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Modeling and Pricing Variance and Volatility Swaps for Stochastic Volatility Models with Jumps

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Modeling and Pricing Variance and Volatility Swaps for Stochastic Volatility Models with Jumps" submitted by Zhao, Lu in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE.

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Abstract

Variance and volatility swaps have been widely studied within the framework of continuous stochastic volatility models. This thesis aims to explore the valuation of variance and volatility swaps for stochastic volatility models with jumps. We examine a variety of continuous stochastic volatility models and extend these by adding jump-diffusion terms. Analytical closed-form formulae are provided to price variance swaps for both continuous stochastic volatility models and extended jump-diffusion models. Explicit valuation via a convexity adjustment is provided for volatility swaps in all continuous models and some stochastic volatility models with jumps. We also compare results from continuous models with those from extended jump-diffusion models. In the end, we show several numerical examples to analyze the impact of jumps as well as the convexity adjustment on swaps' fair values.

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Table of Contents

List of Tables

List of Figures

Introduction

An equity-index's volatility is the simplest measure of its uncertainty. As the word "volatility" means several related but distinct concepts, we shall clarify volatility with an adjective "realized", "implied", or "local". The realized volatility of an equity-index is the annualized standard deviation of its return (usually quoted daily) over the period of interest. The implied volatility of an equity-index is implied by the current price of a particular vanilla option. As a volatility parameter in the Black-Scholes formula, the implied volatility is the value that makes the model value equal to the option price when entered. The local volatility of an equity-index at some future market level and time is the volatility the index must have at that market level and time so that current option prices are fair. Local volatilities can be extracted from the set of all implied volatilities at a given time, and locked in by trading portfolios of currently available options [14]. While the last two concepts are widely considered as people talk about volatility skews and smiles, it is the realized volatility about which we are concerned in this thesis.

The history of exchange-traded, volatility-based derivative products can be traced back to the year of 1996 when volatility futures were launched by OM (the Swedish Exchange) [40]. On January 19, 1998, the Deutsche Terminborse (DTB) became the first exchange to trade volatility futures based on an underlying equity index of implied volatility, namely, the VOLAX future. The VOLAX is based on the implied volatility of DAX index options represented by the VDAX, a set of eight volatility indices introduced by the DTB in 1997 [17]. So far, the development of exchange-traded derivative products with volatility is disappointing. It is possibly due to the fact that volatility can be traded by combining together static positions in options on price and dynamic trading in the underlying [8]. In Carr and Madan's paper (2002), three methods towards trading

realized volatility were reviewed. The first method involves taking static positions in options, with a classic example of a long position in a straddle. The second method is by delta-hedging an option position. The leading determinant of the gain or loss from this method is the difference between realized volatility and the anticipated volatility used in pricing and hedging the option, provided that the price risk is hedged away. The last method for trading realized volatility is to buy and sell volatility contracts with an explicit payoff function of volatility. The simplest examples for such kind of contracts are variance and volatility swaps, developed not from exchange trading, but from the over-the-counter (OTC) market.

Variance and volatility swaps are not the first volatility-based derivative products, but they offer an easier way for traders to take positions in pure volatility. Despite its name, variance and volatility swaps are in fact forward contracts with payoffs depending on the realized volatility of, for example, an equity index, such as the NASDAQ. Investors seem to be interested in variance and volatility swaps in 1998, when the swap market began to grow as the Long Term Capital Management (LTCM) collapsed, accompanied by a dramatic increase in volatility. As with most new products, variance and volatility swaps started almost as a one-sided market, with clients consisting mainly of hedge funds selling volatility to dealers [27]. Now trading variance and volatility swaps becomes popular. A lot of work has been devoted to pricing and hedging variance and volatility swaps. For example, Demeterfi, Derman, Kamal and Zou (1999) [13] showed that variance swaps could be theoretically replicated by a portfolio of standard options with suitably chosen strikes, under the assumption that stock prices move without jumps, and that volatility swaps could be replicated by dynamically trading variance swaps. They also derived an analytical formula for theoretical fair value in the presence of realistic volatility skews. Windcliff, Forsyth and Vetzal (2006) [43] dicussed pricing volatility products by numerical

PIDE methods. They also studied the hedging of volatility swaps using delta and delta-gamma strategies. The more recent papers working on variance and volatility swaps prefer to take stochastic volatility models (SVMs) as a starting point, since SVMs are suitable to fit skews and smiles, and at the same time provide sensible Greeks [20]. (Most derivatives markets exhibit persistent patterns of volatilities. Volatility smiles and skews are the general phenomena of volatilities varying by strike. Before the idea of stochastic volatility models arose, the volatility had been considered in the way such as deterministic functions of the time or the asset's price, but such a way failed to stand the persistence of volatility smiles and skews. The SVMs are better also in the sense that SVMs are general enough to include DVMs as special cases.)

In the article written by Javaheri, Wilmott and Haug (2002) [24], the valuation and hedging of volatility swaps was discussed within the framework of a GARCH(1,1) stochastic volatility model. The first two moments of the realized variance were determined by the partial differential equation approach, and the expected realized volatility was approximated via a convexity adjustment, which was proposed by Brockhaus and Long (2000) [6] via Taylor's expansion to the second order for function $\sqrt{x}$. The working paper by Théoret, Zabré and Rostan (2002) [40] presented an analytical solution for pricing volatility swaps based on a GARCH(1,1) stochastic volatility process, which was developed by Javaheri, Wilmott and Haug (2002) [24]. They applied the result to price a swap on the volatility of the S&P60 Canada index.

In the paper written by Swishchuk (2004) [38], the change of time method was proposed to study variance and volatility swaps with underlying asset and variance following the Heston (1993) [19] model. Covariance and correlation swaps were also priced in this paper.

The paper by Howison, Rafailidis and Rasmussen (2004) [20] provided closed-form

formulae under a risk-neutral probability measure for volatility-average and variance swaps based on a variety of diffusion and jump-diffusion models for volatility. It also described a general partial differential equation framework for derivatives that had an extra dependence on an average of the volatility, and approximate solutions were given for volatility products written on assets for which the volatility process varied on a timescale that is fast compared with the life of contracts.

The recent paper by Benth, Groth and Kufakunesu (2007) [3] discussed the valuation of variance and volatility swaps for a non-gaussian Ornstein-Uhlenbeck stochastic volatility model, and derived a key formula for the realized variance which was able to describe the dynamics of swaps' prices in Laplace transforms. They also gave an example to compare the fast Fourier transform with the Brockhaus and Long approximation.

Thus far, it seems that many papers working on variance and volatility swaps have considered the valuation and hedging within the framework of SVMs without jumps, but little effort has been devoted to study the case of SVMs with jumps. (There is actually one paper, as mentioned above, by Howison, Rafailidis and Rasmussen (2004) [20], who did think of the stochastic volatility process with jump-diffusion, but they considered pricing variance and volatility-average swaps only, where the volatility, not the variance, follows jump-diffusion stochastic process.) There is growing evidence to support SVMs with jumps rather than SVMs without jumps. One obvious example lies in the energy market. Nobody could ignore the huge spikes of volatilities of energy prices observed in the market. Therefore, it is useful and necessary to model and price variance and volatility swaps for SVMs with jumps, and this is the primary purpose of this thesis.

The rest of the thesis is organized as follows:

Chapter 1: we will introduce the basic concepts of variance and volatility swaps and how they work in practice. We also propose ways to price these swaps under the risk-

neutral probability measure. Some existing results for the valuation of variance swaps are reviewed.

Chapter 2: we will take a review of deterministic volatility models (DVMs) as well as SVMs without jumps, and discuss seven continuous SVMs: the Orstein-Uhlenbeck (1930) model [41], the Vasicek (1977) model [42], the Hull-White (1987) model [22], Schwartz's (1997) one-factor model [34], Pilipović's (1998) two-factor model [32], the GARCH(1,1) (2002) model [24], and the Heston (1993) model [19], to price variance and volatility swaps. Analytical closed-form formulae for swaps' fair values are given via probabilistic approach.

Chapter 3: we will introduce SVMs with jumps, and apply those extended from Chapter 2 to the valuation of variance and volatility swaps. Closed-form solutions are provided for all SVMs with jumps except the Heston (1993) model [19]. First moments of variance for all models considered are provided, and compared with expectations of variance for the corresponding continuous SVMs obtained in Chapter 2.

Chapter 4: we will show the fair value of variance swaps for all SVMs with jumps considered in Chapter 3. Closed-form formulae are given to compare with variance swaps' fair values obtained in Chapter 2.

Chapter 5: we will show the fair value of volatility swaps for some SVMs with jumps considered in Chapter 3, and compare with volatility swaps' fair values obtained in Chapter 2.

Chapter 6: we will provide several numerical examples to apply results we've reached to price variance and volatility swaps and examine the impact of jumps as well as the convexity adjustment on swaps' fair values.

Chapter 7: we will conclude by summarizing results obtained in this thesis and proposing future works on the valuation of variance and volatility swaps.

Chapter 1

Variance and Volatility Swaps

The market for variance and volatility swaps started growing in late 1998 with the LTCM meltdown due to an unprecedented level of implied stock index volatility. Different from stock options, these swaps provide pure exposure to the future level of volatility [13], making them attractive to many investment banks and other financial institutions. Investors can buy and sell these tools to directly speculate on future volatility levels, to trade the spread between realized and implied volatility, or to hedge the implied volatility exposure of other positions. They are now actively quoted for various assets such as stock indices, currencies and commodities. In the remainder of this chapter, we will give an overview of variance and volatility swaps, from definitions to valuation strategies, and from theory to practice.

1.1 Volatility Swaps

Definition 1.1-1. A volatility swap is a forward contract on the underlying asset's annualized volatility. Its payoff at maturity is given by

$$N(\sigma_R - K_{vol}) \tag{1.1}$$

where N is the notional amount of the swap in dollars per annualized volatility point, K_{vol} is the annualized volatility delivery price, σ_R is the realized underlying asset's volatility (quoted in annual terms) over the life of the contract.

The measure of realized volatility to be used is defined at the beginning of the contract. One typical formula for this is

$$\sigma_R = \sqrt{\frac{n}{(n-1)T}\sum_{i=1}^{n}\log^2\frac{S_{t_i}}{S_{t_{i-1}}}} \tag{1.2}$$

where S_{t_i} is the underlying asset's price at time t_i, given $t_0 = 0 < t_1 < t_2 < \ldots < t_n = T$ with $t \in [0, T]$ between current time and maturity; the log return $\log\frac{S_{t_i}}{S_{t_{i-1}}}$ is an approximation of $\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}$, assuming that the change of price between t_{i-1} and t_i is relatively small. We also assume that the mean of the return is of order $\frac{1}{n}$ and is negligible. Moreover, the scaling by $\frac{n}{T}$ ensures these quantities are annualized (daily) if the maturity T is expressed in years (days) [6].

In continuous time, we define realized volatility by

$$\sigma_R = \sqrt{\frac{1}{T}\int_0^T {\sigma_t}^2 dt} \tag{1.3}$$

where σ_t is the underlying asset's volatility at time t.

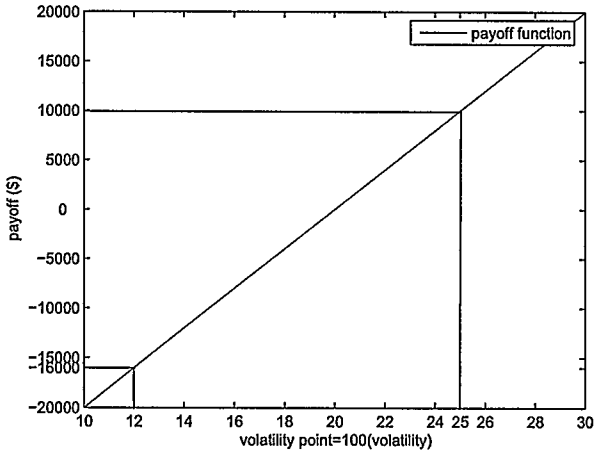
The holder of a volatility swap at maturity receives N dollars for every point by which σ_R has exceeded K_{vol}. For example, if the delivery price K_{vol} is quoted as 20% and the notional amount N is quoted as \$2,000 per volatility point, the holder will receive \$10,000 when the realized underlying asset's volatility σ_R is 25% (σ_R exceeds K_{vol} by five points) and will lose \$16,000 when σ_R is 12% (K_{vol} exceeds σ_R by eight points).

According to the definition, the payoff for volatility swaps is a linear function in realized volatility. To illustrate this, we can show the payoff of the above example in Figure 1.1 on the following page.

1.2 Variance Swaps

Definition 1.2-1. A variance swap, very similar to a volatility swap, is a forward contract on the annualized underlying asset's variance, the square of the realized volatility.

Figure 1.1: Payoff for Volatility Swaps



Its payoff at expiration is given by

$$N(\sigma_R^2 - K_{var}) \tag{1.4}$$

where N is the notional amount of the swap in dollars per annualized volatility point squared, K_{var} is the annualized variance delivery price, σ_R^2 is the realized underlying asset's variance (quoted in annual terms) over the life of the contract.

The typical measure of realized variance is

$$\sigma_R^2 = \frac{n}{(n-1)T} \sum_{i=1}^{n} \log^2 \frac{S_{t_i}}{S_{t_{i-1}}}. \tag{1.5}$$

In continuous time, realized variance is defined by

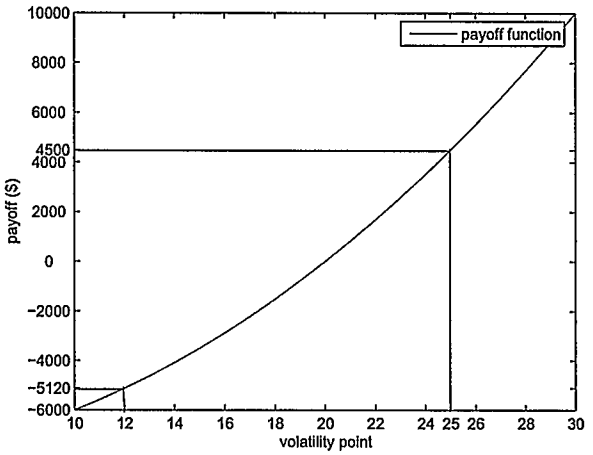
$$\sigma_R^2 = \frac{1}{T} \int_0^T {\sigma_t}^2 dt. \tag{1.6}$$

The holder of a variance swap at expiration receives N dollars for every point by which σ_R^2 has exceeded K_{var}. For example, if the delivery price K_{var} is quoted as $(20\%)^2$ and the notional amount N is quoted as $\$2,000$ per volatility point squared, the holder will receive $\$4,500$ when the realized underlying asset's variance σ_R^2 is $(25\%)^2$ (σ_R^2 exceeds

K_{var} by $2\frac{25}{100}$ points) and will lose $\$5,120$ when σ_R^2 is $(12\%)^2$ (K_{var} exceeds σ_R^2 by $2\frac{56}{100}$ points).

