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On Sharpe-ratio-based Optimal Insurance Design

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by

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Abstract

As an important risk-hedging tool, insurance can increase an individual's expected utility or reduce her risk exposure. However, pursuing both goals is rarely considered in the literature of insurance contracting. This thesis delves into the optimal insurance design problem by striking a balance between the expected utility and the associated risk. To tackle this objective, we resort to the notion of the Sharpe ratio to identify the optimal contract, which is located on the efficient frontier. The focus of this thesis centers on utilizing Value at Risk (VaR) and Tail Value at Risk (TVaR) as risk measures. We derive parametric forms of the optimal indemnity function in scenarios where a decision maker (DM) seeks to maximize end-of-period expected utility subject to a pre-set acceptable risk level. Since the closed-form or analytical solution for such a contract is rather difficult to derive, we present numerous numerical examples to comprehensively explore various aspects of this methodology. As shown by the results, the Shapre-ratio-based contract is relatively robust except in the Pareto case under VaR preference, and increasing the probability level or risk loading factor adversely affects the ratio. Furthermore, we numerically analyze the popular industrial contract specifically the limited excess-of-loss contract, under the framework of VaR. Our findings reveal that the optimal policy is achieved when the upper limit coverage equals VaR minus the deductible amount. This finding bears a strong resemblance to the optimal contract in our proposed model. The results complement the study of Jiang and Ren (2021).

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Table of Contents

List of Figures

List of Tables

List of Symbols, Abbreviations, and Nomenclature

Symbol	Definition
$\mathcal{I}_0$	Set of admissible indemnity functions
X	Random loss
I	Indemnity function
v	Marginal indemnity function
π	Function of premium
π_0	Premium budget
w	Insured's initial wealth
α	Probability level
η	Safety loading factor
VaR	Value at Risk
TVaR	Tail value at Risk

Chapter 1

Introduction

Insurance is a common risk-hedging tool that protects individuals or groups from suffering catastrophic losses. Typically, an insurance policy consists of an indemnity function, which maps the loss to indemnity, and a premium, which is the compensation paid by the insurance buyer to the seller before the policy comes into effect. In practice, the premium is always deterministic while the indemnity is contingent upon the realized loss, which is unknown (or random) at the time that the policy is put in force. Moreover, a larger expected indemnity would result in a higher premium. As such, the design of optimal insurance (or reinsurance) policy is a fascinating topic in actuarial studies. Understandably, different objectives lead to different optimal insurance policies. Given the pivotal role of risk-benefit analysis in the industry, great effort has been put forward on both risk minimization and utility maximization from either the insurance buyer's or seller's perspective. The literature is rich in both veins. To name but a few, Borch (1960) proved that the excess of the loss function can minimize the variance of the insured's total loss under the expected value premium principle. More recent studies have introduced various risk measures, such as Value at Risk (VaR), and Tail Value at Risk (TVaR). See, for example, Cai et al. (2008), Bernard and Tian (2009), Cheung et al. (2014). Furthermore, Arrow (1963) employed expected utility theory to examine models for maximizing the insurer's wealth utility. Building upon his work, Raviv (1979) further developed this concept while considering the participation constraints of insurers. Bernard et al. (2015) devised optimal insurance policies by incorporating rank-dependent expected utility into their framework.

More popular research has focused on the interplay between risk and utility, particularly in the context of expected utility under risk constraints. Huang (2006) explored optimal reinsurance contracts with the aim of maximizing the insured's expected utility while considering VaR constraints. Under a similar idea, Bernard and Tian (2010) studied optimal insurance contracts from both the insured's and insurer's perspectives by taking the insurer's VaR constraint into account. However, these models do not include the concern of moral hazard. In the absence of any constraints on the indemnity function, the function may exhibit discontinuities, leading to potential *ex-post* issues. More specifically, an upward jump in the indemnity function incentivizes the over-reporting of losses by the insured, while a downward jump encourages under-reporting. This issue has gained significant attention among actuarial researchers. For example, Huberman et al. (1983) highlighted that ensuring non-decreasing ceded and retained loss functions, a condition known as incentive compatibility, can mitigate this problem. Tan et al. (2020) further addressed moral hazard by excluding indemnity functions with slopes greater than one or less than zero. It's worth noting that the incentive compatibility condition is also referred to as the no-sabotage condition, as per Carlier and Dana (2003). This principle lays the foundation for further study on the optimal insurance design. For instance, Jiang and Ren (2021) incorporated into their model the incentive compatibility condition, and studied an optimal insurance problem by considering both the expected utility maximization and risk control.

This thesis focuses on the same track as the above-mentioned references and seeks a way to strike the balance between utility improvement and risk control. The notion of the Sharpe ratio is applied within the context of insurance, where the optimal contract is defined to be the maximizer of the ratio between the expected utility improvement and the risk increment (or between the expected utility and the risk). Though the literature on the application of the "Sharpe ratio" in optimal (re)insurance contracting is rather thin, there are numerous relevant studies that mainly focus on the optimization of the risk-adjusted objective. Interested readers can refer to, for example, Chi (2012), Asimit et al. (2013), Cheung and Lo (2017) , and the reference papers cited therein. The existing work that is closest to this thesis is Bølviken and Wang (2019), which proposed the use of the ratio between the achieved expected surplus and risk level to select the optimal insurance. Our model, though bears certain similarity with that of Bølviken and Wang (2019), focuses on the ratio between the achieved expected utility and the corresponding risk level, which is inspired by the notion of the Sharpe ratio – a well-known economics term in Markowitz's portfolio theory. To solve our model, we derive the parametric indemnity function in the first step and then numerically optimize the parameters in the second step. Such a two-step procedure is quite popular in the literature when the closed-form solution is extremely difficult to derive. Our goal is to assess the trade-off between the expected utility and the associated risk. To identify the optimal indemnity function, we convert the problem to a constrained problem and then employ the approach in Jiang and Ren (2021) to get the optimal parametric indemnity function. In the next step, numerical approaches (e.g., the non-convex or non-concave optimization tools that are readily available in modern computing software) are applied to search for the optimal parameters. Our numerical examples reveal that the ratio-based optimal contract could achieve a large expected utility and a small risk simultaneously. Some sensitivity analyses are given to demonstrate the impact of certain parameters on the solution.

This thesis is structured as follows. Chapter 2 reviews important concepts and some preliminaries (e.g., expected utility theory, distortion risk measures, and the Sharpe ratio) for understanding our model. Chapter 3 provides a brief overview of optimal insurance models, which include the models with risk minimization and expected utility maximization as goals, as well as the models that consider risk-to-surplus ratios. Chapter 4 focuses on the model solution, which involves finding the optimal parametric solution as well as the optimal parameters. This chapter also includes ample illustrative numerical examples to better elucidate the implications of our solution. Chapter 5 concludes the thesis and provides avenues for future research. The Python code for the numerical parts is all delegated to the appendix.

Chapter 2

Review of Important Concepts

2.1 Utility theory

Utility theory represents a comprehensive framework designed to offer guidance on decision-making in the face of uncertainty. Addressing this decision-making challenge, one approach is to establish the value of an economic project with uncertain outcomes as its expected value.

Usually, decision-makers do not make choices solely by comparing the expected monetary outcomes. In other words, the value attributed to various potential wealth levels does not directly correspond to the wealth itself. Hence, the concept of utility comes into play: decision-makers employ a utility function denoted as u, where the utility associated with a particular wealth level w is expressed as $u(w)$. This utility function transforms uncertain outcomes, and decisions are typically guided by the expected utility, denoted as $\mathbb{E}[u(w)]$. The decision-maker will prefer w_1 over w_2 if, and only if, $\mathbb{E}[u(w_2)] \leq \mathbb{E}[u(w_1)]$, as long as these expectations exist. For relevant examples of utility functions, such as linear, quadratic, logarithmic, exponential, and power utility functions. We refer the interested readers to Kaas et al. (2008) for a comprehensive review.

2.1.1 Risk aversion

A pivotal concept in insurance economics is risk aversion. Risk aversion is the inclination that leads individuals to seek protection from unpredictability and to purchase insurance, thereby avoiding uncertainty. Decision-makers exhibit risk aversion when they consistently prefer a certain outcome with an expected value of $\mathbb{E}[X]$ over a risky prospect X, regardless of the distribution of X. Their utility function then adheres to the inequality $\mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$ for all X. This condition follows directly from Jensen's inequality if the utility function u is concave. A function is concave if its second-order derivative is non-positive. Hence, risk aversion, represented by the concavity of the utility function u, implies that the marginal increment of utility diminishes with the wealth.

2.1.2 Utility theory in insurance

The role of the insurance company (also known as the insurer) is to mitigate the financial consequences of the damage or destruction of property. The insurer offers policies with the commitment to cover the losses of the policyholder (also known as the insured) in the event of property damage or destruction, where the payment does not exceed the incurred financial loss during the policy period. This payment, contingent on the extent of the loss, is called a claim payment. In exchange for the protection outlined in the policy, the policyholder pays a premium.

Zero-utility premiums

Consider a scenario where an insurer, whose preference is dictated by a utility function v, possesses an initial wealth w_r. The insurer indemnifies the policyholder $I(X)$ for the incurred loss X and sets its price for the coverage as $\pi(I)$. Under the equivalence principle, the minimum premium $\pi(I)$ is sought by solving the following equation:

$$\mathbb{E}[v(w_r - I(X) + \pi(I))] = v(w_r).$$

By setting w_r=0, we arrive at what is known as the zero-utility premium principle, a concept introduced by Bühlmann (1970): the premium $\pi(I)$ calculated according to this principle is the root of the equation

$$\mathbb{E}[v(\pi(I) - I(X))] = v(0).$$

The above equation indicates that incremental expected utility resulting from the random income $\pi(I) - I(X)$ is zero.

Without any loss of generality, based on Denuit et al. (2006), we assume that the function v has been scaled or standardized such that it satisfies the conditions $v(0) = 0$ and $v'(0) = 1$. Since v is non-decreasing and $I(X) \leq X \leq \operatorname{ess\,sup} X$, we have

$$v(0) = \mathbb{E}[v(\pi(I) - I(X))] \geq v[\pi(I) - \operatorname{ess\,sup} X],$$

such that the inequality $\pi(I) \leq \operatorname{ess\,sup} X$ holds trivially, which is also known as the *no-ripoff* principle.

Furthermore, if v is concave then according to Jensen's inequality,

$$v(0) = \mathbb{E}[v(\pi(I) - I(X))] \leq v[\pi(I) - E(I(X))],$$

so that $\pi(I) \geq \mathbb{E}(I(X))$ holds. This is also known as the *non-negative loading* principle. Besides *no-ripoff* and *non-negative loading* principles, more principles regarding premiums can be found in Denuit et al. (2006).

2.2 Risk measures

We denote by $\mathcal{X}$ the set that contains all the loss variables X which are defined on the probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. The risk measure is defined as follows.

Definition 2.1. A risk measure is characterized by a functional ρ that maps the risk-related random variables, e.g. X, to non-negative real numbers $\rho(X)$.

It is understandable that a riskier loss variable X should be equipped with a larger $\rho(X)$. Taking an example in actuarial science, if X is a loss resulting from random events, then $\rho(X)$ can be interpreted as the risk capital of the property.

2.2.1 Properties

In the field of actuarial science, Goovaerts et al. (1984) were among the first who systematically investigated the properties of premium principles. Following their work, numerous avenues are opened up for the further exploration of the properties of risk measures, with a particular focus on their applications and interpretations in economics. In the sequel, we review a list of properties that most risk measures should possess (please refer to Denuit et al. (2006) for further insights). Note that, if the risk measure is monetary, $\rho(X)$ can be interpreted as the risk capital that is needed to make the position $X - \rho(X)$ acceptable Artzner et al. (1999).

1. No-ripoff

   $\rho(X) \leq \operatorname{ess\,sup} X$[1] for all random variables X. Obviously, it is not rational to hold capital greater than maximum loss.

2. Non-negative loading

   $\rho(X) \geq \mathbb{E}(X)$ for all random variables X. That is, the required capital should exceed the expected loss.

[1] It can also be written as max X if X is bounded.

3. Comonotonic additivity

   $\rho(X+Y) = \rho(X) + \rho(Y)$ for all comonotonic random variables X and Y. Given that the comonotonic risks are essentially wagers on the same event and cannot serve as hedges against each other, the risk capital for the combined risk should be equal to the sum of the two individual risk capitals.

4. Subadditivity

   $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for any two loss variables X and Y. Subadditivity is also known as the diversification feature of the risk measure. When an insurer provides coverage for multiple risks together, the overall risk tends to be lower than the sum of the risks if they are to be covered separately, as they may offset each other.

5. Monotonicity

   If $X(\omega) \leq Y(\omega)$ for $\omega \in \Omega$, then $\rho(X) \leq \rho(Y)$. This is straightforward as a riskier position should be charged for a higher capital reserve.

6. Positive homogeneity

   For any positive constant c, $\rho(cX) = c\rho(X)$. Any random variable X and itself are comonotonic, so positive homogeneity can be treated as resulting from the commonotonic additivity property:

$$\rho(cX) = \rho\underbrace{(X + \cdots + X)}_{\text{c terms}} = \rho(X) + \cdots + \rho(X) = c\rho(X).$$

7. Translation invariance

   For any constant c, $\rho(X+c) = \rho(X) + c$. In other words, a certain additional loss amount should be compensated by the same amount in capital reserve.

2.2.2 Distortion risk measures (DRM)

Definition 2.2. Given a non-negative random variable X, its distortion risk measure, denoted by $\rho(X)$, is defined as

$$\rho(X) = \int_0^\infty g(S_X(x))dx, \tag{2.1}$$

where $g(x)$ is called the distortion function, which is a non-decreasing function that satisfies $g(0) = 0$ and $g(1) = 1$.

The DRM is also known as Wang risk measure (Wang et al., 1997). By the properties of Choquet integrals and smoothness of the distortion function, it can be verified that DRM satisfies several properties, such as translation invariance, positive homogeneity, monotonicity, and comonotonic additivity. In addition, Wirch and Hardy (2001) shows that the DRM is coherent in the sense of Artzner et al. (1999) if the distortion function is concave. Furthermore, a concave distortion function can also guarantee non-negative loading for the DRM.

The DRM encompasses many well-known risk measures as its special cases, such as Value at Risk (VaR) and Tail Value at Risk (TVaR). Their definitions are presented in the sequel.

Definition 2.3 (VaR)**.** Given a risk random variable X, whose CDF is $F_X(x)$, for a probability level $\alpha \in (0, 1)$, the $VaR_\alpha(X)$ is defined as

$$VaR_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x : F_X(x) \geq \alpha\}.$$

The distortion function of VaR is given by

$$g_V(x) = \begin{cases} 0, & x \in [0, 1-\alpha), \\ 1, & x \in [1-\alpha, 1]. \end{cases}$$

Note that VaR satisfies most of the properties of distortion risk measure except subadditivity. Take a portfolio consisting of two zero-coupon bonds, A and B, as an example. Assume bond A has a probability of 2% to default, bond B has a probability of 4% to default, and the probability of the portfolio having one bond default is 5%. Suppose each bond has face value 100, and X and Y are loss from bond A and bond B respectively, then we have $P(X = 100) = 2\%$, $P(X = 0) = 98\%$, $P(Y = 100) = 4\%$, $P(Y = 0) = 96\%$, $P(X + Y = 100) = 5\%$, $P(X + Y = 200) = 1\%$, $P(X + Y = 0) = 94\%$.

If we value loss at probability level $\alpha = 0.95$, then $VaR_{0.95}(X)$ and $VaR_{0.95}(Y)$ are both 0, while $VaR_{0.95}(X + Y)$ is 100 because probability of either bond A default or bond B default is 5%. Therefore, we should be very careful when measuring the risk of multiple events by VaR.

Moreover, an important feature of VaR is presented below.

Theorem 2.4 (Dhaene et al. (2002))**.** *Given a random variable X and a function $I(x)$, If $I(x)$ is non-decreasing and left continuous, then*

$$F_{I(X)}^{-1}(\alpha) = I(F_X^{-1}(\alpha)).$$ [2]

Definition 2.5 (TVaR)**.** Given a random variable X, whose cumulative density function is $F_X(x)$, for a

[2] Or equivalently $VaR_\alpha(I(X)) = I(VaR_\alpha(X))$

probability level $\alpha \in (0,1)$, the $TVaR_\alpha(X)$ is defined as

$$TVaR_\alpha(X) = \mathbb{E}[X|X \geq VaR_\alpha(X)] = \frac{1}{1-\alpha}\int_\alpha^1 VaR_t(X)dt.$$

The distortion function of TVaR is given by

$$g_T(x) = \begin{cases} \frac{x}{1-\alpha}, & x \in [0, 1-\alpha), \\ 1, & x \in [1-\alpha, 1]. \end{cases}$$

Here g_T is a non-decreasing distortion function. Furthermore, g_T is continuous and concave. Since g_T is concave, $TVaR_\alpha$ is coherent and satisfies all properties in Section 2.2.1.

It is not difficult to tell the difference between TVaR and VaR based on their definitions. VaR measures the risk of a random loss by focusing on its percentile only, while TVaR considers all the tail values that are greater than the percentile. Apparently, TVaR includes more tail information than VaR does.

2.3 Sharpe ratio

In 1966, the economist William F. Sharpe introduced the Sharpe ratio, which stems from his research on the Capital Asset Pricing Model(CAPM) and is coined as the "reward-to-variability ratio" Sharpe (1966).

Sharpe ratio is derived from the Capital Asset Pricing Model (CAPM), which is a theory developed from Markowitz's portfolio optimization theory. To understand it graphically, the following plot is provided.

