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International Evidence Concerning Macroeconomic Chaos

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "International Evidence Concerning Macroeconomic Chaos" submitted by Richard Anton Stokl in partial fulfillment of the requirements for the degree of Master of Arts.

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ABSTRACT

This thesis will test for the effects of chaos on an economic time series. A positive Lyapunov exponent is an indication of chaotic behaviour. Recent research has derived the asymptotic distribution of the nonparametric neural network estimator of the Lyapunov exponent in a noisy system by Nychka *et al* (1992), this methodology will be summarized and its statistical framework will be used to test the hypothesis of chaos. This framework will be applied to the Real per Capita Gross Domestic Product of ten countries. Chaos theory provides additional research as to whether economist should analyze macroeconomic models and data from an endogenous or exogenous approach. The conclusion that chaos is present in a time series is support of the endogenous approach.

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DEDICATION

For Diana.

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INTRODUCTION

One of the most interesting and often studied topics in macroeconomics is the interpretation of business cycles. A nation's economy may at one time or another experience a deviation above or below its steady growth path. Negative deviations are recessions and the prediction of their occurrence and duration is a topic of interest in the popular media and with macroeconomists. How business cycle fluctuations or perturbations are interpreted is one of the most important issues in economic theory. There are two general approaches – the exogenous and endogenous methodology.

The exogenous approach is based on the work of Slutsky (1927) and Frisch (1933). Their hypothesis is that fluctuations are deviations from the steady state or equilibrium growth path and are seen as external random shocks that are temporary. The market is viewed as the stabilizing mechanism that, in the absence of extraneous changes, will return the economy to equilibrium – its steady growth path.

The endogenous approach, partly based on the research of Kaldor (1940), Hicks (1950) and Goodwin (1951), is based on the hypothesis that there is some internal mechanism to the model that is responsible for deviations and fluctuations. The shocks are of a permanent nature and the economy will not return to its previous steady state growth path with this approach. The differences between these two approaches are that the exogenous approach sees the fluctuations as resulting from external shocks to the model that are temporary and eventually will return to its previous path. The endogenous approach sees these same fluctuations as resulting from changes to the internal parameters of the model that are permanent. These two approaches differ in how they view the source of the shocks and its effects on the model. Knowing which approach is applicable to a model or economy enables policy makers to determine the appropriate actions in response to deviations from the norm.

Chaos theory is the most recent addition to the endogenous approach. It will be shown in Chapter 1 that alterations to the parameters of a mathematical system can alter the nature of this system's path or orbit. The logistic difference equation will be used to introduce the concept of chaos theory and it will be seen that altering this model's internal parameters causes permanent changes to the system's path. Therefore, testing for chaos and proving its occurrence is a justification that the endogenous approach is relevant.

The study of chaos has its origins in the late nineteenth century. However, practical applications and uses were not developed till the groundbreaking work of Edward Lorenz (1963). Originally, the weather was Lorenz's area of interest. He discovered that small changes to the initial conditions of a computerized mathematical weather model had very large effects later in time. This is a basic definition of sensitive dependence on initial conditions (SDIC), which is one of the main characteristics of chaotic behaviour. Lorenz is credited with the beginnings of present day research into chaos theory. It then spread to the natural and physical sciences and eventually to economics.

The initial use of chaos was in the physical sciences due mainly to the fact that examples of its existence were more readily found in nature. The exploration of chaos theory in economic models did not take place till the 1980's. This delay was because there was no formal definition or methodology by which one could test for chaos in either econometric models or time series.

Li and Yorke (1975) developed a theorem that tested for the presence of chaos. If a time series were to exhibit a three period cycle as part of a number of other cycles and random patterns than it is chaotic. This early research provided economists with an applicable definition and testing methodology for chaos. Jess Benhabib and Richard H. Day can be credited for the introduction of chaos theory in economics. Much of their early analysis is based on the Li-Yorke theorem. Their work, therefore, became the foundation used to test economic models for chaotic behaviour.

Day (1982) used the Li-Yorke theorem to determine whether chaos was present in a Solow (1956) growth model. Day (1982) derived the conditions under which a three period cycle was apparent. Consequently, according to the Li-Yorke theorem it is an indication of chaos. He also demonstrated that economic growth does not have to follow a steady growth path. Economic growth could have periods along its path that are chaotic. Benhabib and Day (1982) examined an overlapping generations model based on the work of Samuelson (1958). Benhabib and Day (1982), and Jean-Michel Grandmont (1985) are considered to be the first examples where an overlapping generations model was shown to have instability or chaotic orbits. Once again, the Li-Yorke theorem is used as a definition and methodology to determine chaos. There is a large body of research into the determination of whether economic models are chaotic. Other studies of interest include the Day (1983) examination of a classical agrarian economy and Nishimura and Sorger's (1996) assessment of a two-sector, infinite living household, optimal growth model.

To this point, the study of chaos in economics was restricted to the examination of macroeconomic models using the Li-Yorke theorem as a definition and testing methodology. All the above-mentioned studies mathematically define the model. The models are then reduced to a difference equation and the conditions under which period three cycles exist are determined. This method of analysis worked well with econometric models, however, it was unable to determine the presence of chaos in a times series, such as the gross domestic product of a nation.

Eckmann and Ruelle (1985) were the first to define sensitive dependence on initial condition (SDIC) as a test for chaos. SDIC has become one of the most important characteristics to define chaos in time series data. The Lyapunov exponent calculates the exponential rate of divergence of two points initially close to one another. It is used to determine SDIC. To date, a number of different methods have been used in the calculation of the exponent. The Nychka, Ellner, Gallant and McCaffrey (1992) method has been found to be the most appropriate when dealing with economic data and their method will be examined in detail in Chapter 5. The Lyapunov exponent calculates whether two points that are initially very close to one another will converge or diverge over time. Divergence, indicated by a Lyapunov exponent greater than zero, is an indicator of chaos. The development and general acceptance by the research community of SDIC as a definition for chaos and the use of the Lyapunov exponent had resulted in new research whether deviations or shocks in an economy were due to endogenous or exogenous factors.

Brock and Sayer (1988) used an early form of the Lyapunov exponent to test for chaos in several US data sets. They tested quarterly data from the late 1940's to the mid

1980's of the unemployment rate, employment numbers, real gross national product (GNP), gross private domestic investment and industrial production of several industries. They concluded that there was some evidence of nonlinearity (a characteristics of chaos but not a condition of it), but no evidence of chaos and the Lyapunov exponent calculation methodology used was weak. They postulated that this methodological weakness resulted in no chaotic behaviour being found. Frank, Gencay and Stengos (1988) used a similar Lyapunov exponent calculation to test the real GNP of Germany, Italy and Japan and the real gross domestic product (GDP) of the United Kingdom (UK). These time series were for the logarithmic first differences of seasonally adjusted quarterly data from the early 1960's to the mid 1980's. Their finding did not support any evidence of chaos.

Additional studies in chaos theory have been completed that are not restricted to the singular use of national income time series. Frank and Stengos (1989) examined gold and silver prices but could not detect chaos using an early version of the Lyapunov exponent. Serletis and Gogas (1997,1999) tested for chaos using the Nychka *et al* (1992) method in seven East European foreign exchange black markets and the North American Natural Gas Liquids market, respectively. Their studies were able to produce a point estimate of the Lyapunov exponent. Only the Russian Ruble and East German Mark had a positive Lyapunov exponent indicating that the orbits diverged, therefore specifying chaos. However, in the East European foreign exchange black markets study, the Gencay and Dechert (1992) method was also used to derive the Lyapunov exponent. Using this alternative methodology, no chaos was evident in any currency. This is indicative of the concerns when testing for chaos – the calculation methodology may change the conclusion.

The methodology in all of these studies resulted in a point estimate of the Lyapunov exponent. A point estimate could not actually test for the presence of chaos, as there was no method to construct confidence intervals based on the statistics of the estimate. Whang and Linton (1999) overcame this concern with their development of a statistical framework allowing researchers to test the Lyapunov exponent's level of significance. Shantani and Linton (2000) applied the work of Whang and Linton's (1999) statistical framework to the Nychka *et al* (1992) method of calculating the Lyapunov exponent. Shintani and Linton (2003) applied this same technique to the seasonally adjusted quarterly real GDP of Canada, the UK and the United States (US) from 1957: First Quarter (Q1) to 1999:Q3, for Germany and Italy from 1960:Q1 to 1998:Q4, and for Japan from 1955:Q2 to 1999:Q1. They found no evidence of chaos in any of the countries and were able to state this with the confidence of having tested the null hypothesis of chaos.

This chapter has discussed the difference between the endogenous and exogenous approach to macroeconomic models. It has been shown that the main difference between these two approaches is in the method by which they account for deviations from the steady growth path that most national economies experience. This thesis will test for chaos using the log real per capita GDP and the first differences of the log real per capita GDP, from 1870 to 1985 of the following ten countries; Australia, Canada, Denmark, France, Germany, Italy, Norway, Sweden, UK and the US, using data sourced from Maddison (1982). This thesis will test the null hypothesis of chaos using the Nychka *et* *al* (1992) method of calculating the Lyapunov exponent. It is the most appropriate method to use with economic data. The statistical framework for this Lyapunov exponent estimate (developed by Whang and Linton (1999) and Shintani and Linton (2000)) will be used to statistically confirm the null hypothesis.

This thesis will go beyond the work of other researchers. It will not only provide a point estimate of the Lyapunov exponent but will answer the null hypothesis of chaos and, by doing so, will determine whether an exogenous or endogenous approach is a more appropriate means of describing macroeconomic models. This will add valuable evidence to the appropriate approach that policy makers should use in their handling of national economics.

The remainder of this thesis is organized as follows. Chapter 2 introduces the concept of chaos theory by discussing its interest to economists. The history of chaos theory will be examined and a simple mathematical example that results in a chaotic orbit or path is given. The main characteristics that define chaos are then summarized. Chapter 3 looks at two economic models that exhibit chaotic behaviour. Chapter 4 takes the first step on the road to testing for chaos within a time series. It introduces the data set to be used in this thesis and tests for stationarity. Chapter 5 introduces the Lyapunov exponent as a method of testing for the presence of chaos within a data set, explains its statistical distribution properties, and summarizes the empirical results. Finally, Chapter 6 summarizes this thesis.

CHAPTER 2

Introduction to Chaos Theory

2.1 Introduction

Sir Isaac Newton developed calculus in the 18th century. He simplified the use of mathematical equations that were continuous. This was accomplished through the use of derivatives, which describe the rate of change in the function at a particular point on its path. One of the shortcomings of calculus is its inability to deal with functions that are non-differentiable (or non-continuous), due to an irregular or complex structure. The development of chaos theory is one possible response to this problem. It has provided researchers with a methodology to understand and work with these types of functions. The purpose of this chapter is to introduce the concept of chaos theory.

The remainder of this chapter consists of the following sections. Section 2.2, will explore some of the reasons that a researcher may be inclined to test for chaotic motion. Section 2.3 summarizes the major historical events resulting in the formulation of chaos theory. Since there is no generally accepted definition for chaos this thesis will take the approach that other researchers have – the logistic difference equation will be used to illustrate chaos in a simple dynamic system. This methodology will be the focus of Section 2.4. Section 2.5 will provide some characteristics of chaos theory as an introduction to formalizing its definition. Finally, Section 2.6 concludes the chapter.

2.2 Why Chaos

Many researchers have recently concluded that systems in nature are, in fact, more complex than originally thought. Ockham's Razor states that the simplest solution is usually the correct one. Thus, many researchers use simple linear regression to explain and forecast complex nonlinear curves and paths. Using a linear methodology may be less than appropriate when dealing with relationships better expressed as nonlinear.

This section will explain why many researchers including economists are exploring the uses of chaotic theory. To accomplish this, one must first have an understanding of the composition and make up of economic data. Many economic studies, such as prices, Gross Domestic Product (GDP), real growth, and monetary analysis have been tested for chaos. There are two problems associated with the use of time series data sets. First, a linear equation may not be the best fit for the data. Econometric techniques in use today are predominantly linear in design. This may not be suitable especially if the series is nonlinear. To illustrate, let us examine the natural logarithms of Canada's GDP. The data is graphically represented in Figure 2.1. Superimposing a linear fit, Figure 2.2, on the data it can be seen that this is not a good fit. Figure 2.3 uses an exponential fit that appears to be more appropriate. Finally, a super-smooth fit as seen in Figure 2.4 is an even better representation. This fit is achieved by a higher order nonlinear function. This analysis does not imply that all economic time series data are nonlinear. Rather, one should be open to the usage of nonlinear techniques where applicable.

Nonstationarity is the second problem one may encounter with time series data. The simplest definition of stationarity is that its mean and variance is not dependent on time, i.e. they remain constant. A time series that is not stationary will not adhere to the ordinary least square properties of regression analysis. Testing for its presence involves using procedures developed by Dickey and Fuller (1979, 1981) and Perron (1989). Nonstationarity can be tested using a Dickey-Fuller (DF) or the Augmented Dickey-Fuller (ADF) test. Standardized techniques require that in order to correct for stationarity the time series be transformed by taking the first differences.

Many first difference plots may mimic white noise. This ability to mimic white noise is one of the characteristics of chaos. The graphical plots of white noise and the logistic difference equation, that is representative of chaos, may appear similar to the eye. When comparing the first log differences for Canada's GDP (Figure 2.5) and that of white noise (Figure 2.6) some similarities can be observed. Thus, time series data that are difference stationary or that is made stationary by taking the first differences, may appear to be similar to white noise. Therefore, to determine whether it is truly stochastic noise or a mimic is the main reason for researchers to test for the presence of chaos.

In conclusion, economists have found that because of the problem of stationarity in time series data, chaos may be present and therefore a more appropriate representation for the data. To remove nonstationarity from a time series data set requires taking the first differences that could result in what appears to mimic white noise but is in reality chaotic motion.

2.3 History/Origins of Chaos Theory

Where does chaos originate? Who developed it? These are some of the questions concerning this subject that will be examined in this section.

The concept of chaos has been around in various forms for much of human history. Some believe that its earliest reference can be found in the bible. The darkness that was night is the chaos that God replaced with order. Heisod in his 'Theogong' made reference to the order that the universe brought to the chaotic nothingness.

The development of chaos as a mathematical and scientific theory has its beginnings in the late nineteenth century. In 1889, Oscar II the King of Sweden and Norway offered a prize to the first person that could solve the three-body problem. This question concerns a system where three objects of various sizes orbit around one another. For example, a solar system with a sun, an orbiting planet, and the planet's orbiting moon is a three-body problem. This problem is a first step towards an understanding of the motion of n-bodies in space. Up to this point, science required that a complex system be divided into components, thereby studying two bodies at a time using Newtonian Physics. Once a system of three or more objects is used, the rules developed by Newton could not provided an explicit solution. Only approximations could be calculated. Thus, the motivation for solving the three-body problem was to be able to understand how the system as a whole interacted with all its components.

The French mathematician, Henri Poincare, eventually won the contest (see Briggs and Peat (1989)). He was declared the winner based on a single possible solution or orbit to the three-bodies. Before he could publish his findings, he discovered that his single solution was incorrect, that there were actually numerous possible orbits that could result and that each orbit was dependent on the initial conditions.

The three-body problem is a system where three objects of varying sizes are orbiting around one another in three dimensions. Poincare simplified this problem by restricting the two largest objects (the sun and its planet) to a plane and limiting the distance between them. He could now focus on the orbit of the smallest object, the moon. Poincare found the following possible orbits of the moon:

1. it exits the system; its path goes out to infinity,
2. it falls into a stable orbit around either one of the other two,
3. it develops a stable orbit around both, a figure eight, or
4. it has an orbit that is non-repeating, with any combination of the previous two options; this is chaotic motion.

Poincare concluded that the three bodies' orbits could only be described by the differential equations that formulated the problem. There are an infinite number of possible solutions that could result depending on the starting point or initial conditions of the system of objects.

Henri Poincare discovered one of the most important characteristics of chaos theory - sensitive dependence on initial condition. The moon's orbit is dependent on its initial starting point and velocity. Sensitive dependence can be described in the differential equations of the system. The scientific community was not ready at this time to continue research in this area and much of Poincare's discovery remained unnoticed and forgotten until the late 1950's. This break in research was due partly to the fact that other events were occurring at this time including groundbreaking research in the areas of relativity and quantum theory.

In 1954, three scientists Kolmogorov, Arnold, and Moser expanded on the findings of Poincare. They found that quasi-periodic orbits defined by non-simplistic ratios in the orbits of two bodies could result in stability. However, if a simple ratio were obtained (i.e. 1:1, 2:1, a ratio of two whole numbers) then instability would invade the system causing the smallest of the three bodies to leave orbit (i.e. speed off to infinity). Their ideas explain the gaps of matter found in the rings of Saturn and the ring of asteroids between Mars and Jupiter. These gaps are where the orbit ratios are simple – resulting in instability and the matter disappearing since its orbit has taken off to infinity. Their discovery began the process that would lead to Lorenz's development of the theory of chaos.

By 1960, the scientific world was ready for new developments in the field of nonlinear dynamics and Edward Lorenz provided them. Lorenz was conducting experiments with a computerized weather model, using five nonlinear equations. He rounded off the first number of the series so as to recompile a portion of an earlier longer sequence. After some computing, he discovered that the results were completely different from the original sequence. In fact, the further away from the new starting point he went, the larger the discrepancy. The second smaller data run was initiated by rounding off its initial value from six to three decimal places. He concluded that small differences in initial conditions were grossly magnified over time confirming Poincare's research and determined that long-term weather forecasting was impossible.

Benoit Mandelbrot and the research he completed in the 1970's and 1980's was a valuable addition to chaos theory. As a researcher at IBM's Thomas Watson Laboratory, he is credited with the development of fractals, which explain the geometric structure of chaos. The essence of his discovery is the fractal dimension, which describes the dimension of an irregular curve.

Mandelbrot's research was prompted by the question "What is the length of the coastline of Britain?" He found that as the unit of measure is decreased, the length of the coastline increases. In other words, increasing the detail of the shoreline will result in a more complex and irregular curve. Theoretically, at very small units of measure, infinitely small or close to zero, the coastline will have a length that increases towards infinity. Thus, two islands having different areas will have the same perimeter length, infinity. Mandelbrot's studies showed that the use of a fractal dimension was a more appropriate measure for the comparison of two irregular objects and can be used to geometrically describe chaos.

The Euclidean geometry used up to this time is lacking in its descriptive powers when dealing with irregular curves. It is known that a point has a zero dimension, a straight-line one dimension, and a plane occupies two dimensions. However, when examining the coastline of Britain, one can see a shape that increases in complexity as the level of detail increases or the unit of measure decreases. This more complex curve cannot be described as one-dimensional. It is not a straight line and it does not enclose a surface, therefore it is not two-dimensional. Consequently, it must have a dimension somewhere in-between. Mandelbrot concluded that the more complex and irregular a curve is, the greater its dimension above one. A curve that reaches a high level of complexity and encloses the entire plane will have a dimension very close to two.

An alternative example of a fractal dimension is the Koch Snowflake or Island. In order to construct the snowflake, simply start with a triangle where each side has unit length. Repeat the following; remove the middle third of each side and replace it with the upper two sides of a triangle with a length equal to that of the section removed.

Performing this process once transforms the triangle into the Star of David. Repeating an infinite number of times results in the snowflake. Two important characteristics of fractals can be seen from the construction of the Koch Island. First, it can be shown that as the number of iterations increases, the perimeter length will approach infinity. Each iteration causes the length to increase by a factor of $4/3$. Thus, after an infinite number of iterations, the perimeter approaches $1 \times 4/3 \times 4/3 \times 4/3 \cdots = \infty$. Second, the curve has a self-similar nature to it such that at ever decreasing scales the curve is exactly the same – magnifying subsections of the whole curve resulting in a similar curve. Mandelbrot found that the Koch Curve had a fractal dimension of approximately 1.69, which is somewhere between a line and the plane that it occupies.

Fractals as introduced and developed by Mandelbrot have been shown to be highly complex and simple at the same time. They are highly complex due to their infinitely small detail, unique mathematical properties and simple since they can be formed using an easy iterative process as seen with the Koch snowflake. Fractals can be characterized as: infinitely detailed curves of infinite length, having no slope or derivative, described by a fractal dimension, exhibiting self similarity, and can easily be generated using a simple repeating process. These characteristics include some of those to be later defined as belonging to chaos theory.

In this section, the major researchers in the field of chaos and its geometry have been introduced. Poincare's research over one hundred years ago into the three-body problem instigated the study of nonlinear dynamics. Lorenz is seen by many to be the modern developer of chaos theory. He began the discussion into this new and exciting field of research. Finally, Mandelbrot's discovery of fractals provided chaotic motion with a topological explanation and allowed individuals to view their results. These scientists are by no means the only individuals to conduct research and make significant contributions to the development of chaos theory, however they started the process.

Further research completed after this time deals directly with defining and testing for the presence of chaos in time series data sets. Some of the more important characteristics and definitions will be summarized in section 2.5. Prior to this, an introduction to chaos will be provided in the next section using a simple dynamic system – the logistic difference equation.

2.4 The Logistic Equation: An Introduction to Chaotic Motion

Chaotic motion can be found in both discrete and continuous time series. The continuous case is not covered here as it is beyond the scope of this thesis. In this section, the discrete case will be introduced using the logistic difference equation. It is an easy function to map and explain. It has been frequently used and written about, and economic data, especially in first difference form, can often be described by this function.

Edward Lorenz provided a simple and often quoted example to describe chaos, 'The Butterfly Effect', which states that a butterfly that flaps its wings in China could influence future weather in Calgary or any other location. In this example, the butterfly is a metaphor for an initial condition for a dynamic system that will cause future changes.

The logistic difference equation will be used to explain two of the main characteristics of chaos. The first characteristic is sensitive dependence on initial conditions (SDIC). In chaotic motion SDIC is present when two different time paths or orbits commence very close together and over time deviate away from one another. The logistic equation can be used to demonstrate this characteristic. In Chapter 5 the Lyapunov exponent will be defined. It can be used to calculate and test for divergence, that is, for the presence of chaotic motion. The second characteristic introduced here is that all possible periodic non-repeating orbits can be found in a chaotic time series. This characteristic will be explained in greater detail in Chapter 3 where Sharkovskii's theorem on cycle ordering and its implications for the Li and Yorke (1975) definition of chaos are introduced.

Chaos theory is rooted in the study of dynamic systems of equations. That is, to explain the motion of variables that changes over time. To simplify our understanding and explain the path to chaos, the logistic difference equation is used in its dynamic form:

$$X_t = \alpha X_{t-1}(1 - X_{t-1}) \tag{2.1}$$

This equation has its origins with population growth theory and was introduced by Malthus in a simpler form, as follows:

$$X_t = \alpha X_{t-1}$$

where α is the parameter that represents the birth rate of a species. Depending on the value of α, this equation exhibits continuous growth ($\alpha > 1$), no change ($\alpha = 1$), or extinction ($\alpha < 1$). Malthus' equation is faulty as continuous growth to infinity is a possibility (for $\alpha > 1$). There are other factors that have been omitted, including, availability of food, natural rate of death, and predatory reduction. These, additional factors or variables are required to improve the equation.

P.F. Verhulst in 1845 (Stewart, 1989) added a difference component $(1 - X_{t-1})$ to the Malthus equation. This term has two effects. First, it transforms it from a linear to a nonlinear equation. Second, the difference term adds a component of reality. It represents the impact of all other environmental factors that may affect the population including those mentioned above. Thus, for values of $\alpha > 1$ there can no longer be continuous growth to infinity and depending on its value some very interesting results are possible.

How chaos is found in nonlinear dynamic equations can be efficiently displayed using the logistic difference equation. This can be accomplished by altering the values of the parameter α. The function can be easily iterated and the results presented in two-dimensional graphs. By examining various values of the parameter α, the route to chaos will become evident, supporting the two characteristics previously mentioned.

Prior to examining the logistic difference equation in greater detail, the following definitions are offered.

Definition 2.1: Fixed Point

A fixed-point p for the map f is, a p such that $f(p) = p$, i.e. the function f does not change the outcome, it remains at p.

Definition 2.2: Stable Fixed Point

A stable fixed point, is a point p, such that there is a neighbourhood $N(p)$ so that iterations with initial values $X_0 \in N(p)$ converge to p.

Definition 2.3: Unstable Fixed Point

Unstable fixed point, as in definition 2.2 except they move away from p.

Definition 2.4: Fixed Point Attractor or Fixed Point Sink

If x is an element of $N(p)$ then its limit at the k^{th} iteration, as k approaches infinity, will converge to the point p, or simply, if $x \in N(p)$ then $\lim_{k \to \infty} f^k(x) = p$.