By definition, the payoff of variance swaps is not a linear function in realized volatility. To illustrate this, we can describe the payoff of the above example in the figure below:

Figure 1.2: Payoff for Variance Swaps



1.3 Variance and Volatility Swaps in Practice

Volatility has several characteristics which make trading attractive. First of all, it increases with risk and uncertainty and is likely to increase more after bad news than good news. Secondly, it appears to follow a mean-reverting process: high volatilities will decrease while low ones will increase. Finally, there is a negative correlation between volatility and stock or index level. the volatility will stay high after large downward moves in the market [13].

Variance and volatility swaps allow investors to profit from the risk of an increase or decrease in future volatility, or to hedge against the risk. One example is as follows:

Consider two counter-parties in a volatility swap: a hedge fund, and a dealer who

writes a sponsored option to someone else. The dealer has the obligation to buy back the option when the holder wants to sell. As is known to all, the option price will increase as the volatility level increases. If it happens, the holder of the option will sell the option back to the dealer at a price higher than that when the dealer sold it. In order to protect himself from this loss, the dealer can transfer the risk to a counter-party, the hedge fund in this example, by entering in a volatility swap. Meanwhile, the hedge fund thinks that the volatility will decrease and so he will enter this volatility swap, taking a position to increase his profit with risk.

As the most straightforward instruments for trading volatilities, variance and volatility swaps play and will continue to be an important role in modern financial markets.

1.4 Risk-Neutral Valuation Strategies

Pricing for variance and volatility swaps is no different from any other derivative securities since these swaps are indeed forward contracts. In practice, it is easier to trade volatility swaps than variance swaps due to the reason that the payoff of a volatility swap is a linear function in realized volatility while that of a variance swap is nonlinear (compare Figure 1.1 with Figure 1.2). However, volatility swaps are more difficult to be priced theoretically than variance swaps. It is variance swaps that play a key role in constructing other volatility-based derivative products [13]. In the remainder of this section, we will consider valuation methods for variance and volatility swaps separately.

1.4.1 Valuation of Variance Swaps

The fair value for a variance swap is the delivery price K_{var} that makes the swap of zero value at present. The present value of a variance swap can be expressed as

$$F = E_Q[e^{-rT}N(\sigma_R^2 - K_{var})]$$

$$= e^{-rT}N[E_Q(\sigma_R^2) - K_{var}] \tag{1.7}$$

where r is the risk-free interest rate, T is the time to maturity and E_Q denotes the expectation under the risk-neutral measure Q.

Therefore, $E_Q(\sigma_R^2)$, namely, the mean value of variance, is the only quantity we need in order to calculate the fair value of a variance swap.

1.4.2 Valuation of Volatility Swaps

The fair value for a volatility swap is the delivery price K_{vol} which makes the present value of the swap equal zero. The present value of a volatility swap is given by

$$\begin{aligned} F &= E_Q[e^{-rT}N(\sigma_R - K_{vol})] \\ &= e^{-rT}N[E_Q(\sigma_R) - K_{vol}] \end{aligned} \tag{1.8}$$

where

$$E_Q(\sigma_R) \approx \sqrt{E_Q(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}. \tag{1.9}$$

This can be proved by Taylor's expansion to the second order for function $\sqrt{x}$ [6]. The last term of equation (1.9) is called the convexity adjustment.

If we want to make the approximation more accurate, we can expand $\sqrt{x}$ by Taylor's expansion to the third or fourth order, which requires more calculation.

By Taylor's Formula,

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0) - \frac{1}{8x_0^{\frac{3}{2}}}(x - x_0)^2 + \frac{1}{16x_0^{\frac{5}{2}}}(x - x_0)^3 - \frac{5}{128x_0^{\frac{7}{2}}}(x - x_0)^4. \tag{1.10}$$

Letting $x = \sigma_R^2$ and $x_0 = E_Q(\sigma_R^2)$, we obtain

$$\begin{aligned} \sigma_R \approx\ & \sqrt{E_Q(\sigma_R^2)} + \frac{1}{2\sqrt{E_Q(\sigma_R^2)}}(\sigma_R^2 - E_Q(\sigma_R^2)) - \frac{1}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}(\sigma_R^2 - E_Q(\sigma_R^2))^2 \\ &+ \frac{1}{16(E_Q(\sigma_R^2))^{\frac{5}{2}}}(\sigma_R^2 - E_Q(\sigma_R^2))^3 - \frac{5}{128(E_Q(\sigma_R^2))^{\frac{7}{2}}}(\sigma_R^2 - E_Q(\sigma_R^2))^4. \end{aligned} \tag{1.11}$$

By taking the expectation on both sides, we get

$$\begin{aligned} E_Q(\sigma_R) &\approx \sqrt{E_Q(\sigma_R^2)} - \frac{1}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}} Var(\sigma_R^2) \\ &+ \frac{1}{16(E_Q(\sigma_R^2))^{\frac{5}{2}}} \text{Skewness}(\sigma_R^2)(Var(\sigma_R^2))^{\frac{3}{2}} \\ &- \frac{5}{128(E_Q(\sigma_R^2))^{\frac{7}{2}}} (\text{Kurtosis}(\sigma_R^2) + 3)(Var(\sigma_R^2))^2. \end{aligned} \tag{1.12}$$

Therefore, we need to know central moments (at least of order 2) of σ_R^2 for calculating volatility swaps. The higher orders of central moments we know, the more accurate the approximation will be.

Besides the convexity adjustment approximation discussed above, there is another way to compute $E_Q(\sigma_R)$, by using the integral expression of a square root function $\sqrt{x} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1-e^{-sx}}{s^{\frac{3}{2}}} ds$. Carr and Lee (2005) [9] applied this expression, proof of which could be found in [33], to derive a closed-form formula for volatility swaps. However, this formula is not within the framework of stochastic volatility models (SVMs) we will discuss in this thesis. Even though the integral expression provide an accurate value rather than an approximation, the calculation within the framework of SVMs is very challenging. Therefore, we still prefer the Brockhaus and Long approximation (1.9) to price volatility swaps.

Note that in the remainder of this thesis, we denote E instead of E_Q as the expectation with respect to the risk-neutral measure, since all models we consider are assumed to be in the risk-neutral world.

1.5 Some Existing Results

Much research has been done to study the valuation and hedging of volatility-based derivative products. As to variance and volatility swaps, we are most interested in papers by Demeterfi, Derman, Kamal and Zou (1999) [13], by Howison, Rafailidis and

Rasmussen (2004) [20], by Javaheri, Wilmott and Haug (2002) [24],and by Swishchuk (2004) [38]. We will discuss results from the last two papers later in the next chapter as typical examples of SVMs without jumps, and focus on valuation approaches developed by the first two.

1.5.1 Replication for Variance Swaps

The method to price variance swaps by replicating a variance swap by a hedged portfolio of standard options was proposed by Demeterfi, Derman, Kamal and Zou (1999) [13]. The fair value of the variance swap is equal to the cost of the replicating portfolio. The only assumption for this approach is that stock prices evolve without jumps in the risk-neutral world, which means it is diffusive. Strategies are shown as follows.

Assume that the stock price process is given by

$$\frac{dS_t}{S_t} = \mu(t,\dots,)dt + \sigma(t,\dots,)dZ_t, \tag{1.13}$$

where $\mu(t,\dots,)$ and $\sigma(t,\dots,)$ are any functions of time and other parameter, and Z_t is a standard Wiener process.

By applying Itô's lemma to $\log S_t$, we obtain

$$d(\log S_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dZ_t. \tag{1.14}$$

Subtracting (1.14) from (1.13), we get

$$\frac{dS_t}{S_t} - d(\log S_t) = \frac{1}{2}\sigma^2 dt. \tag{1.15}$$

Therefore, the fair value of the variance swap is given by

$$\begin{aligned} E(\sigma_R^2) &= \frac{1}{T}E(\int_0^T \sigma^2 dt) \\ &= \frac{2}{T}E(\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{T}[rT - (\frac{S_0}{S_*}e^{rT} - 1) - \log\frac{S_*}{S_0} \\
&+ e^{rT}\int_0^{S_*}\frac{1}{K^2}P(K)dK + e^{rT}\int_{S_*}^{\infty}\frac{1}{K^2}C(K)dK], \qquad (1.16)
\end{aligned}
$$

where we have

$$
E(\int_0^T \frac{dS_t}{S_t}) = E_Q[\int_0^T rdt + \int_0^T \sigma(t,\ldots,)dZ_t] = rT, \qquad (1.17)
$$

and

$$
\begin{aligned}
-\log\frac{S_T}{S_0} &= -\log\frac{S_T}{S_*} - \log\frac{S_*}{S_0} \\
&= -\frac{S_T - S_*}{S_*} + \int_0^{S_*}\frac{1}{K^2}P(K)dK + \int_{S_*}^{\infty}\frac{1}{K^2}C(K)dK - \log\frac{S_*}{S_0}, \qquad (1.18)
\end{aligned}
$$

where S_* is an arbitrary parameter to define the boundary between calls and puts, and $P(K)$ and $C(K)$ are the current fair value of puts and call options with strike price K respectively (see [13] for a derivation of (1.18)).

Equation (1.18) represents a portfolio consisting of a short position in $\frac{1}{S_*}$ forward contracts with strike S_*, a long position in $\frac{1}{K^2}$ put options with continuous strike K from 0 to S_* and a long position in $\frac{1}{K^2}$ call options with continuous strike K from S_* to ∞. The last term is a constant, independent of S_T.

Therefore, the fair value for the variance swap can be expressed as a constant amount plus the cost of a portfolio of options, described by

$$
K_{var} = E(\sigma_R^2) = \frac{2}{T}[rT - (\frac{S_0}{S_*}e^{rT} - 1) - \log\frac{S_*}{S_0}] + \frac{2e^{rT}}{T}\Pi, \qquad (1.19)
$$

where the portfolio Π is given by

$$
\Pi = \int_0^{S_*}\frac{1}{K^2}P(K)dK + \int_{S_*}^{\infty}\frac{1}{K^2}C(K)dK. \qquad (1.20)
$$

1.5.2 Diffusion and Jump-diffusion Models for Variance swaps

Howison, Rafailidis and Rasmussen (2004) [20] proposed a variety of diffusion and jump-diffusion models (two types) for volatility to price variance and volatility-average swaps. Closed-form formulae were provided.

TYPE A

The general diffusion and jump-diffusion process for volatility is given by

$$d\sigma_t = (a_1 + a_2\sigma_t)dt + (a_3 + a_4\sigma_t)dW_t + (a_5 + a_6\sigma_t)dN_t \tag{1.21}$$

where N_t is a standard compound Poisson process with constant intensity λ and zero correlation with the Wiener process W_t, and $a_1, \ldots, a_6$ are constant.

Taking expectation on both sides of (1.21), given the initial value σ_0 and the fact that $E(dN_t) = \lambda dt$, we obtain

$$E\sigma_t = \sigma_0 e^{\alpha_1 t} - \frac{\alpha_0}{\alpha_1}(1 - e^{\alpha_1 t}), \tag{1.22}$$

where $\alpha_0 = a_1 + \lambda a_5$ and $\alpha_1 = a_2 + \lambda a_6$.

Therefore, the fair value for volatility-average swaps is given by

$$\begin{aligned} K_{vol-ave} &= E(\frac{1}{T}\int_0^T \sigma_t dt) \\ &= \frac{1}{T}\int_0^T E\sigma_t dt \\ &= \frac{1}{T}\int_0^T [\sigma_0 e^{\alpha_1 t} - \frac{\alpha_0}{\alpha_1}(1 - e^{\alpha_1 t})]dt \\ &= \frac{\alpha_0 + \alpha_1\sigma_0}{T\alpha_1^2}(e^{\alpha_1 T} - 1) - \frac{\alpha_0}{\alpha_1}. \end{aligned} \tag{1.23}$$

From (1.21), we can, by Itô's formula with jumps, easily get

$$\begin{aligned} d\sigma_t^2 &= [2\sigma_t(a_1 + a_2\sigma_t) + (a_3 + a_4\sigma_t)^2]dt + 2\sigma_t(a_3 + a_4\sigma_t)dW_t \\ &+ (a_5 + a_6\sigma_t)[a_5 + (a_6 + 2)\sigma_t]dN_t. \end{aligned} \tag{1.24}$$

Taking expectation on both sides of (1.24), given (1.22) and the initial value σ_0^2, we obtain

$$E\sigma_t^2 = (\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_2})(1 - e^{\beta_2 t}) - \frac{\beta_1(\alpha_1\sigma_0 + \alpha_0)}{\alpha_1(\alpha_1 - \beta_2)}(e^{\beta_2 t} - e^{\alpha_1 t}) + \sigma_0^2 e^{\beta_2 t} \tag{1.25}$$

where α_0 and α_1 are as before, and $\beta_0 = a_3^2 + \lambda a_5^2$, $\beta_1 = 2(a_1 + \lambda a_5) + 2(\lambda a_5 a_6 + a_3 a_4)$ and $\beta_2 = 2(a_2 + \lambda a_6) + \lambda a_6^2 + a_4^2$.

Therefore, the fair value for variance swaps is given by

$$\begin{aligned}
K_{var} &= E(\frac{1}{T}\int_0^T \sigma_t^2 dt) \\
&= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\
&= \frac{1}{T}\int_0^T [(-\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_1} - \frac{\beta_1(\alpha_1\sigma_0 + \alpha_0)}{\alpha_1(\alpha_1 - \beta_2)} + \sigma_0^2)e^{\beta_2 t} \\
&+ \frac{\beta_1(\alpha_1\sigma_0 + \alpha_0)}{\alpha_1(\alpha_1 - \beta_1)}e^{\alpha_1 t} + \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_2}]dt \\
&= \frac{1}{T\beta_2}[\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_2} + \frac{\beta_1(\alpha_1\sigma_0 + \alpha_0)}{\alpha_1(\alpha_1 - \beta_2)} - \sigma_0^2](1 - e^{\beta_2 t}) \\
&- \frac{\beta_1(\alpha_1\sigma_0 + \alpha_0)}{T\alpha_1^2(\alpha_1 - \beta_2)}(1 - e^{\alpha_1 t}) + \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_2}.
\end{aligned} \tag{1.26}$$

TYPE B

Another stochastic process considered for volatility is described as

$$d\sigma_t = (b_1 + b_2\sigma_t)dt + b_3\sqrt{\sigma_t}dW_t \tag{1.27}$$

where W_t is a standard Wiener process and b_1, b_2 and b_3 are constants.