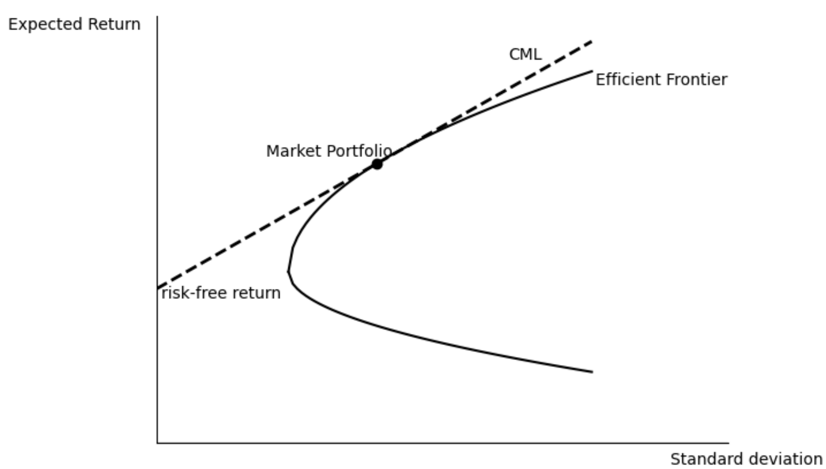

Figure 2.1: Expected return to volatility.

The solution to Markowitz's Model depicts the so-called "efficient frontier" of the portfolio, on which the expected return and standard deviation of the portfolio move along the same direction. The ones that lie on the lower boundary are not favorable as one can always enhance the expected return by choosing the portfolio on the efficient frontier. The general shape of the efficient frontier is almost the same for all investors as the classical model assumes a profit-maximizing investor, who is subject to the constraint on the portfolio's volatility.

In the realm of investment analysis, the CAPM is quite well known. It assesses the relationship between the portfolio return, risk-free asset return, market excess return, and the portfolio's sensitivity with respect to the market excess return:

$$\mathbb{E}[r_p] = r_f + \beta(\mathbb{E}[r_M] - r_f),$$

where

- $\mathbb{E}[r_p]$ is the expected return of a portfolio;
- r_f is the risk-free return;
- β is the sensitivity of the expected excess asset returns to the expected excess market returns;
- $\mathbb{E}[r_M]$ is the expected return of the market.

Despite the debate on the simple linear relationship between the portfolio's return and the market excess return, CAPM is widely applied as it allows a simple comparison between various assets or portfolios.

A notion that is closely tied to CAPM is the Sharpe ratio – another way to measure the performance of portfolios. The definition of the Sharpe ratio is given below.

$$\frac{\mathbb{E}[r_p - r_f]}{\sigma_p},$$

where

- r_p is the portfolio return;
- r_f is the risk-free return;
- $E[r_p - r_f]$ is the expected excess return of the portfolio as compared with the risk-free asset;
- $\sigma_p =$ is the standard deviation of the portfolio.

As the Sharpe ratio is simply the ratio between the portfolio's excess return and the associated standard deviation, it can be easily interpreted as the portfolio's excess return per unit of excess risk. An investor would certainly like to maximize the Sharpe ratio, which yields the tangent line as shown in Fig. 2.1, in which the slope is exactly the maximized Sharpe ratio. Understandably, maximizing the Sharpe ratio can result in a balance between maximizing the excess return and minimizing the excess risk.

The application of the Sharpe ratio can be extended to the optimal insurance problem in Chapter 4, where different insurance policies are compared using a similar ratio-based measure.

Chapter 3

Review of Literature in Optimal Insurance Contracting

This chapter serves as a review of some classical models in the design of optimal (re)insurance. Throughout this chapter and in the subsequent chapters, the notations $(x)_+ = \max\{0, x\}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ will be used frequently.

Assume that a decision maker (DM), who is endowed with the initial wealth w, is faced with a potential loss at the end of the current period, where the loss is denoted by a random variable X. The DM is assumed to be risk-averse, where the risk aversion is captured by a concave utility function. The DM intends to buy an insurance contract that promises to pay $I(X)$ to the DM.

Here the function $I(\cdot)$ is called the indemnity function. In most cases, the contract is priced based on the choice of $I(\cdot)$, which reduces the problem of optimal (re)insurance to the determination of the indemnity function. In practice, commonly applied indemnity functions include, to name but a few, excess-of-loss function, quota-share function, and limited excess-of-loss function. The use of the excess-of-loss function is due to not only its simplicity but also the theoretical foundation laid by Arrow (1974) and Borch (1960), where the former studied a unilateral expected utility maximization problem under the expected value premium principle while the latter designed the optimal reinsurance by minimizing the variance of the DM's end-of-period risk exposure.

The past few decades have witnessed tremendous effort in extending the pioneering works of Arrow (1974) within the frameworks of either the expected utility maximization, risk minimization, or the mixture of both. Plenty of advances have been developed along these three tracks. Interested readers are referred to the book Albrecher et al. (2017) and the review paper Cai and Chi (2020) for a more comprehensive review of the most recent developments. In the remainder of this section, we only review some of the literature that is closely related to the main work of this thesis.

3.1 The literature on risk minimization only

3.1.1 Review of the VaR-based optimal reinsurance by using the geometric approach

Despite the non-subadditivity, VaR is still applied as one of the most popular risk measures in either the banking industry or the insurance industry. Cheung (2010) re-visited the optimal reinsurance problem under the VaR preference, where the insurer intends to minimize her total risk exposure after purchasing a reinsurance contract that is priced by using either the expected value premium principle or Wang's premium principle. The model is formulated as:

$$\min_{I \in \mathcal{C}_0} \ \mathrm{VaR}_\alpha(X - I(X) + \pi(I(X))), \tag{3.1}$$

where $I(X)$ is the indemnity function with respect to loss X and $\mathcal{C}_0$ denotes the set of all non-decreasing convex functions that satisfy $I(0) = 0$ and $I(x) \in [0, x]$ for all $x \geq 0$.

The key idea involved in the geometric approach is to show geometrically the optimal form of the indemnity function. By using the expected value premium principle, the translation invariance and comonotonic additivity properties of VaR, as well as Theorem 2.4, one can re-write the optimization problem as

$$\min_{I \in \mathcal{C}_0} \ \mathrm{VaR}_\alpha(X - I(X) + \pi(I(X))) = \min_{I \in \mathcal{C}_0} \ \mathrm{VaR}_\alpha(X) - I(\mathrm{VaR}_\alpha(X)) + (1+\eta)\mathbb{E}[I(X)], \tag{3.2}$$

where $\eta \geq 0$ is a safety loading factor. Let $a = \mathrm{VaR}_\alpha(X)$. For any indemnity function $I \in \mathcal{C}_0$, if one considers all the functions within a smaller class:

$$\left\{ \tilde{I} : \ \tilde{I} \in \mathcal{C}_0, \ \tilde{I}(a) = I(a) \right\},$$

then it is easy to find, in a geometric way, that the function I always dominates a function h, as shown in Fig. 3.1, where $(a, I(a))$ is a point of tangency between I and h.

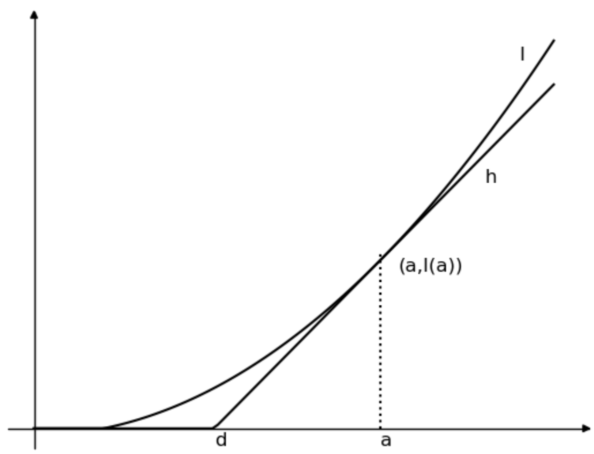

Figure 3.1: The function h is dominated by the function I.

Since I dominates h, we have

$$a - I(a) + (1+\eta)\mathbb{E}[I(X)] \geq a - h(a) + (1+\eta)\mathbb{E}[h(X)],$$

from which we conclude that h outperforms f in terms of minimizing the objective.

It is easy to show that the parametric form of h is $h(x) = c(x-d)_+$, where d is the deductible level and c is the marginal indemnity for the loss greater than the deductible level.

With the optimal parametric form, the rest work is to solve the following finite-dimensional optimization problem

$$\min_{c\in[0,1],d\in[0,\infty)} \; a - c(a-d)_+ + (1+\eta)\mathbb{E}[c(X-d)_+]. \tag{3.3}$$

Apparently, one only needs to resort to basic calculus knowledge to optimize the two parameters in the above problem.

3.1.2 Review of the optimal reinsurance problem under the cost-benefit analysis

Cheung and Lo (2017) examined the indemnity function that minimizes an insurer's risk-adjusted liability that is measured by a distortion risk measure in the presence of budget constraint. Let $T_I(X) = X - I(X) + \pi(I(X))$, the main problem is formulated as:

$$\begin{cases} \inf\limits_{I\in\mathcal{C}_1} \mathbb{E}[T_I(X)] + \delta * \rho(T_I(X) - \mathbb{E}[T_I(X)]), \\ s.t.\ \pi(I(X)) \leq \pi_0, \end{cases} \tag{3.4}$$

where $\mathcal{C}_1$ represents the collection of all the functions that satisfy $I(0) = 0$ and 1-Lipschitz continuity, δ is known as the cost-of-capital rate and $\delta * \rho(T_I(X) - \mathbb{E}[T_I(X)])$ is recognized as the risk margin which is used to cover extra risk beyond the actuarial reserve. When $\pi_0 \to \infty$, the Problem (3.4) becomes a budget-free problem.

A sketch of the idea that solves Problem (3.4) is given below. First, a representation regarding the distortion risk measure of the ceded loss is proved, which lays the foundation for further analysis:

$$\rho(I(X)) = \int_0^\infty g\left(S_X(t)\right) dI(t).$$

With the above representation, we have

$$\mathbb{E}[T_I(X)] + \delta\rho(T_I(X) - \mathbb{E}[T_I(X)]) = \delta\rho(X) + (1-\delta)\mathbb{E}[X] + \int_0^\infty G_X(t)I^{'}(t)dt,$$

where

$$G_X(t) = r(S_X(t)) - [\delta g(S_X(t)) + (1-\delta)(S_X(t))],$$

and

$$\pi(I(X)) = \int_0^\infty r(S_X(t))dI(t).$$

Apparently, only the term $\int_0^\infty G_X(t)I^{'}(t)dt$ includes the decision variable (i.e., the indemnity function). Therefore, minimizing the objective of (3.4) amounts to minimizing $\int_0^\infty G_X(t)I^{'}(t)dt$. By using element-wise optimization, it is easy to derive the solution to the budget-free problem:

$$I^*(x) = \int_0^x \left\{\mathbf{1}_{\{G_X<0\}}(t) + \mathbf{1}_{\{G_X=0\}}(t)h^*(t)\right\} dt, \tag{3.5}$$

where $h^*(\cdot)$ is an arbitrary Lebesgue-measurable function that makes I^* be within $\mathcal{C}_1$.

For the budget-constrained problem, if $\int_{\{G_X\leq 0\}} r(S_X(t))dt \leq \pi_0$ for a given π_0, then the solution to the budget-free problem also solves the budget-constrained problem. Therefore, it is more interesting to investigate the form of $I^*(X)$ when ceding all the excess layers of loss when the benefit of reinsurance exceeds or is equal to its cost violates the budget constraint. For that purpose, the paper defines a benefit-to-cost ratio $H_X : [0, \operatorname{ess\,sup} X] \to \boldsymbol{R}^+$:

$$H_X(t) = \frac{\delta g(S_X(t)) + (1-\delta)(S_X(t))}{r(S_X(t))}. \tag{3.6}$$

It is obvious that $H_X \geq 1$ if and only if $G_X \leq 0$. Note that the numerator and denominator of H_X represent the benefit and cost associated with reinsurance. By transferring the surplus layers of loss with a higher value of H_X, the insurer can attain a more substantial reduction in risk-adjusted liabilities while incurring a smaller reinsurance premium. To minimize the overall liability within the constraint of a limited budget, the above analysis implies that the insurer should start ceding from the point where H_X is at its maximum and stop ceding when all the premiums have been utilized. The paper shows the optimal solutions based on different level sets of H_X and the forms of I^* are similar to that of the function (3.5). For more details, please refer to Cheung and Lo (2017).

3.2 The literature on the mixture of the expected utility maximization and risk control

3.2.1 Review of the optimal insurance design under the insured's VaR constraint

In practice, solvency is what concerns most decision-makers when they pursue the maximized interests. Huang (2006) studied an optimal insurance contracting problem where the insured aims to maximize her expected utility under the VaR constraint, for which the optimization problem could be formulated as

$$\begin{aligned}
\max_{B>0, 0\leq R(x)\leq x} \quad & \mathbb{E}[u(w-B-R(X)], \\
s.t. \quad & B = C(\mathbb{E}[I(X)]), C(0)=0, C^{'}(\mathbb{E}[I(X)])>1, C^{''}(\mathbb{E}[I(X)])>0, \text{ for all } \mathbb{E}[I(X)]>0, \\
& \mathbb{P}\{R(X)\leq K\} \geq 1-\alpha, \text{where } K = VaR_{1-\alpha}(R(X)-\mathbb{E}[R(X)]) + \mathbb{E}[R(X)],
\end{aligned}$$

where B represents the premium, $R(x) := x - I(x)$ is the retained loss schedule and $I(x)$ is the indemnity function.

The paper first assumes a fixed premium B and derives the optimal form of $R(x)$. For that purpose, let $A = F^{-1}(1-\alpha)$, and assume that α and $\mathbb{E}[I(X)]$ are small enough such that $K + \mathbb{E}[I(X)] < A$, and the loss variable X has an upper bound T, then define

$$\begin{aligned}
B_{min} &\in \arg_B \int_K^A (x-K)f(x)dx - C^{-1}(B) = 0, \\
B_A &\in \arg_B \int_K^A (x-K)f(x)dx + \int_A^T (x-A)f(x)dx - C^{-1}(B) = 0, \\
B_K &\in \arg_B \int_K^T (x-K)f(x)dx - C^{-1}(B) = 0.
\end{aligned}$$

The paper shows that B_{min}, B_A, and B_K are uniquely determined by proving their corresponding functions $G_{min}(B) = \int_K^A (x-K)f(x)dx - C^{-1}(B)$, $G_A(B) = \int_K^A (x-K)f(x)dx + \int_A^T (x-A)f(x)dx - C^{-1}(B)$ and $G_K(B) = \int_K^T (x-K)f(x)dx - C^{-1}(B)$ are all positive and always decreasing in B.

The optimal form of $R(x)$ is summarized below, with a discussion of the premium level within different ranges:

$$R^*(x) = \begin{cases} (x \wedge K) \cdot \mathbf{1}_{[0,A]}(x) + (x \wedge \bar{D}) \cdot \mathbf{1}_{[A,T]}(x), & B_{min} \leq B \leq B_A, \\ (x \wedge K) \cdot \mathbf{1}_{[0,A]}(x) + \bar{D} \cdot \mathbf{1}_{[A,T]}(x), & B_A < B < B_K, \\ (x \wedge D^*) \cdot \mathbf{1}_{[0,T]}(x), & B_K \leq B, \\ \text{No solution}, & B < B_{min}, \end{cases}$$

where $\bar{D} \in [A,T]$ for $B_{min} \leq B \leq B_A$ and $\bar{D} \in [K,A]$ for $B_A < B < B_K$. $\bar{D}$ is the upper deductible such that $\int_0^A (x \wedge K)f(x)dx + \int_A^T (x \wedge \bar{D})f(x)dx = \mathbb{E}[X] - C^{-1}(B)$, and D^* is the optimal deductible such that $\int_0^{D^*} xf(x)dx + D^* \int_{D^*}^T f(x)dx = \mathbb{E}[R(X)]$.

Let $R_B(X)$ denote the particular $R(x)$ with respect to a specific B, then the optimal problem can be rewritten as :

$$\max_{B \geq B_{min}} \quad \mathbb{E}[u(w - B - R_B(X)].$$

By specifying the distribution of X and the utility function u, then optimal B can be found numerically within $[B_{min}, B_{max}]$. As will be shown in Chapter 4, we follow this idea to find the optimal premium and the specific indemnity function for our problem. However, the difference is our main problem considers maximizing the ratio between expected utility and the corresponding VaR(TVaR) risk level, we assume a 1- Lipschitz continuous indemnity function that excludes moral hazards, and we reformulate our problem in integral forms to derive optimal forms of indemnity function.

Huang (2006) also discussed the different coverage levels under the common insurance contracts, such as the excess-of-loss insurance, the limited excess-of-loss insurance, and the quota-share insurance. Please refer to Huang (2006) if interested.

3.2.2 Review of the optimal insurance design under the insurer's VaR constraint

Zhou and Wu (2009) studied the optimal insurance policy that maximizes the insured's expected utility under the insurer's VaR constraint. The mathematical problem is written as :

$$\max_{I} \mathbb{E}[u(w_d - X + I(X) - \pi(\mathbb{E}[I(X)])]$$
$$s.t.\ \ \mathbb{P}\{w_r - I(X) + \pi(\mathbb{E}[I(X)]) \geq \underline{w}\} \geq 1 - \alpha,$$

where w_d and w_r are the initial wealth of the insured and insurer respectively, $\pi(\mathbb{E}[I(X)])$ represents the insurance premium, $\pi(x)$ is a strictly increasing function of x with $\pi(0) = 0$, and the floor $\underline{w}$ and $\alpha \in [0, 1]$ are exogenously specified.