Definition 2.5: Repelling Fixed-Point or Source

If x is an element of $N(p)$ than its limit at the k^{th} iteration, as k approaches infinity, will diverge away from the point p, or simply, if $x \in N(p)$ then $\lim_{k \to \infty} f^k(x) \neq p$.

(Summarized from Alligood et al (1997), Cambel (1993) and Smith (1998))

Definitions 2.4 and 2.5 can be extended to periodic situations. Thus, a periodic sink or limit cycle attractor is a dynamic time path that is continually repeating a set pattern. For example, after a possible short interval of randomness, the system will stabilize to a path that is cyclical every k^{th} interval, and is called a k period orbit. These k orbits are defined by $f^k(p) = p$. Here every k^{th} iteration of the function f will return to the same point p, the pattern is repeated every k^{th} period/orbit. For example, $f^2(p) = f(f(p)) = p$, is a period two cycle.

The results of the logistic difference equation for various values of α are illustrated in Figures 2.7 to 2.13 using two types of graphs: the cobweb diagram (upper panel) and a time series plot (lower panel). The time series graph is simply a display of the values of the logistic difference equation as they change after each iteration of the function (the number of iteration is set to one hundred). The cobweb diagram is actually three functions plotted together: (1) a 45 degree line $x = y$; (2) the logistic equation for the given value of α; and (3) the cobweb which plots the movement of the logistic equation between the actual function and the 45° line at each iteration. Together these functions provide an illustration of whether a dynamic system will converge to a fixed point (as in Figures 2.7 and 2.8), converge to a periodic limiting cycle (Figure 2.9 and 2.10), or result in instability (Figure 2.12). Altering α will result in a graphical representation that can explain the equation's orbit.

The parameter α of the logistic difference equation when altered results in an increasing level of complexity as seen here:

1. $0 \leq \alpha \leq 1$: the map of the function will converge to the origin resulting in a sink,
2. $1 < \alpha < 3$: a sink will occur at the point ϑ where $\vartheta = {(\alpha - 1)}/{\alpha}$,
3. $3 \leq \alpha < 4$: here instability occurs i.e. various periodic cycles result with an increasing number of cycles as α increases, and
4. $\alpha \geq 4$: denotes the present of chaos - no point or periodic sink is detected, cycles are of all sizes and non-repeating.

In order to explain the above results, they can be shown graphically with alterations to the parameters. Let α equal to two, this will result in the graphs shown in Figures 2.7 and 2.8. In both cases, the system converges to a sink at 0.5. The only variation between the two figures is the starting point and the number of iterations required to reach the stable sink. The fixed-point attractor or sink for $\alpha = 2$ is $\vartheta = (\alpha - 1)/\alpha = (2 - 1)/2 = 0.5$. It should be noted that for values of α less than four, there is only one possible periodic or point sink regardless of the initial condition. In Figure 2.9 α is increased to three resulting in a period two cycle. For α equal to 3.5 (Figure 2.10), a period four cycle is present. Increasing the parameter α from 3 to 3.5 has resulted in period doubling. This can be seen more clearly with a bifurcation diagram (Figure 2.16 - more on this later). Figure 2.11 examines the value for α equal to 3.83, resulting in a period three cycle. Thus, by examining the results of Figures 2.9 to 2.11, which represent values of the parameter of the logistic difference equation between three and four, it can be seen that the magnitude of the periodic sink increases until chaos is eventually detected at α equal to four.

The logistic difference equation has provided an example of how a simple nonlinear function can be altered and thus leads to chaos. It has been shown that as α increases beyond two, there is the possibility of an increasing number of periodic cycles. A bifurcation diagram summarizes the number and values of periodic orbits found in a dynamic equation. Figure 2.16, graphs the branching process that the logistic equation experiences as the value of α increases. For values of αless than three, the result is a single fixed-point attractor, a point sink. At exactly α equal three the bifurcation diagram splits into two paths, it doubles. The resulting paths indicate the values of the period two sink. A period two cycle continues until α reaches the value of 3.4495. Here the path again doubles to a period four sink (Figure 2.10). This process of doubling continues until there are an infinite number of periods occurring at α equal to 3.56999. At this point, the logistic equation is what is known as transient chaos. This is not complete chaos since the infinite number of periodic cycles does not occupy the complete phase space. Just after this value, the logistic function shows four broad regions, the process has reduced to a period four cycle. The process of period doubling repeats until α is 3.83. At this point, transient chaos returns, followed by another reduction to a periodic cycle (this time with three periods). Period doubling reoccurs, eventually reaching a state of chaos that includes the whole of the phase space at α equal to four, where true chaos occurs. This process illustrates how periodic doubling leads to chaos.

An interesting phenomena occurs when α equals four. Figures 2.12 and 2.13 have the same parameter value of α at four. Their only difference is their initial starting values - X_0 differs. Figure 2.14 offers a closer examination of the resulting orbits. The resulting paths differ, even though the initial values vary by only 0.00001. Figure 2.15 displays the same result in an overlay format. These graphs exhibit one of the main characteristics of chaos theory - sensitive dependence on initial conditions (SDIC). Dynamic chaotic systems that are identical, except for their initial starting point, will have paths that initially overlap but eventually diverge away from one another (Figure 2.15). Thus, a very small initial change of 0.00001 in a chaotic system will eventually result in a completely different solution, path, and orbit. This fact explains why Lorenz's small error in computation resulted in a completely different solution to his weather program.

In this section, a simple nonlinear dynamic equation has been used to demonstrate how chaos can be attained through the adjustment of its parameter or due to small changes to its initial starting point. The logistic difference equation has been shown to have these attributes, especially the chaos characteristics of SDIC and period doubling.

2.5 Characterizing Chaos

This chapter examined the history of chaos theory and used the logistic difference equation as an example to introduce simple chaotic motion. In this section, some of the various characteristics that define chaos will be presented.

Medio and Gallo (1992) provide a brief and concise definition for chaos that highlights its duality. They defined chaos as "stochastic behaviour occurring in a deterministic system." This short statement summarizes the difficulty that many have with the concept of chaos – its ambiguity. On the one hand, it is deterministic implying that there is no exogenous randomness in the system. A deterministic system can be stated with precise mathematical equations such as the logistic difference equation, which in the previous section was demonstrated to exhibit chaotic motion. Alternatively, Medio and Gallo define chaos as stochastic, a random process that over time changes due to a probabilistic process. Medio and Gallo's (1992) definition explains a process that appears to be stochastic or random but can be explained using simple dynamic mathematical formulas. Thus, a chaotic process may appear to be a white noise stochastic process. However, it simultaneously can be represented by a simple mathematical deterministic function.

Cambel (1993) has provided a more detailed definition, which summarizes the main characteristics of chaos theory:

> "There is no one standard starting point to explain chaos theory. It is a heterogeneous amalgam of different techniques of mathematics and science. Systems that upon analysis are found to be nonlinear, nonequilibrium, deterministic, dynamic, and that incorporate randomness

so that they are sensitive to initial conditions, and have strange attractors are said to be chaotic."

According to Cambel (1993), these are necessary, but not sufficient conditions for the occurrence of chaotic behaviour. Not all components of his definition are required for a system to be considered chaotic.

The most accepted method to detect for chaos, supported by Cambel (1993), is with the use of the Lyapunov number and its exponent. This number calculates the divergence of orbits that are initially close together – SDIC. The actual methodology of the Lyapunov exponent will be discussed in greater detail in Chapter 5.

Smith's (1998) definition of chaos simply requires that the Lyapunov exponent (λ) be greater than zero. His method for calculating λ is based on the following:

$$\left|f^n(x) - f^n(y)\right| \approx |x - y| e^{\lambda n},$$

where $f^n(\cdot)$ is the nth iteration. This equation supports SDIC. It states that if two points x and y that are initially very close to one another, will eventually have orbits that exponentially diverge.

Finally, a last definition of chaos provided by Robert L. Devaney (1992) is presented below. Prior to introducing this definition, an understanding of two concepts is required. First, periodic points that are dense in a space are the same as stating that the orbits or motion of some trajectory will eventually cover or pass through the whole of the phase space. Second, a definition of topologically transitive, is:

Definition 2.6: Topologically Transitive

A function f is topologically transitive if, given any two intervals U and V, there is some positive integer k such that $f^k(U) \cap V \neq O$.

This implies that intervals of points will eventually become larger sets that do not stick together. Based on this Devaney's definition of chaos is:

Definition 2.7: Definition of Chaos from Devaney (1992)

A continuous map f defined on the space S is chaotic if f has an invariant set $K \subseteq S$, such that:

1) f is (weakly) sensitive dependent on K,
2) periodic points are dense in K, and
3) f is topologically transitive on K.

It was subsequently determined that (2) and (3) imply (1), therefore SDIC is the only requirement for chaos.

In this section, a sampling of recent and popular attempts at defining chaos theory has been presented. There is no generally accepted definition for chaos theory. Fault can be found with any or all of the explanations put forward, since each researcher has his or her own definition, theories, and ideas on chaos and how to test for it. Some of the more important characteristics that define chaotic behaviour have been shown through the use of various quotes and formal definitions. Chaos is a dynamic process that is nonlinear, which results in a constantly changing system that mimics white noise or a purely random stochastic process. At the same time, this process can be described by deterministic functions. The most important characteristic of a chaotic process is that it exhibits SDIC. Therefore, this allows the Lyapunov exponent, discussed in Chapter 5, to be used as a method of testing for chaos. Through continued research, it is hoped that a standardized and universally accepted definition of chaotic theory will be developed.

2.6 Summary

The purpose of this chapter was to introduce chaos theory and to provide a better understanding of its basic concepts and purpose. To accomplish this task, the origins of chaos have been stated. Initial work in this field and some of the more important contributions of scientists has been reviewed. This review shows that this area of study is relatively new, highly technical and a growing area of research. It is an important area of research for economists since much of economic data has been found to have nonlinear attributes.

Section 4 provided a simple introduction to the use of the logistic difference equation and how it can be altered in such a way that chaotic behaviour is possible. Finally, some simple definitions of chaos were provided in the previous section. These definitions emphasize the main characteristics of chaos theory. SDIC is the most important characteristic of chaos theory. The Lyapunov exponent provides a methodology to test this characteristic and will be defined and used in Chapter 5 to test a time series for chaos. Prior to this, the following chapter will introduce two examples of economic growth models where chaotic behaviour can be observed. This will add to the basic definitions introduced in this chapter.

Figure 2.1

Log Real per Capita GDP

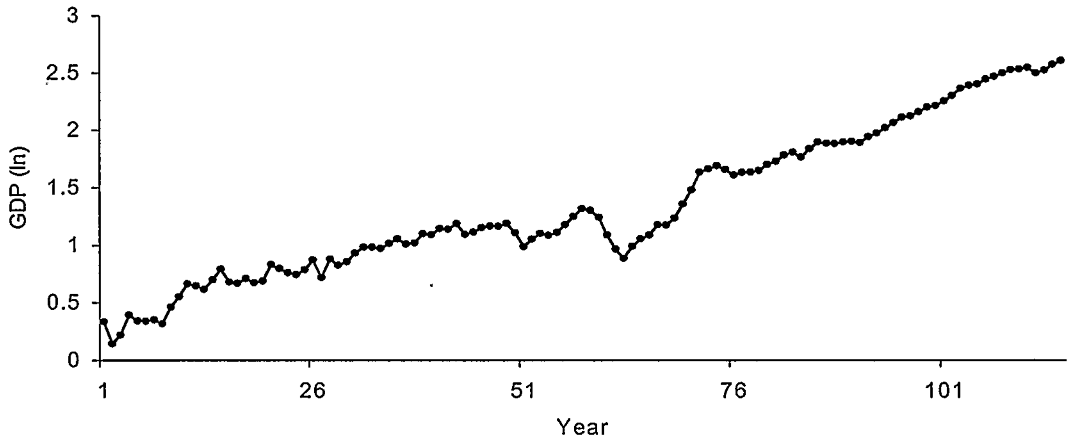


Figure 2.2

GDP - Linear Fit

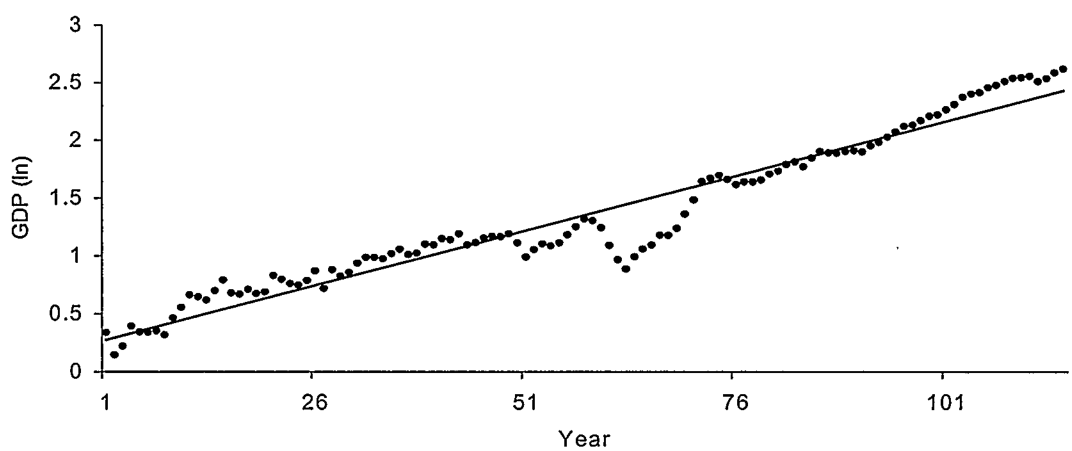


Figure 2.3

GDP – Exponential Fit

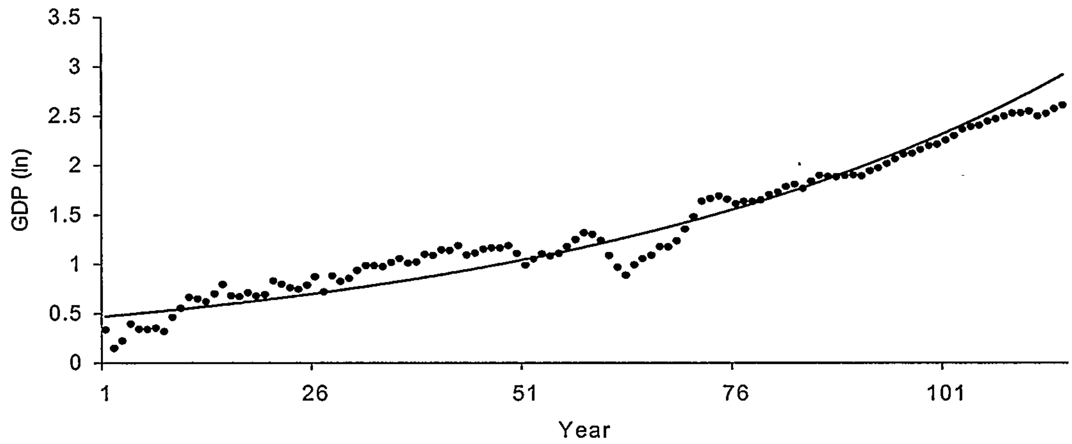


Figure 2.4

GDP – Super Smooth Fit

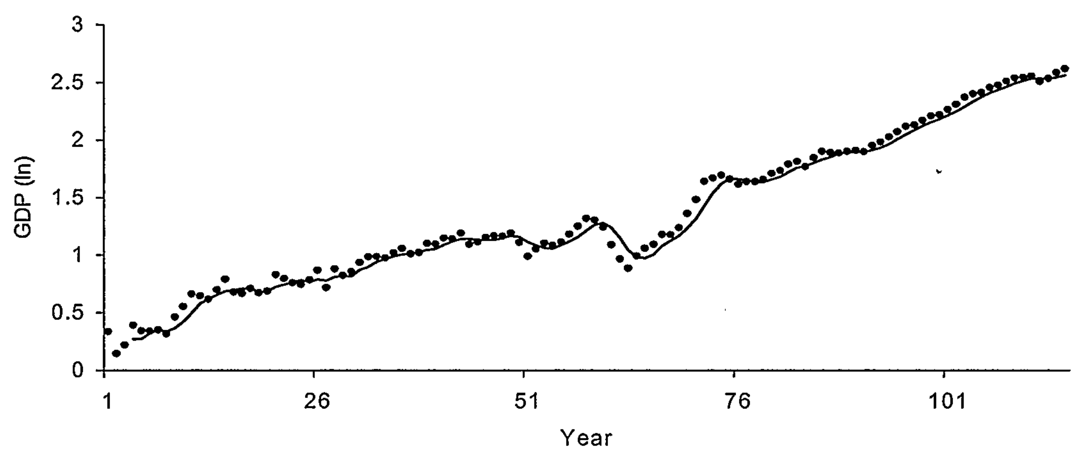


Figure 2.5

GDP - Log First Differences

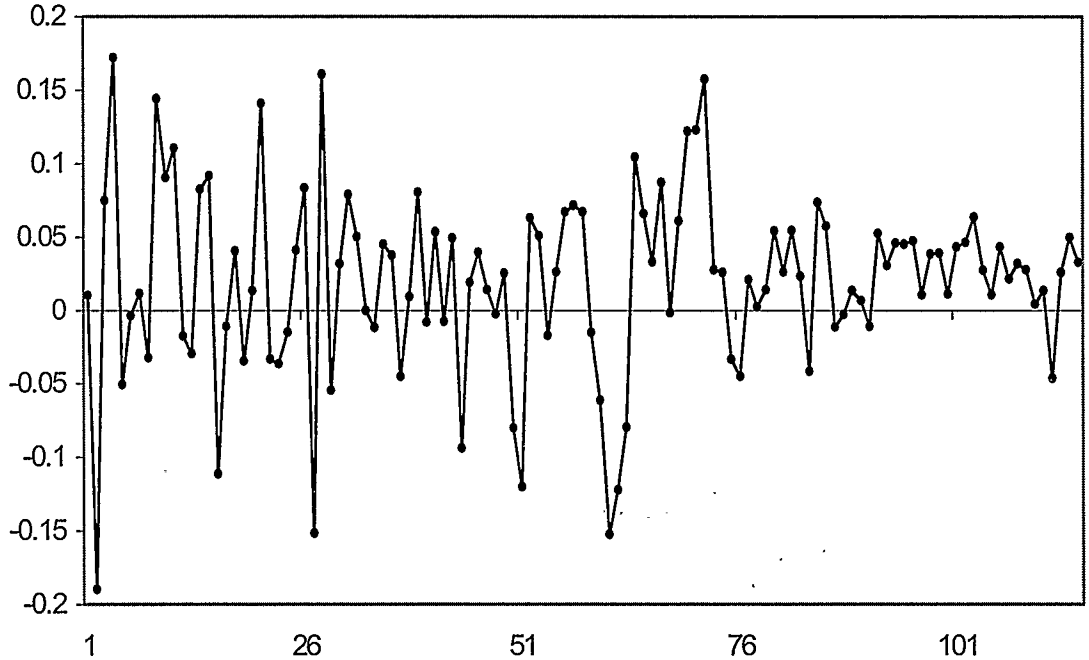
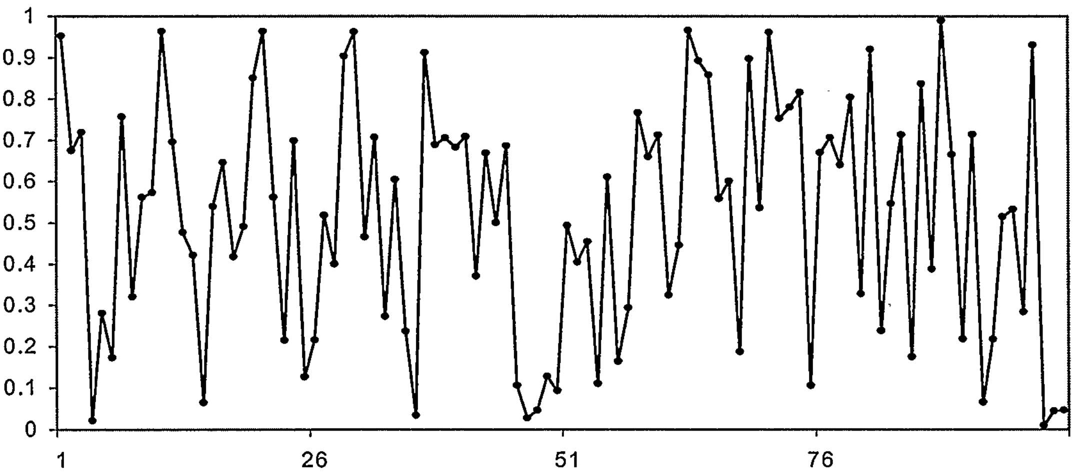


Figure 2.6

Examples of White Noise (n=100)



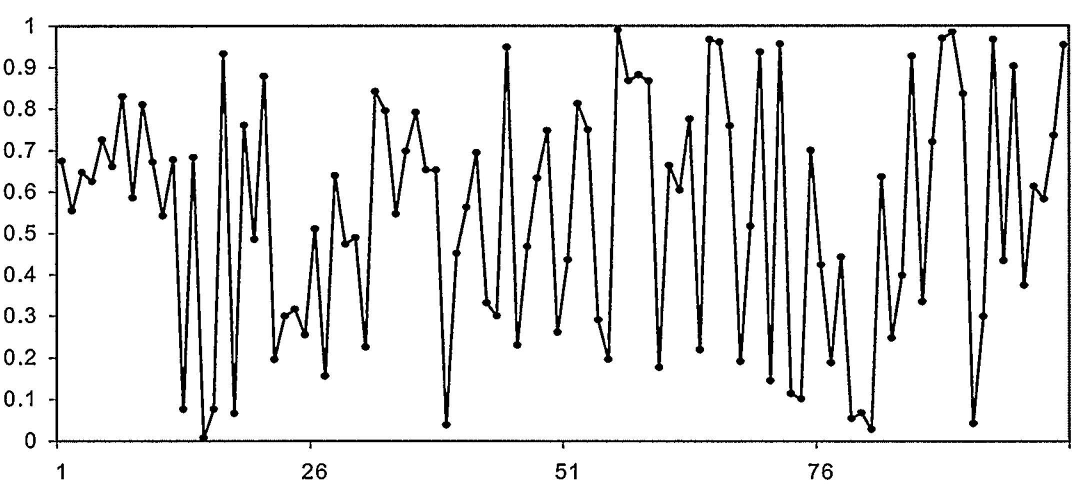
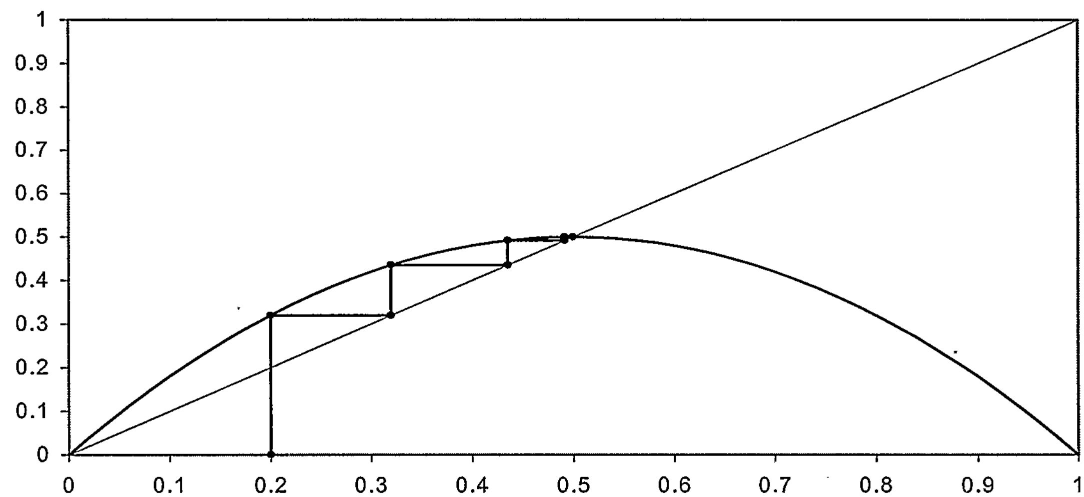


Figure 2.7

Logistic Equation
($\alpha = 2.0$, $X_0 = 0.2$)