Given the initial value σ_0, we can easily get

$$E\sigma_t = \frac{b_1 + b_2\sigma_0}{b_2}e^{b_2 t} - \frac{b_1}{b_2} \tag{1.28}$$

and, therefore, the fair value for volatility-average swaps is given by

$$K_{vol-ave} = E(\frac{1}{T}\int_0^T \sigma_t dt)$$

$$
\begin{aligned}
&= \frac{1}{T}\int_0^T E\sigma_t dt \\
&= \frac{b_1 + b_2\sigma_0}{Tb_2^2}(e^{b_2 T} - 1) - \frac{b_1}{b_2}.
\end{aligned} \tag{1.29}
$$

From (1.27), the process for σ_t^2, by Itô's lemma, yields

$$
d\sigma_t^2 = [(2b_1 + b_3^2)\sigma_t + 2b_2\sigma_t^2]dt + 2b_3(\sigma_t)^{\frac{3}{2}}dW_t. \tag{1.30}
$$

Given (1.28) and initial value σ_0^2, we obtain

$$
E\sigma_t^2 = -\frac{\delta_1}{2b_2} - \frac{\delta_2}{b_2}e^{b_2 t} + (\sigma_0^2 + \frac{\delta_2}{b_2} + \frac{\delta_1}{2b_2})e^{2b_2 t}, \tag{1.31}
$$

where $\delta_1 = -\frac{b_1(2b_2+b_3^2)}{b_2}$ and $\delta_2 = \sigma_0(2b_1 + b_3^2) - \delta_1$.

Therefore, the fair value for variance swaps is given by

$$
\begin{aligned}
K_{var} &= E(\frac{1}{T}\int_0^T \sigma_t^2 dt) \\
&= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\
&= -\frac{\delta_1}{2b_2} - \frac{\delta_2}{Tb_2^2}(e^{b_2 T} - 1) \\
&+ \frac{1}{2Tb_2}(\sigma_0^2 + \frac{\delta_2}{b_2} + \frac{\delta_1}{2b_2})(e^{2b_2 T} - 1).
\end{aligned} \tag{1.32}
$$

The two approaches discussed above yield closed-form formulae for the valuation of variance swaps. However, they fail to provide formulae for volatility swaps and are not within the framework of stochastic volatility models (SVMs). Even though some popular SVMs were discussed in the paper by Howison, Rafailidis and Rasmussen (2004) [20], they're incomplete in two ways: first of all, these SVMs are diffusive process without any jumps considered; secondly, valuation of volatility swaps isn't given. Therefore, we will discuss valuations of both variance and volatility swaps in the frameworks of both SVMs without jumps and SVMs with jumps in the following chapters, with the framework of SVMs without jumps in Chapter 2 as the first step.

Chapter 2

Stochastic Volatility Models without Jumps

In the early 1970's, the Black-Scholes (1973) model [4] was developed to derive pricing formulas for European options with stocks as underlying assets, where the volatility term was assumed to be constant. However, constant implied volatility is not compatible with derivatives' prices observed in financial markets. Lots of work has been done to study the volatility issue. In general, it has been studied within two different frameworks, namely, deterministic volatility models and stochastic volatility models.

2.1 Deterministic Volatility Models (DVMs)

In 1973, Merton [28] extended the term structure of volatility to $\sigma \triangleq \sigma_t$, a deterministic function of time, with the implied volatility for an option of maturity T given by

$$\sigma_T^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt. \tag{2.1}$$

In this approach, the volatility coefficient is independent of the current level of the underlying stochastic process S_t. The special case where σ is constant reduces to the Black-Scholes model, suggesting changes in stock prices are lognormally distributed, but the empirical test by Bollerslev (1986) [5] indicates otherwise.

In Hull's book in 2000 [21], volatility is assumed to be a function of time and the current level of the underlying asset's price S_t, $\sigma \triangleq \sigma(t, S_t)$, where S_t satisfies

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t \tag{2.2}$$

where W_t is a standard Wiener process. In this approach, a time and level-dependent

volatility coefficient makes the pricing more challenging and usually precludes the existence of a closed-form solution.

2.2 Stochastic Volatility Models (SVMs) without Jumps

Besides DVMs addressed above, another approach would be to consider the volatility influenced by some sources of randomness, for example, a Wiener process. In this way, we will obtain a stochastic volatility model, the concept of which was introduced by Hull and White (1987) [22]. Obviously, SVMs are general enough to include DVMs as special cases.

Several types of SVMs have been considered. For example, the time variation of the volatility contains an additional source of randomness besides W_t^1, given by

$$d\sigma_t = a(t, \sigma_t)dt + b(t, \sigma_t)dW_t^2 \tag{2.3}$$

where Wiener processes W_t^1, which governs the price process, and W_t^2 may be correlated (see [7], [19], [22]).

Other examples include: volatility depends on a random parameter x such that $\sigma_t = \sigma(x_t)$, where x_t is some random process (see [16], [18], [36], [37], [39]); the volatility $\sigma(t, S_t)$ depends on $S_t = S(t+\theta)$ for $\theta \in [-\tau, 0]$, namely, stochastic volatility with delay (see [25]); and so on.

In the remainder of this chapter, we will examine seven stochastic volatility models and compute the corresponding fair values for both variance and volatility swaps. Since some results have been appeared in papers such as Javaheri, Wilmott and Haug (2002) [24], Swishchuk (2004) [38] and Howison, Rafailidis and Rasmussen (2004) [20], we will synthesize their results here with our own via a probabilistic approach. As is mentioned in the first chapter, we consider SVMs in a risk-neutral probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$

with filtration $\mathcal{F}_t$, $t \in [0,T]$, and denote the expectation E instead of E_Q with respect to the risk-neutral measure.

In the beginning, we list some results for Itô integrals in the following lemmas.

Lemma 2.2-1. *The expectation of the product of two Itô integrals with exponential integrating functions is given by*

$$E(\int_0^t e^{\alpha s} dW_s \int_0^u e^{\alpha s} dW_s) = \frac{1}{2\alpha}[e^{2\alpha(t\wedge u)} - 1], \tag{2.4}$$

where t and u are positive, $t \wedge u \triangleq min(t,u)$, α is a non-zero constant, and W_s is a standard Wiener process.

The proof can be found in Section A.1 in Appendix A.

Lemma 2.2-2. *The expectation of the product of two exponential functions with powers of Itô integrals is given by*

$$\begin{aligned} & E[\exp(\int_0^t e^{-\kappa(t-s)} dW_s + \int_0^u e^{-\kappa(u-s)} dW_s)] \\ = \ & \exp[\frac{1}{4\kappa}(1 - e^{2\kappa(t\wedge u - t\vee u)}) + \frac{1}{4\kappa}(e^{-\kappa t} + e^{-\kappa u})^2 (e^{2\kappa(t\wedge u)} - 1)], \end{aligned} \tag{2.5}$$

where t and u are positive, $t \wedge u \triangleq min(t,u)$, $t \vee u \triangleq max(t,u)$, κ is a non-zero constant, and W_s is a standard Wiener process.

The proof can be found in Section A.2 in Appendix A.

Lemma 2.2-3. *The expectation of the time integral*

$$\int_0^t e^{(\kappa + \frac{\gamma^2}{2})s + \gamma(W_t + W_u - W_s)} ds$$

is given by

$$\begin{aligned} & E[\int_0^t e^{(\kappa + \frac{\gamma^2}{2})s + \gamma(W_t + W_u - W_s)} ds] \\ = \ & \frac{1}{\kappa - \gamma^2}(e^{(\kappa - \gamma^2)(t\wedge u)} - 1) e^{\frac{\gamma^2}{2}(t\vee u) + \frac{3\gamma^2}{2}(t\wedge u)} \\ & + I_{\{t>u\}} \frac{1}{\kappa}(e^{\kappa t} - e^{\kappa u}) e^{\frac{\gamma^2}{2}(t+u)}, \end{aligned} \tag{2.6}$$

where t and u are positive, $t \wedge u \triangleq min(t,u)$, $t \vee u \triangleq max(t,u)$, κ and γ are non-zero constants with $\kappa - \gamma^2 \neq 0$, W_s is a standard Wiener process, and $I_{\{t>u\}}$ is an indicator function taking value 1 if $t > u$ and 0 if $t \leq u$.

The proof can be found in Section A.3 in Appendix A.

From the above lemma, we can immediately reach the following corollary:

Corollary 2.2-4. *The expectation of the time integral*

$$\int_0^u e^{(\kappa+\frac{\gamma^2}{2})s+\gamma(W_t+W_u-W_s)}ds$$

is given by

$$\begin{aligned} & E[\int_0^u e^{(\kappa+\frac{\gamma^2}{2})s+\gamma(W_t+W_u-W_s)}ds] \\ = \; & \frac{1}{\kappa-\gamma^2}(e^{(\kappa-\gamma^2)(t\wedge u)}-1)e^{\frac{\gamma^2}{2}(t\vee u)+\frac{3\gamma^2}{2}(t\wedge u)} \\ & +I_{\{u>t\}}\frac{1}{\kappa}(e^{\kappa u}-e^{\kappa t})e^{\frac{\gamma^2}{2}(t+u)}, \end{aligned} \tag{2.7}$$

where t and u are positive, $t \wedge u \triangleq min(t,u)$, $t \vee u \triangleq max(t,u)$, κ and γ are non-zero constants with $\kappa - \gamma^2 \neq 0$, W_s is a standard Wiener process, and $I_{\{u>t\}}$ is an indicator function taking value 1 if $u > t$ and 0 if $u \leq t$.

Lemma 2.2-5. *The expectation of the double time integral*

$$\int_0^t \int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)+\gamma(W_t+W_u-W_x-W_y)}dydx$$

is given by

$$\begin{aligned} & E[\int_0^t \int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)+\gamma(W_t+W_u-W_x-W_y)}dydx] \\ = \; & e^{\frac{\gamma^2}{2}t+\frac{\gamma^2}{2}u}[e^{2\kappa(t\wedge u)}(\frac{1}{\kappa(2\kappa-\gamma^2)}+\frac{1}{\kappa(\kappa-\gamma^2)}-\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa^2}) \\ & -e^{\kappa(t\wedge u)}(\frac{2}{\kappa(\kappa-\gamma^2)}-\frac{1}{\kappa^2}) \\ & +e^{\gamma^2(t\wedge u)}(\frac{1}{\kappa(\kappa-\gamma^2)}+\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa(2\kappa-\gamma^2)}) \\ & +\frac{1}{\kappa^2}(e^{\kappa(t+u)}-e^{\kappa(t\vee u)})], \end{aligned} \tag{2.8}$$

where t and u are positive, $t \wedge u \triangleq min(t,u)$, $t \vee u \triangleq max(t,u)$, κ and γ are non-zero constants with $\kappa - \gamma^2 \neq 0$ and $2\kappa - \gamma^2 \neq 0$, W_s is a standard Wiener process.

The proof can be found in Section A.4 in Appendix A.

2.2.1 The Ornstein-Uhlenbeck Model

The Ornstein-Uhlenbeck process, also known as a mean-reverting process, was first introduced by L.S. Ornstein and G.E. Uhlenbeck in 1930 [41] to capture the velocity of a particle in physics. Here we apply this model to capture variance's evolution, assuming that the underlying asset's variance follows the process described below:

$$d\sigma_t^2 = -\alpha\sigma_t^2 dt + \gamma dW_t. \tag{2.9}$$

Equation (2.9) describes a mean-reverting stochastic process with zero mean and reversion speed $\alpha > 0$. γ is a positive constant and W_t is a standard Wiener process.

Solving this stochastic differential equation (SDE) by multiplying $e^{\alpha t}$, given the initial value σ_0^2, we obtain

$$\sigma_t^2 = \sigma_0^2 e^{-\alpha t} + \gamma \int_0^t e^{-\alpha(t-s)} dW_s. \tag{2.10}$$

Thus, the cross product is given by

$$\begin{aligned} \sigma_t^2 \sigma_u^2 &= \sigma_0^4 e^{-\alpha(t+u)} + \gamma^2 \int_0^t e^{-\alpha(t-s)} dW_s \int_0^u e^{-\alpha(u-s)} dW_s \\ &+ \sigma_0^2 e^{-\alpha t}\gamma \int_0^u e^{-\alpha(u-s)} dW_s + \sigma_0^2 e^{-\alpha u}\gamma \int_0^t e^{-\alpha(t-s)} dW_s. \end{aligned} \tag{2.11}$$

Taking expectations on both sides of (2.10) and (2.11), we conclude with

$$E\sigma_t^2 = \sigma_0^2 e^{-\alpha t}, \tag{2.12}$$

$$\begin{aligned} E\sigma_t^2\sigma_u^2 &= \sigma_0^4 e^{-\alpha(t+u)} + \gamma^2 e^{-\alpha(t+u)} E\Big(\int_0^t e^{\alpha s} dW_s \int_0^u e^{\alpha s} dW_s\Big) \\ &= \sigma_0^4 e^{-\alpha(t+u)} + \frac{\gamma^2}{2\alpha} e^{-\alpha(t+u)}[e^{2\alpha(t\wedge u)} - 1], \end{aligned} \tag{2.13}$$

where we apply (2.4) in (2.13).

Consequently, the mean values for σ_R^2 and σ_R^4 are given by

$$\begin{aligned} E\sigma_R^2 &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{\sigma_0^2}{T}\int_0^T e^{-\alpha t}dt \\ &= \frac{\sigma_0^2}{\alpha T}(1-e^{-\alpha T}), \qquad (2.14) \\ E\sigma_R^4 &= \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2 dtdu \\ &= \frac{1}{T^2}\int_0^T\int_0^T (\sigma_0^4 - \frac{\gamma^2}{2\alpha})e^{-\alpha(t+u)} + \frac{\gamma^2}{2\alpha}e^{2\alpha(t\wedge u)-\alpha(t+u)}dtdu \\ &= \frac{2\alpha\sigma_0^4-\gamma^2}{2T^2\alpha^3}(e^{-\alpha T}-1)^2 + \frac{\gamma^2}{T^2\alpha^3}(e^{-\alpha T}-1) + \frac{\gamma^2}{T\alpha^2}. \qquad (2.15) \end{aligned}$$

From (2.14) and (2.15), we obtain the variance for σ_R^2

$$\begin{aligned} Var(\sigma_R^2) &= E\sigma_R^4 - (E\sigma_R^2)^2 \\ &= -\frac{\gamma^2}{2T^2\alpha^3}(e^{-\alpha T}-1)^2 + \frac{\gamma^2}{T^2\alpha^3}(e^{-\alpha T}-1) + \frac{\gamma^2}{T\alpha^2}. \qquad (2.16) \end{aligned}$$

Therefore, the fair value for variance swaps with maturity T is given by (2.14).

Recall (1.9) where we have $E_Q(\sigma_R) \approx \sqrt{E_Q(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}$. It follows that the fair value of volatility swaps can be computed according to (1.9) when substituting (2.14) and (2.16) in.