The methodology proposed by the paper is: first, fix the premium level B, and rewrite the optimization problem. By adopting the convex-duality approach (see Karatzas and Shreve (1998) for example), the optimal $I^*(x)$ is proved to be of the form:

$$I_1^*(x) = (x - d)_+,$$

or

$$I_2^*(x) = \begin{cases} (x-d)_+, & x \leq d + \kappa, \\ \kappa, & d + \kappa < x \leq \bar{x}, \\ x - d, & x > \bar{x}, \end{cases}$$

where $\bar{x}$ be such that $\mathbb{P}(X \leq \bar{x}) = 1 - \alpha$, $\kappa = w_r + B - \underline{w} \geq 0$, $0 \leq B \leq \pi(\mathbb{E}[X])$, and $d \geq 0$ satisfies $\mathbb{E}[I^*(X)] = \pi^{-1}(B)$.

If the risk constraint is binding, i.e. $0 \leq \kappa < \bar{x}$, then optimal indemnity function should be $I_2^*(x)$, otherwise $I_1^*(x)$ is optimal.

Next, since d can be treated as a function of B, the paper figures out the optimal premium or the optimal d using the approach stated below.

If $I_1^*(x)$ is the optimal solution, where $d \geq 0$, considering $u'(d) = 0$, risk constraint $\mathbb{P}(I(X) \leq \kappa) \geq 1 - \alpha$, and definition of $\bar{x}$, it can be proved that $I_1^*(x)$ should satisfy

$$u'(w_d - d - \pi(\mathbb{E}[I_1^*(X)])) = \pi'(\mathbb{E}[I_1^*(X)])\mathbb{E}[u'(w_d - X + I_1^*(X) - \pi(\mathbb{E}[I_1^*(X)])],$$

and

$$w_r - \underline{w} + \pi(\mathbb{E}[I_1^*(X)]) \geq (\bar{x} - d)_+.$$

If $I_2^*(x)$ is the optimal solution, where $d, \kappa \in (0, \infty)$ and $d + \kappa \leq \bar{x}$, then $I_2^*(X)$ should satisfy the insurer's risk constraint with equality and apply the method of Lagrange multipliers, we have

$$\begin{aligned} &u'(w_d - d - \pi(\mathbb{E}[I_2^*(X)]))[1 - \pi'(\mathbb{E}[I_2^*(X)]) + \pi'(\mathbb{E}[I_2^*(X)])F(d)] \\ &= \pi'(\mathbb{E}[I_2^*(X)])\mathbb{E}[u'(w_d - X - \pi(\mathbb{E}[I_2^*(X)]))\mathbf{1}_{\{X \leq d\}}], \end{aligned}$$

and

$$w_r - \underline{w} + \pi(\mathbb{E}[I_2^*(X)]) = \kappa,$$

where $F(\cdot)$ is the CDF of X.

This thesis will apply a similar idea as presented in Zhou and Wu (2009). That is, we first fix the premium at some level B, which allows us to identify the optimal indemnity function, and then we search for the optimal premium within $[0, B_{max}]$.

3.2.3 Review of the optimal reinsurance that optimizes the risk over surplus ratios

Bølviken and Wang (2019) studied the optimal reinsurance problem where the VaR and expected surplus are balanced through their ratio and show how the risk-adjusted surplus is calculated via the optimized indemnity function.

The paper first sets up the form of risk-adjusted surplus

$$\mathcal{L}(I) = \mathcal{G}(I) - \delta\rho(I), \tag{3.7}$$

where $\rho(I)$ is a risk measure and $\delta > 0$ is a price on risk. $\mathcal{G}(I)$ is the expected surplus under a reinsurance policy. To maximize $\mathcal{L}(I)$, it is sufficient to find the minimum $\rho(I)$ as δ varies to define an efficient frontier for a given value of $\mathcal{G}(I)$.

The ratio of risk over expected surplus is defined as follows:

$$\mathcal{C}(I) = \frac{\rho(I)}{\mathcal{G}(I)} \quad \text{with} \quad \mathcal{G}(I) > 0. \tag{3.8}$$

The main idea is to search for the optimal indemnity function I to minimize $\mathcal{C}(I)$. The criterion not only embodies the industrial perspectives but also serves as a tool to minimize the risk relative to the expected surplus in terms of the monetary unit. The paper moves on to demonstrate that the optimal solutions align with the efficient frontier of the Markowitz model and fall within the same category as those that maximize the risk-adjusted surplus. This finding inspires further exploration of the core issue concerning the expected utility of insurance over risk within this thesis.

The paper also looks into scenarios involving large portfolios, approximation methods for determining the optimal portfolio, and theoretical aspects regarding performance deterioration. If readers are interested, please find more from Bølviken and Wang (2019).

3.2.4 Review of the effect of risk constraints on the optimal insurance policy

Jiang and Ren (2021) studied the maximization of a decision maker's expected utility with distortion-risk-measure-type constraints, where a general characterization of the optimal indemnity function is obtained with closed-forms for some specific cases derived, such as the cases where VaR and TVaR are applied.

A few assumptions are presented before introducing the optimal problems. First, to avoid the *ex-post* moral hazards, the admissible indemnity functions I can only be from $\mathcal{S}_0$, where

$$\mathcal{S}_0 := \{I : [0, M] \to [0, M] |\ I(0) = 0 \text{ and } \ 0 \leq I(x_2) - I(x_1) \leq x_2 - x_1 \text{ for } \ 0 \leq x_1 \leq x_2\}.$$

For any function $I \in \mathcal{S}_0$, indemnity functions $I(x)$ and retained loss $X - I(X)$ are comonotonic and non-decreasing. Moreover, I is in 1-Lipschitz forms and is continuous and almost everywhere differentiable, which admits the representation $I(x) = \int_0^x v(t)dt$ where $0 \leq v(x) \leq 1$.

Second, to simplify discussions, the insurance premium is calculated by the expectation premium principle:

$$\pi(I) = (1 + \eta)\mathbb{E}[I(X)],$$

where $\eta \geq 0$ is the safety loading factor.

The main problem is formulated from either the insured's or the insurer's perspective.

(1) The insured's decision problem

$$
\begin{aligned}
&\max_{I\in\mathcal{S}_0} \mathbb{E}[u(w_d - X + I(X) - \pi(I)] \\
&s.t.\ \rho_d(X - I(X) + \pi(I)) \leq A_d, \\
&\quad \rho_r(I(X) - \pi(I)) \leq A_r, \\
&\quad \pi(I) \leq \pi_0.
\end{aligned} \tag{3.9}
$$

(2) The insurer's decision problem

$$
\begin{aligned}
&\max_{I\in\mathcal{S}_0} \mathbb{E}[\tilde{u}(w_r - I(X) + \pi(I)] \\
&s.t.\ \rho_d(X - I(X) + \pi(I)) \leq A_d, \\
&\quad \rho_r(I(X) - \pi(I)) \leq A_r, \\
&\quad \pi(I) \leq \pi_0.
\end{aligned} \tag{3.10}
$$

Then the paper applies the calculus of variations with some modification and reformulates the optimization problem. Now the Problem (3.9) is stated as follows:

$$
\begin{aligned}
&\max_{0\leq v(t)\leq 1} \int_0^M \left\{ \int_t^M u'(w_d - x + I^*(x) - B) dF_X(x) \right\} v(t) dt \\
&s.t.\ \int_0^M g_d(S_X(t)) v(t) dt \geq C_d, \\
&\quad \int_0^M g_r(S_X(t)) v(t) dt \leq C_r, \\
&\quad \int_0^M S_X(t) v(t) dt = \frac{B}{1+\theta},
\end{aligned}
$$

where $C_d = \rho_d(X) + B - A_d$ and $C_r = A_r + B$.

Next, by adopting the Lagrangian dual approach, the objective function of the dual problem is:

$$
L_d(t; I^*, \lambda_1, \lambda_2, \lambda_3) = \int_t^M u^{'}(w_d - x + I^*(x) - B) dF_X(x) + \lambda_1 g_d(S_X(t)) - \lambda_2 g_r(S_X(t)) + \lambda_3 S_X(t),
$$

where λ_1,$\lambda_2 \in \boldsymbol{R}^+$ and $\lambda_3 \in \boldsymbol{R}$. Then a function v^* solves the Problem (3.9) if and only if

$$
v^*(t) = \mathbf{1}_D(t) + \xi(t)\mathbf{1}_E(t),
$$

where

$$D = \{t : L_d(t; I^*, \lambda_1, \lambda_2, \lambda_3) > 0\},$$

$$E = \{t : L_d(t; I^*, \lambda_1, \lambda_2, \lambda_3) = 0\}.$$

$\mathbf{1}_D(t)$ is an indicator function and $\xi(t) \in [0, 1]$ is any function such that $0 \leq v^*(t) \leq 1$.

The Problem (3.10) can be solved analogously. This thesis is built on Jiang and Ren (2021) and aims to extend their results by identifying the optimal risk levels, as well as the associated optimal utilities.

Chapter 4

Main Results

Suppose a risk-averse insured is endowed with the initial wealth w, and has a preference measure that is depicted by an increasing and strictly concave utility function u. The insured faces a random loss of X which follows a certain distribution. Let $f_X(x)$, $F_X(x)$, and $S_X(x)$ denote the probability density function (PDF), cumulative distribution function (CDF), and survival function of X respectively. The insured would consider buying an insurance policy, which mainly composes the premium $\pi(I)$ and a promised indemnity $I(X)$. To reduce the *ex-post* moral hazard, we impose additional assumptions on the indemnity functions following the rationale as stated in Huberman et al. (1983) and Tan et al. (2020). In line with the literature (e.g. Chi and Zhuang (2020) and Cheung and Lo (2017)), we will only consider the indemnity functions from the following class

$$\mathcal{I}_0 := \left\{ I : [0, \operatorname{ess\,sup}(X)) \to [0, \operatorname{ess\,sup}(X)) \,\middle|\, \begin{array}{c} 0 \le I(x) \le x \quad \text{for all} \quad x \ge 0, \\ I(0) = 0, \\ 0 \le I(x_2) - I(x_1) \le x_2 - x_1 \text{ for } \ 0 \le x_1 \le x_2. \end{array} \right\}$$

For any function $I \in \mathcal{I}_0$, the function $I(X)$ and retained loss $X - I(X)$ are non-decreasing and comonotonic. Moreover, any function $I \in \mathcal{I}_0$ possesses Lipschitz continuity property and therefore is differentiable almost everywhere. As a result, according to Zhuang et al. (2016), the indemnity function $I(x) \in \mathcal{I}_0$ can be written in an integral form:

$$I(x) = \int_0^x v(t)dt, \tag{4.1}$$

where $0 \le v(t) \le 1$.

We define the ratio between the expected utility and the associated risk, and the main problem of the thesis is to maximize it:

Problem 4.1 (Main problem)**.**

$$\max_{I\in\mathcal{I}_0} \frac{\mathbb{E}[u(w-X+I(X)-\pi(I)]}{\rho(X-I(X)+\pi(I))}$$
$$s.t. \quad \pi(I)\leq \pi_0.$$

The solution to the above problem is found using two steps. First, we fix the insurance premium $\pi(I)$ at some level $0<B\leq \pi_0$ and later find the optimal premium level B^* numerically within this range. Second, similar to the method applied in Jiang and Ren (2021), we rewrite the problem in integral forms. For the sake of convenience in notation, we use $\int_0^\infty$ to represent $\int_0^{\mathrm{ess\,sup}(X)}$.

To handle the numerator of the problem, we apply the calculus of variations with some adjustments. Consequently, suppose $I^*(X)\in\mathcal{I}_0$ is the optimal solution to the problem, for $\epsilon\in[0,1]$, $\epsilon I^*(X)+(1-\epsilon)I(X)$ should also be within $\mathcal{I}_0$. Define

$$\begin{aligned}
K(\epsilon) &= \mathbb{E}[u(w-X+\epsilon I^*(X)+(1-\epsilon)I(X)-\pi(I))] \\
&= \mathbb{E}[u(w-X+\epsilon I^*(X)+I(X)-\epsilon I(X)-B)] \\
&= \mathbb{E}[u(w-X+\epsilon(I^*(X)-I(X))+I(X)-B)],
\end{aligned}$$

then the first-order derivative $K'(\epsilon)=\mathbb{E}[u'(w-X+\epsilon(I^*(X)-I(X))+I(X)-B)](I^*(X)-I(X))$ and the second-order derivative $K''(\epsilon)=\mathbb{E}[u''(w-X+\epsilon(I^*(X)-I(X))+I(X)-B)](I^*(X)-I(X))^2$. Due to the strict concavity of $u(x)$, it is easy to verify that $K^{''}(\epsilon)<0$. To this end, we can find a sufficient and necessary condition for $I^*(X)$ to be the optimal solution:

$$\begin{aligned}
K'(1) &= \mathbb{E}[u'(w-X+I^*(X)-B)](I^*(X)-I(X))\geq 0 \\
&\Rightarrow \mathbb{E}[u'(w-X+I^*(X)-B)I^*(X)]\geq \mathbb{E}[u'(w-X+I^*(X)-B)I(X)].
\end{aligned}$$

In other words, $I^*(X)$ is also the solution to

$$\max_{I\in\mathcal{I}_0} \mathbb{E}[u'(w-X+I^*(X)-B)I(X)].$$

Therefore, the objective expected utility part can be rewritten as

$$
\begin{aligned}
&\mathbb{E}[u'(w - X + I^*(X) - B)I(X)] \\
&= \int_0^\infty u'(w - X + I^*(X) - B)I(X)f_X(x)dx \\
&= \int_0^\infty u'(w - X + I^*(X) - B)\{\int_0^x v(t)dt\}dF_X(x) \\
&= \int_0^\infty \left\{\int_t^\infty u'(w - X + I^*(X) - B)dF_X(x)\right\} v(t)dt,
\end{aligned}
$$

where the last equation is due to the Fubini's Theorem.

Next, we focus on the denominator of the problem. Due to the comonotonic additivity and translation invariance properties of distortion risk measures, we have:

$$
\begin{aligned}
\rho(X - I(X) + \pi(I)) &= \rho(X - I(X) + B) \\
&= \rho(X) - \rho(I(X)) + B.
\end{aligned}
$$

Recall that the distortion risk measure for a non-negative random variable is $\rho(X) = \int_0^\infty g(S_X(x))dx$, and as proved rigorously in Cheung and Lo (2017), we know

$$
\begin{aligned}
\rho(I(X)) &= \int_0^\infty g(S_X(t))dI(t) \\
&= \int_0^\infty g(S_X(t))v(t)dt.
\end{aligned}
$$

If we were to fix the risk level $\rho(X - I(X) + \pi(I)) = C$ and $C \leq \operatorname{ess\,sup} X$, the denominator of the ratio in Problem (4.1) can be written as follows:

$$
\begin{aligned}
&\rho(X) - \rho(I(X)) + B = C \\
\Longrightarrow\ &\rho(I(X)) = \rho(X) + B - C \\
\Longrightarrow\ &\int_0^\infty g(S_X(t))v(t)dt = \rho(X) + B - C.
\end{aligned}
$$

In line with most literature, we assume that the insurance premium is determined by the expectation premium principle

$$
\pi(I) = (1 + \eta)\mathbb{E}[I(X)]. \tag{4.2}
$$

where $\eta \geq 0$ is the loading factor. By applying the Fubini's theorem again, we also have

$$\begin{aligned}\mathbb{E}[I(X)] &= \int_0^\infty I(X)dF_X(x) \\ &= \int_0^\infty \{\int_0^x v(t)dt\} f_X(x)dx \\ &= \int_0^\infty \{\int_t^\infty f_X(x)dx\} v(t)dt \\ &= \int_0^\infty S_X(t)v(t)dt.\end{aligned}$$

Then the premium constraint is easily written as :

$$\int_0^\infty S_X(t)v(t)dt = \frac{B}{1+\eta}.$$

To sum up, to solve Problem (4.1) we need to solve the Problem (4.3) first, which is much more tractable:

$$\begin{aligned}\max_{0 \leq v(t) \leq 1} \int_0^\infty & \left\{\int_t^\infty u'(w - X + I^*(X) - B)dF_X(x)\right\} v(t)dt \\ s.t. \quad & \int_0^\infty g(S_X(t))v(t)dt = \rho(X) + B - C, \\ & \int_0^\infty S_X(t)v(t)dt = \frac{B}{1+\eta}.\end{aligned} \tag{4.3}$$

After applying the Lagrangian dual approach and point-wise maximization method, the implicit solution of the Problem (4.3) can be stated as follows.

Theorem 4.1. *Suppose $\mathcal{I}_0$ is non-empty and $I^*(X)$ is the optimal indemnity function where $I^*(X) \in \mathcal{I}_0$. Assume θ_1, $\theta_2 \in \boldsymbol{R}$, Let*

$$L(t; I^*, \theta_1, \theta_2) = \int_t^\infty u'(w - X + I^*(X) - B)dF_X(x) + \theta_1 g(S_X(t)) + \theta_2 S_X(t). \tag{4.4}$$

Then the sufficient and necessary condition for $v^*(t)$ being the optimal solution is

$$v^*(t) = \mathbf{1}_A(t) + \zeta(t)\mathbf{1}_D(t), \tag{4.5}$$

where $A = \{t : L(t; I^*, \theta_1, \theta_2) > 0\}$ and $D = \{t : L(t; I^*, \theta_1, \theta_2) = 0\}$. $\mathbf{1}_A(t)$ and $\mathbf{1}_D(t)$ are indicator functions and $0 \leq \zeta(t) \leq 1$ is any function such that $0 \leq v^*(t) \leq 1$.

