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Wavelet-Based Analysis of Fractional Integrated Processes: A Re-examination of Fisher Effect

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ABSTRACT

The past several decades have seen numerous empirical studies of the Fisher equation. This well-known hypothesis, introduced by Irving Fisher (1930), maintains that the nominal interest rate is the sum of the constant real rate and expected decline in the purchasing power of money. In this paper, we use the discrete wavelet transform to decompose inflation and different maturity interest rates data for both U.S. and Canada into five time scales and estimate the Fisher effect at wavelet domain. The basic findings are that DWT can overcome the difficulty of spurious regression resulted in long-memory time series and there is a long-run Fisher effect in both United States and Canada but not a short-run Fisher effect.

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I would like to thank my parents and my wife for their patience and understanding throughout the years that made it possible for me to reach my desired goals. I must admit that without their undying support, it is unlikely that this thesis would have been completed.

Any errors or omissions are my sole responsibility.

Dedication

To my wife: Haiyue

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Chapter 1 Introduction

The past several decades have seen numerous empirical studies of the relationship between the level of interest rates and inflation. It has proved to be one of the most studied topics in economics. A standard view, which is commonly referred to as the Fisher effect, is that the nominal interest rate is the sum of the real interest rate and expected decline in the purchasing power of money. Fisher (1930) hypothesized that the nominal rate of interest was equal to the sum of both the real rate of interest and the expected rate of inflation. He claimed a one-to-one relationship between the rate of interest and expected inflation, with the real rate being independent of the rate of inflation. This is also known as the super-neutrality proposition.

Earlier studies investigating the hypothesis generally conclude in favour of the proposed relationship (see Fama (1975), Nelson and Schwert (1977), Mishkin (1981, 1988), and Fama and Gibbons (1982)). However, several studies later argue that the relationship does not hold strongly for certain periods of time or when the test is performed on other country beside United States (see Barsky (1987), Huizinga and Mishkin (1984, 1986), and Kandel, Ofer and Sarig (1996)).

In modern time series econometrics, it has become standard practice using two steps procedure to estimate stationary ($I(0)$) relationships for a set of non-stationary ($I(1)$) variables. This two-step procedure, choosing between $I(0)$ and $I(1)$ alternatives, pre-tests the orders of integration of the variables entering the system. If variables are found to be integrated of order one, denoted $I(1)$, then the focus shifts towards locating co-integrating relationships between these variables in order to exploit any long-run equilibrium properties of the data.

Granger and Newbold (1974) first use simulation to study the spurious regression. They show that when unrelated data series are close to the integrated processes of order 1 or the $I(1)$ processes, then running a regression with this type of data will yield spurious effects. That is, the null hypothesis of no relationship among the unrelated $I(1)$ or unit root processes will be rejected much too often. More recently, Tsay and Chung (2000) extended the theoretical analysis of spurious regression from unit root processes to the class of long-memory fractionally integrated processes. They found that spurious regression can arise among a wide range of long-memory fractionally integrated processes, even in cases where both the dependent variable and regressors are stationary, as long as their orders of integrating sum up to a value greater than 0.5. Based on this, they conclude that it is the long memory or strong dependence, instead of non-stationary or lack of ergodicity, that causes such spurious effects and that the usual first differencing procedure may not be able to completely eliminate spurious effects if the data series are not only non-stationary but possess strong long memory. A growing body of empirical evidence supports the notion that important economic data series might be fractionally integrated, such as interest rate, GDP, inflation rate, etc.

There are two purposes of this paper. First, we will introduce a new approach to eliminate spurious regression resulting from the long-memory fractionally integrated processes. Instead of differencing the series, as is commonly done in practice, we propose to apply the discrete wavelet transform (DWT) to the series and then estimate the regression in the wavelet domain. It is motivated by the approximate de-correlation property of the DWT for long memory processes. Wavelets have some characteristics that make them particularly suitable as a vehicle for analyzing economic and financial data. Because of the translation and scale properties, non-stationary in the data is not a problem when using wavelets. Further, because of the flexibility in choice of basis function, that is, choice of wavelet function, and because of the property of "narrow" compact support, wavelets are particularly well suited to handle complex signals that involve cusps, discontinuities, and rapid changes in modeling regime.

For the second contribution of this paper, an even more important property of wavelets involves the separation of time scales of variation into a sequence of scales that can be decomposed orthogonally. A key premise underlying the research in this paper is that the relationships between inflation rate and interest rates may be better expressed in terms of restrictions to given time scale rather than over averaged time scales. In the empirical results, we use United States and Canadian data to investigate the famous Fisher effect in five different time scales, obtained from reconstruction of the data by discrete wavelet transform (DWT). In this paper, we want to argue that the regression results across time scales and countries are different and that the difference can be statistically detected through the use of a wavelet decomposition.

The paper is organized as follows. Chapter 2 briefly surveys the literature on the several applications of wavelet on the economic and financial fields. Chapter 3 presents the basic wavelet theory and focuses on the discrete wavelet transform (DWT). Chapter 4 describes the data and the empirical findings. Chapter 5 summaries the conclusions.

Chapter 2 Literature Review

Wavelets are mathematical expansions that transform data from the time domain into different layers of frequency levels. Compared to standard Fourier analysis, they have the advantage of being localized both in time and in the frequency domain, and enable the researcher to observe and analyze data at different scales. This section provides a selective review of the use of wavelets in the analysis of economics and financial data. Part of this survey follows Ramsey (1996).

2.1 Wavelet De-Noising

A convenient model for a uniformly sampled process y_t is that of the standard signal plus noise model; that is,

$$y_t = f_t + \varepsilon_t \tag{2.1}$$

where y_t is the observed signal, f_t is the actual signal, and ε_t is the noise. In economics and finance, the often un-stated assumption is that the signal, f_t is smooth, but the noise is definitely non-smooth. Under these assumptions, the generic answer to approximate f_t is to use some form of averaging over the noise. The principle idea behind "de-noising" is that one can define a "noise threshold" such that variations in the data below the threshold are to be regarded as noise, whereas variations greater than the threshold are regarded as "signal". Utilizing this threshold one may then remove (hard threshold) or shrink toward zero (soft threshold) wavelet coefficients at each level of the decomposition in an attempt to eliminate the noise from the signal. Threshold wavelet coefficients are appealing, since they capture information at different combinations of time and frequency, thus the wavelet-based estimate is locally adaptive.

There are several ways to achieve this result. One pair of ways is that recommended by Donoho and Johnstone in a series of path breaking articles, Donoho and Johnstone (1994, 1995, 1998). They defined soft shrinkage by:

$$\delta_s(w) = \begin{cases} 0 & if|w| \le c \\ \operatorname{sgn}(w)(|w| - c) & if|w| > c \end{cases} \tag{2.2}$$

and hard, $\delta_h(w)$, by:

$$\delta_h(w) = \begin{cases} 0 & if|w| \le c \\ w & if|w| > c \end{cases} \tag{2.3}$$

$\delta_s(w)$, soft shrinkage, returns the amount by which $|w|$ exceeds the threshold, c, zero otherwise. $\delta_h(w)$ returns w itself, or zero in the same circumstances. Yet another alternative is to use Breiman's Garrote, Breiman (1995):

$$\delta_g(w) = \begin{cases} 0 & if|w| \le c \\ w - c^2/w & if|w| > c \end{cases} \tag{2.4}$$

In all three cases, the value returned if the estimated coefficient is below the threshold is the same, zero. The three cases differ in how they treat the value of the coefficient when the threshold is exceeded.

2.2 Time Scale Decompositions of Economic Relationships

It has long been recognized that economic decision-making is dependent on the time scale involved, and economists emphasize the importance of discerning between long-run and short-run behaviour. Wavelets offer the possibility of going beyond this simplifying dichotomy by decomposing a time series into several layers of orthogonal sequences of scales using Mallat's multi-scales analysis. These scales can then be analyzed individually and compared across different series.

An effort along these lines is illustrated in the paper by Davison et al. (1998), who investigated U.S. commodity prices. Even though the differences across scales were not pursued fully, the authors did consider the different properties of the wavelet coefficients across scales and calculated a measure of the relative importance of the coefficients between scales.

In Ramsey and Lampart (1998a, 1998b) two relationships were examined; that between expenditure and income and that between money and income. With respect to the former, the claim that the relationship would vary and that the relevant variables would differ across scales was confirmed. The real interest rate was discovered to be a significant variable only for the longest time scales and only for durable goods. For both durable goods and for non-durable goods, the degree of fit and the strength of the relationship declined monotonically as the scale decreased. At certain scales the relationship between expenditure and income was seemingly more complex than a simple linear relationship.

For the money income relationship, the question of major interest is whether "money cause income" or "income cause money" in the Granger sense. The result of the empirical analysis by Ramsey and Lampart of this long debated and inconclusively resolved issue is that at the shortest scale, income cause money, that at intermediate scales, money cause income, and that at the longest scales, there is a feedback mechanism. All of this not only accords with theory, but parts of this result are aspects of conventional wisdom in the literature.

Recently, in Gencay et al (2002) the empirical results on the money income relationship were confirmed by employing the same techniques, but using slightly different data definitions, for the U.S., United Kingdom, Japan, and Austria. For all countries, except Austria, the Ramsey and Lampart results obtained were almost exactly the same qualitatively. Austria was different in that the first three scales, which had money caused income, but at the longest scales the feedback mechanism prevailed. This substantial confirmation is a remarkable result in econometric analysis.

2.3 Forecasting

Arino (1998) and Arino, Pedro, and Vidakovic (1995) describe a very simple approach for forecasting time series using wavelets. Their methods first decompose the signal into its wavelet coefficients then compute the energy associated with each scale. The dominant scales are defined as those with highest energies. Using the properties of the multi-scales analysis, the time series is then decomposed into two separate series.

Each individual is then fitted using an ARIMA model and the aggregate forecast is obtained by adding up the individual forecasts. Arino shows that his forecasts are preferable to a standard Box and Jenkins approach, but does not discuss the distributional properties of his forecast.

Aussem et al. (1998) have used wavelet transformed financial data as the input to a neural network, which was trained to provide five-days-ahead forecasts for the S&P500 closing prices. They performed the analysis on the wavelet coefficients over the four smallest scales using a B-spline wavelet. In addition, they examined each wavelet series individually to provide separate forecasts for each time scale and recombined these forecasts to form an overall forecast using a dynamic recurrent neural network. Since neural networks need a lot of variation to extract information, only scales with a relatively high frequency can be used.

The main benefit of wavelets in forecasting appears to be their ability to reveal features in the individual scales that are dampened by the overlapping scales. It is therefore easier for ARIMA models or neural networks to extract periodic information in the individual scales.

2.4 Long-memory Processes

In the latter part of the twentieth century, several disciplines came to the realization that naturally occurring phenomena (river flow, atmospheric patterns, telecommunications, financial markets, etc.) exhibit correlations that do not decay at a sufficiently fast rate. This means that observations separated by great periods of time still exhibit significant correlation. These time series are known as long-memory processes and require different approaches to modeling than the so-called short-memory time series. Wavelets have shown great promise in handling long-memory and a combination of short-memory and long-memory processes. Both least squares and maximum likelihood procedures have been established for estimating the model parameters in the case of long-memory dependence.

Jensen (1999) develops a simpler, ordinary least squares (OLS) estimator that is based on the observation that for a mean zero $I(d)$ processes, $|d| < 0.5$, the wavelet coefficients, d_{jk} are asymptotically normally distributed with mean zero and variance $\sigma^2 2^{-2jd}$ as j goes to zero. Taking logs, we can estimate d using the linear relationship:

$$\ln R(j) = \ln \sigma^2 - d \ln 2^{2j} \tag{2.5}$$

where $R(j)$ is the sample estimate of the covariance in each scale. The wavelet estimators have a higher small-sample bias than the GPH estimator, but Monte Carlo experiments show that they have a mean-squared error that is about six times lower.

Wavelet based maximum likelihood estimation procedures have been investigated by McCoy and Walden (1996) and Jensen (2000). Although least squares estimation is popular because of its simplicity to program and compute, it produces much larger mean square errors when compared to maximum likelihood methods. McCoy and Walden (1996) replace the covariance matrix of the process with an approximation using the discrete wavelet transform (DWT) to overcome the difficulty of computing the exact likelihood. This is the ability of the DWT to de-correlate long-memory processes. Jensen (2000) generalizes the long-memory parameter estimator of McCoy and Walden (1996) to estimate simultaneously the ARFIMA model's short and long-memory parameters. He uses the sparse wavelet representation of a matrix operator and approximates the likelihood function of an ARFIMA model with the series' wavelet coefficients and variances. And then, he estimates the wavelet maximum likelihood estimator of the ARFIMA model from maximization of this approximate likelihood function over the short and long-memory parameter space.

Overall, it is apparent that the wavelets are particularly well adapted to the vagaries of the statistical analysis of economic and financial data. But there are still some exciting and revealing analysis waiting for explored by the application of wavelets. The potential benefits of using wavelets are not only restricted to the application of new technology, but will lead to new insight into and novel theories about economic and financial phenomena based on wavelets analysis. We will present basic wavelet theory and technical detail in next chapter.

Chapter 3 Wavelet Analysis

The study of wavelets as a distinct discipline started in the late 1980s. Wavelet theory has since inspired the development of a powerful methodology, which includes a wide range of tools such as wavelet transforms, multi-resolution analysis, time-scale analysis, time-frequency representations with wavelet packets. Signal processing, data compression, medical imaging, turbulence and numerical analysis are only a few examples from a long list of disciplines in which wavelets have been successfully employed. Among others, the wavelet transforms and their modifications are becoming increasingly popular in diverse areas of applied and theoretical science.

While there are many useful properties of wavelets, in this paper, we are interested in only two major facets of wavelet analysis: the ability to handle non-stationary time series data and the resolution of the signal in terms of the timescale of analysis. In this section, we give a brief overview of three basic tools of wavelet analysis: discrete wavelet transform (DWT), multi-resolution analysis (MRA) and fast wavelet transform (FWT).

3.1 The Basic of Wavelet Analysis

Wavelet theory has its roots in Fourier analysis, but there are important differences. Fourier series is a linear combination of sines and cosines. Each of these sines and cosines is itself a function of frequency, and therefore the Fourier transform may be seen as decomposition on a frequency-by-frequency basis. For example, the Fourier series of any real valued function $f(x)$ on the [0,1] interval is expressed as

$$f(x) = b_0 + \sum_{k=1}^{\infty}[b_k \cos kx + a_k \sin kx] \tag{3.1}$$

where the parameters a_k, b_0 and b_k, for $\forall k$ can be solved using least squares.

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx,\ b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx,\ \text{and}\ a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx$$

The Fourier basis functions (sines and cosines) are very appealing when working with stationary time series. However, Fourier analysis has a serious drawback. In transforming to the frequency domain, time information is lost. When looking at a Fourier transform of a signal, it is impossible to tell when a particular event took place. If the signal properties do not change much over time -- that is, if it is what is called a stationary signal – this drawback isn't very important. However, most interesting signals contain numerous non-stationary or transitory characteristics: drift, trends, abrupt changes, and beginnings and ends of events. These characteristics are often the most important part of the signal, and Fourier analysis is not suited to detecting them. In addition, sine and cosine functions are periodic functions that are inherently *non-local*; that is, they go on to plus and minus infinity on both ends of the real line. Therefore, any change at a particular point of the time domain has an effect that is felt over the entire real line.