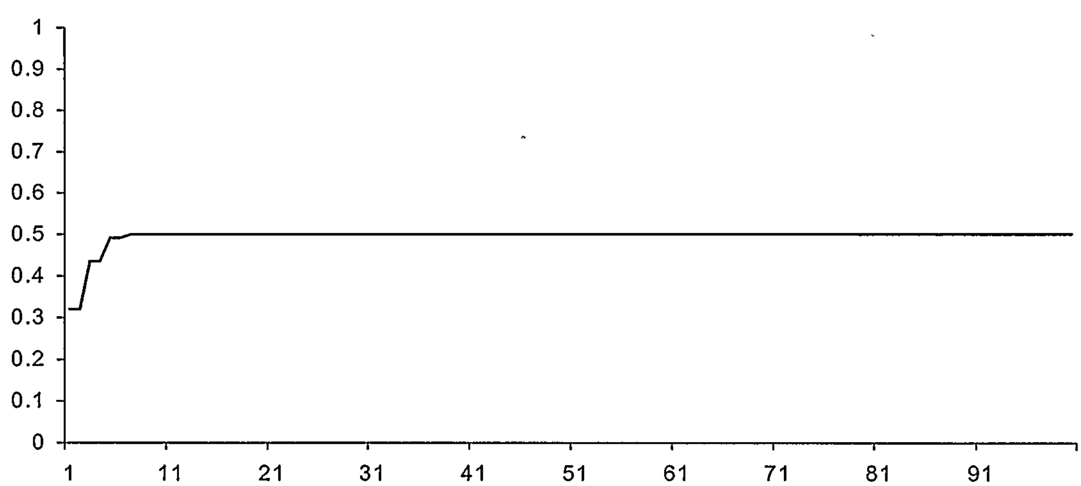
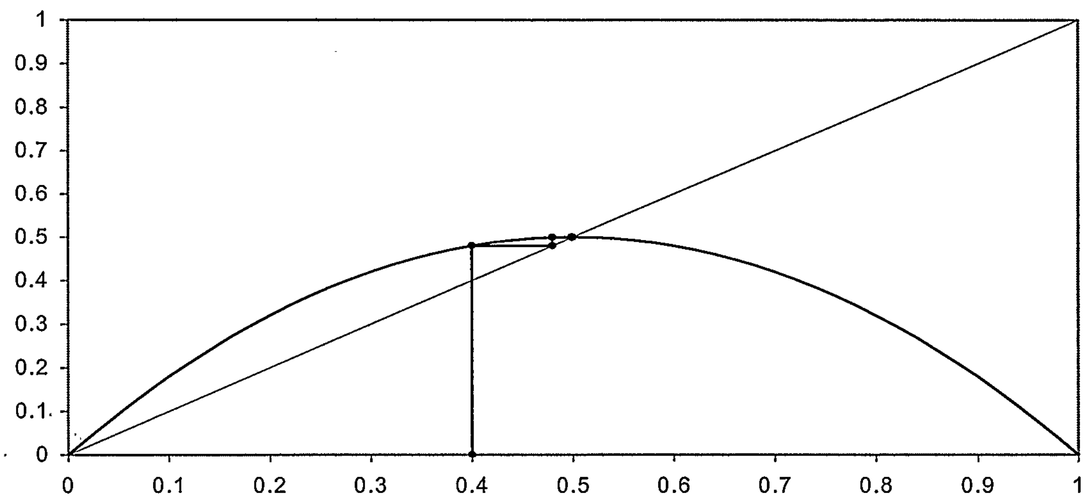


Figure 2.8

Logistic Equation
($\alpha = 2.0,\ X_0 = 0.4$)



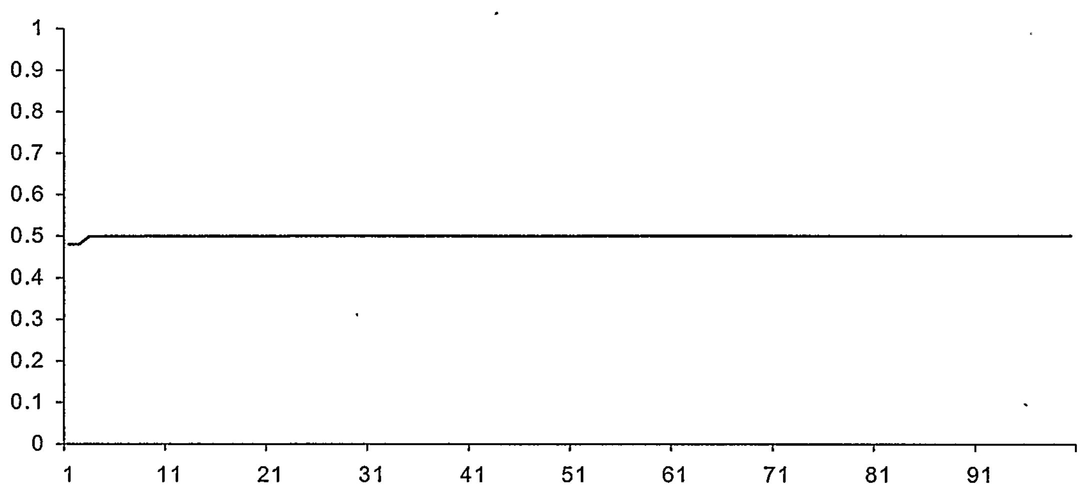
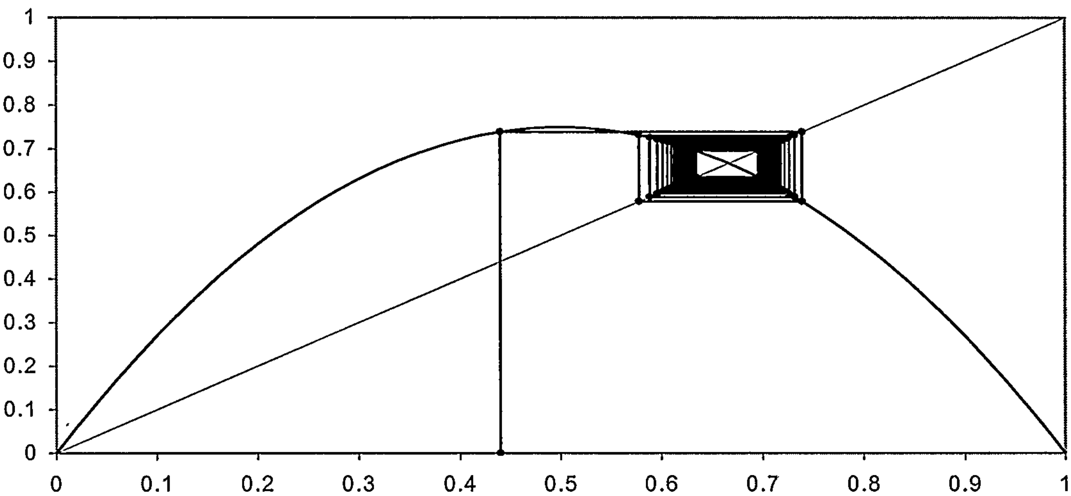


Figure 2.9

Logistic Equation
($\alpha = 3.0,\ X_0 = 0.44$)



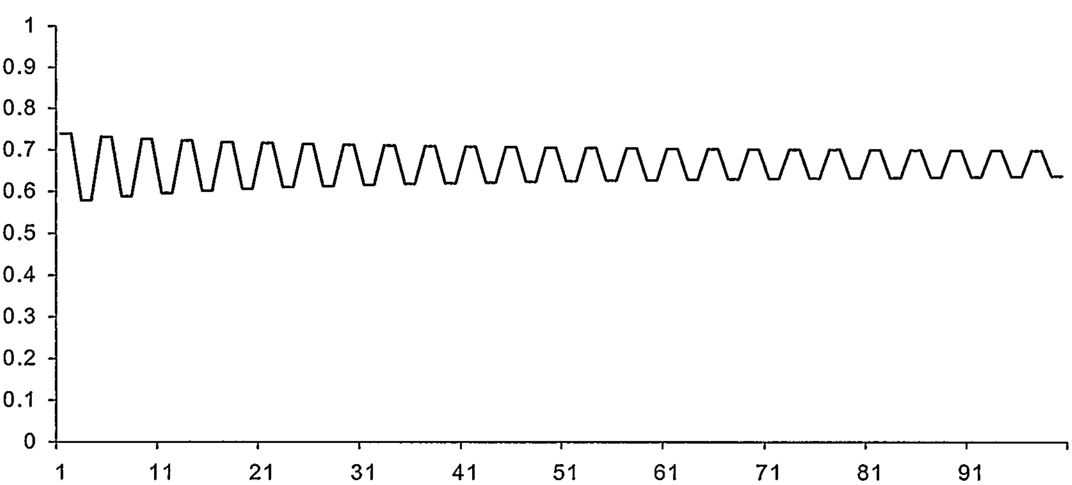
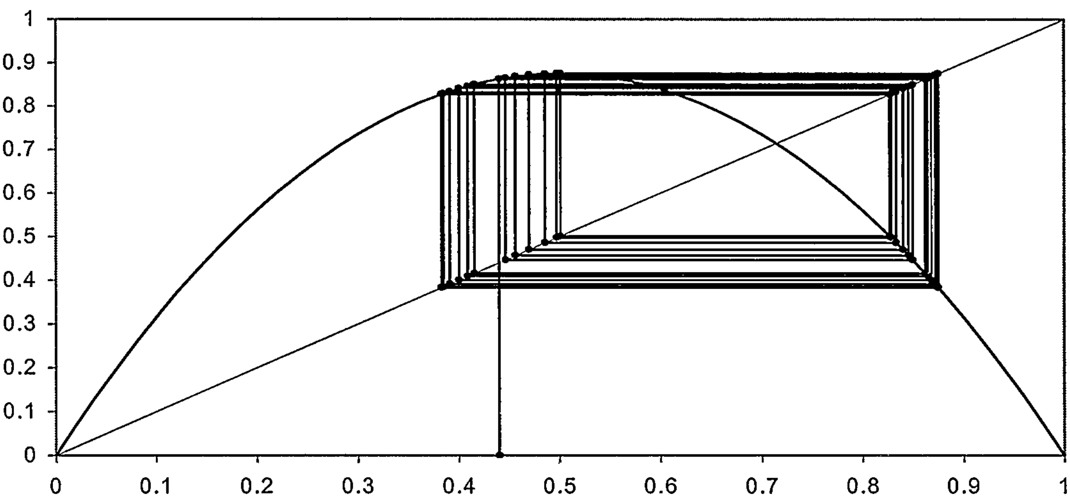


Figure 2.10

Logistic Equation
($\alpha = 3.5,\ X_0 = 0.44$)



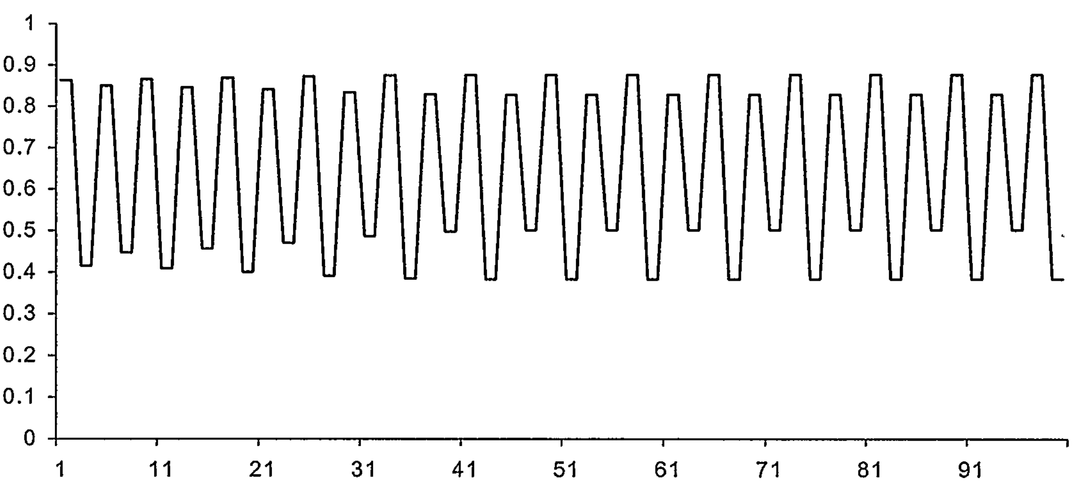
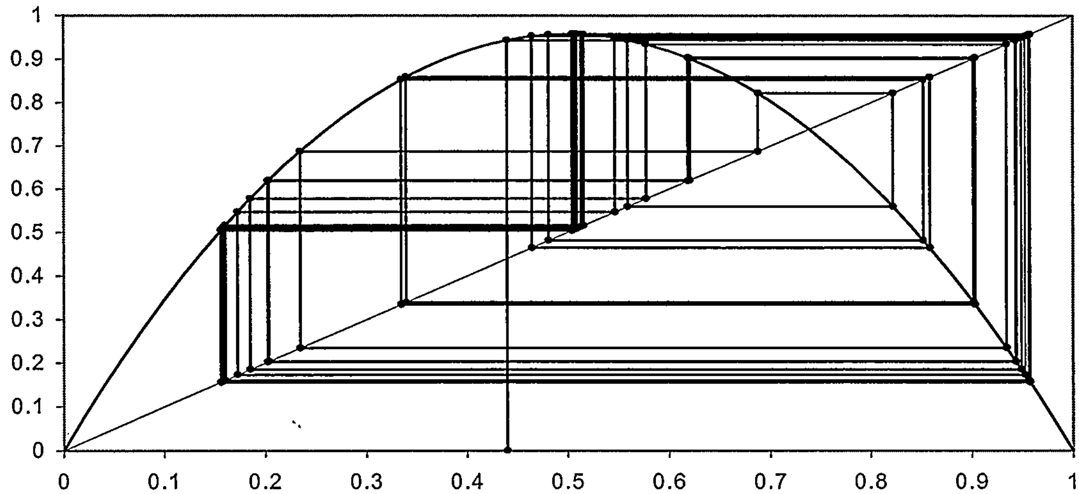


Figure 2.11

Logistic Equation
($\alpha = 3.83$, $X_0 = 0.44$)



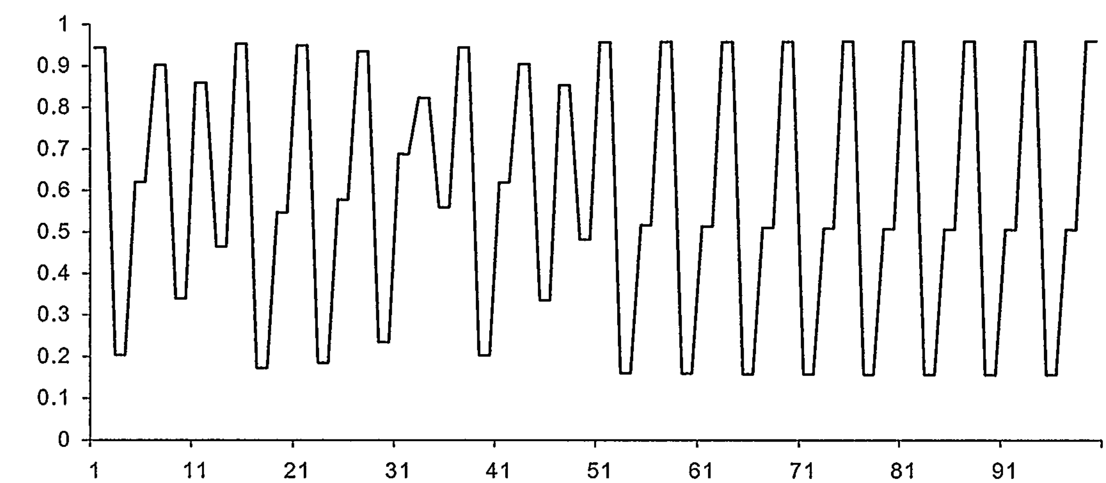
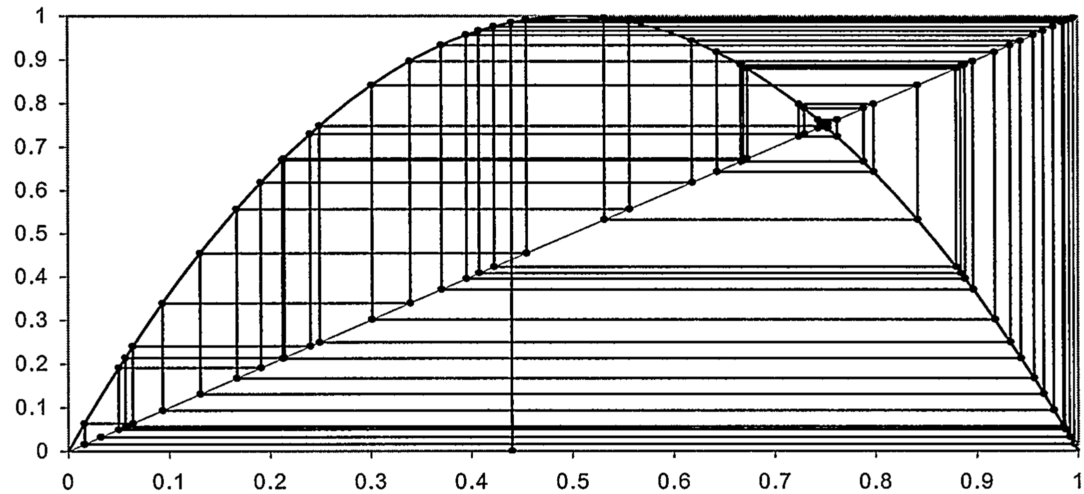


Figure 2.12

Logistic Equation
$(\alpha = 4.0,\ X_0 = 0.44)$



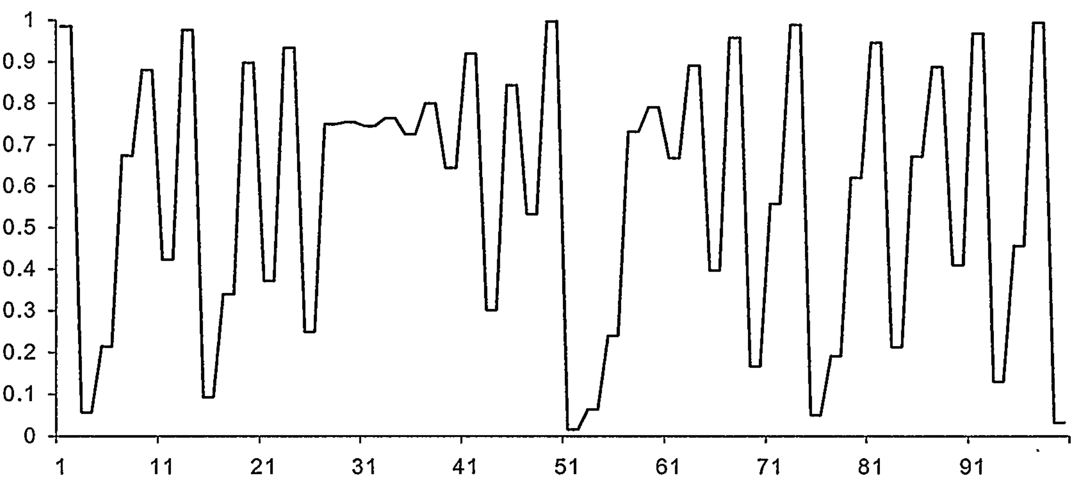
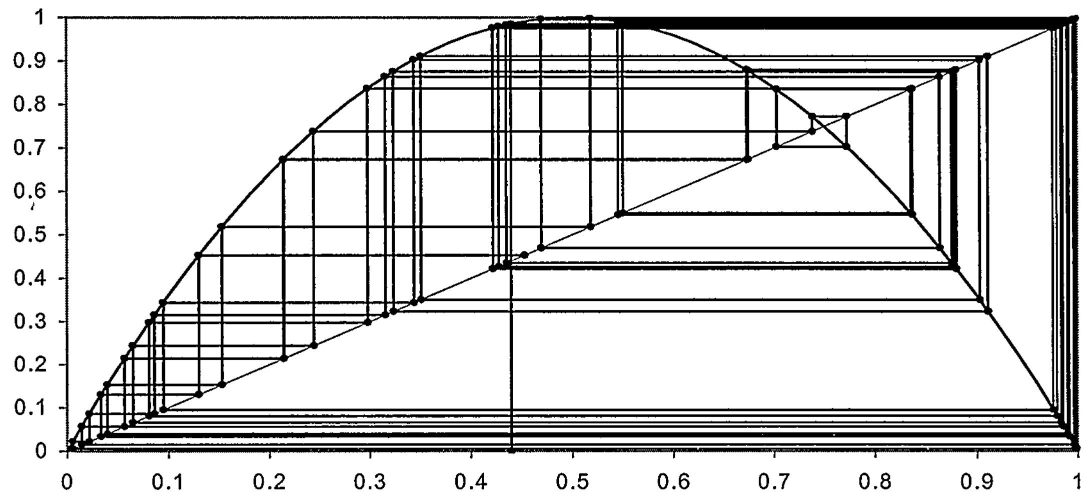


Figure 2.13

Logistic Equation
($\alpha = 4.0,\ X_0 = 0.44001$)



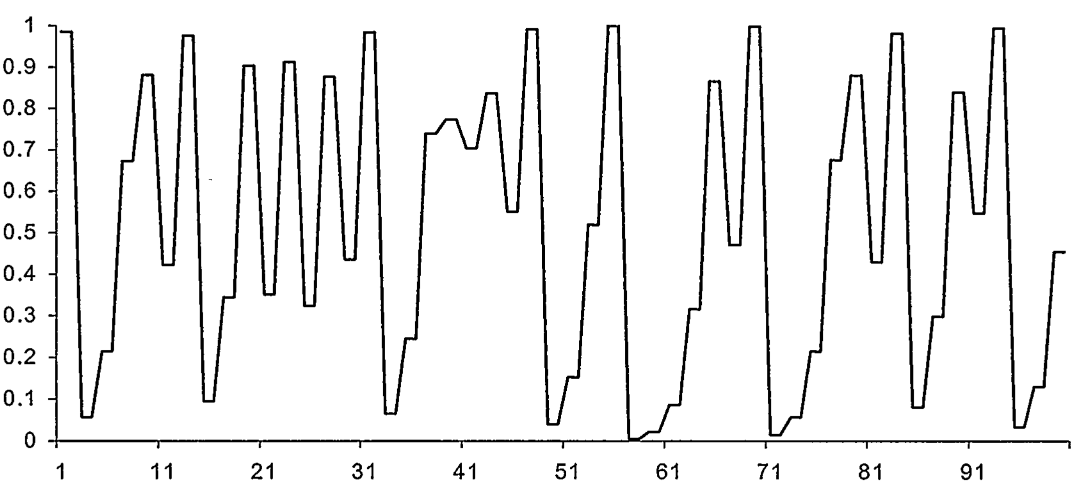


Figure 2.14(a)
(X_0= 0.44000)

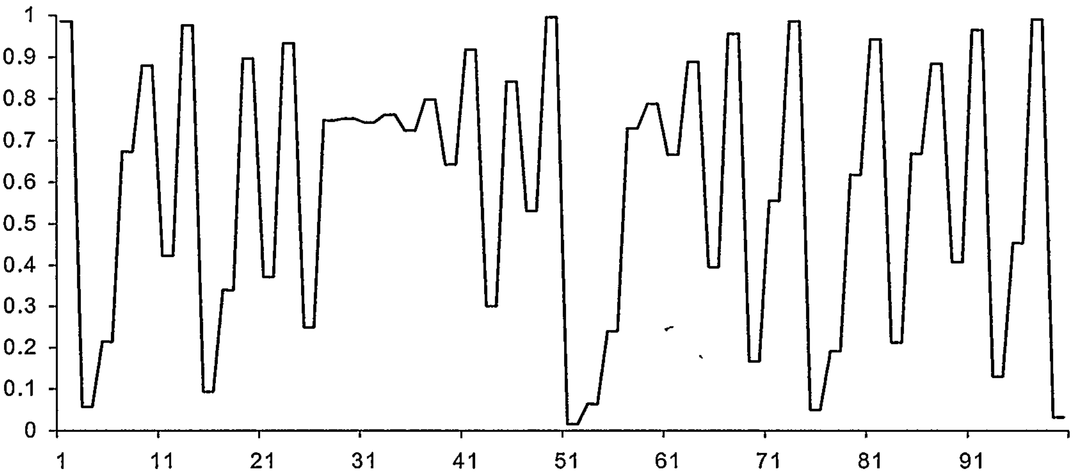


Figure 2.14(b)
(X_0=0.44001)