The parameters θ_1 and θ_2 are established based on the satisfaction of the slackness conditions

$$\theta_1 \left(\int_0^\infty g(S_X(t)) v^*(t) dt - (\rho(X) + B - C) \right) = 0,$$

$$\theta_2 \left(\int_0^\infty S_X(t) v^*(t) dt - \frac{B}{1+\eta} \right) = 0.$$

Proof. To prove the sufficiency, we write the Problem (4.3) in its Lagrangian form:

$$\max_{0 \le v(t) \le 1} \int_0^\infty \left\{ \int_t^\infty u'(w - X + I^*(X) - B) dF_X(x) \right\} v(t) dt$$
$$+ \theta_1 (\int_0^\infty g(S_X(t)) v(t) dt - (\rho(X) + B - C)) + \theta_2 (\int_0^\infty S_X(t) v(t) dt - \frac{B}{1+\eta}).$$

It is not difficult to find that the above equation equals to $\max_{0 \le v(t) \le 1} \int_0^\infty L(t; I, \theta_1, \theta_2) v(t) dt$. The integral is maximized if its integrand function is maximized point-wisely, in other words,

$$v^*(t) = \begin{cases} 1, & L(t; I, \theta_1, \theta_2) > 0, \\ \zeta(t), & L(t; I, \theta_1, \theta_2) = 0, \\ 0, & L(t; I, \theta_1, \theta_2) < 0, \end{cases}$$

where $0 \le \zeta(t) \le 1$ and $\zeta(t)$ is arbitrary as long as it satisfies the requirement of $0 \le v(t) \le 1$.

To prove the necessity. Suppose $v^*(t)$ is equals to Eq. (4.5), then we have $\int_0^\infty L(t; I, \theta_1, \theta_2)(v^*(t) - v(t)) \ge 0$, which indicates that $v^*(t)$ is the optimal solution. Additionally, we show that there exists θ_1 and θ_2 that make $v^*(t)$ meets all the constraints. if θ_2 is given, $\theta_1 \to +\infty$ then $L(t; I, \theta_1, \theta_2) \to +\infty$ which leads to $v^*(t) = 1$, while if $\theta_1 \to -\infty$ then $L(t; I, \theta_1, \theta_2) \to -\infty$ that $v^*(t) = 0$ as a result. We can get the same conclusion by examining θ_2 in the same way. This ends the proof.

Although the solution is implicit, it provides insight into the structure of the optimal indemnity function. It is easy to find that the value of function $L(t; I^*, \theta_1, \theta_2)$ increases as the parameter $\theta_1 \in \boldsymbol{R}$ goes up, which leads to a larger set A and as a result $v^*(t) = 1$ (a higher insurance coverage) than that without buying an insurance policy. We will analyze the explicit solutions from the insured's perspective in the next subsection where the risk measures are Value at Risk (VaR) and Tail Value at Risk (TVaR).

4.1 The optimal $I(x)$ under specific risk measures

VaR and TVaR are popularly used as risk measures to determine the risk capital requirement. We would like to study further the explicit solution of our maximization problem under VaR and TVaR respectively in this section.

4.1.1 $I_V^*(x)$ under VaR

Assume VaR at the probability level of α, and let $a = VaR_\alpha(X)$. Note that

$$\int_t^\infty u'(w - X + I^*(X) - B)dF_X(x) = \mathbb{E}[u'(w - X + I^*(X) - B)\mathbf{1}_{[t,\infty]}(X)],$$

$$S_X(t) = \int_t^\infty f_X(x)dx = \mathbb{E}[\mathbf{1}_{[t,\infty]}(X)].$$

Then apply the distortion function for VaR, we have

$$L_V(t; I^*, \theta_1, \theta_2) = \begin{cases} \mathbb{E}[(u'(w - X + I^*(X) - B) + \theta_2)\mathbf{1}_{[t,\infty]}(X)] + \theta_1, & t \leq a, \\ \mathbb{E}[(u'(w - X + I^*(X) - B) + \theta_2)\mathbf{1}_{[t,\infty]}(X)], & t > a. \end{cases} \tag{4.6}$$

Take the most complex case as an example, for $\theta_1 > 0$, the function $L_V(t; I^*, \theta_1, \theta_2)$ has a downward jump at a and otherwise is continuous on $[0, a] \cup (a, \infty)$, and has an intersection point at $(t_0, 0)$ and $(t_1, 0)$ where $t_0 \leq a \leq t_1$. The shape is illustrated in Fig. 4.1.

Proposition 4.1.1. *For Problem (4.1) under VaR, if we set the premium at B level, and fix the risk level such that $VaR_\alpha(X - I_V^*(X) + B) = C$, then the optimal indemnity function $I_V^*(x)$ can be written by $I_V^*(x) = (x \wedge a - t_0)_+ + (x - t_1)_+$ where $0 \leq t_0 \leq a \leq t_1$, $t_0 = C - B$ and $B = (1 + \eta)\mathbb{E}[I_V^*(X)]$.*

Proof. For brevity, let $L_V(t)$ stands for $L_V(t; I^*, \theta_1, \theta_2)$. Then differentiate Eq. (4.6) gives us:

$$\frac{\partial L_V(t)}{\partial t} = L_V^{'}(t) = -(u'(w - t + I^*(t) - B) + \theta_2)f_X(t), \tag{4.7}$$

and define

$$h := inf\{t : u'(w - t + I^*(t) - B) + \theta_2 \geq 0\}. \tag{4.8}$$

Due to the properties of $I(x)$, $t - I^*(t)$ is non-decreasing for any $I \in \mathcal{I}_0$, so $u'(w - t + I^*(t) - B)$ is also non-decreasing.

Then we should have

$$\begin{cases} L'_V(t) > 0, & t \in [0,h), \\ L'_V(t) \leq 0, & t \in [h,\infty). \end{cases}$$

Next, for the most complicated case that we considered, the relative position of h and a will affect the results of $L_V(t)$. In the following, we will discuss the case of $h \geq a$ and $h < a$ respectively.

When $h \geq a$ and suppose there is only one jump at $t = a$, we obtain

$$\begin{cases} L'_V(t) > 0, & t \in [0,a) \cup (a,h), \\ L'_V(t) \leq 0, & t \in [h,\infty). \end{cases}$$

Hence we know the trend of $L_V(t)$. Like Fig. 4.1 that is illustrated below, because $L_V(t)$ is strictly increasing on $[0,a) \cup (a,h)$ with only one jump at a, there should at most one root t_0 exists on $[0,a)$ and at most one root t_1 exists on $(a,h]$. Moreover, since we suppose $t_0 \in [0,a]$ and $t_1 \in [a,h]$, let $t_0 = 0$ if $L_V(0) > 0$, $t_0 = a$ if $L_V(a) < 0$, $t_1 = a$ if $L_V(a) > 0$ and $t_1 = h$ if $L_V(h) < 0$. We can also deduce that $L_V(t) = \mathbb{E}[(u'(w - X + I^*(X) - B) + \theta_2)\mathbf{1}_{[t,\infty]}(X)] \geq 0$ when $t \in [h,\infty)$, based on Eq. (4.8). Now we have

$$\begin{cases} L_V(t) < 0, & t \in [0,t_0), \\ L_V(t) > 0, & t \in (t_0,a], \\ L_V(t) < 0, & t \in (a,t_1], \\ L_V(t) \geq 0, & t \in (t_1,\infty). \end{cases}$$

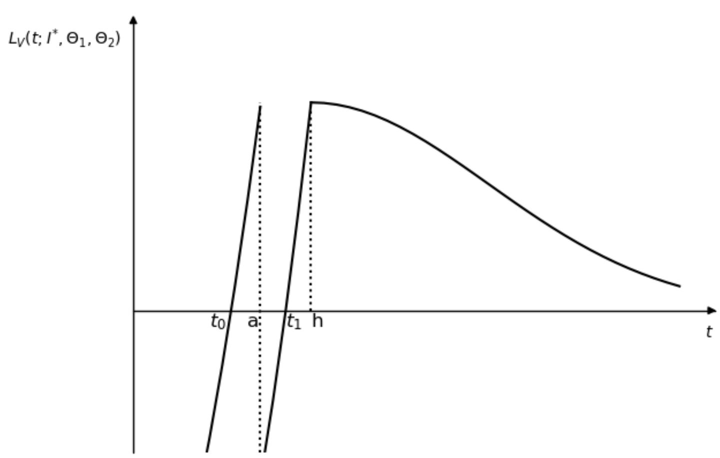

Figure 4.1: Illustration shape of $L_V(t; I^*, \theta_1, \theta_2)$.

Suppose that $L_V(t) = 0$ on some interval Then, according to Eq. (4.7), if $L_V^{'}(t) = 0$ on a certain interval, it implies that $-t + I^*(t)$ is constant on that interval. Consequently, we can deduce that $I^{*'}(t) = 1$ on that interval. Additionally, considering Eq. (4.5) from Theorem 4.1, we conclude that $v^*(t) = I^{*'}(t) = 1$ if $L_V \geq 0$ and $v^*(t) = I^{*'}(t) = 0$ if $L_V < 0$, thereby we have

$$\begin{cases} I^{*'}(t) = 0, & t \in [0, t_0), \\ I^{*'}(t) = 1, & t \in (t_0, a], \\ I^{*'}(t) = 0, & t \in (a, t_1], \\ I^{*'}(t) = 1, & t \in (t_1, \infty). \end{cases}$$

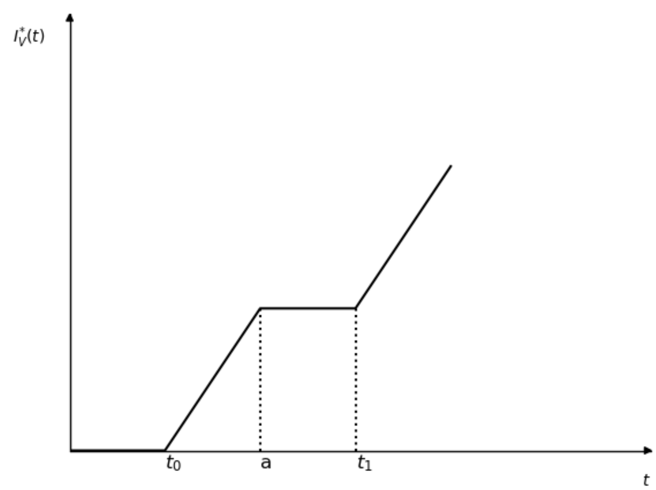

Figure 4.2: Illustrative example of $I_V^*(t)$.

The above figure indicates that the $I_V^*(t)$ should be of the layered form, from which we obtain

$$I_V^*(x) = (x \wedge a - t_0)_+ + (x - t_1)_+. \tag{4.9}$$

The case of $h < a$ is examined similarly.

$$\begin{cases} L'_V(t) > 0, & t \in [0, h), \\ L'_V(t) \leq 0, & t \in [h, a) \cup (a, \infty). \end{cases}$$

$L_V(t)$ now is strictly increasing on $[0, h)$, and it has at most one root t_0. $L_V(t)$ decreases on $[h, \infty)$ but recall that $L_V(t) \geq 0$ on this interval, so there are no other roots here. Therefore,

$$\begin{cases} L_V(t) < 0, & t \in [0, t_0), \\ L_V(t) \geq 0, & t \in (t_0, \infty). \end{cases}$$

This leads to $I^*(x) = (x - t_0)_+$.

Combining these two cases, we conclude that $I_V^*(X) = (x \wedge a - t_0)_+ + (x - t_1)_+$.

Remark 1. *Eq. (4.6) and Fig. 4.1 illustrate that the Lagrangian coefficients θ_1 and θ_2 will affect the intersection points t_0 and t_1, which also intuitively means that optimal t_0 and t_1 can be generated when optimizing θ_1 and θ_2. As a result, we should be able to optimize them directly rather than calculating them through the optimal θ_1 and θ_2. This helps us to focus on t_0 and t_1 and save more effort since they are the parameters of our optimal $I^*(x)$ in Proposition 4.1.1. This reasoning should also apply to the TVaR case.*

From Proposition 4.1.1, we notice that once we fix the premium B, we should be able to find the minimization of risk C and so does the global optimal indemnity function $I^*(x)$ that maximizes expected utility over risk. Furthermore, there should be a global optimal premium $B^* \in (0, \pi_0]$ to determine the optimal solution to the main problem.

4.1.2 $I_T^*(x)$ under TVaR

Now we use TVaR as the risk measure and apply the distortion function (2.5) for TVaR to the function L in Eq. (4.4), we have

$$L_T(t; I^*, \theta_1, \theta_2) = \begin{cases} \mathbb{E}[(u'(w - X + I^*(X) - B) + \theta_2)\mathbf{1}_{[t,\infty]}(X)] + \theta_1, & t \leq a, \\ \mathbb{E}[(u'(w - X + I^*(X) - B) + \theta_2 + \frac{\theta_1}{1-\alpha})\mathbf{1}_{[t,\infty]}(X)], & t > a. \end{cases} \tag{4.10}$$

Similarly, the function is continuous on $[0, \infty)$, and the slope of function $L_T(t; I^*, \theta_1, \theta_2)$ also has a jump at a, with an intersection point occurring at t_0 where $t_0 \leq a$. Again, the solution to the main problem under TVaR is given in the proposition below.

Proposition 4.1.2. *For Problem (4.1) under TVaR, if we set the premium at B level, and fix the risk level such that $TVaR_\alpha(X - I_T^*(X) + B) = C$, then the optimal indemnity function ${I_T}^*(x)$ can be written by ${I_T}^*(x) = (x - t_0)_+$ where $0 \leq t_0$, $TVaR_\alpha(I_T^*(X)) = TVaR_\alpha(X) + B - C$ and $B = (1 + \eta)\mathbb{E}[I_T^*(X)]$.*

Proof. Let $L_T(t)$ short for $L_T(t; I^*, \theta_1, \theta_2)$, similar to the VaR case, we differentiate Eq. (4.10) and get

$$L_T^{'}(t) = \begin{cases} -(u'(w - t + I^*(t) - B) + \theta_2) f_X(t), & t \le a, \\ -(u'(w - t + I^*(t) - B) + \theta_2 + \frac{\theta_1}{1-\alpha}) f_X(t), & t > a. \end{cases}$$

Again $u'(w - t + I^*(t) - B)$ is non-decreasing, define

$$g_1 := \inf \{t : u'(w - t + I^*(t) - B) + \theta_2 \ge 0\} \cap \{t : t \le a\},$$

$$g_2 := \inf \{t : u'(w - t + I^*(t) - B) + \theta_2 + \frac{\theta_1}{1 - \alpha} \ge 0\} \cap \{t : t > a\}.$$

Obviously, $L_T(t)$ is strictly increasing on $[0, g_1) \cup (a, g_2)$ and non-increasing on $[g_1, a] \cup [g_2, \infty)$. It is easy to find that $g_2 = a$ if $g_1 < a$ and $g_1 = a$ if $g_2 > a$. Analyse case of $g_1 < a = g_2$, we have

$$\begin{cases} L'_T(t) > 0, & t \in [0, g_1), \\ L'_T(t) \le 0, & t \in [g_1, \infty). \end{cases}$$

Recall from Eq. (4.10), $L_T(t) \ge 0$ on $t \in [g_1, \infty)$. As such, we obtain

$$\begin{cases} L_T(t) < 0, & t \in [0, t_0), \\ L_T(t) \ge 0, & t \in [t_0, \infty). \end{cases}$$

This indicates that $I_T^*(x) = (x - t_0)_+$. The other cases, such as $g_1 = a < g_2$ can be proved in a similar way and hence omitted. This ends the proof.

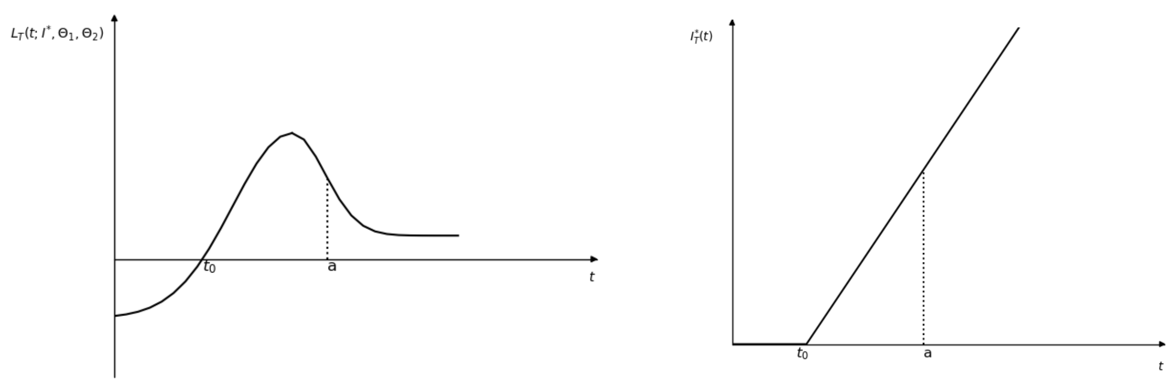

Figure 4.3: Illustration of $L_T(t; I^*, \theta_1, \theta_2)$ and $I_T^*(t)$.

4.2 Numerical study

In previous subsections, we derived the parametric form of the optimal indemnity function. Now we take a closer look at the optimization of the parameters via the ratio of expected utility and risk. More interestingly, we will take a look at the resulting efficient frontier between the expected utility and risk.

