thru 6. Chapter 7 represents a related interest of the author, and as of yet, does not have a major bearing with respect to satisfying the main objective of the thesis. However, the author believes that the direction taken in Chapter 7 will ultimately prove to be invaluable, and so we include the preliminary examinations.

Chapter 1

Some Background

1.1 Notation

For a set X and a subset Y of X, we write $Y \subset X$. If $Y \subset X$ and it is possible that $X = Y$, then we write $Y \subseteq X$ to emphasize this. The notation $x \in X$ is used if x is an element of X. The power set

$$\mathcal{P}(S) = \{T : T \subseteq S\}$$

of a set S is the collection of all subsets of S. The notation 2^S is sometimes used to denote the power set of S. The symbols $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ are used to denote the set of natural numbers, integers, rational numbers, and real numbers respectively. The set of non-negative integers is denoted by ω, the first infinite ordinal. For $m \in \omega$ it is often convenient to use the ordinal representation,

$$m = \{0, 1, \ldots, m-1\}.$$

For non-negative integers i, m satisfying $0 \leq i < m$, we may write $i \in m$.

For a set S and a given $n \in \omega$ we define

$$[S]^n = \{T \subseteq S : |T| = n\}$$

to be the set of all subsets of S of size n. Similarly we define $[S]^{\leq n}$. We denote by $\lceil x \rceil$ the least integer $z \geq x$ and $\lfloor x \rfloor$ is the greatest integer $y \leq x$.

The expression "if and only if" is occasionally abbreviated by "iff".

1.2 Some Familiar Partitions

For a fixed $r \in \omega$ and a set S, a *partition* of S into r classes is a collection $\{S_i \subset S : i \in r\}$ satisfying $\cup_{i \in r} S_i = S$, $S_i \cap S_j = \emptyset$ for $i \neq j$, and for each $i \in r$, $S_i \neq \emptyset$. A partition of S into at most r classes can be viewed as a function $\Delta : S \longrightarrow r$. Such a partition is also called an *r-coloring* of S; usually Δ will denote such a function. Braces are often deleted, as in $\Delta(x, y)$ rather than $\Delta(\{x, y\})$.

Perhaps the simplest of all combinatorial results having a 'Ramsey flavor' is the *pigeon hole principle.* If $r + 1$ pigeons were to roost in r holes, then two pigeons would just have to get acquainted. In general, if $r(m-1)+1$ pigeons were to roost in r holes, then there would be at least one hole with m pigeons in it. Written in the notation of colorings, this says that for any coloring

$$\Delta : r(m-1) + 1 \longrightarrow r$$

there exist m elements of $\{0, 1, 2, \ldots, r(m-1)\}$ which are monochromatic with respect to Δ (colored the same). Similarly, if one were to divide an infinite set into two 'smaller' sets, then one of them must be infinite also (if they were both finite, putting them back together gives just a finite set, obviously not the whole set!). The same argument works for dividing an infinite set into any finite number of pieces. We give this in the form of a lemma, also called the *pigeon hole principle.*

Lemma 1.2.1 *For any fixed $r \in \omega$ and $\Delta : \omega \longrightarrow r$, there exists an $i \in r$ so that $\Delta^{-1}(i)$ is infinite.*

Along the same vein, one can discuss partitioning 'subtrees of an infinite tree' in order to give a very useful result.

A *partially ordered set* (or simply a *poset*) $(P, \leq)$ is a set P together with a relation, $\leq$ which is reflexive ($p \leq p$), antisymmetric ($p \leq q$ and $q \leq p$ implies $p = q$), and transitive ($p \leq q \leq r$ implies $p \leq r$), that is, $\leq$ is a *partial order*. The relation $\leq$ is a *total order*, or *linear order* if for any two elements $p, q \in P$, either $p \leq q$ or $q \leq p$ holds. A poset $(T, \leq)$ is a *tree* if for every $t \in T$, the set $\{x \in T : x \leq t\}$ is a totally ordered set with no infinite descending subsequence. A tree $(T, \leq)$ is said to be *rooted* if there exists a unique vertex (called the *root*) $v \in T$ with the property that $v \leq x$ for every $x \in T$. A vertex $y \neq x$ is a *successor* of x if $x \leq y$ and for any $z \neq x$ satisfying $x \leq z \leq y$, $y = z$ holds. A tree is *locally finite* if every vertex has finitely many successors. A *branch* of a tree is a maximal linearly ordered subset. We recall König's Infinity Lemma [72]:

Lemma 1.2.2 *A locally finite rooted infinite tree has an infinite branch.*

Proof: Let $(T, \leq)$ be a locally finite tree with root v, and let $v_0^1, v_1^1, \ldots, v_i^1, \ldots$, ($i \in I^1$ be a labelling of the successors of v. By Lemma 1.2.1, one of the trees

$$\{(T_i^1, \leq) : T_i^1 = \{x \in T : v_i^1 \leq x\} : i \in I^1\}$$

is infinite, say T_0^1 (with root v_0^1) is one such. Repeat this idea using trees having roots which are successors of v_0^1 to obtain another infinite tree T_0^2. Continue in this manner to get an infinite number of trees, T_0^1, T_0^2, $T_0^3, \ldots$. Then the vertices $v, v_0^1, v_0^2, \ldots$ determine an infinite path. $\Box$

1.3 Ramsey's Theorem

Frank Plumpton Ramsey was a logician, economist, philosopher, and mathematician born in 1903. He died at 26 years of age. Despite his early demise, he was a man of many accomplishments. (See for example, [100],[112],[55] for personal history and further references.) Among these accomplishments was the following theorem [116], known as the 'infinite version of Ramsey's theorem'. The proof given here is an adaptation of that found in [100].

Theorem 1.3.1 *Let a countably infinite set G and positive integers $r,k \in \omega$ be given. For any r-coloring $\Delta : [G]^k \longrightarrow r$, there exists an infinite set $X \subset G$ so that Δ is constant on $[X]^k$. We denote this by $\omega \longrightarrow (\omega)^k_r$.*

Proof: The proof is by induction on k. The case $k = 1$ is trivial by the pigeon hole principle, so assume the theorem is true for some $k \geq 1$. Let $\Delta : [G]^{k+1} \longrightarrow r$ be a given coloring and pick an arbitrary $x_0 \in G$. Then Δ induces a coloring $\Delta_0 : [G \backslash \{x_0\}]^k \longrightarrow r$ by $\Delta_0(H) = \Delta(H \cup \{x_0\})$. So by the induction hypothesis, there exists an infinite set $A_0 \subset G$ so that Δ_0 is constant on $[A_0]^k$, and hence Δ is constant on

$$\{x_0\} \times [A_0]^k = \{\{x_0, y_1, \ldots, y_k\} : \{y_1, \ldots, y_k\} \in [A]^k\} \subset [G]^{k+1},$$

say $\Delta(\{x_0\} \times [A_0]^k) = r_0 \in r$. Now pick any element $x_1 \in A_0$. Repeating the same argument, there exists an infinite set $A_1 \subset A_0$ so that Δ is constant on $\{x_1\} \times [A_1]^k$, say $\Delta(\{x_1\} \times [A_1]^k) = r_1 \in r$. [Note that r_0 and r_1 may be different, while still $\Delta(\{x_0\} \times [A_1]^k) = r_0$.] Continuing in this manner, we get a set $X = \{x_i : i \in \omega\}$ so that for H,$H' \in [X]^{k+1}$, $\Delta(H) = \Delta(H')$ whenever $\min(H) = \min(H')$. [If $i =$

$\min\{j : x_j \in H\}$ we say $\min(H) = x_i$.] This induces an r-coloring Δ^* of X by $\Delta^*(x_i) = \Delta(H)$ for any $H \in [X]^{k+1}$ satisfying $\min(H) = x_i$.

By the pigeon hole principle, there is an infinite set $Y \subset X$ so that Δ^* is constant on Y, and hence Δ is constant on $[Y]^{k+1}$, since

$$[Y]^{k+1} \subseteq \{H \in [X]^{k+1} : min(H) \in Y\}.$$

□

Let us remark that it is sometimes convenient to totally order the set G at the beginning of the above proof, then we need only pick the lowest element in the order for the subsequent element in the formation of X. Such a device also allows us to pick an ordering of some other type rather than just an ω-ordering and obtain extensions of this result, but this does not concern us directly in this thesis. Similar results are also obtained for other infinite cardinals. The infinite version of Ramsey's theorem was first generalized to all cardinals by P. Erdős and R. Rado [40], [38]. (There is an extensive bibliography on the subject of infinite generalizations; for other work see [36], [35], [120].) We are more concerned with obtaining finite results.

Ramsey gave a construction for the proof of the finite version, but we prefer to derive it from the infinite version using König's Lemma, (a method referred to in [41]—perhaps for the first time). We now give the finite version of Ramsey's theorem.

Theorem 1.3.2 *For any $m, k, r \in \omega$, there is a smallest number $n = R(m, k, r)$ so that for any coloring $\Delta : [n]^k \longrightarrow r$, there exists $A \in [n]^m$ so that Δ is constant on $[A]^k$. We express this property of n by writing $n \longrightarrow (m)^k_r$.*

Proof: Assume, in hope of a contradiction, that the theorem does not hold, that is, for every $n \in \omega$ there exists a 'bad' coloring $\Delta : [n]^k \longrightarrow r$ so that for every

$A \in [n]^m$, Δ is not constant on $[A]^k$. The restriction of a bad coloring $\Delta : [n]^k \longrightarrow r$ to a coloring $\Delta^* : [n-1]^k \longrightarrow r$ is again bad. Order all such bad colorings by restriction to form a tree $(T, \leq)$ with the coloring of the empty set as the root. $(T, \leq)$ is locally finite since any coloring of $[n-1]^k$ can only have finitely many 'extensions' to a coloring of $[n]^k$. Thus by König's Lemma (Lemma 1.2.2), $(T, \leq)$ contains an infinite branch, corresponding to a bad coloring of ω. This contradicts Theorem 1.3.1. □

The Ramsey arrow notation $n \longrightarrow (m)^k_r$ is due to Erdős and Rado [39]. The numbers $R(m, k, r)$ are very elusive. Extensive studies of these are given in [128] and [51]. Other comments on these numbers appear later throughout this thesis.

1.4 A Geometric Analogue

A proof of the finite version of Ramsey's theorem (simply known as Ramsey's theorem) was rediscovered by G. Szekeres inspired by an observation of Esther Klein (later to become E. Szekeres) in a completely different setting. (See, for example, [57] for an interesting account; also see [55].) The generalization of E. Klein's result [41] is by Erdős and Szekeres:

Theorem 1.4.1 *For any $n \in \omega$, there exists $m \in \omega$ so that if m points on a plane are placed with no three collinear, then there are n of the m points which determine a convex n-gon.*

This theorem has been generalized (e.g., [6]) and reproven (e.g., [69], [55]). Many modern day proofs use Ramsey's Theorem. The original paper contains a proof using Ramsey's discovery and a second proof using a blend of geometric and combinatorial arguments. (See also [53] for another geometric analogue to Ramsey's theorem. For Ramsey-type theorems for metric spaces, see [71].

1.5 Schur's Lemma

The following theorem is of Ramsey nature, even though it appeared in 1916, when F. P. Ramsey was approximately 13 years old. It is a result of I. Schur, a student of Hilbert and doctoral supervisor of R. Rado. The proof we give [100] uses Ramsey's Theorem. The result is known as "Schur's Lemma" [119], a slight strengthening of the original.

Theorem 1.5.1 *Fix $r \in \omega$. Then there exists $n \in \omega$ so that for any coloring*

$$\Delta : \{1, 2, \ldots, n\} \longrightarrow r,$$

there exist positive integers $x, y \in \{1, \ldots, n\}$ so that

$$\Delta(x) = \Delta(y) = \Delta(x + y).$$

Proof: Using Ramsey's Theorem, let $n \in \omega$ so that $n - 1 \longrightarrow (3)^2_r$. Fix a coloring $\Delta : \{1, \ldots, n\} \longrightarrow r$. Then Δ induces a coloring $\Delta^* : [n]^2 \longrightarrow r$ defined by $\Delta(a, b) = \Delta(b - a)$ for $b > a$. By the choice of n there exist positive integers $u, v, w \in n$, $u < v < w$, so that

$$\Delta^*(u, v) = \Delta^*(u, w) = \Delta^*(v, w),$$

and hence,

$$\Delta(v - u) = \Delta(w - u) = \Delta(w - v).$$

Setting $x = v - u$ and $y = w - v$ concludes the proof. □

Schur was actually trying to reprove the following theorem of Dickson, a modular form of 'Fermat's Last Theorem'. For the proof, a basic knowledge of algebra is assumed.

Theorem 1.5.2 *For every $m \in \omega$ and for all primes p sufficiently large, the equation*

$$x^m + y^m \equiv z^m \pmod{p}$$

has a non-trivial solution.

Proof: Fix $m \in \omega$. Using Schur's Lemma, pick a prime p sufficiently large so that for any coloring $\Delta : p \rightarrow m$, there exist positive integers $x, y \in p$ so that

$$\Delta(x) = \Delta(y) = \Delta(x+y).$$

Let $G = \mathbf{Z}_p^*$ be the multiplicative group of the field $\mathbf{Z}_p$. It is well known that G is cyclic (of order $p-1$), so for convenience write $G = \{a, a^2, \ldots, a^{p-1} = 1\}$, where 1 is the identity of the group. Let

$$H = \{x^m \pmod{p} : x \in G\}$$

and let

$$K = \{x \in G : x^m \equiv 1 \pmod{p}\}.$$

K is the kernel of the obvious homomorphism of G, having H as the image. So $H \cong G/H$, and by Lagrange's theorem, $|K| = |G|/|H| = |G/H|$, the number of distinct cosets xH partitioning G. We wish to use this partition as a coloring, so first we need to count the elements in K.

Let $p-1 = qd$ and $m = rd$ where $d = \gcd(m, p-1)$. Now $a^j \in K$ iff $(p-1) \mid jm$ iff $q \mid jr$. But q and r are relatively prime, so $a^j \in K$ if and only if $q \mid j$. Thus

$$K = \{a^q, a^{2q}, \ldots, a^{(d-1)q}, a^{dq}\},$$

hence $|K| = d \leq m$.

The d cosets xH of G define a coloring $\Delta^* : G \longrightarrow d$ defined by $\Delta^*(b) = \Delta^*(c)$ if and only if b and c are in the same coset, i.e., when $b^{-1}c \in H$. By the choice of p, there exists positive integers $a, b \in p$ so that $\Delta^*(a) = \Delta^*(b) = \Delta^*(a+b)$. In $\mathbf{Z}_p$, $1 + a^{-1}b = a^{-1}(a+b)$ and each of 1, $a^{-1}b$, and $a^{-1}(a+b)$ are m-th powers (i.e., are elements of H). □

1.6 Van der Waerden's Theorem

We conclude this chapter with one last important theorem. B. L. van der Waerden attended a lecture given by Baudet in which he learned of a conjecture by Schur (for his own account of the story see [127]). He managed to give a proof [126] for the so called 'Baudet's conjecture', now becoming eponymous with van der Waerden. We give one form of van der Waerden's theorem:

Theorem 1.6.1 *Fix $r,t \in \omega$. Then there exists a smallest $n = W(t,r)$ so that for any coloring $\Delta : n \longrightarrow r$ there exists a monochromatic arithmetic progression containing t terms in n.*

This will follow from the Hales-Jewett theorem in Section 2.4. A result by Shelah will prove that the function $W(t,r)$ is primitive recursive. Some numbers $W(t,r)$ are known, for example, $W(2,2) = 3$, $W(3,2) = 9$, $W(4,2) = 35$, and $W(5,2) = 178$.

See [19] and [125] for some work done on finding these numbers. For a related theorem using infinitely many colors, see [102].

There are many other significant results proved early in the age of Ramsey Theory; we choose to omit them. See any of the references mentioned in the introduction for a thorough historical exposition.

Chapter 2

The Hales-Jewett Theorem

2.1 Introduction

In this chapter we prove a Ramsey-type theorem which is very general in its combinatorial nature, namely the Hales-Jewett Theorem. Remarkably, this theorem, which is central in Ramsey theory, arose from a generalization of the game 'Tic-Tac-Toe'. It can be stated in terms of 'combinatorial subspaces' or in terms of 'parameter words'. Although we give the development and proof in terms of parameter words, we can give the general idea as follows. Suppose $f : n \longrightarrow A$ is a function corresponding to an ordered n-tuple from a finite alphabet A. We are interested in *combinatorial subspaces* of A^n. For example, the set of functions which agree with f in all but, say, m fixed positions is a particular kind of an m-dimensional subspace of A^n. The Hales-Jewett Theorem says we can make n large enough so that no matter how we partition $A^n = \{f : n \longrightarrow A\}$ into r classes, there exists an m-dimensional subspace of A^n which is contained entirely in one class.

In choosing parameter words as the method of presentation numerous corollaries may be obtained in a rather straightforward manner. We give some of these corollaries, among which are van der Waerden's Theorem and applications suitable for discussion of lattices. Perhaps more importantly, we use the Hales-Jewett Theorem in a later chapter to prove a difficult theorem which is used many times in subsequent chapters. The result we speak of is a Ramsey theorem for ordered hypergraphs.

Also included in this chapter is a result by Shelah which concerns bounds on the number promised by the Hales-Jewett theorem. It is found that this number is bounded above by a primitive recursive function.

2.2 Notation and Preliminaries

Throughout this chapter, A is a finite alphabet and $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ are symbols not in A, called *parameters.* As usual, we use $A^n = \{f : n \longrightarrow A\}$. For $m \leq n$ we define the set of *m-parameter words* of length n over A by

$$[A]\binom{n}{m} = \{f : n \longrightarrow (A \cup \{\lambda_0, \lambda_1, \ldots, \lambda_{m-1}\}) : \\ \forall j \in m, f^{-1}(\lambda_j) \neq \emptyset \text{ and for } i < j, \min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)\}.$$

So $[A]\binom{n}{m}$ can be viewed as a set of ordered n-tuples containing each of the $\lambda_i, i \in m$, at least once and the first occurrence of λ_i must precede the first occurrence of λ_j if $i < j$. For example, if $A = \{a, b, c\}$, $ab\lambda_0\lambda_1$ and $\lambda_0 c\lambda_1\lambda_0$ are in $[A]\binom{4}{2}$ but $a\lambda_1\lambda_0\lambda_1$ and $a\lambda_0\lambda_0 b$ are not. We make the trivial observation that $A^n = [A]\binom{n}{0}$. For $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ we define the composition $f \circ g \in [A]\binom{n}{k}$ by

$$f \circ g = \begin{cases} f(i) & \text{if } f(i) \in A, \\ g(j) & \text{if } f(i) = \lambda_j. \end{cases}$$

For example if $f = a\lambda_0\lambda_1\lambda_0$ and $g = b\lambda_0$, then $f \circ g = ab\lambda_0 b$.

Lemma 2.2.1 *The composition of parameter words is associative.*

Proof: Let $f \in [A]\binom{n}{m}$, $g \in [A]\binom{m}{l}$ and $h \in [A]\binom{l}{k}$. Since $g \circ h \in [A]\binom{m}{k}$ and $f \circ g \in [A]\binom{n}{l}$, both $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are defined and are parameter words in $[A]\binom{n}{k}$. Using the definition we have

$$f \circ (g \circ h)(i) = \begin{cases} f(i) & \text{if } f(i) \in A, \\ g(\alpha) & \text{if } f(i) = \lambda_\alpha \text{ and } g(\alpha) \in A, \\ h(j) & \text{if } f(i) = \lambda_\alpha \text{ and } g(\alpha) = \lambda_j, \end{cases}$$

and

$$(f \circ g) \circ h(i) = \begin{cases} f(i) & \text{if } f(i) \in A \text{ and } f \circ g(i) \in A, \\ g(\alpha) & \text{if } f(i) = \lambda_\alpha \text{ and } f \circ g(i) \in A, \\ h(j) & \text{if } f \circ g(i) = \lambda_j. \end{cases}$$

If $f(i) \in A$ then $f \circ g(i) \in A$, and so trivially $f(i) \in A$ if and only if $f \circ g(i) \in A$. If $f(i) \notin A$, say $f(i) = \lambda_\alpha$ and $g(\alpha) \in A$, then $f \circ g(i) = g(\alpha)$. Similarly, if $f(i) \notin A$, say $f(i) = \lambda_\alpha$, and $g(\alpha) = \lambda_j$, then $f \circ g(i) = \lambda_j$. Hence the conditions given above are equivalent and so $f \circ (g \circ h) = (f \circ g) \circ h$. □

For $f \in [A]\binom{n}{m}$, define the *space* of f,

$$\text{sp}(f) = \{f \circ g : g \in [A]\binom{m}{0}\},$$

sometimes denoted $f \circ [A]\binom{m}{0}$, to be the set of words from $[A]\binom{n}{0}$ which are formed by faithfully replacing parameters in f with elements from A. We define an *m-dimensional (combinatorial) subspace* of A^n to be the space of some word in $[A]\binom{n}{m}$. If $f \in [A]\binom{n}{1}$ then we say sp(f) is a *combinatorial line* in A^n, or simply, a line. For example, if $f = a\lambda_0\lambda_1\lambda_0$, then $\text{sp}(f) = \{ax_0x_1x_0 : x_0, x_1 \in A\} \subset A^4$. In this example sp$(f)$ can be seen as a 2-dimensional subspace of A^4 where $f \in [A]\binom{4}{2}$.

Note that if $A = \{0, 1, \ldots, t-1\}$ and we view A^n as a discrete 'geometric' n-cube, then not all 'geometric lines' are combinatorial lines. For example, $(\{2,0,0\}, \{1,1,0\}, \{0,2,0\})$ is a geometric line with equation $x + y + z = 2$ in the three dimensional cube over $\{0,1,2\}$ but 200, 110, 020 is not a combinatorial line in $[\{0,1,2\}]\binom{3}{0}$. However, there are Ramsey type theorems for affine spaces and vector spaces (e.g. [47], [50] and [123]), although we do not go into these here.

We now give a type of concatenation of parameter words which distinguishes between parameters from respective words. If $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{k}{l}$ we define $f\,\hat{}\,g \in [A]\binom{n+k}{m+l}$ as follows:

$$f\,\hat{}\,g(i) = \begin{cases} f(i) & \text{if } i < n, \\ g(i-n) & \text{if } i \geq n \text{ and } g(i-n) \in A, \\ \lambda_{j+m} & \text{if } i \geq n \text{ and } g(i-n) = \lambda_j. \end{cases}$$

For example, $a\lambda_0 b\lambda_1\,\hat{}\,c\lambda_0\lambda_1\lambda_0 a = a\lambda_0 b\lambda_1 c\lambda_2\lambda_3\lambda_2 a$. Note that

$$\mathrm{sp}(f\,\hat{}\,g) = \{f'\,\hat{}\,g' : f' \in \mathrm{sp}(f), g' \in \mathrm{sp}(g)\}.$$

Counting the number of elements in $[A]\binom{n}{m}$ is interesting. If $|A| = t$, then clearly

$$\left|[A]\binom{n}{0}\right| = t^n.$$

Now examine $f \in [A]\binom{n}{1}$. Since $f(i) \in A \cup \{\lambda_0\}$ and $f^{-1}(\lambda_0) \neq \emptyset$, there are $(t+1)^n - t^n$ choices for f, that is,

$$\left|[A]\binom{n}{1}\right| = (t+1)^n - t^n.$$

Lemma 2.2.2

$$\left|[A]\binom{n}{m}\right| = \frac{1}{m!}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(t+i)^n.$$

Proof: Let $B = (A \cup \{\lambda_0, \lambda_1, \ldots, \lambda_{m-1}\})^n$ be the set of all strings of length n formed from letters of A and the set of m parameters. Setting $|A| = t$, then we easily see that $|B| = (t+m)^n$. Let $C_m \subset B$ be the set of strings each of which contains exactly m distinct parameters. (So $[A]\binom{n}{m} \subset C_m$.) For each $j \in m$, let

$$B_j = \{f \in B : f^{-1}(\lambda_j) = \emptyset\} \subset B$$

be those strings not containing the parameter λ_j. Then, by the Inclusion-Exclusion principle,

$$\begin{aligned} |C_m| &= |B \backslash (B_0 \cup \ldots \cup B_{m-1})|, \\ &= (t+m)^n - \binom{m}{1}(t+m-1)^n + \binom{m}{2}(t+m-2)^n - \ldots, \\ &= \sum_{i=0}^{m}(-1)^{m-i}(t+i)^n. \end{aligned}$$

Any element of C_m contains m parameters occurring first in any of $m!$ possible orders, and $[A]\binom{n}{m}$ determines those elements using a specific order of first occurrences, so we have that

$$m!\left|[A]\binom{n}{m}\right| = |C_m| = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(t+i)^n.$$

Division by $m!$ concludes the proof. □

One can obtain a recursive formula for $\left|[A]\binom{n}{m}\right|$.

Lemma 2.2.3 *Setting* $|A| = t$*,*

$$\left|[A]\binom{n+1}{m}\right| = (t+m)\left|[A]\binom{n}{m}\right| + \left|[A]\binom{n}{m-1}\right|.$$

Proof:

$$\left|[A]\binom{n+1}{m}\right| = \frac{1}{m!}\sum_{i=0}^{m}(-1)^{m-1}\binom{m}{i}(t+i)^{n+1},$$

$$= \frac{1}{m!}\left[\sum_{i=0}^{m-1}\left((-1)^{m-i}\binom{m}{i}(t+m-(m-i))(t+i)^n\right) + (t+m)^{n+1}\right],$$

$$= \frac{1}{m!}\left[\sum_{i=0}^{m-1}\left((t+m)(-1)^{m-i}\binom{m}{i}(t+i)^n - (-1)^{m-i}(m-i)\binom{m}{i}(t+i)^n\right)\right.$$
$$\left. + (t+m)^{n+1}\right],$$
$$= \frac{t+m}{m!}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(t+i)^n - \frac{1}{m!}\sum_{i=0}^{m-1}(-1)^{m-i}(m-i)\binom{m}{i}(t+i)^n,$$
$$= \frac{t+m}{m!}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(t+i)^n + \frac{m}{m!}\sum_{i=0}^{m-1}(-1)^{m-1-i}\binom{m-1}{i}(t+i)^n,$$
$$= (t+m)\left|[A]\binom{n}{m}\right| + \left|[A]\binom{n}{m-1}\right|. \ \square$$

This recursion formula can be made sense of by looking at how $f \in [A]\binom{n+1}{m}$ could be constructed. If $f(n) = \lambda_{m-1}$ and $f(i) \neq \lambda_{m-1}$ for $i < n$, then $f = g \hat{} \lambda_{m-1}$ for some $g \in [A]\binom{n}{m-1}$. If $f(i) = \lambda_{m-1}$ for some $i < n$ then f can be formed from some $g \in [A]\binom{n}{m}$ by affixing $f(n)$, of which there are $t+m$ choices, to the end of g.

The expression for $\left|[A]\binom{n}{m}\right|$ agrees with that for "non-central Stirling numbers of the second kind" [100]. The recursion formula developed here for such numbers resembles one for Stirling numbers of the second kind. The insight for such a relation came from the study of parameter words, and thus it is possible that parameter words have use in the study of partitions and factorial polynomials.

2.3 The Main Theorem

The following result, known as the Hales-Jewett Theorem [62], is one of the main tools used in a later chapter. This theorem captures the intrinsic nature of Ramsey-type theorems.