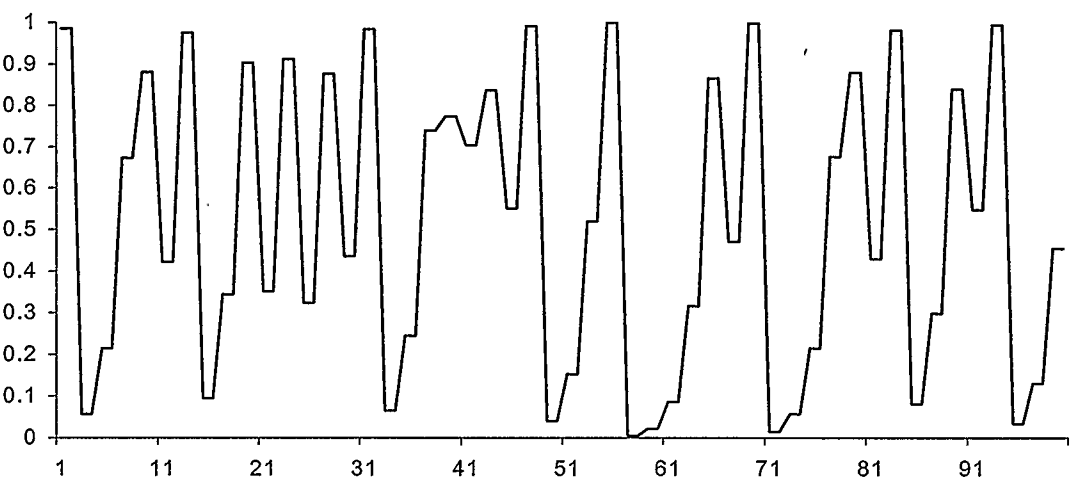


Figure 2.15

Comparison of Logistic equation
(X_0= 0.44000 vs. X_0= 0.44001)

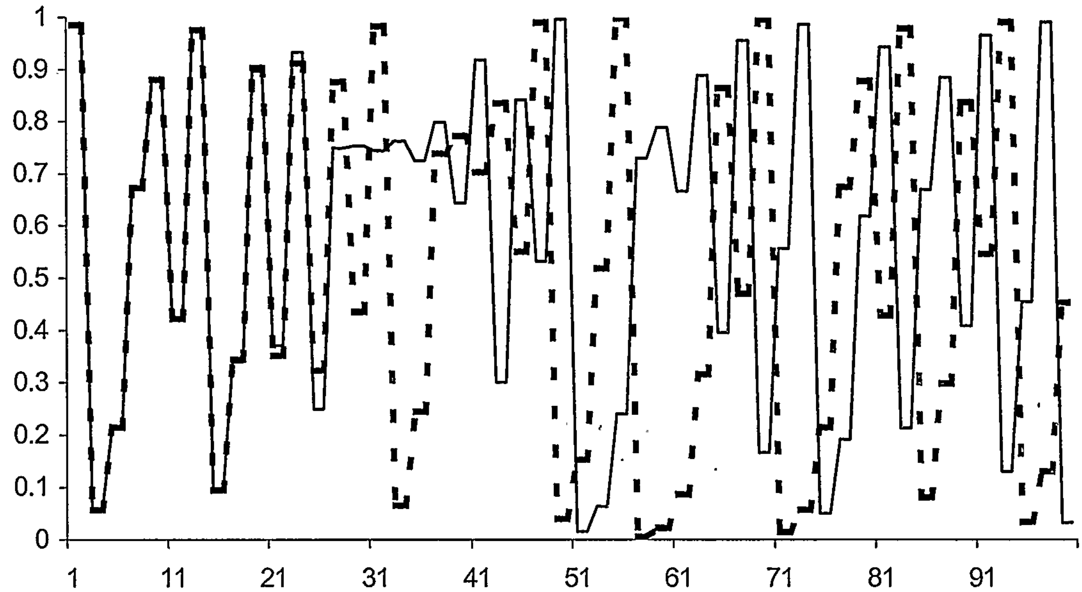
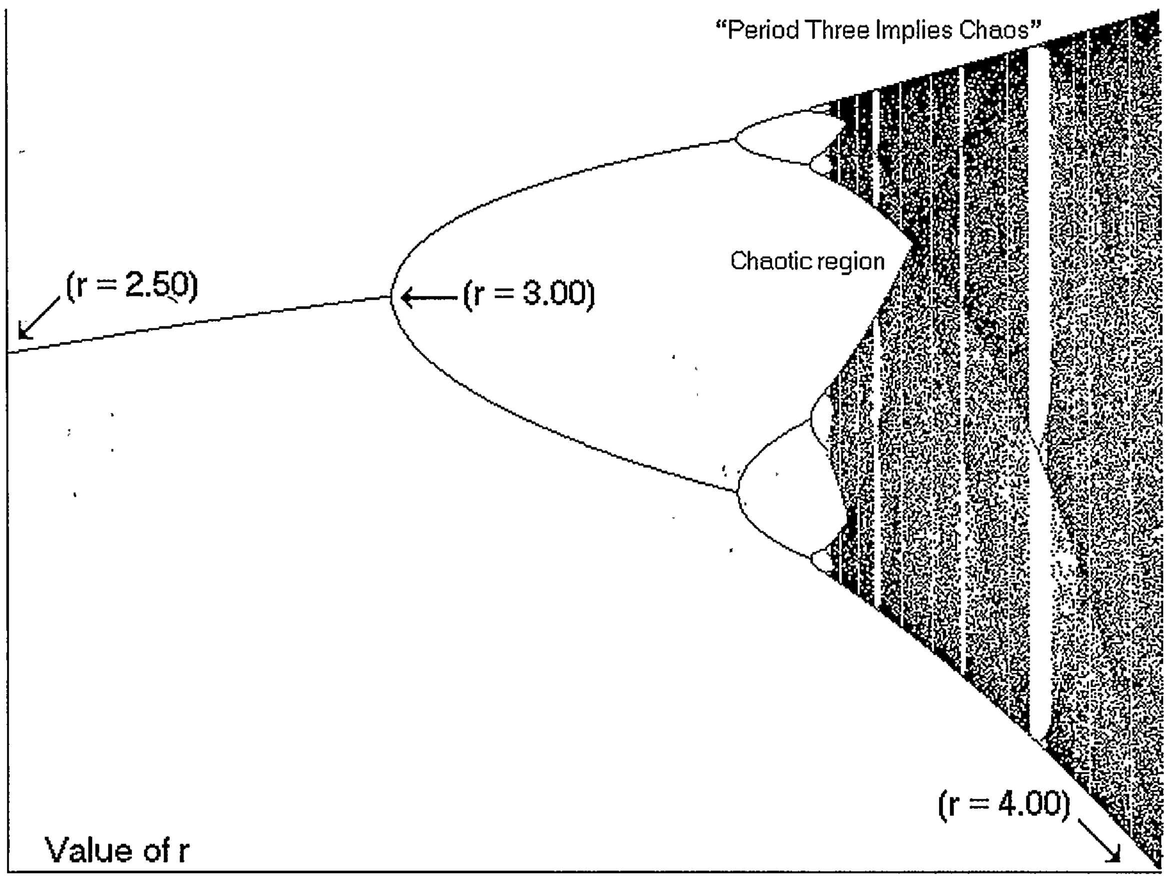


Figure 2.16

Bifurcation Diagram
(r = α)



Chapter 3

OPTIMAL CYCLES AND CHAOS

3.1 Introduction

How relevant is chaos theory to economics? Chapter 2 provided a summary and an introduction to chaos. The aim of this chapter is to offer two examples from the many that exist in the economic literature where chaotic motion has been detected. Section 3.2 describes the Li-Yorke theorem that is used to define and test for chaotic orbits. A Solow (1956) type descriptive growth model will be explained in Section 3.3. Section 3.4 examines an overlapping generations model. Both Sections 3.3 and 3.4 will explain and derive the economic models and show how specific parameters can be adjusted so that chaos results. Finally, Section 3.5 summarizes this chapter.

3.2 The Li-Yorke Theorem

It has been shown in Chapter 2 that no one definition can be attributed to chaos theory. Over the last thirty years, research has produced a variety of definitions and testing methodologies. This can be attributed to increased interest, research and development of the mathematical techniques employed with chaotic behaviour.

The Li-Yorke definition of chaos was initially stated as a theorem in the 1975 article "Period Three Implies Chaos". This section restates this theorem and its implications to the study of chaos theory.

In order to understand the Li-Yorke theorem, one must first examine the Sharovskii (1964) theorem, which developed an ordering for periodic cycles, as follows:

Theorem 3.1: Sharovskii Theorem

Consider the following ordering of integers:

$$
\begin{aligned}
&3 \prec 5 \prec 7 \prec 9 \prec \ldots \\
&\prec 3\cdot 2 \prec 5\cdot 2 \prec 7\cdot 2 \prec 9\cdot 2 \prec \ldots \\
&\prec 3\cdot 2^2 \prec 5\cdot 2^2 \prec 7\cdot 2^2 \prec 9\cdot 2^2 \prec \ldots \\
&. \\
&. \\
&\prec 3\cdot 2^n \prec 5\cdot 2^n \prec 7\cdot 2^n \prec 9\cdot 2^n \prec \ldots \\
&\prec 32 \prec 16 \prec 8 \prec 4 \prec 2 \prec 1
\end{aligned}
$$

where '$\prec$' means precede. If f is a map which has a point x that leads to a p-cycle, then it must have a point leading to a q-cycle for every q that follows p – as per the above ordering.

This theorem provides an order to which periodic cycles will occur in a dynamic system and concludes that if $p \prec q$ (p precedes q) then a system that has a p-cycle will also have a q-cycle. In the extreme, the Sharovskii theorem concludes that if a period three cycle were to be found in a system, then one could conclude that all other cycles are possible, i.e. will also be found in the system. Sharovskii's theorem is supported and expanded on in the Li-Yorke theorem.

Theorem 3.2: Li-Yorke Theorem, Li and Yorke (1975) P.987

Let J be an interval and let F: $J \rightarrow J$ be continuous.

Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfying:

$$d \le a < b < c \; (or \; d \ge a > b > c). \qquad (1)$$

Then

T1: for every $k = 1,2, \ldots$ there is a periodic point in J having period k.

Furthermore,

T2: there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:

(A) For every p, $q \in S$ with $p \neq q$,

$$\limsup_{n\to\infty} \left| F^n(p) - F^n(q) \right| > 0$$

and

$$\liminf_{n\to\infty} \left| F^n(p) - F^n(q) \right| = 0$$

(B) For every $p \in S$ and periodic point $q \in J$,

$$\limsup_{n\to\infty} \left| F^n(p) - F^n(q) \right| > 0 .$$

This theorem reaches the following two conclusions. First, there exist cycles of every magnitude in J. This implies that a time series that is chaotic will exhibit all possible periodic cycles (as defined by the Sharovskii theorem) at different segments of its path. However, the system will not remain in any one periodic cycle for any length of time. It will change to either a different cycle or become irregular. Thus, chaos is a continually changing system with all possible periodic cycles. A small subset of a chaotic system may appear periodic, however examining the whole process results in a different conclusion, many varying cycles. Second, the trajectories of S can move very close to one another but will never overlap and will eventually move away from each other.

The Li-Yorke theorem and specifically equation (1), provides a sufficient condition to test for the presence of chaos in time series data. With $d = F^3(a)$ defining a period three cycle, this implies that when such a cycle is found as part of a larger irregular series (including both periodic cycles and complex patterns) one can conclude that the model is chaotic.

The following two sections will examine economic models that test for chaos using this theorem as a definition and testing methodology.

3.3 Solow's Neoclassical Model

The standard neoclassical growth model as developed by Robert Solow (1956) without depreciation, $\delta = 0$, is:

$$k_{t+1} = s(k_t) \cdot f(k_t)/(1+v), \tag{3.1}$$

where $s(k_t)$ is the savings function per capita, $f(k_t)$ is the production function, and v is the rate of growth of the population.

Richard Day in his 1982 article "Irregular Growth Cycles" uses this model to test for chaos using the Li-Yorke theorem. Day's approach was to define k^m as the maximum attainable capital labour ratio:

$$k^m = s(k^*) \cdot f(k^*)/(1+v), \tag{3.2}$$

where k^* maximizes $s(k) \cdot f(k)$, the steady state.

Day (1982) has assumed that $k^m > k^*$ and takes the smallest root of $s(k) \cdot f(k)/(1+v) = k^*$, calling it k^c. Based on this the Li and Yorke (1975) sufficient condition for chaos, equation (1) of Theorem 3.2, as adapted by Day (1982) becomes:

$$s(k^m) \cdot f(k^m)/(1+v) \le k^c < k^* < k^m. \tag{3.3}$$

This equation provides the condition to test for the presence of chaos in the optimal growth model. Simply put, there is some maximum attainable capital labour ratio, k^m, where all other non-repeating orbits are possible, as is the case with period three implying chaos from the Li-Yorke theorem. Day (1982) examined two cases where chaos occurred using alternative savings and production functions.

The first case introduces a pollution effect term to a Cobb-Douglas production function. This term causes an increase in the concentration of capital according to the following, $(m-k)^\gamma$. This term will approach one as γ approaches zero, so long as *m* and *k* are not equal and it decreases in value when *m* approaches *k*. With this modification, the standard Cobb-Douglas production function becomes:

$$f(k) = Bk^\beta (m-k)^\gamma .$$

Assuming a constant saving rate (*s*) the Solow (1956) model becomes:

$$k_{t+1} = sBk_t^\beta (m-k_t)^\gamma / (1+v). \quad (3.4)$$

There are two possibilities where chaos can occur. This requires that the parameters of equation (3.4) attain certain values. First, if the values of β and γ are positive and nonzero then the sufficient condition of the Li-Yorke theorem is feasible. Day (1982) found that satisfying equation (3.3) is possible thus proving chaos. The second case examines the situation where β = γ = m = 1. Equation (3.4) can now be reduced to:

$$k_{k+1} = sk_t (1-k_t)/(1+v).$$

This equation appears similar to the logistic difference equation that was introduced in Chapter 2, where $\alpha = s/(1+v)$. Thus, based on our knowledge of the logistic difference equation, one could naturally conclude that for the appropriate values of $\alpha = s/(1+v)$ a chaotic orbit would result. Day (1982) found irregular investment cycles for values of $\alpha = s/(1+v)$ between 3.57 and 4.00. Overall, he concluded that chaos is possible for this case.

The second case uses a variable saving rate, which depends on income (y), wealth (k), and the real interest rate (r). It is characterized by the following equation:

$$s(k)y = a\left(1 - \tfrac{b}{r}\right)k. \tag{3.5}$$

This function states that per capita savings is proportional to wealth and that savings will increase with real interest rate supporting standard economic theory. The real interest rate is defined as $r = \beta y/k$. Substituting the real interest rate into the savings function and replacing y with the Cobb-Douglas production function, will produce the following difference equation:

$$k_{t+1} = (a/(1+v))\left[1 - \left(b/\beta B\right)k_t^{1-\beta}\right]k_t \tag{3.6}$$

where $\alpha = a/(1+v)$, as per the logistic difference equation of Chapter 2.

It can be shown that increasing the savings parameter, a, will result in higher ordered cycles. An interval can be found for a where the sufficient conditions for chaos, as defined by equation (3.3), are present. Day (1982) stated, without specifics, that there are ranges of values that affect the savings function and can result in chaotic growth. He thus provided a second case where chaos can be detected in the Solow (1956) growth model.

In this section, the standard neoclassical growth model as developed by Solow (1956) has been examined. Day (1982) has found that there are modifications that can be performed to the parameters of this model that will result in chaotic behaviour. He concludes the following:

- With constant savings and a Cobb-Douglas production function – no irregular cycles were found – no chaos.
- Adding a pollution effect to the production function causes changes to the technology parameter (β), specific values for which can be found to exhibit irregular cycles or chaos.
- By altering the savings rate to that of a variable function, Day stated that there are some values for the savings parameter, a, which can cause irregular growth cycles – chaos.

Both cases where chaos was discovered were reduced to a difference equation of the form introduced in Chapter 2, the logistic difference equation. An alternative proof that chaos is possible is that under the appropriate conditions the logistic difference equation will result in chaos (shown in Chapter 2). Day (1982) has shown that the Solow (1956) optimal growth model can be reduced to a difference equation and chaos can be found under specific conditions.

3.4 The Overlapping Generations Model

Samuelson (1958) originally introduced the overlapping generations (OLG) model. The advantage of this model, over a Solow type model, is that it represents an infinite time horizon for individuals with finite two period lives. As an economic representation of consumption, the OLG model has been studied extensively using various approaches.

In this section, the model as developed by Benhabib and Day (1982), will be examined. It will be shown that for specific utility functions chaotic behaviour can be detected.

The OLG model starts with the following utility:

$$U\left(c_0(t), c_1(t+1); r_t, \omega_0, \omega_1\right)$$

where: $U\left(c_0(t), c_1(t+1)\right)$ = the utility function of an agent over the two periods of their life, youth and old age,

ν = population growth rate,

$c_0(t)$ = a representative agents consumption for the t^{th} time period in their youth where they are employed (assumed $c_0(t) \geq 0$),

$c_1(t+1)$ = a representative agents consumption for the $(t+1)^{th}$ time period in their old age where retirement occurs, (assume $c_1(t+1) \geq 0$ and agents live two periods, t and $t+1$)

ω_0 = initial endowment in their youth,

ω_1 = initial endowment in their old age, and

r_t = an interest factor that represents the rate of exchange of present for future consumption.

The OLG model will have the following budget constraint:

$$c_1(t+1) = \omega_1 + r_t\left[\omega_0 - c_0(t)\right] \tag{3.7}$$

where $\omega_0 - c_0(t)$ is the amount saved in ones youth. The savings of the working youth are spent on consumption in old age. There are no bequests for the next generation.

The market clearing equilibrium or the materials balance constraint as stated by Benhabib and Day (1982) is:

$$(1+v)[\omega_0 - c_0(t)] + \omega_1 - c_1(t) = 0 \tag{3.8}$$

where $\omega_1 - c_1(t)$ is the savings of the old in the t^{th} period and $\omega_0 - c_0(t)$ is the savings of the youth for the same period, adjusted for population growth. Together the above two equations define the feasible programs of the economy.

The consumption vector $(c_0^*(t), c_1^*(t+1))$ defines all consumption, which will maximize the utility of the t^{th} generation subject to the budget constraint (3.7). Thus, one can define the following optimality condition:

$$U(c_0^*(t), c_1^*(t+1); r_t, \omega_0, \omega_1) = U^*(r_t, \omega_0, \omega_1). \tag{3.9}$$

This leads to the following definition as stated in Benhabib and Day (1982):

Definition 3.1:

A pure exchange equilibrium trajectory is a sequence of vectors $(r_t, \omega_0, \omega_1)_{t=1}^{\infty}$ such that, $\forall t$, $U(c_0(t), c_1(t+1); r_t, \omega_0, \omega_1) = U^*(r_t, \omega_0, \omega_1)$ and (3.8) holds.

There are two possibilities that a youth in this model can follow. The Classical approach is where the youth borrow from the old in order to increase their consumption. This borrowed amount will be repaid in their old age. Second, is the Samuelson case the youth will save to increase their consumption in old age. This will effectively smooth one's consumption over their life. The discount factor r is the market clearing condition that ensures the savings (dissavings) of the youth are equal to the dissavings (savings) of old age. Gale (1973) concluded that the Samuelson case is the more logically consistent of the two possibilities, since the goal of most individuals is to save for retirement and old age. However, the Classical case is actually more representative of the real world.

One might question dividing an individuals life into just two periods representing working and nonworking years. Many consumers in their youth are initially borrowers but later become savers. This leads one to conclude that more than two periods may be more appropriate to describe the savings structure of a representative consumer. However, the additional time period will increase the complexity and underlining mathematics of the model, thus, staying with the two periods.

Prior to stating the actual problem, the following two assumptions are required (these are assumptions 1 and 2 of Lemma 3.1). First, it is assumed that the utility function has the following properties: it is strictly concave, twice differentiable, is increasing in its arguments, and the function will be either separable or homothetic. Second, the utility maximizing problem will be an interior solution.

The consumer's problem can now be stated as:

$$\max_{c_0, c_1} \quad U\left(c_0(t), c_1(t+1); r_t, \omega_0, \omega_1\right),$$

$$\text{subject to} \quad c_1(t+1) = \omega_1 + r_t\left[\omega_0 - c_0(t)\right]. \tag{3.7}$$

The Lagrangian (L) becomes:

$$L = U\left(c_0(t), c_1(t+1)\right) + \lambda\left[c_1(t+1) - \omega_1 - r_t\left(\omega_0 - c_0(t)\right)\right].$$

This results in the first order conditions (FOC):

$$U_0\left(c_0(t), c_1(t+1)\right) + \lambda\left[r_t\right] = 0, \tag{1}$$

$$U_1\left(c_0(t), c_1(t+1)\right) + \lambda[1] = 0, \tag{2}$$

where $U_i\left(c_0(t), c_1(t+1)\right)$ is the i^{th} argument derivative (for $i = 1,2$) of the utility function.

By solving the FOC (1) and (2) above, we obtain the following Euler equation:

$$r_t = \frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))}.$$

Substituting the above into the budget constraint (3.7) we get:

$$c_1(t+1) = \omega_1 + \frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))}[\omega_0 - c_0(t)] .$$

and therefore

$$\frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))} = \frac{\omega_1 - c_1(t+1)}{c_0(t) - \omega_0}. \tag{3.10}$$

Gale (1973) developed a lemma that assists with solving (3.10) and is stated here in the form used by Benhabib and Day (1982). The purpose of this lemma is to provide the tools necessary to calculate a difference equation for intergenerational consumption.

Lemma 3.1: (from Gale (1973))

In the classical case using the assumptions 1 and 2, condition (3.10) can be solved uniquely for $c_1(t+1)$. Call this function $c_1(t+1) = G(c_0(t); \omega_0, \omega_1)$.

Define the constrained marginal rate of substitution (CMRS) to be:

$$V(c_0(t); \omega_0, \omega_1) = \frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))}$$

Substitute this equation into the material balances constraint (3.8), using the following procedure. The material budget constraint can be written as:

$$(1+v)[\omega_0 - c_0(t)] + \omega_1 - c_1(t) = 0,$$

which when t is replaced by t+1 becomes:

$$(1+v)[\omega_0 - c_0(t+1)] + \omega_1 - c_1(t+1) = 0.$$

Isolate c_1 on the left-hand side and we get:

$$c_1(t+1) = (1+\nu)[\omega_0 - c_0(t+1)] + \omega_1,$$

then substituting this into the budget constraint (3.7) results in the following:

$$c_0(t+1) = \omega_0 + \frac{1}{1+\nu}[r_t(c_0(t) - \omega_0)],$$

$$c_0(t+1) = \omega_0 + \frac{1}{1+\nu}[V(c_0(t);\omega_0,\omega_1)(c_0(t) - \omega_0)] = f(c_0(t)).$$

Thus, a difference equation has been derived, similar to the logistic difference equation introduced in Chapter 2. This equation shows that once intergenerational trade or transfer of consumption has been started it will continue uninterrupted. The difference equation derived above characterizes the exchange equilibrium trajectory for the Classical case.

The Samuelson case, where the youth save for their old age, will not be examined here, as it will eventually converge to a situation where there is no trade equilibrium. As such, chaos is not possible.

Benhabib and Day (1982) examined three cases where chaos can be found. Each case is dependent on the utility function used. The first case uses the following concave utility function:

$$U(c_0, c_1) = A - e^{a[1-((c_0-\omega_0)/a)]} + c_1$$

where: ω_0 is the youth endowment and a, A are constants greater than zero. The exponential term is the MRS (marginal rate of substitution), and $\nu = 0$, i.e. there is no population growth.

The consumer's problem can be restated as:

$$\max_{c_0, c_1} \qquad U(c_0, c_1) = A - e^{a[1-((c_0-\omega_0)/a)]} + c_1$$

subject to $$c_1(t+1) = \omega_1 + r_t[\omega_0 - c_0(t)]. \tag{3.7}$$

The Lagrangian for this problem is:

$$L = \{\, A - e^{a[1-((c_0-\omega_0)/a)]} + c_1 \} + c_1(t+1) - \omega_1 - r_t[\omega_0 - c_0(t)],$$

and the FOC are:

$$\lambda = -\tfrac{1}{r_t} e^{a[1-((c_0-\omega_0)/a)]}, \tag{1}$$

$$\lambda = -1. \tag{2}$$

By equating (1) and (2) above, the Euler equation is found to be:

$$r_t = e^{a[1-((c_0-\omega_0)/a)]},$$

which, when substituted into the budget constraint (3.7) yields:

$$c_1(t+1) = \omega_1 + e^{a[1-((c_0-\omega_0)/a)]}[\omega_0 - c_0(t)]. \tag{3}$$

Rewrite the material budget constraint with no population growth ($\nu = 0$):

$$[\omega_0 - c_0(t)] + \omega_1 - c_1(t) = 0,$$

or, more conveniently,

$$c_1(t) - \omega_1 = \omega_0 - c_0(t).$$

This equation can be substituted into equation (3) above and results in:

$$c_0(t+1) - \omega_0 = e^{a[1-((c_0-\omega_0)/a)]}(c_0(t) - \omega_0).$$

To simplify this equation into a form that is identifiable, let $r = e^{a t}$ and $x_t = [c_0(t) - \omega_0]$, and write it as:

$$x_{t+1} = r x_t e^{-x_t},$$

This is a difference equation. Benhabib and Day (1982) found chaotic behaviour as characterized by the Li and Yorke (1975) theorem for values of $a > 2.692$ (or $r > 14.765$).