2.2.2 The Vasicek Model

As an extension of the Ornstein-Uhlenbeck process, the Vasicek model was first introduced by Vasicek (1977) [42] to capture the evolution of interest rates in a financial market. It describes the instantaneous interest rate r_t as follows

$$dr_t = \alpha(\beta - r_t)dt + \gamma dW_t, \qquad (2.17)$$

where W_t is a standard Wiener process, and α, β and γ are positive parameters. Obviously, if we set $\beta = 0$, then (2.17) reduces to an Ornstein-Uhlenbeck process described in (2.9).

(2.17) is an one factor mean-reverting process with mean β and mean reversion speed α. Given an initial value r_0, the SDE (2.17) has a solution of the form

$$r_t = (r_0 - \beta)e^{-\alpha t} + \beta + \gamma \int_0^t e^{-\alpha(t-s)} dW_s. \tag{2.18}$$

By taking expectations on both sides of (2.18), we obtain

$$Er_t = (r_0 - \beta)e^{-\alpha t} + \beta. \tag{2.19}$$

Taking the limit of (2.19) yields

$$\lim_{t \to \infty} Er_t = \beta. \tag{2.20}$$

Now we apply the Vasicek model to describe the evolution of variance, given by

$$d\sigma_t^2 = \alpha(\beta - \sigma_t^2)dt + \gamma dW_t. \tag{2.21}$$

Solving this SDE by multiplying $e^{\alpha t}$, given the initial value σ_0^2, we obtain

$$\sigma_t^2 = (\sigma_0^2 - \beta)e^{-\alpha t} + \beta + \gamma \int_0^t e^{-\alpha(t-s)} dW_s. \tag{2.22}$$

The cross product is given by

$$\begin{aligned} \sigma_t^2 \sigma_u^2 &= \beta^2 + (\sigma_0^2 - \beta)^2 e^{-\alpha(t+u)} + \beta(\sigma_0^2 - \beta)(e^{-\alpha t} + e^{-\alpha u}) \\ &+ \gamma\beta\left(\int_0^t e^{-\alpha(t-s)} dW_s + \int_0^u e^{-\alpha(u-s)} dW_s\right) \\ &+ \gamma(\sigma_0^2 - \beta)\left(e^{-\alpha t} \int_0^u e^{-\alpha(u-s)} dW_s + e^{-\alpha u} \int_0^t e^{-\alpha(t-s)} dW_s\right) \\ &+ \gamma^2 \int_0^t e^{-\alpha(t-s)} dW_s \int_0^u e^{-\alpha(u-s)} dW_s. \end{aligned} \tag{2.23}$$

Taking expectations of (2.22) and (2.23), we get

$$
\begin{aligned}
E\sigma_t^2 &= (\sigma_0^2-\beta)e^{-\alpha t}+\beta, && (2.24)\\
E\sigma_t^2\sigma_u^2 &= \beta^2+(\sigma_0^2-\beta)^2e^{-\alpha(t+u)}+\beta(\sigma_0^2-\beta)(e^{-\alpha t}+e^{-\alpha u})\\
&+ \gamma^2e^{-\alpha(t+u)}E(\int_0^t e^{\alpha s}dW_s\int_0^u e^{\alpha s}dW_s)\\
&= \beta^2+(\sigma_0^2-\beta)^2e^{-\alpha(t+u)}+\beta(\sigma_0^2-\beta)(e^{-\alpha t}+e^{-\alpha u})\\
&+ \frac{\gamma^2e^{-\alpha(t+u)}}{2\alpha}[e^{2\alpha(t\wedge u)}-1], && (2.25)
\end{aligned}
$$

where we apply (2.4) in (2.25). It follows that the mean value of σ_R^2 and σ_R^4, and the variance of σ_R^2 can be expressed as

$$
\begin{aligned}
E\sigma_R^2 &= \frac{1}{T}\int_0^T E\sigma_t^2dt\\
&= \frac{1}{T}\int_0^T(\sigma_0^2-\beta)e^{-\alpha t}+\beta dt\\
&= \frac{\sigma_0^2-\beta}{\alpha T}(1-e^{-\alpha T})+\beta, && (2.26)\\
E\sigma_R^4 &= \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2dtdu\\
&= \beta^2+\frac{\gamma^2}{T\alpha^2}+(\frac{2\beta(\sigma_0^2-\beta)}{T\alpha}-\frac{\gamma^2}{T^2\alpha^3})(1-e^{-\alpha T})\\
&+ \frac{2\alpha(\sigma_0^2-\beta)^2-\gamma^2}{2T^2\alpha^3}(1-e^{-\alpha T})^2, && (2.27)\\
Var(\sigma_R^2) &= E\sigma_R^4-(E\sigma_R^2)^2\\
&= \frac{\gamma^2}{T\alpha^2}-\frac{\gamma^2}{T^2\alpha^3}(1-e^{-\alpha T})-\frac{\gamma^2}{2T^2\alpha^3}(1-e^{-\alpha T})^2. && (2.28)
\end{aligned}
$$

Therefore, the fair value for variance swaps with maturity T is given by (2.26).

Recall (1.9), where we have $E_Q(\sigma_R)\approx\sqrt{E_Q(\sigma_R^2)}-\frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}$, so that the fair value for volatility swaps can be expressed via (1.9), substituting in (2.26) and (2.28).

2.2.3 The Hull-White Model

In 1987, Hull and White [22] first proposed the model to price options, described as follows.

$$\begin{cases} dS_t = \mu S_t dt + \sigma_t S_t d\hat{W}_t \\ d\sigma_t^2 = \alpha\sigma_t^2 dt + \gamma\sigma_t^2 dW_t \end{cases} \tag{2.29}$$

where $\alpha > 0$ is the volatility return, $\gamma > 0$ is the volatility of volatility, $\hat{W}_t$ and W_t are independent standard Wiener processes. Obviously the variance follows a log-normal process.

We can solve the SDE for the variance by letting $Y_t = \ln \sigma_t^2$ to obtain

$$Y_t = Y_0 + (\alpha - \frac{\gamma^2}{2})t + \gamma W_t. \tag{2.30}$$

Applying the initial value $Y_0 = \ln \sigma_0^2$, we have

$$\sigma_t^2 = \sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})t + \gamma W_t]. \tag{2.31}$$

The cross product is given by

$$\sigma_t^2 \sigma_u^2 = \sigma_0^4 \exp[(\alpha - \frac{\gamma^2}{2})(t+u) + \gamma(W_t + W_u)]. \tag{2.32}$$

Taking expectations on both sides of (2.31) and (2.32), we reach

$$E\sigma_t^2 = \sigma_0^2 e^{\alpha t} \tag{2.33}$$

$$\begin{aligned} E\sigma_t^2\sigma_u^2 &= \sigma_0^4 \exp[(\alpha - \frac{\gamma^2}{2})(t+u) + \gamma(W_t + W_u)] \\ &= \sigma_0^4 \exp[\alpha(t+u) + \gamma^2(t \wedge u)], \end{aligned} \tag{2.34}$$

where we apply the property of Wiener process $E[\exp(\gamma W_t)] = e^{\frac{t\gamma^2}{2}}$.

It follows that

$$E\sigma_R^2 = \frac{1}{T}\int_0^T E\sigma_t^2 dt$$

$$= \frac{\sigma_0^2}{\alpha T}(e^{\alpha T} - 1), \tag{2.35}$$

$$E\sigma_R^4 = \frac{1}{T^2}\int_0^T \int_0^T E\sigma_t^2 \sigma_u^2 dtdu$$

$$= \frac{2\sigma_0^4}{\alpha T^2(\alpha+\gamma^2)(2\alpha+\gamma^2)}[\alpha e^{(2\alpha+\gamma^2)T} - (2\alpha+\gamma^2)e^{\alpha T} + (\alpha+\gamma^2)], \tag{2.36}$$

$$Var(\sigma_R^2) = E\sigma_R^4 - (E\sigma_R^2)^2$$

$$= \frac{\sigma_0^4}{\alpha T^2(\alpha+\gamma^2)(2\alpha+\gamma^2)}[2\alpha e^{(2\alpha+\gamma^2)T} - \frac{(\alpha+\gamma^2)(2\alpha+\gamma^2)}{\alpha}e^{2\alpha T}$$

$$+ \frac{2\gamma^2}{\alpha}(2\alpha+\gamma^2)e^{\alpha T} - \frac{\gamma^2}{\alpha}(\alpha+\gamma^2)]. \tag{2.37}$$

Therefore, the fair value of variance swaps with maturity T is given by (2.35).

Recall (1.9), where we have $E_Q(\sigma_R) \approx \sqrt{E_Q(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}$, so that the fair value of volatility swaps is obtained by substituting (2.35) and (2.37) in (1.9).

2.2.4 Schwartz's One-Factor Model

Three models were proposed by Schwartz (1997) [34] to describe the stochastic behavior of commodity prices. Here we introduce the most commonly-used one-factor model, which assumes that the asset price S_t follows the process below:

$$dS_t = \alpha(L - \ln S_t)dt + \sigma S_t dW_t, \tag{2.38}$$

where $\alpha > 0$ is the drift speed, L is the equilibrium level and σ is the volatility of asset price. Closed-form solutions as well as European option pricing formulae can be found in [34].

Now we apply the Schwartz's one-factor model to capture the behavior of variance as follows:

$$d\sigma_t^2 = \kappa(\alpha - \ln \sigma_t^2)\sigma_t^2 dt + \gamma\sigma_t^2 dW_t, \tag{2.39}$$

where κ, α and γ are positive constants, and W_t is a standard Wiener process.

By the same transformation as we used in the Hull-White model, $Y_t = \ln \sigma_t^2$, we obtain

$$dY_t = \kappa(\alpha - \frac{\gamma^2}{2\kappa} - Y_t)dt + \gamma dW_t. \tag{2.40}$$

The solution of (2.40) is given by integrating after multiply $e^{\kappa t}$ on both sides:

$$Y_t = e^{-\kappa t}\ln\sigma_0^2 + \kappa(\alpha - \frac{\gamma^2}{2\kappa})\int_0^t e^{-\kappa(t-s)}ds + \int_0^t \gamma e^{-\kappa(t-s)}dW_s. \tag{2.41}$$

Thus

$$\sigma_t^2 = \exp[e^{-\kappa t}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1-e^{-\kappa t}) + \gamma\int_0^t e^{-\kappa(t-s)}dW_s]. \tag{2.42}$$

The cross product is

$$\begin{aligned}\sigma_t^2\sigma_u^2 &= \exp[(e^{-\kappa t}+e^{-\kappa u})\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(2-e^{-\kappa t}-e^{-\kappa u}) \\ &+ \gamma(\int_0^t e^{-\kappa(t-s)}dW_s + \int_0^u e^{-\kappa(u-s)}dW_s)].\end{aligned} \tag{2.43}$$

It follows that the expectations of σ_t^2 and $\sigma_t^2\sigma_u^2$ are given by

$$E\sigma_t^2 = \exp[e^{-\kappa t}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1-e^{-\kappa t}) + \frac{\gamma^2}{4\kappa}(1-e^{-2\kappa t})], \tag{2.44}$$

$$\begin{aligned}E\sigma_t^2\sigma_u^2 &= \exp[(e^{-\kappa t}+e^{-\kappa u})\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(2-e^{-\kappa t}-e^{-\kappa u})] \\ &\times E[\exp(\gamma\int_0^t e^{-\kappa(t-s)}dW_s + \gamma\int_0^u e^{-\kappa(u-s)}dW_s)] \\ &= \exp[(e^{-\kappa t}+e^{-\kappa u})\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(2-e^{-\kappa t}-e^{-\kappa u})] \\ &\times \exp[\frac{\gamma^2}{4\kappa}(1-e^{2\kappa(t\wedge u - t\vee u)}) + \frac{\gamma^2}{4\kappa}(e^{-\kappa t}+e^{-\kappa u})^2(e^{2\kappa(t\wedge u)}-1)],\end{aligned} \tag{2.45}$$

where we apply the simple factor

$$\begin{aligned}E(\exp[\gamma\int_0^t e^{-\kappa(t-s)dW_s}]) &= \exp[\frac{\gamma^2}{2}\int_0^t e^{-2\kappa(t-s)}ds] \\ &= \exp[\frac{\gamma^2}{4\kappa}(1-e^{-2\kappa t})]\end{aligned} \tag{2.46}$$

in (2.44), and apply (2.5) in (2.45).

It follows that

$$\begin{aligned}
E\sigma_R^2 &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\
&= \frac{1}{T}\int_0^T \exp[e^{-\kappa t}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1-e^{-\kappa t}) + \frac{\gamma^2}{4\kappa}(1-e^{-2\kappa t})]dt \\
&= \frac{1}{T}\exp(\alpha - \frac{\gamma^2}{4\kappa})\int_0^T \exp[(\ln\sigma_0^2 - \alpha + \frac{\gamma^2}{2\kappa})e^{-\kappa t} - \frac{\gamma^2}{4\kappa}e^{-2\kappa t}]dt, \qquad (2.47) \\
E\sigma_R^4 &= \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2 dtdu \\
&= \frac{1}{T^2}\exp(2\alpha - \frac{3\gamma^2}{4\kappa})\int_0^T\int_0^T \exp[(\frac{\gamma^2}{2\kappa} - \alpha + \ln\sigma_0^2)(e^{-\kappa t} + e^{-\kappa u}) \\
&- \frac{\gamma^2}{4\kappa}e^{2\kappa(t\wedge u - t\vee u)} + \frac{\gamma^2}{4\kappa}(e^{2\kappa(t\wedge u)} - 1)(e^{-\kappa t} + e^{-\kappa u})^2]dtdu, \qquad (2.48) \\
Var(\sigma_R^2) &= E\sigma_R^4 - (E\sigma_R^2)^2 \\
&= \frac{1}{T^2}\exp(-\frac{\gamma^2}{4\kappa})\int_0^T\int_0^T \exp[\frac{\gamma^2}{4\kappa}(e^{-2\kappa t} + e^{-2\kappa u}) - \frac{\gamma^2}{4\kappa}e^{2\kappa(t\wedge u - t\vee u)} \\
&+ \frac{\gamma^2}{4\kappa}(e^{2\kappa(t\wedge u)} - 1)(e^{-\kappa t} + e^{-\kappa u})^2]dtdu. \qquad (2.49)
\end{aligned}$$

Therefore, the fair value for variance swaps with maturity T is given by (2.47).

Recall (1.9), where we have $E_Q(\sigma_R) \approx \sqrt{E_Q(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}$, so that the fair value for volatility swaps is yielded from (1.9) by substituting in (2.47) and (2.49).