Theorem 2.3.1 *Let m,$r \in \omega$ and a finite alphabet A be given. Then there exists a smallest number $n = HJ(|A|, m, r) \in \omega$ so that for every coloring $\Delta : [A]\binom{n}{0} \longrightarrow r$ there is an $f \in [A]\binom{n}{m}$ for which $sp(f)$ is monochromatic.*

Proof: Put $t = |A|$. The proof will be by induction on t and will be based on the following two inequalities:

$$\begin{aligned} \mathrm{HJ}(t, m+1, r) &\leq \mathrm{HJ}(t, 1, r) + \mathrm{HJ}(t, m, r^{t^{HJ(t,1,r)}}), &(2.1)\\ \mathrm{HJ}(t+1, 1, r+1) &\leq \mathrm{HJ}(t, 1 + \mathrm{HJ}(t+1, 1, r), r+1). &(2.2) \end{aligned}$$

Observe that for all m and r, $\mathrm{HJ}(1, m, r) = m$ since the space of any word in $[\{a\}]\binom{n}{m}$ is unique (for any $n \geq m$). Let $P(t, m, r)$ denote the assertion that the theorem is true for t,m, and r. It follows by induction on m that, for a fixed t, (2.1) implies

$$\forall r[P(t, 1, r)] \Rightarrow \forall m, r[P(t, m, r)]. \qquad (2.3)$$

Now suppose for the moment that t_0 is smallest so that there exists r_0 with

$$\neg[P(t_0 + 1, 1, r_0 + 1)].$$

Fix r_0 smallest. Then for all r, both $P(t_0, 1, r)$ and $P(t_0 + 1, 1, r_0)$ hold. So by (2.3),

$$\forall m, r[P(t_0, m, r)]$$

holds and using $m = 1 + \mathrm{HJ}(t_0 + 1, 1, r_0)$ we then have, by (2.2), that

$$P(t_0 + 1, 1, r_0 + 1)$$

holds, a contradiction. So, essentially, induction on t is done by first decreasing r by 1. Thus it remains to prove (2.1) and (2.2).

Throughout the proof fix t, m, and r.

Proof of (2.1): Set $M = \mathrm{HJ}(t,1,r)$ and $N = \mathrm{HJ}(t,m,r^{t^M})$. Fix a coloring $\Delta : [A]\binom{M+N}{0} \longrightarrow r$ and define $\Delta_N : [A]\binom{N}{0} \longrightarrow r^{t^M}$ by

$$\Delta_N(f) = \langle \Delta(g\,\hat{}\,f) : g \in [A]\binom{M}{0}\rangle,$$

i.e., each f is colored with a sequence induced by Δ. Let $f_N \in [A]\binom{N}{m}$, guaranteed by the choice of N, be so that $\mathrm{sp}(f_N)$ is monochromatic with respect to Δ_N. Define $\Delta_M : [A]\binom{M}{0} \longrightarrow r$ by

$$\Delta_M(g) = \Delta(g\,\hat{}\,(f_N \circ h))$$

for any $h \in [A]\binom{m}{0}$. Note that by our choice of f_N this does not depend on A.So there exists $f_M \in [A]\binom{M}{1}$ so that $\mathrm{sp}(f_M)$ is monochromatic with respect to Δ_M.

Setting

$$f = f_M\,\hat{}\,f_N \in [A]\binom{M+N}{1+m},$$

we claim that $\mathrm{sp}(f)$ is monochromatic with respect to Δ. For any $l \in [A]\binom{1}{0}$, $h \in [A]\binom{m}{0}$,

$$f \circ (l\,\hat{}\,h) = (f_M \circ l)\,\hat{}\,(f_N \circ h)$$

and so $\Delta(f \circ (l\hat{}h)) = \Delta_M(f_M \circ l)$ is constant. Hence we have shown (2.1).

Proof of (2.2): In this part, we put $M = \mathrm{HJ}(t+1,1,r)$ and $N = \mathrm{HJ}(t, 1+M, r+1)$. With $|A| = t$, choose $b \notin A$ and put $B = A \cup \{b\}$. Fix a coloring

$$\Delta : [B]\binom{N}{0} \longrightarrow r+1.$$

We wish to show there exists a monochromatic line in B^N, i.e., an $h \in [B]\binom{N}{1}$ so that $h \circ [B]\binom{1}{0}$ is monochromatic with respect to Δ.

Define

$$\Delta_A : [A]\binom{N}{0} \longrightarrow r+1,$$

the restriction of Δ to A, in the natural way (i.e.,$\Delta_A(f) = \Delta(f)$ for any $f \in A^N$). By the theorem, there is $f_A \in [A]\binom{N}{1+M}$ which has $f_A \circ [A]\binom{1+M}{0}$ monochromatic with respect to Δ_A. Without loss, let $\Delta_A(f) = r$ for every $f \in f_A \circ [A]\binom{1+M}{0}$. Now if there is a $g \in [B]\binom{M}{0}$ so that also $\Delta(f_A \circ (\langle b \rangle \hat{\ } g)) = r$, (i.e., if there is a word in B^N containing b's which occur in the same positions as λ_0 occurs in f_A and is colored the same as f_A) construct h by replacing the occurrence of each b in $f_A \circ (\langle b \rangle \hat{\ } g)$ with λ_0. Clearly $h \in [B]\binom{N}{1}$ and

$$h \circ \langle x \rangle \in f_A \circ [A]\binom{1+M}{0}$$

whenever $x \in A$, and so in this case $\Delta(h \circ \langle x \rangle) = r$. Also

$$h \circ \langle b \rangle = f_A \circ (\langle b \rangle \hat{\ } g)$$

and so $\Delta(h \circ \langle b \rangle) = r$. We have shown that in this case $h \circ [B]\binom{1}{0}$ is monochromatic with respect to Δ.

Now suppose there is no such $g \in [B]\binom{M}{0}$ satisfying $\Delta(f_A \circ (\langle b \rangle \hat{\ } g)) = r$. Define $\Delta_M : [B]\binom{M}{0} \longrightarrow r$ by

$$\Delta_M(g) = \Delta(f_A \circ (\langle b \rangle \hat{\ } g)).$$

By the theorem, there is $f_M \in [B]\binom{M}{1}$ with $f_M \circ [B]\binom{1}{0}$ monochromatic with respect to Δ_M. Look at

$$h = f_A \circ (\langle b \rangle \hat{\ } f_M) \in [B]\binom{N}{1}.$$

Then $h \circ [B]\binom{1}{0}$ is monochromatic with respect to Δ, finishing the proof. □

A natural question arises. Can the word guaranteed by the Hales-Jewett theorem be chosen so that for each i, $|f^{-1}(\lambda_i)| = 1$? The following example (given by N. Sauer - oral communication) answers this in the negative. Let $A = \{0, 1, \ldots, 9\}$, $m = 1$, $r = 2$, and fix $n = \mathrm{HJ}(10, 1, 2)$. Now define the coloring $\Delta : [A]\binom{n}{0} \longrightarrow 2$ by

$$\Delta(f) = \sum_{i \in n} f(i) \pmod 2.$$

One need only observe that if $\mathrm{sp}(f)$ is a monochromatic line in A^n, then f must have the parameter in an even number of places, not just one.

It is interesting to note that Graham and Rothschild [52] proved a much stronger partition result for parameter words generalizing the Hales-Jewett theorem to higher dimensions. Although we do not give the proof here, we include the statement of the Graham-Rothschild theorem for completeness:

Theorem 2.3.2 *Let k,m,$r \in \omega$ and a finite alphabet A be given. Then there exists a smallest number $n = GR(|A|, k, m, r) \in \omega$ so that for every r-coloring $\Delta : [A]\binom{n}{k} \longrightarrow r$, there exists $f \in [A]\binom{n}{m}$ so that $f \circ [A]\binom{m}{k}$ is monochromatic.*

Observe that the Hales-Jewett theorem is the case $k = 0$. See [98], [105], [111] for extensive discussion and extensions fo the Graham-Rothschild theorem.

2.4 Some Applications

We are now ready to give one proof of van der Waerden's theorem.

Proof of Theorem 1.6.1: Let $A = \{0, 1, 2, \ldots, t-1\}$ and define a map $\psi : [A]\binom{n}{0} \longrightarrow t^n$ by $\psi(f) = \sum_{i \in n} f(i) t^i$. Observe that ψ is one to one on A^n. For an element $f \in [A]\binom{n}{1}$, $\mathrm{sp}(f)$ determines an arithmetic progression of t terms under the mapping ψ. In this setting, Theorem 2.3.1 immediately gives the result. □

The above proof occurred in [55], however another idea [100] is to set $\psi(f) = \sum_{i\in n} f(i)$. Then, as above, if $f \in [A]\binom{n}{1}$, then $\mathrm{sp}(f)$ determines an arithmetic progression. If we denote an arithmetic progression by $\{a_0 + id : i \in t\}$, then the given proof of van der Waerden's theorem does not enable us to specify d. However, some conditions on d may be imposed. For example, suppose for some given $x \in \omega$ we need d to be of the form $d = \sum_{i\in I\subset\omega} x^i$ and we need $t' \leq x$ terms. Apply the proof with $t = x$ obtaining f with

$$\psi(f) = \sum_{i\in n\setminus I} a_0 x^i + \sum_{i\in I} \lambda_0 x^i.$$

Now only pick, say, the first t' terms of the progression determined by $\mathrm{sp}(f)$. Observe that the alternate idea of setting $\psi(f) = \sum f(i)$ does not allow any non-trivial conditions on d. We now turn to a generalization of van der Waerden's theorem.

Let $\mathbf{R}$ denote the set of real numbers. For $X \subseteq \mathbf{R}^m$, a function $f : X \to \mathbf{R}^m$ is *homothetic* if there is $\mathbf{a} \in \mathbf{R}^m$ and $d \in \mathbf{R}$ so that for each $\mathbf{x} \in X$, $f(\mathbf{x}) = \mathbf{a} + d\mathbf{x}$. Gallai —see [114] (alias Grünwald) and Witt [129] independently proved the following:

Theorem 2.4.1 *Given a finite $X \subset \mathbf{R}^m$ and $r \in \omega$, there exists a finite $Y \subset \mathbf{R}^m$ so that for every coloring of Y with r colors, X has a monochromatic homothetic image in Y.*

Proof: Fix a finite set $X \subset \mathbf{R}^m$ and set $n = \mathrm{HJ}(|X|, 1, r)$. Put

$$Y = \{\sum_{i\in n} f(i) : f \in [X]\binom{n}{0}\},$$

where the sums are component wise, and let $\Delta : Y \longrightarrow r$ be a coloring. Examine the induced coloring $\chi : [X]\binom{n}{0} \longrightarrow r$ defined by $\chi(f) = \Delta(\sum_{i\in n} f(i))$. By the Hales-Jewett theorem, there is a $g \in [X]\binom{n}{1}$ with $g \circ [X]\binom{1}{0} = \mathrm{sp}(g)$ being a monochromatic line. Supposing λ_0 is the parameter of g occurring $d = |\{i : g(i) = \lambda_0\}|$ times, set

$$\mathbf{a} = \sum_{g(i)\neq\lambda_0} g(i).$$

So

$$\{\sum_{i\in n} f(i) : f \in \mathrm{sp}(g)\} = \{\mathbf{a} + d\mathbf{x} : \mathbf{x} \in X\}$$

is monochromatic with respect to Δ and we are done. □

An extension of this theorem can be found in [101]. Many other related observations regarding sums also occur (cp. [55],[43]).

We now turn to lattices. For $f \in [\emptyset]\binom{n}{m}$ and $g \in [\emptyset]\binom{n}{k}$ we say $f \leq g$ if and only if there exists an $h \in [\emptyset]\binom{m}{k}$ so that $f \circ h = g$. Observe that there is a one to one correspondence between elements of $[\emptyset]\binom{n}{k}$ and partitions of $n = \{0, 1, \ldots, n-1\}$ into k parts. Furthermore, if $f \leq g$ then the partition determined by f is a refinement of that determined by g.

For example, if $f = \lambda_0\lambda_1\lambda_0\lambda_2 \in [\emptyset]\binom{4}{3}$ and $g = \lambda_0\lambda_1\lambda_0\lambda_1 \in [\emptyset]\binom{4}{2}$, then $f \leq g$ with $h = \lambda_0\lambda_1\lambda_0 \in [\emptyset]\binom{3}{2}$ as a witness. In this light, we see that $\cup_{k\leq n}[\emptyset]\binom{n}{k}$ together with the relation $\leq$ is isomorphic to the lattice of partitions. Although we do not discuss them here, one can deduce partition theorems for these lattices from the Hales-Jewett theorem. Interpretations using a two letter alphabet also yield partition theorems for Boolean lattices, distributive lattices, and posets in general. See [100], pp. 47-57, for a more comprehensive discussion of these and other classes. The interested reader may consult [76] and [92] for other extensions. Many other results in combinatorics have been discussed using parameter words—see e.g. [103], [106], [107], and [110].

2.5 Shelah Bound

By 1988, S. Shelah had discovered a truly remarkable proof that the function $\mathrm{HJ}(t, m, r)$ is primitive recursive. Shelah then relayed this information to W. Deuber, who lectured on this in Germany later that year (and also in Norwich,1989 [24]). N. Sauer attended this lecture and gave the outline for the case $m = 1$ at the University of Calgary in September of 1988. We give here a version of these handed down arguments for the case $m = 1$. The author noticed that essentially the same proof works for arbitrary m, yet Rödl (oral communication) pointed out that a (well known) lemma gives another way of proving the general case from the case $m = 1$. The paper by Shelah [121] was discovered only after learning the proof we give here; in his paper, Shelah proves much more, and in a more condensed fashion.

As usual, $[n]^i = \{P \subseteq n = \{0, 1, 2, \ldots, n-1\} : |P| = i\}$. In what we refer to as "Shelah's Lemma" we will be interested in 'strings' of the form

$$(\{x_0, y_0\}, \{x_1, y_1\}, \ldots, \{x_{i-1}, y_{i-1}\}, \{x_i\}, \{x_{i+1}, y_{i+1}\}, \ldots, \{x_{m-1}, y_{m-1}\})$$

and coloring such strings. The string given will be said to be an element from $([n]^2, [n]^2, \ldots, [n]^2, [n]^1, [n]^2, \ldots, [n]^2)$, where $[n]^1$ is in the i'th position. It will sometimes be convenient to put subscripts in to mark positions, such as in $[n]^1_i$. We now give Shelah's Lemma.

Lemma 2.5.1 *Given m,$r \in \omega$, then there exists a smallest $n = Sh(m, r)$ so that for* *every family* $\Delta_i, (i \in m)$, *of* m *functions*

$$\Delta_i : ([n]^2, [n]^2, \ldots, [n]^2, [n]^1_i, [n]^2, \ldots, [n]^2) \longrightarrow r$$

there exists $x_0 < y_0$, $x_1 < y_1$, ..., $x_{m-1} < y_{m-1}$ *so that for each* $i \in m$

$$\Delta_i(\{x_0, y_0\}, \{x_1, y_1\}, \ldots, \{x_{i-1}, y_{i-1}\}, \{x_i\}, \{x_{i+1}, y_{i+1}\}, \ldots, \{x_{m-1}, y_{m-1}\}) =$$
$$\Delta_i(\{x_0, y_0\}, \{x_1, y_1\}, \ldots, \{x_{i-1}, y_{i-1}\}, \{y_i\}, \{x_{i+1}, y_{i+1}\}, \ldots, \{x_{m-1}, y_{m-1}\}).$$

Proof: $\mathrm{Sh}(1, r) = r + 1$ trivially by the pigeon hole principle. Assume that $\mathrm{Sh}(m, r)$ exists. We claim $\mathrm{Sh}(m + 1, r) \leq n$ where

$$n = 1 + r^{\binom{Sh(m,r)}{2}^m}.$$

Consider any $m + 1$ colorings $\Delta_i : ([n]^2_0, \ldots, [n]^1_i, \ldots, [n]^2_m) \longrightarrow r$. Examine the restriction of Δ_m

$$\Delta^*_m : ([\mathrm{Sh}(m, r)]^2_0, \ldots, [\mathrm{Sh}(m, r)]^2_{m-1}, [n]^1_m) \longrightarrow r.$$

For each $x \in [n]^1_m$, there are $\binom{Sh(m,r)}{2}^m$ strings of the form

$$\{x_0, y_0\}, \{x_1, y_1\}, \ldots, \{x_{m-1}, y_{m-1}\}, \{x\},$$

where $\{x_i, y_i\} \in [Sh(m, r)]^2_i$ and $x_i < y_i$, each of which are colored by Δ^*_m with one of r colors. So for each $x \in [n]^1$, there are $r^{\binom{Sh(m,r)}{2}^m}$ possible choices for colorings of $([\mathrm{Sh}(m, r)]^2_0, \ldots, [\mathrm{Sh}(m, r)]^2_{m-1}, \{x\})$. Thus by the pigeon hole principle, there is

$$\{x_m, y_m\} \in [n]^2, \ x_m < y_m,$$

so that for each choice of pairs $\{x_i, y_i\} \in [\mathrm{Sh}(m, r)]^2$, $i \in m$, we have

$$\Delta^*_m(\{x_0, y_0\}, \{x_1, y_1\}, \ldots, \{x_{m-1}, y_{m-1}\}, \{x_m\}) =$$
$$\Delta^*_m(\{x_0, y_0\}, \{x_1, y_1\}, \ldots, \{x_{m-1}, y_{m-1}\}, \{y_m\}).$$

Fix such a pair $\{x_m, y_m\}$.

We now look at the remaining $m-1$ maps Δ_i, $i = 0, 1, \ldots, m-1$. Define restrictions

$$\Delta_i^* : ([Sh(m,r)]^2, \ldots, [Sh(m,r)]_i^1, \ldots, [Sh(m,r)]_{m-1}^2, \{x_m, y_m\}) \longrightarrow r.$$

By the induction hypothesis, for each $i = 0, 1, \ldots, m-1$ we can select a pair $\{x_i, y_i\} \in [Sh(m,r)]^2$, $x_i < y_i$, which satisfy the required conditions. This completes the proof by induction. □

Let us now examine the bounds on the function $\mathrm{HJ}(t, m, r)$. First we must discuss what it means for a function to be *primitive recursive.* The exact definition lies in a detailed explanation from recursion theory, a discussion we choose not to go into here. If $f_i : \omega^{m_i} \to \omega$, $i \in I$, are primitive recursive functions, then $f : \omega^m \to \omega$ is primitive recursive if $f(x_1, \ldots, x_m)$ can be written as a finite collection of symbols from x_1, ..., $+$, $-$, $\cdot$, $^{-1}$, parentheses, and $f_i(y_1, \ldots, y_{m_i})$, where the y_j's are chosen from $\{x_1, \ldots, x_m\}$. For example, exponential functions and $n!$ are both primitive recursive.

One example of a function which is not primitive recursive is an 'Ackermann function' [2]. We use $f^{(n)}$ to denote the n'th iterate of f under composition. If we define functions $f_i : \omega \to \omega$ by $f_1(x) = 2x$ and $f_{n+1}(x) = f_n^{(x)}(x)$, then the function F defined by $F(n) = f_n(2)$ is an Ackermann function. Such a function eventually dominates any one primitive recursive function. (See [55], p.51 for additional discussion.)

The inequalities (2.1) and (2.2) in the proof of Theorem 2.3.1 do not yield primitive recursive bounds for $\mathrm{HJ}(t, m, r)$. However, Shelah's lemma can be used to show that HJ$(t,1,r)$ has a primitive recursive upper bound (viewed as a function of t). We first observe that the function Sh(m,r) has a primitive recursive upper bound. One obvious bound,

$$Sh(m,r) \leq (2mr)^{(2mr)^{\cdot^{\cdot^{2mr}}}}, \text{ a tower of height } m,$$

follows from the inequality

$$Sh(m+1,r) \leq 1 + r^{\binom{Sh(m,r)}{2}^{m-1}}.$$

We can now give Shelah's result [121] which shows HJ$(t,1,r)$ is primitive recursive.

Theorem 2.5.2 $HJ(t+1,1,r) \leq HJ(t,1,r) \cdot Sh(HJ(t,1,r), r^{(t+1)^{HJ(t,1,r)-1}})$

Proof: Let $m = HJ(t,1,r)$ and $n = Sh(m, r^{(t+1)^{m-1}})$. Fix an alphabet A with $|A| = t$ and let $a \in A$ and $b \notin A$ be given. Set $B = A \cup \{b\}$. For each $x,y \in n$, $x < y < n$, define words $Lxy \in [B]\binom{n}{1}$ by $Lxy(i) = a$ if $i \in x$, $Lxy(i) = \lambda$ if $x \leq i < y$ and $Lxy(i) = b$ if $y \leq i < n$. That is, Lxy is of the form $aa \ldots a\lambda\lambda \ldots \lambda bb \ldots b$ where there are a's in the first x positions, λ's in the next $y - x$ positions, and b's in the remaining $n - y$ spots. Also, for each $x \in n$ define words $Rx \in [B]\binom{n}{0}$ by $Rx(i) = a$ if $i \in x$ and $Rx(i) = b$ if $x \leq i < n$. Let $\Delta : [B]\binom{mn}{0} \longrightarrow r$ be given. We need to show the existence of a monochromatic line in $[B]\binom{mn}{0}$.

For the moment, let us examine restricted collections from $[B]\binom{mn}{m-1}$. As a local definition, we shall say that $f \in [B]\binom{mn}{m-1}$ is of type i if f is of the form

$$f = Lx_0y_0\hat{\ }Lx_1y_1\hat{\ }\ldots\hat{\ }Lx_{i-1}y_{i-1}\hat{\ }Rx_i\hat{\ }Lx_{i+1}y_{i+1}\hat{\ }\ldots\hat{\ }Lx_{m-1}y_{m-1},$$

(where $\lambda = \lambda_j$ in Lx_jY_j). Restrict Δ to

$$\Delta_i : \{f \circ g : f \in [B]\binom{mn}{m-1}, f \text{ is of type } i, g \in [B]\binom{m-1}{0}\} \longrightarrow r.$$

For each f of type i, substitutions of the $m-1$ parameters (from $t+1$ letters), Δ_i produces a vector of colors, one of $r^{(t+1)^{m-1}}$ such possible. That is, Δ_i induces a coloring of each f of type i using $r^{(t+1)^{m-1}}$ colors. But each such f is determined by an element from $([n]^2, \ldots, [n]^1, \ldots, [n]^2)$ determining the indices x_i, y_i. So by Shelah's Lemma, there exists a fixed choice of the x_i's and y_i's, say

$$x_0 < y_0, x_1 < y_1, \ldots, x_{m-1} < y_{m-1},$$

so that

$$\Delta_i(Lx_0y_0\hat{\ }Lx_1y_1\hat{\ }\ldots\hat{\ }Lx_{i-1}y_{i-1}\hat{\ }Rx_i\hat{\ }Lx_{i+1}y_{i+1}\hat{\ }\ldots\hat{\ }Lx_{m-1}y_{m-1}) =$$
$$\Delta_i(Lx_0y_0\hat{\ }Lx_1y_1\hat{\ }\ldots\hat{\ }Lx_{i-1}y_{i-1}\hat{\ }Ry_i\hat{\ }Lx_{i+1}y_{i+1}\hat{\ }\ldots\hat{\ }Lx_{m-1}y_{m-1}).$$

Furthermore, this does not depend on i; the set of indices is a uniform choice.

Now look at the m-parameter word

$$S = Lx_0y_0\hat{\ }Lx_1y_1\hat{\ }\ldots\hat{\ }Lx_{m-1}y_{m-1}.$$

The coloring Δ induces a coloring $\Delta_S : [B]\binom{m}{0} \longrightarrow r$ by $\Delta_S(g) = \Delta(S \circ g)$. For any $h \in [B]\binom{m}{1}$, we claim that $\Delta_S(h \circ a) = \Delta_S(h \circ b)$. Observe that $Lx_iy_i \circ a = Ry_i$ and $Lx_iy_i \circ b = Rx_i$. Since Δ_i is insensitive to changes between Rx_i and Ry_i, we have that $\Delta_S(h \circ a)$ and $\Delta_S(h \circ b)$ must agree, settling the claim. This gives us that the coloring $\Delta_S : [B]\binom{m}{0} \longrightarrow r$ is induced by a coloring $\Delta^* : [A]\binom{m}{0} \longrightarrow r$, omitting the element b. But m was chosen large enough so that there exists a monochromatic line in $[A]\binom{m}{0}$, i.e., since $\mathrm{HJ}(t, 1, r) = m$, there is $f \in [A]\binom{m}{1}$ so that $\mathrm{sp}(f)$ is monochromatic with respect to Δ. This is clear, since Δ_S was determined by those words in the space of f. So the word $S \circ f \in [B]\binom{mn}{1}$ is as desired. $\square$

One might observe that this shows the function $W(t,r)$ asserted by van der Waerden's Theorem is also primitive recursive. Apparently, this was previously not known. See [55], pp. 52-3 for discussion related to these bounds. Some recent work in this area (not using Shelah's work) can be found starting with [58].

An observation, somewhat astonishing, is that in the proof of Theorem 2.5.2 nowhere was it relied on that we were doing the case for only lines, i.e., nowhere was it used that $m = 1$, except in some details near the end. Well, in fact, the proof works for arbitrary m, but if this assertion is not convincing, there is a more direct [and hence more satisfying?] proof of the general case. This alternate method (see e.g. [55] p.37) simply relates the Hales-Jewett number for m-spaces to that for lines.

Theorem 2.5.3 *The function $HJ(k,t,r)$ is primitive recursive.*

Proof: We claim that $t \cdot \mathrm{HJ}(k^t, 1, r) \geq \mathrm{HJ}(k,t,r)$. In order to prove this claim we use the idea that if $A = B^t$ where B is an alphabet, then a combinatorial line in A^n is a t-space in B^{nt}.

So let B be an alphabet with $|B| = k$ and set $A = \{f : t \to B\}$. Fix $n = \mathrm{HJ}(k^t, 1, r)$. If $g \in [B]\binom{nt}{0}$, then setting

$$g_i = g(it)g(it+1)\dots g((i+1)t-1)$$

for each $i \in n$, we have

$$g = g_0\hat{\ }g_1\hat{\ }\dots\hat{\ }g_{n-1} \in [A]\binom{n}{0},$$

showing $[B]\binom{nt}{0} \subseteq [A]\binom{n}{0}$. In fact, $[B]\binom{nt}{0}$ can be viewed as precisely $[A]\binom{n}{0}$. Fix a coloring $\Delta : [B]\binom{nt}{0} \longrightarrow r$ (which is also a coloring of $[A]\binom{n}{0}$). By the choice of n, there exists $g \in [A]\binom{n}{1}$ so that Δ is constant on $g \circ [A]\binom{n}{0}$. But since the parameter of g can be replaced by any $f : t \to B$ we have that $g \in [B]\binom{nt}{t}$ and Δ is constant on $g \circ [B]\binom{nt}{0}$. Thus $nt \geq \mathrm{HJ}(k, t, r)$, proving the claim and hence the theorem. $\square$

The powerful techniques of Shelah have put an end to a very difficult outstanding question. It may be of no surprise to some that, in the same paper, Shelah completely settled the same question for the function $\mathrm{GR}(|A|, k, m, r)$ (as given in the statement of Theorem 2.3.2) among other related functions.

Chapter 3

Finite Graph Ramsey Theory

3.1 Introduction to Graphs

Many visions come to mind when one hears the word "graph", but this discussion deals with only certain types of graphs — graphs in the combinatorial sense. Before we give the definition of a graph, let us give the general idea. If one puts dots on a page and then connects some pairs of them with lines, then one has a graph in the simplest sense, sometimes referred to as an *ordinary* graph. The dots are called *vertices* (or *points*) and the lines are called *edges*. Edges ordinarily 'include' only two vertices each, but we need not restrict ourselves so. We could use any number of vertices in one edge, even though this is hard to imagine (and draw!); such a *hypergraph* is a very useful structure. The generic term "graph" is often used to denote hypergraphs in general although it is usually reserved for ordinary graphs.