The second case is for the following quadratic type utility equation:

$$U(c_0(t), c_1(t+1)) = ac_0 - \tfrac{1}{2}bc_0^2 + c_1,$$

where *a, b > 0,* $0 \le c_0 \le \frac{a}{b}$, $(\omega_0, \omega_1) = (0, \overline{\omega})$, $\overline{\omega} > a/b$, $c_0 < \overline{\omega}$, and there is no population growth ($\nu = 0$). Using the same methodology of deriving the Lagrangian (*L*) as in the first case results in the following difference equation:

$$c_0(t+1) = ac_0(t)[1 - (a/b)c_0(t)].$$

Benhabib and Day (1982) found chaos for $a \in [3.53, 4]$.

The third and final utility function is of the form:

$$U(c_0, c_1) = \frac{\lambda(c_0 + b)^{1-a}}{1-a} + c_1,$$

where $a, b \ge 0$, $a \ne 0$,and $\lambda > 0$. Once again the process used in the first case is employed resulting in the following difference equation:

$$c_0(t+1) - \omega_0 = \frac{\lambda}{1+\nu} \frac{1}{([c_0(t) - \omega_0] + k)^a} (c_0(t) - \omega_0),$$

where $k = b + \omega_0$, and ν = population growth. In this final case, Benhabib and Day (1982) found chaos for values of $a \ge 5$ and $\lambda \ge 50$.

In this section, the OLG model has been introduced and shown to have chaotic behaviour. The research of Benhabib and Day (1982) has derived and explained how chaos can be found in economic models dependent on their parameters and the utility functions used to describe the problem. Three different utility functions were shown to produce solutions that were difference equations. Benhabib and Day (1982) also concluded that the parameters involved would have certain values that resulted in chaotic paths.

Thus, as was stated in Chapter 2, a logistic difference equation under the appropriate conditions will yield a chaotic orbit. This section has shown that the OLG model can be reduced to a simple difference equation and with certain parameter values will result in chaotic behaviour.

3.5 Summary

This chapter has accomplished two goals. First, it has provided the Li and Yorke (1975) definition of chaotic orbits. Their theorem states that irregular paths that exhibit a period three cycle as part of a time series are considered to be chaotic. Second, two examples of economic growth models have been introduced and explained. The Solow (1956) and Overlapping Generations Model have been shown to have chaotic paths for specific parameters as per the Li-Yorke theorem. These models were reduced to a difference equation and found to have chaotic behaviour under the appropriate conditions.

Some examples of chaos found in economic models have been examined and stated. The remainder of this thesis will examine a specific data set and will attempt to determine if it is chaotic. The specific data set and methodology used will be presented and will be the focus of the following two chapters.

Chapter 4

Unit Roots

4.1 Introduction

One of the assumptions of the Lyapunov exponent, stated in the next chapter, is that the time series being tested for chaos is stationary, i.e. does not have a unit root. Nonstationarity or a unit root is a common problem found in economic time series. Having a unit root will skew the testing results of the parameters of an ordinary least squares regression. One possible solution to this problem is to use the first differences of the time series. Chapter 2 demonstrated that differencing a series, which was illustrated using the logistic difference equation, might appear to be similar to chaos. Thus, prior to testing for chaos one should test for stationarity. If a unit root is found than, chaos should be tested using the first differences of the time series.

This chapter includes the following sections. Section 4.2 will define a unit root and the testing procedure for its detection. Section 4.3 will describe the Gross Domestic Product (GDP) time series to be used in the remainder of this thesis. Section 4.4 will use the methodology introduced in Section 4.2 to test this GDP series for a unit root and summarizes the results. Finally, Section 4.5 concludes the chapter.

4.2 Defining Stationarity

This section will introduce and define stationarity. To accomplish this, we start with the following basic definitions. Gujarati (1995) defines a stationary stochastic process as one where the mean and the variance of the observations of a data set are not dependent on time and the covariance between two periods depends only on the distance or lag between these periods. According to Verbeek (2000), this definition is that of a weak stationary process. He provides a definition for strict stationarity, which not only requires that the mean and variance not change over time, but also that their individual distributions are identical.

The definition used in this chapter is weak stationarity. Thus, nonstationarity occurs when the mean or variance have changed over time. For example, a time series is nonstationary if the mean of some initial subset of observations differs from that of latter values. This simple approach, however, may not be the most scientific method to test for stationarity. The generally accepted test to determine stationarity was developed by Dickey and Fuller (1979). To understand their methodology begin with the following autoregressive equation:

$$Y_t = \rho Y_{t-1} + \varepsilon_t \,, \tag{4.1}$$

where Y_t is the observation for the t^{th} period, ε_t is white noise ($\sim N\left(0,\sigma^2\right)$, white noise is normally distributed with mean zero and constant variance, σ^2), and ρ is the coefficient to be determined using regression analysis. By subtracting Y_{t-1} from both sides, the following results:

$$Y_t - Y_{t-1} = \rho Y_{t-1} - Y_{t-1} + \varepsilon_t \,,$$

$$\Delta Y_t = (\rho - 1) Y_{t-1} + \varepsilon_t \,,$$

$$\Delta Y_t = \delta Y_{t-1} + \varepsilon_t \,, \tag{4.2}$$

where Δ is the difference operator and $\delta = (\rho - 1)$. The above conversions can be done without affecting the equations properties or its results; they are invariant to linear transformations.

Nonstationarity is present in the time series $\{Y_t\}$ if either $\rho = 1$ for equation (4.1) or $\delta = 0$ in (4.2). If either of these conditions are tested and found to occur, then one may conclude a time series is nonstationary.

Alternative descriptive terms for nonstationarity include: unit root (i.e. $\rho = 1$) or integrated of order *d*, *I(d)*, where $d > 0$. Order of integration defines the number of times differencing is required for a series to become stationary. Therefore, an *I(1)* series will be stationary when differenced once and a process that is integrated of order zero, *I(0)*, is stationary.

The order of integration has an important affect on how an exogenous shock affects a time series. A system that is integrated of order zero, will experience no lasting effect due to an external shock, it is temporarily. Whereas, a process that is integrated of order one, will be permanently altered by an exogenous shock. This *I(1)* process has an error term with a variance directly related to the number of observations in the series, i.e. equal to $n\sigma^2$. Therefore, exogenous shocks to an economic time series will permanently affect a nonstationary time series, whereas, a stationary series will experience a temporary affect.

The unit root test requires that one regress the variable Y_t or ΔY_t on Y_{t-1}, this can be accomplished using any standard statistical programming package. These packages will produce an estimate of the coefficient on Y_{t-1}, ρ or δ depending on which of the equations (4.1) or (4.2) are used. The contribution of Dickey and Fuller (1979) to the problem of stationarity was their discovery that for the null hypothesis, $H_0 : \hat{\rho} = 1$ or $H_0 : \hat{\delta} = 0$, the standard t test ratio does not have a t distribution. This fact is especially true if the data is nonstationary. Thus, comparing the calculated t statistic to its critical value is inappropriate and will skew the results. Dickey and Fuller (1979) developed their own critical values, which are used to test the null hypothesis of a unit root. Failure to use the Dickey-Fuller (DF) critical values will result in an increased frequency of rejection of the null hypothesis.

Summarizing to this point, to test for the presence of nonstationarity or a unit root in a time series, one simply regresses Y_t on Y_{t-1}, then compare the DF critical values with the t test calculated value, i.e. tests the null hypothesis, $H_0 : \hat{\rho} = 1$. Rejecting the null hypothesis implies that the time series is stationary. Failure to reject the null hypothesis indicates that the series nonstationary.

There are two additional potential problems that may be encountered when testing for stationarity. First, the trend of the time series may be due to a deterministic process rather than a unit root. A series with this characteristic is known as a trend stationary (TS) process. The addition of a trend variable, t, transforms the data set to a stationary series. Alternatively, a series that requires differencing to achieve stationarity are known as a difference stationary (DS) process. A second potential problem is that the error term (ε_t) associated with the regression may not be white noise (i.e. ε_t is not distributed according to $N(0, \sigma^2)$). This second problem is overcome by adding some optimal number of lagged independent variable (Y_{t-i}) to the regression equation. The exact number of lags is dependent on the methodology used to determine optimality.

These problems can be simultaneously solved through the use of the Augmented Dickey-Fuller (1981) test (ADF), which is defined by the following equation:

$$\Delta Y_t = \beta_1 + \beta_2 t + \delta Y_{t-1} + \alpha_i \sum_{i=1}^{m} \Delta Y_{t-i} + \varepsilon_t, \qquad (4.3)$$

where t is the trend variable and m is the number of lags. Many statistical packages have specific methods for determining the optimal lag length. Shazam chooses the optimal lag length between one and the square root of the number of observations for the lowest Akaike Information Criteria (AIC). The AIC will be discussed further in Section 4.4 when two different methodologies will be compared.

The ADF equation solves the problem of whether a time series is trend or difference stationary. Adding m lagged independent variables ensures that the statistical properties of the error term are standard normal (with mean zero and constant variance). Rejection of the null hypothesis implies that the time series tested is a trend stationary (TS) process. Whereas, failure to reject the null hypothesis indicates that it is a difference stationary (DS) process.

This section has defined the methodology, originally developed by Dickey and Fuller (1979,1981) for detecting a unit root in a time series. Using the Augmented Dickey Fuller equation (4.3), one can determine if a time series is stationary. Rejection of the null hypothesis implies that the process is trend stationary. Whereas, if the null hypothesis is not rejected, the time series is difference stationary and taking the first differences will transform the data set into a stationary series.

4.3 Data

The data used in this thesis was originally sourced from Maddison (1982), and is for the following ten countries: Australia, Canada, Denmark, France, Germany, Italy, Norway, Sweden, the United Kingdom (UK) and the United States of America (US). It consists of 115 observations of the log real per capita GDP and the first differences of the log real per capita GDP, from 1870 to 1985.

The log real per capita GDP are graphically shown at the end of this chapter for: Australia (Figure 4.1), Canada (Figure 4.2), Denmark (Figure 4.3), France (Figure 4.4), Germany (Figure 4.5), Italy (Figure 4.6), Norway (Figure 4.7), Sweden (Figure 4.8), the United Kingdom (Figure 4.9) and the United States of America (Figure 4.10). The first differences of the log real per capita GDP are summarized graphically in Figure 4.11.

This data set was also used by Serletis (1994) to test for unit roots. The remainder of this chapter will support his conclusions that all of the ten series have a unit root and that the first differences are stationary.

4.4 Empirical Results

This section will summarize the results of the Augmented Dickey Fuller (ADF) test that has been performed on the log real per capita GDP for the ten countries described in the previous section.

The GDP data has been tested using the statistical package Shazam. The results for the ADF tests are summarized in Tables 4.1 at the end of this chapter. There is a discrepancy between the Shazam results and those stated in Serletis (1994). The optimal lag lengths chosen are not the same. This difference is due to variations in methodologies of the statistical programs.

Serletis' (1994) criteria for the optimal lag length was accomplished by working backwards from a maximum of twelve lags (for the ADF equation) and choosing the lag with the t statistic in the autoregression that was, in absolute value, greater than 1.6 and that had a next higher ordered autoregression less than 1.6.

The criteria used in the Shazam program is to choose the highest significant lag, using a 95 percent confidence interval, based on the autocorrelation function or the partial autocorrelation function of the first differenced series. Shazam evaluates all lags between zero and k ($k = \sqrt{n}$ and n = number of observations).

Serletis (1994) has concluded that one could not reject the null hypothesis of a unit root in any of the ten countries. Shazam supports these results and derives the optimal lag length where one cannot reject the same null hypothesis. One may conclude that, for all ten countries, the log real per capita GDP is a nonstationary time series.

It can also be shown for each of the ten countries, evaluated between one and twelve lagged lengths that a unit root is present. This result supports the methodology used by Serletis (1994). For the data of all countries except the US, the null hypothesis is not rejected for all twelve lags, at levels of significance - one, five, and ten percent. The US result shows that the longer the lag length, the more likely the null hypothesis cannot be rejected. That is to say, as the lag increases, the calculated t statistic moves further into the acceptance region. Overall, one cannot reject the null hypothesis for the US. Thus, the US time series is nonstationary. One may conclude that for all lags from one to twelve, the time series of all ten countries are nonstationary.

Table 4.2 shows the results of the ADF tests on the first differenced log real per capita GDP for all ten nations. If the time series is actually a difference stationary process, then these ADF tests should reject the null hypothesis of the first differences. Table 4.2 concludes that the null hypothesis may be rejected for all countries except Italy – which is close to the rejection region for a 90 percent confidence interval. The time series for Italy may not be a difference stationary process but a trend stationary process – analysis that is beyond the scope of this thesis.

Overall, the log real per capita GDP for the ten countries is integrated of order one. By using the ADF test, it has been shown that these countries have a difference stationary process and by taking their first differences, they can be made stationary.

4.5 Summary

This chapter has provided an introduction to the concept of stationarity and introduced a method by which one can test for its presence. The DF and ADF equations were stated and the testing criterion for the null hypothesis was explained. In either case, the failure to reject the null hypothesis implies that a unit root is present. Using the ADF test, one is able to differentiate between a trend stationary and a difference stationary process. The ADF test ensures that the error term has the appropriate statistical properties. With the ADF test, failure to reject the null hypothesis implies that the time series is a difference stationary process.

The results of using the ADF test on the log real per capita GDP data set taken from Maddison (1982) proved that all ten countries have a unit root and are a difference stationary process. The correct procedure to overcome this problem is to conduct analysis on the first differences of the time series.

Figure 4.11 summarizes the first differences of all ten nations. Close examination of these plots reminds one of the logistic difference plot introduced in Chapter 2, especially plots where the coefficient was close to four. As stated in Chapter 2, chaos mimics white noise and the first differences in Figure 4.11 may exhibit this characteristic. It remains to be determined whether the first differences of the log real per capita GDP of the ten countries is white noise or whether it is mimicking it and actually is chaos. To answer this question, the next chapter will introduce the Lyapunov exponent as a technique to test for chaos.