Wavelets, on the other hand, are defined over a finite domain. Unlike the Fourier transform, they are localized both in time and in scale. They provide a convenient and efficient way of representing complex signals. The wavelet is long in time when capturing low frequency events, and hence has good frequency resolution. Conversely, the wavelet is short in time when capturing high frequency events and therefore has good time resolution for these events. By combining several combinations of shifting and stretching of the father and mother wavelet, the wavelet transform is able to capture all the information in a time series and associate it with specific time horizons and locations in time. This makes the wavelet transform an ideal tool for studying non-stationary time series.

Given a real valued function ψ called wavelet "mother", the wavelet ψ_{jk} is obtained from ψ by dilating by the factor 2^j and translating by $2^{-j}k$ as follows:

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k) \tag{3.2}$$

The coefficient $2^{j/2}$ is a scaling factor. Under suitable conditions on ψ (see Chui, 1992) the set $\{\psi_{jk}\}$ forms an orthonormal basis for the space of the square integrable functions in $L^2(R)$ (the space of all square integrable functions). This means that any element in $L^2(R)$ may be represented as a linear combination of these basis functions. Let y be a data vector with 2^n elements that can be represented by a piece constant function $f(x)$ on $[0,1]$. The wavelet series representation of a square integrable function $f(x)$ is

$$f(x) = \sum_{j,k} c_{jk}\psi_{jk}(x) \tag{3.3}$$

Where c_{jk} are the wavelet coefficients of $f(x)$ with respect to the basis $\{\psi_{jk}\}$, which can be found by:

$$c_{jk} = \int_{-\infty}^{+\infty} f(x)\psi_{jk}(x)dx \tag{3.4}$$

The simplest example of wavelet basis is the Haar basis generated by the "mother" wavelet ψ defined by

$$\psi_{jk}(x) = \begin{cases} 1 & for 0 \le x < 1/2 \\ -1 & for 1/2 \le x < 1 \\ 0 & otherwise \end{cases} \tag{3.5}$$

The parameters j and k dilate and translate the function. Increasing j makes the Haar function finer, while k shifts it from the left to the right. ψ_{jk} is an orthonormal basis, because it

1. is orthogonal $\int(\psi_{jk} * \psi_{lm}) = 0, ((j \neq l) \vee (k \neq m))$, and
2. has an L^2 norm of unity $\int(2^{j/2}\psi(2^j x - k))^2 dx = 1$

The scaling factor $2^{j/2}$ helps achieve the latter result.

3.2 Discrete Wavelet Transform (DWT)

There are two types of wavelets defined on different normalization rules: father wavelets ϕ and mother wavelets ψ. The father wavelet integrates to 1 and the mother wavelet integrates to 0:

$$\int \phi(t)dt = 1$$

$$\int \psi(t)dt = 0$$

Roughly speaking, the father wavelets are good at representing the smooth and low frequency parts of a signal, and the mother wavelets are useful in describing the detail and high-frequency components. Thus, they are used in pairs within a family of wavelet functions, with father wavelets used for the trend components and the mother wavelets for all the deviations from the trend.

Any function $f(x)$ in $L^2(R)$ to be represented by a wavelet analysis can be built up as a sequence of projections onto father and mother wavelets generated from ϕ and ψ through scaling and translation as follows:

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) = 2^{j/2}\phi(\frac{x - 2^{-j}k}{2^{-j}}) \tag{3.6}$$

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) = 2^{j/2}\psi(\frac{x - 2^{-j}k}{2^{-j}}) \tag{3.7}$$

The wavelet representation of the signal or function $f(x)$ in $L^2(R)$ can now be given as:

$$f(x) = \sum_k A_{J,k}\phi_{J,k}(x) + \sum_k d_{J,k}\psi_{J,k}(x) + \sum_k d_{J-1,k}\psi_{J-1,k}(x) + \cdots + \sum_k d_{1,k}\psi_{1,k}(x)$$

where J is the number of multi-resolution components, and k ranges from 1 to the number of coefficients in the specified component. The coefficients $A_{J,k}, d_{J,k}, \cdots, d_{1,k}$, are the wavelet transform coefficients given by the projections

$$A_{J,k} = \int \phi_{J,k}(x) f(x) dx \tag{3.8}$$

$$d_{j,k} = \int \psi_{j,k}(x) f(x) dx \text{, for } j = 1,2,\cdots,J \tag{3.9}$$

The magnitude of these coefficients reflects a measure of the contribution of the corresponding wavelet function to the total signal. The basic functions $\phi_{J,k}(x)$ and $\psi_{j,k}(x)$ are the approximating wavelet functions generated as scaled and translated versions of ϕ and ψ, with scale factor 2^j and translation parameter $2^{-j}k$, respectively. The scale factor 2^j is also called the dilation factor and the translation parameter $2^{-j}k$ refers to the location. Here 2^j is a measure of the scale or width of the functions $\phi_{J,k}(x)$ and $\psi_{j,k}(x)$. That is, the larger the index j, the larger the scale factor 2^j, and hence the function get shorter and more spread out. The translation parameter $2^{-j}k$ is matched to the scale parameter 2^j in that as the functions $\phi_{J,k}(x)$ and $\psi_{j,k}(x)$ get wider, their translation steps are correspondingly larger.

3.3 Multi-resolution Analysis

The discrete wavelet transformation (DWT) calculates the coefficients of the wavelet representation for the discrete signals $f_1,\cdots,f_n$ for finite extent. The DWT maps the vector $f=(f_1,f_2,\cdots,f_n)'$ to a vector of n wavelet coefficients $w=(w_1,w_2,\cdots,w_n)'$. The vector w contains the coefficients $A_{J,k},d_{J,k},\cdots,d_{1,k}$ of the wavelet series representation. The coefficients $A_{J,k}$ are called the smooth coefficients or approximation coefficients, representing the underlying smooth behaviour of the signal at the coarse scale 2^J. On the other hand, $d_{j,k}$ are called the detailed coefficients representing deviations from the smooth behaviour, where $d_{J,k}$ describe the coarse scale deviations and $d_{J-1,k},\cdots,d_{1,k}$ provide progressively finer scale deviations.

In case when n is divisible by 2^J, there are $n/2$ coefficients $d_{1,k}$ at the finest scale $2^1=2$. At the next finest scale $2^2=4$, there are $n/4$ coefficients $d_{2,k}$. Likewise, at the coarsest scale, there are $n/2^J$ coefficients each for $d_{J,k}$ and $A_{J,k}$. Summing up, we have a total of n coefficients:

$$n = n/2 + n/4 + \cdots + n/2^{J-1} + n/2^J + n/2^J$$

The number of coefficients at a scale is related to the width of the wavelet function. At scale 2, the translation steps are $2k$, and so $n/2$ terms are required in order for the functions $\psi_{1,k}(x)$ to cover the interval $1 \le t \le n$. By similar reasoning, a summation involving $\psi_{j,k}(x)$ requires just $n/2^j$ terms, and the summation involving $\phi_{J,k}(x)$ requires only $n/2^J$ terms. The string of coefficients can be ordered from coarse scales to fine scales as:

$$w = \begin{pmatrix} A_J \\ d_J \\ d_{J-1} \\ \vdots \\ d_1 \end{pmatrix}$$

Each of the sets of coefficients in w is called a 'crystal', and the wavelet associated with each coefficient is referred to as an 'atom'.

The multi-resolution decomposition of a signal can now be defined by using the product of the crystals and the corresponding wavelet atoms, namely:

$$A_J(x) = \sum_k A_{J,k}\phi_{J,k}(x) \tag{3.10}$$

$$D_j(x) = \sum_k d_{j,k}\psi_{J,k}(x), \text{ for } j = 1,2,\cdots,J \tag{3.11}$$

The functions are called the smooth signal and the detail signals, respectively, which constitute a decomposition of a signal into orthogonal components at different scales. Similarly to the wavelet representation of a signal in $L^2(R)$, a signal $f(x)$ can now be expressed in terms of these signals:

$$f(x) = A_J(x) + D_J(x) + D_{J-1}(x) + \cdots + D_1(x) \tag{3.12}$$

As each term in represent components of the signal $f(x)$ at different resolutions, it is called a multi-resolution decomposition (MRD).

The coarsest scale signal $A_J(x)$ represents a coarse scale smooth approximation to the signal. Adding the detail signal $D_J(x)$ gives a scale 2^{J-1} approximation to the signal, $A_{J-1}(x)$, which is a refinement of the coarsest approximation $A_J(x)$. Further refinement can sequentially be obtained as:

$$A_{j-1}(x) = A_j(x) + D_{j-1}(x) = A_J(x) + D_J(x) + D_{J-1}(x) + \cdots + D_j(x) \tag{3.13}$$

The collection $\{A_J, A_{J-1}, A_{J-2}, \cdots A_1\}$ provides a set of multi-resolution approximations of the signal $f(x)$.

3.4 Fast Wavelet Transform (FWT)

Calculating wavelet expansions directly by matrix inversion is computationally intensive. A big breakthrough came in the mid-1980s when Mallat introduced methods from signal processing theory to wavelet. Using a technique called quadrature mirror filtering, he showed that any discrete wavelet transformation could be calculated rapidly using a cascade-like algorithm.

Except in some special cases, there is no analytical formula for computing a wavelet function. Instead, wavelets are derived using a special two-scale dilation equation. For a father wavelet $\phi(x)$, the dilation equation is defined by

$$\phi(x) = \sqrt{2}\sum_{k} l_k \phi(2x-k) \tag{3.14}$$

The mother wavelet $\psi(x)$ can similarly be obtained from the father wavelet by the relationship

$$\psi(x) = \sqrt{2}\sum_{k} h_k \phi(2x-k) \tag{3.15}$$

The coefficients l_k and h_k are the low-pass and high-pass filter coefficients defined as:

$$l_k = \frac{1}{\sqrt{2}}\int \phi(x)\phi(2x-k)dt \tag{3.16}$$

$$h_k = \frac{1}{\sqrt{2}}\int \psi(x)\phi(2x-k)dt \tag{3.17}$$

or $h_k = (-1)^k l_k$

The simplest example is provided by the Haar wavelet for $N = 2$

$$l_k = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

$$h_k = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

The low pass filter coefficients, $\{l_k\}$, are a moving average filter that smoothes the high frequency traits (jumps, cusps, singularities) of a series. On the other hand, the high band-pass filter coefficients, $\{h_k\}$, act as a differencing operator that captures the details filtered out by the low pass filter.

In theory the wavelet coefficient, $c_{j,k}$, equals the convolution of $f(x)$ with $\psi_{j,k}$. Since empirically $f(x)$ is only known on a discrete set of points, $c_{j,k}$ is calculated by a two-channel filter bank. Thus, one never needs to explicitly calculate ψ, in stead all the calculations are performed in terms of filter bank coefficients, l_k.

Defining the dilations and translations of ϕ as

$$\phi_{j,k} = 2^{j/2}\phi(2^j x - k) \tag{3.18}$$

where $j,k \in Z$, the filter bank definition of $\psi_{j,k}$ can be written as:

$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k) = 2^{j/2}\sum_{k=0}^{2J-1} l_k \phi(2^{(j-1)}x - 2n - k) \tag{3.19}$$

Using this high band pass filter definition of $\psi_{j,k}$, it follows that

$$\begin{aligned} c_{j,k} &= 2^{j/2}\sum_{k=0}^{2J-1} l_k \int x(x)\phi(2^{(j-1)}t - 2n - k)dx \\ &= 2^{1/2}\sum_{k=0}^{2J-1} l_k A_{j-1,2n+k} \end{aligned} \tag{3.20}$$

where $A_{j,k} = 2^{j/2}\int x(x)\phi(2^j - k)dx$ is the scaling coefficient. Thus, computing $c_{j,k}$ requires knowledge of $\{A_{j-1,k}\}_{k \in Z}$.

Calculation of the scaling coefficient, $A_{j,k}$, can be performed by writing $\phi_{j,k}$ in terms of the low pass filter as

$$\phi_{j,k} = 2^{j/2}\phi(2^j x - k) = 2^{(j-1)/2}\sum_{k=0}^{2J-1} h_k \phi(2^{(j-1)}x - 2n - k) \tag{3.21}$$

Convoluting $x(t)$ with the above equation we find that $A_{j,k}$ equals

$$A_{j,k} = 2^{(j-1)/2} \sum_{k=0}^{2J-1} h_k \int x(t)\phi(2^{(j-1)}t - 2n - k)dt$$
$$= \sum_{k=0}^{2J-1} h_k A_{j-1,2n+k} \tag{3.22}$$

Thus, both $A_{j,k}$ and $c_{j,k}$ are calculated recursively from the smallest to the largest scale with the simple multiplication and addition operators of a two-channel filter bank.

The recursive algorithm known as the fast wavelet transform (FWT) is illustrated in Figure 1.

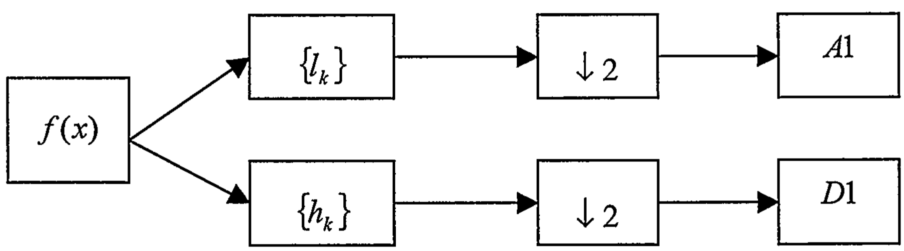


Figure 1: One Stage Filtering: Approximation and Detail

The original signal, $f(x)$, passes through two complementary filters $\{l_k\}$ and $\{h_k\}$, and emerges as two signals $A1$ and $D1$. The box $[\downarrow 2]$ in Figure 1 represents the decimation of the output from the filter by 2, i.e., discarding the odd sampled observation. By their definition the ideal low and high band-pass filter, ϕ and ψ, include this decimation. Because the filters coefficients $\{l_k\}$ and $\{h_k\}$ are both applied to approximation coefficients, twice as many observations as the length of $A_{j,k}$ are created. Only half of this output is needed to completely represent or recover $A_{j,k}$.

The decomposition process can be iterated, with successive approximations being decomposed in turn, so that one signal is broken down into many lower resolution components. This is called the wavelet decomposition tree. In Figure 2, we present this decomposition processes. Since the analysis process is iterative, in theory it can be continued indefinitely. In reality, the decomposition can proceed only until the individual details consist of a single sample. In practice, we can select a suitable number of levels based on the nature of the signal, such as the economic theory, or on a suitable criterion such as entropy.

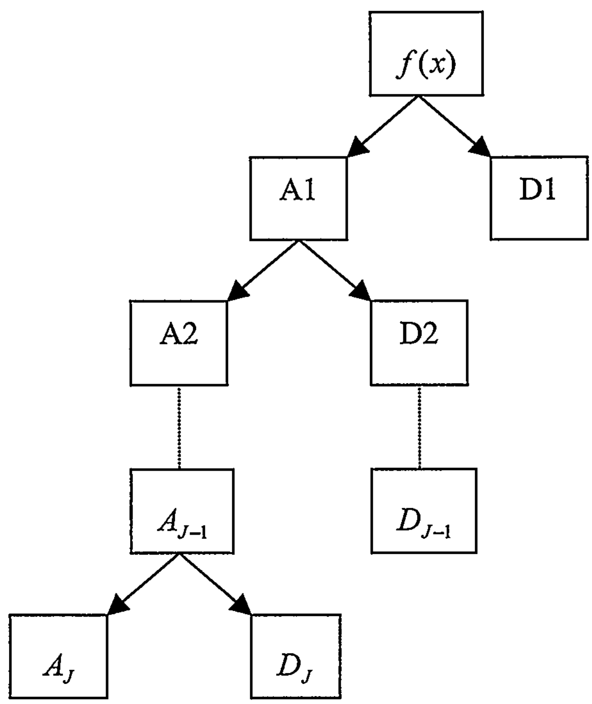


Figure 2: Schematic Representation of Fast Wavelet Transform

Because of the orthogonality of the filter banks for the wavelet, the arrows in Figure 2 can be reserved to synthesize $f(x)$ from its wavelet transform. In this wavelet synthesis, adding the details of D_J to the smoothed series A_J provides us with the representation of $f(x)$ at the next degree of resolution A_{J-1} with the highest resolution being the series $f(x)$.