One could look at graphs in another sense. Suppose we are given a set X and finitary relations R_0, R_1, ..., R_{n-1},which are *symmetric* (i.e., $R_i(y_0, \ldots, y_{m_i-1}) = R_i(y_{\sigma(0)}, \ldots, y_{\sigma(m_i-1)})$ for any permutation σ of m_i). We could say $\{x_0, \ldots, x_{m-1}\} \subset X$ is an edge if $R_i(x_0, \ldots, x_{m-1})$ holds for some $i \in n$. These relations determine a hypergraph on the vertex set X. If there is only one binary relation then the hypergraph is just an ordinary graph (providing $R(x,x)$ never holds). If all the relations are of the same arity, then the graph has all edges the same 'size' and hence the hypergraph is said to be *uniform*.

A last look at how hypergraphs can be envisioned is as follows. Given a set X, we might be interested in a collection of subsets of X. These subsets are simply edges of a hypergraph on X and thus hypergraphs are a useful, in fact equivalent, way of examining many combinatorial problems. Since Ramsey theory is a branch of combinatorics, it is natural to consider the theory in the setting of graphs. The language and notation used in graph theory is well established and hence presents a natural environment for Ramsey theory. When couched in graph theoretical language, the original questions and results in Ramsey theory correspond to only certain types of graphs (namely the 'complete' ones). Recently, the same types of questions have been asked of arbitrary graphs, and to emphasize this, "graph Ramsey theory" is used to denote this stimulating, although sometimes difficult, field.

3.2 Notation and Preliminaries

The basic notions in this and subsequent chapters are common to many of the popular books (e.g. [3],[7],[63]) in graph theory. To preserve uniformity we review some basic definitions and notation.

As mentioned in the introduction, there are two kinds of graphs we wish to consider here, ordinary graphs and hypergraphs. Usually ordinary graphs are referred to as simply "graphs", although "graphs" may refer to hypergraphs in general. We hope that the context will remain clear throughout regarding the kind of graph we are discussing. We give most of the definitions in their general form, that is, for hypergraphs, of which ordinary graphs form a special case.

A *hypergraph* G is a pair $G = (V(G), E(G))$ where $V(G)$ is the vertex set and

$E(G) \subseteq 2^{[V(G)]}$ is the edge set. We restrict ourselves to finite hypergraphs, i.e., $|V(G)|$ is finite. By this definition, no edge appears more than once in $E(G)$, i.e., for any given set $X \subseteq V(G)$, X determines at most one edge. This is often expressed by saying the graph has no *multiple edges.* (If 2 towns are connected by 2 distinct roads, the corresponding map would contain multiple edges — we do not discuss such cases here.) Each edge is considered to be an unordered set, quite unlike the case of 'directed graphs'. Ramsey properties of infinite directed graphs has also been studied (for example, see [16],[17],[18], and [27]), but surprisingly there seems to have been little accomplished in the finite case ([5] is an exception). The *size* of an edge $e \in E(G)$ is merely the number of vertices contained in e. If all edges of a hypergraph G have the same size, say k, then G is called *k-uniform*, i.e., when $E(G) \subset [V(G)]^k$. A k-uniform graph is often referred to as a k-graph. Later we shall see that edges can be further classified into *types* or *multiplicities*; we need not give the formal definition here. A *loop* is an edge consisting of a single vertex.

An *ordinary graph*, or simply, a *graph* $G = (V(G), E(G))$ on a finite vertex set $V(G)$ has edge set $E(G) \subseteq [V(G)]^2$. Note that under this definition, a graph has no loops or multiple edges, is undirected and is finite. If $E(G) = [V(G)]^2$ we say G is *complete* and if $E(G) = \emptyset$, we say G is an *empty* graph. A 'graph' with no vertices is called a *null* graph. We use standard notation for ordinary graphs: K_n denotes a complete graph on n vertices. A *path* is a graph on vertices x_0, x_1, ..., x_n, where $(x_i, x_i + 1)$, $i \in n$ are edges. We use P_n to denote the path of length n on $n + 1$ vertices. A *star* is a graph with one vertex connected to all others by an edge and no other edges present. A star with one central vertex and n end vertices is denoted by S_n.

A *cycle* in an ordinary graph is a sequence of vertices x_0, x_1, ..., x_n, where $(x_i, x_i + 1)$, $i \in n$ and (x_n, x_0) are edges. A cycle on n vertices is denoted by C_n. A cycle of length p in a k-uniform hypergraph $G = (V, E)$ (where $E \subset [V]^k$) is a sequence $e_0, e_1, \ldots, e_{p-1}$ of different edges in E so that there are distinct vertices $v_0, v_1, \ldots, v_{p-1} \in V$ so that $v_i \in e_i \cap e_{i+1}$ for $i < p - 1$ and $v_{p-1} \in e_{p-1} \cap e_0$.

A hypergraph $G = (V, E)$ is *connected* if for any two vertices $v, w \in V$ there is a path in G containing both v and w. A (hyper)graph which is not connected is called *disconnected.* A graph is *n-connected* if between any two vertices there are n vertex-disjoint (except at endpoints) paths joining them. It is easy to see that a graph is 2-connected if and only if the graph can not be made disconnected by the removal of any single vertex, that is, its smallest *cutset* contains at least two elements. See [4], ch. 9 for detailed explanation of 'connectivity'. We also say, in this case, that the graph has no cutpoints. (A *cutpoint* is a vertex whose removal disconnects the graph.)

For two hypergraphs H and G, we say an injection $f : V(H) \to V(G)$ is a (graph) *embedding* just in case $f(X) \in E(G)$ if and only $X \in E(H)$ holds, in which case we say H embeds in G (or, G embeds H). If $f : V(H) \to V(G)$ is a bijective embedding (graph isomorphism), H and G are *isomorphic*, denoted by $H \cong G$.

When $V(H) \subset V(G)$ and $E(H) \subseteq 2^{[V(H)]} \cap E(G)$ then H is a *weak subhypergraph* of G and we write $H \subseteq G$. If $H \subseteq G$ and $E(H) = 2^{[V(H)]} \cap E(G)$ then we say H is an *induced subhypergraph* of G, denoted by $H \preceq G$. So H embeds into G if there is $H' \preceq G$ so that $H' \cong H$. If we speak of a sub(hyper)graph we shall take it to be *induced.* Graph Ramsey theory for weak subgraphs is also studied, but we emphasize that our interest here is primarily in the induced, or 'strong' Ramsey theory. Some interesting results arise from the weak cases which are relevant in our study, so some references are given later. We sometimes confuse issues by saying "H is a subgraph of G" when it is actually meant that H embeds into G. We hope these situations are clear when they present themselves.

A *clique* in an ordinary graph G is a maximal complete subgraph of G. We define the *clique number*, $\mathrm{cl}(G)$, to be the maximum size of a clique in G. The *chromatic number*, $\chi(G)$, of a hypergraph G is the least positive integer n so that there exists a partition of $V(G)$ into n classes with no edge contained entirely in any one class. If no such integer exists, (as in the case of a graph with loops) then we say $\chi(G)$ is infinite. For example, $\chi(K_3) = 3$ and $\chi(G) \leq n$ for any n-partite graph G (without loops) and the chromatic number of an empty graph is 1. The *girth* of a hypergraph is the size of the smallest cycle contained in it as a subhypergraph. For a family $\mathcal{F}$ of hypergraphs, we let $\mathrm{Forb}(\mathcal{F})$ be the class of all hypergraphs which do not contain any *induced* subhypergraph isomorphic to an element of $\mathcal{F}$.

For hypergraphs we define the binomial coefficient

$$\binom{G}{H} = \{H' \preceq G : H' \cong H\}.$$

This notation is apparently due to Leeb (–see [84]). For hypergraphs F, G and H, and a fixed $r \in \omega$ the standard Ramsey arrow notation $F \longrightarrow (G)^H_r$ denotes the fact that for any coloring

$$\Delta : \binom{F}{H} \longrightarrow r,$$

there exists $G' \in \binom{F}{G}$ so that Δ is constant on $\binom{G'}{H}$. We use the analogous notation for ordered graphs. This notation will be used extensively throughout this and later chapters since our main objective is to study the triples G,H,r for which there is an

F satisfying $F \longrightarrow (G)^H_r$. With this goal in mind, it is convenient to introduce the special notation

$$\mathcal{R}[(G)^H_r] = \{F : F \longrightarrow (G)^H_r\},$$

the *Ramsey class* for G in coloring of H's with r colors. This notation was developed by the author (and may not yet be standard) to eliminate many cumbersome statements.

Another notation used in Ramsey theory is used when we have two (or more) graphs of which we require only one to be monochromatic. Most common is the case of edge colorings with two colors. Then $F \longrightarrow (G, H)$ means that in any red-blue edge coloring of F, there is either a red copy of G or a blue copy of H. More often than not, this notation was used for the weak subgraph Ramsey statements and was very commonly referred to as *Generalized Ramsey Theory.* The reason it is referred to as 'generalized', is that this study stemmed from the examination of Ramsey numbers, diagonal and otherwise. There is extensive literature on generalized Ramsey theory (e.g.,[32],[44], [64], [65]). The (diagonal) Ramsey numbers correspond to the size of the smallest graph F satisfying $F \longrightarrow (K_n, K_m)$ where $m = n$. (The off-diagonal numbers are when $m \neq n$.) One can also assign Ramsey numbers to the 'generalized' cases (i.e., when $F \longrightarrow (G, H)$ for arbitrary graphs G and H). Determining such numbers (and their existence) is difficult work. Although no complete solution exists, significant progress has been made (e.g., [20], [29], [30], [45]). Cases dealing with only certain families of graphs, such as stars, matchings and forests have been looked at extensively (e.g. [11], [12], [13], and [68]).

A familiar Ramsey statement is: $6 \longrightarrow (3, 3)$. This says that if we color the pairs of a six element set with two colors then we are guaranteed the existence of a three element subset, all of whose two element subsets are colored the same. Translated into the language of graph theory, this statement reads: $K_6 \longrightarrow (K_3)_2^{K_2}$.

Theorem 1.3.2, the finite version of Ramsey's theorem [116] can be stated as follows:

Theorem 3.2.1 *For any $m,k,r \in \omega$, there exists an $n \in \omega$ so that $K_n \longrightarrow (K_m)_r^{K_k}$, i.e., $\mathcal{R}[(K_m)_r^{K_k}] \neq \emptyset$.*

An *ordered hypergraph* $(G, \leq)$ is a hypergraph G together with a total order $\leq$ on $V(G)$. Two ordered hypergraphs are isomorphic just in case there is an order preserving graph isomorphism between them. Definitions analogous to those given above hold for ordered hypergraphs as well. For a hypergraph H we let

$$ORD(H) = \{(H, \leq_0), (H, \leq_1), \ldots, (H, \leq_{k-1})\},$$

be the set of (distinct) isomorphism types of orderings of H. It is often convenient to abuse the notation and deliberately confuse an isomorphism type with a hypergraph of that given type.

In some theorems we use 'partite' graphs. Let $V_0, V_1, \ldots, V_{n-1}$ be a system of pairwise disjoint nonempty sets and let $E \subset \mathcal{P}([\cup_{i \in n} V_i])$ be so that for any $e \in E$, $|e \cap V_i| \leq 1$ for each $i \in n$. Then $G = (V_0, V_1, \ldots, V_{n-1}, E) = ((V_i)_{i \in n}, E)$ is called an *n-partite hypergraph* with *parts* (or *coordinates*) $V_i = V_i(G)$. For $n = 2$, the graph is called *bipartite.* All the remaining notation we give here regarding partite graphs also applies to hypergraphs but we state it only for ordinary graphs.

Given an n-partite graph $G = ((V_i)_{i \in n}, E)$ and graph $H = (W, D)$ then H embeds (partite-wise) into G if there exists a partition of $W = W_0 \cup W_1 \cup \cdots \cup W_{m-1}$ with

$D \cap [W_i]^2 = \emptyset$ for each $i \in m \leq n$ (i.e., H is m-partite) and there exists an injection $f : W \to \cup_{i \in n} V_i$ so that for each $i \in m$ there is $\sigma(i) \in n$ with $f(W_i) \subseteq V_{\sigma(i)}$ and $f(D) = E \cap [\cup_{i \in m} f(W_i)]^2$. This says that H is isomorphic to some m-partite induced subgraph of G, that is, $H \cong H' = (V', E')$ where $V' \subset \cup_{i \in n} V_i$, $V_i \cap V' \neq \emptyset$ for exactly m values of i, and $E' = [V']^2 \cap E$. In this case we write $H' \preceq_{part} G$ to emphasize the fact that H and G are partite graphs. As before, for a partite graph G, let $\binom{G}{H}_{part} = \{H' \cong H : H \preceq_{part} G\}$. Note that when the partite structure is ignored, $\binom{G}{H}$ may contain more elements than does $\binom{G}{H}_{part}$.

For partite graphs F, G, H, and $r \in \omega$, we use the partite Ramsey arrow notation $F \longrightarrow_{part} (G)^H_r$ to mean that for every coloring $\Delta : \binom{F}{H}_{part} \longrightarrow r$ there exists $G' \in \binom{F}{G}_{part}$ so that Δ is constant on $\binom{G}{H}_{part}$. Observe that if F, G, H are l, m, n-partite respectively, we need $l \geq m \geq n$ for this statement to be non-trivial.

3.3 Vertex Partitions

A coloring of the vertices of a graph can be viewed as a partition. In this section, we are interested in expressions of the form $F \longrightarrow (G)^{K_1}_r$ where K_1 denotes a vertex.

In general, if we are given two graphs G and H and a number $r \in \omega$, it is quite difficult to ascertain whether or not there is a graph $F \in \mathcal{R}[(G)^H_r]$. One of the earlier successes [46] is the following

Theorem 3.3.1 *For any ordinary graph G, $\mathcal{R}[(G)^{K_1}_2] \neq \emptyset$.*

Proof: Let the graph G be given and define the lexicographic product $F = G \otimes G$

on $V(F) = V(G) \times V(G)$ by

$$((u_0, v_0), (u_1, v_1)) \in E(F) \text{ iff } \begin{cases} (u_0, u_1) \in E(G), \quad \text{or,} \\ u_0 = u_1 \text{ and } (v_0, v_1) \in E(G). \end{cases}$$

It can be verified that F satisfies $F \longrightarrow (G)_2^{K_1}$ (since if there is no monochromatic copy of G in any of the 'coordinates' of F, then there is certainly one straddling the coordinates). □

We now turn the discussion to the existence of hypergraphs with high chromatic number *and* containing no small cycles. Naively, one would think that since short cycles force the chromatic number up, short cycles are necessary to do so. In fact, quite the opposite is true. According to [87], the study of such questions began with Tutte and Zykov [21] in the 1940's. The question was finally answered by P. Erdős [28] and Erdős and A. Hajnal [33] in the 1960's using *probabilistic* methods. The first constructive proof was given by L. Lovász [73]. The author is not familiar with Lovász's construction (of uniform hypergraphs having arbitrarily large chromatic number and girth); the construction of Nešetřil and Rödl [87] which we give here is apparently an extension of an idea of Tutte [21].

The method we use here is called *partite amalgamation* a method generally attributed to Nešetřil and Rödl. If we think of this process as 'gluing' partite graphs together but only at certain parts or coordinates, the process we use here consists of gluing only at a single part. In the next section (Theorem 3.4.4) we employ a similar technique but involving the amalgamation along two parts. In a more general situation, the method will be again dressed up for the occasion to prove a most powerful theorem later in this manuscript (Theorem 4.5.2). Because of the difficulty of the process of partite amalgamation, the proof we give of the existence of *sparse* (large girth) highly chromatic hypergraphs will serve as a nice introduction to the idea.

As usual, we let $\chi(G)$ denote the chromatic number of G.

Theorem 3.3.2 *For positive integers $k \geq 2$, n, p there exists a k-uniform hypergraph G so that $girth(G) > p$ and $\chi(G) > n$.*

Proof: For an a-partite k-uniform hypergraph $G = ((V_i)_{i \in a}, E)$ with $|V_r| = l$ for some $r \in a$, and an l-uniform hypergraph $H = (X, D)$, we define $H *_r G = ((V_i')_{i \in a}, E')$, an a-partite k-uniform hypergraph as follows.

For $i \neq r$, set $V_i' = V_i \times D$ and set $V_r' = X$. For each $d \in D$ fix an injection $\psi_d : \cup_{i \in a} V_i \to \cup_{i \in a} V_i'$ taking V_r to $d \subset V_r'$ and for $i \neq r$, $psi_d(V_i) = \{(v, d) : v \in V_i\}$. Define

$$E' = \{\{\psi_d(v_1), \ldots, \psi_d(v_k)\} : \{v_1, \ldots, v_k\} \in E, d \in D\}.$$

An edge $e \in E'$ will be denoted $\psi_d(e)$ for some $e \in E$, and $d \in D$. So $H *_r G$ is formed by taking $|D|$ copies of G and identifying the copies of V_r with edges of H. So we have *amalgamated* copies of G together along the r-th part, using H as a 'template' for the new r-th part.

We now prove the theorem by induction on p. For $p = 1$, observe that any loopless hypergraph G satisfies $girth(G) \geq 2$, and for each k, trivial examples of k-graphs exist with $\chi(G) \geq n$. So assume that for fixed p the theorem holds for every q satisfying $1 \leq q < p$ and for all edge sizes k'. Put $a = (k-1)n + 1$ (note that $a \longrightarrow (k)^1_n$), and let

$$G^0 = ((V_i^0)_{i \in a}, E^0)$$

be such an a-partite k-uniform hypergraph so that for every set $A \subset [a]^k$ there is an edge $e \in E^0$ with $e \cap V_i' \neq \emptyset$ for every $i \in A$ and has girth$(G^0) > p$. (We can take

G^0 to be a collection of disjoint k-edges.)

Define inductively a-partite k-graphs $G^j = ((V_i^j)_{i\in a}, E^j)$, $1 \leq j \leq a$, as follows. Having defined $G^m = ((V_i^m)_{i\in a}, E^m)$, $(m < a)$, put $|V_m^m| = l_m$ and let $H^m = (X^m, D^m)$ be an l_m-graph which satisfies girth$(H^m) \geq p$ and $\chi(H^m) > n$. (Such a hypergraph exists by induction hypothesis using a different value for k.) Put

$$G^{m+1} = H^m *_m G^m = ((V_i^{m+1})_{i\in a}, E^{m+1}).$$

We claim the graph $G^a = ((V_i^a)_{i\in a}, E^a)$ satisfies the theorem.

To see that girth$(G^j) > p$ for each $j \leq a$, we use induction on j; suppose that girth$(G^j) > p$ for a fixed j. In E^{j+1}, pick a sequence of vertices

$$C = \{\psi_{d_0}(v_0), \ldots, \psi_{d_{q-1}}(v_{q-1})\}$$

of minimal length q which determine a cycle. If all the d_i, $i \in q$ were equal, by the induction hypothesis there are no small cycles in a copy of G^j and so $q > p$ would hold as desired. So suppose that not all the d_i are equal. Then in this case, the only way that C can be a cycle is if it uses vertices from V_j^{j+1}, the j-th part of G^{j+1}. Now use the fact that, by induction hypothesis, girth$(H^j) \geq p$ and conclude that $q > p$ (in fact, q would in this case be at least $2p$) as desired.

To see the proof of $\chi(G^a) > n$, fix a coloring $\Delta : V(G^a) \longrightarrow n$. The restriction of Δ to X_{a-1}^a, the last part of G^a, imposes a coloring on X^{a-1}, the vertices of H^{a-1}. But $\chi(H_{a-1}) > n$ by the inductive hypothesis, and so there exists a monochromatic edge $d_{a-1} \in D_{a-1}$. Setting $Z_{a-1} = d_{a-1} \in [X_{a-1}^a]^{l_{a-1}}$ to be the last part of a new graph $F^{a-1} \preceq G^{a-1}$, i.e.,

$$F^{a-1} = ((X_i^{a-1})_{i\in a-1}, Z_{a-1}, E^{a-1} \cap [\cup_{i\in a-1} X_i^{a-1} \cup Z_{a-1}]^k),$$

we shall look only at how Δ colors $V(F^{a-1})$. Certainly Δ is constant on Z_{a-1} by design. Repeat in this manner, using the vertices of a monochromatic edge of H^{a-2} as a new part, a subset of X^{a-1}_{a-2}, create F^{a-2}, Δ being constant on the second last part thereof. Continuing in this manner, we get $F^0 = ((Z^0_i)_{i\in a}, E(F^0))$, a copy of G^0, the vertices of which have colors depending only on the part from whence they came. By the choice of a, there exist k parts all colored the same. By the design of G^0, there exists a k-edge determined by those parts, guaranteed now to be monochromatic. □

We will later observe how closely this proof follows that of a Ramsey theorem (Theorem 3.3.1) for edge partitions. This is not surprising since one can interpret Theorem 3.3.2 as a Ramsey result. For a k-uniform graph G, let a copy of an edge of G (all of which are isomorphic) be denoted by E_G. Also, let C_i be a k-uniform hypergraph which is a cycle of length i (they are unique even in this hypergraph setting). Then Theorem 3.3.2 can be restated as follows.

Theorem 3.3.3 *For positive integers $k \geq 2$, n, p there exists a k-uniform hypergraph $G \in Forb(C_2, C_3, \ldots, C_p)$ so that $G \longrightarrow (E_G)^{K_1}_n$.*

Imposing ordinary graphs on hyperedges is a common trick used in graph theory. It is interesting to play with this idea using Theorem 3.3.3. Nešetřil and Rödl [81] noticed the following strengthening of Theorem 3.3.1. (The proof we give here captures the simplicity of that given in [100].) We shall say a graph is *2-connected* if it is connected and can not be made disconnected by removing one vertex.

Theorem 3.3.4 *Let $\mathcal{A}$ be a finite family of 2-connected graphs and $r \in \omega$. If $G \in Forb(\mathcal{A})$, then there is $F \in Forb(\mathcal{A})$ so that $F \longrightarrow (G)^{K_1}_r$.*

Proof: Let $p = \min\{|V(A)| : A \in \mathcal{A}\}$. If $p = 1$, $Forb(\mathcal{A})$ is empty and if $p = 2$, $Forb(\mathcal{A})$ consists only of empty graphs or complete graphs and the theorem reduces to the pigeon hole principle. So assume $p > 2$. Set $q = \max\{|V(A)| : A \in \mathcal{A}\}$ and let $G = (V, E) \in Forb(\mathcal{A})$. Using Theorem 3.3.2, let H be a k-uniform hypergraph with girth$(H) > q$ and $\chi(H) > r$. Define an ordinary graph F which is constructed from H as follows. Let $V(F) = V(H)$. We wish to embed a copy of G in each edge of H (and then 'forget' the k-edges of H). For each $e \in E(H)$, fix an injection $f_e : V(G) \to e$. Since girth$(H) > 2$, hyperedges of H intersect in at most one vertex and so each f_i is a graph embedding, i.e., the copies of G are embedded consistently. Since girth$(H) > q$ no copies of any element in $\mathcal{A}$ are formed in F, so $F \in Forb(\mathcal{A})$. The chromatic number of H now ensures that F has the Ramsey property for G, i.e., $F \longrightarrow (G)_2^{K_1}$. □

Here we see the utility of Theorem 3.3.2. This theme is repeated in other applications, for example, in Theorem 5.7.2.

Before we leave this section, we mention that many results for vertex colorings are extendable to the infinite (e.g. [61]). However, we must limit ourselves to the finite cases here. Any partial list of contributors and accomplishments in the field of infinite Ramsey theory, that we could supply here, could do no justice.

3.4 Edge Partitions

The next obvious problem for graphs is to consider edge (K_2) partitions, as Henson [66] first did, rather than vertex partitions. A similar result to that of Theorem 3.3.1 holds for edge colorings. We wish to prove such a theorem (following [88]) but we must first prove some lemmas.

For d,$k \in \omega$, define $B(d,k)$ to be the bipartite graph $(X,[X]^k,D)$ where X is a set with $|X| = d$, and for $x \in X$, $A \in [X]^k$, $\{x, A\} \in D$ holds if and only if $x \in A$.

Lemma 3.4.1 ([88]) *For any bipartite graph $G = (V_1, V_2, E)$ there exists d,$m \in \omega$ so that $G \preceq B(d,m)$.*

Proof: Suppose $|V_1| = m$ and $|V_2| = n$. We will construct $G' = (V_1', V_2, E')$ with $G \preceq G'$ so as to increase the degree of each vertex in V_2 to $m+1$. This will be done by adding $m+n$ new vertices to V_1 (forming V_1') and adding sufficiently many edges. The addition of m vertices to V_1 will serve to bring the degree of vertices in V_2 to m, and the remaining n new vertices will bring the degree to $m+1$.

Let A and B be disjoint sets with $|A| = m$, $|B| = n$, $(A \cup B) \cap V_1 = \emptyset$ and set $V_1' = V_1 \cup A \cup B$ (where $|V_1'| = 2m+n$). For each vertex $y_i \in V_2$ let $c_i = |\{x : (x, y_i) \in E\}|$ be the degree of y_i, and fix a (possibly empty) set $A_i \subseteq [A]^{m-c_i}$. Let $f : V_2 \to B$ be a bijection. To complete the definition of G', define

$$E' = E \cup \{(x, y_j) : x \in A_j, y_j \in V_2\} \cup \{(f(y_i), y_i) : y_i \in V_2\}.$$

It is clear that $G \preceq G'$. We now claim $G' \preceq B(2m+n, m+1)$. Suppose $B(2m+n, m+1) = (W, [W]^{m+1}, D)$. Let $g : V_1' \to W$ and extend g to $h : V_1' \cup V_2 \to W \cup [W]^{m+1}$ defined as follows. Set $h(x) = g(x) \in W$ for each $x \in V_1'$. For $y_i \in V_2$, let $h(y_i) = \{h(x) : (x, y_i) \in E'\}$. Straightforward verification shows h is an embedding required by the claim. (Edges of the form $(f(y_i), y_i)$ in E' ensure that h is an injection.) □

The values $k = m$ (cp. $k = m+1$ in above proof) and $d = 2m+n$ are given as sufficient in [88]. This is not quite accurate [the author could not find other references], however, we can prove a special case of the stated (unproved) claim by induction.

Lemma 3.4.2 *Let $G = (V_1, V_2, E)$ be a bipartite graph with $|V_1| = m > n = |V_2|$. Then $G \preceq B(2m+n, m)$.*

Proof: We use induction on m. For $m = 2$, the result is clear since $m > n$. So assume the lemma is true for a fixed $m \geq 2$ and let $|V_1| = m+1$, $|V_2| = n$. In fact, let $V_1 = \{a_1, \ldots, a_m, a_{m+1}\}$, $V_2 = \{b_1, \ldots, b_n\}$. Let $G^* \preceq G$ be the subgraph induced by $V^* = \{a_1, \ldots, a_m\}$, i.e., $G^* = G \backslash \{a_{m+1}\}$. Let $B(2m+n, m) = (X, [X]^m, F)$ where $X = \{x_1, \ldots, x_{2m+n}\}$. If we let $Y = X \cup \{y_1, y_2\}$, then use $(Y, [Y]^{m+1}, F')$ to denote $B(2(m+1)+n, m+1)$. By the induction hypothesis, $G^* \preceq B(2m+n, m)$ with, say, f^* the embedding of G^* into $B(2m+n, m)$. We wish to show that $G \preceq B(2(m+1)+n, m+1)$.