4.2.1 Case 1: Exponential distribution

Presented below is the setting of the first numerical example.

1. The insured's initial wealth w is 8600.

2. The premium is determined by the expectation principle with the risk loading $\eta = 0.2$.

3. The loss variable X follows the exponential distribution with $\lambda = 1000$. i.e.

$$F_X(x) = 1 - \exp(-\frac{x}{\lambda}) = 1 - \exp(-\frac{x}{1000}).$$

   Under this distribution, the maximum premium $\pi_0 = (1+\eta)\mathbb{E}[X]$=1200.

4. The insured's preference is captured by a quadratic utility function:

$$u(x) = -\frac{1}{2}\sigma x^2 + x.$$

   We set $\sigma = \frac{1}{10000}$ based on the equivalence principle $\mathbb{E}[u(w - X)] = u(w - (1+\eta)\mathbb{E}[X])$. Moreover, we notice that the quadratic function is an increasing concave utility function only when $x \leq \frac{1}{\sigma}$. As a result, $\frac{1}{\sigma}$ is the saturation point which represents the maximum wealth that can be attained by the insured.

5. The probability level $\alpha = 0.95$, and $VaR_\alpha(X) = 2996$ and $TVaR_\alpha(X) = 3996$.

VaR as risk measure

In this case, the denominator $C = VaR_\alpha(X - I_V^*(X) + B)$. We have from Proposition 4.1.1 that $I_V^*(x) = (x \wedge a - t_0)_+ + (x - t_1)_+$ and $t_0 = C - B$, so $I_V^*(x)$ can also be written as $I_V^*(x) = (x \wedge a - (C - B))_+ + (x - t_1)_+$. Note that $(x - t_1)_+$ is monotone decreasing in t_1, while $(x \wedge a - (C - B))_+$ is monotone decreasing with respect to C. Therefore, if one fixes the premium level B, t_1 and C should change along the different directions. Furthermore, since $t_0 = C - B$ and $\frac{B}{(1+\eta)} = \mathbb{E}[I_V^*(X)]$, when premium and risk loading factor are known, both t_0 and t_1 can be treated as a function of the parameter C. The range of t_0 and t_1 given by Proposition

4.1.1 yields that C should also be limited to a certain range when premium B is fixed, and we search for the optimal C within that range. Recall that $a \leq t_1$, C attains its upper bound, which is denoted by r_u, when t_1 approaches a, while the lower bound of C, which is denoted as r_l, is attained when t_1 approaches to infinity. With $0 \leq t_0 \leq a$, the equation $t_0 = C - B$ implies that $B \leq C \leq a + B$, from which we can identify the range of C, that is $\max\{r_l, B\} \leq C \leq \min\{r_u, a + B\}$.

The left panel of Fig. 4.4 is the efficient frontier of expected utility versus the risk when the premium level is within $(0, \pi_0]$. The expected utility and risk without any insurance contracts are $u(x) = 4662$ and $\rho(X) = VaR_\alpha(X) = 2996$, which is plotted in the figure as well. The efficient frontier shows that when the risk is reduced via an insurance contract, the expected utility can be improved simultaneously. However, a large reduction of risk may yield the decrement of expected utility as well. This is consistent with Markowitz's portfolio theory, where the portfolio's average return rate does not always move in the same direction as the volatility of return.

To identify the Sharpe ratio, a tangent line is plotted based on the zoomed figure (the right panel of Fig. 4.4). The optimal premium is found numerically to be $B^* = 933$. We also find $r_l = 1122$ and $r_u = 1185$. Recall that $a = VaR_\alpha(X) = 2996$, $\max(1122, 933) \leq C \leq \min(1185, 3929)$, which yields the range for the risk level $C \in [1122, 1185]$. Through numerical search, we get $C^* = 1122$ and the corresponding expected utility 4668. Therefore, the optimal ratio of the expected utility and risk is around 4.16.

Finally, we get the optimized parameters $t_0 = 190$, $t_1 = 9400$. The optimal indemnity function is given by $I_V^*(x) = (x \wedge 2996 - 190)_+ + (x - 9400)_+$. The indemnity function tells that the insured cares more about the left and right tails while leaving the majority of the loss that is beyond its VaR uncovered. This is consistent with most insureds' selection behavior, as it is more likely to incur small losses, and insureds tend to have their extremely large losses covered in case of the occurrence of some rare adverse events.

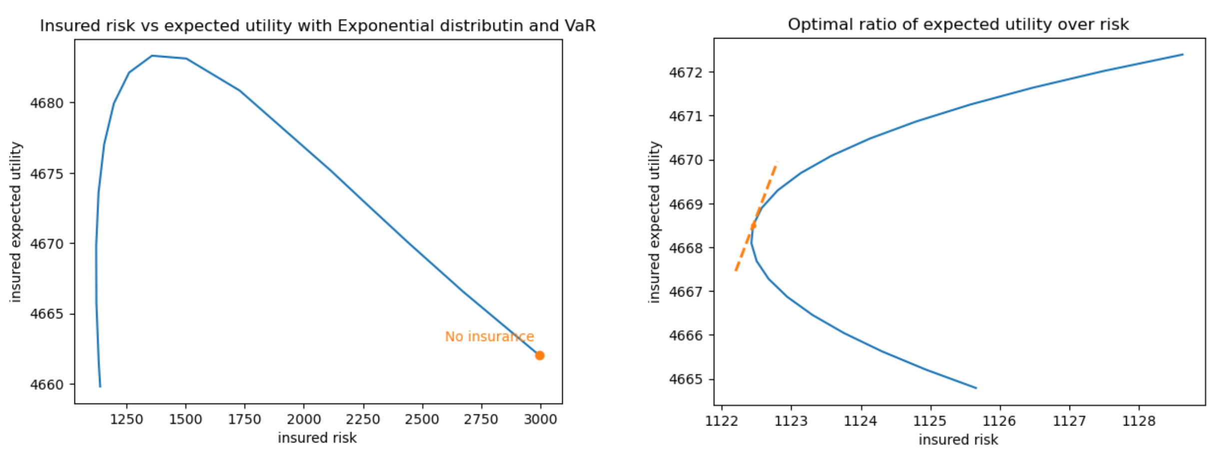

Figure 4.4: Exponential distribution under VaR.

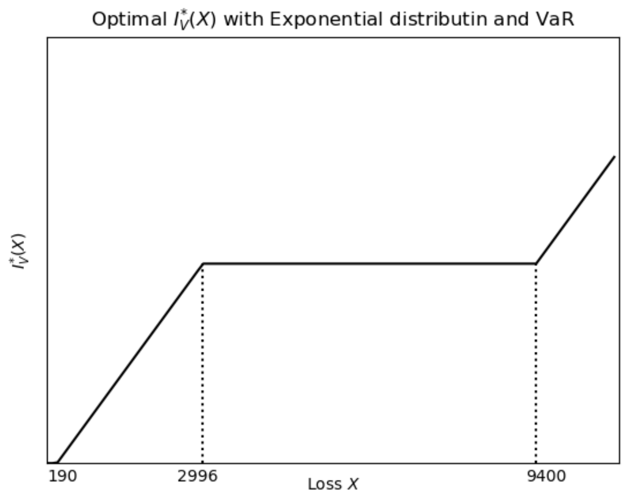

Figure 4.5: Case 1 optimal indemnity function under VaR.

TVaR as risk measure

Here the denominator $C = TVaR_\alpha(X - I_T^*(X) + B)$. Applying the comonotonic additivity and translation invariance properties of TVaR, we obtain $C = TVaR_\alpha(X) - TVaR_\alpha((X - t_0)_+) + B$. Note that under TVaR, the excess-of-loss function is optimal, so we can calculate t_0 explicitly if the premium level B is fixed. And then the expected utility and risk can be identified. It is not difficult to get the negative correlation between t_0 and B. We know $0 \leq t_0$, so maximum B should be equal to $\pi_0 = (1+\eta)\mathbb{E}[X] = 1200$. Then we search for the best B^* within $(0, 1200]$ that maximizes the ratio of expected utility and risk.

The left panel of Fig. 4.6 is the efficient frontier of the expected utility and the associated risk. The general shape of the frontier is quite similar to that under VaR. We can observe that the expected utility achieves a maximum value of around 4688 when the associated risk $C = 1547$. The expected utility and risk without insurance contracts are $u(x) = 4662$ and $\rho(X) = TVaR_\alpha(X) = 3996$ respectively.

The Sharpe ratio is calculated to be 3.96, where $B^* = 962$, $C^* = 1181$, and the expected utility is 4672. From $\frac{B}{(1+\eta)} = \mathbb{E}[I_T^*(X)] = \mathbb{E}[(X - t_0)_+]$, t_0 is calculated to be 221 and therefore $I_T^*(x) = (x - 221)_+$. Since TVaR cares about all the losses beyond the VaR, it is not surprising to see that the whole right tail is covered by the contract.

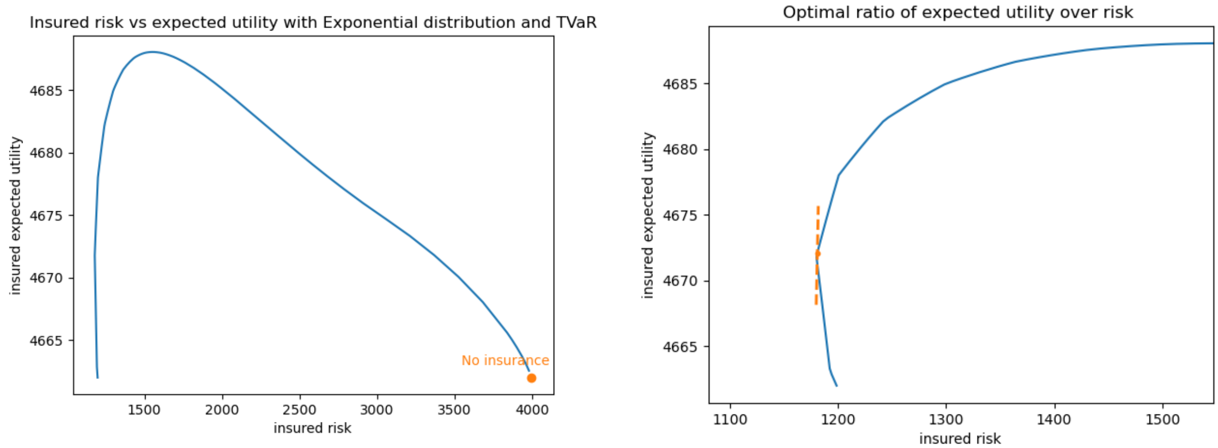

Figure 4.6: Exponential distribution under TVaR.

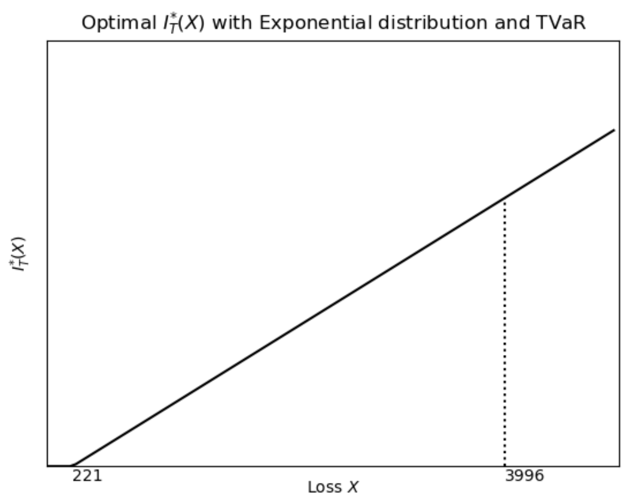

Figure 4.7: Case 1 optimal indemnity function under TVaR.

Sensitivity test

In this section, we conduct some sensitivity tests to analyze how the model setting affects the Sharpe ratio, as well as the corresponding optimal parameters.

In the first part, we only change the probability level α in our model setting. We notice that our ratio tends to be smaller with a higher probability level. The expected utility also increases with a very small incremental amount with α. Meanwhile, the deductible amount t_0 and the first layer t_1 are not quite sensitive to α. These non-significant changes indicate that the Sharpe-ratio-based contract is robust in probability levels under the VaR preference.

α	B^*	C^*	EU	ratio	t_0	t_1	$I_V^*(X)$
0.95	933	1122	4668.49	4.16	190	9400	$(X \wedge 2996 - 190)_+ + (X - 9400)_+$
0.96	946	1134	4668.90	4.12	189	9399	$(X \wedge 3219 - 189)_+ + (X - 9399)_+$
0.97	954	1146	4669.54	4.07	193	9403	$(X \wedge 3507 - 193)_+ + (X - 9403)_+$
0.975	957	1153	4669.89	4.05	196	9406	$(X \wedge 3689 - 196)_+ + (X - 9406)_+$
0.98	964	1158	4670.10	4.03	195	9405	$(X \wedge 3912 - 195)_+ + (X - 9405)_+$
0.99	977	1170	4670.54	3.99	194	9404	$(X \wedge 4605 - 194)_+ + (X - 9404)_+$

Table 4.1: Results with probability level α changes under VaR.

In the second part, we only change the safety loading factor in our model setting. The ratio goes down with an increase of η, and t_0 and t_1 both increase by around 100 with a 0.1 increment of η. The results in the table imply that the insured bears less utility gain per unit of risk if the contract becomes more expensive.

η	B^*	C^*	EU	ratio	t_0	t_1	$I_V^*(X)$
0.1	942	1040	4686	4.50	98	9309	$(X \wedge 2996 - 98)_+ + (X - 9309)_+$
0.2	933	1122	4668	4.16	190	9400	$(X \wedge 2996 - 190)_+ + (X - 9400)_+$
0.3	918	1198	4654	3.89	280	9490	$(X \wedge 2996 - 280)_+ + (X - 9490)_+$

Table 4.2: Results with risk loading η changes under VaR.

In the third part, we only change the parameter in the loss distribution. When λ becomes larger, the Sharpe ratio decreases, which indicates that the insured gains less utility per unit of risk borne by herself if the loss becomes riskier.

λ	B^*	C^*	EU	ratio	t_0	t_1	$I_V^*(X)$
990	920	1111	4672	4.20	192	9300	$(X \wedge 2966 - 192)_+ + (X - 9300)_+$
1000	933	1122	4668	4.16	190	9400	$(X \wedge 2996 - 190)_+ + (X - 9400)_+$
1010	938	1134	4666	4.12	196	9508	$(X \wedge 3026 - 196)_+ + (X - 9508)_+$

Table 4.3: Results with Exponential parameter λ changes under VaR.

The next two tables summarize the sensitivity test results when we switch the risk measure to TVaR. The Sharpe ratio is reduced as compared with that under VaR preference. As TVaR is concerned more about the right tail, the decrement in the Sharpe ratio indicates that the insured needs to sacrifice some utility to reduce the risk if a right-tail-covering risk measure is applied.

η	B^*	C^*	EU	ratio	t_0	$I_T^*(X)$
0.1	1058	1095	4689	4.28	39	$(X - 39)_+$
0.2	962	1181	4672	3.96	221	$(X - 221)_+$
0.3	1041	1261	4651	3.69	222	$(X - 222)_+$

Table 4.4: Results with risk loading η changes under TVaR.

λ	B^*	C^*	EU	ratio	t_0	$I_T^*(X)$
990	982	1168	4674	4.00	189	$(X-189)_+$
1000	962	1181	4672	3.96	221	$(X-221)_+$
1010	950	1193	4670	3.91	246	$(X-246)_+$

Table 4.5: Results with Exponential parameter λ changes under TVaR.

4.2.2 Case 2: Gamma distribution

We change some of the example settings in Case 1:

1. The insured's initial wealth w is now 10600.

2. The loss variable X follows Gamma distribution with the shape parameter $k = 5$ and rate parameter $\beta = \frac{1}{200}$. i.e.

$$F_X(x) = \frac{1}{\Gamma(k)}\gamma(k, \beta x) = \frac{1}{\Gamma(5)}\gamma(5, \frac{1}{200}x),$$
$$f_X(x) = \frac{\beta^k}{\Gamma(k)}x^{k-1}e^{-\beta x} = \frac{(\frac{1}{200})^5}{\Gamma(5)}x^4 e^{-\frac{1}{200}x}.$$

   In this distribution, the expected value $\mathbb{E}[X]$ equals $\frac{k}{\beta} = 1000$, mirroring the conditions in Case 1, and the maximum premium π_0 is still 1200 given that the risk loading factor remains unchanged.

3. The probability level $\alpha = 0.95$, and $VaR_\alpha(X) = 1831$ and $TVaR_\alpha(X) = 2134$.

VaR as risk measure

Once again, the left panel of Fig. 4.8 illustrates the efficient frontier (depicted by the blue line) representing expected utility versus risk for premium levels within the range $(0, \pi_0]$. Points lying along the tan line are evidently less efficient, as we can consistently achieve higher expected utility at the same risk level. Compared with Case 1, this efficient frontier is right-skewed with a narrower risk range. Notably, values within the expected utility range generally exceed those in Case 1, and the risk range is a subset of the risk range in Case 1. This implies that the Sharpe ratio in this scenario tends to be larger. The expected utility and risk without the protection of insurance are $u(x) = 4982$, $\rho(X) = VaR_\alpha(X) = 1831$, as indicated in the plot again.