2.2.5 Pilipović's Two Factor Model

Pilipović (1998) [32] proposed mean-reverting one-factor and two-factor models to price energy derivatives. In her one-factor model, the equilibrium price, the long-run average, is assumed to be constant. In her two-factor model, the long-run average follows a geometric Brownian motion process. The two factor model is described as follows:

$$\begin{cases} dS_t = \kappa(\alpha_t - S_t)dt + \gamma S_t dW_t^1 \\ d\alpha_t = \mu\alpha_t dt + \sigma\alpha_t dW_t^2 \end{cases} \qquad (2.50)$$

where κ, γ, μ and σ are positive constants, and W_t^1, W_t^2 are independent Wiener processes.

If we set $\mu = \sigma = 0$, then (2.50) reduces to the one-factor model. The solution of (2.50) has been given by Pilipović (1998) [32] as follows:

$$\begin{aligned} S_t &= S_0 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\ &+ \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \int_0^t e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds. \end{aligned} \tag{2.51}$$

Now we apply Pilipović's two-factor model to the evolution of the underlying asset's variance, described in the following

$$\begin{cases} d\sigma_t^2 = \kappa(\alpha_t - \sigma_t^2)dt + \gamma\sigma_t^2 dW_t^1 \\ d\alpha_t = \mu\alpha_t dt + \sigma\alpha_t dW_t^2 \end{cases} \tag{2.52}$$

with solution given by

$$\begin{aligned} \sigma_t^2 &= \sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\ &+ \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \int_0^t e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds, \end{aligned} \tag{2.53}$$

which yields the cross product

$$\begin{aligned} \sigma_t^2\sigma_u^2 &= \sigma_0^4 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)} e^{\gamma(W_t^1+W_u^1)} \\ &+ \kappa^2\alpha_0^2 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)} e^{\gamma(W_t^1+W_u^1)} \int_0^t \int_0^u e^{(\mu-\frac{\sigma^2}{2})(x+y)+\sigma(W_x^2+W_y^2)} e^{(\kappa+\frac{\gamma^2}{2})(x+y)-\gamma(W_x^1+W_y^1)} dxdy \\ &+ \kappa\alpha_0\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)} e^{\gamma(W_t^1+W_u^1)} \int_0^u e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds \\ &+ \kappa\alpha_0\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)} e^{\gamma(W_t^1+W_u^1)} \int_0^t e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds. \end{aligned} \tag{2.54}$$

It follows expectations of (2.53) and (2.54) are given by

$$\begin{aligned} E\sigma_t^2 &= \sigma_0^2 e^{-\kappa t} + \frac{\kappa\alpha_0}{\kappa+\mu}(e^{\mu t} - e^{-\kappa t}), \\ E\sigma_t^2\sigma_u^2 &= \sigma_0^4 e^{-\kappa(t\vee u)} e^{(\gamma^2-\kappa)(t\wedge u)} \end{aligned} \tag{2.55}$$

$$
\begin{aligned}
&+ \kappa^2\alpha_0^2 e^{-\kappa(t+u)}[(\frac{1}{(\mu+\kappa+\sigma^2)(2\mu+2\kappa+\sigma^2-\gamma^2)}+\frac{1}{(\mu+\kappa-\gamma^2)(\mu+\kappa+\sigma^2)}\\
&- \frac{1}{2\mu+2\kappa+\sigma^2-\gamma^2}-\frac{1}{(\mu+\kappa)(\mu+\kappa+\sigma^2)})e^{(2\mu+2\kappa+\sigma^2)(t\wedge u)}\\
&+ (\frac{1}{(\mu+\kappa)(\mu+\kappa+\sigma^2)}-\frac{2}{(\mu+\kappa+\sigma^2)(\mu+\kappa-\gamma^2)})e^{(\mu+\kappa)(t\wedge u)}\\
&+ (\frac{1}{2\mu+2\kappa+\sigma^2-\gamma^2}+\frac{1}{(\mu+\kappa+\sigma^2)(\mu+\kappa-\gamma^2)}\\
&- \frac{1}{(\mu+\kappa+\sigma^2)(2\mu+2\kappa+\sigma^2-\gamma^2)})e^{\gamma^2(t\wedge u)}\\
&+ \frac{1}{(\mu+\kappa)(\mu+\kappa+\sigma^2)}e^{(\mu+\kappa+\sigma^2)(t\wedge u)+(\mu+\kappa)(t\vee u)}\\
&- \frac{1}{(\mu+\kappa)(\mu+\kappa+\sigma^2)}e^{(\mu+\kappa)(t\vee u)}]\\
&+ \frac{2\kappa\alpha_0\sigma_0^2}{\mu+\kappa-\gamma^2}(e^{(\mu+\kappa-\gamma^2)(t\wedge u)}-1)e^{-\kappa(t+u)+\gamma^2(t\wedge u)}\\
&+ I_{\{u>t\}}\frac{\kappa\alpha_0\sigma_0^2}{\mu+\kappa}(e^{(\mu+\kappa)u}-e^{(\mu+\kappa)t})e^{-\kappa(t+u)}\\
&+ I_{\{t>u\}}\frac{\kappa\alpha_0\sigma_0^2}{\mu+\kappa}(e^{(\mu+\kappa)t}-e^{(\mu+\kappa)u})e^{-\kappa(t+u)}, \qquad (2.56)
\end{aligned}
$$

where we apply (2.6), (2.7)and (2.8) in (2.56).

Thus the mean of σ_R^2 is given by

$$
\begin{aligned}
E\sigma_R^2 &= \frac{1}{T}\int_0^T E\sigma_t^2 dt\\
&= \frac{1}{T}\int_0^T \frac{\kappa\alpha_0}{\kappa+\mu}e^{\mu t}+(\sigma_0^2-\frac{\kappa\alpha_0}{\kappa+\mu})e^{-\kappa t}dt\\
&= \frac{\kappa\alpha_0}{\mu T(\kappa+\mu)}(e^{\mu T}-1)+(\frac{\alpha_0}{T(\kappa+\mu)}-\frac{\sigma_0^2}{\kappa T})(e^{-\kappa T}-1), \qquad (2.57)
\end{aligned}
$$

which is the fair value of variance swaps with maturity T.

The expectation of σ_R^4 could be computed from (2.56) by $E\sigma_R^4 = \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2 du dt$ and follows the variance of σ_R^2 by $Var(\sigma_R^2) = E\sigma_R^4 - (E\sigma_R^2)^2$. Recall (1.9), where we have $E_Q(\sigma_R) \approx \sqrt{E_Q(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}$, so that the fair value for volatility swaps can be done via (1.9) after substituting in (2.57) and $Var(\sigma_R^2)$.

2.2.6 A GARCH (1,1) Model

In the paper written by Javaheri, Wilmott and Haug (2002) [24], the valuation and hedging of variance and volatility swaps were studied within the frame of a GARCH (1,1) stochastic volatility model, a mean-reverting process which defined the variance as follows:

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \gamma\sigma_t^2 dW_t, \tag{2.58}$$

where κ is the speed of mean reversion, θ is the mean reversion level, γ is the volatility of variance, and W_t is a standard Wiener process.

Javaheri, Wilmott and Haug use the partial differential equation (PDE) approach to determine the first two moments of realized variance in [24]. Here we employ another way to solve the problem, namely, by a probabilistic approach.

(2.58) can be solved via a variation of constants technique, giving

$$\sigma_t^2 = \sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t+\gamma W_t} + \kappa\theta \int_0^t e^{-(\kappa+\frac{\gamma^2}{2})(t-s)+\gamma(W_t-W_s)}ds, \tag{2.59}$$

from which it follows that the cross product is given by

$$\begin{aligned}
\sigma_t^2\sigma_u^2 &= \sigma_0^4 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)+\gamma(W_t+W_u)} \\
&+ \kappa\theta\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)} \int_0^u e^{(\kappa+\frac{\gamma^2}{2})s+\gamma(W_t+W_u-W_s)}ds \\
&+ \kappa\theta\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)} \int_0^t e^{(\kappa+\frac{\gamma^2}{2})s+\gamma(W_t+W_u-W_s)}ds \\
&+ \kappa^2\theta^2 e^{-(\kappa+\frac{\gamma^2}{2})(t+u)} \int_0^t \int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)+\gamma(W_t+W_u-W_x-W_y)}dydx.
\end{aligned} \tag{2.60}$$

It follows that the expectations of σ_t^2 and $\sigma_t^2\sigma_u^2$ are obtained by

$$\begin{aligned}
E\sigma_t^2 &= (\sigma_0^2 - \theta)e^{-\kappa t} + \theta, \qquad (2.61) \\
E\sigma_t^2\sigma_u^2 &= \sigma_0^4 e^{-\kappa(t\vee u)} e^{(\gamma^2-\kappa)(t\wedge u)}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\kappa\theta\sigma_0^2}{\kappa-\gamma^2}(e^{(\kappa-\gamma^2)(t\wedge u)}-1)e^{\gamma^2(t\wedge u)-\kappa(t+u)} \\
&+ \theta\sigma_0^2 e^{-\kappa(t+u)}(e^{\kappa u}-e^{\kappa t})(I_{\{u>t\}}-I_{\{t>u\}}) \\
&+ \kappa^2\theta^2 e^{-\kappa(t+u)}[e^{2\kappa(t\wedge u)}(\frac{1}{\kappa(2\kappa-\gamma^2)}+\frac{1}{\kappa(\kappa-\gamma^2)}-\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa^2}) \\
&- e^{\kappa(t\wedge u)}(\frac{2}{\kappa(\kappa-\gamma^2)}-\frac{1}{\kappa^2}) \\
&+ e^{\gamma^2(t\wedge u)}(\frac{1}{\kappa(\kappa-\gamma^2)}+\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa(2\kappa-\gamma^2)}) \\
&+ \frac{1}{\kappa^2}(e^{\kappa(t+u)}-e^{\kappa(t\vee u)})], \qquad (2.62)
\end{aligned}
$$

where we apply (2.6), (2.7) and (2.8) in (2.62).

Therefore, the mean of σ_R^2 is given by

$$
\begin{aligned}
E\sigma_R^2 &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\
&= \frac{\sigma_0^2-\theta}{\kappa T}(1-e^{-\kappa T})+\theta, \qquad (2.63)
\end{aligned}
$$

which is the fair value for variance swaps with maturity T.

The expectation of σ_R^4 could be computed from (2.62) by $E\sigma_R^4 = \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2 dudt$ and follows the variance of σ_R^2 by $Var(\sigma_R^2) = E\sigma_R^4 - (E\sigma_R^2)^2$. Recall (1.9); where we have $E_Q(\sigma_R) \approx \sqrt{E_Q(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}$, so that the fair value for volatility swaps can be found via (1.9) after substituting in (2.63) and $Var(\sigma_R^2)$.

2.2.7 The Heston Model

The last model we examine here is a very popular one, the Heston (1993) model [19], which has already been studied by Swishchuk (2004) [38]. It assumes that the underlying asset and its variance follow the process below:

$$
\begin{cases}
dS_t = r_t S_t dt + \sigma_t S_t dW_t^1 \\
d\sigma_t^2 = \kappa(\theta^2-\sigma_t^2)dt + \gamma\sigma_t dW_t^2
\end{cases} \qquad (2.64)
$$

where r_t is the deterministic interest rate, $\kappa > 0$ is a reversion speed, $\gamma > 0$ is the volatility of volatility, and W_t^1 and W_t^2 are independent standard Wiener processes.

In the paper by Swishchuk (2004) [38], a closed-form expression for valuation of variance and volatility swaps is found by the change of time method (see [23]). We will simply review the results in order to compare with results for SVMs with jumps later.

The asset's variance follows a Cox-Ingersoll-Ross (1985) (CIR) process [12], which was first suggested to model the behavior of the instantaneous interest rate. Due to the independence of W_t^1 and W_t^2, we only need to consider the CIR process for variance as follows

$$d\sigma_t^2 = \kappa(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dW_t. \tag{2.65}$$

Obviously the CIR process is a mean-reverting process, where $\kappa > 0$ is the reversion speed, $\theta > 0$ is the long-run volatility and $\gamma > 0$ is the volatility of volatility.

By the change of time method, the solution of (2.65) is given by

$$\sigma_t^2 = e^{-\kappa t}(\sigma_0^2 - \theta^2 + W(\phi_t^{-1})) + \theta^2, \tag{2.66}$$

where $W(t)$ is an one-dimensional Wiener process, and ϕ_t^{-1} is the inverse function of

$$\phi_t = \gamma^{-2}\int_0^t [e^{\kappa\phi_s}(\sigma_0^2 - \theta^2 + W(s)) + \theta^2 e^{2\kappa\phi_s}]^{-1}ds. \tag{2.67}$$

By the properties of $W(\phi_t^{-1})$ (see [23]), we reach

$$E\sigma_t^2 = (\sigma_0^2 - \theta^2)e^{-\kappa t} + \theta^2, \tag{2.68}$$

$$\begin{aligned} E\sigma_t^2\sigma_u^2 &= \gamma^2 e^{-\kappa(t+u)}[\frac{\sigma_0^2 - \theta^2}{\kappa}(e^{\kappa(t\wedge u)} - 1) + \frac{\theta^2}{2\kappa}(e^{2\kappa(t\wedge u)} - 1)] \\ &+ e^{-\kappa(t+u)}(\sigma_0^2 - \theta^2)^2 + e^{-\kappa t}\theta^2(\sigma_0^2 - \theta^2) \\ &+ e^{-\kappa u}\theta^2(\sigma_0^2 - \theta^2) + \theta^4. \end{aligned} \tag{2.69}$$

We obtain the mean of σ_R^2 and σ_R^4 as well as the variance of σ_R^4 below:

$$\begin{aligned}
E\sigma_R^2 &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\
&= \frac{1}{T}\int_0^T [(\sigma_0^2-\theta^2)e^{-\kappa t}+\theta^2]dt \\
&= \frac{\sigma_0^2-\theta^2}{\kappa T}(1-e^{-\kappa T})+\theta^2, \qquad (2.70) \\
E\sigma_R^4 &= \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2 dudt \\
&= \frac{\gamma^2}{T^2}\int_0^T\int_0^T e^{-\kappa(t+u)}[\frac{\sigma_0^2-\theta^2}{\kappa}(e^{\kappa(t\wedge u)}-1)+\frac{\theta^2}{2\kappa}(e^{2\kappa(t\wedge u)}-1)]dudt \\
&+ \frac{(\sigma_0^2-\theta^2)^2}{\kappa^2T^2}(1-2e^{-\kappa T}+e^{-2\kappa T})+\frac{2(\sigma_0^2-\theta^2)\theta^2}{\kappa T}(1-e^{-\kappa T})+\theta^4, \qquad (2.71) \\
Var(\sigma_R^2) &= \frac{\gamma^2}{T^2}\int_0^T\int_0^T e^{-\kappa(t+u)}[\frac{\sigma_0^2-\theta^2}{\kappa}(e^{\kappa(t\wedge u)}-1) \\
&+ \frac{\theta^2}{2\kappa}(e^{2\kappa(t\wedge u)}-1)]dudt \\
&= \frac{\gamma^2 e^{-2\kappa T}}{2\kappa^3T^2}[(2e^{2\kappa T}-4e^{\kappa T}\kappa T-2)(\sigma_0^2-\theta^2) \\
&+ (2e^{2\kappa T}\kappa T-3e^{2\kappa T}+4e^{\kappa T}-1)\theta^2]. \qquad (2.72)
\end{aligned}$$

Therefore, the fair value of variance swaps with maturity T is obtained by (2.70).