List $[X]^m = \{A_\beta : \beta \in \binom{2m+n}{m}, |A_\beta| = m\}$ and for each $A_\mu \in [X]^m$, let $\mathcal{A}_\mu = \{A_{\mu,i} \in [Y]^{m+1} : A_\mu \subset A_{\mu,i}\} \subset [Y]^{m+1}$. Note that $|\mathcal{A}_\mu| = m+n+2$. For a fixed $\beta \in \binom{2m+n}{m}$, $y_1 \in A_{\beta,i}$ for exactly one i and $y_2 \in A_{\beta,j}$ for exactly one $j \neq i$. So without loss, for each β, let $y_1 \in A_{\beta,1}$ and $y_2 \in A_{\beta,2}$. Furthermore, we can rename the A_β's so that $f^*(b_i) = A_i$ for each $i \leq n$.

We can now describe f, an embedding of G into $B(2(m+1)+n, m+1)$. Set $f(a_i) = f^*(a_i)$ for $i \leq m$ and $f(a_{m+1}) = y_1$, say. For $b_\alpha \in Y_2$, we choose $f(b_\alpha) = A_{\alpha,1}$ if $(a_{m+1}, b_\alpha) \in F'$ and if $(a_{m+1}, b_\alpha) \notin F'$ then choose $f(b_\alpha) = A_{\alpha,2}$. This completes the construction of f and we see that f is an injection satisfying $f(a_i, b_j) \in F'$ if and only if $(a_i, b_j) \in E$. $\square$

Note that actually we created two copies of G, one for each of y_1 and y_2. One easily sees that $G \preceq B(k,l)$ for all $k \geq 2m+n$ and $l \geq m$ in the proof. As noted by E. C. Milner (oral communication), the statement of Lemma 3.4.2 fails if $m = n$, for one only need examine $G = K_{2,2}$, the complete bipartite graph on four vertices.

We now prove our first of a sequence of four Ramsey results for edge coloring; it is called the "Bipartite Lemma". The statement and proof we give here is a generalization of the case $r = 2$ whose proof is outlined in [88].

Lemma 3.4.3 *For every bipartite (ordinary) graph G and $r \in \omega$, there exists a Ramsey bipartite graph F satisfying $F \longrightarrow_{part} (G)^{K_2}_r$.*

Proof: By the above lemmas, we may assume that $G = B(d,k)$ for some fixed $d,k \in \omega$. Put $l = r(k-1)+1$ and let Y be a set so large that $|Y| \longrightarrow (ld)^l_{r\binom{l}{k}}$. Let $F = (Y, [Y]^l, D) = B(|Y|, l)$. We claim that $F \longrightarrow_{part} (G)^{K_2}_r$. Regarding notation, impose a fixed total order on $Y = \{y_1 < y_2 < \cdots < y_{|Y|}\}$. Any subset of Y will be taken as an ordered tuple respecting the underlying order on Y. Let $\Delta : D \longrightarrow r$ be a given coloring of the edges of F.

We produce a coloring $\Delta^* : [Y]^l \longrightarrow r\binom{l}{k}$ induced by Δ, as follows. For $B \in [Y]^l$, let $B(i)$ denote the i-th element of the (ordered) B. We first define $\Delta_B : [B]^k \longrightarrow r$. Let $\Delta_B(x) = \Delta(\{x, B\})$ for $x \in B$. Since $l = r(k-1)+1$, the pigeon hole principle gives us that there is a k-tuple $C = C(B)$ in B colored entirely the same. Since there are $\binom{l}{k}$ possible positions for such a C, and there are r colors such a C could take, we could color each B by the position of the first (in say, a lexicographic ordering of $[B]^k$) occurrence of such a Δ_B-monochromatic C, thereby defining the induced coloring Δ^*.

By the choice of $|Y|$, there exists $R \in [Y]^{ld}$ which is monochromatic with respect to Δ^*, i.e., every $B \in [R]^l$ has a k-tuple colored the same under Δ_B, say 0, *and* in the same position, say $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{k-1})$. So, for all $i \in k$, $\Delta_B(B(\beta_i)) = 0$, that is, $\Delta(\{B(\beta_i), B\}) = 0$ for every $B \in [R]^l$ and every $i \in k$.

We now set out to find a monochromatic element of $\binom{F}{G}$. Enumerate $R = \{x_1, x_2, \ldots, x_{ld}\}$ respecting the order of Y. Fix

$$X' = \{x_{\beta_0}, x_{\beta_0+l}, x_{\beta_0+2l}, \ldots, x_{\beta_0+(d-1)l}\} \subset R,$$

where $|X'| = d$. Note that $\beta_0 \leq l - k < l$ and so X' is defined.

If for every $A \in [X']^k$ we could choose exactly one $B(A) \in [R]^l$, $A \subset B(A)$, satisfying $\Delta(\{x, B\}) = 0$ for each $x \in A$, then the bipartite graph

$$G' = (X', \{B(A) : A \in [X']^k\}, \{\{x, B(A)\} : x \in A\}) \preceq F$$

is isomorphic to G and hence would be the monochromatic element of $\binom{F}{G}$ as required. So we only have to find such a $B(A)$ for each $A \in [X']^k$.

Since X' is 'spread out' sufficiently in R, there is $B(A) \in [R]^l$ so that A occurs in $B(A)$ in precisely the position $\boldsymbol{\beta}$, i.e., there is $B(A) = B$ so that

$$A = \langle B(\beta_0), B(\beta_1), \ldots, B(\beta_{k-1})\rangle$$

. Then by the choice of R, $A \subset B(A)$ is monochromatic with respect to Δ_B, in fact, for every $x \in A$, $\Delta(\{x, B(A)\}) = 0$. So $B(A)$ behaves as a $k-$tuple with respect to coloring as required, and we are done. □

Before we leave (at least for now) bipartite graphs, let us just mention that 'weak' Ramsey statements are considered for bipartite graphs in [44], where the notions of 'achievement' and 'avoidance' games are employed for edge colorings.

We are now ready to prove the Ramsey theorem for (ordinary) graphs under edge colorings. It was first proved independently by Deuber [22], Erdős, Hajnal, and Posa [37] and Rödl [117], proofs appearing first in the mid 1970's. In 1978, Nešetřil and Rödl [83] gave a proof using the fact that every finite graph is an induced subgraph of a direct product of complete graphs. Again, in 1981 Nešetřil and Rödl [88] proved the edge coloring case using a 'partite amalgamation' construction. We give this proof (with minor technical corrections) for two reasons.

Firstly, many additional related results follow quite easily as a result of the partite construction used in the proof of the main theorem. Most of the many consequences of the construction were proved years earlier, but one would be hard set to find a common theme to all of the associated proofs. We present four of these related theorems here in one unified setting.

Secondly, this proof can serve as an introduction to the method of partite amalgamation. Partite amalgamation may very well be the consummate combinatorial construction, but to the uninitiated, it appears inelegant and is hence quite difficult. A constructive proof of a theorem which gives sparse hypergraphs having high chromatic number (Theorem 3.3.2) used this method as well but with amalgamation defined on a single part (or coordinate). The proof of the edge coloring Ramsey question given here uses amalgamation along two parts. As we have already mentioned, this method will be used again later (Theorem 4.5.2) in greater generality where amalgamation occurs across many coordinates.

Theorem 3.4.4 *For any ordinary graph G, $\mathcal{R}[(G)_2^{K_2}] \neq \emptyset$.*

Proof: Let $G = (V(G), E(G))$ with $|V(G)| = m$. By Ramsey's theorem, choose s

so large that $s \longrightarrow (m)^2_r$ and fix a set $H = \{v_0, v_1, \ldots, v_{s-1}\}$. Let $\{e_1, e_2, \ldots, e_{\binom{s}{2}}\} = [H]^2$ be a list of all pairs in H. Enumerate $[H]^m = \{A_0, A_1, \ldots, A_{\binom{s}{m}-1}\}$ as well. Define the s-partite graph

$$P^0 = ((V_i^0)_{i \in s}, E^0)$$

on $s\binom{s}{m}$ vertices as follows. For each $i \in s$ define the part

$$V_i^0 = \{(v_i, j) : j \in \binom{s}{m}\}.$$

For each $A_j \in [H]^m$, fix a graph embedding

$$f_j : V(G) \to \{(v_i, j) : v_i \in A_j\},$$

and define

$$E^0 = \{\{(v_i, j), (v_{i'}, j)\} : f_j^{-1}(v_i, v_{i'}) \in E(G)\}.$$

So we have 'strung a copy' of G across each 'level' of $H \times \binom{s}{m}$, each copy 'touching' a different m-subset of parts.

Now suppose we have defined $P^n = ((V_i^n)_{i \in s}, E^n)$ for some $n \in \binom{s}{2}$. Examine $e_{n+1} = \{v_{x_0}, v_{x_1}\} \in [H]^2$, $(x_0, x_1 \in s)$. Let B be the bipartite graph induced by the vertex set $V_{x_0}^n \cup V_{x_1}^n$. Using Lemma 3.4.3, let $R(B)$ be a bipartite graph which is Ramsey for B using r colors. List $\binom{R(B)}{B}_{part} = \{B_0, B_1, \ldots, B_{q-1}\}$. We define $P^{n+1} = ((V_i^{n+1})_{i \in s}, E^{n+1})$ as follows.

For $i \neq x_0, x_1$, set $V_i^{n+1} = V_i^n \times q$, and put $V_{x_i}^{n+1} = V_i(R(B))$ for $i \in 2$. For each $i \in q$, fix natural embeddings $\phi_i : B \to B_i \preceq R(B)$ and extend each to an injection $\psi_i : V^n \to V^{n+1}$ defined by $\psi_i(v) = \phi_i(v)$ if $v \in V(B)$ and $\psi_i(v) = (v, i)$ if $v \notin V(B)$. We can now define

$$E^{n+1} = \{(\psi_i(u_1), \psi_i(u_2)) : (u_1, u_2) \in E^n, i \in q\}.$$

See Figure 3.1 for rough idea. Observe that each copy of P^n in P^{n+1} is induced.

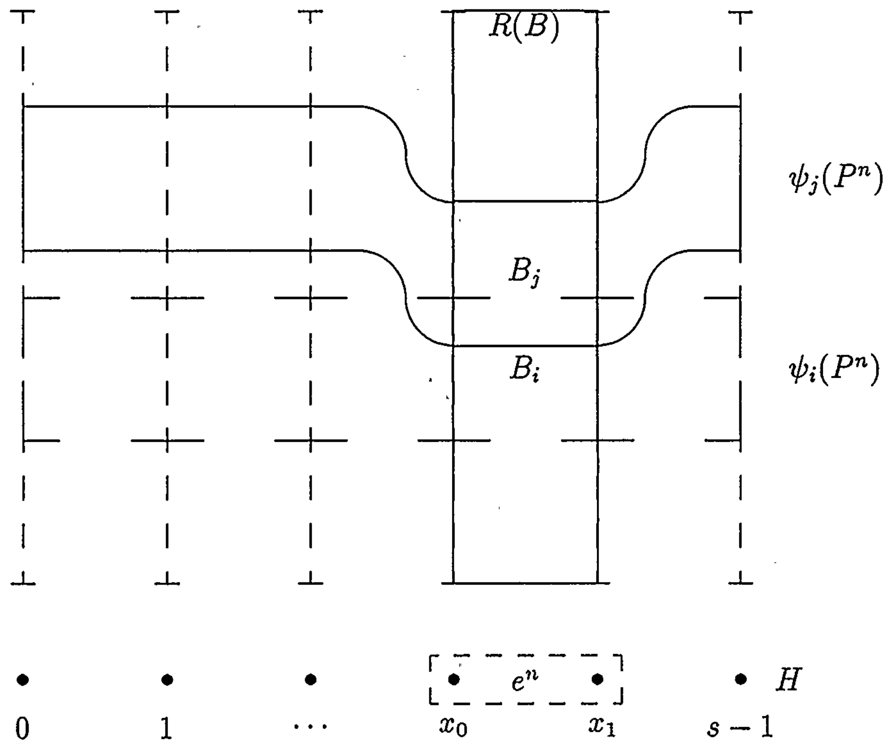


Figure 3.1: Amalgamation along two coordinates.

We now claim that $P^{\binom{s}{2}} \in \mathcal{R}[(G)_r^{K_2}]$. Let $\Delta : E(P^{\binom{s}{2}}) \longrightarrow r$ be a given coloring and suppose $e_{\binom{s}{2}} = (v_{n_1}, v_{n_2})$. Then there exists a copy of $P^{\binom{s}{2}-1}$, call it $P_*^{\binom{s}{2}-1}$, embedded in $P^{\binom{s}{2}}$, that has all of its edges which occur between $V_{n_1}^{\binom{s}{2}-1}$ and $V_{n_2}^{\binom{s}{2}-1}$ colored the same. Next we consider $P_*^{\binom{s}{2}-1}$. Again, there is $P_*^{\binom{s}{2}-2} \preceq P_*^{\binom{s}{2}-1}$ so that all edges of $P_*^{\binom{s}{2}-2}$ occurring between some fixed but different pair of parts are monochromatic. Continuing in this manner, we get P_*^0, a copy of P^0, $P_*^0 \preceq P_*^{\binom{s}{2}}$, which has the property that the color of an edge in P_*^0 depends only on the pair of parts determining it. But by the choice of s, there exist m parts $(W_i)_{i\in m}$ of P_*^0 so that $E(P_*^0) \cap [\bigcup_{i \in m} W_i]^2$, the edges induced in P_*^0 are monochromatic. However, a copy of G existed across every such collection of m parts by the construction of P^0 and so we are done. $\square$

Two of the papers giving proofs of Theorem 3.4.4, [37] and [22], were presented at a conference in 1973 at Keszthely, Hungary. Yet at the same conference Nešetřil and Rödl submitted a paper [77] which solved a related problem which looked only at triangle-free graphs:

Theorem 3.4.5 *Fix $r \in \omega$. Then for any triangle-free graph G, there exists a triangle-free $F \in \mathcal{R}[(G)_r^{K_2}]$.*

Nešetřil and Rödl [80] produced a more general result in 1974 (published in 1976) using 'types', a notion also used in [79], but introduced in [34]. This next theorem extends Theorem 3.4.5 and answers a question posed by F. Galvin (see e.g. [34]). Since Erdős, Folkman, and Hajnal knew certain cases, this theorem was known as a solution to 'the EFGH problem', also known as the 'Galvin-Ramsey' property. Again, we give the theorem for any number of colors, the proof for which is an easy extension of the case for two colors found in [88]. Recall that we use $\mathrm{cl}(G)$ to denote the clique number of the graph G, the size of the largest complete subgraph contained in G.

Theorem 3.4.6 *For any ordinary graph G and $r \in \omega$ there exists $F \in \mathcal{R}[(G)_r^{K_2}]$ with $cl(F) = cl(G)$.*

Proof: In the construction used for the proof of Theorem 3.4.4, observe that $\mathrm{cl}(G) = \mathrm{cl}(P^0)$ and this number is never increased (i.e., $\mathrm{cl}(P^n) = \mathrm{cl}(P^{n+1})$ for each $n \in \binom{s}{2}$) thereafter.

This result has been extended even further. In triangle-free graphs, we may say that a triangle is 'forbidden' (as are larger complete graphs). Graphs with certain clique numbers are similarly determined by forbidden subgraphs. Not always does the forbidden subgraph need to be complete in order to make sense of this notion. Extensions of Theorem 3.4.6 are possible where the forbidden family of graphs is more extensive (the first of which can be found in [82],[86], [88], for examples). Some of these extensions will be corollaries of proofs we present both in this chapter (cp. Theorem 3.4.8) and elsewhere.

This next theorem [88] settles a question of Erdős given in [29]. The theorem says something about how much graphs need to be intermeshed in order to produce a Ramsey graph — not very much!

Theorem 3.4.7 *Fix a positive integer $m \geq 3$. Then there exists $F \in \mathcal{R}[(K_m)_2^{K_2}]$ so that any two elements of $\binom{F}{K_m}$ intersect in at most two vertices.*

Proof: Applying the construction used for the proof of Theorem 3.4.4 with $G = K_m$, then certainly no two copies of K_m in P^0 intersect at all. We need only observe if P^n satisfies the required conditions, then so does P^{n+1}, since each copy of K_3 in P^{n+1} belongs to exactly one copy of P^n. (The copies of P^n in P^{n+1} are disjoint except at two parts, and then the partite nature prevents any triangles existing along just two parts.) □

In order to state the last of the four related theorems, we need some more notation. We say $C \subset V(G)$ is a *cutset* of a graph G if the deletion of C (and all associated edges) disconnects G. If we use $G[C]$ to denote the subgraph of G induced by $C \subset V(G)$, then for $n \geq 2$ we say G is *n-chromatically connected* if the chromatic number $\chi(C[G]) \geq n$ for any cutset C. We now can give the theorem.

Theorem 3.4.8 *Let $\mathcal{F}$ be a set of 3-chromatically connected graphs. Then for every $G \in Forb(\mathcal{F})$, $\mathcal{R}[(G)_2^{K_2}] \cap Forb(\mathcal{F}) \neq \emptyset$.*

Proof: Using the construction given in the proof of Theorem 3.4.4, we claim that if $P^n \in Forb(\mathcal{F})$, then so also $P^{n+1} \in Forb(\mathcal{F})$. Suppose $K \in Forb(\mathcal{F})$ and $K \not\leq P^n$. In P^{n+1}, copies of P^n intersect in a bipartite graph, and bipartite graphs have chromatic number 2. So any 'newly created' subgraphs will have vertices from a bipartite graph as a cutset, and hence no copy of K can be formed in the construction of P^{n+1}. □

This concludes the results found in [88]. In [31] Erdős looks at the size of K_4-free Ramsey graphs F satisfying $F \longrightarrow (K_3)_2^{K_2}$. (The existence of such an F for any number of colors was guaranteed by Nešetřil and Rödl in [80]; Folkman [46] proved the case for 2 colors.) In [124] and [67] Spencer showed that three billion vertices suffice, claiming a reward offered by Erdős. Some bounds on the size of the Ramsey F for edges are discussed in [86] (sections 1,2, and 10) as well as some interesting references regarding this.

3.5 Some Related Facts

Another difficult result proven by Deuber [23] and Nešetřil and Rödl [78], which is an extension of Theorems 3.3.1 and 3.4.4, is the following:

Theorem 3.5.1 *For any ordinary graph G and fixed $r, n \in \omega$ $\mathcal{R}[(G)_r^{K_n}] \neq \emptyset$.*

This will be proved in the next chapter (see Corollary 5.3.2) by use of Theorem 4.5.2, but let us make a few remarks. A simple proof using the Graham-Rothschild theorem (Theorem 2.3.2) was given in [93] which uses the fact that any graph can be embedded into the complement of some power set lattice.

So far, we have been very careful to prove graph Ramsey results for a specific number of colors. This is not always necessary since, in many cases, the general theorem follows from the situation using only two colors. Specifically, it is an easy exercise to prove the following lemma.

Lemma 3.5.2 *Fix a hypergraph H and a family of hypergraphs $\mathcal{G}$. If for every $G \in \mathcal{G}$,*

$$\emptyset \neq \mathcal{R}[(G)^H_2] \subset \mathcal{G}$$

holds, then $\mathcal{R}[(G)^H_r] \neq \emptyset$ for any $G \in \mathcal{G}$.

So, for example, Theorem 3.5.1 need only be proved for $r = 2$.

Chapter 4

The Ordered Hypergraph Ramsey Theorem

4.1 Introduction

The primary purpose of this chapter is to give a Ramsey theorem for ordered hypergraphs. The theorem says that for *any* two ordered hypergraphs, i.e., hypergraphs with a fixed orientation of the vertices, say $(G, \leq)$ and $(H, \leq)$, and a number $r \in \omega$, there exists $(F, \leq)$ so that $(F, \leq) \longrightarrow (G, \leq)_r^{(H,\leq)}$, the usual Ramsey arrow. This was first proved by Nešetřil and Rödl [82], [90] and independently by Abramson and Harrington [1]. Many more proofs of this result have appeared since (see e.g. [89], [95]), and for the special case of ordered graphs (see e.g. [98] for a proof using a generalization of the Graham-Rothschild theorem). A technique employed by Nešetřil and Rödl is called partite amalgamation; most recent proofs follow this idea. Much progress has been made in extremal graph theory due to this amalgamation technique, a technique known to Nešetřil and Rödl since 1976 [95].

An acquaintance with the amalgamation technique was acquired with the proofs of Theorem 3.3.2 (existence of a highly chromatic graph with no short cycles) and Theorem 3.4.4, (Ramsey theorem for ordinary graphs coloring edges). The Nešetřil Rödl technique is usually broken into two parts, the first being a Ramsey theorem for 'partite' graphs, all with the same number of 'parts', and the second, a more general partite graph Ramsey result for which an amalgamation is defined. These two steps can be compared to Lemma 3.4.3 and Theorem 3.4.4 respectively. The first part has had many proofs and variations, but we give here perhaps the most powerful, that which employs the Hales-Jewett theorem.

We follow closely the proof given in [95], where the result is given for 'systems'. A system is really an ordered hypergraph in the most general sense. As an added bonus in this proof, certain conditions concerning induced subsystems (or induced ordered sub-hypergraphs) can be given. The next section helps us to interpret a hypergraph as a system (using the notation in [95]). The following section gives the first of the two steps necessary for the proof of the main theorem, namely a 'partite lemma'. Another section is used to define the type of amalgamation used in the main theorem; this is done rigorously, however, this may be the most streamlined (yet complete) definition given so far. The main theorem follows in another section and some easy corollaries (re: ordered graphs) of the powerful theorem are given in the final section.

4.2 Preliminaries

A *type* $\mathbf{\Delta} = (n_\delta : \delta \in \Delta)$ is a collection of positive integers indexed by some set $\Delta \subset \omega$. For the remainder of this chapter, $\mathbf{\Delta}$ is fixed. A *system* G *of type* $\mathbf{\Delta}$ is a pair $(X, \mathcal{G})$ where X is a finite linearly ordered set, and $\mathcal{G} = \{\mathcal{G}_\delta : \delta \in \Delta\}$ where $\mathcal{G}_\delta \subset [X]^{n_\delta}$, and $\mathcal{G}_\delta \cap \mathcal{G}_{\delta'} = \emptyset$ for $\delta \neq \delta'$. Elements of each $\mathcal{G}_\delta$ shall be called *edges of type* δ (of size n_δ). We see that the index set Δ might just as well be an initial segment of ω, that is, an ordinal.

Although we have fixed the type $\mathbf{\Delta}$, one may note that for any two systems of putatively different types, one can find a single type common to both. The type $\mathbf{\Delta}$

will be taken to be so inclusive in this discussion, and so "system" will now be taken to indicate a system of such a general type.

A simple (no loops or multiple edges) ordered ordinary graph could be seen as a system of type $(2,2)$ where 'non-edges' are of type 0 and edges are of type 1. Using this notation of systems, one can easily see that a system is merely the (most?) general form of an ordered hypergraph. For example, if an ordered hypergraph $(H,\leq)$ is defined on the vertex set $\{0,1,2,3\}$ with hyperedges $(0,1),(0,2),(1,2,3),(0,1,2)$, the first having 'multiplicity' 1 (or 'type' or 'kind') and the remainder with multiplicity 2, then $(H,\leq)$ can seen as a system H of type $(2,2,3)$ where the given edges are of type 1, 2, 3 and 3 respectively.

Two systems $H=(Y,\mathcal{H})$ and $G=(X,\mathcal{G})$ are *isomorphic* if there exists a monotone bijection $f:X\rightarrow Y$ taking $\mathcal{G}$ onto $\mathcal{H}$, in which case we write $G\cong H$. A system $F=(Z,\mathcal{F})$ is a *subsystem* of the system $G=(X,\mathcal{G})$ if $Z\subseteq X$ (with the induced order) and $\mathcal{F}_\delta=\mathcal{G}_\delta\cap\mathcal{P}(Z)$ for each $\delta\in\Delta$ (i.e., consider only induced subsystems). A system $A=(V,\mathcal{A})$ is called *irreducible* if every pair of vertices in V is contained in some edge of A (having non-zero type). $H=(Y,\mathcal{H})$ is called *complete* if for the set of non-zero types Δ^*

$$\bigcup_{\delta\in\Delta^*}\mathcal{H}_\delta=[Y]^{n_\delta}.$$

4.3 The Partite Lemma

Let $G=(X,\mathcal{G})$ be a system with $\mathcal{G}=\{\mathcal{G}_\delta:\delta\in\Delta\}$ where each $\mathcal{G}_\delta\subseteq[X]^{n_\delta}$ and X is a totally ordered set. For a partition $X=\cup_{i\in a}X_i$ (each $X_i\neq\emptyset$), satisfying $X_0<X_1<\ldots<X_{a-1}$, we say $G=((X_i)_{i\in a},\mathcal{G})$ is an *a-partite system* if for each

$i \in a$, $|e \cap X_i| \leq 1$ for every edge $e \in \cup\mathcal{G} = \cup_{\delta\in\Delta}\mathcal{G}_\delta$. ($X < Y$ denotes that for every $x \in X$, $y \in Y$, $x < y$ holds.) The sets X_i are called *parts* (or *coordinates*) of G.

Given a subset $Y \subseteq X$, let

$$\mathrm{sh}(Y) = \{i : Y \cap X_i \neq \emptyset\}$$

denote the *shadow* (sometimes called the *trace*) of Y. The a-partite system $G = ((X_i)_{i\in a}, \mathcal{G})$ is *transversal* if for each $i \in a$ we have $|X_i| = 1$. A complete a-partite system has every subset of vertices (which intersects each part in at most one vertex) as an edge.

The a-partite system $H = ((Y_i)_{i\in a}, \mathcal{H})$ is (partite) isomorphic to $G = ((X_i)_{i\in a}, \mathcal{G})$ if there is a (system) isomorphism $f : \cup X_i \longrightarrow \cup Y_i$ preserving parts, i.e., $f(X_i) = Y_i$ for each $i \in a$. If necessary to emphasize the partite structure, we denote this by $G \cong_{part} H$. We say $F = ((Z_i)_{i\in b}, \mathcal{F})$ is a partite subsystem of G if there is a monotone injection $\sigma : b \longrightarrow a$ so that $Z_i \subset X_{\sigma(i)}$ for each $i \in b$ and $\mathcal{F}_\delta = \mathcal{G}_\delta \cap [\cup_{i\in a} Z_i]^{n_\delta}$ for each $\delta \in \Delta$, denoted by $F \preceq_{part} G$.

For the partite situation, we use

$$\binom{G}{H}_{part} = \{H' \preceq_{part} G : H' \cong_{part} H\}.$$

Also analogous to the graph case, we use $F \longrightarrow_{part} (G)^H_r$ to denote the corresponding partite Ramsey statement for systems. We now are ready to state an analogue to Lemma 3.4.3, referred to as "The Partite Lemma".

Lemma 4.3.1 *Let $r \in \omega$ be given and suppose G and H are a-partite systems with H transversal. Then there exists an a-partite system F so that*

$$F \longrightarrow_{part} (G)^H_r.$$

Before we give the proof, let us just mention that in the proof given in [95], it was stated that H could be assumed to be complete without loss. We shall briefly outline the trick. Let $G = ((X_i)_{i\in a}, \mathcal{G})$, and set $X = \cup X_i$ If H was not complete, add "dummy edges" to form $\overline{H}$ complete. Then, (as outlined by V.Rödl—oral communication) if an edge e was added to H in the formation of $\overline{H}$, look at each subset $e' \subset X$ with $|e' \cap X_i| \leq 1$ which satisfies $\mathrm{sh}(e) = \mathrm{sh}(e')$. If e' is an edge of G, delete it; if e' is not an edge of G, add it as an edge. Continue for each dummy edge of $\overline{H}$, thereby producing $\overline{G}$. Then take $\overline{F}$ satisfying $\overline{F} \longrightarrow (\overline{G})_r^{\overline{H}}$ and in $\overline{F}$, 'undo' the complementing. All the $\overline{G}$'s return to G's and all the $\overline{H}$'s return to H's. There seems to be a problem with this. If across, say X_0 and X_1 there are two (or more) different types of edges (of size 2), then the 'undoing' process does not seem to be well defined. However, this trick works just fine for hypergraphs $(X, \mathcal{E})$ with $\mathcal{E} \subset \mathcal{P}(X)$, i.e., for hypergraphs with only one 'type of edge' for each size. To avoid this difficulty, let *all* the non-edges of H be given new types, types not used for edges in G and 'fill in' G accordingly (introducing only new 'transversal' edges). When we are done finding the necessary F, dismiss all such edges added with the new types and all remains the same. Using this idea, it suffices to prove the theorem for H complete and G a-partite complete. With no significant modifications the proof given in [95] still works.