In the next chapter, we will present the empirical results. With each set of macroeconomic data, the idea is to examine in detail the relationship between the variables when the variation in each variable has been restricted to a specific scale. Instead of looking at the relationship between inflation and interest rate "averaged" over all timescales, we examine the relationship at each timescale separately. Let $I(A_J), R(A_J), I(D_j)$ and $R(D_j)$ represent "inflation" and "interest rate" at each scale, as determined by the wavelet decomposition. Then, we can estimate a sequence of regressions using

$$R(A_J) = \alpha_J + \beta_J I(A_J) + \varepsilon_J \qquad (3.23)$$

or

$$R(D_j) = \alpha_j + \beta_j I(D_j) + \varepsilon_j, \quad j = 1,2,\cdots J \qquad (3.24)$$

Chapter 4 Empirical Results

4.1 The Data

The empirical analysis makes use of monthly data on inflation rate and different maturity interest rates of United States and Canadian post-war data, which are taken from Federal Reserve Economic Data and CANSIM database, respectively. For the U.S. we use the Federal funds rate, the 90-Day Treasure bill rate and as well as the 1, 3, 5 and 10 years Government bond rates. For Canada, we use the Bank of Canada rate and the 91-Day Treasure Bill rate, as well as the 1-3, 3-5, 5-10 and 10+ years Bond Yield. The interest rate is expressed in percentage per annum. In order to be compared with interest rates, we measured inflation rate as the month-to-month change in the consumer price index multiplied by 1200. The data is monthly and runs from Sep 1959 to Apr 2002. The details of the data used and their sources are listed in Table 1 and Table 2 for United States and Canada, respectively.

Table 1: The Data Used in the Analysis (U.S.)

Name	Dates	Observations	Source	Code
CPI Inflation Rate (CPI)	1959:09-2002:04	512	FRED	cpiaucsl
Federal Funds Rate (FED)	1959:09-2002:04	512	FRED	fedfunds
90-Day Treasure Bills Rate (TB90)	1959:09-2002:04	512	FRED	tb3ms
1 Years Gov'n Bond Rate (GS1)	1959:09-2002:04	512	FRED	gs1
3 Years Gov'n Bond Rate (GS3)	1959:09-2002:04	512	FRED	gs3
5 Years Gov'n Bond Rate (GS5)	1959:09-2002:04	512	FRED	gs5
10 Years Gov'n Bond Rate (GS10)	1959:09-2002:04	512	FRED	gs10

Table 2: The Data Used in the Analysis (Canada)

Name	Dates	Observations	Source	Code
CPI Inflation Rate (CPI)	1959:09-2002:04	512	CANSIM	P100000
Bank of Canada Rate (BOC)	1959:09-2002:04	512	CANSIM	B14006
91-Day Treasure Bills Rate (TB91)	1959:09-2002:04	512	CANSIM	B14001
1-3 Years Bond Yield (BY1)	1959:09-2002:04	512	CANSIM	B14009
3-5 Years Bond Yield (BY3)	1959:09-2002:04	512	CANSIM	B14010
5-10 Years Bond Yield (BY5)	1959:09-2002:04	512	CANSIM	B14011
10+ Years Bond Yield (BY10)	1959:09-2002:04	512	CANSIM	B14013

We plot the U.S. inflation rate and Federal funds rate in Figure 3 and their Canadian counterparts in Figure 4 to make some sense about our data.

4.2 Estimation of Fractional Integration Order

In this part, we check the long-memory property of those data and estimate the fractional integration order of each time series using wavelet OLS estimator suggested by Jensen (1999). Let x_t be the fractional integrated process, $I(d)$, defined by,

$$(1-L)^d x_t = \varepsilon_t \tag{4.1}$$

Where $\varepsilon_t \to i.i.d(0,\sigma^2)$, d is any real value and $(1-L)^d$ is fractional integrating operator defined by the binominal expansion,

$$\begin{aligned}(1-L)^d &= 1+\sum_{j=1}^{\infty}\frac{\Gamma(d+1)(-L)^j}{\Gamma(d-j+1)\Gamma(j+1)} \\ &= 1-dL+\frac{d(d-1)}{2!}L^2-\frac{d(d-1)(d-2)}{3!}L^3+\cdots\cdots \end{aligned} \tag{4.2}$$

Where L denotes the lag operator and Γ the gamma operator. When $d=0$, the process x_t is simply equal to ε_t, so $x_t \sim N(0,\sigma^2)$, or $x_t \sim I(0)$. When $d=1$, however, x_t

follows a unit root process (without drift), implying it has a zero mean with infinite variance.

More generally, if we allow d taking non-integer values, the process x_t is said to be fractionally integrated, making (4.1) an ARFIMA process (i.e., fractionally integrated ARMA). As shown by Hosking (1981), when $0 < d < 1/2$, the autocovariance function of x_t declines hyperbolically to zero. If $1/2 \le d < 1$, x_t has an infinite variance; however, it will still revert to its mean (or trend) in the very long run. Table 3 below summarizes the different values of d and the corresponding consequences for the mean (or trend), variance and duration of a shock.

Table 3: Summary of Fractional Integration Parameter Values

d	Mean (or trend) and variance	Shock duration
$-0.5 < d < 0$	Short-run mean anti-persistent Finite variance	Long-lived
$d = 0$	Short-run mean-reversion Finite variance	Short-lived
$0 < d < 0.5$	Long-run mean-reversion Finite variance	Long-lived
$0.5 \le d < 1$	Long-run mean-reversion Infinite variance	Long-lived
$d = 1$	No mean-reversion Infinite variance	Infinite
$d > 1$	No mean-reversion Infinite variance	Infinite; effect increases as time moves forward

Jensen (1999) demonstrates that, for an $I(d)$ process x_t with $|d| < 1/2$, use of the auto-covariance function implies that the wavelet coefficients $c_{j,k}$ in (3.4) are distributed as $N(0,\sigma^2 2^{-2jd})$. If $R(j)$ denotes the wavelet coefficient's variances at scale j, then after taking logarithms, an estimate of d can be obtained from

$$LnR(j) = Ln\sigma^2 - dLn2^{2j} \tag{4.3}$$

Thus, the wavelet transform is applied to the auto-covariance function of a particular series, not to the series itself. The wavelet is used only in the estimation of the d coefficient with the observed auto-covariance function. Furthermore, because of the form of the wavelet expansion (3.4), it should be noted that the number of observations for the underlying process x_t must be a factor of 2.

In order to decide the robustness, we consider the un-smooth Haar wavelet, and three different degrees of smoothing for the Daubechies wavelet, where the smoothness increases with the order of the wavelet. We present the results in Table 4 and Table 5 for U.S. and Canada, respectively, and put the standard errors into the parentheses. The wavelet estimators show some interesting results. First, the inflation rate has smaller fractional order of integration than all the interest rates. It suggests that the autocorrelation function of inflation rate will decay more quickly than the interest rate. Second, for U.S., all rates on securities with maturities of one year or less are at least two standard errors below 1.0, implying that they are not strict unit root processes. The implication is that they have a tendency to revert back to their means in the very long run. Finally, the fractional integration parameters increase as the term to maturity increases. We therefore find that the longest rates, the 5- and 10-year rates, are the rates that display properties that most closely resemble unit root processes for both U.S. and Canada.

Table 4: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Haar	Daubechies-4	Daubechies-12	Daubechies-20
Inflation Rate	0.4701	0.5233	0.5343	0.5274
(π)	(0.0721)	(0.0600)	(0.0663)	(0.0728)
Federal Funds Rate	0.7105	0.8612	0.8520	0.8119
(FED)	(0.1000)	(0.0592)	(0.0624)	(0.0889)
90-Day Treasure	0.7420	0.8522	0.8662	0.8051
Bills (TB90)	(0.0927)	(0.0480)	(0.0616)	(0.0872)
1 Years Gov'n	0.7638	0.8642	0.8950	0.8260
Bond Rate (GS1)	(0.0894)	(0.0436)	(0.0616)	(0.0877)
3 Years Gov'n	0.8274	0.8912	0.9187	0.8304
Bond Rate (GS3)	(0.0671)	(0.0458)	(0.0671)	(0.1068)
5 Years Gov'n	0.8566	0.9141	0.9311	0.83221
Bond Rate (GS5)	(0.0608)	(0.0500)	(0.0700)	(0.1204)
10 Years Gov'n	0.9005	0.9458	0.9508	0.8182
Bond Rate (GS10)	(0.0566)	(0.0557)	(0.0762)	(0.1659)

Table 5: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Haar	Daubechies-4	Daubechies-12	Daubechies-20
Inflation Rate	0.3650	0.4150	0.4182	0.4220
(π)	(0.0831)	(0.0927)	(0.0947)	(0.1000)
Bank of Canada	0.7934	0.8689	0.8657	0.7685
Rate (BOC)	(0.0693)	(0.0557)	(0.0728)	(0.1296)
91-Day Treasure	0.8269	0.8939	0.8886	0.7415
Bills (TB91)	(0.0640)	(0.0656)	(0.0866)	(0.1735)
1-3 Years Bond	0.8150	0.8520	0.8448	0.6715
Yield (BY1)	(0.0529)	(0.0574)	(0.0825)	(0.1700)
3-5 Years Bond	0.8305	0.8624	0.8565	0.6793
Yield (BY3)	(0.0510)	(0.0566)	(0.0831)	(1706)
5-10 Years Bond	0.8559	0.8872	0.8750	0.6861
Yield (BY5)	(0.0510)	(0.0574)	(0.0837)	(0.1789)
10+ years Bond	0.8913	0.9207	0.8984	0.7061
Yield (BY10)	(0.0539)	(0.0600)	(0.0906)	(0.1836)

From the above estimation, we argue that both the inflation rate and interest rates are fractional integration with long-memory property for both countries, especially for inflation rates, which are obviously not unit root processes and displaying different characteristics with interest rates. In addition, the orders of integration for inflation rate and interest rate sum up to a value greater than 0.5. As shown by Tsay and Chung (2000), when we regress a long memory fractionally integrated process on another unrelated long memory fractionally integrated process, no matter these processes are stationary or not, the t ratios will become divergent and spurious effects occur. Based on this, they conclude that it is the long memory or strong dependence, instead of non-stationary or lack of ergodicity, that causes such spurious effects and that the usual first differencing procedure may not be able to completely eliminate spurious effects if the data series are not only non-stationary but possess strong long memory, such as fractionally integrated processes.

In the next section, we will introduce a new approach to eliminate spurious regression between long memory fractionally integrated processes suggested by Fan and Whitcher (2001). Instead of differencing the series, as is commonly done in practice, we apply the discrete wavelet transform (DWT) to decompose the series into five different time scales and then estimate the regression in the wavelet domain with each individual scale.

4.3 Discrete Wavelet Transform (DWT)

We use the discrete wavelet transform (DWT) method that we described in last chapter to decompose our data into five different time scales. As for examples, Figure 5 and Figure 6 show the wavelet-based multi-resolution analysis of Federal Funds Rate (FED) for U.S. and Bank of Canada Rate data, respectively. We use the Haar wavelet to perform this transformation. The sub-series D_j and A_J in a multi-resolution analysis form an additive decomposition of the original time series:

$$f(x) = A_J + \sum_{j=1}^{J} D_j \qquad (4.4).$$

Each sub-series D_j is associated with changes at scale $\lambda_j = 2^j$, while A_J is associated with weighted averages over scales of 2^J: see Percial and Mofjeld (1997) for more details. The first scale of wavelet coefficients D_1 is filtering out the high-frequency fluctuations by essentially looking at adjacent differences in the data. As we can see from the original data that there is a large group of rapid fluctuations around end of 1970's for the Federal funds rate. We also observe some small variations in the magnitude of individual detail coefficients at the same period. In addition, when the time scales increase, the detail coefficients become more volatile. Since the Federal funds rate exhibit low-frequency oscillations, the higher scale (low-frequency) vectors of wavelet coefficients D_4 and D_5 indicate large variations from zero. The same is true for the scaling coefficients, A_5, which are associated with average of scale 32 months or longer.

For the Bank of Canada rate decomposition, there has exhibited very similar results as the Federal funds rate series. The low-frequency detail components represent much more energy than high-frequency un-smooth components. Here the finest scale crystal D_1 represents short-term variations due to shocks occurring within two month, and the next finest component D_2 accounts for variations at a time scale of $2^2 = 4$ months. Such observation indicates that movements in interest rate are mainly caused by long-term fluctuations.

We report the summary of statistics of DWT coefficients from Table 6.1 to Table 6.7 for United States, and from Table 7.1 to Table 7.7 for Canadian counterparts. We leave these tables in the appendix for reference. From the larger standard deviation of CPI inflation detail coefficients, we argue that the inflation rate is more volatile than interest rate.

In this section, we estimate the fractional integration orders of the wavelet detail coefficients and approximation coefficients separately. The results are presented from Table 8.x to Table 9.x for United States and Canada, respectively. In order to decide the robustness, we use Haar and three Daubechies wavelet for this estimation. We list the tables of CPI inflation and Bank of Canada rate for Canada here and the readers can refer to the other tables in the appendix.

Those wavelet estimators show some informative results. First, all the fractional integration orders of the detail coefficients are smaller than the original time series. This is even more obvious for the estimations of the Bank of Canada rate and most

Table 9.1: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
Inflation Rate	512	0.3650	0.4150	0.4182	0.4220
(π)		(0.0831)	(0.0927)	(0.0947)	(0.1000)
D1	256	0.1794	0.1468	0.1923	0.2147
		(0.0975)	(0.0894)	(0.0883)	(0.0927)
D2	128	0.3276	0.3437	0.3755	0.4033
		(0.0775)	(0.1039)	(0.0980)	(0.1000)
D3	64	-0.1004	0.0532	-0.0044	-0.1851
		(0.0883)	(0.0872)	(0.0954)	(0.1421)
D4	32	0.1002	-0.2998	-0.4647	0.0241
		(0.1658)	(0.2534)	(0.4551)	(0.2821)
D5	16	0.3459	-0.0207	0.2870	0.3041
		(0.1241)	(0.3688)	(0.0819)	(0.4309)
A5	16	0.3530	0.8472	0.7595	0.1720
		(0.3952)	(0.2823)	(0.4194)	(0.5378)

Table 9.2: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
BOC Rate	512	0.7934	0.8689	0.8657	0.7685
(BOC)		(0.0693)	(0.0557)	(0.0728)	(0.1296)
D1	256	0.0171	0.0809	-0.0285	-0.0633
		(0.0656)	(0.0469)	(0.0608)	(0.0616)
D2	128	0.0592	0.0078	-0.1770	-0.1192
		(0.1200)	(0.1319)	(0.2302)	(0.1562)
D3	64	0.0167	0.0277	-0.0362	-0.0195
		(0.0949)	(0.2140)	(0.1852)	(0.0917)
D4	32	-0.2025	-0.3540	-0.3908	-0.2145
		(0.1533)	(0.1170)	(0.2173)	(0.1670)
D5	16	0.0976	-0.1443	-0.1398	0.0617
		(0.4968)	(0.2713)	(0.3481)	(0.6391)
A5	16	0.4692	0.6205	-0.5129	0.7786
		(0.3461)	(0.3121)	(1.0099)	(0.3276)

of them are down from 0.8 to 0.1 range. This makes the discrete wavelet transform (DWT) an ideal tool for handling the fractional integration process. After filtering, the shock occurred to the DWT data will decay much faster than to the original time series. Second, the standard error of the estimations increase from high-frequency to low-frequency. It is mostly caused by the smaller sample size associated with the low-frequency components. As we can see, at the coarsest detail and approximation coefficients, D_5 and A_5, there are only 16 observations in the sample compared with 256 observations in the finest detail coefficients $D1$. Third, all the fractional integration orders of the detail coefficients are between –0.5 and 0.5, which make them as stationary time series with finite variance. Also, the sum up of the integration order is below 0.5. When we regress the detail coefficient of inflation rate with respect to the interest rate, we can successfully avoid the spurious regression resulted by long memory property.