Figure 4.1

Log Real Per Capita GDP - Australia

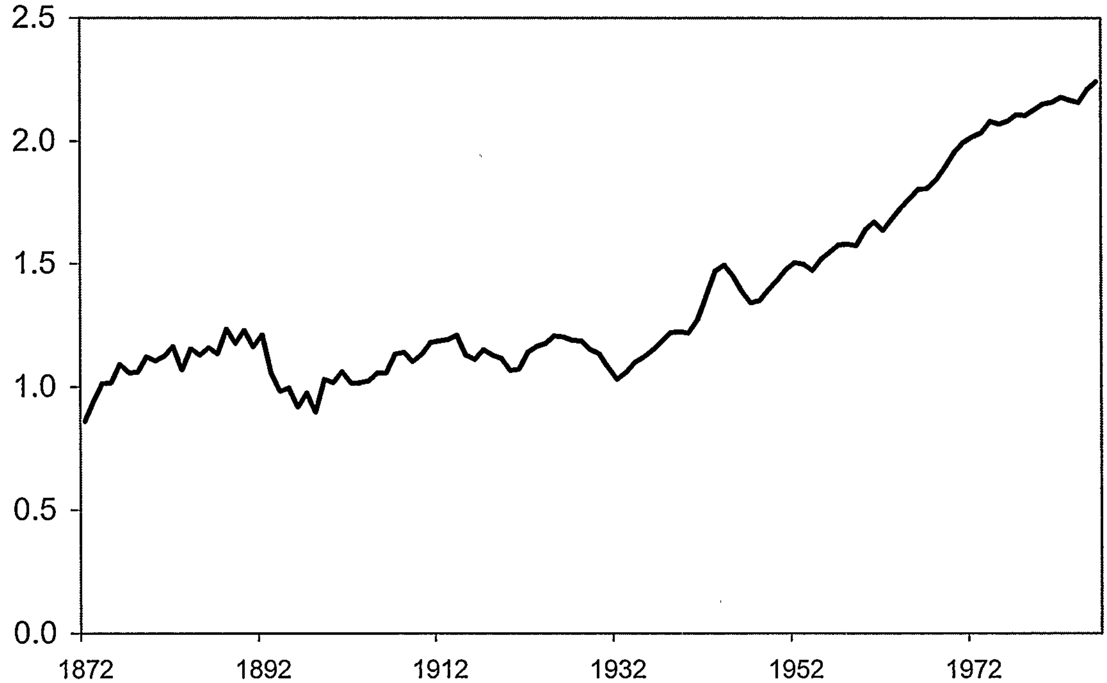


Figure 4.2

Log Real Per Capita GDP - Canada

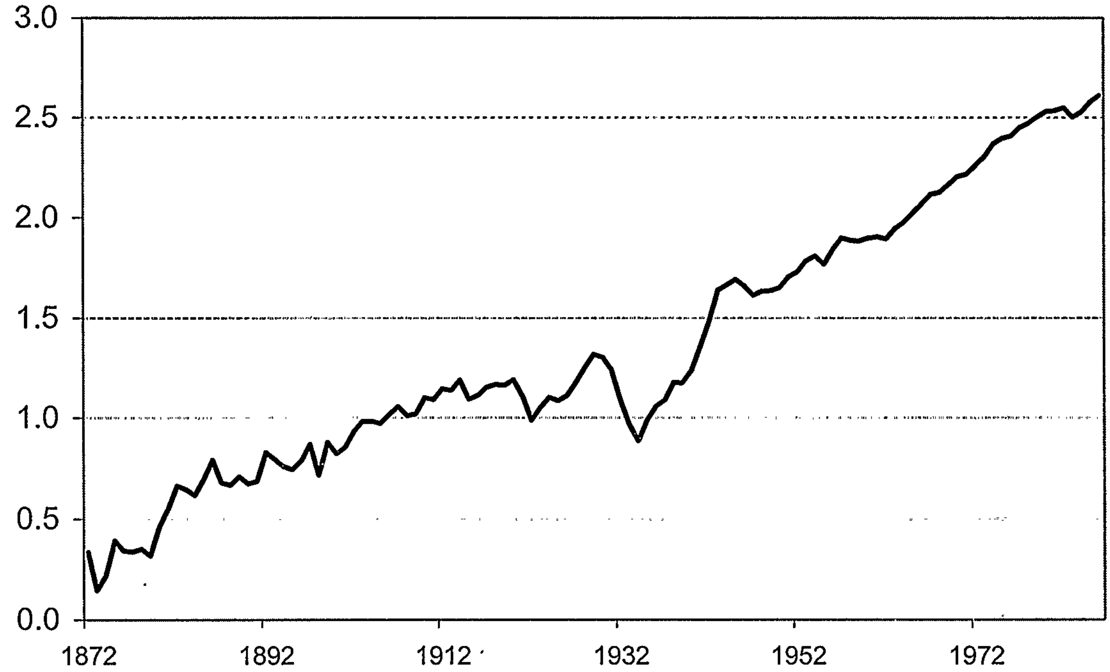


Figure 4.3

Log Real Per Capita GDP – Denmark

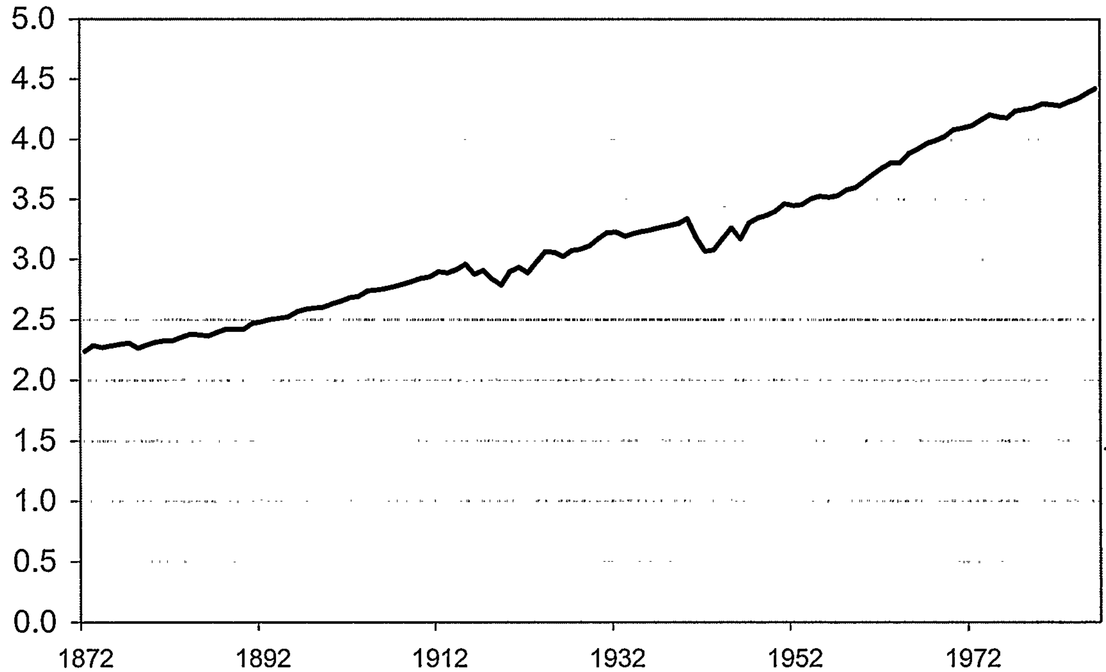


Figure 4.4

Log Real Per Capita GDP – France

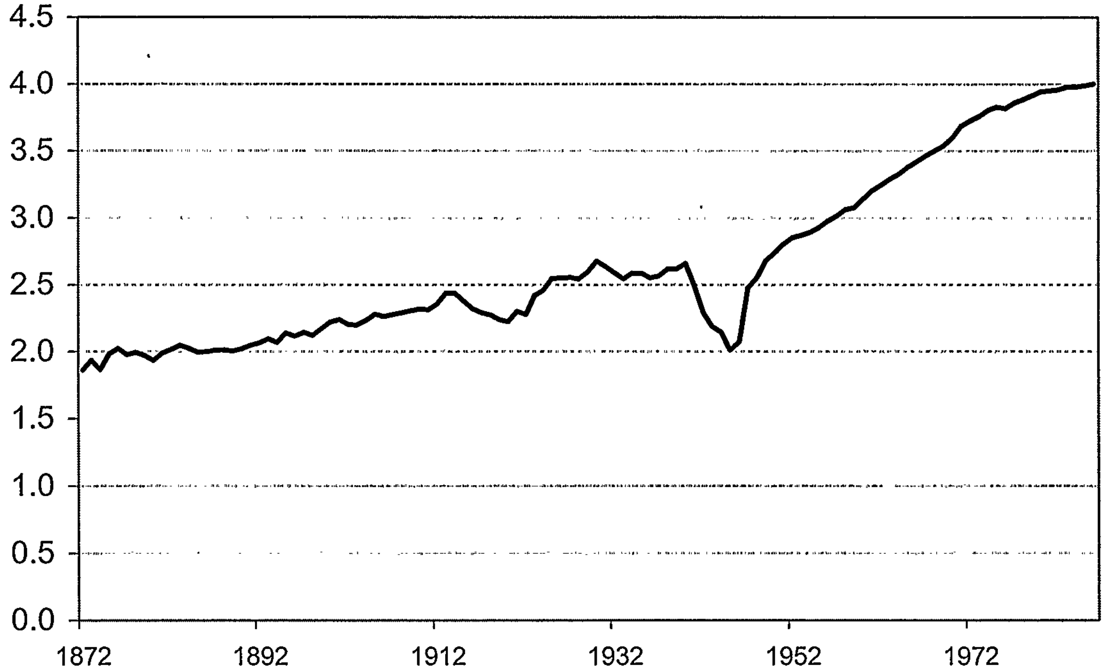


Figure 4.5

Log Real Per Capita GDP – Germany

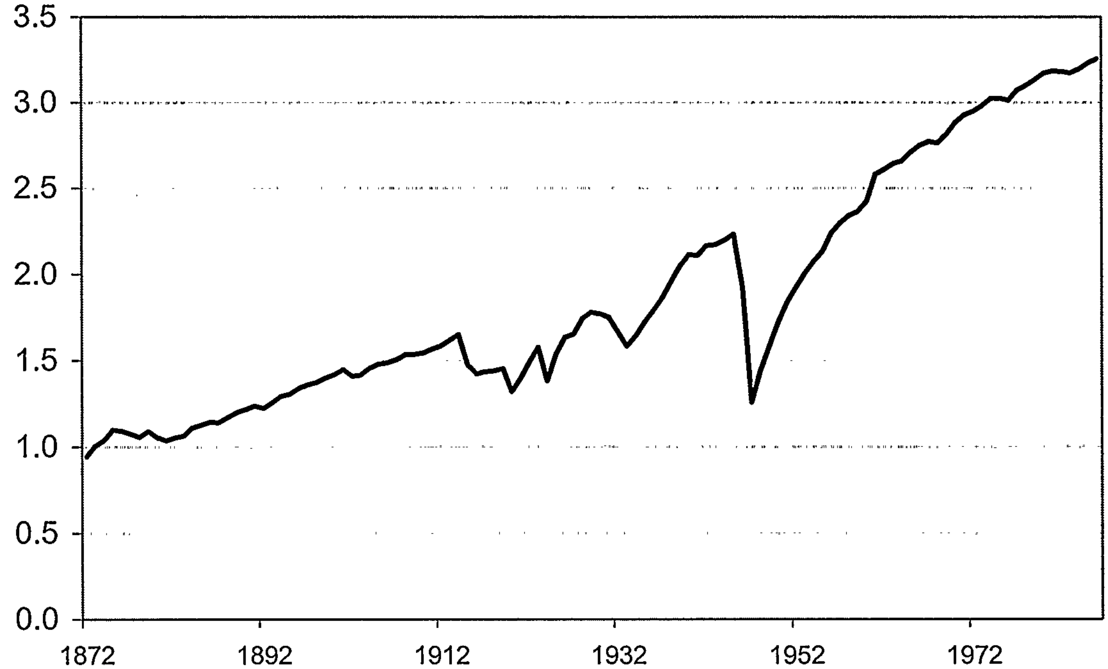


Figure 4.6

Log Real Per Capita GDP – Italy

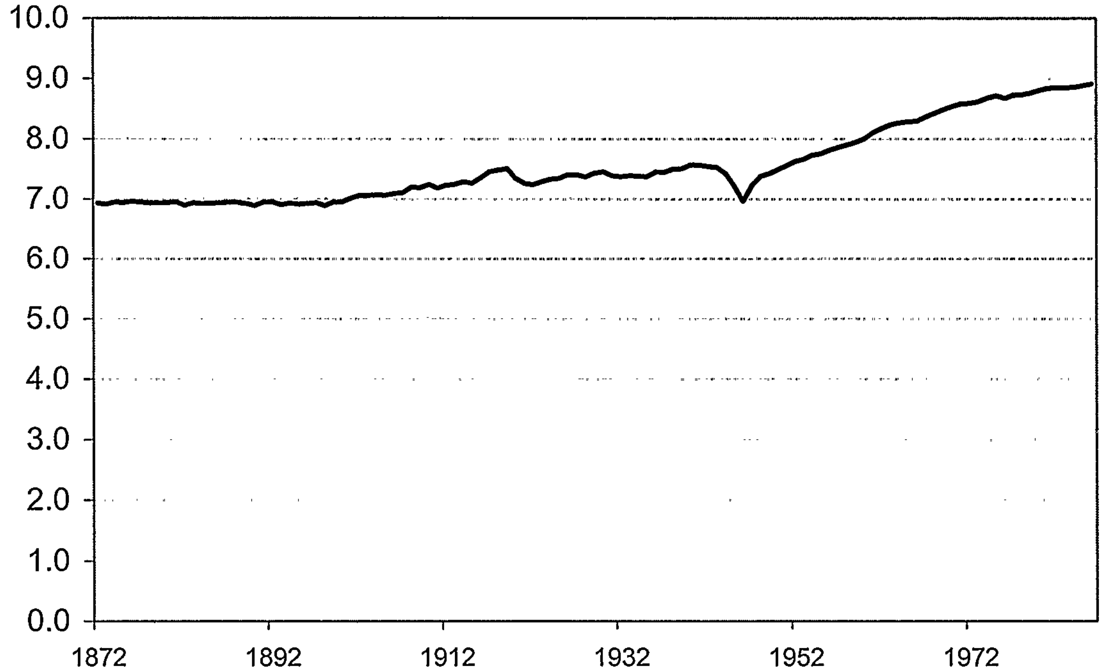


Figure 4.7

Log Real Per Capita GDP – Norway

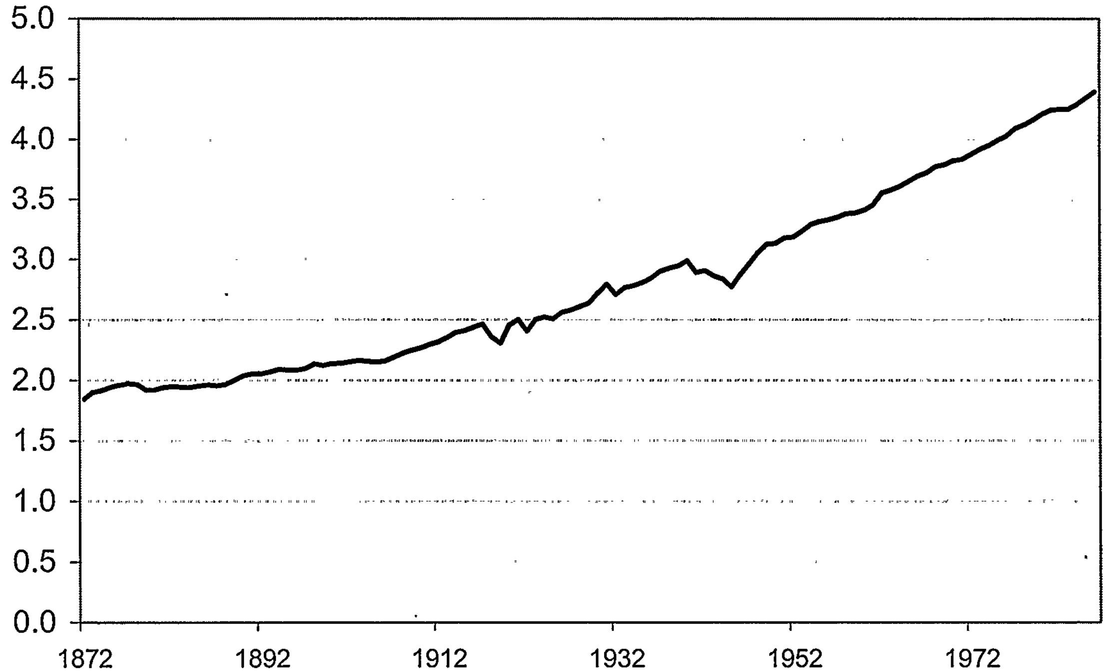


Figure 4.8

Log Real Per Capita GDP – Sweden

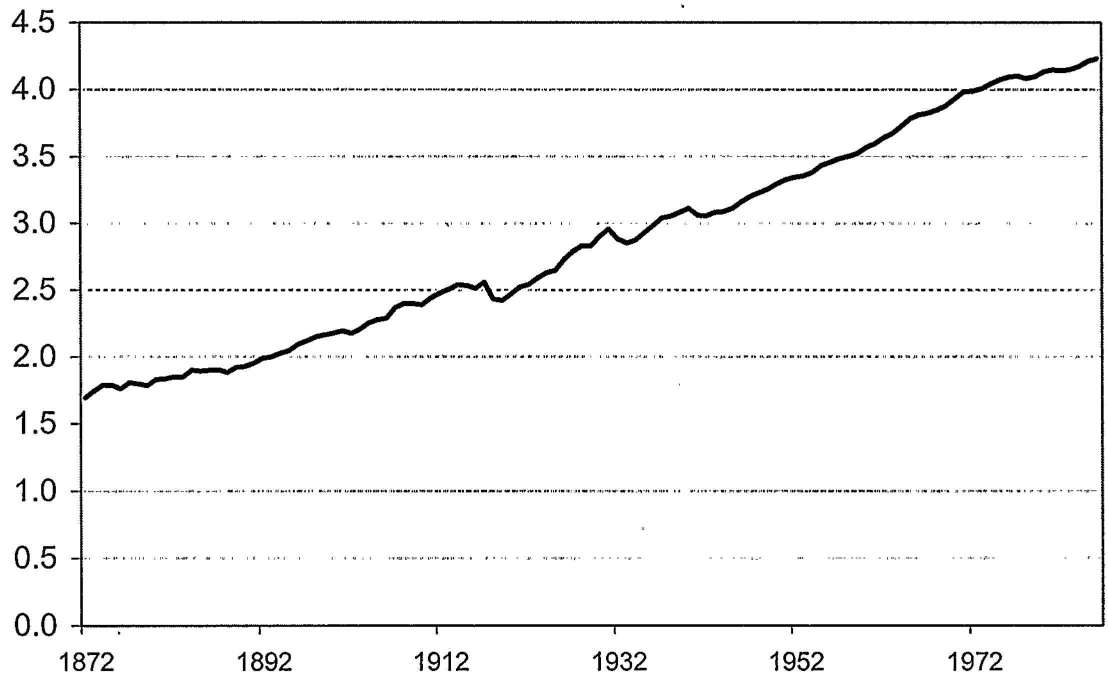


Figure 4.9

Log Real Per Capita GDP – United Kingdom

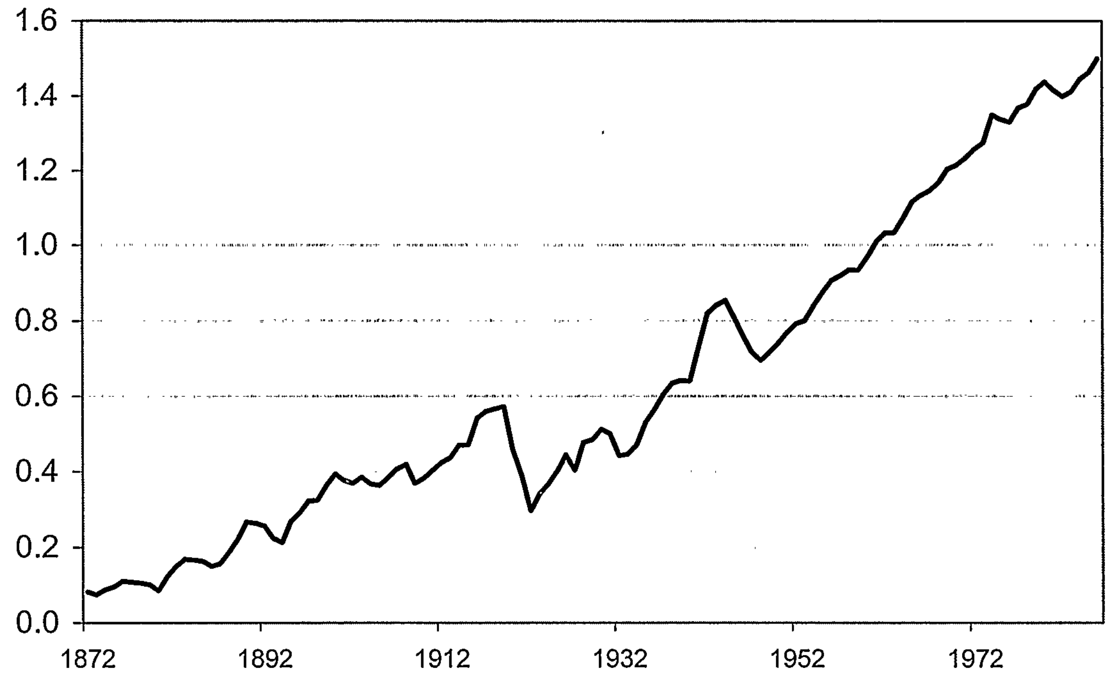


Figure 4.10

Log Real Per Capita GDP – United States

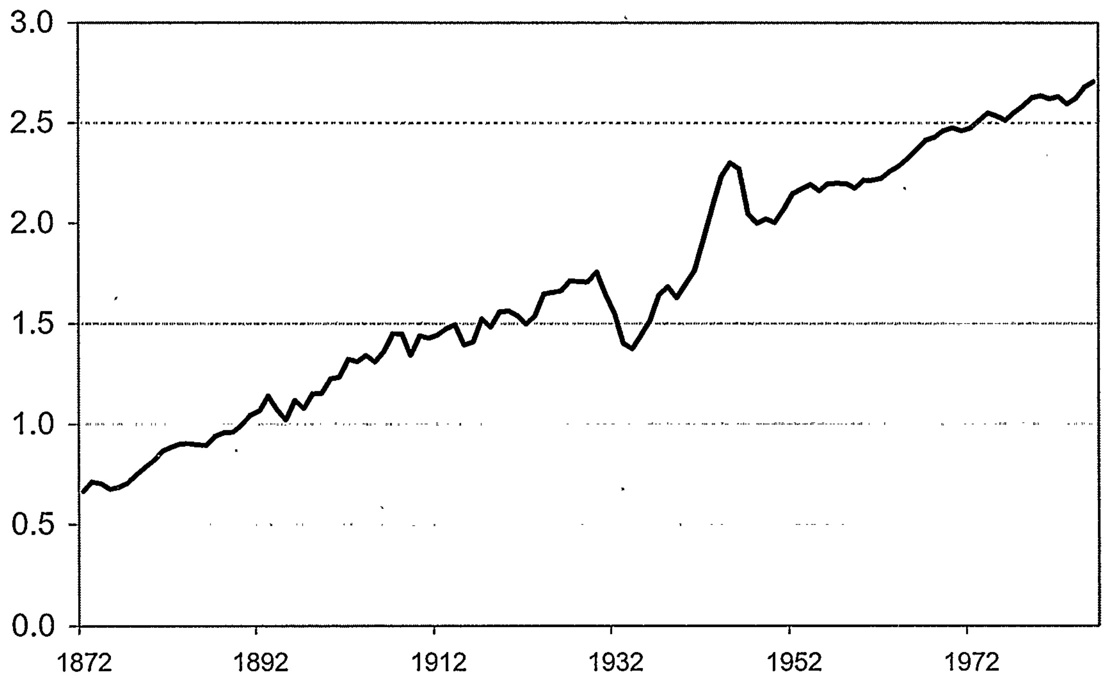


Figure 4.11

First Differences

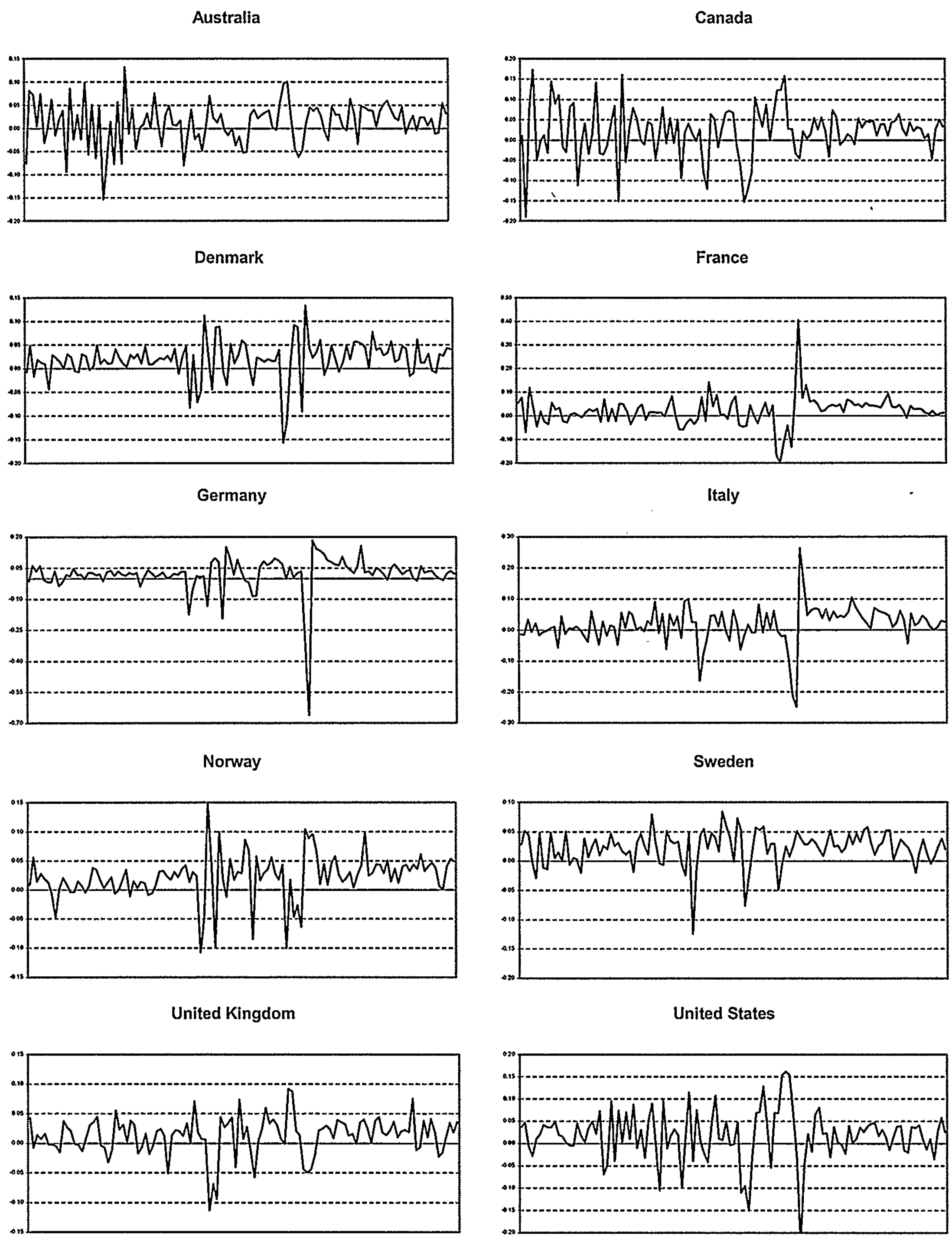


Table 4.1

Summary of ADF Test Results

Critical Values for all ADF Tests:

	@10%	@5%	@1%
Critical Value	**-3.1300**	**-3.41**	**-3.96**

All countries - letting Shazam Pick Optimal Lag

Country	No. of Lags	T stat	AIC	SC
Australia	2	-1.2040	-6.2450	-6.1230
Canada	9	-0.8843	-5.6540	-5.3500
Denmark	2	-1.3151	-6.4480	-6.3260
France	1	-1.6898	-5.6260	-5.5290
Germany	1	-2.5443	-4.8630	-4.7660
Italy	1	-1.5249	-5.6890	-5.5920
Norway	0	-0.9097	-6.5580	-6.4860
Sweden	9	-2.4916	-7.0610	-6.7570
UK	4	-1.1820	-6.9850	-6.8130
USA	9	-3.0331	-5.7820	-5.4780

Table 4.2

ADF Test of 1st Differences for all Countries letting Shazam Pick Optimal Lag

	@10%	@5%	@1%
Critical Value	**-3.13**	**-3.41**	**-3.96**

Country	No. of Lags	T stat	AIC	SC	Significant@ (%)
Australia	10	-3.7964	-6.1660	-5.8350	10,5
Canada	10	-3.2280	-5.6640	-5.3330	10
Denmark	6	-4.2265	-6.3670	-6.1440	10,5,1
France	8	-3.2217	-5.5640	-5.2880	10
Germany	3	-5.5297	-4.7900	-4.6440	10,5,1
Italy	10	-2.7163	-5.4950	-5.1640	Nil
Norway	8	-4.4980	-6.5030	-6.2270	10,5,1
Sweden	8	-3.4436	-7.0190	-6.7430	10,5
UK	10	-3.4813	-6.9580	-6.6280	10,5
USA	5	-4.6525	-5.7390	-5.5420	10,5,1

Chapter 5

Lyapunov Exponent as a Direct Test For Chaos

5.1 Introduction

The previous chapter accomplished two tasks. First, it introduced the GDP data set and second it tested this data for stationarity. It will be shown later in this chapter that in order to use the Lyapunov exponent a sample should be stationary. This chapter will examine the Lyapunov exponent as a method by which a data set may be found to have the characteristic of chaotic behaviour.

Eckmann and Ruelle (1985) were the first to use the characteristic of sensitive dependence on initial conditions (SDIC) as a definition for chaos. The calculation of the Lyapunov exponent provides the average exponential divergence (if >0) or convergence (<0) between the trajectories that start from initial points that are infinitesimally close together. The Lyapunov exponent calculates a value stating whether an orbit or path has SDIC, the main characteristic of chaotic behaviour. Thus, the Lyapunov exponent and its ability to test for the presence of SDIC have made it the accepted method for the determination of chaotic behaviour.

There are three methods by which one can derive the Lyapunov exponent. First, the direct method, developed by Wolf *et al* (1985), was found to work well with data sets that were large in size and with no stochastic noise. This method was found to have a tendency to over estimate the presence of chaos. These shortcomings led to the development of other methodologies. Gencay and Dechert (1992) developed a method using neural networks to calculate all possible Lyapunov exponents. The third method, by Nychka *et al* (1992) expanded on the previous two. This approach employed the Jacobian method, based on a nonparametric neural network estimator and was found to work better with small data sets that contained stochastic noise. The method derived by Nychka *et al* (1992) will be used in this thesis.

The remainder of this chapter is organized as follows. The next section explains how to calculate the estimate of the Lyapunov exponent. Section 5.3 derives the asymptotic distribution properties of this estimate and the resulting test statistic. Section 5.4 uses the results from the previous two sections, applies them to the GDP data set and discusses the results. Finally, section 5.5 concludes this chapter.

5.2 Lyapunov Exponent

This section will introduce the estimation procedure used in this thesis to derive the Lyapunov exponent. Eckmann and Ruelle (1985) suggested using a nonparametric regression, known as the Jacobian method. This approach is an alternative to the direct method of Wolf *et al.* (1985). The Jacobian method has two advantages over the direct method. It is more suitable with data sets containing stochastic noise and of small sample size. The Jacobian method can, by definition, use any nonparametric regression estimator. Nychka *et al.* (1992) and Gencay and Dechert (1992) further developed the Jacobian method to include using neural networks as their nonparametric regression estimators. This approach is the most commonly used method of deriving the point estimators of the Lyapunov exponent. Its main advantage over other estimators is that neural nets are less sensitive to increases in the dimension of the model. The only shortcoming was that confidence intervals could not be derived. Since, the asymptotic distribution properties of the Lyapunov exponent were not known. This limitation will be examined in the next section.

Since economic series are small in size and include stochastic noise, the remainder of this chapter and thesis will only focus on the Jacobian method of Nychka *et al* (1992) using neural networks as their nonparametric estimators.

In order to derive the Lyapunov exponent let $\{X_t\}_{t=1}^{T}$, where T is the sample size of a random scalar sequence that is generated by the following nonlinear autoregressive model:

$$X_t = \theta(X_{t-1}, X_{t-2}, \ldots, X_{t-m}) + u_t, \tag{5.1}$$

where $\theta : R^m \rightarrow R$ is a nonlinear dynamic map and $\{u_t\}_{t=1}^{T}$ is a random sequence of identically and independently distributed random samples, with $E(u_t) = 0$, $E(u_t^2) = \sigma^2 < \infty$ and σ^2 is a constant. If $\sigma^2 = 0$, (5.1) reduces to a deterministic system and if the Lyapunov exponent is greater than zero (an indication of chaos), then Nychka *et al* (1992) referred to this as deterministic chaos. Otherwise, moderate values of σ^2 with a Lyapunov exponent greater than zero are referred to as noisy chaos.

It has been assumed that θ satisfies a smoothness condition (θ is a target function in the parameter space with a finite absolute first moment) and $Z_t = (X_t, \ldots, X_{t-m+1}) \in R^m$ is a strictly stationary and continuous (additional assumptions beyond the scope of this thesis can be found in Shintani and Linton (2000)). Shintani and Linton (2000) allow Equation (5.1) to be expressed in terms of the map:

$$F(Z)_t = (\theta(X_{t-1}, X_{t-2}, \ldots, X_{t-m}), X_{t-1}, \ldots, X_{t-m+1})', \tag{5.2}$$

which results in:

$$Z_t = F(Z_{t-1}) + U_t,$$

where $U_t = (u_t, 0, \ldots, 0)$

Let the Jacobian (J_t) of the map F in (5.2) evaluated at Z_t be defined as:

$$J_t = \frac{\partial F(Z_t)}{\partial Z'} = \begin{bmatrix} \Delta\theta_{1t} & \Delta\theta_{2t} & \cdots & \Delta\theta_{m-1,t} & \Delta\theta_{mt} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{5.3}$$

for $t = 0,1,\ldots,M-1$ and where

$$\Delta\theta_{jt} = D^{e_j}\theta(Z_t)$$

for $j = 0,1,\ldots,m$ and where $e_j = (0,\ldots,1,\ldots,0) \in R^m$, is the j-th elementary vector of dimension m. The right hand side of the above equation can be expanded into the following:

$$D^{e_j}\theta(Z_t) = \frac{\partial^{e_j}\theta(Z_t)}{\partial Z_1^{e_1},\cdots,\partial Z_m^{e_m}},$$

which is defined to be the $e_j - th$ order derivative of $\theta(Z_t)$.