Proof of Lemma 4.3.1 : Let $H = ((Y_i)_{i\in a}, \mathcal{H})$ with $\cup_{i\in a} Y_i = Y$ and $G = ((X_i)_{i\in a}, \mathcal{G})$ with $\cup_{i\in a} X_i = X$ be given, and let H be transversal. By the preceding comments, we can assume that H is complete. Also without loss of generality, we can assume that every vertex of G is contained in some H-subsystem, for if G^* is the system induced by $\binom{G}{H}_{part}$ and $F^* \longrightarrow_{part} (G^*)^H_r$, then enlarging every copy of G^* in F^* can be done to produce a system F satisfying $F \longrightarrow_{part} (G)^H_r$. (See [84] for general comment.)

Set $|\binom{G}{H}_{part}| = t$ and using the Hales-Jewett theorem (Theorem 2.3.1), set $N =$ HJ$(t, 1, r)$. For each $\delta \in \Delta$, put

$$\mathcal{G}_\delta = \mathcal{G}'_\delta \cup \mathcal{G}''_\delta$$

where $\mathcal{G}'_\delta$ is the set of all edges of $\mathcal{G}_\delta$ which belong to a copy of H in G.

Define an a-partite system $F = ((Z_i)_{i \in a}, \mathcal{F})$, $\mathcal{F} = \{\mathcal{F}_\delta : \delta \in \Delta\}$ with $Z = \cup_{i \in a} Z_i$ as follows. For each $i \in a$, set $Z_i = (X_i)^N$, the direct product N times, that is, each vertex of Z_i has the form

$$\mathrm{x}^k = \langle x^k_0, x^k_1, \ldots, x^k_{N-1} \rangle : x^k_j \in X_i, j \in N.$$

For each $j \in N$, define a projection $\pi_j : Z \to X$ by

$$\pi_j(\langle x^k_0, x^k_1, \ldots, x^k_{N-1} \rangle) = x^k_j.$$

Each projection is onto and preserves parts. For any set $\Gamma \subset Z$, we use the notation

$$\pi_j(\Gamma) = \{\pi_j(\mathrm{x}) : \mathrm{x} \in \Gamma\},$$

and likewise, we can speak of a projection of a subsystem in F to be a system.

For each $\delta \in \Delta$, define $\mathcal{F}_\delta$ in the following manner. For $\Gamma \in [Z]^{n_\delta}$, $\Gamma \in \mathcal{F}_\delta$ if and only if one of the following conditions holds:

1. $\pi_j(\Gamma) \in \mathcal{G}'_\delta$ for every $j \in N$.

2. There exists $J \subset N$, $J \neq \emptyset$, and $B \in \mathcal{G}''_\delta$ so that whenever $j \in J$, $\pi_j = B$, and whenever $j \notin J$, $\pi_j(\Gamma) \in \mathcal{G}'_\eta$ for some $\eta \in \Delta$.

Notice that if every edge of G was an edge of some H-subsystem, then condition (2) would never be invoked. Setting $\mathcal{F} = \{\mathcal{F}_\delta : \delta \in \Delta\}$ completes the definition of F.

Let $\Psi : \binom{F}{H}_{part} \longrightarrow r$ be a given coloring. We wish to show the existence of $G' \in \binom{F}{G}_{part}$ so that Ψ is constant on $\binom{G'}{H}_{part}$.

Suppose that we have some $H' \in \binom{F}{H}_{part}$ induced by the vertex set $Y' \subset Z$. By the product construction of F, it is clear that for every $j \in N$, $\pi_j(Y')$ induces an element of $\binom{G}{H}$. Similarly, if $Y^* \in [Z]^a$ is so that for every $j \in N$, $\pi_j(Y^*)$ induces a copy of H in G, then condition 1 gives us that Y^* induces a copy of H in F. We abbreviate these facts by saying $H' \in \binom{F}{H}$ if and only if $\pi_j(H') \in \binom{G}{H}$ for every $j \in N$.

List $\binom{G}{H} = \{H_0, H_1, \ldots, H_{t-1}\}$. Now employing the notation of parameter sets, examine $[\binom{G}{H}]\binom{N}{0}$, words of length N over the alphabet $\{H_0, H_1, \ldots, H_{t-1}\}$. For each $f \in [\binom{G}{H}]\binom{N}{0}$, define

$$V(f) = \{\mathrm{x} \in Z : \pi_j(\mathrm{x}) \text{ induces } f(j) \in \binom{G}{H}, j \in N\}.$$

So each $V(f)$ is a set of vertices of F whose each projection determines a member of the alphabet. For $g \in [\binom{G}{H}]\binom{N}{1}$, put

$$V(g) = \cup_{f \in sp(g)} V(f).$$

(For example,when $N = 5$, if $g = H_1 H_2 H_1 H_1 \lambda$, then the set of vertices are all those determining H_1,H_2,H_1 and H_1 in the first four projections respectively, and in the last projection, there are no restrictions.)

Now since $H' \in \binom{F}{H}$ if and only if $\pi_j(H') \in \binom{G}{H}$ for every $J \in N$, Ψ induces a coloring

$$\Psi^* : [\binom{G}{H}]\binom{N}{1} \longrightarrow r.$$

By the choice of N, let $h \in [\binom{G}{H}]\binom{N}{1}$ be so that $\mathrm{sp}(h)$ is monochromatic with respect to Ψ^*. By virtue of condition 2 and the fact that H is complete, $V(h)$ induces a copy of G in F (along those projections $\{\pi_j : j \in f^{-1}(\lambda)\}$—all other positions are fixed), call it G'. It is now easy to see that $\binom{G'}{H}$ is monochromatic with respect to Ψ. □

4.4 $*_J$-Amalgamation Defined

In the proof of the Ramsey theorem for systems, the main tool we use is that of $*_J$-*amalgamation.* To streamline the already challenging proof, we first define in detail the amalgamation process used.

Let $A = ((X_i)_{i\in a}, \mathcal{A})$ and $B = ((Y_i)_{i\in b}, \mathcal{B})$ with $Y = \cup_{i\in a} Y_i$ be partite systems with $b \geq a$. Fix a subset $J \in [b]^a$.

For each $\delta \in \Delta$, let

$$\mathcal{B}_{\delta,J} = \mathcal{B}_\delta \cap \mathcal{P}(\cup_{j\in J} Y_j)$$

be those edges e of $\mathcal{B}_\delta$ with $\mathrm{sh}(e) \subseteq J$, and set

$$\mathcal{B}_J = \{\mathcal{B}_{\delta,J} : \delta \in \Delta\}.$$

The system $B = ((Y_j)_{j\in J}, \mathcal{B}_J)$ is the a-partite system of B induced by $Y_J = \cup_{j\in J} Y_j$. Enumerate

$$\binom{A}{B_J}_{part} = \{B_J^0, B_J^1, \ldots, B_J^{s-1}\},$$

and for each $i \in s$, set $B_J^i = ((Y_j^i)_{j\in J}, \mathcal{B}_J^i)$ with $Y_J^i = \cup_{j\in J} Y_j^i$.

We are now prepared to define the b-partite system

$$B *_J A = ((Z_i)_{i\in b}, \mathcal{E})$$

which is formed by copies of B amalgamated (along parts J) on A. For each $i \in J$, set

$$Z_i = Y_i \times s = \{(y_\alpha, j) : y_\alpha \in Y_i, j \in s\}$$

and let $Z = \cup_{i\in b} Z_i$. For each $i \in s$, let $\phi_i : Y_J \to Y_J^i$ be an embedding of B_J into B_J^i, and extend each ϕ_i to

$$\psi_i : Y \to \{(y_\alpha, i) : y_\alpha \in Y_i\}$$

defined by

$$\psi_i(y) = \begin{cases} \phi_i(y) \text{ if } y \in Y_J. \\ (y, i) \text{ if } y \notin Y_J. \end{cases}$$

For each $\delta \in \Delta$, define

$$\mathcal{E}_\delta = \{\{\psi_i(u) : u \in e\} : e \in \mathcal{B}_\delta\} \cup \mathcal{A}_\delta.$$

Now set $\mathcal{E} = \{\mathcal{E}_\delta : \delta \in \Delta\}$ to complete the definition of $B *_J A$. See Figure 4.1 for the idea.

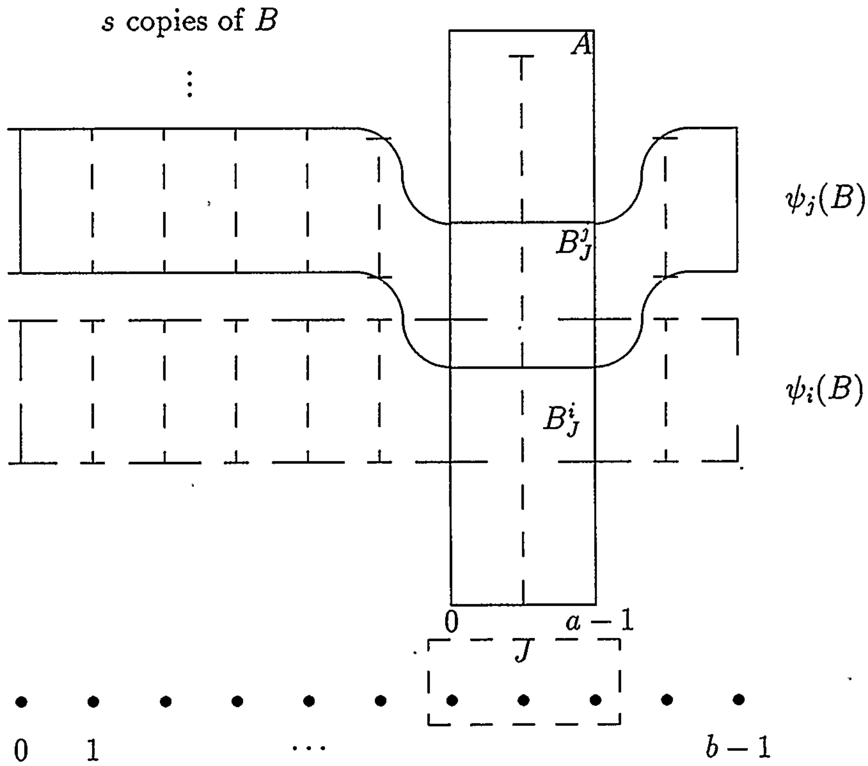


Figure 4.1: $B *_J A$: Amalgamation along a parts.

In our application it is not necessary that we keep the 'non-essential' edges of A, (i.e., those which are not included in some copy of B_J) for usually we use each copy of A as a Ramsey graph for B.

We note that variants of this amalgamation are used by different authors. In [100], [108], and [109], 'left-rectified' amalgamation is used. It has been noted by myself and independently by V. Rödl (—oral communication, Aug. '89) that applications of this variant may be more restricted than previously thought.

The amalgamation defined here has had many applications. In [94], a version is used to give a Ramsey theorem for Steiner systems. Amalgamation also appears in the proofs of Theorem 3.3.2 [87] and Theorem 3.4.4 [88]. It was also used in [91] to show restricted Ramsey results for graphs.

4.5 The Ubiquitous Theorem

We are now ready for the theorem on which this thesis thrives, namely *the* Ramsey theorem for systems [95] and so the proof is complete. Recall that a system is an ordered structure.

Theorem 4.5.1 *For a given $r \in \omega$ and systems G and H, there exists a system F so that $F \longrightarrow (G)^H_r$. Moreover, if G and H do not contain an irreducible system A, then F can be chosen with the same property.*

Proof: Let $G = (X, \mathcal{G})$ and $H = (Y, \mathcal{H})$ where $|X| = b$ and $|Y| = a$ and let s be minimal so that $s \longrightarrow (b)^a_r$. We consider G and H as transversal b-partite and a-partite systems respectively.

Choose an s-partite system $P^0 = ((V_i^0)_{i \in s}, \mathcal{E}^0)$ with the property that for each $I \in [s]^b$ the system induced by parts $\{V_i^0 : i \in I\}$ contains an induced (transversal) copy of G. This may be found easily as a disjoint union of copies of G. (Each V_i^0 can be chosen so that $|V_i| \leq \binom{s-1}{b-1}$, since $\binom{s}{b}$ disjoint copies of G contains $b\binom{s}{b} = s\binom{s-1}{b-1}$ vertices.)

Set $\binom{s}{a} = q$ and list $[s]^a = \{J_0, J_1, \ldots, J_q\}$. We will define inductively s-partite systems P^i for $i \leq q$. Suppose that for some $n < q$ the system $P^n = ((V_i^n)_{i \in s}, \mathcal{E}^n)$ has been defined. Let $P^n_{J_{n+1}}$ be the a-partite system induced by $\{(V_i^n) : i \in J_{n+1}\}$, and select (by Partite Lemma 4.3.1) an a-partite system T^{n+1} satisfying

$$T^{n+1} \longrightarrow_{part} (P^n_{J_{n+1}})^H_r.$$

Furthermore, let T^{n+1} be minimal so as to not contain edges which are not in some copy of $P^n_{J_{n+1}}$. Set $P^{n+1} = T^{n+1} *_{J_{n+1}} P^n$. Notice that any irreducible subsystem of P^{n+1} is also a subsystem of P^n. Set $F = P^q$ and notice, by downward induction, that any irreducible subsystem of F is contained in G.

We claim $F \in \mathcal{R}[(G)^H_r]$. Let $\Psi : \binom{F}{H} \longrightarrow r$ be a given coloring. By the construction of P^q, there is

$$P^{q-1}_* \in \binom{F}{P^{q-1}}$$

so that every $H' \in \binom{P^{q-1}_*}{H}$ satisfying $\mathrm{sh}(H') = J_q$ is colored the same. Now by the construction of P^{q-1}_*, there is

$$P^{q-2}_* \in \binom{P^{q-1}_*}{P^{q-2}}$$

so that every $H' \in \binom{P^{q-2}_*}{H}$ satisfying $\mathrm{sh}(H') = J_{q-1}$ is colored the same. Continuing in this manner, we get $P^0_* \in \binom{F}{P^0}$ so that the color of any $H' \in \binom{P^0_*}{H}$ depends only on $\mathrm{sh}(H')$. This induces a coloring on $[s]^a$ and by the choice of s, there exists $I \in [s]^b$ so that every $H' \in \binom{P^0_*}{H}$ satisfying $\mathrm{sh}(H') \subset I$ is monochromatic. By the choice of P^0, there exists $G' \in \binom{P^0}{G}$ with $\mathrm{sh}(G') = I$, and so we are done (finally!). □

This theorem for systems can be rewritten in terms of hypergraphs (which may have edges of multiple types of the same size). It is in this form that we most refer to it in this thesis.

Theorem 4.5.2 *Given $r \in \omega$ and ordered hypergraphs $(G, \leq)$ and $(H, \leq)$,*

$$\mathcal{R}[(G, \leq)^{(H,\leq)}_r] \neq \emptyset.$$

As in Chapter 3, there are a number of extensions [95] of Theorem 4.5.1 which are related by the constructions used in the proofs in this chapter. We choose not to include them here for the sake of brevity.

4.6 The Ordering Property for Graphs

For a given ordered (ordinary) graph $(H, \leq)$ if $G = (V(G), E(G))$ is an unordered (ordinary) graph with the property that $(H, \leq) \preceq (G, \leq^*)$ for every ordering of G, then we write $G \longrightarrow_{ord} (H, \leq)$. The following well known theorem is proved in [100], for example.

Theorem 4.6.1 *For every ordered graph $(H, \leq)$, there exists an (unordered) graph G so that $G \longrightarrow_{ord} (H, \leq)$.*

Proof: Let $v_0 < v_1 < \ldots < v_{m-1}$ be an enumeration of $V(H)$ respecting the order of $(H, \leq)$. Furthermore, assume that pairs of the form (v_i, v_{i+1}) are edges, for if not, introduce a new vertex between v_i and v_{i+1} connected to both. Let $(H, \leq^{-1})$ be a copy of H with $\leq^{-1}$ the inverse of $\leq$. Form an ordered graph $(H^*, \leq^*)$ by taking the ordered sum (in fact any disjoint union preserving each order will do) of $(H, \leq)$ and $(H, \leq^{-1})$. By Theorem 4.5.2, select an ordered graph $(G, \leq')$ satisfying the Ramsey statement

$$(G, \leq') \longrightarrow (H, \leq^*)_2^{K_2}.$$

We claim that G, the unordered version of $(G, \leq')$, satisfies $G \longrightarrow_{ord} (H, \leq^*)$. Let $(G, \leq'')$ be an arbitrary ordering of G and define the coloring $\Psi : E(G) \longrightarrow 2$ by $\Psi(\{x, y\}) = 0$ if the orders $\leq'$ and $\leq''$ agree on x and y, and $\Psi(\{x, y\}) = 1$ otherwise.

By the choice of $(G, \leq')$, there exists an ordered $(H^*, \leq^*)$-subgraph monochromatic with respect to Ψ. Depending on the color of this subgraph, the connections in $(H, \leq)$ ensure that either the $(H, \leq)$-portion or the $(H, \leq^{-1})$-portion yields the desired copy of $(H, \leq)$. Removal of any 'new' vertices completes the proof. □

Chapter 5

Some Results

5.1 Notation

This chapter contains material which was assembled in the form of a joint paper [59] by the author together with Professors N. W. Sauer and V. Rödl. Theorems 5.5.1 thru 5.7.4 are new results from that paper.

For this discussion we introduce some new notation. For a hypergraph H we let

$$ORD(H) = \{(H, \leq_0), (H, \leq_1), \ldots, (H, \leq_{k-1})\},$$

be the set of (distinct) isomorphism types of orderings of H. It is often convenient to abuse the notation and deliberately confuse an isomorphism type with a hypergraph of that given type. For a given (unordered) hypergraph H and an ordered hypergraph $(G, \leq^*)$ define

$$DO(H, G, \leq^*) = \{(H, \leq_i) \in ORD(H) : \binom{G, \leq^*}{H, \leq_i} \neq \emptyset\}.$$

Set $mdo(H, G) = \min\{|DO(H, G, \leq^j)| : (G, \leq^j) \in ORD(G)\}$, denoting the smallest number of orderings of H in any one ordered G . For example, if an ordinary graph H is complete, then $mdo(H, G) \leq 1$ for any choice of G. The number $mdo(H, G)$ will be of particular interest throughout this chapter.

Observe that for these Ramsey type statements to be non-trivial we usually only consider pairs G, H so that $mdo(H, G) \geq 1$.

5.2 Introduction

Could it be that for *every* triple G, H and r we have a Ramsey $F \in \mathcal{R}[(G)_r^H]$? This can be answered in the negative by the following well known example (e.g. [100], p.192):

Example 5.2.1 $\mathcal{R}[(C_4)_2^{P_2}] = \emptyset$.

Proof: Let $\leq^*$ be any total order of $V(C_4)$. Then it is easy to see that $(C_4, \leq^*)$ contains at least two distinct orderings of P_2, namely one with the middle vertex highest in the order, and one with the middle vertex lowest in the order. (These two 'middle' vertices correspond to the two vertices on the 'ends' of the order in $(C_4, \leq^*)$.)

Now fix any (ordinary) graph F. Impose an arbitrary order $\leq$ on $V(F)$. We will produce a coloring of $\binom{F}{P_2}$ which ensures that every copy of C_4 in F is multicolored. Simply color the copies of P_2 according to their orientation; if one is 'pointed' upwards, color it red and if one is pointed downwards, color it blue. We can color the other ordered P_2's arbitrarily. Now since each ordered C_4 contains one of each kind of P_2 it receives two colors. □

It is not difficult to see that $mdo(P_2, C_4) = 2$. In subsequent sections we rely heavily on this idea of ordering graphs so that we can find particular colorings. Pairs like C_4 and P_2 are not anomalous; there are 'many' such cases, as given in the following theorem [78].

We call a graph *trivial* if it is either complete or empty.

Theorem 5.2.2 *For any non-trivial graph H there is a graph G so that*

$$\mathcal{R}[(G)^H_2] = \emptyset.$$

A proof of this theorem uses an idea similar to the one used for Example 5.2.1, that is, the idea of coloring ordered graphs. It is an easy application of Theorem 4.6.1 and we omit it.

5.3 The Question

For a fixed $r \in \omega$ and given graphs G and H, how can we tell if $\mathcal{R}[(G)^H_r] \neq \emptyset$? Oddly enough, this is completely answered in the case of ordered hypergraphs. This might seem counterintuitive since ordered graphs are rigid and so any Ramsey structure would have to be larger, in some sense, than in the unordered case so as to contain the necessary richness of substructures required. Nevertheless we recall Theorem 4.5.2 which states that for $r \in \omega$ and ordered hypergraphs $(G, \leq)$ and $(H, \leq)$, the Ramsey class $\mathcal{R}[(G, \leq)^{(H,\leq)}_r]$ is not empty. An immediate application of this powerful theorem is the following:

Corollary 5.3.1 *Fix $r \in \omega$. If H and G are (unordered) hypergraphs which satisfy $mdo(H, G) = 1$ then*

$$\mathcal{R}[(G)^H_r] \neq \emptyset.$$

Proof: Let $mdo(H, G) = 1$ and fix an ordering $\leq$ of G so that every induced H-subgraph of G is $\leq$-order-isomorphic to say $(H, \leq)$. Apply Theorem 4.5.2 to obtain $(F, \leq) \in \mathcal{R}[(G, \leq)^{(H,\leq)}_r]$. We claim the unordered F also satisfies $F \longrightarrow (G)^H_r$.

Fix a coloring $\Delta : \binom{F}{H} \longrightarrow r$ and order F according to $\leq$. Then Δ induces a coloring $\Delta^* : \binom{F,\leq}{H,\leq} \longrightarrow r$ and so there exists a $(G', \leq) \in \binom{F,\leq}{G,\leq}$ so that Δ^* is constant on $\binom{G',\leq}{H,\leq}$. But since $|DO(H,G,\leq)| = 1$, Δ^* assigns a color to every copy of H in G'. Hence $\binom{G'}{H}$ is monochromatic with respect to Δ^* and so also with respect to Δ.
□

Now another otherwise difficult result proven by Deuber [22] and Nešetřil and Rödl [78], (which is an extension of Theorems 3.3.1 and 3.4.4) is simply an obvious

Corollary 5.3.2 *For any ordinary graph G and fixed $r, n \in \omega$ $\mathcal{R}[(G)_r^{K_n}] \neq \emptyset$.*

One might hope to find some necessary conditions on G, H and r so that $\mathcal{R}[(G)_r^H] \neq \emptyset$, however at least one straightforward restriction must be respected.

Lemma 5.3.3 *Fix hypergraphs G and H where $|ORD(H)| = r$. If $mdo(H,G) \geq 2$ then $\mathcal{R}[(G)_r^H] = \emptyset$.*

Proof: In hope of a contradiction, suppose F is so that $F \longrightarrow (G)_r^H$ and impose an arbitrary ordering $\leq$ on $V(F)$. Now define an r-coloring $\Delta : \binom{F}{H} \longrightarrow r$ by

$$\Delta(H') = i \text{ if } (H', \leq) \cong (H, \leq_i) \in ORD(H).$$

Since $mdo(H,G) \geq 2$, every copy of G in F is two-colored. □

5.4 Counterexample

In a corollary ([104] p.54) Prömel and Voigt state that for all hypergraphs G and H, $mdo(H,G) = 1$ if and only if $\mathcal{R}[(G)_2^H] \neq \emptyset$.

After trying to prove this 'corollary' we discovered the following counterexample.

Theorem 5.4.1 *There exist graphs G and H so that $mdo(H,G) = 2$ but*

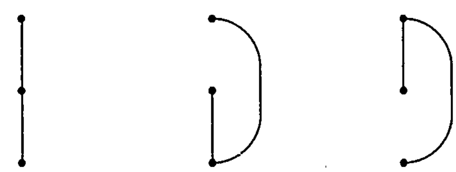
$$\mathcal{R}[(G)_2^H] \neq \emptyset.$$

Proof: Let T and $P = P_2$ be the graphs given in Figure 5.1.

Label $ORD(P) = \{(P, \leq_0), (P, \leq_1)(P, \leq_2)\}$ as in Figure 5.2.

T P

Figure 5.1: Graphs used for counterexample.

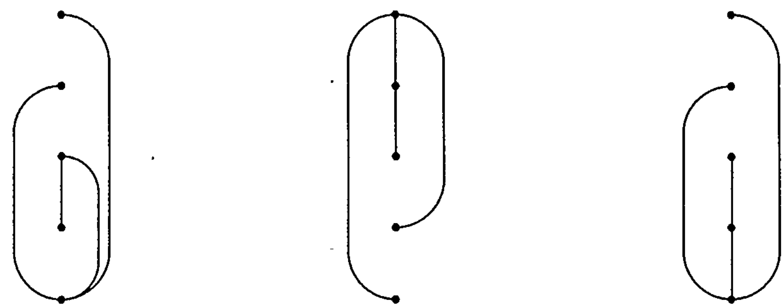


$(P, \leq_0)$ $(P, \leq_1)$ $(P, \leq_2)$

Figure 5.2: Orderings of P

It is easy to verify that $mdo(P,T) = 2$. Fix three orderings of T as shown in Figure 5.3.

We point out that for each $i \in 3$, $(T, \leq^i)$ contains as induced P-subgraphs exactly $\{(P, \leq_j) : j \in 3, j \neq i\}$. Let $(B, \leq)$ be the ordered (disjoint) sum of $(T, \leq^0)$, $(T, \leq^1)$, and $(T, \leq^2)$, taken in a fixed, but arbitrary order.



$(T, \leq^0)$ $(T, \leq^1)$ $(T, \leq^2)$

Figure 5.3: Three orderings of T.

Using Theorem 4.5.2 successively choose ordered graphs $(C, \leq)$, $(D, \leq)$ and $(F, \leq)$ so that

$$(C, \leq) \longrightarrow (B, \leq)_2^{(P, \leq_0)}, \tag{5.1}$$

$$(D, \leq) \longrightarrow (C, \leq)_2^{(P, \leq_1)}, \text{and} \tag{5.2}$$

$$(F, \leq) \longrightarrow (D, \leq)_2^{(P, \leq_2)}. \tag{5.3}$$

We claim that F, the unordered version of $(F, \leq)$, actually satisfies $F \longrightarrow (T)_2^P$. Fix a 2-coloring $\Delta : \binom{F}{P} \longrightarrow 2$. By (5.3), there is $(D', \leq) \in \binom{F, \leq}{D, \leq}$ so that Δ is constant on $\binom{D', \leq}{P, \leq_3}$, say

$$\Delta[\binom{D', \leq}{P, \leq_2}] = s_2 \in 2.$$

Now by (5.2) there also exists $(C', \leq) \in \binom{D', \leq}{C, \leq}$ so that Δ is constant on $\binom{C', \leq}{P, \leq_1}$, say $\Delta[\binom{C', \leq}{P, \leq_1}] = s_1$ (while of course Δ is still constant on $\binom{C', \leq}{P, \leq_2}$). Similarly, by (5.1) we choose $(B', \leq) \in \binom{C', \leq}{B, \leq}$ with $\Delta[\binom{B', \leq}{P, \leq_0}] = s_0 \in 2$ while still being constant on $\binom{B', \leq}{P, \leq_2}$ and $\binom{B', \leq}{P, \leq_1}$. So in $(B', \leq)$ all copies of P are colored with two colors, only depending on their orientation. Since $\{s_0, s_1, s_2\} \subset 2$, at least two of s_0, s_1, s_2 agree. If, say,

$s_0 = s_1$ then the $(T, \leq^2)$ part of $(B', \leq)$ has all its P-subgraphs colored the same. In any case, at least one monochromatic copy of T will exist as an induced subgraph of F. □

In general, the idea is easy to apply if we can find an H with $|ORD(H)| = 3$, and G so that $mdo(H, G) \geq 2$ and yet there are 3 orderings of G witnessing the fact, each containing a different pair of (distinct) elements from $ORD(H)$ as induced subgraphs. This recipe can be generalized to reveal the essence of the method, as we see in the next section.