4.4 The Theory of Fisher Effect

The relationship between interest rates and inflation, first put forward by Fisher (1930), postulates that the nominal interest rate in any period is equal to the sum of the real interest rate and the expected rate of inflation. This is termed as the Fisher effect. Fisher (1930) hypothesized that the nominal interest rate could be decomposed into two components, a real rate plus an expected inflation rate. He claimed a one-to-one relationship between inflation and interest rates in a world of perfect foresight, with real interest rates being unrelated to the expected rate of inflation and determined entirely by the real factors in an economy, such as the productivity of capital and investor time preference. This is an important prediction of the Fisher Hypothesis for, if real interest rates are related to the expected rate of inflation, changes in the real rate will not lead to full adjustment in nominal rates in response to expected inflation. In empirical studies, if nominal interest rate is used as a dependent variable, the following equation is estimated:

$$R_t = \alpha + \beta E_{t-1}(\pi_t) + \mu_t \qquad (4.5)$$

where

$E_{t-1}(\pi_t)$ =the inflation rate from $t-1$ to t, expected by the bond market at time $t-1$.

R_t =the nominal interest rate for a one period bond, maturing at time t.

If expectations are rational as in Fama (1975), the realized future inflation rate can be written as:

$$\pi_t = E_{t-1}(\pi_t) + \varepsilon_t \qquad (4.6)$$

where ε_t is the forecast error of inflation, a zero mean disturbance, and is independent of all information known at time $t-1$. We can replace the $E_{t-1}(\pi_t)$ with equation (4.6) and get the empirical model for Fisher effect:

$$R_t = \alpha + \beta\pi_t + \eta_t \qquad (4.7)$$

where $\eta_t = -\beta\varepsilon_t + \mu_t$. The OLS estimate of β in equation (4.7) is a consistent estimator of equation (4.5). The error term η_t is orthogonal to π_t, assuring consistency of the β estimate. In empirical studies equation (4.7) is the one, which is used in testing for the Fisher effect.

From the last section, we have argued that inflation and interest rates are long memory processes. As we have mentioned above that the long memory or strong dependence will cause spurious effects and that the usual first differencing procedure may not be able to completely eliminate spurious effects if the data series are not only non-stationary but possess strong long memory, such as fractionally integrated processes. Discrete wavelet transform (DWT) is an ideal tool to eliminate the long memory property of time series data and to analyze it at different timescales. Instead of looking at the relationship between inflation and interest rate "averaged" over all timescales, we examine the relationship at each timescale separately. Let $I(A_J), R(A_J), I(D_j)$ and $R(D_j)$ represent "inflation rate" and "interest rate" at each scale, as determined by the wavelet decomposition. Then, we can estimate a sequence of regressions using the two equations we mentioned in last chapter:

$$R(A_J) = \alpha_J + \beta_J I(A_J) + \varepsilon_J \qquad (4.8)$$

or

$$R(D_j) = \alpha_j + \beta_j I(D_j) + \varepsilon_j, \quad j = 1,2,\cdots J \qquad (4.9)$$

2.5 Empirical Results

In order to investigate whether and to what extent the inflation rate movements are transmitted to the interest rate, we start with a simple regression of different maturities interest rate on inflation rate both for United States and Canada for the individual wavelet coefficients we mentioned in last section. Table 10.x and Table 11.x show the coefficient estimates from running a sequence of least square regressions of interest rate on inflation rate using the data described in the previous section for United States and Canada, respectively.

We list the empirical results of Bank of Canada rate on inflation rate for Canada and FED rate on inflation rate for U.S., respectively. We leave all the others regression results in the appendix. The empirical results are very similar but with a little bit difference between Canada and United States. For Canada, a review of Table 11.1 indicates that the degree of fit of the regression of Bank of Canada rate on inflation rate increases as we move to longer time scales, and the slope coefficient rises to 0.78 as the time scale of analysis increases. There is an exception to this pattern in that at the higher level of time scale analysis, D_4, the slop coefficient is smaller than the previous level and is even negative, which is wrong sign as Fisher effect suggested. The intercept terms are very small and statistically insignificant except at A_5. Recognize that at each scale level one is seeing the isolated effect of the given scale; rather at each scale, the tables demonstrate the relationship between inflation rate and interest rate where the variation in both variables has been restricted to the indicated scale. For Canada, the table indicates clearly that except for the highest scale, D_5 and A_5, there is only a weak relationship between the variables; indeed, below D_5 there seems to be little of significance. In the longest time scale, the slope coefficients are statistically close to 1 as Fisher (1930) suggested one-to-one relationship between inflation and interest rates. The regression results reveal that the Fisher effect cannot be identified at the short time scale, but there is

Table 10.1: Regressions of Federal Funds Rate on Inflation Rate

for Individual Crystals (U.S.)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	-0.0042 (0.0238)	0.0101 (0.0104)	0.0037	0.969
D2	128	-0.0583 (0.0746)	0.1329 (0.0303)	0.1321	4.379
D3	64	-0.0988 (0.1962)	0.1753 (0.0673)	0.0985	2.603
D4	32	0.3572 (0.3785)	0.6182 (0.0975)	0.5729	6.343
D5	16	-0.2250 (0.6950)	1.7388 (0.2088)	0.8321	8.329
A5	16	18.5975 (5.5696)	0.7458 (0.1953)	0.5103	3.820

Table 11.1: Regressions of Bank of Canada Rate on Inflation Rate

for Individual Crystals (Canada)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	-0.0165 (0.0247)	0.0019 (0.0060)	0.0004	0.318
D2	128	-0.0077 (0.0696)	0.0140 (0.0170)	0.0054	0.826
D3	64	0.0181 (0.1888)	0.0306 (0.0455)	0.0073	0.673
D4	32	0.7233 (0.5363)	-0.0286 (0.1367)	0.0015	-0.209
D5	16	1.7519 (1.1006)	0.7757 (0.2305)	0.4472	3.366
A5	16	23.5280 (5.8695)	0.7505 (0.1951)	0.5138	3.846

a positive relationship at the long time scale.

Same as Canada, there has very weak relationship between inflation and interest rate at the lowest scale level, D_1, for U.S. However, after D_1, the slope coefficients are all statistically significant for both detail and approximation coefficients rather than like Canada waiting for till D_5 detail coefficients to become significant. It suggests that Fisher effect does not exist in the very short-run but prevail from intermediate time scale. Not same as Canada, we cannot reject the relationship between inflation and interest rate after two months period for U.S. In addition, at the time scales, A_5, we cannot reject the null hypothesis that the slope coefficients are equal to one. It means that the super-neutrality position prevails in this period.

As for examples to check more clearly the relationship between inflation and interest rate, we plot the individual detail and approximation coefficients of Bank of Canada rate on inflation rate in Figure 7 from the finest scale $D1$ to coarsest scale $D5$ and $A5$, respectively. We leave the U.S. counterpart Figure 8 in appendix. Figure 7 illustrates the relationship between interest rate and inflation rate at scales varying from $2^1 = 2$ months to $2^5 = 32$ months using father and mother wavelet. The graphs indicate clearly the same results as in Table 11.1 that there is very weak relationship between the variables for the lowest scale, but for the highest scales, $D5$ and $A5$, an obvious trend exists in it. We argue that there will have no Fisher effect in Canada up till three years or 32 months. It is consistent with Mishkin (1992), in which he ended up with the same conclusion for the United States.

Figure 7: Plots of CPI Inflation on BOC Rate using Crystals

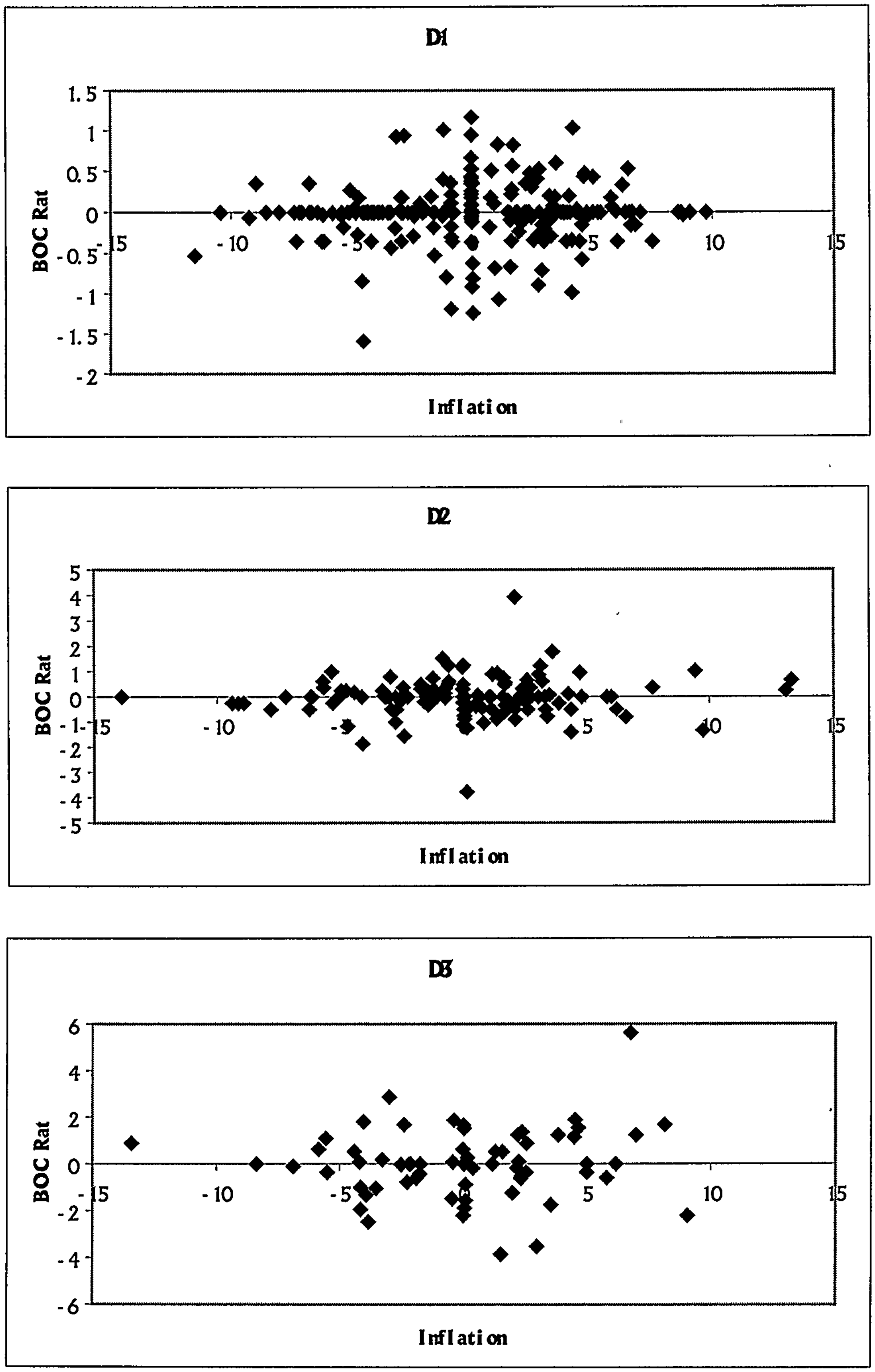
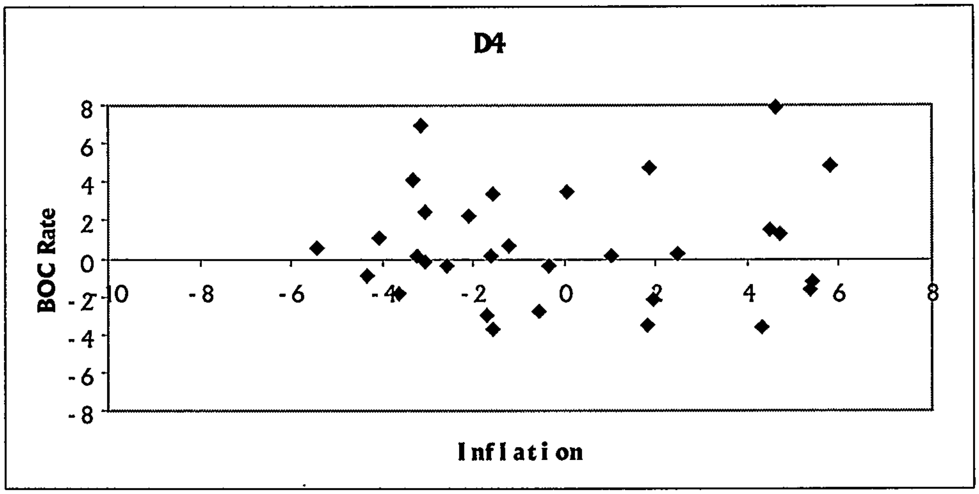
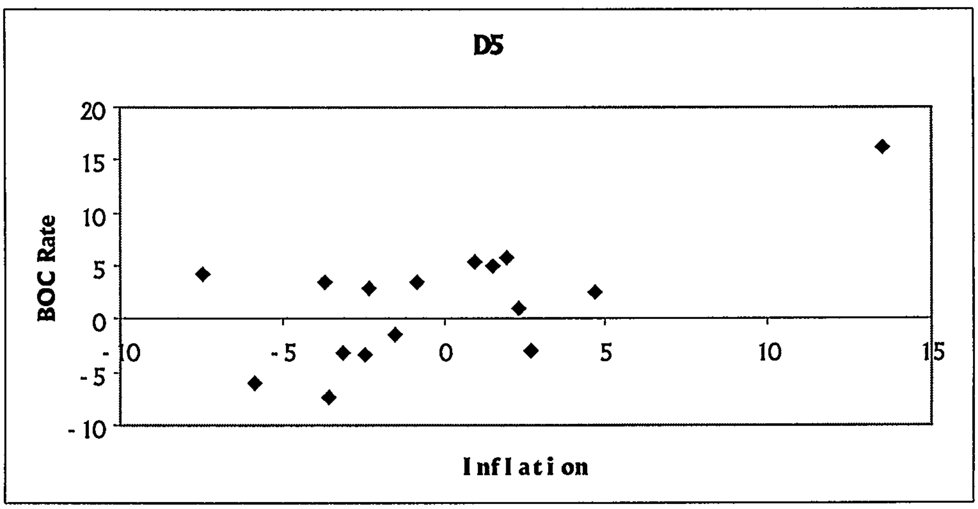
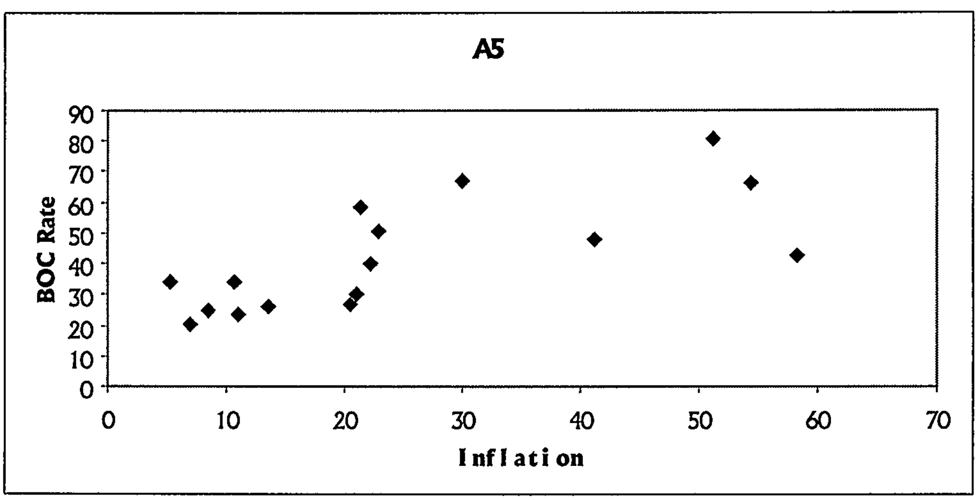