The dominant Lyapunov exponent of the system stated in (5.1) can now be defined as:

$$\lambda \equiv \lim_{M\to\infty} \frac{1}{2M} \ln v_1(T_M' T_M) \tag{5.4}$$

where:

$$T_M = \prod_{t=1}^{M} J_{M-t} = J_{M-1} \cdot J_{M-2} \cdot\cdots\cdot J_0 \tag{5.5}$$

and $v_i(\cdot)$ is the i-th largest eigenvalue of a matrix. One should note that m, as seen in (5.1) to (5.3), is the dimension of the time series, that is, the number of lags in the autoregressive process. Whereas, M used in the definition of the Lyapunov exponent (5.4), is the block length. The block length is the number of points that are used to evaluate the Lyapunov exponent, where $M \leq T$ and $T=$ the number of observations.

The estimate of the Lyapunov exponent in this thesis will be based on the Jacobian method. This method uses a nonparametric regression analysis, which was introduced by Eckmann and Ruelle (1985). This formulation requires that the θ of the Jacobian in (5.3) be replaced by its nonparametric estimator $\hat{\theta}$. The Lyapunov exponent estimator of λ is:

$$\hat{\lambda} = \frac{1}{2M} \ln v_1\left(\hat{T}_M' \hat{T}_M\right) \tag{5.6}$$

where

$$\hat{T}_M = \prod_{t=1}^{M} \hat{J}_{M-t} = \hat{J}_{M-1} \cdot \hat{J}_{M-2} \cdot \cdots \cdot \hat{J}_0 \tag{5.7}$$

with

$$\hat{J}_t = \frac{\partial \hat{F}(Z_t)}{\partial Z'} = \begin{bmatrix} \Delta\hat{\theta}_{1t} & \Delta\hat{\theta}_{2t} & \cdots & \Delta\hat{\theta}_{m-1,t} & \Delta\hat{\theta}_{mt} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{5.8}$$

for $t = 0,1,\ldots,M-1$ and where

$$\Delta\hat{\theta}_{jt} = D^{e_j}\hat{\theta}(Z_t)$$

for $j = 0,1,\ldots,m$ and where $e_j = (0,\ldots,1,\ldots,0) \in R^m$, is the *j*-th elementary vector of dimension *m*.

The neural net estimator $\hat{\theta}$ can be derived by minimizing the least square criterion:

$$S_T(\theta_T) = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}(X_t - \theta_T(Z_{t-1}))^2 \qquad (5.9)$$

where, θ_T, the neural network sieve, $\theta_T : R^m \to R$, is an approximation function defined as:

$$\theta_T(z) = \beta_0 + \sum_{j=1}^{k}\beta_j\Psi(a'_j z + b_j) \qquad (5.10)$$

where: *k* is the number of hidden units (derived by minimizing the Bayesian Information Criterion (BIC)) and is an activation function. The activation function used here is:

$$\Psi(u) = \frac{u(1+|u/2|)}{2+|u|+u^2/2} \quad . \qquad (5.11)$$

The function (5.11) is a sigmoid function used to derive the neural network estimator in the program FUNFITS, developed by Nychka *et al.* (1996). The selection of the minimized BIC criterion determines the optimal number of hidden units, and is chosen by minimizing:

$$BIC = \ln\hat{\sigma}^2 + \frac{\ln T}{T}[1 + k(d+2)],$$

where

$$\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}(Y_t - \hat{\theta}(X_{t-1}))^2 .$$

This section has discussed the most commonly used methodology to calculate the point estimator of the Lyapunov exponent. This method works well with economic time series since they are often small in size with stochastic noise. The next section will explore the most recent research into the asymptotic distribution of the Lyapunov exponent estimator.

5.3 Asymptotic Distribution of the Lyapunov Exponent

To this point, most research into chaos theory was only able to derive a point estimate of the Lyapunov exponent. This section will explain the asymptotic behaviour and provide a consistent variance estimator that allows for the construction of a test statistic. Thus, allowing for the null hypothesis of chaos to be tested.

Shintani and Linton (2000) found that under the appropriate assumptions, some of which are stated in the previous section and the remainder of which can be found in their article, that the Lyapunov exponent estimator $(\hat{\lambda})$ is asymptotically normal, i.e., $N(0,\Phi_i)$, where

$$\Phi_i = \lim_{M\to\infty} \operatorname{var}\left[\frac{1}{\sqrt{M}}\sum_{t=1}^{M}\eta_{it}\right]$$

is positive and finite, for $1 \le i \le d$, and where $\hat{\eta}_{it} = \hat{\xi}_{it} - \hat{\lambda}_i$, with $\hat{\xi}_{it} = \frac{1}{2}\ln\left(\frac{v_i\left(\hat{T}_t'\hat{T}_t\right)}{v_i\left(\hat{T}_{t-1}'\hat{T}_{t-1}\right)}\right)$

for $t \ge 2$ and $\hat{\xi}_{i1} = \frac{1}{2}\ln v_1\left(\hat{T}_1'\hat{T}_1\right)$.

Shintani and Linton (2000) constructed a one sided test for the null hypothesis of chaos, $H_0 : \lambda \geq 0$, against the alternative of no chaos, $H_1 : \lambda < 0$. Their multidimensional test statistic is:

$$\hat{t}_i = \frac{\hat{\lambda}_{iM}}{\sqrt{\hat{\Phi}_i / M}}. \tag{5.12}$$

The null hypothesis is rejected if $\hat{t}_i \leq -z_\alpha$, where z_α is the critical values of a standard normal distribution. If the null hypothesis cannot be rejected, it increases the confidence in the Lyapunov exponent estimator.

The test statistic is determined using the covariance estimator:

$$\hat{\Phi}_i = \sum_{j=-M+1}^{M-1} w(j/S_M)\hat{\gamma}(j) \tag{5.13}$$

where $\omega(\cdot)$ is a kernel function, S_M denotes a lag truncation parameter and $\hat{\gamma}(j) = \frac{1}{M} \sum_{t=|j|+1}^{M} \hat{\eta}_t \hat{\eta}_{t-|j|}$, where $\hat{\eta}_{it} = \hat{\xi}_{it} - \hat{\lambda}_i$, with $\hat{\xi}_{it} = \frac{1}{2} \ln\left(\frac{v_i\left(\hat{T}_t' \hat{T}_t\right)}{v_i\left(\hat{T}_{t-1}' \hat{T}_{t-1}\right)} \right)$ for $t \geq 2$ and $\hat{\xi}_{i1} = \frac{1}{2} \ln v_1\left(\hat{T}_1' \hat{T}_1\right)$.

This section has derived a one-sided test statistic for the Lyapunov exponent estimator. Applying the above, the next section will employ this test statistics with the GDP data set introduced in Chapter 4.

5.4 Empirical Results Using GDP Data Set

The previous section provided a summary of the mathematics used in the derivation of the Lyapunov exponent. This section will apply this methodology with the intent of testing for the presence of chaotic behaviour to the Maddison (1982) data set of log real per capita gross domestic product (GDP) and the first differences of the log real per capita GDP. The calculations are completed using the statistical software packages, 'R' and Gauss and the results are summarized at the end of this chapter.

The results of the log real per capita GDP for Australia (Table 5.1), Canada (Table 5.2), Denmark (Table 5.3), France (Table 5.4), Germany (Table 5.5), Italy (Table 5.6), Norway (Table 5.7), Sweden (Table 5.8), UK (Table 5.9) and the USA (Table 5.10) are listed at the end of this chapter. This is followed by the first differences of the log real per capita GDP results for Australia (Table 5.11), Canada (Table 5.12), Denmark (Table 5.13), France (Table 5.14), Germany (Table 5.15), Italy (Table 5.16), Norway (Table 5.17), Sweden (Table 5.18), UK (Table 5.19) and the USA (Table 5.20). Each table lists the largest estimated Lyapunov exponent, its student t statistic in parentheses, the p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets, and the Bayesian Information Criterion (BIC) value. All countries' results are presented for dimensions (or lag lengths - m) one to six and for one to three hidden units (k). The minimized value of the BIC criterion determines the optimal number of hidden units (k) in the neural network (equation 5.10). The results presented for all countries are only for the full sample, i.e., no block or subspace Lyapunov exponents were calculated, since such work is beyond the scope of this thesis.

The reported Lyapunov exponents are for the most part negative. There are, however, exceptions. Each country, for both the log and first differences of the real per capita GDP, has at least one case (i.e. a particular dimension and number of hidden units) where the Lyapunov exponent is positive. Some countries have a large number of positive cases. For example, the log real per capita GDP of Norway and Denmark have more than ten instances where the Lyapunov exponent is greater than zero. Chaotic behaviour is not indicated exclusively by the point estimate of a positive Lyapunov exponent and therefore further analysis is required.

The optimal number of hidden units can be determined using the BIC criterion. Its minimum value for each lag length determines the appropriate number of hidden units. Adding this criterion to the analysis will eliminate many of the positive Lyapunov exponents from consideration for chaotic behaviour by reducing the number of positive exponents from consideration. Therefore, the only countries that still have possibilities for chaotic behaviour in their data sets are Denmark, France and Germany; for both the log and first differences of the real per capita GDP; and Italy, Norway, Sweden and the UK for the log real per capita GDP.

The previous section discussed the work of Shintani and Linton (2000). They derived the statistics and distribution of the Lyapunov exponent and developed a methodology to construct its confidence interval and its *p*-values. Prior to their work, one could have concluded that there was the possibility of chaotic behaviour in all of the ten countries except Australia, Canada and the USA, since these are the only nations with no positive estimates of the Lyapunov exponent. Thus, the importance of Shintani and Linton's (2000) contribution is that it increased the certainty with which one could say that there is a positive Lyapunov exponent, indicating chaos.

The *p*-values can now be used to determine chaotic behaviour with more certainty. The *p*-value is the probability, under the null hypothesis, of obtaining a value greater than the observed amount. The number of countries can now be narrowed down to three if a

p-value of greater than 90 percent is used (this implies a 90 percent confidence interval with a corresponding 10 percent rejection region of the null hypothesis). There are three countries for the log real per capita GDP where the null hypothesis $H_0 : \lambda \geq 0$ cannot be rejected. First, France has two possible cases, for lag length (m) of 4 and 6 both with 3 hidden units (k) each. In both cases, the p-value is greater than 90 percent. Second, Germany has one case for $m = 6$ and $k = 2$, where the null hypothesis cannot be rejected. Finally, Norway has three possibilities, all with $k = 1$ for $m = 4,5$, and 6. In fact, Norway has one case where the p-value is greater than 95 percent.

However, as stated earlier in this chapter, the data employed in the derivation of the Lyapunov exponent should be strictly stationary. The log real per capita GDP data set was found, in the previous chapter, to be nonstationary. Therefore, the above conclusion of chaotic behaviour is questionable. Under the first differences of the log real per capita GDP, there is only one country that comes close to our stated conditions; Denmark. However, the p-value in this case is only 87.4 percent. Unless one wishes to decrease the level of confidence, below the 90 percent requirement one must accept the alternative hypothesis that there is no chaotic behaviour for any of the nations tested.

This section has examined and tested the GDP of ten nations for chaos. The standard used for determination of chaos is a Lyapunov exponent estimate with a value greater than zero, a p-value greater than 90 percent and the BIC minimization criterion to determine the optimal number of hidden units. These conditions were found in three countries, France, Germany and Norway, but only for their log GDP data sets. Of these, only Norway was found to have a p-value that exceeded 95 percent. The log GDP time series was tested and found to be nonstationary in Chapter 4. This contradicts the assumptions in Nychka *et al* (1992) that requires a stationary time series be employed in the derivation of the estimate of the Lyapunov exponent. One cannot definitively conclude that France, Germany and Norway are chaotic, as the stationarity condition was not adhered to. There were no instances of chaos for the stationary time series of the first differences of the real per capita GDP data set. However, Denmark could be a possible chaotic data set, if one were to lower the confidence intervals.

Overall, the null hypothesis of chaos cannot be accepted for any of the ten nations tested, for either the log or the log first differences time series.

5.5 Summary

This chapter examined a method by which one can test a time series for chaotic behaviour and has applied this methodology to the GDP data.

The second section of this chapter focused on Nychka *et al*'s (1992) method of calculating the largest estimate of the Lyapunov exponent. This is a Jacobian method based on a nonparametric neural network estimator and is the preferred method with data sets that are small and may contain stochastic noise. As these conditions are characteristic of many economic time series, the Nychka *et al* (1992) approach is used in this thesis.

Section 5.3 summarized the work of Shintani and Linton (2000). They were able to derive the statistics of the Lyapunov exponent. Their development has enabled one to test the null hypothesis of chaos and determine the level of significance of the largest estimate of the Lyapunov exponent.

The methodologies of Nychka *et al* (1992) and Shintani and Linton (2000) were applied to the Maddison (1982) data of log real per capita GDP and its first differences. The results are summarized in the tables at the end of this chapter. Although chaotic behaviour can be found in three countries data sets using the log real per capita GDP, this series was found to violate the stationary assumption. Taking the first differences corrected for this shortcoming. The first differences of the log real per capita GDP were found to accept the alternative hypothesis that chaotic behaviour was not present.

Overall, there is no evidence that chaotic behaviour exists in the real per capita GDP of any of the countries examined.

Table 5.1 Lyapunov Exponent Estimates for Australia Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.139505	-0.086 (-2.162) [0.015]	-6.025731	-0.087 (-2.139) [0.016]	-5.900423	-0.084 (-2.037) [0.021]
$m=2$	-6.091758	-0.066 (-2.026) [0.021]	-5.945484	-0.092 (-2.143) [0.016]	-5.787109	-0.061 (-1.749) [0.040]
$m=3$	-6.090989	-0.068 (-1.956) [0.025]	-5.959926	-0.121 (-2.988) [0.001]	-5.866319	-0.097 (-2.697) [0.004]
$m=4$	-6.056569	-0.055 (-1.859) [0.032]	-5.911466	-0.077 (-2.245) [0.012]	-5.740114	-0.144 (-3.858) [<0.001]
$m=5$	-6.022085	-0.033 (-1.501) [0.067]	-5.818954	-0.096 (-2.439) [0.007]	-5.590231	-0.115 (-3.047) [0.001]
$m=6$	-5.971767	-0.030 (-1.479) [0.070]	-5.773864	-0.072 (-2.014) [0.022]	-5.888779	0.018 (0.697) [0.757]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.2 Lyapunov Exponent Estimates for Canada Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k = 1$		$k = 2$		$k = 3$	
Lags (m)	BIC		BIC		BIC	
$m = 1$	-5.377454	-0.015 (-1.032) [0.151]	-5.256560	-0.020 (-3.436) [<0.001]	-5.139333	-0.031 (-1.531) [0.063]
$m = 2$	-5.473573	-0.028 (-1.317) [0.094]	-5.385823	-0.070 (-4.479) [<0.001]	-5.286684	-0.068 (-1.343) [0.090]
$m = 3$	-5.426673	-0.020 (-1.746) [0.040]	-5.307855	-0.021 (-0.970) [0.166]	-5.235689	-0.075 (-2.274) [0.011]
$m = 4$	-5.394661	-0.010 (-1.095) [0.137]	-5.225149	-0.027 (-0.971) [0.016]	-5.129941	-0.081 (-2.416) [0.008]
$m = 5$	-5.390237	-0.012 (-1.178) [0.119]	-5.152397	-0.024 (-1.137) [0.128]	-5.087509	-0.044 (-1.680) [0.046]
$m = 6$	-5.364669	-0.015 (-1.237) [0.108]	-5.164325	-0.013 (-0.553) [0.290]	-4.951726	0.028 (0.849) [0.802]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.3 Lyapunov Exponent Estimates for Denmark Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.301850	-0.004 (-36.810) [<0.001]	-6.196569	-0.002 (-0.185) [0.423]	-6.086869	0.000 (0.042) [0.517]
$m=2$	-6.242897	-0.004 (-0.335) [0.369]	-6.121513	0.000 (-0.022) [0.491]	-5.961018	0.006 (0.476) [0.683]
$m=3$	-6.260960	0.001 (0.286) [0.612]	-6.089678	-0.002 (-0.174) [0.431]	-5.892109	0.000 (0.028) [0.511]
$m=4$	-6.206640	0.001 (0.055) [0.522]	-5.997583	-0.001 (-0.061) [0.476]	-5.776318	-0.018 (-0.602) [0.274]
$m=5$	-6.158009	0.003 (0.323) [0.627]	-5.899484	0.002 (0.111) [0.544]	-5.844267	0.013 (0.409) [0.659]
$m=6$	-6.113596	0.004 (0.552) [0.709]	-5.819311	0.002 (0.124) [0.549]	-5.830225	-0.008 (-0.277) [0.391]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.4 Lyapunov Exponent Estimates for France Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-5.445765	-0.042 (-2.555) [0.005]	-5.320945	-0.045 (-1.532) [0.063]	-5.197031	-0.039 (-3.146) [0.001]
$m=2$	-5.494947	-0.061 (-1.777) [0.038]	-5.327830	-0.075 (-1.726) [0.042]	-5.220109	-0.067 (-2.078) [0.019]
$m=3$	-5.489968	-0.099 (-2.116) [0.017]	-5.362700	-0.062 (-2.110) [0.017]	-5.559438	0.008 (0.194) [0.577]
$m=4$	-5.458078	-0.070 (-1.896) [0.029]	-5.313420	-0.054 (-1.882) [0.030]	-5.490441	0.053 (1.505) [0.934]
$m=5$	-5.411352	-0.047 (-1.989) [0.023]	-5.225974	-0.042 (-1.360) [0.087]	-5.428160	0.019 (0.705) [0.760]
$m=6$	-5.405402	-0.030 (-1.459) [0.072]	-5.559314	-0.032 (-1.020) [0.154]	-5.568099	0.049 (1.578) [0.943]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

**Table 5.5 Lyapunov Exponent Estimates
for Germany
Log Real per Capita GDP**

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-4.659987	-0.028 (-1.536) [0.062]	-4.545430	-0.026 (-2.223) [0.013]	-4.428993	-0.023 (-1.262) [0.103]
$m=2$	-4.681353	-0.061 (-1.787) [0.037]	-5.110635	-0.024 (-0.697) [0.243]	-5.276625	0.023 (0.467) [0.680]
$m=3$	-4.658353	-0.037 (-2.681) [0.004]	-5.021157	-0.033 (-0.842) [0.200]	-5.142477	0.025 (0.415) [0.661]
$m=4$	-4.605632	-0.038 (-2.260) [0.012]	-4.927266	-0.018 (-0.526) [0.299]	-5.009731	-0.004 (-0.114) [0.455]
$m=5$	-4.558335	-0.039 (-2.265) [0.012]	-4.834725	-0.003 (-0.101) [0.460]	-4.920948	0.009 (0.169) [0.567]
$m=6$	-4.512350	-0.034 (-2.040) [0.021]	-4.754939	0.041 (1.277) [0.899]	-4.521853	0.108 (1.355) [0.912]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.6 Lyapunov Exponent Estimates for Italy Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-5.541839	-0.025 (-91.387) [<0.001]	-5.416366	-0.029 (-2.156) [0.016]	-5.310580	0.004 (0.479) [0.684]
$m=2$	-5.555841	-0.048 (-1.396) [0.082]	-5.518586	-0.057 (-1.671) [0.048]	-5.533644	-0.047 (-0.919) [0.179]
$m=3$	-5.510909	-0.034 (-1.660) [0.048]	-5.583138	-0.040 (-1.221) [0.111]	-5.607380	-0.048 (-1.742) [0.041]
$m=4$	-5.471702	-0.024 (-1.220) [0.111]	-5.519781	-0.058 (-2.034) [0.021]	-5.586373	-0.017 (-1.036) [0.150]
$m=5$	-5.420008	-0.019 (-1.018) [0.154]	-5.396298	-0.062 (-2.207) [0.014]	-5.504175	0.004 (0.166) [0.566]
$m=6$	-5.384952	-0.012 (-0.721) [0.235]	-5.324638	-0.015 (-0.854) [0.197]	-5.359935	0.038 (1.478) [0.930]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.7 Lyapunov Exponent Estimates for Norway Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.412967	-0.002 (-1.074) [0.142]	-6.301486	0.008 (2.865) [0.998]	-6.181929	0.008 (2.694) [0.996]
$m=2$	-6.372682	0.004 (0.688) [0.754]	-6.226765	0.011 (1.987) [0.977]	-6.059432	0.010 (1.716) [0.957]
$m=3$	-6.343439	0.005 (1.044) [0.852]	-6.214063	0.009 (0.600) [<0.726]	-6.069773	0.003 (0.389) [0.651]
$m=4$	-6.305651	0.009 (1.278) [0.899]	-6.185702	0.024 (1.800) [0.964]	-5.955327	0.020 (2.733) [0.997]
$m=5$	-6.249176	0.009 (1.285) [0.901]	-6.092328	0.024 (1.962) [0.975]	-5.866697	0.012 (1.268) [0.898]
$m=6$	-6.251536	0.012 (1.675) [0.953]	-6.033839	0.025 (2.573) [0.995]	-5.790538	0.028 (2.819) [0.998]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.8 Lyapunov Exponent Estimates for Sweden Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.918420	-0.10 (-3.467) [<0.001]	-6.82861	-0.016 (-1.017) [0.155]	-6.706856	-0.018 (-19.605) [<0.001]
$m=2$	-6.886672	-0.006 (-0.507) [0.306]	-6.775977	-0.005 (-0.312) [0.377]	-6.612504	-0.008 (-0.508) [0.306]
$m=3$	-6.854651	-0.004 (-0.485) [0.314]	-6.740620	0.002 (0.142) [0.556]	-6.533196	-0.001 (-0.096) [0.462]
$m=4$	-6.816721	0.000 (-0.013) [0.495]	-6.650396	0.003 (0.170) [0.568]	-6.391324	0.003 (0.220) [0.587]
$m=5$	-6.785276	-0.001 (-0.113) [0.455]	-6.591181	0.005 (0.335) [0.631]	-6.354137	0.006 (0.288) [0.613]
$m=6$	-6.768958	0.002 (0.281) [0.611]	-6.531526	0.008 (0.960) [0.755]	-6.186378	0.006 (0.481) [0.678]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.9 Lyapunov Exponent Estimates for United Kingdom Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.751487	-0.010 (-6.901) [<0.001]	-6.636952	-0.006 (-0.643) [0.260]	-6.513207	-0.006 (-0.657) [0.256]
$m=2$	-6.810048	-0.014 (-0.861) [0.195]	-6.657519	-0.021 (-1.133) [0.129]	-6.525304	-0.006 (-0.306) [0.380]
$m=3$	-6.757695	-0.10 (-0.692) [0.244]	-6.564491	-0.008 (-0.431) [0.333]	-6.405818	-0.017 (-1.247) [0.106]
$m=4$	-6.767572	-0.005 (-0.405) [0.343]	-6.551553	0.000 (0.002) [0.501]	-6.427741	-0.003 (-0.257) [0.399]
$m=5$	-6.766462	0.000 (-0.002) [0.499]	-6.518897	0.002 (0.162) [0.564]	-6.339529	0.006 (0.436) [0.669]
$m=6$	-6.733488	-0.003 (-0.245) [0.403]	-6.450987	0.005 (0.268) [0.606]	-6.313505	-0.001 (-0.093) [0.463]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