5.5 A Characterization

Let $K = (X, \mathcal{K})$ be a hypergraph and recall that the *chromatic number*, $\chi(K)$, of K is the least integer n so that there is an n-coloring of the vertex set X yielding no monochromatic edge $E \in \mathcal{K}$. For a given pair of hypergraphs G and H, let us define a new hypergraph $K_{H,G}$ on the vertex set $ORD(H)$ with edge set $E(K_{H,G}) = \{DO(H, G, \leq^j) : (G, \leq^j) \in ORD(G)\}$. Since for each edge there corresponds an ordering of G we may, by abuse of notation, refer to the edges as orderings of G, i.e., we could say $E(K_{H,G}) = ORD(G)$, and a vertex $(H, \leq_i)$ is contained by an edge $(G, \leq^j)$ if and only if $(H, \leq_i) \preceq (G, \leq^j)$. We now give a characterization of those triples H, G and r for which there exists a Ramsey graph.

Theorem 5.5.1 *Let G and H be hypergraphs. Then $\mathcal{R}[(G)^H_r] \neq \emptyset$ if and only if $\chi(K_{H,G}) > r$.*

The proof in one direction is based on the construction given in the proof of the counterexample and the other direction is by simple contradiction. It might be helpful to keep in mind that if $\chi(K_{H,G}) > r$ this would mean that for every r-coloring $\chi : ORD(H) \longrightarrow r$ there exists an order $\leq^*$ of G so that $DO(H, G, \leq^*)$ is monochromatic. This fact will be used to show that the graph we construct in the first part of the proof is indeed in $\mathcal{R}[(G)^H_r]$. Throughout the proof we fix $r \in \omega$, hypergraphs G, H and $K = K_{G,H}$.

Proof:($\Leftarrow$) Assume $\chi(K) > r$. Enumerate

$$ORD(H) = \{(H, \leq_0), (H, \leq_1), \ldots, (H, \leq_{t-1})\} \text{ and}$$

$$ORD(G) = \{(G, \leq^0), (G, \leq^1), \ldots, (G, \leq^{s-1})\}.$$

Construct the graph $(B, \leq) = \dot{\bigcup}_{j \in s}(G, \leq^j)$, the (disjoint) ordered sum of the orderings of G. (It is not necessary that all the vertices of one ordering of G be entirely below all vertices of another,—though it helps to imagine it this way—only that the order of each is preserved and they remain disjoint, but yet form a new ordered graph.) By Theorem 4.5.2 choose $(B_0, \leq)$ satisfying

$$(B_0, \leq) \longrightarrow (B, \leq)^{(H, \leq_0)}_r,$$

and for $i = 1, \ldots, t-1$ choose (again by Theorem 4.5.2) successively $(B_i, \leq)$ so that

$$(B_i, \leq) \longrightarrow (B_{i-1}, \leq)^{(H, \leq_i)}_r.$$

We claim that B_{t-1}, the unordered version of $(B_{t-1}, \leq)$, satisfies $B_{t-1} \longrightarrow (G)^H_r$. Fix a coloring

$$\Delta : \binom{B_{t-1}}{H} \longrightarrow r.$$

As in the proof of Theorem 5.4.1, construction guarantees the existence of

$$(B', \leq) \in \binom{B_{t-1}, \leq}{B, \leq}$$

so that for any fixed i, all the induced $(H, \leq_i)$-subgraphs of $(B', \leq)$ are monochromatic. This coloring of ordered H's in $(B', \leq)$ induces a r-coloring χ of the vertices of K and hence (by the remark preceding the proof) there exists a $(G, \leq^j)$ in the edge set of K which is monochromatic (since $\chi(K) > r$) with respect to χ. Thus, there exists $G^* \in \binom{B_{t-1}}{G}$ monochromatic with respect to Δ.

($\Rightarrow$) Assume $\chi(K) \leq r$. So choose a coloring $\chi : ORD(H) \longrightarrow r$ so that each element in $ORD(G)$ is multi-colored. Choose any F and impose an arbitrary (but fixed) ordering $\leq^*$ on $V(F)$. This naturally imposes an order on each $H' \in \binom{F}{H}$, so color $\binom{F}{H}$ according to χ. That is, define $\Delta : \binom{F}{H} \longrightarrow r$ by $\Delta(H') = \chi((H', \leq^*))$ for each $H' \in \binom{F}{H}$, where $(H', \leq^*) \in ORD(H)$ is the $\leq^*$-ordered H-subgraph. Then since each element in $ORD(G)$ is multi-colored with respect to χ, so also is each $G' \in \binom{F}{G}$ with respect to Δ. $\square$

This theorem is a strengthened version of a general comment made in the first page of [86], concerning the work of de Bruijn and Erdős [8].

Theorem 5.5.1 also yields the following characterization which was also suggested to the present author by Xuding Zhu (oral communication).

Corollary 5.5.2 *For given hypergraphs G and H, $mdo(H, G) = 1$ if and only if for every $r \in \omega$ $\mathcal{R}[(G)^H_r] \neq \emptyset$.*

Proof: One direction is simply Corollary 5.3.1, so assume that for some fixed G and H and every $r \in \omega$, $\mathcal{R}[(G)^H_r] \neq \emptyset$. Then by Theorem 5.5.1 the chromatic number of the associated $K_{H,G}$ is infinite. Since $K_{H,G}$ is finite, this means that there is a hyperedge consisting of only one point. A single vertex edge would correspond to an ordering of G witnessing $mdo(H, G) = 1$. $\square$

Additionally, one can now derive the following corollary of Theorem 5.5.1 by only examining a particular $K_{H,G}$ with known chromatic number. This gives sufficient conditions (which can be tested directly) on pairs of hypergraphs G and H for which $\mathcal{R}[(G)_r^H] \neq \emptyset$ holds.

Corollary 5.5.3 *Let G and H be hypergraphs with*

$$mdo(H, G) = l \leq k = |ORD(H)|$$

and fix $r \in \omega$. If there exists an $s \in \omega$, $(l \leq s < k)$, so that both $k \geq rs - 1$ and *for each $J \subseteq ORD(H)$ with $|J| = s$ there exists $(G, \leq^j) \in ORD(G)$ so that $DO(H, G, \leq^j) = J$, then $\mathcal{R}[(G)_r^H] \neq \emptyset$.*

5.6 A Special Hypergraph

We will, in the next section, give an infinite family of pairs of graphs H and G so that $mdo(H, G) \geq 2$ however $\mathcal{R}[(G)_2^H]$ is non-empty. To do this we must first find a (very large) hypergraph with certain properties.

Let us recall the following definition of a hypergraph with no short cycles (cf. [33] p.94). For the r-uniform hypergraph $E = (X, \mathcal{E})$ has $girth(E) > l$ if for every sequence of distinct edges $f_0, f_1, \ldots, f_{j_0-1} \in \mathcal{E}$ with $j_0 \leq l$

$$|\bigcup_{j \in j_0} f_j| \geq j_0(r-1) + 1 \tag{5.4}$$

holds. If (5.4) fails to be true, then a cycle of length $\leq j_0$ exists among the edges $f_0, f_1, \ldots, f_{j_0-1}$.

For a hypergraph on a vertex set X partitioned by

$$X = X_0 \,\dot{\cup}\, X_1 \,\dot{\cup} \ldots \dot{\cup} X_{n-1},$$

we use the notation $((X_i)_{i\in n}, \mathcal{E})$, commonly used for n-partite graphs.

Theorem 5.6.1 *For a given integers $n \geq 1$ and $l \geq 2$ there exists a $2n$-uniform hypergraph $E = ((X)_{i\in n}, \mathcal{E})$ with $|X_0| = |X_1| = \ldots = |X_{n-1}|$, which enjoys the following properties:*

1. *For each edge $e \in \mathcal{E}$, $|e \cap X_i| = 2$ for all $i \in n$.*
2. *$Girth(E) > l$*
3. *For each choice of $X'_0, X'_1, \ldots, X'_{n-1}$ with $X'_i \subset X_i$ and $|X'_i| \geq \frac{1}{n}|X_i|$ for all $i \in n$, $\mathcal{E} \cap [\bigcup_{i\in n} X'_i]^{2n} \neq \emptyset$, i.e., there exists an edge in the hypergraph induced by $\bigcup_{i\in n} X'_i$.*

For this proof we use the *probabilistic method*, due to P. Erdős [28], in a manner similar to that used in [33]. We use the notation $f(n) = o(g(n))$ if

$$\lim_{n\to\infty} f(n)/g(n) = 0$$

for functions f and g. Note that if $f(n) = o(n)$ then so also $f(n) = o(kn)$ for fixed $k > 0$.

Proof: The case $n = 1$ is trivial so consider only when $n \geq 2$ (while of course $l \geq 2$ as well). We use elementary asymptotic formulae (see [42]) throughout; it will be tacitly assumed that numbers used are large enough so these formulae hold. Let $N = N(n, l)$ be a large positive integer. Consider n pairwise disjoint sets $X_0, X_1, \ldots, X_{n-1}$ of cardinality N.

Let $\mathcal{U}$ be the set of all $2n$-element sets f with the property $|f \cap X_i| = 2$ for every $i \in n$. Fix $\epsilon = \frac{1}{2l}$ and set $M = \lceil N/n \rceil$. Let $\mathcal{E}$ be random subset of $\mathcal{U}$, where for

$f \in \mathcal{U}$

$$\text{Prob}[f \in \mathcal{E}] = p = M^{\epsilon-2n+1}.$$

Let $X_i' \subset X_i, i = 0, 1, \ldots, n-1$ be subsets so that $|X_i'| \geq M$. Then

$$\begin{aligned}
&\text{Prob}[|\mathcal{E} \cap [\bigcup_{i \in n} X_i']^{2n}| \leq M] && (5.5)\\
&= \sum_{j=0}^{M} \binom{\binom{M}{2}^n}{j} p^j (1-p)^{\binom{M}{2}^n - j}, && (5.6)\\
&\leq M \binom{\binom{M}{2}^n}{M} p^M (1-p)^{\binom{M}{2}^n - M}, && (5.7)\\
&\sim M(\frac{e}{2^n} M^{2n-1})^M (M^{\epsilon-2n+1})^M \exp\left[-M^{\epsilon-2n+1}\left(\binom{M}{2}^n - M\right)\right], &&\\
&< M^{\epsilon M} \exp\left[-\frac{1}{2^n} M^{1+\epsilon}\right], &&\\
&= o(1). &&
\end{aligned}$$

The first equality (5.6) merely sums probabilities according to the binomial distribution. In such a distribution, it is well known that the occurrence with highest probability occurs close to the expected value, which is, in this case,

$$\binom{M}{2}^n M^{\epsilon-2n+1} = \frac{N^{1+\epsilon}}{2^n}.$$

For large enough N, we see that this can be made larger than M, and so the inequality (5.7) holds. These equations show us that for sufficiently large N the graph induced by $\bigcup_{i \in n} X_i'$ has more than N/n edges with probability close to 1.

Now we will examine short cycles; if (5.4) fails to be true (with $r = 2n$) for some $j_0 \leq l$, then there exists an l-tuple of edges $f_0, f_1, \ldots, f_{l-1} \in \mathcal{E}$ and a set $Y \subset \bigcup_{i \in n} X^i$, $|Y| = l(2n-1)$ so that $\bigcup_{j \in l} f_j \subset Y$. The number of choices for each set Y is bounded by

$$\binom{nN}{l(2n-1)} < c_1 N^{l(2n-1)}$$

and given Y, the number of choices for $f_0, f_1, \ldots, f_{l-1}$ is easily bounded by

$$\binom{l(2n-1)}{2n}^l < c_2$$

where $c_1 = c_1(n,l)$, $c_2 = c_2(n,l)$ are independent of the choice of N. Thus the expected number of cycles of length at most l can be bounded from above by

$$c_1 c_2 N^{l(2n-1)} p^l \leq c_1 c_2 N^{l(2n-1)} \left(\frac{2N}{n}\right)^{l(\epsilon-2n+1)} = c_3\sqrt{N} = o(N) = o(M),$$

where $c_3 = c_3(n,l)$ is a constant independent of N.

Summarizing, with large probability $((X_i)_{i\in n}, \mathcal{E})$ has the following properties: (i) $|[\bigcup_{i\in n} X_i']^{2n} \cap \mathcal{E}| > M$ whenever $X_i' \subset X_i$ and $|X_i'| \geq \frac{1}{n}|X_i|$ for all $i \in n$; and (ii): the number of cycles of length at most l is $o(M)$.

Let $((X_i)_{i\in n}, \mathcal{E}')$ be a hypergraph satisfying both (i) and (ii) (while still satisfying condition 1 in the statement of the theorem). Delete an edge from each cycle of length at most l to obtain a hypergraph $E = ((X_i)_{i\in n}, \mathcal{E})$. We deleted at most $o(M)$ edges, and thus due to (i),

$$|[\bigcup_{i\in n} X_i']^{2n} \cap \mathcal{E}| > M - o(M) > 0$$

whenever $X_i' \subset X_i$ and $|X_i'| \geq \frac{1}{n}|X_i|$.□

The hypergraph constructed in the above theorem has a very special property; with the help of this next useful lemma, we shall find it. For a given order $\leq^*$ on a set A, we use $C \leq^* D$ to denote $c \leq^* d$ for all $c \in C \subset A$ and $d \in D \subset A$, where no relations in C or D are specified.

Lemma 5.6.2 *For given n, $N \in \omega$, let $(X, \leq^*)$ be a totally ordered set with $|X| = nN$. Let $X = X_0 \dot\cup X_1 \dot\cup \ldots \dot\cup X_{n-1}$ be a partition of X with $|X_i| = N$ for each $i \in n$. Then there exists a subfamily $X'_0, X'_1, \ldots, X'_{n-1}$, where for each i $X'_i \subseteq X_i$ and $|X'_i| \geq N/n$, together with a permutation $\sigma : n \to n$ so that*

$$X'_{\sigma(0)} <^* X'_{\sigma(1)} <^* \ldots <^* X'_{\sigma(n-1)}.$$

Proof: Since the case $n = 1$ is trivial, assume $n \geq 2$ and let X_i $(i \in n)$ and $(X, \leq^*)$ be given with $x_0 \leq^* x_1 \leq^* \ldots \leq^* x_{nN-1}$ an enumeration of X. First we select the smallest $k_0 \in (nN-1)$ so that for some $i \in n$,

$$|\{x_0, \ldots, x_{k_0-1}\} \cap X_i| = \lceil N/n \rceil,$$

and set $\sigma(0) = i$. Note that, by the pigeon hole principle,

$$k_0 \leq n(\lceil N/n \rceil - 1) + 1 \leq N.$$

Also observe that for each $j \neq \sigma(0)$

$$|\{x_{k_0}, \ldots, x_{nN-1}\} \cap X_j| > N - \lceil N/n \rceil,$$

since at most $\lceil N/n \rceil - 1$ elements of X_j $(j \neq \sigma(0))$ occurred in $\{x_0, \ldots, x_{k_0-1}\}$ and k_0 was chosen smallest. We set

$$X'_{\sigma(0)} = \{x_0, \ldots, x_{k_0-1}\} \cap X_{\sigma(0)}$$

. We repeat the procedure with $\{x_{k_0}, \ldots, x_{nN-1}\}$ and $\{X_i : i \neq \sigma(0)\}$. In general, suppose we have found

$$J = \{\sigma(0), \ldots, \sigma(t-1)\}$$

and $\{X'_j : j \in J\}$ so that

$$X'_{\sigma(0)} <^* \ldots <^* X'_{\sigma(t-1)}$$

where $\max(X'_{\sigma(t-1)}) = x_{k_{(t-1)}-1}$ with $t < n$. Then for $\nu \notin J$,

$$|\{x_{k_{t-1}}, \ldots, x_{nN-1}\} \cap X_\nu| \geq [N - t(\lceil N/n \rceil - 1)] > (n-t)(\lceil N/n \rceil - 1),$$

where the first inequality is because we could have 'used' only so many at each step and the second inequality holds since $N > n(\lceil N/n \rceil - 1)$. Thus we can continue, finding $\sigma(t) \in n \backslash J$ and a minimal k_t so that

$$|X_{\sigma(t)} \cap \{x_i : k_{t-1} \leq i < k_t\}| \geq \lceil N/n \rceil$$

and so we set

$$X'_{\sigma(t)} = X_{\sigma(t)} \cap \{x_i : k_{t-1} \leq i < k_t\}. \ \square$$

Let $E = ((X_i)_{i \in n}, \mathcal{E})$ be the hypergraph guaranteed by Theorem 5.6.1. Since for each $e \in \mathcal{E}$, $|e \cap X_i| = 2$ for each $i \in n$, let us denote each edge by $e = \{x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}\}$ where $x_i, y_i \in X_i$ for each $i \in n$.

Lemma 5.6.3 *For $E = ((X_i)_{i \in n}, \mathcal{E})$ and $<^*$ a total order on $\bigcup_{i \in n} X_i$, then there exists $e = \{x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}\} \in \mathcal{E}$ and a permutation σ of n so that the vertices of e satisfy*

$$x_{\sigma(0)} <^* y_{\sigma(0)} <^* x_{\sigma(1)} <^* y_{\sigma(1)} <^* \ldots <^* x_{\sigma(n-1)} <^* y_{\sigma(n-1)},$$

where $\{x_{\sigma(i)}, y_{\sigma(i)}\} \subset X_{\sigma(i)}$ for each $i \in n$. That is, there remains at least one edge which keeps vertices from the same coordinate X_i 'together' in the order $<^*$.

Proof: Let $<^*$ be a given order on $\bigcup_{i \in n} X_i$; then by Lemma 5.6.2 there exists a subfamily $X'_0, X'_1, \ldots, X'_{n-1}$, where for each i $X'_i \subseteq X_i$ and $|X'_i| \geq N/n$, and a permutation $\sigma : n \to n$ so that

$$X'_{\sigma(0)} <^* X'_{\sigma(1)} <^* \ldots <^* X'_{\sigma(n-1)}.$$

Now by condition 3 of Theorem 5.6.1, the desired edge exists.

5.7 Infinite Family of Counterexamples

In this section, we produce infinitely many pairs H and G so that
$mdo(H,G) \geq 2$ and yet $\mathcal{R}[(G)^H_2] \neq \emptyset$. We do this by choosing H of a particular nature (of which there are infinitely many) and, using the large hypergraph of Theorem 5.6.1, produce a corresponding G satisfying the sought after conditions. We first give a simple observation.

Lemma 5.7.1 *All connected non-trivial ordinary graphs contain a copy of P_2 as an induced subgraph.*

Proof: Let $H = (V(H), E(H))$ be connected. Choose $a, b \in V(H)$ so that $\{a,b\} \notin E(H)$. Since H is connected there exist $x_1, x_2, \ldots, x_m \in V(H)$ determining a path $ax_1x_2 \ldots x_m b$. Assume that no copy of P_2 occurs as an induced subgraph of the graph induced by $\{a, x_1, x_2, \ldots, x_m\}$. Then we must have $\{a, x_2\} \in E(H)$ (otherwise a,x_1 and x_2 determine a copy of P_2). Similarly, $\{a, x_3\}, \ldots, \{a, x_m\}$ must also be edges. In this case a,x_m and b determine a copy of P_2. □

We now come to the main theorem of this section. This result ([60]) yields infinitely many desired examples. Recall that a graph is 2-connected if it is connected and has no cutpoints.

Theorem 5.7.2 *If H is a non-trivial 2-connected (ordinary) graph, then there exists a graph G so that $mdo(H,G) \geq 2$ and $\mathcal{R}[(G)^H_2] \neq \emptyset$.*

Proof: Let $H = (V(H), E(H))$ be given with $|V(H)| = n$. By Lemma 5.7.1 fix a copy of P_2 across $\{h_0, h_1, h_2\} \subset V(H)$ with enumeration $V(H) = \{h_0, h_1, \ldots, h_{n-1}\}$. Form a new graph $K \cong H$, $V(H) \cap V(K) = \emptyset$, with $\psi : V(H) \to V(K) = \{k_o, k_1, \ldots, k_{n-1}\}$, the isomorphism defined by $\psi(h_i) = k_{i+1(\text{mod}3)}$ for $i = 0$, 1 and 2 and $\psi(h_i) = k_i$ otherwise. We have simply relabeled H using a permutation of the first three vertices. Order each of $V(H)$ and $V(K)$ in the natural way (i.e., $h_i < h_j$ and $k_i < k_j$ if and only if $i < j$) producing $(H, \leq)$ and $(K, \leq)$. Note that $(H, \leq) \ncong (K, \leq)$.

Select a hypergraph $E = ((X_i)_{i \in n}, \mathcal{E})$ satisfying the conditions in Theorem 5.6.1 with girth$(E) > n$. Construct a new (ordinary) n-partite graph G on the vertex set $\bigcup_{i \in n} X_i$ by disjointly embedding a copy of $H \dot{\cup} K$ into each hyperedge of E (as in Theorem 3.3.4) in the following manner: For each hyperedge $e = \{x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}\}$ (where $x_i, y_i \in X_i$ for each $i \in n$) in $\mathcal{E}$ define embeddings $f_e : V(H) \dot{\cup} V(K) \to e$ by $f_e(h_i) = x_i$ and $f_e(k_i) = y_i$ for each $i \in n$. So $\{a, b\} \in E(G)$ if and only if $\{f_e^{-1}(a), f_e^{-1}(b)\} \in E(H) \cup E(K)$ for some $e \in \mathcal{E}$.

Since girth$(E) > 2$, hyperedges of E intersect in at most one point and so these embeddings are well defined. Essentially, $G = ((X_i)_{i \in n}, E(G))$ is a graph formed by 'stringing out' copies of H across its coordinates; the H-subgraphs can sit in one of two ways. We claim that $mdo(H, G) \geq 2$.

Let $\leq$ be an order on $V(G) = V(E)$. By Lemma 5.6.3, there exists a hyperedge $e \in \mathcal{E}$ which respects grouping of vertices along coordinates, and hence we find a copy of H and a copy of K which satisfy $\{f_e(h_i), f_e(k_i)\} \subset X_{\sigma(i)}$ for each i and some permutation σ. We need only notice that permuting the order of the vertices of both $(H, \leq)$ and $(K, \leq)$ in the same way produces again two non-isomorphic orderings of H. So any ordering of G produces two non-isomorphic ordered H's, i.e., $mdo(H, G) \geq 2$. We have yet to demonstrate that $\mathcal{R}[(G)_2^H] \neq \emptyset$.

Rename $(H, \leq) = (H, \leq^0)$ and $(K, \leq) = (H, \leq^1)$ and fix $(H, \leq^2)$, a third ordering of H defined by

$$h_2 \leq^2 h_0 \leq^2 h_1 \leq^2 h_3 \leq^2 h_4 \leq^2 \ldots \leq^2 h_{n-1},$$

agreeing with the first two except by a cyclical permutation on h_0, h_1 and h_2. Let $\leq_0$, $\leq_1$ and $\leq_2$ be three total orders on $V(G)$ which preserve coordinates and agree except that the first three coordinates are permuted, e.g.:

$$\begin{aligned} &X_0 \leq_0 X_1 \leq_0 X_2 \leq_0 X_3 \leq_0 \ldots \leq_0 X_{n-1}, \\ &X_1 \leq_1 X_2 \leq_1 X_0 \leq_1 X_3 \leq_1 \ldots \leq_1 X_{n-1}, \\ &X_2 \leq_2 X_0 \leq_2 X_1 \leq_2 X_3 \leq_2 \ldots \leq_2 X_{n-1}. \end{aligned}$$

Now each of $(G, \leq_0)$, $(G, \leq_1)$ and $(G, \leq_2)$ can be seen to contain a different pair of $(H, \leq^0)$, $(H, \leq^1)$ and $(H, \leq^2)$ as induced subgraphs. We claim that these are the *only* induced H-subgraphs of the given ordered G's.

In the construction of G no edges were added between hyperedges of E. Since hyperedges of E intersect in at most one point and H is 2-connected, no new copy of H is introduced by two hyperedges intersecting. (If one was newly formed, the point of intersection would be a cutpoint, contrary to being 2-connected.) The introduction of more hyperedges intersecting the first two might help to construct a new copy of H except that the condition 'girth$(E) > n$' prevents any such occurrence. So no other copies of H exist in G other than those produced explicitly in the construction. Recall now Theorem 5.5.1, and using the three orderings of G and the three orderings of H, we see $\chi(K_{H,G}) \geq 3$. Hence $\mathcal{R}[(G)_2^H] \neq \emptyset$. □

We denote the complement of an ordinary graph G by $\overline{G}$. Given a collection $\mathcal{G}$ of graphs, we define $\overline{\mathcal{G}} = \{\overline{G} : G \in \mathcal{G}\}$. Let $\mathcal{M}$ be the collection of all graphs G containing a cutpoint which is connected to all other vertices except at most one. It is not difficult to derive the following:

Lemma 5.7.3 *$H \in \mathcal{M} \cup \overline{\mathcal{M}}$ if and only if neither H nor $\overline{H}$ is 2-connected.*

This was proved in [118]. Using this terminology, we obtain an immediate corollary of Theorem 5.7.2.

Corollary 5.7.4 *Let H be a non-trivial ordinary graph. If $H \notin \mathcal{M} \cup \overline{\mathcal{M}}$ then there exists a graph G so that $mdo(H, G) \geq 2$ and $\mathcal{R}[(G)_2^H] \neq \emptyset$.*

Proof: If $H \notin \mathcal{M} \cup \overline{\mathcal{M}}$ then either H or $\overline{H}$ is 2-connected. If H is 2-connected, we are done by Theorem 5.7.2. Suppose H is not 2-connected but $\overline{H}$ is. By Lemma 5.7.1, $\overline{H}$ contains a copy of P_2. So we duplicate the construction in Theorem 5.7.2 to produce G and F with $F \longrightarrow (G)_2^{\overline{H}}$. We claim that $\overline{F} \longrightarrow (\overline{G})_2^H$. Fix a coloring $\Delta : \binom{\overline{F}}{H} \longrightarrow 2$. This induces a coloring $\overline{\Delta} : \binom{F}{\overline{H}} \longrightarrow 2$ by $\overline{\Delta}(\overline{H}) = \Delta(H)$. Since $F \longrightarrow (G)_2^{\overline{H}}$ there exists a $G' \in \binom{F}{G}$ so that $\binom{G'}{\overline{H}}$ is monochromatic with respect to $\overline{\Delta}$. Hence $\binom{\overline{G'}}{H}$ is monochromatic with respect to Δ.

So we only need show $mdo(H, \overline{G}) \geq 2$. Choose an ordering $(\overline{G}, \leq)$ of $\overline{G}$. If $(H, \leq^0)$, $(H, \leq^1) \in DO(H, \overline{G}, \leq)$ are non-isomorphic, then certainly we have that $(\overline{H}, \leq^0)$, $(\overline{H}, \leq^1) \in DO(\overline{H}, G, \leq)$ are non-isomorphic as well. $\square$

We conclude with some remarks. Many generalizations of these results to hypergraphs are possible. The notions of connectivity and complement must be extended however. One may also choose a subhypergraph which plays the role of P_2 in Theorem 5.7.2, but with cautionary heed to extra assumptions. Are there 'elegant' extensions of this type?