Figure 7: Plots of CPI Inflation on BOC Rate using Crystals (Continued)







The conclusion from the preceding empirical analysis is that it is reasonable to assume that there is a long-run Fisher effect in both United States and Canada but not a short-run Fisher effect, especially for Canada. The super-neutrality proposition of interest rate exists for some periods in United States and Canada. As Mishkin and Simon (1994) stated that these findings have important implications for policy makers. They indicate that the level of short-term interest rates can be an inappropriate guide for monetary policy because a high interest rate that has persisted for some time is an indication that expected inflation is high. Hence the high level of the interest rate is not an indicator that monetary policy is tight, indeed it might indicate the reverse. This suggests that looking solely at the level of short-term interest rates can produce a misleading picture of the stance of monetary police. On the other hand the evidence that a short-run Fisher effect does not exist in Canada, suggests that short-run changes, at least for one and half year ($2^4 = 16$ months), in the short-term interest rate reflect changes in the real interest rate rather than expected inflation. Thus changes in short-term interest rates can reflect the stance of monetary policy.

Chapter 5 Conclusions

Wavelet analysis is a comparatively new and powerful mathematical tool for signal processing. In particular, the discrete wavelet transform (DWT) is very useful in decomposing time series data into an orthogonal set of components with different frequencies and eliminating the long-memory property existed in it. In this paper, we use the DWT to decompose inflation and different maturity interest rates data for both United States and Canada into five time scales, starting from 2 months up till 32 months and longer, and estimate the Fisher effect at wavelet domain. We study the Fisher effect at each isolated time scale rather look at the relationship between inflation and interest rates averaged over all timescales.

The overall conclusions are that timescale decompositions are very important for understanding economic phenomena, and that those differences can be revealed, at least in part, by the application of a wavelet decomposition. There are several remarks for the Fisher effect in both United States and Canada:

1. Inflation and interest rates data are fractional integrated process in both countries. The discrete wavelet transform (DWT) can overcome the difficulty of spurious regression resulted in long-memory time series.
2. The conclusion from the preceding empirical analysis is that it is reasonable to assume that there is a long-run Fisher effect in both United States and Canada but not a short-run Fisher effect, especially for Canada.

3. Studies for the longest time scale appear to suggest a positive relationship between interest rates and inflation and they do establish a one-to-one relationship in both United States and Canada as postulated by Fisher (1930).

All in all, the DWT is a powerful mathematical tool, enabling econometrians and economists to examine much more complicated processes by separating features on a scale-by-scale basis. However, there are still some potential research fields opening to those wishing to model macro data successfully with wavelet approach. The immediate requirements for further work involve first the confirmation, possibly using data from other countries, of the qualitative results presented in this paper. The second step is to explore the potential causes of the changes that were observed in the time series at the various scales.

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Figure Appendix

Figure 3: Inflation and FED Rate. (U.S.)
Sample: 1959:09-2002:04

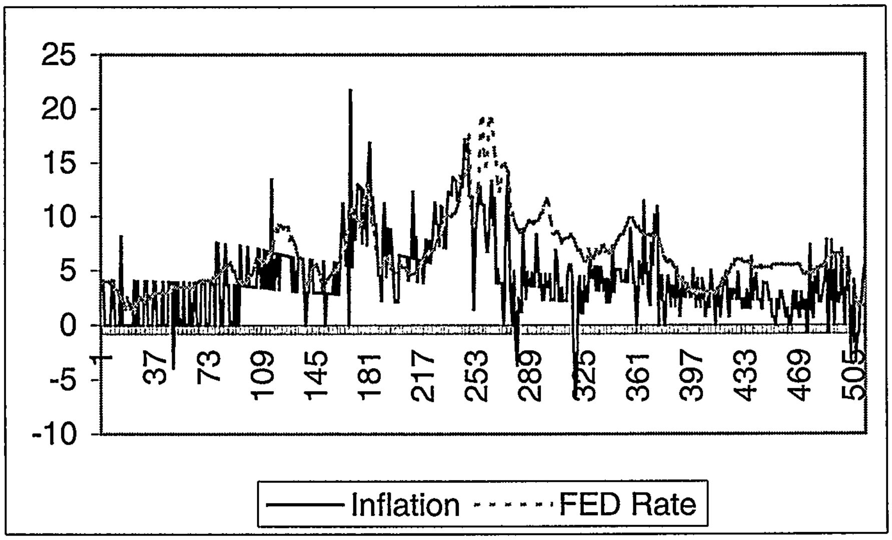


Figure 4: Inflation and Bank of Canada Rate. (Canada)
Sample: 1959:09-2002:04

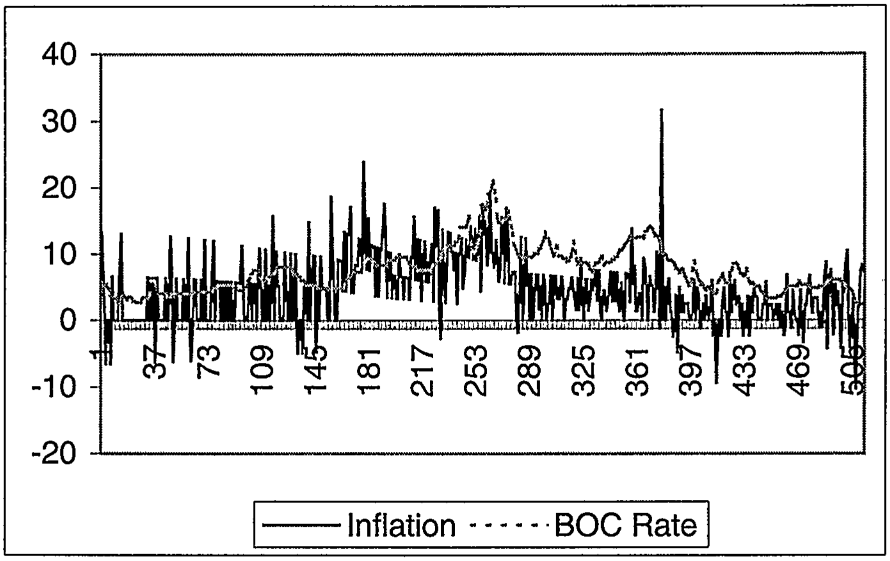
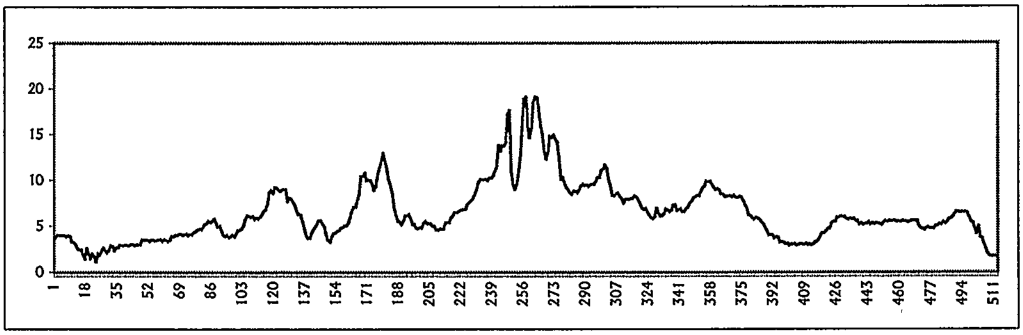
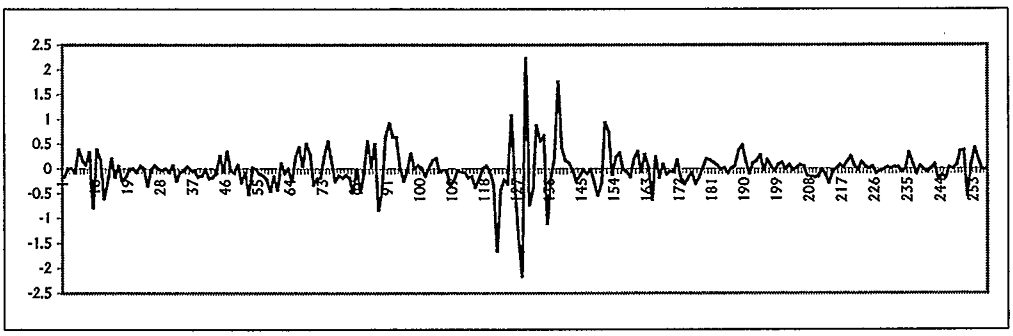


Figure 5: Multi-resolution Decomposition of FED Rate (U.S.)

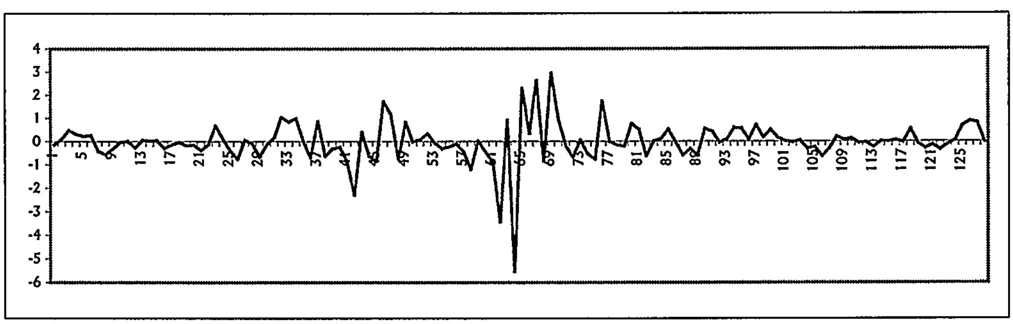
Data



D1



D2



D3

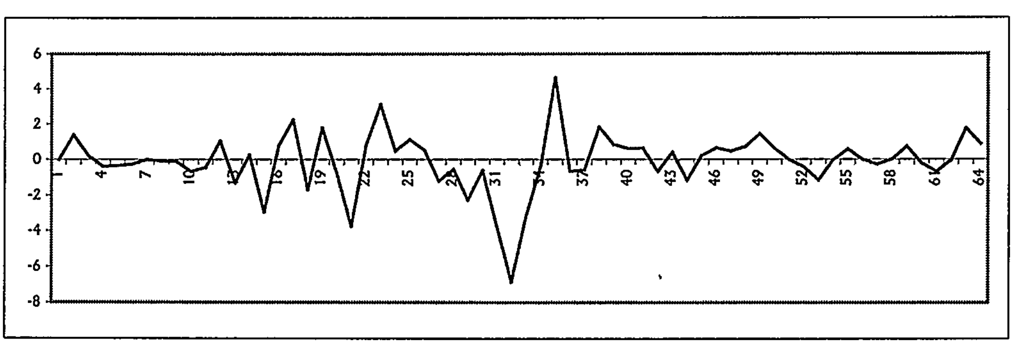
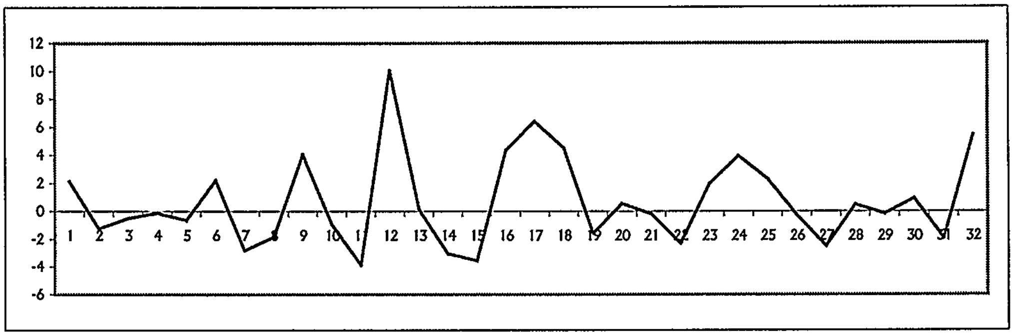
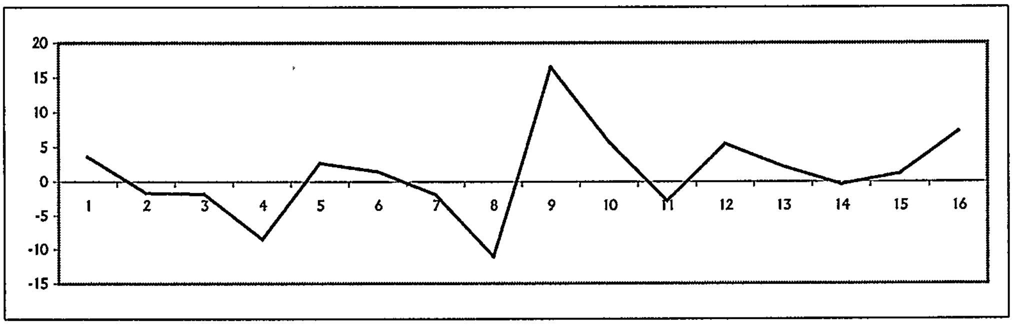


Figure 5: Multi-resolution Decomposition of FED Rate (U.S.)
(continued)