Similar to the case with the exponential distribution, we plot the tangent line based on the magnified view presented in the right panel of Fig. 4.8. We identify the optimal premium as $B^* = 514$, with lower and upper risk bounds $r_l = 1094$ and $r_u = 1112$, respectively. Recall $\max(r_l, B) \leq C \leq \min(r_u, a+B)$, hence the range of C is deduced to be $[1094, 1112]$.Through numerical optimization, we determine the optimal value for

C^* to be 1094 and the corresponding expected utility to be 4988. As a result, the optimal ratio of expected utility over risk stands at approximately 4.56.

In this case, we have $t_0 = 580$, $t_1 = 3253$, then the optimal indemnity function is expressed as $I_V^*(x) = (x \wedge 1831 - 580)_+ + (x - 3253)_+$. The structure of the indemnity function suggests that the insured prioritizes coverage for significant losses while accepting a slightly higher burden for smaller losses. This configuration emerges as the preferred choice when the insured anticipates facing substantial losses.

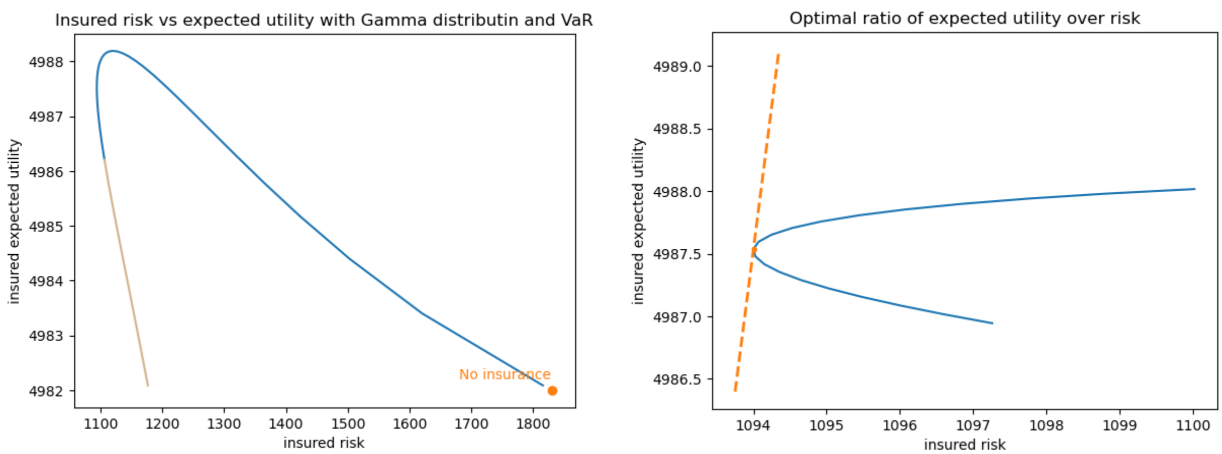

Figure 4.8: Gamma distribution under VaR.

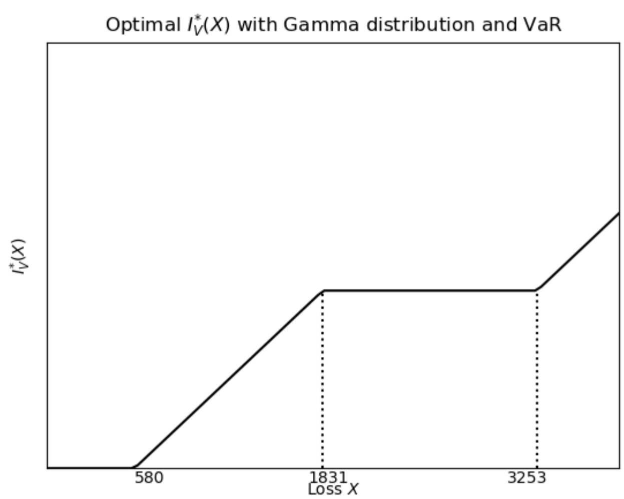

Figure 4.9: Case 2 optimal indemnity function under VaR.

TVaR as risk measure

The left side of Fig. 4.10 depicts the efficient frontier (indicated by the blue line) that showcases the expected utility and the corresponding risk. The overall shape of this frontier closely resembles that observed under

VaR. The Sharpe ratio in this scenario also tends to be larger compared with Case 1, as it does under VaR preference. Moreover, the trend highlights that the expected utility attains its maximum value of 4989 at a risk level of $C = 1145$. The orange point on the graph represents the scenario without insurance contracts, denoted as (2134, 4982).

By drawing the tangent line, we obtain the optimal values: $C^* = 1112$, $\mathbb{E}[u]^* = 4988$, and numerically found B^* to be 577 this time. Consequently, the optimal ratio of expected utility to risk is around 4.49. Recall that $\frac{B}{(1+\eta)} = \mathbb{E}[(X - t_0)_+]$, we deduce that $t_0 = 536$, and $I_T^*(x) = (x - 536)_+$. TVaR covers the entire right tail once again.

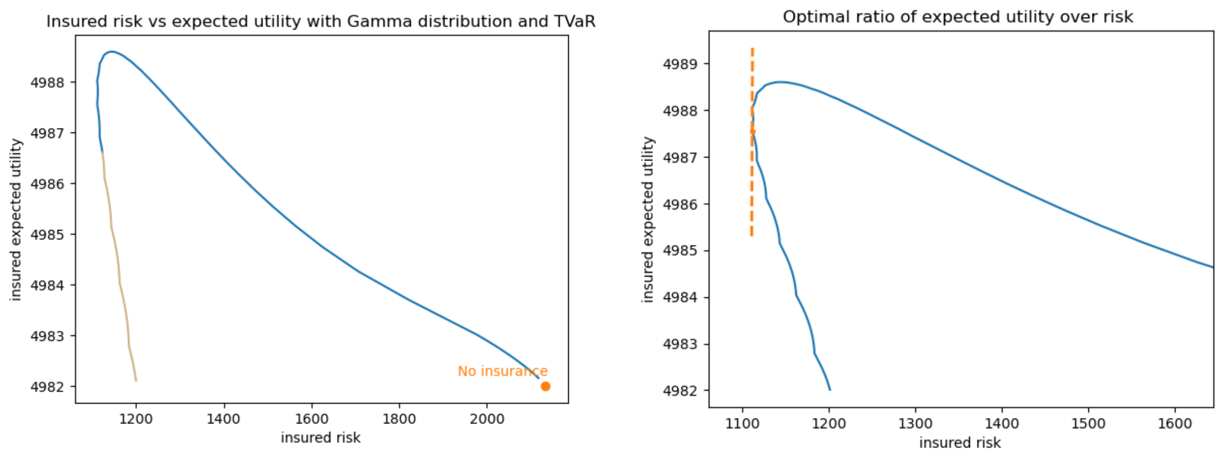

Figure 4.10: Gamma distribution under TVaR.

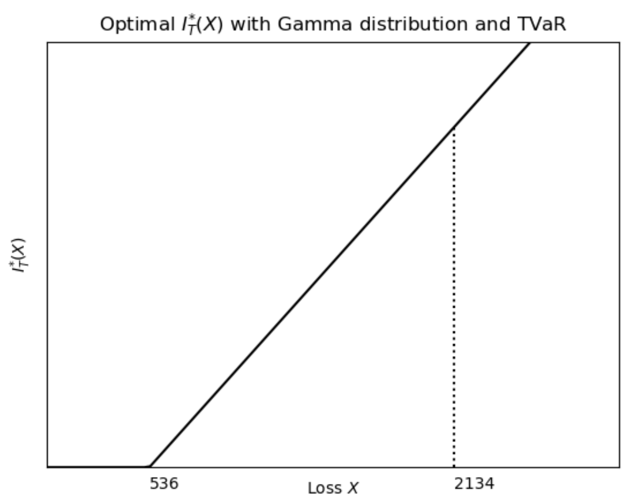

Figure 4.11: Case 2 optimal indemnity function under TVaR.

Sensitivity test

In this section, we replicate the sensitivity tests performed in Case 1 to assess the impact on the Sharpe ratio and optimal parameters with the introduction of new model settings.

Initially, our attention is directed towards the variation in the probability parameter, denoted as α. Similar to the findings in Case 1, we observe a decrease in the Sharpe ratio accompanied by an increase in expected utility as the probability level rises. Additionally, there is a moderate variation in both the deductible amount t_0 and the first layer t_1, underscoring the resilience of the Sharpe-ratio-based contract across different probability levels under the VaR preference.

α	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	t_1	$I_V^*(X)$
0.95	514	1094	4987.54	4.56	580	3253.10	$(X \wedge 1831 - 580)_+ + (X - 3253)_+$
0.96	520	1098	4987.59	4.54	578	3252.54	$(X \wedge 1902 - 578)_+ + (X - 3253)_+$
0.97	523	1102	4987.66	4.53	579	3252.75	$(X \wedge 1992 - 579)_+ + (X - 3253)_+$
0.975	527	1103	4987.68	4.52	577	3252.21	$(X \wedge 2048 - 577)_+ + (X - 3252)_+$
0.98	527	1105	4987.72	4.51	579	3252.66	$(X \wedge 2116 - 579)_+ + (X - 3253)_+$
0.99	527	1109	4987.80	4.50	582	3253.54	$(X \wedge 2321 - 582)_+ + (X - 3254)_+$

Table 4.6: Results with probability level α changes under VaR.

When we solely modify the risk loading factor within the Gamma distribution scenario, we also observe a decline in both the ratio and expected utility as the parameter η increases. This aligns with intuitive reasoning, indicating that as the cost of the contract rises, the insured experiences a reduced utility gain per unit of risk.

η	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	t_1	$I_V^*(X)$
0.1	574	1047	4989	4.77	473	3231	$(X \wedge 1831 - 473)_+ + (X - 3231)_+$
0.2	514	1094	4988	4.56	580	3253	$(X \wedge 1831 - 580)_+ + (X - 3253)_+$
0.3	480	1134	4986	4.39	653	3273	$(X \wedge 1831 - 653)_+ + (X - 3273)_+$

Table 4.7: Results with risk loading η changes under VaR.

Loss distribution following the Gamma distribution has two parameters. If we elevate the shape parameter k while adjusting the rate parameter β to ensure a consistent expected value of $\frac{k}{\beta} = 1000$, the result is a reduction in loss variance. Notably, we observe that both the ratio and expected utility experience an upturn with these adjustments. This suggests that the insured can achieve greater utility per unit of risk when confronted with lower risk variance while maintaining the same expected losses.

k	β	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	t_1	$I_V^*(X)$
4	$\frac{1}{250}$	570	1101	4986.91	4.53	531	3670	$(X \wedge 1938 - 531)_+ + (X - 3670)_+$
5	$\frac{1}{200}$	514	1094	4987.54	4.56	580	3253	$(X \wedge 1831 - 580)_+ + (X - 3253)_+$
6	$\frac{6}{1000}$	477	1088	4987.95	4.58	612	2966	$(X \wedge 1752 - 612)_+ + (X - 2966)_+$

Table 4.8: Results with Gamma parameter k and β changes under VaR.

The following tables provide a summary of the sensitivity test outcomes when transitioning to TVaR as the risk measure. Again, the Sharpe ratio experiences a reduction compared to its performance under VaR preference. The consistent takeaway is that opting for a right-tail-covering risk measure requires the insured to compromise some utility in order to mitigate risk.

η	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	$I_T^*(X)$
0.1	606	1062	4989.58	4.70	456	$(X-456)_+$
0.2	577	1112	4987.54	4.49	536	$(X-536)_+$
0.3	475	1153	4986.54	4.32	679	$(X-679)_+$

Table 4.9: Results with risk loading η changes under TVaR.

k	β	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	$I_T^*(X)$
4	$\frac{1}{250}$	584	1123	4987.39	4.44	538	$(X-538)_+$
5	$\frac{1}{200}$	577	1112	4987.54	4.49	536	$(X-536)_+$
6	$\frac{6}{1000}$	490	1104	4988.19	4.52	614	$(X-614)_+$

Table 4.10: Results with Gamma parameter k and β changes under TVaR.

4.2.3 Case 3: Pareto distribution

Similar to Case 2, we adhere to the settings outlined in Case 1 while making adjustments to a select set of example settings:

1. the insured's initial wealth w is now 6933.

2. Loss X follows Pareto II distribution where the scale $k = 4000$ and shape $\beta = 5$. ie.

$$F_X(x) = 1 - (\frac{k}{x+k})^\beta = 1 - (\frac{4000}{x+4000})^5,$$
$$f_X(x) = \frac{\beta k^\beta}{(x+k)^{\beta+1}} = \frac{5 \cdot 4000^5}{(x+4000)^6}.$$

   Within this distribution, the expected value, denoted as $\mathbb{E}[X]$, aligns with the conditions in Case 1, maintaining a value of $\frac{k}{\beta-1} = 1000$. Furthermore, the maximum premium π_0 remains at 1200 since the risk loading factor remains constant.

3. The probability level $\alpha = 0.95$, so $VaR_\alpha(X) = 3282$ and $TVaR_\alpha(X) = 5103$.

VaR as risk measure

Utilizing the left panel of Fig. 4.12, we can identify the efficient frontier portraying the trade-off between expected utility and risk within the premium level range $(0, \pi_0]$. The overall shape closely resembles that of the other two cases under VaR. This reiterates the idea that it is possible to enhance expected utility while concurrently mitigating risk, except for cases involving a substantial reduction in risk. Remarkably, values in the range of expected utility typically fall below those in Case 1 and Case 2, while the risk range is a superset of the ranges in Case 1 and Case 2. This suggests a tendency for a smaller Sharpe ratio in this context. Without the protective coverage of insurance, the expected utility and risk are $u(x) = 4090$ and $\rho(X) = VaR_\alpha(X) = 3282$, respectively, as indicated in the figure.

Similar to the approach in Case 1 and Case 2, we construct a tangent line based on the magnified illustration (the right panel of Fig. 4.12) to identify the Sharpe ratio. Utilizing numerical search methods, we determine the optimal values as $B^* = 914$, $r_l = 1077$, and $r_u = 1196$. Considering that the range of C is confined within limits when premium B is fixed, it is established to be $[1077, 1196]$. Subsequently, the optimal C^* is identified as 1077, yielding an associated expected utility of approximately 4089. Therefore, the optimal ratio of expected utility over risk stands at around 3.80.

Ultimately, we obtain the optimized parameters $t_0 = 163$ and $t_1 = 38048$, and the optimal indemnity function is expressed as $I_V^*(x) = (x \wedge 3282 - 163)_+ + (x - 38048)_+$. This indemnity function reveals that the insured is likely more concerned about minor losses compared with Case 1, given the substantial first layer amount (significantly exceeding VaR), which can be interpreted as an exceptionally rare event.

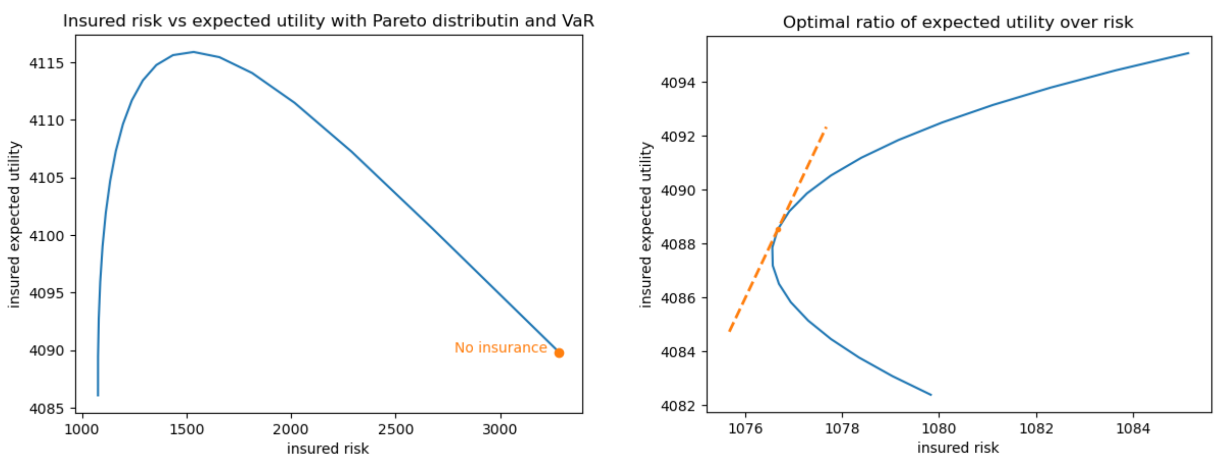

Figure 4.12: Pareto distribution under VaR.

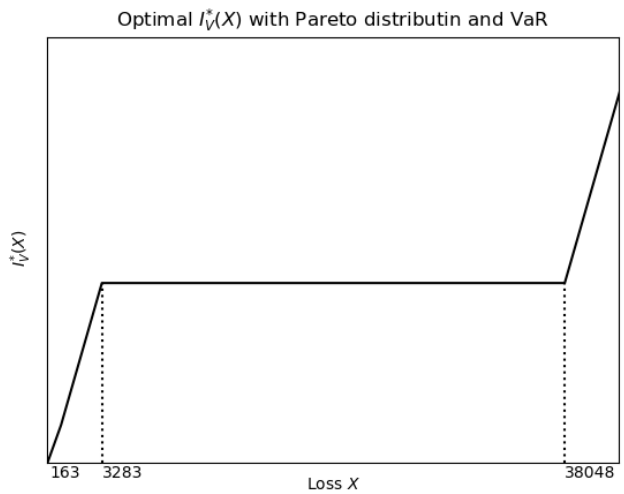

Figure 4.13: Case 3 optimal indemnity function under VaR.

TVaR as risk measure

Not surprisingly, the efficient frontier of Pareto distribution under TVaR shown in the left panel of Fig. 4.14 has a similar shape as it is under VaR. The maximum expected utility is found to be around 4136 at risk $C = 1328$. The orange point on the graph represents the scenario without insurance contracts, denoted as (5103,4090).