Hitherto, even a partial classification of ordinary graphs for which the statement "$mdo(H, G) = 1$ if and only if $\mathcal{R}[(G)_2^H] \neq \emptyset$" has not been given. However, the case for $H = P_2$ is settled in the next chapter.

Alternately, how much can the conditions on H be weakened so that this statement fails; Theorem 5.4.1 shows that H need not be 2-connected. Numerous other interesting questions should suggest themselves to the reader now. If one is to complete the classification of ordinary graphs with respect to Ramsey properties, it is believed that pursuits in directions similar to those taken here may be of assistance.

Chapter 6

Ramsey Graphs for Coloring P_2 's

6.1 Introduction

When one studies the pairs H, G for which $\mathcal{R}[(G)_r^H]$ is non-empty, it is natural to begin by classifying the smallest non-trivial cases. In this chapter, we are particularly interested in finding those graphs G for which $\mathcal{R}[(G)_r^{P_2}] \neq \emptyset$ (where P_2 is the path on three vertices). We use standard notation along with specialized notation given in previous chapters. Recall that $\mathrm{DO}(H, G, \leq)$ is the collection of different ordered induced H-subgraphs of $(G, \leq)$, and $\mathrm{mdo}(H, G)$ is the minimum (taken over all orderings of G) number of distinct orderings of H in any one ordered $(G, \leq)$. We also use the graph $K_{H,G}$ as defined in the previous chapter.

Recall that a *forest* is an ordinary graph containing no cycles, and a connected forest is a *tree*. For $x \in V(G)$, we use $N(x) = N_G(x) = \{y : (x, y) \in E(G)\}$ to denote the *neighborhood* of the vertex x, and let $N^*(x) = N_G^*(x) = N(x) \cup \{x\}$ be the extended neighborhood.

6.2 Colored Forests

Attempting to determine those graphs G for which $\mathcal{R}[(G)_2^{P_2}]$ is non-empty, we find that members of a very large class appear. This class was discovered [60] primarily as a result of examining the proof of Theorem 5.4.1.

Theorem 6.2.1 *If G is a forest, then $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$.*

Proof: If $\binom{G}{P_2} = \emptyset$ then the result is trivial. If $P_3 \not\preceq G$, then every connected component of G is a star. Clearly then $\mathrm{mdo}(P_2, G) = 1$ and hence $\chi(K_{P_2,G}) = \infty$ giving the result by Theorem 5.5.1. So assume $P_3 \preceq G$. We will produce three orderings of G, namely $(G, \leq^0)$, $(G, \leq^1)$ and $(G, \leq^2)$, so that each of $\mathrm{DO}(P_2, G, \leq^i)$, $i \in 3$, is a unique pair from $\mathrm{ORD}(P_2)$. Hence $\chi(K_{P_2,G}) \geq 3$, for in this case $K_{P_2,G}$ will contain a triangle.

Fix a representation of G as a collection of rooted trees with at least one of these roots being an inner vertex of some copy of $P_3 \preceq G$. Let

$$V(G) = L_1 \dot{\cup} L_2 \dot{\cup} \cdots \dot{\cup} L_n$$

be a partition of $V(G)$ into 'levels', that is, each L_j is the union of the j-th levels of all the rooted trees comprising G, where L_1 is the set of all the roots. Note that we have insisted that a copy of P_3 begins in L_2, goes 'down' to L_1, then back 'up' through L_2 to L_3. Impose an order $\leq^2$ on $V(G)$ which respects

$$L_1 \leq^2 L_2 \leq^2 L_3 \leq^2 \cdots \leq^2 L_n,$$

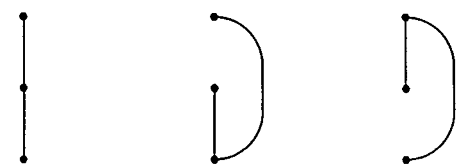
and let $\leq^1$ be the inverse order of $\leq^2$. Lastly, fix an order $\leq^0$ of $V(G)$ which 'folds' at levels, i.e.,

$$\cdots \leq^0 L_5 \leq^0 L_3 \leq^0 L_1 \leq^0 L_2 \leq^0 L_4 \leq^0 L_6 \leq^0 \cdots,$$

continuing until all levels are exhausted. Let $\mathrm{ORD}(P_2)$ be enumerated as in Figure 6.1.

Straightforward verification shows that

$$\mathrm{DO}(P_2, G, \leq^i) = \{(P_2, \leq_j) : j \neq i\}$$



$(P_2, \leq_0)$ $(P_2, \leq_1)$ $(P_2, \leq_2)$

Figure 6.1: ORD(P_2)

for each $i \in 3$ as required. □

Notice that we can not conclude from this proof that the resulting Ramsey graph is also a forest, even if it is minimal in some sense.

It is natural to ask whether or not any converse of Theorem 6.2.1 holds, i.e., if G is so that $\mathcal{R}[(G)_2^{P_2}]$ is non-empty, is G necessarily a forest? As it turns out, we can not conclude that G is a forest, however G may look like a forest with 'exploded vertices'.

6.3 Exploded Vertices

If $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$, G need not be a forest. If G is a triangle (a K_3), we trivially have $\mathcal{R}[(K_3)_2^{P_2}] \neq \emptyset$, just choose $F = G = K_3$. Furthermore, the orderings of the two graphs, G_1 and G_2, in Figure 6.2 show $\mathrm{mdo}(P_2, G_i) = 1$ for $i = 1, 2$ and hence each $\mathcal{R}[(G_i)_2^{P_2}]$ is non-empty. Note that G_1 consists of n copies of K_3 attached at a single vertex, while G_2 is n copies of K_3 all sharing a common edge. Alternatively, we could say G_1 was constructed by starting with a star S_n, replacing each end-vertex with a copy of K_2 (edge) and then joining vertices of each K_2 in the same manner the original vertex was. Similarly, G_2 could be conceived by replacing the central

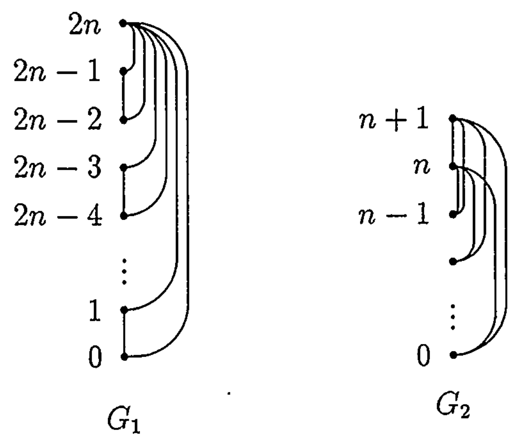


Figure 6.2: Vertices of a star exploded into edges.

vertex of S_n with an edge in a like manner. Observe that $\mathrm{mdo}(S_n, P_2) = 1$ and so $\mathcal{R}[(S_n)_2^{P_2}] \neq \emptyset$ also holds.

This method of replacing a vertex by a K_2 works in general. We first give a definition which generalizes that for a lexicographic product. Let G be a graph with a fixed enumeration $x_0, x_1, \ldots, x_{k-1}$ of $V(G)$. Let $M_0, M_1, \ldots, M_{k-1}$ be (vertex disjoint) graphs and define the product $G \otimes (M_0, M_1, \ldots, M_{k-1})$ on the vertex set $\dot{\cup}_{i \in k} V(M_i)$ by

$$E(G \otimes (M_0, \ldots, M_{k-1})) = \dot{\cup}_{i \in k} E(M_i) \cup \{(y_i, y_j) : y_i \in M_i, y_j \in M_j, (x_i, x_j) \in E(G)\}.$$

In this product, we replace each vertex by a graph (possibly empty) and connect each vertex of a 'replacement' graph to each vertex of another 'replacement' graph if and only if the replaced vertices were originally connected. If we let K_0 denote a null structure (a 'graph' with no vertices), and K_1 a single vertex, the graph

$$G \otimes (K_0, K_1, K_1, \ldots, K_1) = G \setminus \{x_0\}$$

is the graph formed by deletion of a vertex and all edges containing that vertex. Such a graph is often denoted by $G \setminus x_0$. The product $G \otimes M$ is just the standard lexicographic product (cp. Theorem 3.3.1). If, in using the definition, all the graphs M_i are complete (or null), we shall write $G \otimes (n_0, n_1, \ldots, n_{k-1})$ rather than $G \otimes (K_{n_0}, K_{n_1}, \ldots, K_{n_{k-1}})$. For example, $K_4 \otimes (0,1,1,1) = K_3$ and $K_3 \otimes (0,1,2) = K_3$.

In applying the definition of this product, we tacitly assume there is a fixed enumeration of $V(G)$; our arguments do not depend on which enumeration. We remark that if $G' = G \otimes (n_0, n_1, \ldots, n_{k-1})$, then

$$G \otimes (n_0 + 1, n_1, \ldots, n_{k-1}) = G' \otimes (2, 1, 1, 1, \ldots, 1)$$

for some appropriate enumeration of $V(G')$. Using this type of inductive step, it is not hard to prove the following lemma:

Lemma 6.3.1 *If for each $i \in k$,$n_i, m_i \in \omega$ and $n_i \leq m_i$, then*

$$G \otimes (n_0, n_1, \ldots, n_{k-1}) \preceq G \otimes (m_0, m_1, \ldots, m_{k-1})$$

In the last chapter we were assured of infinitely many pairs H, G so that both $\text{mdo}(H, G) > 1$ yet $\mathcal{R}[(G)_2^H] \neq \emptyset$ held. This was a difficult construction using probabilistic arguments. In Theorem 5.4.1 a small example of such a pair was also given with $H = P_2$. Using this example, the next theorem ([59]) actually gives us a construction for infinitely many G's so that $\text{mdo}(P_2, G) > 1$ while $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$.

Theorem 6.3.2 *Let G be a graph satisfying $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$. If $|V(G)| = k$, then for any collection n_0, n_1, ..., $n_{k-1} \in \omega$ of non-negative integers,*

$$\mathcal{R}[(G \otimes (n_0, n_1, \ldots, n_{k-1}))_2^{P_2}] \neq \emptyset$$

also holds.

Proof: We first show the result for the case when each $n_i > 0$. In this case we use induction on $\sum_{i \in k} n_i$, the size of the vertex set of the product graph. The base step $n_0 = n_1 = \ldots = n_{k-1} = 1$ is the assumption. Fix positive integers n_0, n_1, ..., n_{k-1}; set

$$G' = G \otimes (n_0, n_1, \ldots, n_{k-1})$$

and

$$G'' = G \otimes (n_0 + 1, n_1, \ldots, n_{k-1}),$$

and assume $\mathcal{R}[(G')_2^{P_2}] \neq \emptyset$. It will suffice to show that $\mathcal{R}[(G'')_2^{P_2}] \neq \emptyset$.

Fix an ordered $(G', \leq')$. Extend $\leq'$ to $\leq''$, an ordering of G'' by inserting the 'new' vertex arbitrarily in the order. It is easy to see that the construction of $(G'', \leq'')$ from $(G', \leq')$ introduces only copies of P_2 of order type identical to those already present. This fact (together with an application of Lemma 6.3.1) shows

$$DO(P_2, G', \leq') = DO(P_2, G'', \leq''),$$

regardless of which extension $\leq''$ is chosen, and hence $K_{P_2,G'}$ and $K_{P_2,G''}$ have the same hyperedges. Theorem 5.5.1 now yields that $\mathcal{R}[(G'')_2^{P_2}]$ is non-empty.

The case where some of the n_i's are zero follows from the fact that for any graph G, the vertex deleted graph $G \otimes (0, 1, 1, \ldots, 1)$ is an induced subgraph of G. If F is so that $F \longrightarrow (G)_2^{P_2}$, then so also $F \longrightarrow (G \otimes (0, 1, 1, \ldots, 1))_2^{P_2}$. $\square$

Following the proof of Theorem 6.3.2, a stronger theorem ([60]) is available which no longer restricts us to P_2-colorings.

Theorem 6.3.3 *Let G and H be graphs which satisfy $\mathcal{R}[(G)_2^H] \neq \emptyset$ and let $G \otimes (n_0, n_1, \ldots, n_{k-1})$ be defined. If for all pairs $(h, k) \in E(H)$ there exists $y \in V(H) \setminus$*

$\{h, k\}$ so that exactly one of $(y, h) \in E(H)$ or $(y, k) \in E(H)$ holds, then

$$\mathcal{R}[(G \otimes (n_0, n_1, \ldots, n_{k-1}))_2^H] \neq \emptyset$$

also holds.

Proof: The proof follows identically the proof of Theorem 6.3.2. We have only to check that in the construction of $G'' = G \otimes (n_0 + 1, n_1, \ldots, n_{k-1})$ from $G' = G \otimes (n_0, n_1, \ldots, n_{k-1})$ we have

$$\mathrm{DO}(H, G', \leq') = \mathrm{DO}(H, G'', \leq'')$$

for each order $\leq'$ on $V(G')$ and $\leq''$ respecting this order. ($\leq^*$ arises from inserting an element in the order $\leq$.)

Fix an ordering $\leq'$ of G' and let x and y be vertices of G'' which replace x_0 in G', keeping x in the same position as x_0 was and introducing y arbitrarily in the order $\leq'$ to determine $\leq''$. Any copy of H in $(G'', \leq'')$ which does not contain (x, y) as an edge is of the same order type as one found in $(G', \leq')$. The assumed condition on H is so as to prevent any new copies of H being formed which might contain (x, y) as an edge since the construction guarantees that x and y are of the same type with respect to the remainder of G''. Thus we have shown that

$$\mathrm{DO}(H, G', \leq') \supseteq \mathrm{DO}(H, G'', \leq''),$$

and the proof now duplicates the proof of Theorem 6.3.2. □

One may observe that if H is a connected triangle-free graph with $|V(H)| \geq 2$ the condition of Theorem 6.3.3 is met.

Although we have not yet classified all those graphs G for which $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$, a great deal many have been found so far.

Let $\mathcal{F}$ be the collection of graphs arising from forests whose vertices have been (possibly) 'exploded' into complete subgraphs. That is,

$$\mathcal{F} = \{F \otimes (n_0, \ldots, n_{k-1}) : F \text{ a forest}, k \geq 1, i \in k, n_i \in \omega\}.$$

As a direct consequence of Theorems 6.2.1 and 6.3.2, we have the following [60].

Theorem 6.3.4 *If* $G \in \mathcal{F}$ $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$.

But there are others; consider the graph on five vertices formed by adding two pendant edges to a triangle (V. Rödl—oral communication).

6.4 Chordal and Comparability Graphs

An ordinary graph is *chordal* if every cycle of length ≥ 4 has a chord, i.e., a chordal graph is that which contains no cycle on ≥ 4 vertices as an *induced* subgraph. (A chordal graph is sometimes called a *rigid circuit* graph (cp. [26]) or a *triangulated* graph (cp. [25],[75]). An easy lemma [60] is as follows.

Lemma 6.4.1 *If a graph G is so that $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$ then G is chordal.*

Proof: This proof mimics that of Example 5.2.1. If G is not chordal, then there exists an induced cycle of length ≥ 4 in G. Then any ordering of G produces two distinct ordered P_2's as induced subgraphs, namely $(P_2, \leq_1)$ and $(P_2, \leq_2)$ (the ones which have the middle vertex at either end of the order).

Fix any graph F and impose an order $\leq$ on $V(F)$. Let $\Delta : \binom{F}{P_2} \longrightarrow 2$ be a coloring which satisfies

$$\Delta(P_2') = 0 \text{ if } (P_2', \leq) \cong (P_2, \leq_1),$$

and

$$\Delta(P_2') = 1 \ \text{ if } \ (P_2', \leq) \cong (P_2, \leq_2),$$

where $(P_2', \leq)$ is a copy of $P_2 \preceq F$ with the order $\leq$ imposed. Thus every $G' \in \binom{F}{G}$ is multicolored and so $F \notin \mathcal{R}[(G)_2^{P_2}]$. Hence we have shown that if G is not chordal, then $\mathcal{R}[(G)_2^{P_2}]$ is empty. □

A vertex x in an ordinary graph G is *simplicial* if its neighbors, $N_G(x)$, induce a complete subgraph of G. We will need the following result of Dirac [26] (Thm. 4; also see [48]) concerning simplicial vertices, (which we give without proof).

Theorem 6.4.2 *Every chordal graph contains a simplicial vertex, and upon removal, produces another chordal graph.*

Given a partially ordered set $(Q, \leq)$, construct the graph $G(Q)$ on vertex set Q, where $(x, y) \in E(G)$ if and only if $x < y$ or $y < x$. Such a graph $G(Q)$ is called the *comparability graph for* $(Q, \leq)$. One can think of $G(Q)$ as the undirected version of the transitive closure of the directed graph associated with $(Q, \leq)$.

Given a partial order $(Q, \leq)$, $(Q, \leq^*)$ is a *linear extension* of $(Q, \leq)$ if $\leq^*$ is a linear (total) order and $a \leq b$ implies $a \leq^* b$. Such a linear extension always exists; one can construct it by first choosing a minimal element of $(Q, \leq)$ to be the smallest element in $(Q, \leq^*)$, then choosing the minimal element of the remaining, and so on.

An interesting characterization of comparibility graphs is the following, which appears to be part of the folklore.

Lemma 6.4.3 *G is a comparability graph if and only if G has an ordering $\leq^0$ so that $(P_2, \leq_0) \npreceq (G, \leq^0)$.*

Proof: Let $G = G(Q)$ be a comparability graph for some poset $(Q, \leq)$. A linear extension $(Q, \leq^*)$ of $(Q, \leq)$ gives rise to the ordered graph $(G, \leq^*)$ in the following manner: for $x \leq^* y$, $(x, y) \in E(G, \leq^*)$ if and only if $x \leq y$. If (x, y) and (y, z) determine a weak $(P_2, \leq_0)$-subgraph of $(G, \leq^*)$, then transitivity of $\leq$ gives (y, z) to be an edge also, preventing an induced copy of $(P_2, \leq_0)$.

Now suppose that G has an ordering $\leq^0$ so that $(P_2, \leq_0) \not\leq (G, \leq^0)$. Look at the relational structure $(Q, \leq)$ defined by $x \leq y$ if and only if $(x, y) \in E(G)$ and $x \leq^0 y$. If (x, y) and (y, z) are (ordered) edges of $(G, \leq^0)$, (x, z) is also, since $(G, \leq^0)$ does not contain a copy of $(P_2, \leq_0)$. Thus $x \leq z$ and the transitivity condition is satisfied for $(Q, \leq)$ to be a partial order and $G = G(Q)$ is a comparability graph.

6.5 Complete classification

We are now ready to give the complete classification [60] of those graphs G for which $\mathcal{R}[(G)_2^{P_2}]$ is non-empty.

Theorem 6.5.1 $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$ *if and only if either G is both chordal and a comparability graph, or there is an ordering $\leq^*$ of G so that $DO(P_2, G, \leq^*) = \{(P_2, \leq_0)\}$.*

Proof: First assume that G is chordal and is a comparability graph. We define three orderings of G as follows.

By Theorem 6.4.2 there exists a simplicial vertex $s_0 \in V(G)$. By the same theorem, there is $s_1 \in V(G) \setminus \{s_0\}$, again simplicial. Continue, exhausting $V(G)$ and let $\leq^1$ be an ordering of $V(G)$ given by $s_0 \leq^1 s_1 \leq^1 \ldots \leq^1 s_{|V(G)|-1}$. Observe that $(P_2, \leq_1) \not\leq (G, \leq^1)$, because each upper (right) neighborhood of each vertex is complete. Similarly define $(G, \leq^2)$ where $\leq^2 = (\leq^1)^{-1}$. Then $(P_2, \leq_2) \not\leq (G, \leq^2)$. Now let $(G, \leq^0)$ be an ordered graph guaranteed by Lemma 6.4.3 so that $(P_2, \leq_0) \not\leq (G, \leq^0)$. So by Theorem 5.5.1, $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$. (cp. comment after 5.4.1).

If $\leq^*$ is an ordering of $V(G)$ so that $\mathrm{DO}(P_2, G, \leq^*) = \{(P_2, \leq_0)\}$, then by Theorem 5.5.1 or 5.3.1 we have $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$ as well.

Now suppose that $\mathcal{R}[(G)_2^{P_2}] \neq \emptyset$. Then by Lemma 6.4.1, G must chordal. It remains to show that G must be a comparability graph. Use of Theorem 5.5.1 together with the two orderings given by chordality in the first part of the proof, we see that either G must have an ordering which 'omits' $(P_2, \leq^0)$ (in which case, by Lemma 6.4.3 we are done) or there exists an ordering $\leq^*$ of $V(G)$ so that $\mathrm{DO}(P_2, G, \leq^*) = \{(P_2, \leq_0)\}$. □

For a good survey on comparability graphs, see [70], and [97] for many other references. In [75] there are examples of chordal graphs and comparability graphs which are not both. An excellent bibliography is given in [25] concerning chordal graphs and related topics. It would be very interesting to classify those graphs which are both. For this, methods of Diestel and Galvin might be useful combined with those of Kelly and Möhring.

It is not known whether or not those graphs G which have an ordering admitting only 'flat' copies of P_2 are also comparability graphs; work with small examples seems to indicate that these two classes of graphs are in fact the same, but no proof has been discovered. We note that the graphs having an ordering admitting only flat P_2's are chordal.

Work on the case for perhaps the next simplest case, the star S_3, has been difficult, but it appears that such a classification is possible, although not nearly as elegant as Theorem 6.5.1.

Chapter 7

Minimal Ramsey Graphs

7.1 Development

We have already seen that the notion of ordering plays a significant role in graph Ramsey theory. In an attempt to actually produce some examples of ordered Ramsey graphs, it may seem natural to restrict the search to small, or minimal, such graphs. Given the importance of ordered graphs in graph Ramsey theory, it seems surprising that there are few results concerning the minimality of ordered Ramsey structures. A great deal of work has been done to find *unordered* minimal Ramsey graphs and a number of results are known. However, even for the unordered case, much waits to be discovered.

In this chapter we restrict ourselves to edge colorings of ordinary graphs (ordered and unordered) with two colors. Some known results are surveyed for the unordered case and we examine possible directions for the related study of ordered graphs. Some trivial cases are looked at, and one non-trivial minimal ordered Ramsey graph is given for the first time. Proofs will be omitted, or at most outlined. This is not meant to be a complete study, only a brief look at the problem.

The notation

$$F \longrightarrow (G_0, G_1, \ldots, G_{n-1})$$

means that for any edge coloring of F with 2 colors, there exists $i \in n$ so that there is a monochromatic *induced* copy of G_i in F. (This is not to be confused with the similar notation for more than two colors.) The associated notation

$$F \longrightarrow_{weak} (G_0, G_1, \ldots, G_{n-1})$$

means there is a weak subgraph G_i of F which is monochromatic. These are known respectively as the strong and weak Ramsey arrows. We say F is a *strong Ramsey graph for* $(G_0, G_1, \ldots, G_{n-1})$ if

$$F \longrightarrow (G_0, G_1, \ldots, G_{n-1}),$$

and we say F is a *weak Ramsey graph* for $(G_0, G_1, \ldots, G_{n-1})$ if the arrow is weak. We extend the notation for ordered graphs in the natural way. When it causes no confusion, the mention of the graphs $(G_0, G_1, \ldots, G_{n-1})$ will be suppressed and we shall only say that F is a Ramsey graph—specifying "weak" or "strong" only as needed. (Note that if F is a Ramsey graph for a complete graph, then it is both a weak and a strong Ramsey graph.)

If F is a Ramsey graph so that for any other Ramsey graph F', $|V(F)| \leq |V(F')|$ holds, then we say F is a *vertex-minimal* Ramsey graph. Given a graph F and a vertex $x \in V(F)$, define the vertex deleted subgraph

$$F \setminus x = (V(F) \setminus \{x\}, E(F) \cap [V(F) \setminus \{x\}]^2).$$

A Ramsey graph F is *vertex-critical* if every vertex deleted subgraph fails to be Ramsey. Clearly every vertex-minimal Ramsey graph is also vertex-critical, but a vertex-critical Ramsey graph need not be vertex-minimal.

Similarly, a Ramsey graph F is *edge-minimal* if for any other Ramsey graph F', $|E(F)| \leq |E(F')|$ holds. For an edge $e \in E(F)$ define

$$F \setminus e = (V(F), E(F) \setminus \{e\}).$$

A Ramsey graph F is *edge-critical* if $F \setminus e$ fails to be Ramsey for any edge e of F. As before, edge-minimal Ramsey graphs are edge-critical, but the converse does not necessarily hold.

If G has no isolated vertices, and F is an edge-critical Ramsey graph for G, then F is also vertex-critical. If F is a weak Ramsey graph for $(G_0, \ldots, G_{n-1})$, then some authors ([12], [13], [15]) say F is "$(G_0, \ldots, G_{n-1})$-minimal" if F is edge-critical, and in some cases ([12]) "edge-minimal" is used to denote edge-critical. Also, in [85], F is said to be a "critical" Ramsey graph for G if F is a strong Ramsey graph for G which is edge-critical. If a strong Ramsey graph F is edge-critical, then for any weak subgraph F' of F, F' fails to be Ramsey. It is worth noting that the existence of a Ramsey graph (weak or strong) for $(G_0, \ldots, G_{n-1})$ is guaranteed by Theorem 3.4.4.

A great deal of the work done in the area of minimal Ramsey graphs has been to show whether or not there are infinitely many such (e.g. [29], [12], [13], [14], [15], [93]). There has also been a great interest in finding Ramsey numbers, that is, the sizes of vertex-minimal weak Ramsey graphs (e.g. [9], [10], [11], [20], [29], [30], [31], [32], [44], [45], [49], [51], [68], [74], [122]). Many (e.g. [96]) have contributed in closely related areas.

7.2 Some Unordered Results

Some questions concerning minimal Ramsey graphs were raised as early as 1975 [29]. Soon after, Burr, Erdős, and Lovász (see reference in [85]) proved the following theorem.

Theorem 7.2.1 *For each $n \in \omega$ there exist infinitely many edge-critical Ramsey graphs for K_n.*

Theorem 7.2.1 also appears as a corollary to a theorem proved in [85]. Also in [85], Nešetřil and Rödl proved the following three theorems.

Theorem 7.2.2 *If G is a forest with $P_3 \preceq G$, then there are infinitely many edge-critical strong Ramsey graphs for G.*

Theorem 7.2.3 *If G is a graph with $\chi(G) \geq 3$, then G has infinitely many edge-critical strong Ramsey graphs.*

For the next theorem, we need a definition. Recall that a graph is 2-connected if the graph is connected and the removal of any vertex does not disconnect the graph. A graph is *2.5-connected* if the removal of any two adjacent vertices does not disconnect the graph.

Theorem 7.2.4 *If G is a 2.5-connected graph, then G has infinitely many edge-critical strong Ramsey graphs.*

It was also conjectured by Nešetřil and Rödl that if G and H were graphs with $|E(G)| \geq 2$ and $G \notin \mathcal{R}[(H)_2^{K_2}]$, then there are infinitely many vertex-critical Ramsey graphs for H which contain G as a subgraph. This conjecture was partially settled by the following theorem [15].

Theorem 7.2.5 *Let G and H be 2.5-connected graphs and suppose F is not a Ramsey graph for (G, H). Then there are infinitely many edge-critical strong Ramsey graphs for (G, H) which contain F as an induced subgraph.*

We now take another look at minimal Ramsey graphs. Define a *minimal-Ramsey* graph to be a vertex-minimal Ramsey graph with fewest edges. It is not clear that a minimal-Ramsey graph is edge minimal, or even edge critical! It is precisely these questions which promoted further research.