Recall (1.9), where we have $E_Q(\sigma_R) \approx \sqrt{E_Q(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8(E_Q(\sigma_R^2))^{\frac{3}{2}}}$, so that the fair value of volatility swaps follows from (1.9) by substituting in (2.70) and (2.72).

Thus far we have overlooked the volatility issue for both DVMs and SVMs without jumps. We take a look at seven typical continuous SVMs and compute the fair value for both variance and volatility swaps. However, it is far from enough to study swaps only within the continuous framework. Evidence in practice, especially from the energy market, has shown the only randomness, Wiener process, is not enough to capture the evolution of underlying asset's variance. We next discuss adding a jump-diffusion term, described by compound Poisson process, to SVMs without jumps to make the model fit real markets better.

Chapter 3

Stochastic Volatility Models with Jumps

The class of jump-diffusion models was first introduced by Merton [29] to capture "discontinuities" in stock prices, an extension of the Black-Scholes model. The idea has gained acceptance recently, especially in papers devoted to pricing assets in energy markets. For example, Cartea and Figueroa (2005) [10] presented a mean-reverting jump-diffusion model for the electricity spot price and derived the corresponding forward prices in closed-form. There are applications in other areas: Swishchuk and Kalemanova (2000) [39] studied the stochastic stability of interest rates and their generalization for the case of random jump changes; Duffie, Pan and Singleton (2000) [15] proposed transform analysis and asset pricing within the frame of affine jump-diffusions, which provided an analytical treatment of a range of valuation and econometric problems.

In this chapter, we consider adding jumps to existing continuous SVMs and pricing variance and volatility swaps within these frames. We first provide an introduction to jump-diffusions.

3.1 Jump-diffusion Description and General Assumptions

We describe a jump-diffusion process for the variance as follows.

Assume that the underlying asset's variance fluctuates according to a continuous stochastic process on intervals $[\tau_i, \tau_{i+1})$, $i = 0, 1, 2, \ldots$, where $\tau_0 = 0$, while at random times τ_i, $i = 1, 2, \ldots$, the variance jumps, namely

$$\sigma^2_{\tau_i} = J_i \sigma^2_{\tau_i -} \tag{3.1}$$

where the total number of jumps on the interval $[0,t]$, denoted by N_t, is assumed to be a Poisson process with intensity $\lambda > 0$, and the jumps J_i, $i = 0,1,2,\ldots$, except $J_0 = 1$, form a sequence of independent identically distributed random variables with values in $(0,+\infty)$, the mean of which is denoted as $A = EJ_i$, $i = 1,2,\ldots$.

From the assumptions above, we immediately reach the following lemma:

Lemma 3.1-1. *For independent identically-distributed nonnegative random variables J_i, $i = 1,2,\ldots$, if $A = EJ_i < +\infty$, then*

$$E(\prod_{i=1}^{N_t} J_i) = e^{\lambda t(A-1)}, \tag{3.2}$$

where N_t is a Poisson process with intensity $\lambda > 0$, independent of J_i, and define $\prod_{i=1}^{0} J_i \triangleq 1$.

The proof of this is straightforward:

$$\begin{aligned} E(\prod_{i=1}^{N_t} J_i) &= P(N_t = 0) + P(N_t = 1)EJ_1 + P(N_t = 2)EJ_1EJ_2 + \ldots \\ &+ P(N_t = n)EJ_1 \ldots EJ_n + \ldots \\ &= \sum_{n=0}^{\infty} A^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(A\lambda t)^n}{n!} = e^{(A-1)\lambda t}. \end{aligned} \tag{3.3}$$

The proof of this can also be found in [39].

We now discuss seven SVMs extended from the previous chapter, in other words, adding jumps to continuous SVMs, and examine the valuation of variance and volatility swaps.

3.2 The Orstein-Uhlenbeck Model with Jumps

Extended from the continuous Orstein-Uhlenbeck process, the OU model with jumps can be described as

$$d\sigma_t^2 = -\alpha\sigma_t^2 dt + \gamma dW_t + (J-1)\sigma_{t-}^2 dN_t. \tag{3.4}$$

This describes a mean-reverting jump-diffusion stochastic process with zero mean and reversion speed $\alpha > 0$. γ is a positive constant and W_t is a standard Wiener process, independent of the Poisson process N_t with intensity $\lambda > 0$, i.e.,

$$dN_t = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1-\lambda dt. \end{cases} \tag{3.5}$$

The jump-diffusion term $(J-1)\sigma_{t-}^2 dN_t$, with the assumptions above, will be used in the following models in this chapter.

We have solved the continuous OU process with solution

$$\sigma_t^2 = \sigma_0^2 e^{-\alpha t} + \gamma \int_0^t e^{-\alpha(t-s)} dW_s. \tag{3.6}$$

Thus, we obtain the following proposition:

Proposition 3.2-1. *The solution of the OU model with jumps is given by*

$$\begin{aligned} \sigma_t^2 &= (J_1 J_2 \dots J_{N_t})\sigma_0^2 e^{-\alpha t} + \sum_{n=0}^{N_t-1} (J_{n+1} J_{n+2} \dots J_{N_t}) \gamma e^{-\alpha(t-\tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\ &+ \gamma e^{-\alpha(t-\tau_{N_t})} \int_{\tau_{N_t}}^{t} e^{\alpha s} dW_s \end{aligned} \tag{3.7}$$

where we define $J_1 J_2 \dots J_{N_t} = 1$ *and* $\sum_{n=0}^{N_t-1} = 0$ *for* $N_t = 0$.

A proof using mathematical induction can be found in Section B.1 in Appendix B.

Taking expectations on both sides of (3.7), we get the mean value for σ_t^2:

$$E\sigma_t^2 = E[(J_1 J_2 \dots J_{N_t})\sigma_0^2 e^{-\alpha t}] + 0 + 0$$

$$
\begin{aligned}
&= \sigma_0^2 e^{-\alpha t} \sum_{n=0}^{\infty} P(N_t = n) A^n \\
&= \sigma_0^2 e^{-\alpha t} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(A\lambda t)^n}{n!} \\
&= \sigma_0^2 e^{(A\lambda - \lambda - \alpha)t}. \qquad (3.8)
\end{aligned}
$$

Equation (3.8) can also be derived from directly taking expectations on both sides of (3.4) and solving the corresponding ordinary differential equation.

Compared with (2.12), where $E\sigma_t^2 = \sigma_0^2 e^{-\alpha t}$ (the expectation of σ_t^2 in continuous case), (3.8) involves an extra "jump term" $e^{(A-1)\lambda t}$. If we set the mean jump size to $A = 1$, then (3.8) reduces to (2.12); if $A > 1$, then the mean value for σ_t^2 within the frame of SVMs with jumps is greater than that within the continuous frame since the "jump term" is greater than 1; if $A < 1$, then the mean value within the continuous frame is larger, since $e^{(A-1)\lambda t} < 1$.

3.3 The Vasicek Model with Jumps

Based on the continuous Vasicek model, the Vasicek model with jumps is given by the following:

$$d\sigma_t^2 = \alpha(\beta - \sigma_t^2)dt + \gamma dW_t + (J-1)\sigma_{t-}^2 dN_t, \qquad (3.9)$$

where α, β and γ are positive parameters, W_t is a standard Wiener process, independent of the jump part which was defined in the previous section.

The solution of the continuous Vasicek model has been given as

$$\sigma_t^2 = (\sigma_0^2 - \beta)e^{-\alpha t} + \beta + \gamma \int_0^t e^{-\alpha(t-s)} dW_s, \qquad (3.10)$$

from where we get the solution for Vasicek model with jumps in the following proposition.

Proposition 3.3-1. *The solution of the Vasicek model with jumps is given by*

$$\begin{aligned}\sigma_t^2 &= (J_1 J_2 \dots J_{N_t})(\sigma_0^2 - \beta)e^{-\alpha t} + \beta \sum_{n=0}^{N_t-1} (J_{n+2} J_{n+3} \dots J_{N_t})(J_{n+1} - 1)e^{-\alpha(t-\tau_{n+1})} \\ &+ \sum_{n=0}^{N_t-1} (J_{n+1} J_{n+2} \dots J_{N_t})\gamma e^{-\alpha(t-\tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\ &+ \gamma e^{-\alpha(t-\tau_{N_t})} \int_{\tau_{N_t}}^{t} e^{\alpha s} dW_s + \beta, \end{aligned} \tag{3.11}$$

where we define $J_1 J_2 \dots J_{N_t} = 1$ *and* $\sum_{n=0}^{N_t-1} = 0$ *for* $N_t = 0$.

A proof by mathematical induction can be found in Section B.2 in Appendix B. Obviously, we can easily see that if the parameter β is set to be 0, then (3.11) reduces to (3.7). The same correspondence occurs between the continuous Ornstein-Uhlenbeck model and the Vasicek model.

It is hard to compute the expectation from (3.11), but the mean value of σ_t^2 can be reached by directly taking expectations of both sides of (3.9), yielding

$$E\sigma_t^2 = (\sigma_0^2 + \frac{\alpha\beta}{A\lambda - \lambda - \alpha})e^{(A\lambda - \lambda - \alpha)t} - \frac{\alpha\beta}{A\lambda - \lambda - \alpha}. \tag{3.12}$$

Recall (2.24), where $E\sigma_t^2 = (\sigma_0^2 - \beta)e^{-\alpha t} + \beta$ (the expectation of σ_t^2 in continuous case), (3.12) appears more complicated. However, if we set the mean jump size to $A = 1$, then (3.12) reduces to (2.24).

3.4 The Hull-White Model with Jumps

The original Hull-White model contains two continuous stochastic process, where the underlying asset price and variance follow two geometric Brownian motions. Since we only care about the evolution of variance, we thus add jumps only to the process for variance, shown as follows:

$$\begin{cases} dS_t = \mu S_t dt + \sigma_t S_t d\hat{W}_t \\ d\sigma_t^2 = \alpha\sigma_t^2 dt + \gamma\sigma_t^2 dW_t + (J-1)\sigma_{t-}^2 dN_t \end{cases} \tag{3.13}$$

where $\alpha > 0$ is the volatility return, $\gamma > 0$ is the volatility of volatility, $\hat{W}_t$ and W_t are independent standard Wiener processes, and the independent jump term is defined as before.

We have the solution for continuous Hull-White model

$$\sigma_t^2 = \sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})t + \gamma W_t], \tag{3.14}$$

yielding the solution for Hull-White model with jumps as the proposition below.

Proposition 3.4-1. *The solution of the Hull-White model with jumps is given by*

$$\sigma_t^2 = (J_1 J_2 \dots J_{N_t})\sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})t + \gamma W_t], \tag{3.15}$$

where we define $J_1 J_2 \dots J_{N_t} = 1$ *for* $N_t = 0$.

A proof by mathematical induction can be found in Section B.3 in Appendix B.

The expectation of σ_t^2 in (3.15) is given by

$$\begin{aligned} E\sigma_t^2 &= E[(J_1 J_2 \dots J_{N_t})]\sigma_0^2 e^{\alpha t} \\ &= \sigma_0^2 e^{\alpha t} \sum_{n=0}^{\infty} P(N_t = n) A^n \\ &= \sigma_0^2 e^{(A\lambda - \lambda + \alpha)t}, \end{aligned} \tag{3.16}$$

which coincides with the expectation of σ_t^2 obtained by directly solving from (3.13).

Compared with the expectation in the continuous framework described in (2.33), where $E\sigma_t^2 = \sigma_0^2 e^{\alpha t}$, (3.16) includes an extra "jump term" $e^{(A-1)\lambda t}$. If we set the mean jump size to $A = 1$, then (3.16) reduces to (2.33); if the value $A > 1$, then the mean value of σ_t^2 obtained from the jump-diffusion process is greater than that from the continuous process since the "jump term" is greater than 1; if $A < 1$, then the expectation from the continuous process is larger.

3.5 Schwartz's One-Factor Model with Jumps

Schwartz's one-factor model with jumps has already been used for valuation in electricity markets in the paper by Cartea and Figueroa (2005) [10] to capture the electricity spot price process. Closed-form formulae are provided for the corresponding forward prices. Here we apply this model to describe the variance process as follows:

$$d\sigma_t^2 = \kappa(\alpha - \ln \sigma_t^2)\sigma_t^2 dt + \gamma \sigma_t^2 dW_t + (J-1)\sigma_{t-}^2 dN_t, \tag{3.17}$$

where κ, α and γ are positive constants, and W_t is a standard Wiener process. The independent jump-diffusion part is defined as before, with more assumptions on the jump size J: the mean jump size is $A = EJ = 1$ and $\ln J$ is normally distributed with mean $-\frac{\sigma_J^2}{2}$ and variance σ_J^2.

The solution comes by directly solving this SDE, which needs the transformation $Y_t = \ln \sigma_t^2$. Firstly we have

$$\begin{aligned} Y_t &= e^{-\kappa t} \ln \sigma_0^2 + \kappa(\alpha - \frac{\gamma^2}{2\kappa}) \int_0^t e^{-\kappa(t-s)} ds \\ &+ \int_0^t \gamma e^{-\kappa(t-s)} dW_s + \int_0^t e^{-\kappa(t-s)} \ln J dN_t. \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned} \sigma_t^2 &= \exp[e^{-\kappa t} \ln \sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1 - e^{-\kappa t}) \\ &+ \gamma \int_0^t e^{-\kappa(t-s)} dW_s + \int_0^t e^{-\kappa(t-s)} \ln J dN_t]. \end{aligned} \tag{3.19}$$

Taking expectations of (3.19), we obtain

$$\begin{aligned} E\sigma_t^2 &= \exp[e^{-\kappa t} \ln \sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1 - e^{-\kappa t}) + \frac{\gamma^2}{4\kappa}(1 - e^{-2\kappa t})] \\ &\times \exp[\lambda \int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-\kappa(t-s)} - \frac{1}{2})^2 - \frac{\sigma_J^2}{8}} - 1) ds], \end{aligned} \tag{3.20}$$

where $E[\exp(\int_0^t e^{-\kappa(t-s)} \ln J dN_t)] = \exp[\lambda \int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-\kappa(t-s)} - \frac{1}{2})^2 - \frac{\sigma_J^2}{8}} - 1) ds$, the proof of which can be found in paper by Cartea and Figueroa (2005) [10].