The Sharpe ratio is computed to be approximately 3.46, with optimized parameters $B^* = 1082$, $C^* = 1184$, and $\mathbb{E}[u]^* = 4098$. Applying the same methodology as before, considering that $\frac{B}{(1+\eta)} = \mathbb{E}[(X - t_0)_+]$, we determine $t_0 = 105$. Consequently, the optimal indemnity function is defined as $I_T^*(x) = (x - 105)_+$.With a minimal deductible, the coverage extends across the entire right tail.

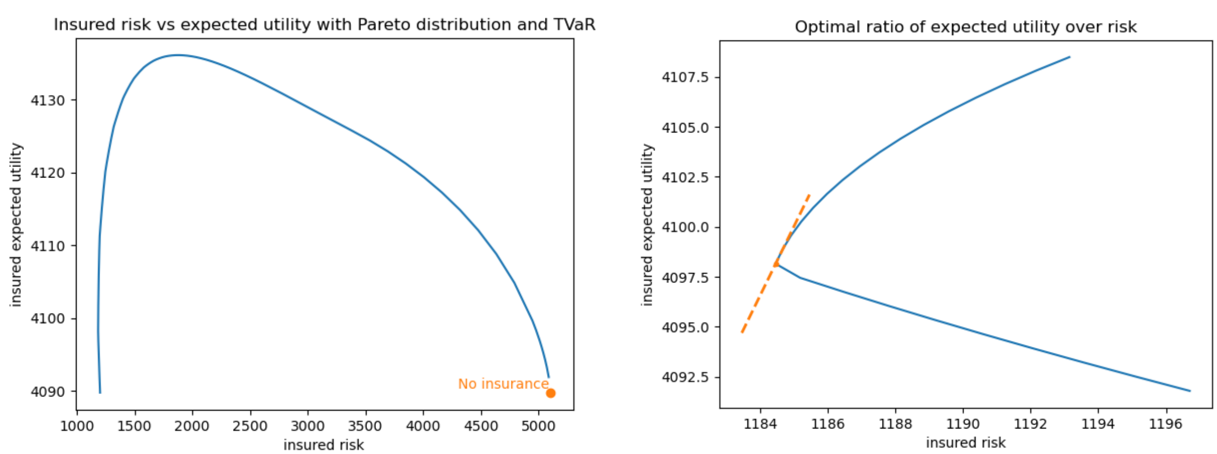

Figure 4.14: Pareto distribution under TVaR.

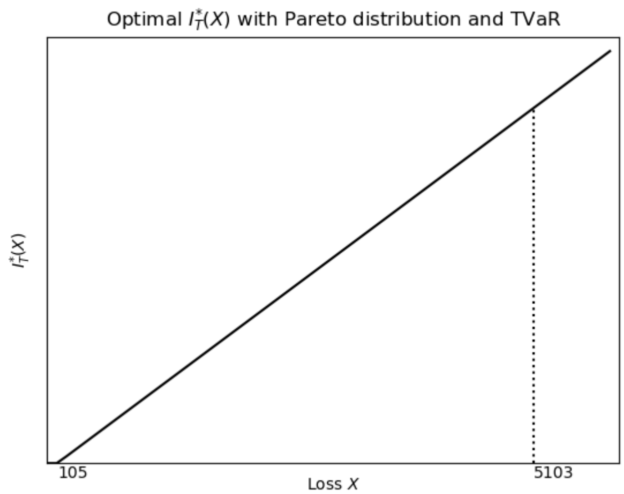

Figure 4.15: Case 3 optimal indemnity function under TVaR.

Sensitivity test

In this section, we duplicate the sensitivity tests conducted in Case 1 and Case 2 to evaluate the impact on the Sharpe ratio and optimal parameters following the incorporation of Pareto loss distribution settings.

First, we modify only the probability level α in our model setup. Consistent with the patterns observed in earlier cases, we note a reduction in the Sharpe ratio along with an increase in expected utility as the probability level increases. Moreover, the deductible amount t_0 and the first layer t_1 exhibit insensitivity to the probability level, reaffirming the robustness of such insurance contracts under VaR.

α	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	t_1	$I_V^*(X)$
0.95	914	1077	4089	3.80	163	38048	$(X \wedge 3282 - 163)_+ + (X - 38048)_+$
0.96	928	1095	4090	3.74	167	38098	$(X \wedge 3615 - 167)_+ + (X - 38098)_+$
0.97	945	1113	4092	3.68	168	38110	$(X \wedge 4066 - 168)_+ + (X - 38110)_+$
0.975	955	1123	4093	3.64	169	38119	$(X \wedge 4365 - 169)_+ + (X - 38119)_+$
0.98	965	1134	4094	3.61	169	38117	$(X \wedge 4747 - 169)_+ + (X - 38117)_+$
0.99	994	1156	4096	3.54	162	38036	$(X \wedge 6048 - 162)_+ + (X - 38036)_+$

Table 4.11: Results with probability level α changes under VaR.

Second, we also witness a decrease in both the ratio and expected utility with the escalation of the parameter η when we exclusively adjust the risk loading factor. This suggests that as the expense of the contract increases, the insured encounters a diminished utility gain per unit of risk.

η	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	t_1	$I_V^*(X)$
0.1	910	996	4118	4.13	87	37084	$(X \wedge 3282 - 87)_+ + (X - 37084)_+$
0.2	914	1077	4089	3.80	163	38048	$(X \wedge 3282 - 163)_+ + (X - 38048)_+$
0.3	914	1151	4063	3.53	238	38993	$(X \wedge 3282 - 238)_+ + (X - 38993)_+$

Table 4.12: Results with risk loading η changes under VaR.

The Pareto distribution is characterized by scale parameter k and shape parameter β. When we raise k and adjust β to maintain a constant expected value of 1000, unlike the scenario with the Gamma distribution, we observe a modest decline in the ratio and a substantial alteration in the first layer t_1. This indicates that the robustness observed for Sharpe-ratio-based contracts does not persist for the Pareto loss distribution under VaR.

k	β	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	t_1	$I_V^*(X)$
4	3000	904	1060	4079	3.85	156	52103	$(X \wedge 3344 - 156)_+ + (X - 52103)_+$
5	4000	914	1077	4089	3.80	163	38048	$(X \wedge 3282 - 163)_+ + (X - 38048)_+$
6	5000	920	1086	4093	3.77	167	27814	$(X \wedge 3238 - 167)_+ + (X - 27814)_+$

Table 4.13: Results with Pareto parameter k and β changes under VaR.

The subsequent tables offer a synopsis of the sensitivity test results when shifting to TVaR as the risk measure. Similar to the observations in Case 1 and Case 2, the Sharpe ratio undergoes a decrease compared to its performance under VaR preference.

η	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	$I_T^*(X)$
0.1	993	1094	4137	3.78	104	$(X-104)_+$
0.2	1082	1185	4098	3.46	105	$(X-105)_+$
0.3	997	1271	4077	3.21	274	$(X-274)_+$

Table 4.14: Results with risk loading η changes under TVaR.

k	β	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	$I_T^*(X)$
4	3000	1026	1184.55	4102	3.463	161	$(X-161)_+$
5	4000	1082	1184.46	4098	3.460	105	$(X-105)_+$
6	5000	992	1186.96	4104	3.458	194	$(X-194)_+$

Table 4.15: Results with Pareto parameter k and β changes under TVaR.

4.2.4 Comparison of the three cases

All three cases are designed to yield the same expected loss of 1000. Among these, Case 2 stands out with the highest ratio of expected utility to risk, coupled with the lowest optimal premium, making it the optimal policy among the three. Notably, Case 2 features the largest deductible amount and the lowest first layer's amount, indicating its particular suitability for mitigating large losses. In the event of a minor loss, the insured maintains relatively low coverage; however, for substantial losses, the potential payout is significantly higher.

Policies in both Case 1 and Case 3 prioritize losses in both the left and right tails. Comparatively, Case 1 exhibits a better expected utility per unit of risk than Case 3. Therefore, Case 3 emerges as a less attractive option for the insured due to its expensive premium and substantial first-layer amount.

No.Cases	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	t_1	$I_V^*(X)$
Case1:Exponential	933	1122	4668	4.16	190	9400	$(X \wedge 2996 - 190)_+ + (X - 9400)_+$
Case2:Gamma	514	1094	4988	4.56	580	3253	$(X \wedge 1831 - 580)_+ + (X - 3253)_+$
Case3:Pareto	914	1077	4089	3.80	163	38048	$(X \wedge 3282 - 163)_+ + (X - 38048)_+$

Table 4.16: Results in 3 Cases under VaR.

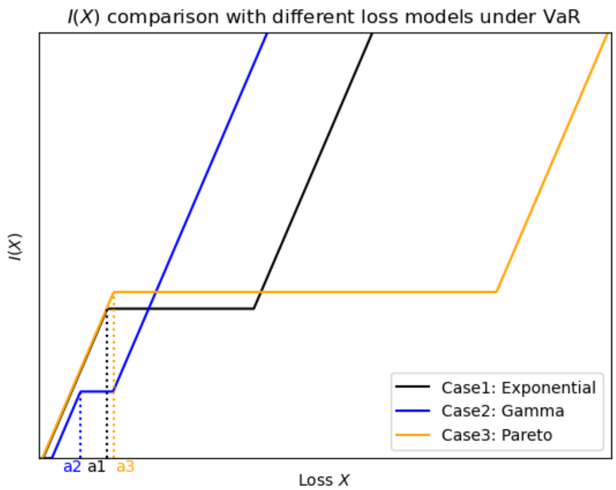

Figure 4.16: Indemnity functions in three Cases under VaR.

From the TVaR summary Table 4.17 and indemnity functions Fig. 4.17 below, we can see that policy in Case 2 produces the highest expected utility per unit of risk. It's also noteworthy that Case 2 boasts the most affordable premium. However, this reduced premium comes at the expense of a higher deductible amount. In practical terms, this implies that the insured must shoulder a more substantial portion of the loss to secure a more economical insurance policy.

For individuals with an expectation of frequent minor losses, alternative considerations may be warranted. In such cases, exploring Case 1 or Case 3 becomes relevant, as they offer lower deductible amounts albeit with higher associated costs. These considerations align with reality when making decisions about insurance purchases.

No.Cases	B^*	C^*	$\mathbb{E}[u]^*$	ratio	t_0	$I_T^*(X)$	$a' = TVaR_\alpha(X)$
Case1:Exponential	962	1181	4672	3.96	221	$(X-221)_+$	3996
Case2:Gamma	577	1112	4988	4.49	536	$(X-536)_+$	2134
Case3:Pareto	1082	1184	4098	3.46	105	$(X-105)_+$	5103

Table 4.17: Results in 3 Cases under TVaR.

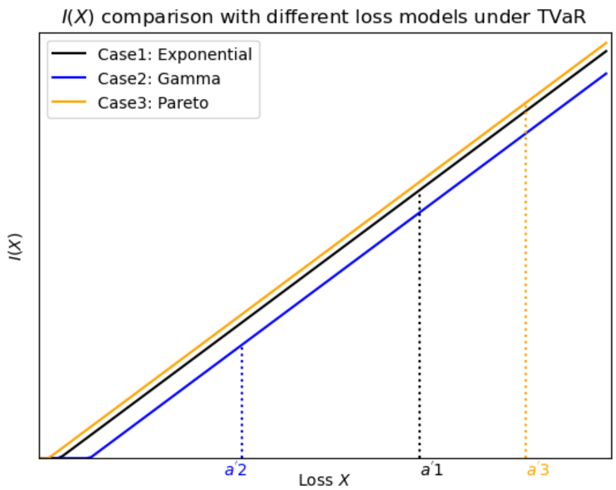

Figure 4.17: Indemnity functions in three Cases under TVaR.

4.3 Comparative study

A commonly recognized industry practice is to provide compensation for losses that exceed the deductible amount while also imposing an upper limit on the coverage. The correlated indemnity function is written as $I_I^*(x) = (x - t_0)_+ \wedge t_1$, excluding $(x - t_1)_+$ and omitting comparison with a parts in contrast to our model under VaR risk measure.

In this section, we will analyze the industry model and compare it with the Case 1 model based on VaR preference. Assuming identical numerical settings as in Case 1, we explore the optimal premium, the Sharpe ratio, and the corresponding indemnity function. Reminding that risk is defined as $C = VaR_\alpha(X - I(X) + B) = VaR_\alpha(X) - I(VaR_\alpha(X)) + B = a + B - I(a)$, where $a = VaR_\alpha(X)$. Given the known form of $I_I^*(x)$ as $I(a) = (a - t_0)_+ \wedge t_1$, we need to discuss the relationship among t_0, t_1 and a. There are three possible scenarios in total:

1. $0 \le t_0 \le a$ and $0 \le t_0 + t_1 \le a$

   In this particular situation, $I(a) = t_1$, leading to $C = a + B - t_1$. It is also evident that determining the range of t_1 implies constraining C to a specific range when the premium B is fixed. Consequently, C should have a lower bound r_l when $t_1 = a - t_0$ and an upper bound $a + B$ when $t_1 = 0$. It's worth noting that when $t_1 = a - t_0$, $C = t_0 + B \ge B$, allowing us to conclude that the range of C should be restricted as $\max(r_l, B) \le C \le a + B$. Additionally, considering that t_0 and t_1 have negative correlations, B should still fall within the interval $(0, \pi_0]$.

   Employing the same method as in Case 1, we draw the tangent line and determine that the optimal premium B^* is 933 with the corresponding optimal risk C^* equalling 1122. The expected utility is approximately 4668.43, only a slight decrease from the expected utility in our model, which stands at 4668.49. The calculated optimal ratio here is around 4.1592035, slightly lower than our model's 4.1592039. The deductible amount of 189.7 closely aligns with our $t_0 = 189.8$ in Case 1. In addition, we observe that the upper limit in the industrial model is situated around 2806, which is exactly equal to $a - t_0$. This implies that our optimal ratio is achieved when the upper limit coverage equals VaR minus the deductible amount. Consequently, the indemnity function takes the form $I_I^*(x) = (x - 189.7)_+ \wedge 2806$.

In the context of Fig. 4.18, a key distinction between the two models, in this case, is that the industry model, with its maximum coverage, does not surpass a, while our model extends compensation beyond the initial layer payment. It's evident that under our model, the insured receives additional compensation when a loss exceeds VaR, making it a more attractive policy, even though the ratio is not significantly different.

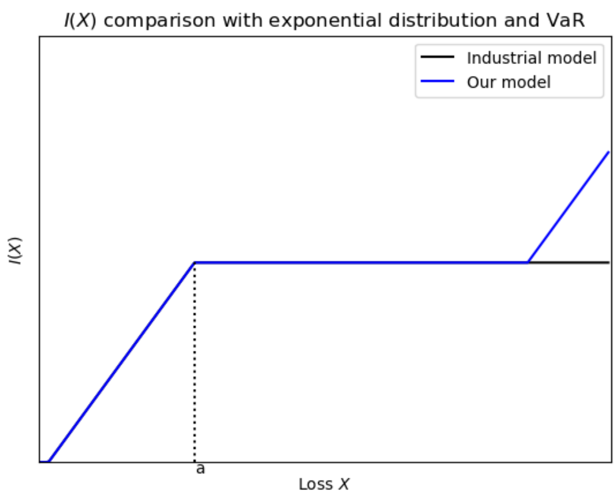

Figure 4.18: Comparison of I^* in Case 1 model and the industry model under VaR.

2. $0 \leq t_0 \leq a$ and $t_0 + t_1 > a$

   In this instance, $I(a) = a - t_0$, resulting in $C = t_0 + B$. Subsequently, considering the range of t_0 and t_1, it follows that C should have a lower bound r_l when $t_0 = a - t_1$, that is, $t_0 + t_1 = a$ again. Therefore, we can deduce that $\max(r_l, B) \leq C \leq a + B$. Clearly, the second case is encompassed within the first one and hence omitted here.

3. $t_0 > a$ and $t_1 \geq 0$

   When $t_0 > a$, $(a - t_0)_+ = 0$, resulting in $I(a)$ consistently equating to 0, and thus, C=a+B, solely dependent on B. The existence of $\frac{B}{1+\eta} = E[I_I^*(X)]$ should persist. If only t_0 increases, B should decrease. As a result, as t_0 approaches a, we should establish an upper bound for B and a lower bound of 0 when t_0 is close to infinite. In the scenario where t_1 is close to 0, B also becomes 0. If t_1 tends toward infinity, we will revisit the discussion on the impact of $(x - t_0)_+$ on B, a topic previously addressed.

The numerical exploration reveals the range of B as [0,60). Within this range, we search for the optimal ratio and determine that $B^* = 0$, risk $C^* = a$, the optimal expected utility is 4662, and the calculated optimal ratio is around 1.6. Additionally, t_0 is expected to converge to the maximum loss, and t_1 is expected to approach 0, indicating that $I_I^* = 0$. Consequently, the optimal decision for the insured is not purchasing insurance when $t_0 > a$.

In summary, to optimize utility per unit of risk, it is advisable for the insured to acquire insurance when the deductible amount is less than or equal to VaR. However, choosing to bear the loss personally is a more favorable option when the deductible amount exceeds VaR.

Chapter 5

Conclusion

In this article, we study the optimal insurance problem by balancing the expected utility and the associated risk via the Sharpe ratio. We first re-formulate the problem, which makes it more tractable. That allows us to characterize the solution implicitly and makes it possible to work out some parametric forms of the optimal indemnity function when focusing on specific risk measures.