D4



D5



A5

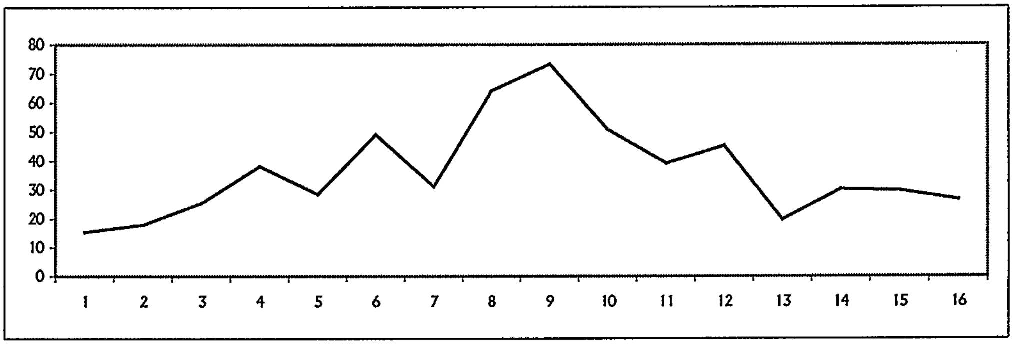
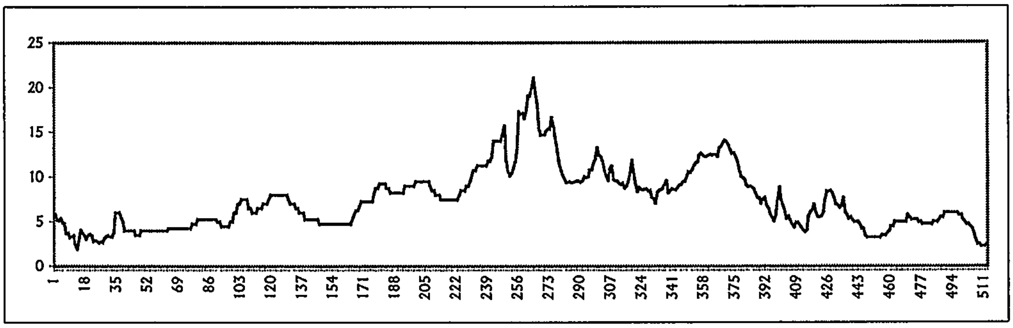
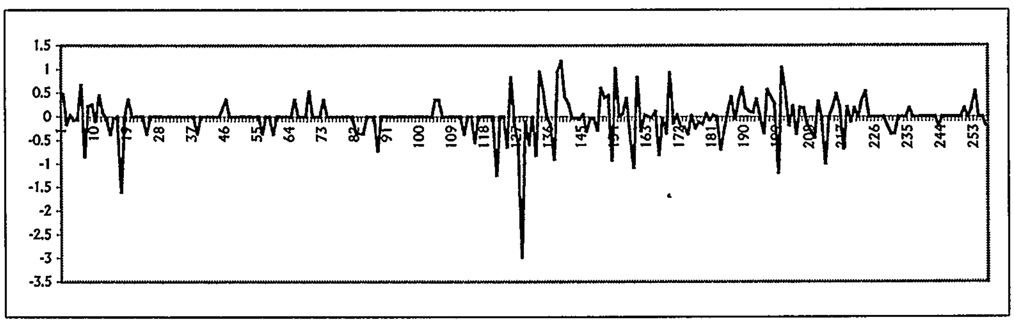


Figure 6: Multi-resolution Decomposition of Bank of Canada Rate (Canada)

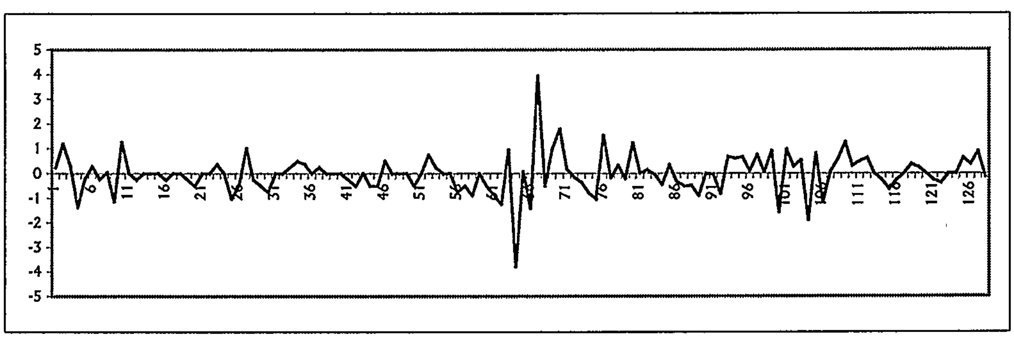
Data



D1



D2



D3

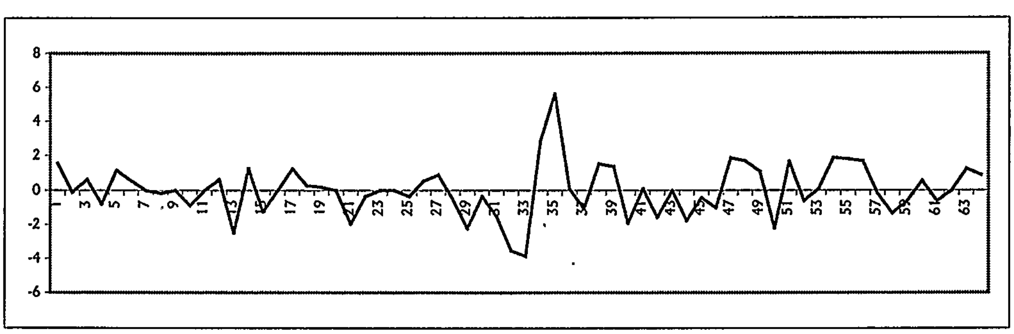
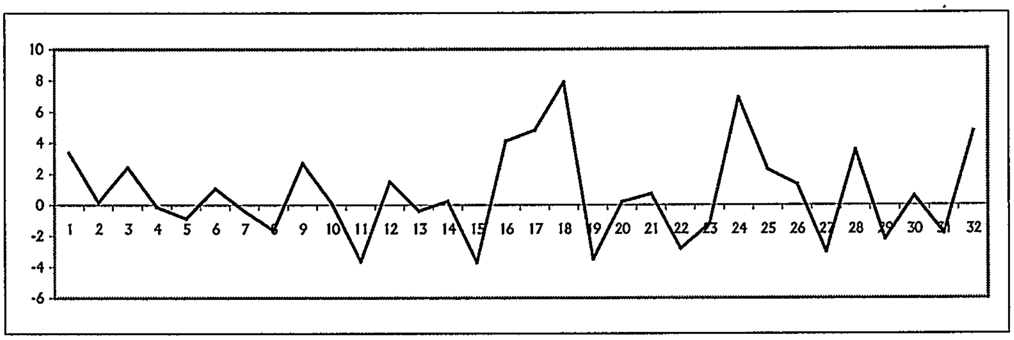
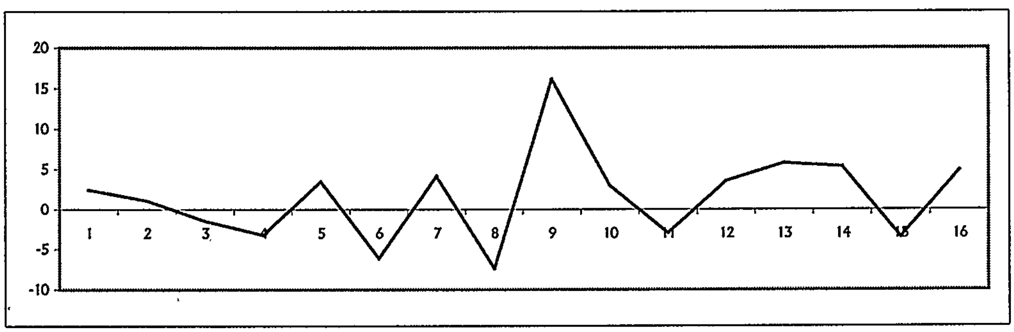


Figure 6: Multi-resolution Decomposition of Bank of Canada Rate (Canada) (continued)

D4



D5



A5

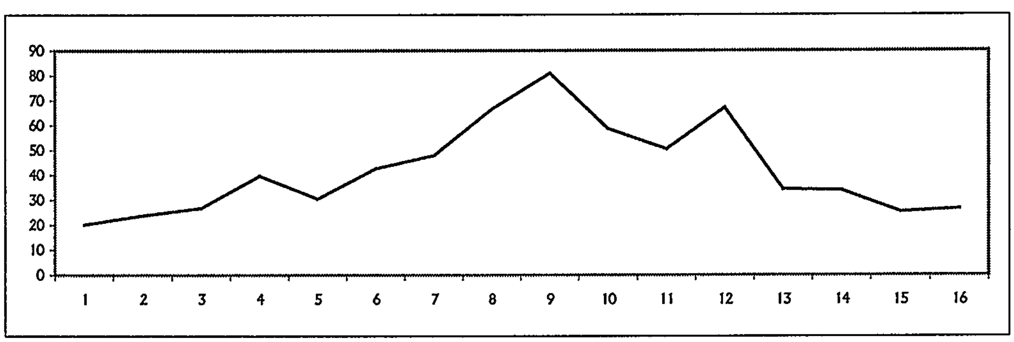
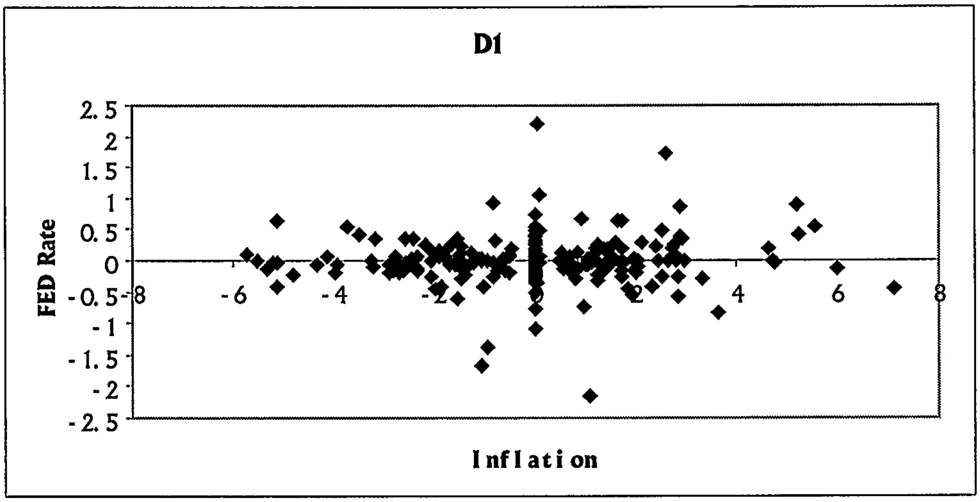
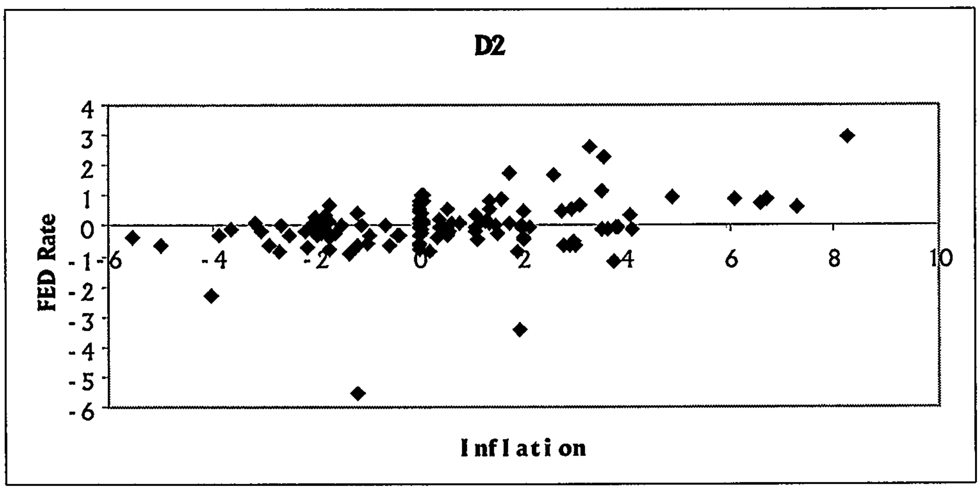


Figure 8: Plots of CPI Inflation on FED Rate using Individual Crystals





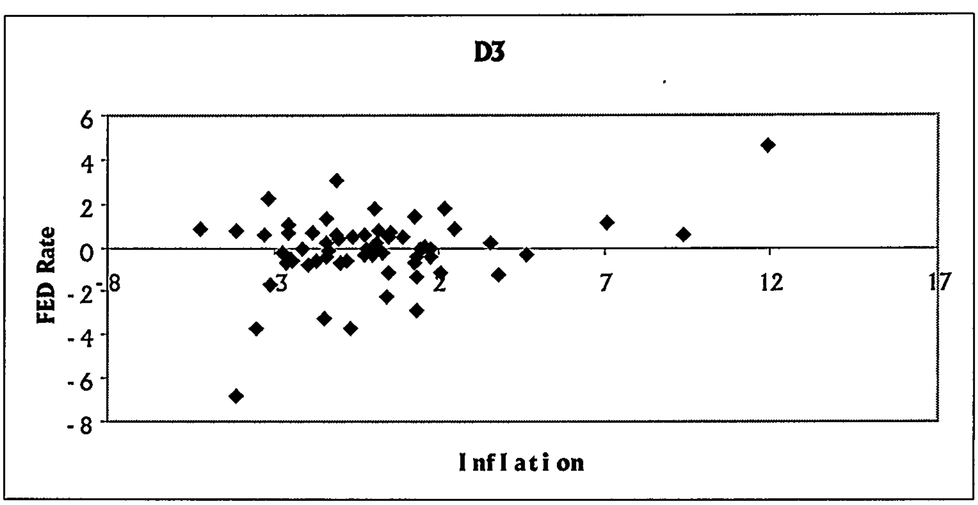
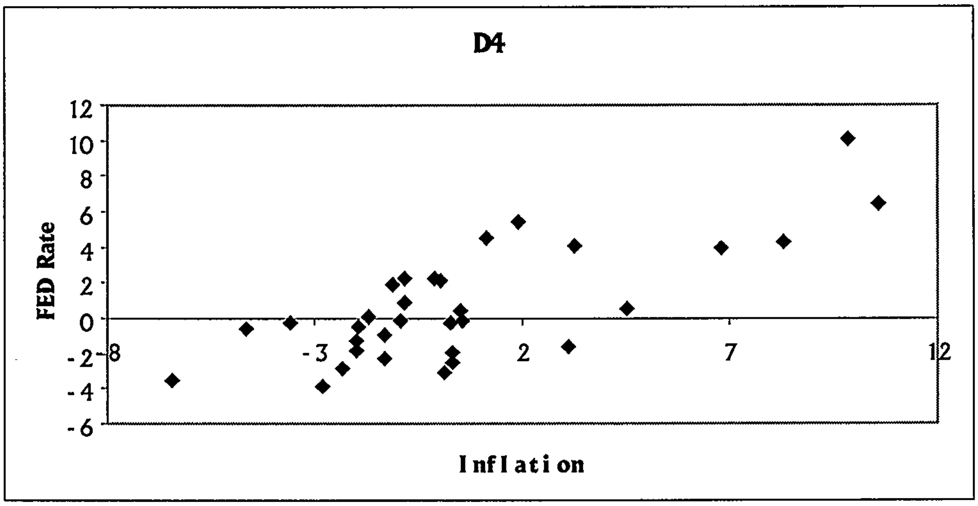
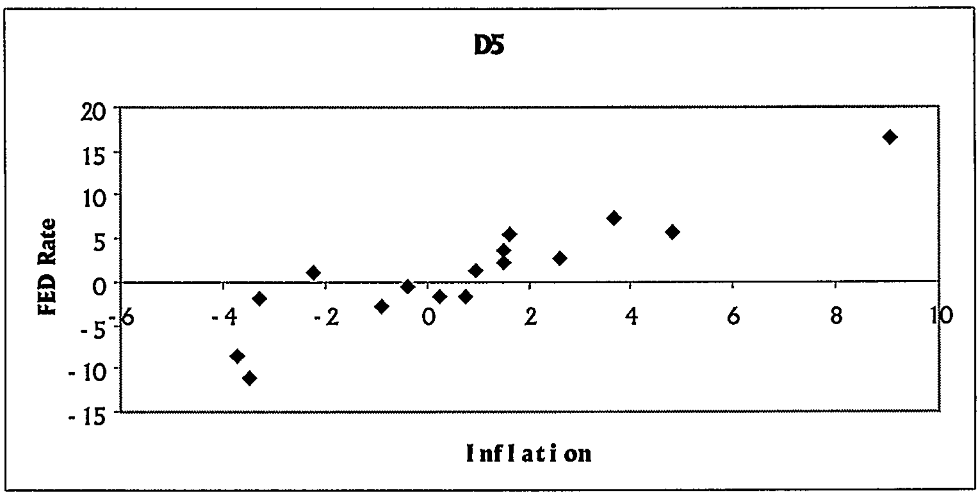