Recall $E\sigma_t^2$ in the continuous case described by (2.44), where $E\sigma_t^2 = \exp[e^{-\kappa t}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1-e^{-\kappa t}) + \frac{\gamma^2}{4\kappa}(1-e^{-2\kappa t})]$. We can easily distinguish (3.20) from (2.44) by the extra "jump term" $\exp[\lambda\int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-\kappa(t-s)}-\frac{1}{2})^2 - \frac{\sigma_J^2}{8}} - 1)ds]$. However, in contrast with jump-diffusion models discussed in previous sections, (3.20) can't reduce to (2.44) even with the assumption that the mean of jumps' sizes is $A = 1$, since (2.44) depends not only on A, but also is affected by the jumps' variance.

3.6 Pilipović's Two-Factor Model with Jumps

Extended from the continuous Pilipović's two-factor model, Pilipović's two-factor model with jumps can be expressed by

$$\begin{cases} d\sigma_t^2 = \kappa(\alpha_t - \sigma_t^2)dt + \gamma\sigma_t^2 dW_t^1 + (J-1)\sigma_{t-}^2 dN_t \\ d\alpha_t = \mu\alpha_t dt + \sigma\alpha_t dW_t^2 \end{cases} \tag{3.21}$$

where κ, γ, μ and σ are positive constants, and W_t^1, W_t^2 are Wiener processes with zero correlation, also independent of the jump term defined as before.

The solution in continuous case has been given as

$$\begin{aligned} \sigma_t^2 &= \sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\ &+ \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \int_0^t e^{(\mu-\frac{\sigma^2}{2})s + \sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s - \gamma W_s^1} ds, \end{aligned} \tag{3.22}$$

which yields the solution of Pilipović's two-factor model with jumps in the following proposition.

Proposition 3.6-1. *The solution of Pilipović's two-factor model with jumps is given by*

$$\begin{aligned} \sigma_t^2 &= (J_1 J_2 \dots J_{N_t})\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\ &+ \kappa\alpha_0 \sum_{n=0}^{N_t-1} [(J_{n+1}J_{n+2}\dots J_{N_t}) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)} e^{\gamma(W_t^1 - W_{\tau_n}^1)} \end{aligned}$$

$$e^{(\mu-\frac{\sigma^2}{2})\tau_n+\sigma W^2_{\tau_n}}\int_{\tau_n}^{\tau_{n+1}} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W^2_s}e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W^1_s}ds]$$

$$+ \quad \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{N_t})}e^{\gamma(W^1_t-W^1_{\tau_{N_t}})}$$

$$e^{(\mu-\frac{\sigma^2}{2})\tau_{N_t}+\sigma W^2_{\tau_{N_t}}}\int_{\tau_{N_t}}^{t} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W^2_s}e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W^1_s}ds, \tag{3.23}$$

where we define $J_1J_2\ldots J_{N_t}=1$ *and* $\sum_{n=0}^{N_t-1}=0$ *for* $N_t=0$.

A proof by the mathematical induction can be found in Section B.4 in Appendix B.

The mean of σ_t^2 can be extracted easily by directly taking expectations of (3.21) and solving for the corresponding ODE, yielding

$$E\sigma_t^2=(\sigma_0^2-\frac{\kappa\alpha_0}{\mu+\kappa+\lambda-A\lambda})e^{(A\lambda-\lambda-\kappa)t}+\frac{\kappa\alpha_0}{\mu+\kappa+\lambda-A\lambda}e^{\mu t}. \tag{3.24}$$

Recall the expectation of σ_t^2 in continuous case, described by (2.55) where $E\sigma_t^2=\sigma_0^2e^{-\kappa t}+\frac{\kappa\alpha_0}{\mu+\kappa}(e^{\mu t}-e^{-\kappa t})$. We easily find out that the parameters in (3.24) contains information about the jumps. If we set the mean jump size to $A=1$, then (3.24) reduces to (2.55).

3.7 GARCH (1,1) Model with Jumps

Extending the GARCH (1,1) model from a continuous process to a mean-reverting jump-diffusion process, we get the evolution of variance as follows:

$$d\sigma_t^2=\kappa(\theta-\sigma_t^2)dt+\gamma\sigma_t^2dW_t+(J-1)\sigma_{t-}^2dN_t, \tag{3.25}$$

where κ is the speed of mean reversion, θ is the mean reversion level, γ is the volatility of variance, and W_t is a standard Wiener process, independent of the jump term defined as before.

As is discussed in the previous chapter, the solution within the frame of continuous GARCH (1,1) is given by

$$\sigma_t^2=\sigma_0^2e^{-(\kappa+\frac{\gamma^2}{2})t+\gamma W_t}+\kappa\theta\int_0^t e^{-(\kappa+\frac{\gamma^2}{2})(t-s)+\gamma(W_t-W_s)}ds, \tag{3.26}$$

which induces the solution for GARCH (1,1) model with jumps, contained in the following proposition.

Proposition 3.7-1. *The solution of the GARCH (1,1) model with jumps is given by*

$$\begin{aligned}\sigma_t^2 &= (J_1 J_2 \dots J_{N_t})\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t+\gamma W_t} \\ &+ \kappa\theta \sum_{n=0}^{N_t-1} (J_{n+1}J_{n+2}\dots J_{N_t}) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)+\gamma(W_t-W_{\tau_n})} \int_{\tau_n}^{\tau_{n+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds \\ &+ \kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{N_t})+\gamma(W_t-W_{\tau_{N_t}})} \int_{\tau_{N_t}}^{t} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds, \end{aligned} \tag{3.27}$$

where we define $J_1 J_2 \dots J_{N_t} = 1$ *and* $\sum_{n=0}^{N_t-1} = 0$ *for* $N_t = 0$.

A proof by the mathematical induction can be found in Section B.5 in Appendix B. It's not hard to see that (3.27) can be considered a special case of (3.23) when we set $\mu = \sigma = 0$ in (3.23) and define $\alpha_0 = \theta$. Actually, if μ and σ are set to 0, then Pilipović's two-factor model reduces to a one-factor model in the same form as the GARCH (1,1) model.

The expectation of σ_t^2 can be obtained directly from (3.25) by taking expectations on both sides of (3.25), and is given by

$$E\sigma_t^2 = (\sigma_0^2 + \frac{\kappa\theta}{A\lambda - \lambda - \kappa})e^{(A\lambda-\lambda-\kappa)t} - \frac{\kappa\theta}{A\lambda-\lambda-\kappa}. \tag{3.28}$$

Compared with the mean value of σ_t^2 in the continuous case described by (2.61), where $E\sigma_t^2 = (\sigma_0^2 - \theta)e^{-\kappa t} + \theta$, the parameters of (3.28) involve information about the jumps. If we set the mean jump size $A = 1$, then (3.28) reduces to (2.61).

Recall (3.24), the mean of σ_t^2 obtained within Pilipović's two-factor model with jumps. We can see that if $\mu = 0$ and we define $\alpha_0 = \theta$ in (3.24), then (3.24) will reduce to (3.28).

3.8 The Heston Model with Jumps

Extended from the continuous Heston model, the jump-diffusion Heston model is described as follows:

$$\begin{cases} dS_t = r_t S_t dt + \sigma_t S_t dW_t^1 \\ d\sigma_t^2 = \kappa(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dW_t + (J-1)\sigma_{t-}^2 dN_t \end{cases} \tag{3.29}$$

where r_t is deterministic interest rate, $\kappa > 0$ is a reversion speed, $\gamma > 0$ is the volatility of volatility, and W_t^1 and W_t are independent standard Wiener process, also independent of the jump term defined as before.

Swishchuk (2004) [38] has already provided a closed-form solution for the CIR process for variance by the change of time method. However, it is hard to induce the solution for the Heston model with jumps from that formula.

As to the expectation of variance, we can get it from (3.29) by taking expectation directly:

$$E\sigma_t^2 = (\sigma_0^2 + \frac{\kappa\theta^2}{A\lambda - \lambda - \kappa})e^{(A\lambda-\lambda-\kappa)t} - \frac{\kappa\theta^2}{A\lambda - \lambda - \kappa}. \tag{3.30}$$

Compared with the expectation of σ_t^2 in the continuous case described by (2.68), where $E\sigma_t^2 = (\sigma_0^2 - \theta^2)e^{-\kappa t} + \theta^2$, (3.34) will reduce to (2.68) if we set the mean of jumps' size to $A = 1$.

Recall $E\sigma_t^2$ obtained within the frame of GARCH(1,1) with jumps, described in (3.28). We find that if the long term means, θ in GARCH(1,1) and θ^2 in Heston model, are set to be the same, then there is no difference for the mean values of σ_t^2 within these two models.

So far, we have discussed seven SVMs with jumps extended from Chapter 2, and compared expectations of variance yielded from them with those obtained in the previous chapter. The interesting thing we have discovered is that, in most cases, if the mean value

of jump size A is set to 1, then expectations of variance for the SVMs with jumps are the same as those for the corresponding SVMs without jumps. The only exception is for Schwartz's one-factor model. Furthermore, we will see in the next chapter how the fair values for variance swaps coincide for two corresponding models if we set $A = 1$.

Chapter 4

Variance Swaps for SVMs with Jumps

Risk-neutral valuation strategies for variance swaps have been discussed in Section 1.4.1 of Chapter 1. The first moment of realized variance $E\sigma_R^2$, which in continuous-time cases is defined as $\frac{1}{T}\int_0^T E\sigma_t^2 dt$, is the only variable we need in order to compute the fair value for variance swaps. It therefore follows that the fair value for variance swaps is determined as long as $E\sigma_t^2$ is known. In this chapter, we will examine the valuation of variance swaps for SVMs with jumps discussed in the previous chapter, and compare results with those obtained from SVMs without jumps.

4.1 Variance Swaps for Orstein-Uhlenbeck Model with Jumps

The mean value of σ_t^2 has been given by (3.8) where $E\sigma_t^2 = \sigma_0^2 e^{(A\lambda-\lambda-\alpha)t}$, so that the fair value for variance swaps with maturity T is obtained by

$$\begin{aligned} K_{var} &= E\sigma_R^2 \\ &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{1}{T}\int_0^T \sigma_0^2 e^{(A\lambda-\lambda-\alpha)t} dt \\ &= \frac{\sigma_0^2}{(A\lambda-\lambda-\alpha)T}(e^{(A\lambda-\lambda-\alpha)T}-1), \end{aligned} \tag{4.1}$$

where A is the mean jump size, λ is the intensity of Poisson process defined for the jump-diffusion term, and α is the mean-reversion speed.

Comparing (4.1) with the fair value of variance swaps for the continuous OU model obtained by (2.14), where $K_{var} = E\sigma_R^2 = \frac{\sigma_0^2}{\alpha T}(1-e^{-\alpha T})$, we find that, if $A = 1$, then (4.1)

reduces to (2.14), which means that if the mean value of jumps' size equals 1, then there is no difference for the variance swaps' fair values within the continuous OU model or the OU model with jumps.

4.2 Variance Swaps for Vasicek Model with Jumps

From the first moment of σ_t^2 obtained by (3.12), where $E\sigma_t^2 = (\sigma_0^2 + \frac{\alpha\beta}{A\lambda-\lambda-\alpha})e^{(A\lambda-\lambda-\alpha)t} - \frac{\alpha\beta}{A\lambda-\lambda-\alpha}$, we can easily compute the fair value for variance swaps with maturity T as follows:

$$\begin{aligned} K_{var} &= E\sigma_R^2 \\ &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{1}{T}\int_0^T [(\sigma_0^2 + \frac{\alpha\beta}{A\lambda-\lambda-\alpha})e^{(A\lambda-\lambda-\alpha)t} - \frac{\alpha\beta}{A\lambda-\lambda-\alpha}]dt \\ &= \frac{1}{T}(\frac{\sigma_0^2}{A\lambda-\lambda-\alpha} + \frac{\alpha\beta}{(A\lambda-\lambda-\alpha)^2})(e^{(A\lambda-\lambda-\alpha)T} - 1) - \frac{\alpha\beta}{A\lambda-\lambda-\alpha}, \quad (4.2) \end{aligned}$$

where A is the mean jump size, λ is the intensity of Poisson process defined for the jump-diffusion term, α is the reversion speed and β is the long-term mean.

Compared with the fair value for the continuous Vasicek model obtained by (2.26) where $K_{var} = E\sigma_R^2 = \frac{\sigma_0^2-\beta}{\alpha T}(1 - e^{-\alpha T}) + \beta$, (4.2) contains some other jump information and thus looks more complicated. However, if we set $A = 1$, then (4.2) reduces to (2.26).

Recall (4.1), the fair value obtained within OU model with jumps. We find that if the long-term mean $\beta = 0$, then (4.2) reduces to (4.1).

4.3 Variance Swaps for the Hull-White Model with Jumps

The first moment of σ_t^2 is given by (3.16) where $E\sigma_t^2 = \sigma_0^2 e^{(A\lambda-\lambda+\alpha)t}$, from where we obtain the fair value for variance swaps with maturity T, described as

$$\begin{aligned} K_{var} &= E\sigma_R^2 \\ &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{1}{T}\int_0^T \sigma_0^2 e^{(A\lambda-\lambda+\alpha)t} dt \\ &= \frac{\sigma_0^2}{(A\lambda-\lambda+\alpha)T}(e^{(A\lambda-\lambda+\alpha)T}-1), \end{aligned} \tag{4.3}$$

where A is the mean jump size, λ is the intensity of Poisson process defined for the jump-diffusion term, and α is the coefficient of the drift term.

It is easy to see that if $A = 1$, then (4.3) coincides with (2.35) obtained as variance swaps' fair value for the continuous Hull-White model, where $K_{var} = E\sigma_R^2 = \frac{\sigma_0^2}{\alpha T}(e^{\alpha T}-1)$.

4.4 Variance Swaps for Schwartz's One-Factor Model with Jumps

Based on the mean value for σ_t^2 given by (3.20) where $E\sigma_t^2 = \exp[e^{-\kappa t}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1-e^{-\kappa t}) + \frac{\gamma^2}{4\kappa}(1-e^{-2\kappa t}) + \lambda\int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-\kappa(t-s)}-\frac{1}{2})^2-\frac{\sigma_J^2}{8}} - 1)ds]$, the fair value for variance swaps with maturity T can be expressed as

$$\begin{aligned} K_{var} &= E\sigma_R^2 \\ &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{1}{T}\int_0^T \exp[e^{-\kappa t}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1-e^{-\kappa t}) + \frac{\gamma^2}{4\kappa}(1-e^{-2\kappa t}) \\ &+ \lambda\int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-\kappa(t-s)}-\frac{1}{2})^2-\frac{\sigma_J^2}{8}} - 1)ds]dt \\ &= \frac{e^{\alpha-\frac{\gamma^2}{4\kappa}}}{T}\int_0^T \exp[(\frac{\gamma^2}{2\kappa} - \alpha + \ln\sigma_0^2)e^{-\kappa t} - \frac{\gamma^2}{4\kappa}e^{-2\kappa t} \end{aligned}$$

$$+ \quad \lambda \int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-\kappa(t-s)}-\frac{1}{2})^2-\frac{\sigma_J^2}{8}} - 1)ds]dt, \tag{4.4}$$

where λ is the intensity of Poisson process defined for the jump-diffusion term, κ is the reversion speed, γ is the volatility of volatility, and σ_J^2 is the variance of $\ln J$, the log of jump sizes.