Notice that $S_3 \longrightarrow (P_2)_2^{K_2}$, and S_3 is minimal-Ramsey. (Recall S_3 is a star with one central vertex and 3 end vertices.) Suppose we have a copy of C_5 and we attach to each vertex of C_5 a pendant edge to form a graph on 10 vertices. Call the resulting graph $*F$ (looks like a starfish?).

Lemma 7.2.6 $*F \longrightarrow (P_3)_2^{K_2}$

Proof: Let the $V(*F) = \{x_0, \ldots, x_4, y_0, \ldots, y_4\}$ where x_0 thru x_4 forms a cycle and for each $i \in 5$, $\{x_i, y_i\}$ is an edge. Fix a coloring $\Delta : (E(G)) \longrightarrow 2$. Without loss, we can assume that $\Delta(\{x_4, x_0\}) = \Delta(\{x_0, x_1\}) = 1$, say. If any of the four edges $\{x_1, y_1\}$, $\{x_1, x_2\}$, $\{x_4, y_4\}$, $\{x_4, x_3\}$ are also colored 1, then we are done. So assume not. Then any coloring of the three remaining edges $\{x_3, y_3\}$, $\{x_2, x_3\}$, $\{x_2, y_2\}$ yields a monochromatic copy of P_3. $\square$

It is easy to see that $*F$ is an edge-critical and vertex-critical strong Ramsey graph for P_3. The author knows of no proof verifying that $*F$ is vertex-minimal. (In fact, the author believes Lemma 7.2.6 is original.)

7.3 Minimal Ordered Ramsey Graphs

In Figure 7.1, some minimal Ramsey ordered graphs are given. For the most part, these are all trivial examples.

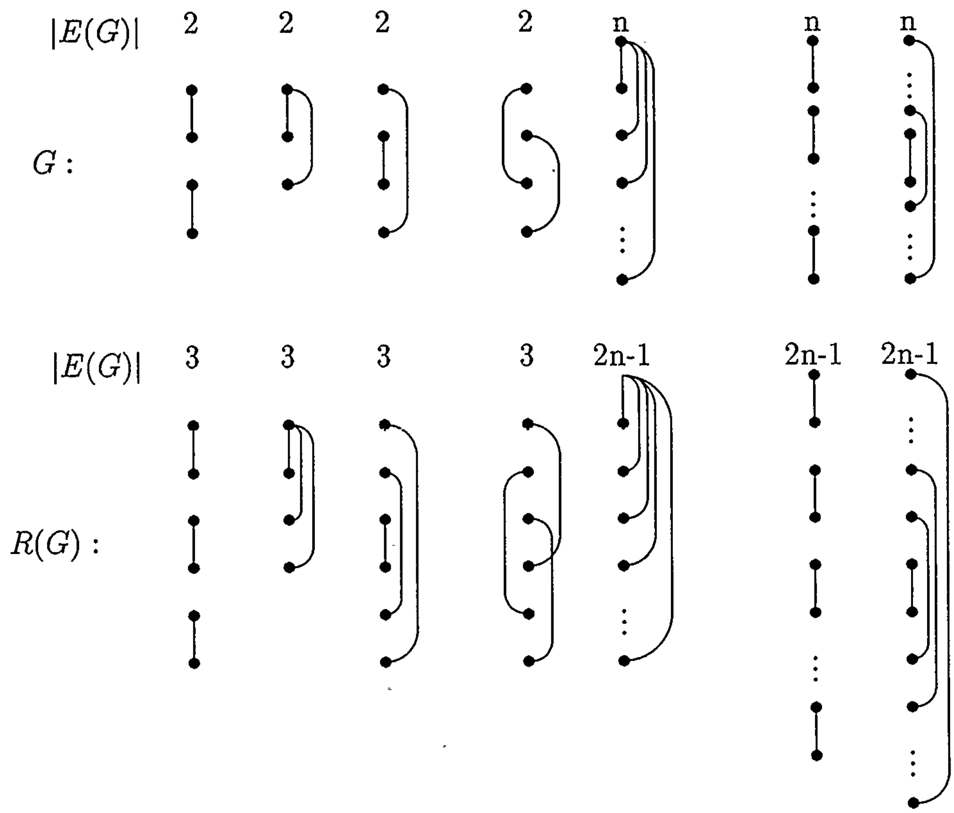


Figure 7.1: Some Minimal Ordered Ramsey Graphs

Figure 7.2 shows the first non-trivial ordered Ramsey graph, that for the linearly order P_2. It is easy to see that the graph is both vertex *and* edge-critical. It can be shown that the graph in Figure 7.2 on 7 vertices is both vertex-minimal and edge-minimal. (Contrast this to the unordered case, where S_3 suffices.) The proof is by (many) cases and we omit it. A computer program was written to verify the claims of minimality, which took 39 days to complete on a Commodore 64. An ordered Ramsey graph for a linearly ordered P_3 appears to have over 20 vertices—a computer search does not seem feasible. (Compare this with ≤ 10 vertices given in the unordered case by Lemma 7.2.6. Due to the work on ordered graphs, the author arrived at some interesting questions.

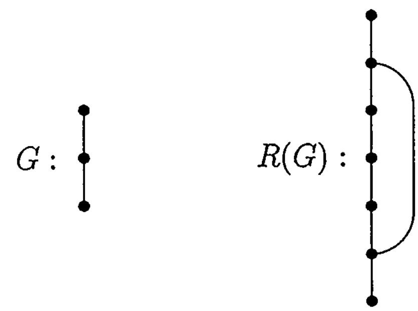


Figure 7.2: A non-trivial example

7.4 A Conjecture

Suppose that G and G' are graphs on the same number of vertices, and that F is a strong Ramsey graph for G. Furthermore, suppose G is a weak subgraph of G'. If one was to prove that F is in fact Ramsey for G, one might try to color F edge by edge, at all times trying to avoid a monochromatic copy of G, only to be forced to create one. While attempting this coloring, any copies of G contained in a copy of G' need not be worrisome, as we are looking for monochromatic *induced* copies of G. Hence, if F were to be minimal, one would expect to find as few copies of G' in F as possible. This same reasoning holds for ordered graphs, as well.

Conjecture: Suppose $F \longrightarrow (G)_2^{K_2}$ and G is a proper subgraph of some G' satisfying $|V(G)| = |V(G')|$. If F is vertex-minimal or edge-minimal, then $G' \not\leq F$.

The similar conjecture for ordered graphs seems to offer more hope of proof. In ordered graphs, the notion of 'forcing' a bad coloring is easier.

Bibliography

[1] F. G. Abramson and L.A. Harrington, Models without discernibles, *J. Symbolic Logic* **43** (1978), 572–600.

[2] W. Ackermann, Zum Hilbertschen Aufbau der reelen Zahlen, *Math. Ann.* **99** (1928), 118–133.

[3] C. Berge, *Graphs and Hypergraphs*, North Holland, Amsterdam, 1973.

[4] C. Berge, *Graphs*, 2nd rev. ed., North Holland, Amsterdam, 1985.

[5] J. C. Bermond, Some Ramsey numbers for directed graphs, *Discrete Math.* **9** (1974), 313–321.

[6] T. Bisztriczky and G. Fejes Tóth, A generalization of the Erdős-Szekers convex n-gon theorem, *J. reine angew. Math.* **395** (1989), 167–170.

[7] B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978.

[8] N. G. de Bruijn and P. Erdős, A colour problem for infinite graphs and a problem in the theory of relations, *Indag. Math.* **13** (1951), 371–373.

[9] S. A. Burr, Generalized Ramsey theory for graphs—a survey, in *Graphs and Combinatorics*, R. A. Bari and F. Harary eds., Lecture Notes in Math. **406** (1974) Springer Verlag, 52–75.

[10] S. A. Burr, A survey of noncomplete Ramsey theory for graphs, in *Topics in Graph Theory*, New York Acad. of Sci., (1979), 58–75.

[11] S. A. Burr and J. A. Roberts, On Ramsey Numbers for stars, *Utilitas Mathematica* **4** (1973), 217–220.

[12] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey-minimal graphs for matchings, in *The theory and applications of graphs* (4-th int'l conference, May 6-9, 1980, Western Michigan Univ., Kalamazoo, Mich.), G. Chartrand et al eds., John Wiley and Sons, New York, 1981.

[13] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey-minimal graphs for star-forests. *Discrete Math.* **33** (1981), 227–237.

[14] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey-minimal graphs for forests. *Discrete Math.* **38** (1982), 23–32.

[15] S. A. Burr, J. Nešetřil and V. Rödl, On the use of senders in generalized Ramsey theory for graphs, *Discrete Math.* **54** (1985), 1–13.

[16] G. Cherlin, *Homogeneous directed graphs I: The imprimitive case*, manuscript, (c. 1986).

[17] G. Cherlin, *Homogeneous directed graphs II: Primitive graphs omitting I_∞*, manuscript, (c. 1986).

[18] G. Cherlin, *Homogeneous directed graphs III: Primitive graphs embedding* I_∞, manuscript, (c. 1986).

[19] V. Chvátal, Some unknown van der Waerden numbers, in *Combinatorial structures and their applications*, R. K. Guy et al eds., Gordon and Breach, New York (1970), 31–33.

[20] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III: Small off-diagonal numbers, *Pacific J. of Math.* **41** 2 (1972), 336–345.

[21] B. Descartes, Solution to advanced problem No. 4526, *Amer. Math. Monthly* **61** (1954), 352.

[22] W. Deuber, Generalizations of Ramsey's theorem, in *Infinite and finite sets*, A. Hajnal et al eds., *Coll. Math. Soc. J. Bolyai* **10**, North Holland (1975), 323–332.

[23] W. Deuber, Partitionstheoreme für Graphen, *Comment. Math. Helvetici* **50** (1975), 311–320.

[24] W. Deuber, Developments based on Rado's dissertation "Studien zur Kombinatorik", in *Surveys in Combinatorics, 1989*, J. Siemons ed., (Proc. of 12'th British Comb. Conf.), London Math. Soc. Lecture Notes **141**, Cambridge Univ. Press (1989), 52–74.

[25] R. Diestel, Simplicial decompositions of graphs: A survey of applications, *Discrete Math.* **75** (1989), 121–144.

[26] G. A. Dirac, On rigid circuit graphs. *Abhandlungen aus dem Mathenatischen Seminar der Universität Hamburg* bf 25 (1961), 71–76.

[27] M. El-Zahar and N. W. Sauer, *On the divisibility of homogeneous directed graphs*, Univ. of Calgary Yellow Series, Research Paper No. 642, April 1987.

[28] P. Erdős, Graph theory and probability, *Canad. J. Math.* **11** (1959), 34–38.

[29] P. Erdős, Problems and results in finite and infinite graphs, in *Recent advances in graph theory* (Proceedings of the symposium held in Prague, June 1975), Academia Praha (1975), 183–192.

[30] P. Erdős, Problems and results in Graph Theory, *The theory and applications of graphs* (4-th int'l conference, May 6-9,1980,Western Michigan Univ., Kalamazoo, Mich.), G. Chartrand et al eds., John Wiley and Sons, New York, 1981.

[31] P. Erdős, Problems and results on chromatic numbers in finite and infinite graphs, *Graph theory with applications to algorithm and computer science* (Kalamazoo, Mich., 1984), Wiley Intersci Publ., Wiley, New York, 201–213.

[32] P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Generalized Ramsey theory for multiple colors, *J. Combin. Theory, Ser. B* **20** (1976), 250–264.

[33] P. Erdős and A. Hajnal, On chromatic number of graphs and set systems, *Acta. Math. Acad. Sci. Hungar.* **17** (1966), 61–99.

[34] P. Erdős and A. Hajnal, Problems and results in finite and infinite combinatorial analysis, *Ann. of N.Y. Acad. of Sci.* **175** (1970), 115–124.

[35] P. Erdős, A. Hajnal, A. Máté, and R. Rado, *Combinatorial set theory: partitions relations for cardinals*, North Holland, Amsterdam, 1984.

[36] P. Erdős, A. Hajnal and E. C. Milner, Set mappings and polarized partition relations, in *Combinatorial theory and its applications,* North Holland (1970), 327–363.

[37] P. Erdős, A. Hajnal and L. Pósa, Strong embeddings of graphs into colored graphs, in *Infinite and finite sets*, A. Hajnal et al eds. *Coll. Math. Soc. J. Bolyai* **10**, North Holland (1975), 585–595.

[38] P. Erdős, A. Hajnal and R. Rado, Partition relations for cardinals, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 61–99.

[39] P. Erdős and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **25** (1950), 249–255.

[40] P. Erdős and R. Rado, Combinatorial theorem on classifications of subsets of a given set, *Proc. London Math. Soc.* **3** (1952), 417–439.

[41] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compos-ito Math.* **2** (1935), 463–470.

[42] P. Erdős and J. Spencer, *Probabilistic methods in combinatorics*, Academic Press New York, 1974.

[43] P. Erdős and J. Spencer, Monochromatic sumsets, *J. Combin. Theory Ser. A* **50** (1989), 162–163.

[44] M. Erikson, F. Harary, Generalized Ramsey theory XV: Achievement and avoidance games for bipartite graphs, in *Graph Theory, Singapore 1983*, Lecture Notes in Mathematics **1073** (1984), Springer Verlag, 212–216.

[45] R. J. Faudree, C. C. Rousseau and R. H. Schelp, Generalizations of a Ramsey result of Chvátal, in *The theory and applications of graphs* (4-th int'l conference, May 6-9,1980,Western Michigan Univ., Kalamazoo, Mich.), G. Chartrand et al eds., John Wiley and Sons, New York, 1981.

[46] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, SIAM *J. Appl. Math.* **18** (1970), 19–29.

[47] P. Frankl, R. L. Graham, and V. Rödl, Induced Restricted Ramsey theorems for spaces, *J. Combin. Theory Ser. A* **44** (1987), 120–128.

[48] D. R. Fulkerson and O. A. Gross, Incidence matrices and interval graphs, *Pacific J. Math.* **15** (1965), 835–855.

[49] M. Gardner, Mathematical Games in which joining sets of points by lines lead into diverse (and diverting) paths, *Scientific American* **237** (Nov. 1977), 18–28.

[50] R. L. Graham, K. Leeb and B. L. Rothschild, Ramsey's theorem for a class of categories, *Adv. Math.* **8** (1972), 417–433.

[51] R. L. Graham and V. Rödl, Numbers in Ramsey theory, in *Surveys in combinatorics 1987*, (Invited papers for the 11'th British Combinatorial Conf.), C. Whitehead ed., London Math Society Lecture Notes Series **123**, Cambridge Univ. Press, Cambridge (1987), 111–153.

[52] R. L. Graham and B. L. Rothschild, Ramsey's theorem for n-parameter sets, *Trans. Amer. Math. Soc.* **159** (1971), 257–292.

[53] R. L. Graham and B. L. Rothschild, Rota's geometric analog to Ramsey's theorem, *Proc. Sympos. Pure Math.* **19** (1971), 101–104.

[54] R. L. Graham and B. L. Rothschild, Some recent developments in Ramsey theory, in *Combinatorics, Part 2*, M. Hall Jr. and J. H. Lint, eds., Mathematical Centre Tracts **56**, (1974), 61–76.

[55] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey theory*, Wiley – Interscience Ser. in Discrete Math., New York, 1980.

[56] R. L. Graham and J. H. Spencer, A general product theorem, *Proc. Amer. Math. Soc.* **73** (1979), 137–139.

[57] R. L. Graham and J. H. Spencer, Ramsey Theory, *Scientific American* **263**, (Jul. 1990), 112–117.

[58] R. N. Greenwell and B. M. Landman, On the existence of a reasonable upper bound for the van der Waerden numbers, *J. Combin. Theory Ser. A* **40** (1989), 82–86.

[59] D. S. Gunderson, V. Rödl, and N. W. Sauer, Some results in finite graph Ramsey theory, (21'st S.E. Int,l CGTC, Boca Raton, FL, Feb.12–16, 1990), *Congressus Numerantium* (1990) **79**, 232–243.

[60] D. S. Gunderson, V. Rödl, and N. W. Sauer, *Finite Graph Ramsey Theory: Partitions of subgraphs*, Manuscript, 1990.

[61] A. Hajnal and P. Komjáth, Embedding graphs into colored graphs, *Trans. Amer. Math. Soc.* **307** (1) (1988), 395–409.

[62] A. W. Hales and R. I. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 22–229.

[63] F. Harary, *Graph theory*, Addison-Wesley, Reading, Mass., 1969.

[64] F. Harary, A survey of generalized Ramsey theory, in *Graphs and Combinatorics* (George Washington Univ. 1973), Lecture Notes in Mathematics **406** (1974) Springer Verlag.

[65] F. Harary, Generalized Ramsey I to XII: Achievement and avoidance numbers, in *The theory and applications of graphs* (4-th int'l conference, May 6–9,1980, Western Michigan Univ., Kalamazoo, Mich.), G. Chartrand et al eds., John Wiley and Sons, New York, 1981.

[66] C. W. Henson, Edge partition properties of graphs, *Canad. J. Math.* **25** (1973), 603–610.

[67] , Erratum, *J. Combin. Theory Ser. A*, **50** (1989), 323.

[68] Michael S. Jacobson, On the Ramsey numbers for stars and a complete graph. *ARS Combinatoria* **17** (1984), 167–172.

[69] S. Johnson, A new proof of the Erdős Szekeres convex k-gon result, J. Combin. Theory Ser. A **42** (1986), 318–319.

[70] D. Kelly, Comparability Graphs, in *Graphs and Order*, I. Rival ed., D. Reidel Publishing Company, Boston (1985),3–40.

[71] P. Komjath, Ramsey-type results for metric spaces, *J. Combin. Theory Ser. A*, **45** (1987), 323–324.

[72] D. König, Über eine Schußweise aus dem Endlichen ins Unendliche, *Acta Litt. Acad. Sci. Hungar.* (Szeged) **3** (1927), 121–130.

[73] L. Lovász, On chromatic number of finite set systems, *Acta. Math. Acad. Sci. Hung.* **19** (1968), 59–67.

[74] R. Mathon, Lower bounds for Ramsey numbers and association schemes, *J. Combin. Theory Ser. B*, **42** (1987), 122–127.

[75] R. H. Möhring, Algorithmic aspects of comparability graphs, in *Graphs and Order*, I. Rival ed., D. Reidel Publishing Company, Boston (1985), 41–101.

[76] J. Nešetřil, H. J. Prömel, V. Rödl, and B. Voigt, Canonizing ordering theorems for Hales-Jewett structures, *J. Combin. Theory Ser. A* **40** (1985), 394–408.

[77] J. Nešetřil and V. Rödl, A Ramsey graph without triangles exists for any graph without triangles, in *Infinite and finite sets* A. Hajnal et. al eds., *Coll. Math. Soc. J. Bolyai* **10**, North Holland (1975), 1127–1132.

[78] J. Nešetřil and V. Rödl, Partitions of subgraphs, in *Recent Advances in Graph Theory*, M. Fiedler ed., Academica Praha, Prague (1975), 405–412.

[79] J. Nešetřil and V. Rödl, Type theory of partition properties of graphs, in *Recent Advances in Graph Theory* (M. Fiedler ed.), Academica Praha, Prague (1975), 413–423.

[80] J. Nešetřil and V. Rödl, The Ramsey property for graphs with forbidden complete subgraphs, *J. Combin. Theory Ser. B* **20** (1976), 243–249.

[81] J. Nešetřil and V. Rödl, Partitions of vertices, *Comment. Math. Univ. Carolin.* **17** (1976), 85-95.

[82] J. Nešetřil and V. Rödl, Partitions of finite relational and set systems *J. Combin. Theory Ser. A* **22** (1977), 289–312.

[83] J. Nešetřil and V. Rödl, A simple proof of the Galvin-Ramsey property of the class of all finite graphs and a dimension of a graph, *Discrete Math.* **23** (1978), 49–55.

[84] J. Nešetřil and V. Rödl, Partition (Ramsey) theory—a survey, in *Combinatorics*, Coll. Math. Soc. J. Bolyai **18** (Keszthely,Hungary, 1976), North Holland (1978), 759–792.

[85] J. Nešetřil and V. Rödl, The structure of critical Ramsey graphs, *Acta Math. Acta. Hungar.* **32** (1978), 295–300.

[86] J. Nešetřil and V. Rödl, Partition theory and its application, in *Surveys in Combinatorics*, B. Bollobás ed., London Math. Soc. Lecture Note Series **38**, Cambridge Univ. Press, London (1979), 97–156.

[87] J. Nešetřil and V. Rödl, A short proof of the existence of highly chromatic hypergraphs without short cycles, *J. Combin. Theory Ser. B* **27** (1979), 225–227.

[88] J. Nešetřil and V. Rödl, Simple proof of the existence of restricted Ramsey graphs by means of a partite construction, *Combinatorica* **1** (2)(1981), 199–202.

[89] J. Nešetřil and V. Rödl, Two proofs of the partition property of set systems, *Europ. J. Combinat.* **3** (1982), 347–352.

[90] J. Nešetřil and V. Rödl, Ramsey classes of set systems, *J. Combin. Theory Ser. A*.**34** (1983), 183–201.

[91] J. Nešetřil and V. Rödl, Sparse Ramsey graphs, *Cominatorica* **4** (1) (1984), 71–78.

[92] J. Nešetřil and V. Rödl, Combinatorial partitions of finite posets and lattices—Ramsey lattices, *Alg. Universalis* **19** (1984), 106–119.

[93] J. Nešetřil and V. Rödl, Two remarks on Ramsey's theorem, *Discrete Math.* **54** (1985), 339–341.

[94] J. Nešetřil and V. Rödl, Strong Ramsey theorems for Steiner systems, *Trans. Amer. Math. Soc.* **303**(1) (1987), 183–192.

[95] J. Nešetřil and V. Rödl, The partite construction and Ramsey set systems, *Discrete Math.* **75** (1989), 327–334.

[96] A. D. Polemeni, J. Yellen, and H. J. Straight, Arrowing results for trees versus complete graphs,in *The theory and applications of graphs* (4-th int'l conf., May 6-9,1980, West. Michigan Univ., Kalamazoo, Mich.), G. Chartrand et.al. eds., John Wiley and Sons, 1981, 493–506.

[97] M. Pouzet and I. Rival, Is there a diagram invariant? *Discrete Math.* **73** (1988/89), 181–188.

[98] H. J. Prömel, Induced partition properties of combinatorial cubes, *J. Combin. Theory Ser. A* **39** (1985), 177–208.

[99] H. J. Prömel, Partition properties of q-hypergraphs, *J. Combin. Theory Ser. B* **41** (3) (1986), 356–385.

[100] H. J. Prömel, *Ramsey theory for discrete structures*, (Manuscript, 1987), 354 pages.

[101] H. J. Prömel and V. Rödl, An elementary proof of the canonizing version of Gallai-Witt's theorem, *J. Combin. Theory Ser. A* **42** (1) (1986), 144–149.

[102] H. J. Prömel and B. L. Rothschild, A canonical restricted version of van der Waerden's theorem, *Combinatorica* **7** (1) (1987).

[103] H. J. Prömel and B. Voigt, Partition theorems for parameter systems and graphs, *Discrete Math.* **36** (1981), 83–96.

[104] H. J. Prömel and B. Voigt, Canonizing Ramsey theorems for finite graphs and hypergraphs, *Discrete Math.* **54** (1985), 49–59.

[105] H. J. Prömel and B. Voigt, *Graham-Rothschild parameter sets*, Report No. 85403-OR, Forschungsinstitut für Diskrete Mathematik, Institut für Ökonometrie und operations research, Universität Bonn, 40 pages.

[106] H. J. Prömel and B. Voigt, Baire sets of k-parameter words are Ramsey, *Trans. Amer. Math. Soc.* **291** (1985), 189–209.

[107] H. J. Prömel and B. Voigt, *Some remarks on Natural orders for combinatorial cubes*, Report No. 85398-OR, Forschungsinstitut für Diskrete Mathematik, Institut für Ökonometrie und operations research, Universität Bonn, (Nov. 1985), 13 pages.

[108] H. J. Prömel and B. Voigt, *Ramsey Theorems for finite graphs*, Report No. 86447-OR, Forschungsinstitut für Diskrete Mathematik, Institut für Ökonometrie und operations research, Universität Bonn, 27 pages.

[109] H. J. Prömel and B. Voigt, *A short proof of the restricted Ramsey theorem for finite set systems*, Report No. 86449-OR, Forschungsinstitut für Diskrete Mathematik, Institut für Ökonometrie und operations research, Universität Bonn, 9 pages.

[110] H. J. Prömel and B. Voigt, Hereditary attributes of surjections and parameter sets, *Europ. J. Combinatorics* **7** (1986), 161–170.

[111] H. J. Prömel and B. Voigt, *A sparse Graham-Rothschild Theorem*, Report No. 87649-OR, Forschungsinstitut für Diskrete Mathematik, Institut für Ökonometrie und operations research, Universität Bonn, (1987), (To appear in *Trans. Amer. Math. Soc.*).

[112] H. J. Prömel and B. Voigt, *Aspects of Ramsey Theory I: Sets*, Report No. 87495-OR, Forschungsinstitut für Diskrete Mathematik, Institut für Ökonometrie und operations research, Universität Bonn, (Mar. 1989), 123 pages.

[113] H. J. Prömel and B. Voigt, *Aspects of Ramsey theory II: Arithmetic Progressions*, Report No 87497, Forschungsinstitut für Diskrete Mathematik, Institut für Ökonometrie und operations research, Universität Bonn, (Mar. 1989), 99 pages.

[114] R. Rado, Note on combinatorial analysis, *Proc. London Math. Soc.* **48** (1943), 122–160.

[115] R. Rado, The partition calculus, *Recent progress in combinatorics*, W. T. Tutte ed., Academic Press, New York, 1969.

[116] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.*, **30** (1930), 264–286.

[117] V. Rödl, A generalization of Ramsey theorem, in *Graphs, Hypergraphs and Block Systems*, Sielona Gora (1976), 211–220.

[118] V. Rödl, N. Sauer, *The Ramsey property and families of graphs characterized by excluding a graph*, (Manuscript).

[119] I. Schur, Uber die Kongruenz $x^m + y^m \equiv z^m \pmod p$, *Jber. Deutsch. Math.-Verein* **25** (1916), 114–116.

[120] S. Shelah, Notes on Partition Calculus, *Infinite and finite sets*, A. Hajnal et al eds., *Coll. Math. Soc. J. Bolyai* **10**, North Holland (1975), 1257–1277.

[121] S. Shelah, Primitive recursive bounds for van der Waerden numbers. *J. Amer. Math. Soc.* **1** (3) (1988), 683–697.

[122] Lower bounds for small diagonal Ramsey numbers, *J. Combin. Theory Ser. A*, **42** (1986), 302–304.

[123] J. H. Spencer, Ramsey's theorem for spaces, *Trans. Amer. Math. Soc.*, **249** (1979), 363–371.

[124] J. H. Spencer, Three hundred million points suffice, *J. Combin. Theory Ser. A*, **49** (1988), 210–217.

[125] R. S. Stevens and R. Shantaram, Computer generated van der Waerden partitions, *Math. Comp.* **17** (1978), 635–636.

[126] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieiw. Arch. Wisk.* **15** (1927), 212–216.

[127] B. L. van der Waerden, How the proof of Baudet's Conjecture was found, in *Studies in Pure Mathematics*, L. Mirsky ed., Academic Press, London (1971), 251–260.

[128] J. A. Winn, Jr., *Asymptotic bounds for classical Ramsey numbers*, Polygonal Publ. House, Washington, N.J., (1988), 101 pages.

[129] E. Witt, Ein Kombinatorischer Satz der Elementargeometrie, *Math. Nachr.*, **6** (1951), 261–262.