Figure 8: Plots of CPI Inflation on FED Rate using Individual Crystals
(continued)





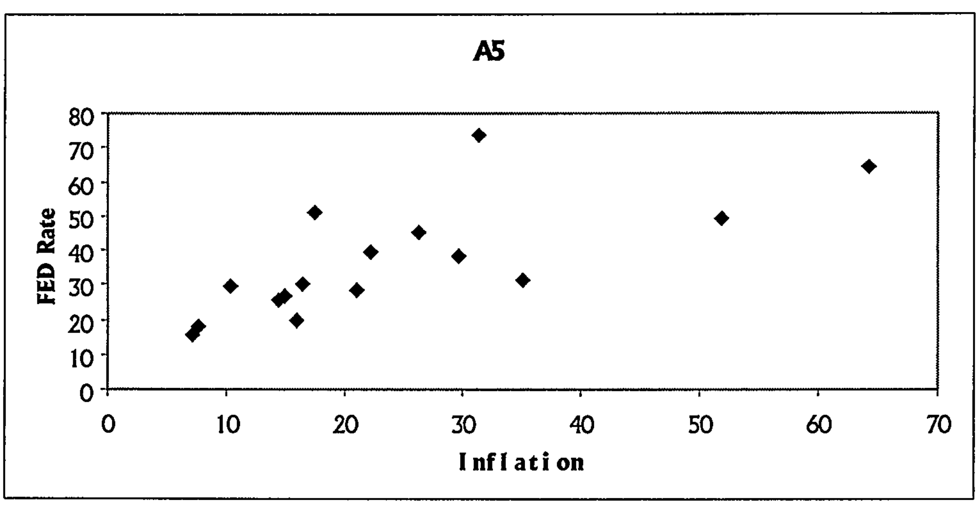


Table Appendix

Table 6.1: Summary of DWT Coefficients for

CPI Inflation Rate (U.S.)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
CPI Inflation	512	4.2669	3.6054	3.7052	-6.5634	21.7195
D1	256	-0.1750	0.0051	2.2860	-15.3580	7.1368
D2	128	0.3985	0.0536	2.4371	-5.5442	8.2428
D3	64	0.0001	-0.2243	2.9368	-4.1423	11.9372
D4	32	0.5550	-0.0309	3.9053	-6.4306	10.5747
D5	16	0.7957	0.8500	3.3385	-3.7349	9.0715
A5	16	24.1371	19.2670	15.6989	7.1576	64.3164

Table 6.2: Summary of DWT Coefficients for

Federal Funds Rate (U.S.)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
FED Rate	512	6.4700	5.5600	3.2197	1.1700	19.1000
D1	256	-0.0060	-0.0141	0.3789	-2.1567	2.2274
D2	128	-0.0054	-0.0150	0.8910	-5.5350	2.9450
D3	64	-0.0988	-0.0106	1.6399	-6.8377	4.6457
D4	32	0.7003	-0.1263	3.1895	-3.8225	10.0350
D5	16	1.1585	1.3303	6.3640	-11.0026	16.5074
A5	16	36.6001	30.7123	16.3904	15.5440	73.3517

Table 6.3: Summary of DWT Coefficients for

90 Days Treasure Bill Rate (U.S.)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
90 Days	512	5.8743	5.3200	2.6317	1.6500	16.3000
D1	256	0.0109	0.0000	0.3067	-1.2445	1.8880
D2	128	-0.0294	-0.0575	0.6819	-3.6650	3.2650
D3	64	-0.0924	-0.0442	1.3488	-6.4594	4.5785
D4	32	0.4639	-0.0175	2.2183	-3.5625	4.3150
D5	16	1.3282	1.2869	5.0138	-8.3386	13.3077
A5	16	33.2300	29.6269	13.6043	15.9948	64.2760

Table 6.4: Summary of DWT Coefficients for

1 Years Rate (U.S.)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
1 Yrs Rate	512	6.4906	5.8450	2.7768	2.1600	16.7200
D1	256	0.0093	0.0000	0.3174	-1.3152	1.7819
D2	128	-0.0111	-0.0425	0.6581	-2.5100	3.3200
D3	64	-0.0349	0.0195	1.4169	-5.8690	4.8932
D4	32	0.4907	-0.1913	1.9677	-2.7850	3.8675
D5	16	1.3167	1.5539	4.9355	-7.2832	12.2241
A5	16	36.7164	34.0667	14.6331	19.6275	70.6205

Table 6.5: Summary of DWT Coefficients for

3 Years Rate (U.S.)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
3 Yrs Rate	512	6.8900	6.4100	2.6229	3.1400	16.2200
D1	256	0.0086	0.0141	0.2732	-1.3859	1.4354
D2	128	0.0020	-0.0425	0.5605	-1.6900	2.4750
D3	64	-0.0006	0.0867	1.2382	-4.3452	4.3699
D4	32	0.3294	-0.0738	1.5590	-2.1375	4.0125
D5	16	1.0089	2.0091	4.2679	-6.7918	9.2348
A5	16	38.9759	35.9608	14.1226	21.1743	72.4183

Table 6.6: Summary of DWT Coefficients for

5 Years Rate (U.S.)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
5 Yrs Rate	512	7.0660	6.6700	2.5653	3.4700	15.9300
D1	256	0.0077	0.0141	0.2488	-1.3152	1.1526
D2	128	0.0098	-0.0575	0.5007	-1.3000	2.1800
D3	64	0.0135	-0.0319	1.1415	-3.5461	4.1012
D4	32	0.2706	-0.0675	1.4024	-2.1600	3.7525
D5	16	0.8119	1.4275	3.9594	-6.7051	7.9921
A5	16	39.9714	36.4903	13.9578	21.9345	72.6181

Table 6.7: Summary of DWT Coefficients for

10 Years Rate (U.S.)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
10 Yrs Rate	512	8.2857	8.0800	2.6202	4.79	17.6600
D1	256	0.0077	0.0141	0.2488	-1.3152	1.1526
D2	128	0.0098	-0.0575	0.5007	-1.3000	2.1800
D3	64	0.0135	-0.0319	1.1415	-3.5461	4.1012
D4	32	0.2706	-0.0675	1.4204	-2.1600	3.7525
D5	16	0.8119	1.4275	3.9594	-6.7051	7.9921
A5	16	39.9714	36.4903	13.9578	21.9345	72.6181

Table 7.1: Summary of DWT Coefficients for

CPI Inflation Rate (Canada)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
CPI Inflation	512	4.4075	4.0850	5.0348	-10.2700	31.5500
D1	256	0.2110	0.0116	4.1076	-11.5141	22.3062
D2	128	-0.0501	0.0390	4.1136	-13.8825	13.3139
D3	64	0.0354	0.0566	4.1817	-13.4281	9.0704
D4	32	-0.4618	-1.3978	3.9574	-12.1814	5.7949
D5	16	-0.2062	-1.1956	4.9268	-7.4513	13.5325
A5	16	24.9328	21.1747	17.3817	5.3093	58.3203

Table 7.2: Summary of DWT Coefficients for

Bank of Canada Rate (Canada)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
BOC Rate	512	7.4669	7.0000	3.4406	1.9300	21.0300
D1	256	-0.0160	0.0000	0.2942	-2.9698	1.1667
D2	128	-0.0083	0.0000	0.7863	-3.7700	3.9350
D3	64	0.0191	0.0000	1.5035	-3.8608	5.6286
D4	32	0.7365	0.2225	2.9658	-3.6875	7.9000
D5	16	1.5920	2.7020	5.7150	-7.3698	16.1680
A5	16	42.2389	37.0666	18.1985	20.1932	81.0804

Table 7.3: Summary of DWT Coefficients for

91 Days Bond Rate (Canada)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
91 Days	512	7.0689	6.3500	3.5031	1.9100	20.8500
D1	256	-0.0149	-0.0071	0.3524	-2.4324	1.1950
D2	128	-0.0028	-0.0400	0.7720	-3.4900	3.6050
D3	64	0.0230	0.0027	1.4411	-3.6628	5.5296
D4	32	0.8191	0.3588	2.9510	-3.705	8.1625
D5	16	1.4828	2.6800	5.7231	-7.7411	16.092
A5	16	39.9876	35.0705	18.6455	18.9186	80.1806

Table 7.4: Summary of DWT Coefficients for

1-3 Years Rate (Canada)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
1-3 Yrs Rate	512	7.4641	6.8600	3.0025	2.8800	18.9400
D1	256	0.0077	0.0071	0.3625	-1.2657	1.5274
D2	128	-0.0299	-0.0750	0.7220	-2.5900	3.7400
D3	64	-0.0124	-0.0230	1.3832	-4.7588	5.3280
D4	32	0.6023	0.2988	1.8672	-2.7950	6.0075
D5	16	1.1421	1.2843	4.5122	-7.8471	11.6036
A5	16	42.2235	38.3332	16.2074	22.0494	77.9126

Table 7.5: Summary of DWT Coefficients for

3-5 Years Rate (Canada)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
3-5 Yrs Rate	512	7.7380	7.2750	2.7835	3.6800	18.7700
D1	256	0.0186	0.0141	0.3237	-1.0748	1.5132
D2	128	-0.0161	-0.0625	0.6453	-2.2250	3.4650
D3	64	-0.0395	-0.0654	1.2305	-3.9174	4.8826
D4	32	0.5143	0.4075	1.5877	-1.9975	5.5950
D5	16	0.8248	0.6594	3.9627	-7.3539	10.1488
A5	16	43.7729	40.8275	15.1678	25.3604	77.9179

Table 7.6: Summary of DWT Coefficients for

5-10 Years Rate (Canada)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
5-10 Yrs Rate	512	7.9689	7.6150	2.6518	4.1600	17.94
D1	256	0.0172	0.0036	0.2838	-0.8697	1.3152
D2	128	-0.0149	-0.0600	0.5558	-2.0200	2.6500
D3	64	-0.0195	-0.0301	1.1124	-3.5709	4.3876
D4	32	0.4013	0.2813	1.4028	-1.6450	4.8500
D5	16	0.6728	1.0139	3.6129	-7.0728	8.4977
A5	16	45.0790	43.7408	14.5744	26.9531	77.8153

Table 7.7: Summary of DWT Coefficients for

10+ Years Rate (Canada)

Variables	Observations	Mean	Median	SD	Minimum	Maximum
10+ Yrs Rate	512	8.2857	8.0800	2.6202	4.7900	17.6600
D1	256	0.0147	0.0000	0.2446	-1.1243	1.1738
D2	128	-0.0058	-0.0250	0.4769	-1.9250	2.3650
D3	64	-0.0201	-0.0283	0.9459	-3.1608	3.6593
D4	32	0.2780	0.1575	1.2239	-1.4875	4.0300
D5	16	0.3593	0.5542	3.4853	-7.1028	6.8748
A5	16	46.8708	46.7725	14.5473	28.9684	79.1765

Table 8.1: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
Inflation Rate	512	0.4701	0.5233	0.5343	0.5274
(π)		(0.0721)	(0.0600)	(0.0663)	(0.0728)
D1	256	-0.0151	-0.0419	-0.0315	-0.0113
		(0.0872)	(0.0480)	(0.0520)	(0.0458)
D2	128	0.0996	-0.3268	-0.2747	-0.0215
		(0.0678)	(0.2796)	(0.2394)	(0.0819)
D3	64	0.1681	-0.1706	-0.1112	0.0375
		(0.0794)	(0.1744)	(0.1581)	(0.1241)
D4	32	0.1624	-0.1935	-0.9040	-0.0543
		(0.1421)	(0.2737)	(0.6107)	(0.1304)
D5	16	0.3228	-0.1866	-0.1564	-0.0241
		(0.2035)	(0.1718)	(0.1847)	(0.2439)
A5	16	0.3592	0.8119	0.7121	-0.0120
		(0.2958)	(0.1490)	(0.6030)	(0.5416)

Table 8.2: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
FED Rate	512	0.7105	0.8612	0.8520	0.8119
(FED)		(0.1000)	(0.0592)	(0.0624)	(0.0889)
D1	256	0.1243	0.0480	-0.0742	-0.1148
		(0.0894)	(0.0700)	(0.1000)	(0.1360)
D2	128	0.2051	0.0920	-0.0191	-0.0504
		(0.0656)	(0.0458)	(0.0943)	(0.0872)
D3	64	0.0727	0.2840	0.1100	0.0380
		(0.1175)	(0.1175)	(0.1063)	(0.0877)
D4	32	-0.1545	-1.3060	-0.9895	-0.5886
		(0.0316)	(0.6710)	(0.3843)	(0.1729)
D5	16	0.3227	0.0999	-0.0727	0.0281
		(0.3703)	(0.2168)	(0.3928)	(0.2951)
A5	16	0.0907	0.4836	0.1964	0.3349
		(0.3795)	(0.1497)	(0.4345)	(0.3881)

Table 8.3: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
90-Days Rate	512	0.7420	0.8522	0.8662	0.8051
(TB90)		(0.0927)	(0.0480)	(0.0616)	(0.0872)
D1	256	-0.0546	-0.0815	-0.2198	-0.1830
		(0.0877)	(0.1044)	(0.1446)	(0.0959)
D2	128	0.2338	0.0953	-0.0264	-0.0312
		(0.0539)	(0.0632)	(0.0866)	(0.0686)
D3	64	0.1612	0.2819	0.1235	0.0771
		(0.0954)	(0.0781)	(0.0854)	(0.0933)
D4	32	-0.2329	-0.3873	-0.6098	-0.4867
		(0.1020)	(0.0700)	(0.1960)	(0.1987)
D5	16	0.3254	0.0661	-0.1310	-0.0401
		(0.4343)	(0.2760)	(0.4337)	(0.2997)
A5	16	0.1584	0.4933	0.1737	0.4217
		(0.3583)	(0.1386)	(0.4935)	(0.3619)

Table 8.4: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
1 Yrs Rate	512	0.7638	0.8642	0.8950	0.8260
(GS1)		(0.0894)	(0.0436)	(0.0616)	(0.0877)
D1	256	-0.0802	-0.0270	-0.1808	-0.1846
		(0.0843)	(0.0872)	(0.1319)	(0.1183)
D2	128	0.1678	0.0958	-0.0120	-0.0064
		(0.0616)	(0.0424)	(0.0480)	(0.0500)
D3	64	0.1094	0.2256	0.0622	0.0195
		(0.1131)	(0.0794)	(0.1058)	(0.1265)
D4	32	-0.2715	-0.3168	-0.5733	-0.5277
		(0.0557)	(0.0700)	(0.1703)	(0.2623)
D5	16	0.3240	0.1494	-0.0599	0.0618
		(0.5685)	(0.2780)	(0.5586)	(0.2632)
A5	16	0.2165	0.5382	0.2295	0.5128
		(0.3704)	(0.1375)	(0.5364)	(0.3682)