Identifying the optimal point on the efficient frontier is a longstanding problem in economics. Common approaches include the use of, for example, Nash cooperative equilibrium or Sharpe ratio method. In the context of our problem, we apply the latter to find the optimal contract that lies on the efficient frontier. Though the closed-form or analytical form of the solution is rather difficult to obtain, we provide ample numerical examples to study a variety of aspects of such a method. More specifically, our numerical examples are carried out under, for example, VaR and TVaR preferences with the loss variable following the Exponential, Gamma, and Pareto distributions. In our numerical examples, we identify the optimal contract based on the largest Sharpe ratio. In addition, we notice that the deductible point in the indemnity function is relatively stable in sensitivity tests except for the Pareto case under the VaR preference, and increasing the probability level or risk loading factor has a negative impact on the ratio. Moreover, within a setting where the expected loss remains constant, the loss variable, when following the Gamma distribution, yields the highest Sharpe ratio in comparison to the Exponential and Pareto distributions. In the end, upon scrutinizing the industry contract under VaR preference, we conclude that acquiring insurance is advantageous only when the deductible amount is equal to or less than VaR. The optimal industry policy is attained when the upper limit coverage equals VaR minus the deductible amount, which has a strong similarity to the optimal contract in our model.

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Appendix A

Python code

A.1 Numerical case under VaR

```
########exp+VAR########

########import libraries (OK)

import numpy as np

np.set_printoptions(suppress=True)

import pandas as pd

from scipy.stats import expon

from scipy.optimize import fsolve,root_scalar

from scipy import optimize

from sympy import *

import sympy

import array as arr


# Plotting

import matplotlib.pyplot as plt

from mpl_toolkits.axisartist.axislines import AxesZero

import matplotlib as mpl


########numerial case-initial assumptions (OK)

sigma=1/10000 #sig_res when do sensitiveity test
```

```
w=8600 #initial wealth
lam=1000 #parameter in exp distr
theta=0.2 #risk loading
pi_0=(1+theta)*lam
# premium budget (1+theta)*lam
alpha=0.95 #confidence level

#quadratic utility function
def u(x):
    return -0.5*sigma*x**2+x

# VaR_alpha(X)
a=expon.ppf(alpha,scale=lam)

#expectated utility without insurance but has loss X (OK)
def EU3(x):
    return u(w-x)*(1/lam*sympy.exp(-1/lam*x)) #pdf of x =(1/lam*sympy.exp(-1/lam*x))
x= symbols('x')
no_ins=sympy.integrate(EU3(x),(x,0,sympy.oo))
print(no_ins)
print(u(w-pi_0)) #for sigma sensitivity test only

#####premium B as parameter of function of risk C (OK)
###T(B) funtion###
#opt I=max(min(x,a)-t0,0)+max(x-t1,0)
#risk measure denominator denoted by C=VaR(x)-I(VaR(x))+B=a-I(a)+B=a-(a-t0)+B=t0+B

B = sympy.symbols('B',real=True,positive=True)#premium
C = sympy.symbols('C',real=True,positive=True)
def T(B): # B is premium
    # expectation of I
    def E2(C):
        t0 ,t1 ,x= symbols('t0 t1 x')
```

```
    t0=C-B
    f = sympy.Piecewise((0,x<=t0),((x-t0)*(1/lam*sympy.exp(-1/lam*x)),And(x>t0,x<=a)),
                    ((a-t0)*(1/lam*sympy.exp(-1/lam*x)),And(x>a,x<=t1)),
                    ((a-t0+x-t1)*(1/lam*sympy.exp(-1/lam*x)),True)) #pdf of x =(1/lam*sympy.exp(-1/l
    return sympy.integrate(f,(x,0,sympy.oo))

#expected utility numerator
def EU2(C):
    #calculate t1
    eq= sympy.Eq(B/(1+theta), E2(C))
    t1= sympy.nsolve(eq,0)     #t1
    t0 ,x= symbols('t0  x')
    t0=C-B
    f = sympy.Piecewise((u(w-x-B)*(1/lam*sympy.exp(-1/lam*x)),x<=t0),
                    (u(w-t0-B)*(1/lam*sympy.exp(-1/lam*x)),And(x>t0,x<=a)),
                    (u(w-x+a-t0-B)*(1/lam*sympy.exp(-1/lam*x)),And(x>a,x<=t1)),
                    (u(w+a-t0-t1-B)*(1/lam*sympy.exp(-1/lam*x)),True))
    return sympy.integrate(f,(x,0,sympy.oo)),t0,t1

#lower bound of C
def E3(C):
    t0 ,x= symbols('t0  x')
    t0=C-B
    f = sympy.Piecewise((0,x<=t0),((x-t0)*(1/lam*sympy.exp(-1/lam*x)),And(x>t0,x<=a)),
                    ((a-t0)*(1/lam*sympy.exp(-1/lam*x)),True))
    return sympy.integrate(f,(x,0,sympy.oo))

eq3= sympy.Eq(B/(1+theta), E3(C))
root1= sympy.nsolve(eq3,0)
#root1=round(root1,0)

#uper bound of C
def E4(C):
```

```
        t0 ,x= symbols('t0  x')
        t0=C-B
        f = sympy.Piecewise((0,x<=t0),((x-t0)*(1/lam*sympy.exp(-1/lam*x)),And(x>t0,x<=a)),
                        ((x-t0)*(1/lam*sympy.exp(-1/lam*x)),True))
        return sympy.integrate(f,(x,0,sympy.oo))
    eq4= sympy.Eq(B/(1+theta), E4(C))
    root2= sympy.nsolve(eq4,0)
    #root2=round(root2,0)

    #range of C
    up= min(a+B,root2)
    lower=max(root1,B)

    def obj_new(C):
        C=float(C)
        return C/EU2(C)[0]

    np.random.seed(2023)
    initials=lower+0.5*(up-lower) # initial guesses
    #if (B==1200):opt_ratio=optimize.minimize(obj_new,initials,bounds=((lower,up),))
    if (root1<B):opt_ratio=optimize.minimize(obj_new,initials,bounds=((lower,up),))
    else:opt_ratio=optimize.minimize(obj_new,initials,bounds=((lower+0.1,up),))
    return opt_ratio.x[0],EU2(opt_ratio.x[0])[0],lower,up, EU2(opt_ratio.x[0])[1],EU2(opt_ratio.x[0])[2

#optimal B test local value
#import timeit

#start = timeit.default_timer()

def obj_B(B):
    B=float(B)
    result=T(B)
    return result[0]/result[1]
```

```
np.random.seed(2023)
initials=50 # initial guesses
opt_ratio_B2=optimize.minimize(obj_B,initials,bounds=((1,pi_0),))
opt_ratio_B2

#stop = timeit.default_timer()

#print('Time: ', stop - start)

def obj_B(B):
    B=float(B)
    result=T(B)
    return result[0]/result[1]

#bnds=[(1,pi_0)]
bnds=[(850,950)] #adjust
opt_ratio_B=optimize.differential_evolution(obj_B,bounds=bnds,seed=np.random.seed(2023))
opt_ratio_B

result=T(opt_ratio_B.x[0])
print("C=",result[0])
print("B=",opt_ratio_B.x[0])
print("utility=",result[1])
print("ratio=",opt_ratio_B.fun)
print("object ratio=",1/opt_ratio_B.fun)
print("lowerC=",result[2])
print("upC=",result[3])
print("t_0=",result[4])
print("t_1=",result[5])

#plot curve with multiple B - show no insurance position
r0=np.trim_zeros(np.arange(0,2,1))
```

```
r1=np.trim_zeros(np.arange(0,100,25))
r2=np.arange(100,pi_0,100)
r3=np.arange(pi_0-60,pi_0,60)
r=np.append(r0,r1)
r=np.append(r,r2)
r=np.append(r,r3)
risk=[0]*len(r)
utility=[0]*len(r)
output=[0]*len(r)
for i in range((len(r))):
    output[i]= T(r[i])
    risk[i]= output[i][0]
    utility[i]=output[i][1]


plt.plot(risk, utility)
plt.plot(a,no_ins, marker="o",color="C1")
plt.annotate("No insurance", (a-400, no_ins+1),color="C1")

# naming the x axis
plt.xlabel('insured risk')
# naming the y axis
plt.ylabel('insured expected utility')

# giving a title to my graph
plt.title('Insured risk vs expected utility with Exponential distributin and VaR')

# function to show the plot
plt.show()

#plot curve with multiple B - tangent line

r=np.arange(opt_ratio_B.x[0]-100,opt_ratio_B.x[0]+100,10)
```

```
risk=[0]*len(r)
utility=[0]*len(r)
output=[0]*len(r)
for i in range((len(r))):
    output[i]= T(r[i])
    risk[i]= output[i][0]
    utility[i]=output[i][1]

# Choose point to plot tangent line
x_tan = result[0]

#tangent line
def line(x):
    slope=result[1]/x_tan
    return slope*x

slope=result[1]/x_tan

#Define x y data range for tangent line
xrange = np.arange(x_tan-0.25, x_tan+0.5, 0.2)

plt.plot(risk, utility)
plt.plot(xrange, line(xrange), 'C1--', linewidth = 2)
plt.plot(x_tan,line(x_tan), marker="o",color="C1",markersize=3)

# naming the x axis
plt.xlabel('insured risk')
# naming the y axis
plt.ylabel('insured expected utility')

# giving a title to my graph
plt.title('Optimal ratio of expected utility over risk')
```

```
# function to show the plot
plt.show()


def I(x):
    if(x <= int(result[4])): return 0
    elif (x>=int(result[4])) and (x<=int(a)): return x-int(result[4])
    elif(x>=int(a)) and (x<=int(result[5])):return int(a)-int(result[4])
    else: return x-(int(result[4])+int(result[5])-int(a))


x=np.arange(0, 11000, 100)
y = np.zeros(len(x))
for i in range(len(x)):
    y[i]=I(x[i])


fig = plt.figure()
ax = fig.add_subplot(axes_class=AxesZero)


# Turn off tick labels and marks
ax.set_xticks([])
ax.set_yticks([])


ax.plot(x, y,color="black")


ax.text(result[4]-200, -250, int(result[4]+1), fontsize=10)
ax.text(a-500, -250, int(a+1), fontsize=10)
ax.text(result[5]-200, -250, int(result[5]), fontsize=10)


plt.vlines(x=a, ymin=0, ymax=a-result[4], ls=':',color="black")
plt.vlines(x=result[5], ymin=0, ymax=a-result[4], ls=':',color="black")
#plt.hlines(y=a-result[4], xmin=0, xmax=a, ls=':',color="black")


# Adding axis title
ax.set_xlabel('Loss $X$',fontsize=12)
```

```
ax.xaxis.set_label_coords(0, 0)
ax.set_ylabel('$I^{*}_{V}(X)$', fontsize=12)

# sets axes labels on both ends
#ax.annotate('Loss X', xy=(1,-0.05), ha='left', va='top',
#             xycoords='axes fraction')
#ax.annotate('Optimal indemnity function $I^{*}_{V}(X)$', xy=(-0.1, 1), ha='left', va='top', xycoords='a

# Adding title
ax.set_title('Optimal $I^{*}_{V}(X)$ with Exponential distributin and VaR', fontsize=12)



plt.xlim(0, 11000)
plt.ylim(0, 6000)

plt.show()
```

A.2 Numerical case under TVaR

```
####exp+TVAR######
########import libraries (OK)
import numpy as np
import pandas as pd
from scipy.stats import expon
from scipy.optimize import fsolve,root_scalar
from scipy import optimize
from sympy import *
import sympy
import array as arr
from mpmath import mp
mp.dps = 20
```

```
# Plotting
import matplotlib.pyplot as plt
from mpl_toolkits.axisartist.axislines import AxesZero
import matplotlib as mpl

########numerial case-initial assumptions (OK)
sigma=1/10000  #sig_res when do sensitiveity test
w=8600 #initial wealth
lam=1000 #parameter in exp distr
theta=0.2 #risk loading
pi_0=(1+theta)*lam # premium budget (1+theta)*lam
alpha=0.95 #confidence level

#quadratic utility function
def u(x):
    return -0.5*sigma*x**2+x

a=(-np.log(1-alpha)+1)/(1/lam) #TVaR_alpha #3995.73

#expectated utility without insurance but has loss X (OK)
def EU3(x):
    return u(w-x)*(1/lam*sympy.exp(-1/lam*x)) #pdf of x =(1/lam*sympy.exp(-1/lam*x))
x= symbols('x')
no_ins=sympy.integrate(EU3(x),(x,0,sympy.oo))
print(no_ins)
print(u(w-pi_0))

######T(B) funtion#######
# opt I=max((x-t0),0) with TVaR(opt I)=a+B-C

#risk measure denominator denoted by C=a+B-TVaR(opt I)

B = sympy.symbols('B',real=True,positive=True)#premium
```

```
# expectation of I
def E1(t0):
    x= symbols('x')
    f = sympy.Piecewise(((x-t0)*(1/lam)*sympy.exp(-1/lam*x),x>t0),(0,True)) #pdf of x =(1/lam*sympy.exp
    return sympy.integrate(f,(x,0,sympy.oo))

def EU_t0(B):
    t0= symbols('t0 ')
    eq= sympy.Eq(B/(1+theta), E1(t0))
    t0= sympy.nsolve(eq,0)
    return t0

#expected utility numerator
def EU1(B):
    x= symbols('x')
    t0= EU_t0(B)
    f = sympy.Piecewise((u(w-t0-B)*(1/lam*sympy.exp(-1/lam*x)),x>t0),
                        (u(w-x-B)*(1/lam*sympy.exp(-1/lam*x)),True))
    return sympy.integrate(f,(x,0,sympy.oo))

def TVAR_I(B):
    x= symbols('x')
    s= sympy.exp(-1/lam*x) #survival function
    g= sympy.Piecewise((x/(1-alpha),And(0<=x,x<1-alpha)),(1,And(1-alpha<=x,x<=1)))
    g_of_s = g.subs(x, s)
    g_of_s_numeric = sympy.lambdify(x, g_of_s, 'mpmath')
    result=mp.quad(g_of_s_numeric, [EU_t0(B), mp.inf])
    return float(result)

def risk_fun(B):
    return float(a+B.flat[0]-TVAR_I(B.flat[0]))

def obj_B(B):
```

```
    return float(risk_fun(B.flat[0])/EU1(B.flat[0]))

bnds=[(1,pi_0)]
opt_ratio_B=optimize.differential_evolution(obj_B,bounds=bnds,seed=np.random.seed(2023))
opt_ratio_B

print("C=",risk_fun(opt_ratio_B.x[0]))
print("B=",opt_ratio_B.x[0])
print("utility=",EU1(opt_ratio_B.x[0]))
print("ratio=",opt_ratio_B.fun)
print("object ratio=",1/opt_ratio_B.fun)
print("t_0=",EU_t0(opt_ratio_B.x[0]))

#plot curve with multiple B - show no insurance position

# plotting the points
B1=np.arange(1, 10)
B2=np.arange(10, pi_0+10, 10)
B=np.append(B1,B2)
risk=[0] * len(B)
utility=[0] * len(B)
for i in range (len(B)):
    risk[i]=a+B[i]-TVAR_I(B[i])
    utility[i]=EU1(B[i])# y axis values


plt.plot(risk, utility)
plt.plot(a,no_ins, marker="o",color="C1")
plt.annotate("No insurance", (a-450, no_ins+1),color="C1")

# naming the x axis
plt.xlabel('insured risk')
# naming the y axis
```

```
plt.ylabel('insured expected utility')

# giving a title to my graph
plt.title('Insured risk vs expected utility with Exponential distribution and TVaR')

# function to show the plot
plt.show()

# Choose point to plot tangent line
x_tan = risk_fun(opt_ratio_B.x[0])

#tangent line
def line(x):
    slope=EU1(opt_ratio_B.x[0])/x_tan
    return slope*x

slope=EU1(opt_ratio_B.x[0])/x_tan

#Define x y data range for tangent line
xrange = np.arange(x_tan-1, x_tan+1, 0.1)


plt.xlim(int(min(risk))-100, int(risk[np.argmax(utility)]))
plt.plot(risk, utility)
plt.plot(x_tan,line(x_tan), marker="o",color="C1",markersize=3)
plt.plot(xrange, line(xrange), 'C1--', linewidth = 2)


# naming the x axis
plt.xlabel('insured risk')
# naming the y axis
plt.ylabel('insured expected utility')
```

```
# giving a title to my graph
plt.title('Optimal ratio of expected utility over risk ')

# function to show the plot
plt.show()

t0=EU_t0(opt_ratio_B.x[0])
def I(x):
    if(x <= t0): return 0
    else: return x-t0

x=np.arange(0, 5000, 50)
y = np.zeros(len(x))
for i in range(len(x)):
    y[i]=I(x[i])

fig = plt.figure()
ax = fig.add_subplot(axes_class=AxesZero)


# Turn off tick labels and marks
ax.set_xticks([])
ax.set_yticks([])

ax.plot(x, y,color="black")

ax.text(t0, -200, int(t0), fontsize=10)
ax.text(a, -200, int(a+1), fontsize=10)

# Adding axis title
ax.set_xlabel('Loss $X$',fontsize=12)
ax.set_ylabel('$I^{*}_{T}(X)$', fontsize=12)
```

```
# Adding title
ax.set_title('Optimal $I^{*}_{T}(X)$ with Exponential distribution and TVaR', fontsize=12)



plt.xlim(0, 5000)
plt.ylim(0, 6000)
plt.vlines(x=a, ymin=0, ymax=a-t0, ls=':',color="black")

plt.show()
```