**Table 5.10 Lyapunov Exponent Estimates
for United States
Log Real per Capita GDP**

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-5.580460	-0.019 (-11.790) [<0.001]	-5.463826	-0.007 (-4.038) [<0.001]	-5.344257	-0.005 (-1.075) [0.141]
$m=2$	-5.595199	-0.018 (-1.226) [0.110]	-5.507990	-0.017 (-1.286) [0.099]	-5.383213	-0.013 (-0.758) [0.224]
$m=3$	-5.547867	-0.019 (-1.454) [0.073]	-5.433450	-0.015 (-0.946) [0.172]	-5.308979	0.015 (0.707) [0.760]
$m=4$	-5.519498	-0.016 (-1.480) [0.069]	-5.368106	-0.016 (-0.831) [0.203]	-5.208319	-0.051 (-2.409) [0.008]
$m=5$	-5.514782	-0.012 (-1.180) [0.119]	-5.463070	-0.029 (-0.811) [0.209]	-5.280742	-0.038 (-2.035) [0.021]
$m=6$	-5.482173	-0.011 (-1.040) [0.149]	-5.413071	-0.001 (-0.018) [0.493]	-5.214693	0.029 (1.017) [0.845]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.11 Lyapunov Exponent Estimates for Australia First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.055815	-2.962	-6.005040	-3.331	-5.904799	-1.599
		(-19.130)		(-26.012)		(-15.957)
		[<0.001]		[<0.001]		[<0.001]
$m=2$	-6.136241	-1.962	-6.064468	-0.735	-5.969497	-0.296
		(-16.445)		(-4.121)		(-3.244)
		[<0.001]		[<0.001]		[0.001]
$m=3$	-6.185426	-1.855	-6.062425	-1.164	-5.981852	-0.189
		(-16.177)		(-12.057)		(-1.901)
		[<0.001]		[<0.001]		[0.029]
$m=4$	-6.132483	-1.144	-6.017132	-0.605	-5.977578	-0.036
		(-14.534)		(-5.146)		(-0.495)
		[<0.001]		[<0.001]		[0.310]
$m=5$	-6.096794	-1.089	-5.943268	-0.205	-5.938323	0.156
		(-12.589)		(-2.181)		(3.233)
		[<0.001]		[0.015]		[0.999]
$m=6$	-6.054781	-0.652	-5.862808	-0.115	-5.861383	0.270
		(-7.545)		(-2.662)		(4.139)
		[<0.001]		[0.004]		[1.000]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.12 Lyapunov Exponent Estimates for Canada First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-5.369789	-2.383	-5.304103	-3.832	-5.203242	-1.277
		(-24.127)		(-20.922)		(-10.824)
		[<0.001]		[<0.001]		[<0.001]
$m=2$	-5.444845	-1.323	-5.404505	-1.159	-5.311035	-0.637
		(-22.757)		(-9.442)		(-5.305)
		[<0.001]		[<0.001]		[<0.001]
$m=3$	-5.434361	-1.106	-5.280491	-0.541	-5.220702	-0.381
		(-9.670)		(-9.767)		(-3.274)
		[<0.001]		[<0.001]		[0.001]
$m=4$	-5.452090	-0.628	-5.335749	-0.179	-5.252061	-0.50
		(-7.987)		(-1.829)		(-0.651)
		[<0.001]		[0.034]		[0.257]
$m=5$	-5.433619	-0.698	-5.384124	-0.228	-5.208350	-0.008
		(-7.414)		(-2.069)		(-0.106)
		[<0.001]		[0.019]		[0.458]
$m=6$	-5.380950	-0.681	-5.316920	-0.009	-5.190948	0.109
		(-7.719)		(-0.099)		(2.181)
		[<0.001]		[0.461]		[0.985]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.13 Lyapunov Exponent Estimates for Denmark First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.386872	-4.332 (-49.388) [<0.001]	-6.318239	-2.943 (-18.598) [<0.001]	-6.200188	-2.123 (-16.170) [<0.001]
$m=2$	-6.294233	-0.792 (-13.861) [<0.001]	-6.371197	-0.887 (-5.672) [<0.001]	-6.349300	-0.376 (-5.343) [<0.001]
$m=3$	-6.299074	-0.545 (-7.240) [<0.001]	-6.208520	-0.229 (-3.152) [0.001]	-6.380449	-0.124 (-1.300) [0.097]
$m=4$	-6.267981	-0.245 (-3.545) [<0.001]	-6.269987	-0.636 (5.921) [<0.001]	-6.225951	0.127 (2.002) [0.977]
$m=5$	-6.253605	-0.265 (-4.231) [<0.001]	-6.271527	-0.089 (-1.668) [0.048]	-6.061430	0.129 (4.122) [1.000]
$m=6$	-6.199057	-0.288 (-4.704) [<0.001]	-6.269292	0.150 (1.146) [0.874]	-6.201732	0.394 (3.780) [1.000]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.14 Lyapunov Exponent Estimates for France First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-5.535034	-1.316 (-22.142) [<0.001]	-5.422918	-1.432 (-30.806) [<0.001]	-5.769397	-3.172 (-12.592) [<0.001]
$m=2$	-5.494313	-1.714 (-28.312) [<0.001]	-5.874422	-1.316 (-10.561) [<0.001]	-5.819022	-0.386 (-5.387) [<0.001]
$m=3$	-5.465779	-1.272 (-22.317) [<0.001]	-5.841875	-0.473 (-5.992) [<0.001]	-5.876843	-0.406 (-2.602) [0.005]
$m=4$	-5.487070	-0.575 (-11.112) [<0.001]	-5.955747	-0.558 (-5.128) [<0.001]	-5.738757	-0.538 (-4.058) [<0.001]
$m=5$	-5.480745	-0.416 (-7.076) [<0.001]	-5.898238	-0.269 (-6.140) [<0.001]	-5.816632	-0.012 (-0.199) [0.421]
$m=6$	-5.714382	-0.498 (-6.313) [<0.001]	-5.874521	-0.173 (-2.801) [0.003]	-5.980599	0.032 (0.707) [0.760]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.15 Lyapunov Exponent Estimates for Germany First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-4.769824	-0.735 (-15.995) [<0.001]	-4.750830	-1.462 (-10.739) [<0.001]	-5.349624	-0.720 (-16.122) [<0.001]
$m=2$	-4.724956	-1.052 (-19.622) [<0.001]	-5.269785	-1.397 (-10.325) [<0.001]	-5.273393	-1.055 (-9.818) [<0.001]
$m=3$	-4.672642	-1.053 (-19.255) [<0.001]	-5.199899	-0.445 (-5.043) [<0.001]	-5.146923	-1.048 (-9.066) [<0.001]
$m=4$	-4.621637	-0.866 (-14.473) [<0.001]	-5.208389	-0.104 (-0.719) [0.236]	-5.192353	-0.279 (-4.013) [<0.001]
$m=5$	-4.570331	-0.646 (-16.290) [<0.001]	-5.509783	0.018 (0.141) [0.556]	-4.995184	0.155 (2.134) [0.984]
$m=6$	-4.518627	-0.393 (-11.254) [<0.001]	-5.318238	0.097 (0.912) [0.819]	-5.312569	0.325 (4.515) [1.000]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.16 Lyapunov Exponent Estimates for Italy First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-5.585078	-1.339 (-32.091) [<0.001]	-5.906213	-1.201 (-20.852) [<0.001]	-5.919597	-0.599 (-5.949) [<0.001]
$m=2$	-5.536392	-0.725 (-7.533) [<0.001]	-5.930691	-0.655 (-7.491) [<0.001]	-5.872053	-1.231 (-9.508) [<0.001]
$m=3$	-5.497291	-0.604 (-12.961) [<0.001]	-5.955835	-0.962 (-7.696) [<0.001]	-5.844676	-0.584 (-5.761) [<0.001]
$m=4$	-5.449157	-0.499 (-8.923) [<0.001]	-5.885021	-0.350 (-4.039) [<0.001]	-5.807742	-0.223 (-3.104) [0.001]
$m=5$	-5.513138	-0.656 (-8.008) [<0.001]	-5.775620	-0.386 (-4.350) [<0.001]	-5.672326	0.032 (0.488) [0.687]
$m=6$	-5.346006	-0.456 (-8.036) [<0.001]	-5.728063	-0.317 (-4.014) [<0.001]	-5.621808	0.199 (1.620) [0.947]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.17 Lyapunov Exponent Estimates for Norway First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.419816	-3.804 (-24.070) [<0.001]	-6.448638	-0.707 (-9.435) [<0.001]	-6.421035	-1.081 (-8.994) [<0.001]
$m=2$	-6.413890	-1.869 (-34.768) [<0.001]	-6.303916	-1.696 (-25.374) [<0.001]	-6.501379	-0.993 (-6.094) [<0.001]
$m=3$	-6.330811	-0.667 (-8.067) [<0.001]	-6.315689	-0.636 (-6.888) [<0.001]	-6.421653	-0.405 (-3.166) [0.001]
$m=4$	-6.397338	-1.001 (-7.656) [<0.001]	-6.304517	-0.308 (-2.923) [0.002]	-6.337840	-0.005 (-0.145) [0.442]
$m=5$	-6.361848	-0.809 (-11.589) [<0.001]	-6.395773	-0.148 (-1.620) [0.053]	-6.305687	0.372 (3.877) [1.000]
$m=6$	-6.315860	-0.695 (-9.596) [<0.001]	-6.239760	0.029 (0.436) [0.669]	-6.124592	0.314 (4.842) [1.000]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.18 Lyapunov Exponent Estimates
for Sweden
First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.955246	-3.645 (-37.706) [<0.001]	-6.858761	-0.880 (-8.583) [<0.001]	-6.762853	-1.739 (-11.576) [<0.001]
$m=2$	-6.953597	-1.282 (-12.402) [<0.001]	-6.923728	-0.320 (-8.942) [<0.001]	-6.972631	-0.305 (-3.762) [<0.001]
$m=3$	-6.918391	-1.036 (-12.657) [<0.001]	-6.766091	-0.817 (-9.787) [<0.001]	-7.036931	-0.215 (-1.875) [0.030]
$m=4$	-6.878600	-0.765 (-11.583) [<0.001]	-6.854320	-0.440 (-3.717) [<0.001]	-7.029050	-0.050 (-0.453) [0.325]
$m=5$	-6.952603	-0.622 (-4.853) [<0.001]	-6.839671	0.175 (3.165) [0.999]	-6.880567	0.149 (2.343) [0.990]
$m=6$	-6.968521	-0.445 (-3.443) [<0.001]	-6.904140	-0.069 (-1.188) [0.117]	-6.922331	0.252 (3.428) [1.000]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Table 5.19 Lyapunov Exponent Estimates
for United Kingdom
First Differences of Log Real per Capita GDP

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-6.853635	-1.480 (-27.513) [<0.001]	-6.730135	-1.454 (-32.812) [<0.001]	-6.657804	-1.316 (-13.516) [<0.001]
$m=2$	-6.815085	-1.331 (-20.435) [<0.001]	-6.708806	-1.162 (-24.266) [<0.001]	-6.654814	-0.966 (-7.198) [<0.001]
$m=3$	-6.874215	-0.606 (-14.094) [<0.001]	-6.781385	-0.285 (-3.012) [0.001]	-6.599146	-0.209 (-3.098) [0.001]
$m=4$	-7.123799	-0.768 (-6.142) [<0.001]	-6.969911	-0.435 (-4.443) [<0.001]	-6.837002	-0.096 (-1.142) [0.127]
$m=5$	-6.901854	-0.298 (-5.604) [<0.001]	-6.876218	-0.249 (-3.035) [0.001]	-6.781533	0.125 (1.273) [0.898]
$m=6$	-6.919426	-0.218 (-3.416) [<0.001]	-6.684615	-0.128 (-2.643) [0.004]	-6.739502	0.121 (1.380) [0.916]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

**Table 5.20 Lyapunov Exponent Estimates
for United States
First Differences of Log Real per Capita GDP**

	Number of Hidden Units (k)					
	$k=1$		$k=2$		$k=3$	
Lags (m)	BIC		BIC		BIC	
$m=1$	-5.691950	-2.573	-5.627510	-1.068	-5.544468	-1.197
		(-19.733)		(-12.417)		(-12.638)
		[<0.001]		[<0.001]		[<0.001]
$m=2$	-5.653760	-2.457	-5.572861	-1.995	-5.600368	-0.931
		(-20.359)		(-12.957)		(-5.975)
		[<0.001]		[<0.001]		[<0.001]
$m=3$	-5.600342	-0.438	-5.674151	-0.546	-5.540167	-0.471
		(-11.381)		(-6.594)		(-5.249)
		[<0.001]		[<0.001]		[<0.001]
$m=4$	-5.618928	-0.620	-5.577025	-0.169	-5.533173	-0.021
		(-7.979)		(-2.098)		(-0.500)
		[<0.001]		[0.018]		[0.308]
$m=5$	-5.595415	-0.688	-5.749877	-0.084	-5.545158	-0.225
		(-6.891)		(-1.102)		(-1.730)
		[<0.001]		[0.135]		[0.042]
$m=6$	-5.599330	-0.316	-5.580037	0.135	-5.422253	0.184
		(-4.510)		(0.982)		(2.981)
		[<0.001]		[0.837]		[0.999]

Note: The Lyapunov exponent estimate are presented for the full sample ($T = 115$) only. The largest estimated Lyapunov exponents are stated with its t statistic in parentheses and its p-value for the hypothesis test $H_0 : \lambda \geq 0$ in brackets.

Chapter 6

CONCLUSION

The purpose of this thesis was to provide the reader with an introduction to the field of chaos theory. It also explained and discussed some examples of where chaos can be found in economics. It introduced the most current and relevant techniques in order to test for chaos. These were applied to an economic time series to test for the presence of chaos.

Edward Lorenz triggered modern research into chaotic behaviour in the early 1960's. Using a simple computer model of the weather, he found that extremely small differences in the initial starting parameters resulted in ever-larger deviations over time. Lorenz revealed one of the main characteristics of chaos theory – SDIC. This property is the foundation for the use of the Lyapunov exponent as a mathematical tool to test for chaotic orbits. The logistic difference equation, a simple mathematical process, can be shown to have chaotic paths through the manipulation of its parameters. Consequently, under the appropriate conditions any model or system represented by a mathematical equation may be altered so that it exhibits chaotic behaviour.

In Chapter 3, the Li and Yorke (1975) theorem was introduced as a method by which one could determine if an orbit was chaotic. The theorem concludes that a time series that is chaotic will exhibit various periodic cycles at different times and it will not remain in any one cycle for any length of time. A three period cycle that is part of the whole time series is an indication of chaos.

SDIC and period three cycles implying chaos (Li-Yorke theorem) are two properties and alternative methods by which one can identify and test for chaos. However, they are not the most suitable when testing an economic time series. The presence of SDIC can be detected in a data set by using the Lyapunov exponent, which has come to be known as the most appropriate test for the recognition of chaotic behaviour. For this reason, the Lyapunov exponent was used in this thesis.

Chapter 5 summarized three techniques to calculate the Lyapunov exponent. It has been shown that the Nychka *et al* (1992) method was the most appropriate for economic data. This method can better handle time series with a small number of observations that contain stochastic noise. A common problem, at least till the late 1990s, to all three methods was that only a point estimate of the Lyapunov exponent was determined. Shintani & Linton (2000) have developed the statistical framework of the Lyapunov exponent. This framework allows one to test the null hypothesis of chaos using the variance of the estimate of the Lyapunov exponent, with confidence intervals.

This thesis applies the above methodologies to calculate the Lyapunov exponent and to test the null hypothesis of chaos. The data tested was for the ten countries including: Australia, Canada, Denmark, France, Germany, Italy, Norway, Sweden, United Kingdom and United States of America. It consists of 115 observations (1870 to 1985) of the log real per capita GDP and their first differences. The detailed results to the testing of these economies are provided in tables at the end of Chapter 5. Each table lists the estimate of the largest Lyapunov exponent, its t statistic, the p-value for the null hypothesis test of chaos, $H_0 : \lambda \geq 0$, and the BIC value. These results are provided for each nation using one to six dimensions (or lags) and for one to three hidden units. The

BIC value determines the optimal number of hidden units. The full sample length was tested; no sub sample or blocks were analyzed.

The data analysis indicates that all countries had at least one instance resulting in a positive Lyapunov exponent. Previous studies were only able to calculate the point estimate of the Lyapunov exponent. These studies did not know its statistical framework. This point estimate is misleading - researchers could not make conclusions regarding the null hypothesis of chaos. Utilizing the statistical framework developed by Shintani & Linton (2000), the *t* statistic and its *p*-value were calculated and listed in the tables. The BIC criterion reduces the number of potential countries with chaos by three. Applying this criterion eliminates Australia, Canada and the US as data sets with chaos. Of the remaining nations, Denmark, France and Germany exhibit chaos ($\lambda \geq 0$) in both the log real per capita GDP and their first differences. Italy, Norway, Sweden and the UK showed potential chaotic results in only the log real per capita GDP data set.

The potential for chaos in the remaining seven countries is further reduced when a rather lenient confidence interval of 90 percent is applied. Now only France, Germany and Norway remain – indicating that for these nations chaos in the data sets cannot be rejected. However, these results are for the log real per capita GDP data set that has been shown in Chapter 4 to be nonstationary. Stationarity is a requirement of the testing methodology developed by Shintani & Linton (2000) and as such, the first differences data set is more appropriate when testing for chaos. Examining the analysis of the first differences of the log real per capita GDP, the null hypothesis of chaos is rejected at 90 percent. The closest that one comes to concluding that chaos exists is with Denmark at 87.4 percent.

Overall, applying the BIC criterion, a 90 percent confidence interval and ensuring stationarity of data, the null hypothesis of chaos is rejected for all ten countries. The rejection of the concept of chaos is an indication that the endogenous approach to economic modelling is not appropriate for the ten countries tested. Thus, an exogenous approach would be more applicable when determining testing methodologies and for use by policy makers. The analysis completed here has calculated the estimate of the largest Lyapunov exponent and constructed their confidence intervals. This application of the statistical framework has provided a more scientific methodology to testing for the presence of chaos in the data of a nation's economy and determining the correct approach to use in modelling. The definitive conclusion reached is that there is no evidence of chaos in any of the GDP time series tested. This conclusion has added to previous research in the area where only a point estimate was employed.

This thesis provided a small part of the continuing body of research that has and should persist on chaos theory – further research is expected and recommended. Extensions to the testing completed here fall into three areas. First, applying quarterly data to the testing methodology could increase the size of the time series used and may change the outcome. Second, a more plausible enhancement would be to block the data sets. This approach would involve taking subsets of the full series and testing each block for the null hypothesis of chaos. Day (1982) concluded that chaotic paths were interspersed between periods of steady growth. Blocks could be constructed with one dedicated to the period around the Great Depression of the 1930s. The detection of chaos in any of the blocks, especially during the Great Depression period, could lead to conclusion that this event was endogenous to the economy. If it is not an exogenous shock, it is validation of the theory put forward by Day (1982). Finally, testing for chaotic behaviour should be conduct on additional time series. A large amount of analysis has been done in the past twenty years on various economic data sets. Much analysis has been done using the two alternative methods of calculating the Lyapunov exponent that are not appropriate with a small number of observations containing stochastic noise and where only a point estimate is determined. Applying the Nychka *et al* (1992) method and the statistical framework of Shintani & Linton (2000) to other series should be considered. This should be considered for other series that define an economy such as unemployment rates, employment amounts and industrial production. In addition, these techniques can be applied to foreign exchange rates, rates of return, natural gas liquids prices.

In summation, this thesis has provided a definition and examples of where chaos is found in economics, stated the best testing methodology for the Lyapunov exponent and its statistical framework when dealing with economic data. The economies of ten countries has been tested and shown that the null of chaos cannot be accepted. The main implications are that the exogenous approach to modelling is more appropriate and that further analysis involving data blocking, quarterly GDP figures and using alternative data sets is suggested.

References

Alligood, K.T., T.D. Sauer, and James A. Yorke, 1997. Chaos An Introduction to Dynamical Systems, Springer-Verlag, New York

Benhabib, Jess and Richard H. Day, 1982. A Characterization of Erratic Dynamics in the Overlapping Generations Model, Journal of Economic Dynamic and Control 4, 37-55.

Brock, W.A. and C.L. Sayer, 1988, Is the Business Cycle Characterized by Deterministic Chaos, Journal Of Monetary Economics 22, 71-80.

Briggs, John and F. David Peat, 1989. Turbulent Mirror: An Illustrated Guide to Chaos Theory and the Science of Wholeness, Harper & Row, New York.

Cambel, A.B., 1993. Applied Chaos Theory: A Paradigm For Complexity, Academic Press, New York.

Day, Richard H., 1982. Irregular Growth Cycles, American Economic Review 72, 406-414.

Day, Richard H., 1983. The Emergence of Chaos from Classical Economic Growth, Quarterly Journal of Economics 98, 201-213.

Dickey, D.A. and W.A. Fuller, 1979. Distribution of the Estimators for Autoregressive Time Series with a Unit Root, Journal of the American Statistical Association 74, 427-431.

Dickey, D.A. and W.A. Fuller, 1981. Likelihood Ratio Statistics for Autoregressive Time Series with a Unit Root, Econometrica 49, 1057-72.

Eckmann, J.P. and D. Ruelle, 1985. Ergodic Theory of Chaos and Strange Attractors, Reviews of Modern Physics 57, 617-656.

Frank, Murray, Ramazan Gencay and Thanasis Stengos, 1988. International Chaos, European Economic Review 32, 1569-1584.

Frank, Murray and Thanasis Stengos, 1989. Measuring the Strangeness of Gold and Silver Rates of Return, Review of Economic Studies 56, 553-567.

Frisch, R., 1933. Propagation Problems and Impulse Problems in Dynamic Economics, In: Economic Essays in honor of Gustav Cassel, George Allen and Unwin, London, 1933.

Gale, David, 1973. Pure Exchange Equilibrium of Dynamic Economic Models, Journal of Economic Theory 6, 12-36.

Gencay, R. and W.D. Dechert, 1992. An Algorithm for the n Lyapunov Exponents of an n-dimensional Unknown Dynamical System, Physica D 59, 142-157.

Gleick, James, 1987. Chaos: Making a New Science, Penguin Books, New York.

Goodwin, Richard M., 1951. The Nonlinear Accelerator and the Persistence of Business Cycles, Econometrica 19, 1-17.

Grandmont, Jean-Micheal, 1985. On Endogenous Competitive Business Cycles, Econometrica 53, 995-1046.

Gujarati, Damodar N., 1995. Basic Econometrics, McGraw-Hill Inc., New York.

Hicks, J.R., 1950. A Contribution to the Theory of the Trade Cycle, Clarendon Press, Oxford.

Kaldor, N., 1940. A Model of the Trade Cycle, Economic Journal 50, 78-92.

Li, Tien-Yien and James A. Yorke, 1975. Period Three Implies Chaos, American Mathematical Monthly 82, 985-992.

Lorenz, Edward, 1963. Deterministic Nonperiodic Flow, Journal of Atmospheric Sciences 20, 130-141.

Maddison, A., 1982. Phases of Capitalist Development, Oxford University Press, London.

Medio, Alfredo and Giampaolo Gallo, 1992. Chaotic Dynamics: Theory and Applications To Economics, Cambridge University Press, Cambridge.

Nishimura, Karuo and Gerhard Sorger, 1996. Optimal Cycles and Chaos: A Survey, Studies in Nonlinear Dynamics and Econometrics 1, 11-28.

Nychka, D., S. Ellner, A. Ronald Gallant, and D. McCaffrey, 1992. Finding Chaos in Noisy Systems, Journal of the Royal Statistical Society B 54, 399-426.

Nychka, D., B. Bailey, S. Ellner, P. Haaland and M. O'Connell, 1996. FUNFITS: data Analysis and Statistical Tools for Estimating Function, North Carolina Institute of Statistics Mimeoseries No. 2289.

Perron, P., 1989. The Great Crash, the Oil Price Shock, and the Unit Root Hypothesis, Econometrica 57, 1361-401.

Samuelson, P.A., 1958. An Exact Consumption-loan Model of Interest With or Without the Social Contrivance of Money, Journal of Political Economy 66, 467-482.

Sharovskii, A.N., 1964. Coexistence of Cycles of a Continuous Map of a Line into Itself, Ukranichkii Mathematicheskii 16, 61-71.

Serletis, Apostolos, 1994. International Evidence on Breaking Trend Functions in Macroeconomic Variables, Applied Economics 26, 175-179.

Serletis, Apostolos and Periklis Gogas, 1997. Chaos in East European Black Market Exchange Rates, Research in Economics 51, 359-385.

Serletis, Apostolos and Periklis Gogas, 1999. The North American Natural Gas Liquids Markets are Chaotic, The Energy Journal 20, 83-103.

Shintani, Mototsugu and Oliver Linton, 2000. Nonparametric Neural Network Estimation of Lyapunov Exponents and a Direct Test for Chaos, mimeo, Vanderbilt University and London School of Economics.

Shintani, Mototsugu, and Oliver Linton, 2003. Is There Chaos in the World Economy? A Nonparametric Test Using Consistent Standard Errors, International Economic Review 44, 331-358.

Slutzky, E., 1927/1937. The Summation of Random Causes as the Source of Cyclic Processess, Econometrica 5, 105-146 (revised and translated version from the original Russian version in: Problems of Economic Conditions, ed. By The Conjuncture Institute, Moskva (Moscow), Volume 3, Number 1, 1927).

Smith, Peter, 1998. Explaining Chaos, Cambridge University Press, Cambridge.

Solow, Robert M., 1956. Contributions to the Theory of Economics Growth, Quarterly Journal of Economics 70, 65-94.

Stewart, Ian, 1989. Does God Play with Dice: The Mathematics of Chaos, Penguin Books, London.

Verbeek, Marno, 2000. A Guide to Modern Econometrics, Wiley, Chichester, New York.

Whang, Yoon-Jae, and Oliver Linton, 1999. The Asymptotic Distribution of nonparametric estimates of the Lyapunov exponent for stochastic time series, Journal of Econometrics 91, 1-42.

Wolf, Alan, J.B. Swift, H.L. Swinney, and J.A. Vastano, 1985. Determining Lyapunov Exponents From a Time Series, Physica D 16, 285-317.