Table 8.5: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
3 Yrs Rate	512	0.8274	0.8912	0.9187	0.8304
(GS3)		(0.0671)	(0.0458)	(0.0671)	(0.1068)
D1	256	-0.1673	0.0218	-0.0870	-0.1399
		(0.1670)	(0.0678)	(0.0990)	(0.1334)
D2	128	0.1331	0.1206	0.0124	0.0155
		(0.0755)	(0.0283)	(0.0283)	(0.0332)
D3	64	0.1052	0.1869	0.0440	0.0035
		(0.1179)	(0.0678)	(0.0954)	(0.1466)
D4	32	-0.1679	-0.1525	-0.3318	-0.3452
		(0.1428)	(0.1039)	(0.1817)	(0.2711)
D5	16	0.2267	0.2184	-0.0317	0.1096
		(0.9464)	(0.2987)	(0.7078)	(0.2983)
A5	16	0.5112	0.5534	0.0646	0.7247
		(0.3228)	(0.2170)	(0.8016)	(0.3289)

Table 8.6: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
5 Yrs Rate	512	0.8566	0.9141	0.9311	0.8322
(GS5)		(0.0608)	(0.0500)	(0.0700)	(0.1204)
D1	256	-0.1328	0.0375	-0.0585	-0.1128
		(0.1568)	(0.0600)	(0.0794)	(0.1229)
D2	128	0.1256	0.1297	0.0279	0.0315
		(0.0768)	(0.0283)	(0.0283)	(0.0300)
D3	64	0.1154	0.1728	0.0441	0.0121
		(0.1077)	(0.0748)	(0.0794)	(0.1292)
D4	32	-0.0771	-0.0490	-0.1624	-0.1999
		(0.1924)	(0.1086)	(0.1682)	(0.2872)
D5	16	0.1915	0.2104	-0.0435	0.1141
		(1.0551)	(0.3391)	(0.7776)	(0.3103)
A5	16	0.6244	0.5756	-0.1754	0.8170
		(0.3153)	(0.2674)	(1.0327)	(0.3167)

Table 8.7: Estimate the Fractional Integration Order

Wavelet Estimate for d (U.S.)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
10 Yrs Rate	512	0.9005	0.9458	0.9508	0.8182
(GS10)		(0.0566)	(0.0557)	(0.0762)	(0.1459)
D1	256	-0.1124	0.0787	0.0015	-0.0471
		(0.1709)	(0.0700)	(0.0843)	(0.1217)
D2	128	0.1440	0.1507	0.0480	0.0479
		(0.0787)	(0.0412)	(0.0424)	(0.0469)
D3	64	0.1614	0.1893	0.0938	0.0784
		(0.1039)	(0.0768)	(0.0557)	(0.1086)
D4	32	-0.0842	0.0767	-0.0100	-0.1000
		(0.3437)	(0.1643)	(0.2335)	(0.3626)
D5	16	0.2375	0.1577	-0.1069	0.0658
		(0.8663)	(0.4214)	(0.8857)	(0.3400)
A5	16	0.7542	0.6057	-0.2649	0.9402
		(0.3116)	(0.3357)	(1.2017)	(0.3061)

Table 9.3: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
91-Days Rate	512	0.8269	0.8939	0.8886	0.7415
(TB91)		(0.0640)	(0.0656)	(0.0866)	(0.1735)
D1	256	0.0406	0.0823	-0.0068	-0.0195
		(0.0436)	(0.0656)	(0.0686)	(0.0490)
D2	128	0.0587	0.0624	-0.0420	-0.0653
		(0.1025)	(0.0632)	(0.0686)	(0.0872)
D3	64	0.0175	0.0095	-0.1139	-0.0385
		(0.1315)	(0.2112)	(0.2687)	(0.1439)
D4	32	-0.2601	-0.3919	-0.4826	-0.3381
		(0.1616)	(0.1876)	(0.2330)	(0.2191)
D5	16	0.1035	-0.1073	-0.1301	0.0820
		(0.5311)	(0.2919)	(0.3910)	(0.6496)
A5	16	0.5081	0.5373	-0.3976	0.8039
		(0.2958)	(0.3228)	(0.9275)	(0.2860)

Table 9.4: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
1-3 Yrs Rate	512	0.8150	0.8520	0.8448	0.6715
(BY1)		(0.0529)	(0.0574)	(0.0825)	(0.1700)
D1	256	-0.0241	-0.0208	-0.0712	-0.1044
		(0.0632)	(0.1082)	(0.1221)	(0.1487)
D2	128	-0.1455	-0.0694	-0.1934	-0.2063
		(0.0608)	(0.0490)	(0.0624)	(0.0721)
D3	64	0.0028	0.1106	-0.0681	-0.0406
		(0.1020)	(0.1652)	(0.2474)	(0.1556)
D4	32	-0.1537	-0.0701	-0.1708	-0.1501
		(0.3274)	(0.4029)	(0.3755)	(0.3936)
D5	16	0.2123	0.0746	-0.0995	0.0827
		(0.6534)	(0.3720)	(0.7531)	(0.4204)
A5	16	0.6160	0.5700	-0.1590	0.9010
		(0.2890)	(0.3375)	(0.8843)	(0.3212)

Table 9.5: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
3-5 Yrs Rate	512	0.8305	0.8624	0.8565	0.6793
(BY3)		(0.0510)	(0.0566)	(0.0831)	(0.1706)
D1	256	-0.1090	-0.0680	-0.0774	-0.0791
		(0.1233)	(0.0812)	(0.0812)	(0.0872)
D2	128	-0.0812	-0.0862	-0.1839	-0.1697
		(0.0964)	(0.0387)	(0.0583)	(0.0728)
D3	64	0.0207	0.1159	-0.0422	-0.1509
		(0.1643)	(0.1407)	(0.2037)	(0.2926)
D4	32	-0.0440	0.0006	-0.1324	-0.0559
		(0.3114)	(0.2764)	(0.4621)	(0.3212)
D5	16	0.2744	0.0827	-0.0701	0.0575
		(0.6704)	(0.3783)	(0.7124)	(0.3926)
A5	16	0.6531	0.5706	-0.0873	0.9279
		(0.2857)	(0.3350)	(0.8832)	(0.3291)

Table 9.6: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
5-10 Yrs Rate	512	0.8559	0.8872	0.8750	0.6861
(BY5)		(0.0510)	(0.0574)	(0.0837)	(0.1789)
D1	256	-0.0606	-0.0366	-0.0514	-0.0658
		(0.0943)	(0.0938)	(0.0954)	(0.1091)
D2	128	-0.0304	0.0109	-0.0720	-0.0724
		(0.1015)	(0.0616)	(0.0762)	(0.0849)
D3	64	0.0489	0.1300	-0.0270	-0.0546
		(0.1459)	(0.1691)	(0.2317)	(0.2152)
D4	32	0.0255	0.0811	-0.0323	0.0105
		(0.3557)	(0.2506)	(0.4399)	(0.3114)
D5	16	0.2918	0.1045	-0.0607	0.0689
		(0.6572)	(0.3971)	(0.7218)	(0.3626)
A5	16	0.7128	0.5909	0.0670	0.9696
		(0.2901)	(0.3432)	(0.8776)	(0.3282)

Table 9.7: Estimate the Fractional Integration Order

Wavelet Estimate for d (Canada)

Variables	Observations	Haar	Daubechies-4	Daubechies-12	Daubechies-20
10+ Yrs Rate	512	0.8913	0.9207	0.8984	0.7061
(BY10)		(0.0539)	(0.0600)	(0.0906)	(0.1836)
D1	256	-0.1515	-0.0015	-0.0114	-0.0225
		(0.1780)	(0.0656)	(0.0608)	(0.0800)
D2	128	-0.0672	-0.0050	-0.0702	-0.0827
		(0.1616)	(0.0707)	(0.0794)	(0.1153)
D3	64	0.0577	0.1310	0.0190	-0.0451
		(0.1685)	(0.1652)	(0.1965)	(0.2258)
D4	32	0.1239	0.1604	0.0555	0.0901
		(0.3059)	(0.2408)	(0.3647)	(0.3370)
D5	16	0.2970	0.0778	-0.0952	0.0649
		(0.5652)	(0.4821)	(0.8181)	(0.3500)
A5	16	0.7627	0.6285	0.1750	1.0220
		(0.3050)	(0.3617)	(0.8802)	(0.3980)

Table 10.2: Regressions of 90 Days Bond Rate on Inflation Rate

for Individual Crystals (U.S.)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0145 (0.0190)	0.0207 (0.0083)	0.0238	2.487
D2	128	-0.0590 (0.0591)	0.0743 (0.0240)	0.0704	3.090
D3	64	-0.0924 (0.1522)	0.2044 (0.0522)	0.1981	3.914
D4	32	0.2325 (0.2736)	0.4168 (0.0704)	0.5385	5.917
D5	16	0.2456 (0.5656)	1.3606 (0.1699)	0.8208	8.009
A5	16	18.9527 (4.8280)	0.5915 (0.1693)	0.4659	3.495

Table 10.3: Regressions of 1 Years Bond Rate on Inflation Rate

for Individual Crystals (U.S.)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0122 (0.0198)	0.0168 (0.0086)	0.0147	1.946
D2	128	-0.0398 (0.0570)	0.0722 (0.0232)	0.0715	3.116
D3	64	-0.0349 (0.1560)	0.2346 (0.0535)	0.2365	4.382
D4	32	0.3182 (0.2812)	0.3107 (0.0724)	0.3804	4.291
D5	16	0.2831 (0.6281)	1.2989 (0.1887)	0.7719	6.884
A5	16	21.7682 (5.3108)	0.6193 (0.1862)	0.4414	3.326

Table 10.4: Regressions of 3 Years Bond Rate on Inflation Rate

for Individual Crystals (U.S.)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0108 (0.0171)	0.0126 (0.0075)	0.0111	1.691
D2	128	-0.0213 (0.0487)	0.0587 (0.0198)	0.0651	2.961
D3	64	-0.0006 (0.1291)	0.2369 (0.0443)	0.3157	5.349
D4	32	0.2431 (0.2607)	0.1555 (0.0671)	0.1517	2.317
D5	16	0.1706 (0.6442)	1.0536 (0.1935)	0.6792	5.445
A5	16	26.2767 (5.5629)	0.5261 (0.1950)	0.3421	2.698

Table 10.5: Regressions of 5 Years Bond Rate on Inflation Rate

for Individual Crystals (U.S.)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0096 (0.0156)	0.0108 (0.0068)	0.0098	1.582
D2	128	-0.0104 (0.0436)	0.0506 (0.0177)	0.0607	2.855
D3	64	0.0135 (0.1165)	0.2280 (0.0340)	0.3441	5.703
D4	32	0.2092 (0.2456)	0.1107 (0.0632)	0.0927	1.751
D5	16	0.0700 (0.6521)	0.9324 (0.1959)	0.6180	4.760
A5	16	27.9612 (5.6172)	0.4976 (0.1969)	0.3132	2.527

Table 10.6: Regressions of 10 Years Bond Rate on Inflation Rate

for Individual Crystals (U.S.)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0086 (0.0133)	0.0075 (0.0058)	0.0066	1.303
D2	128	-0.0030 (0.0367)	0.0437 (0.0149)	0.0638	2.931
D3	64	-0.0010 (0.0973)	0.2096 (0.0334)	0.3887	6.279
D4	32	0.1498 (0.2297)	0.0568 (0.0591)	0.0299	0.961
D5	16	-0.0278 (0.6312)	0.7949 (0.1896)	0.5567	4.193
A5	16	29.5419 (5.7218)	0.4712 (0.2006)	0.2827	2.349

Table 11.2: Regressions of 91 Days Bond Yield on Inflation Rate

for Individual Crystals (Canada)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	-0.0142 (0.0221)	-0.0032 (0.0054)	0.0014	-0.590
D2	128	-0.0025 (0.0685)	0.0069 (0.0167)	0.0013	0.410
D3	64	0.0213 (0.1798)	0.0480 (0.0433)	0.0194	1.107
D4	32	0.8071 (0.5337)	-0.0261 (0.1361)	0.0012	-0.192
D5	16	1.6445 (1.0937)	0.7841 (0.2291)	0.4557	3.423
A5	16	21.8094 (6.3259)	0.7291 (0.2103)	0.4620	3.467

Table 11.3: Regressions of 1-3 Years Bond Yield on Inflation Rate

for Individual Crystals (Canada)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0080 (0.0227)	-0.0014 (0.0055)	0.0003	-0.261
D2	128	-0.0296 (0.0640)	0.0066 (0.0156)	0.0014	0.424
D3	64	-0.0154 (0.1685)	0.0844 (0.0406)	0.0651	2.078
D4	32	0.5957 (0.3377)	-0.0142 (0.0861)	0.0009	-0.165
D5	16	1.2647 (0.8889)	0.5946 (0.1862)	0.4215	3.194
A5	16	26.7426 (5.5927)	0.6209 (0.1859)	0.4434	3.340

Table 11.4: Regressions of 3-5 Years Bond Yield on Inflation Rate

for Individual Crystals (Canada)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0187 (0.0203)	-0.0007 (0.0049)	0.0001	-0.146
D2	128	-0.0159 (0.0573)	0.0034 (0.0140)	0.0005	0.247
D3	64	-0.0423 (0.1494)	0.0786 (0.0360)	0.0713	2.182
D4	32	0.5060 (0.2870)	-0.0180 (0.0732)	0.0020	-0.246
D5	16	0.9299 (0.7937)	0.5099 (0.1662)	0.4020	3.068
A5	16	29.5750 (5.3159)	0.5694 (0.1767)	0.4258	3.222

Table 11.5: Regressions of 5-10 Years Bond Yield on Inflation Rate

for Individual Crystals (Canada)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0174 (0.0178)	-0.0009 (0.0043)	0.0002	-0.204
D2	128	-0.0148 (0.0493)	0.0015 (0.0120)	0.0001	0.129
D3	64	-0.0217 (0.1361)	0.0634 (0.0328)	0.0568	1.932
D4	32	0.3903 (0.2533)	-0.0240 (0.0646)	0.0046	-0.371
D5	16	0.7615 (0.7579)	0.4302 (0.1587)	0.3441	2.710
A5	16	31.9381 (5.2429)	0.5271 (0.1743)	0.3951	3.024

Table 11.6: Regressions of 10+ Years Bond Yield on Inflation Rate

for Individual Crystals (Canada)

Crystal	Observations	Intercept (standard error)	Slope (standard error)	R^2	t-value
D1	256	0.0146 (0.0153)	0.0004 (0.0037)	0.0001	0.120
D2	128	-0.0057 (0.0423)	0.0007 (0.0103)	0.0000	0.068
D3	64	-0.0216 (0.1170)	0.0432 (0.0282)	0.0365	1.532
D4	32	0.2731 (0.2213)	-0.0106 (0.0564)	0.0012	-0.189
D5	16	0.4329 (0.7794)	0.3569 (0.1632)	0.2546	2.187
A5	16	33.6968 (5.2181)	0.5284 (0.1735)	0.3986	3.046