Compared with the variance swaps' fair values obtained by continuous Schwartz's one-factor model described by (2.47), where $K_{var} = E\sigma_R^2 = \frac{e^{\alpha-\frac{\gamma^2}{4\kappa}}}{T}\int_0^T \exp[(\frac{\gamma^2}{2\kappa}-\alpha+\ln\sigma_0^2)e^{-\kappa t}-\frac{\gamma^2}{4\kappa}e^{-2\kappa t}]dt$, (4.4) contains an extra integrating term $\exp[\lambda\int_0^t(e^{\frac{\sigma_J^2}{2}(e^{-\kappa(t-s)}-\frac{1}{2})^2-\frac{\sigma_J^2}{8}} - 1)ds]$. Similar to the property of $E\sigma_t^2$, even if we set the mean jump size to $A = 1$, this extra term can't be eliminated and therefore (4.4) cannot reduce to (2.47).

4.5 Variance Swaps for Pilipović's Two-Factor Model with Jumps

The expectation of σ_t^2 has been shown in (3.24) where $E\sigma_t^2 = (\sigma_0^2 - \frac{\kappa\alpha_0}{\mu+\kappa+\lambda-A\lambda})e^{(A\lambda-\lambda-\kappa)t} + \frac{\kappa\alpha_0}{\mu+\kappa+\lambda-A\lambda}e^{\mu t}$, and thus we can obtain the fair value for variance swaps with maturity T as follows:

$$\begin{aligned}
K_{var} &= E\sigma_R^2 \\
&= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\
&= \frac{1}{T}\int_0^T [(\sigma_0^2 - \frac{\kappa\alpha_0}{\mu+\kappa+\lambda-A\lambda})e^{(A\lambda-\lambda-\kappa)t} + \frac{\kappa\alpha_0}{\mu+\kappa+\lambda-A\lambda}e^{\mu t}]dt \\
&= (\frac{\sigma_0^2}{(A\lambda-\lambda-\kappa)T} - \frac{\kappa\alpha_0}{T(A\lambda-\lambda-\kappa)(\mu+\kappa+\lambda-A\lambda)})(e^{(A\lambda-\lambda-\kappa)T}-1) \\
&+ \frac{\kappa\alpha_0}{\mu T(\mu+\kappa+\lambda-A\lambda)}(e^{\mu T}-1),
\end{aligned} \tag{4.5}$$

where A is the mean jump size, λ is the intensity of Poisson process defined for the jump-diffusion term, κ is the mean-reverting speed, μ is the coefficient of drift term for the long-term average process and α_0 is the initial value of the long-term average.

Note that if we define $A = 1$ (4.5) will reduce to (2.57) where $K_{var} = E\sigma_R^2 = (\frac{\alpha_0}{T(\mu+\kappa)} - \frac{\sigma_0^2}{\kappa T})(e^{-\kappa T} - 1) + \frac{\kappa\alpha_0}{\mu T(\mu+\kappa)}(e^{\mu T} - 1)$, meaning no difference between the variance swaps' fair values for the continuous Pilipović's two-factor model and that for Pilipović's two-factor model with jumps.

4.6 Variance Swaps for GARCH(1,1) Model with Jumps

From the mean value of σ_t^2 obtained by (3.28) where $E\sigma_t^2 = (\sigma_0^2 + \frac{\kappa\theta}{A\lambda-\lambda-\kappa})e^{(A\lambda-\lambda-\kappa)t} - \frac{\kappa\theta}{A\lambda-\lambda-\kappa}$, we can compute the fair value for variance swaps with maturity T by

$$\begin{aligned} K_{var} &= E\sigma_R^2 \\ &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{1}{T}\int_0^T [(\sigma_0^2 + \frac{\kappa\theta}{A\lambda-\lambda-\kappa})e^{(A\lambda-\lambda-\kappa)t} - \frac{\kappa\theta}{A\lambda-\lambda-\kappa}]dt \\ &= \frac{1}{T}(\frac{\sigma_0^2}{A\lambda-\lambda-\kappa} + \frac{\kappa\theta}{(A\lambda-\lambda-\kappa)^2})(e^{(A\lambda-\lambda-\kappa)T} - 1) - \frac{\kappa\theta}{A\lambda-\lambda-\kappa}, \quad (4.6) \end{aligned}$$

where A is the mean jump size, λ is the intensity of Poisson process defined for the jump-diffusion term, κ is the mean-reverting speed, and θ is the long-term mean.

Recall (2.63), the fair value of variance swaps within the frame of continuous GARCH (1,1), where $K_{var} = E\sigma_R^2 = \frac{\sigma_0^2-\theta}{\kappa T}(1 - e^{-\kappa T}) + \theta$, we note that if $A = 1$, then (4.6) reduces to (2.63).

Compared with (4.5), where we consider variance swaps' fair value within Pilipović's two-factor model, (4.6) could be deduced from (4.5) by setting $\mu = 0$ and defining $\theta = \alpha_0$ in (4.5).

4.7 Variance Swaps for Heston Model with Jumps

The expectation of σ_t^2 is given by (3.30) where $E\sigma_t^2 = (\sigma_0^2 + \frac{\kappa\theta^2}{A\lambda-\lambda-\kappa})e^{(A\lambda-\lambda-\kappa)t} - \frac{\kappa\theta^2}{A\lambda-\lambda-\kappa}$, from where we obtain the fair value for variance swaps with maturity T as follows:

$$\begin{aligned} K_{var} &= E\sigma_R^2 \\ &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{1}{T}\int_0^T [(\sigma_0^2 + \frac{\kappa\theta^2}{A\lambda-\lambda-\kappa})e^{(A\lambda-\lambda-\kappa)t} - \frac{\kappa\theta^2}{A\lambda-\lambda-\kappa}]dt \\ &= \frac{1}{T}(\frac{\sigma_0^2}{A\lambda-\lambda-\kappa} + \frac{\kappa\theta^2}{(A\lambda-\lambda-\kappa)^2})(e^{(A\lambda-\lambda-\kappa)T} - 1) - \frac{\kappa\theta^2}{A\lambda-\lambda-\kappa}, \quad (4.7) \end{aligned}$$

where A is the mean jump size, λ is the intensity of Poisson process defined for the jump-diffusion term, κ is the mean-reverting speed, and θ^2 is the long-term mean.

Again, variance swaps' fair values for the Heston model with jumps are no different from those for the continuous Heston model if the mean jump size A is defined as 1 so that (4.7) reduces to (2.70), where $K_{var} = E\sigma_R^2 = \frac{\sigma_0^2-\theta^2}{\kappa T}(1 - e^{-\kappa T}) + \theta^2$.

Moreover, if we define the long-term means, θ in GARCH(1,1) model and θ^2 in Heston model respectively, to be the same value, then the variance swaps' fair values within these two models with jumps are identical, which can be checked out by simply comparing (4.6) with (4.7).

Just as mentioned in the end of previous chapter, fair values of variance swaps for SVMs with jumps in most cases coincide with those for continuous SVMs obtained in Chapter 2, as long as the mean value of jump size $A = 1$. It is verified after we examined all models considered in the previous two chapters, with Schwartz's one-factor model as the only exception. The assumption $A = 1$ can be thought of a model with no jumps in average.

From the discussion above, we can see that the computation of the fair value for variance swaps is quite straightforward. Whatever complicated model it is, we know

variance swaps' fair value as longs as $E\sigma_t^2$ is given. We even don't need to know the explicit solution of variance for a particular stochastic volatility model. However, it's not the case when the question for valuation of volatility swaps arises. We will discuss this issue in the next chapter.

Chapter 5

Volatility Swaps for SVMs with Jumps

Risk-neutral valuation strategies have been discussed in Section 1.4.2 of Chapter 1. Since we cannot obtain the first moment of realized volatility $E\sigma_R$ directly within the frame of SVMs, pricing for volatility swaps becomes difficult and we need to approximate the fair value by a convexity adjustment $K_{vol} = E\sigma_R \approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}}$, which requires the first two moments of realized variance, $E\sigma_R^2$ and $Var(\sigma_R^2)$. Therefore, in order to price volatility swaps, we need not only the mean of σ_t^2 to compute $E\sigma_R^2$, but also the mean of cross product $\sigma_t^2\sigma_u^2$ to compute $Var(\sigma_R^2)$. Things will become relatively straightforward as long as explicit formulae for σ_t^2 are given. Fair values for volatility swaps within the frame of continuous SVMs have been given in Chapter 2, according to the strategies discussed above. However, it is still a challenging task to price volatility swaps for SVMs with jumps due to the complexity of explicit formulae for σ_t^2.

In this chapter, we examine volatility swaps' fair values for two SVMs with jumps discussed in Chapter 3. Specifically, we compare the fair value of volatility swaps for the Hull-White model with jumps with that for the continuous Hull-White model, where the difference depends on the mean of the jumps' sizes $A = EJ$ and the mean of the jumps' sizes squared $B = EJ^2$.

5.1 Volatility Swaps for Hull-White Model with Jumps

The solution of the Hull-White model with jumps is given by (3.15) where $\sigma_t^2 = (J_1 J_2 \dots J_{N_t})\sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})t + \gamma W_t]$, yielding the cross product as follows:

$$\sigma_t^2\sigma_u^2 = (J_1 J_2 \dots J_{N_t})(J_1 J_2 \dots J_{N_u})\sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\gamma(W_t+W_u)}. \tag{5.1}$$

As to the expectation of (5.1), we have the following theorem.

Theorem 5.1-1. *The expectation of the cross product (5.1) is given by*

$$E\sigma_t^2\sigma_u^2 = \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(t\vee u)+\frac{3\gamma^2}{2}(t\wedge u)+\lambda(A-1)(t\vee u)+\lambda(B-A)(t\wedge u)}, \tag{5.2}$$

where A is the mean jump size, λ is the intensity of the Poisson process for the jump-diffusion term and we define $B = EJ^2$ to be the mean of jump sizes' squared.

A proof for this can be found in Section C.1 in Appendix C.

Therefore, the mean value of σ_R^4 is given by

$$\begin{aligned} E\sigma_R^4 &= \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2 dudt \\ &= \frac{1}{T^2}\int_0^T\int_0^t e^{nt}e^{mu}dudt + \frac{1}{T^2}\int_0^T\int_t^T e^{mt}e^{nu}dudt \\ &= \frac{2\sigma_0^4}{T^2}[\frac{1}{m(m+n)}e^{(m+n)T} - \frac{1}{mn}e^{nT} + \frac{1}{n(m+n)}], \end{aligned} \tag{5.3}$$

where $m = \alpha + \gamma^2 + \lambda(B-A)$ and $n = \alpha + \lambda(A-1)$.

Recall (4.3) where $E\sigma_R^2 = \frac{\sigma_0^2}{(\alpha+\lambda(A-1))T}(e^{(A\lambda-\lambda+\alpha)T} - 1)$. It follows that the variance for σ_R^2 is obtained by

$$\begin{aligned} Var(\sigma_R^2) &= E\sigma_R^4 - (E\sigma_R^2)^2 \\ &= \frac{\sigma_0^4}{T^2}[\frac{2}{m(m+n)}e^{(m+n)T} - \frac{1}{n^2}e^{2nT} + 2(\frac{1}{n^2} - \frac{1}{mn})e^{nT} \\ &+ \frac{2}{n(m+n)} - \frac{1}{n^2}], \end{aligned} \tag{5.4}$$

where $m = \alpha + \gamma^2 + \lambda(B-A)$ and $n = \alpha + \lambda(A-1)$.

Thus, the fair value for volatility swaps with maturity T can be obtained by substituting (4.3) and (5.4) in the convexity adjustment approximation $K_{vol} = E\sigma_R \approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}}$, yielding

$$K_{vol} = E\sigma_R$$

$$
\begin{aligned}
&\approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}} \\
&= \frac{\sigma_0}{\sqrt{nT}}\sqrt{e^{nT}-1} \\
&- \frac{\sigma_0 n^2[\frac{2}{m(m+n)}e^{(m+n)T} - \frac{1}{n^2}e^{2nT} + 2(\frac{1}{n^2} - \frac{1}{mn})e^{nT} + \frac{2}{n(m+n)} - \frac{1}{n^2}]}{8\sqrt{nT}(e^{nT}-1)^{\frac{3}{2}}},
\end{aligned} \tag{5.5}
$$

where $m = \alpha + \gamma^2 + \lambda(B - A)$ and $n = \alpha + \lambda(A - 1)$.

Recall what we have done within the framework of the continuous Hull-White model in Chapter 2. We have the first two moments of realized variance, given by

$$
E\sigma_R^2 = \frac{\sigma_0^2}{\alpha T}(e^{\alpha T} - 1), \tag{5.6}
$$

$$
\begin{aligned}
Var(\sigma_R^2) &= \frac{\sigma_0^4}{\alpha T^2(\alpha+\gamma^2)(2\alpha+\gamma^2)}[2\alpha e^{(2\alpha+\gamma^2)T} - \frac{(\alpha+\gamma^2)(2\alpha+\gamma^2)}{\alpha}e^{2\alpha T} \\
&+ \frac{2\gamma^2}{\alpha}(2\alpha+\gamma^2)e^{\alpha T} - \frac{\gamma^2}{\alpha}(\alpha+\gamma^2)],
\end{aligned} \tag{5.7}
$$

yielding the fair value for volatility swaps with maturity T as follows:

$$
\begin{aligned}
K_{vol} &= E\sigma_R \\
&\approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}} \\
&= \frac{\sigma_0}{\sqrt{\alpha T}}\sqrt{e^{\alpha T}-1} \\
&- \frac{\frac{\sigma_0\alpha}{(\alpha+\gamma^2)(2\alpha+\gamma^2)}[2\alpha e^{(2\alpha+\gamma^2)T} - \frac{(\alpha+\gamma^2)(2\alpha+\gamma^2)}{\alpha}e^{2\alpha T}]}{8\sqrt{\alpha T}(e^{\alpha T}-1)^{\frac{3}{2}}} \\
&- \frac{\frac{\sigma_0\alpha}{(\alpha+\gamma^2)(2\alpha+\gamma^2)}[\frac{2\gamma^2}{\alpha}(2\alpha+\gamma^2)e^{\alpha T} - \frac{\gamma^2}{\alpha}(\alpha+\gamma^2)]}{8\sqrt{\alpha T}(e^{\alpha T}-1)^{\frac{3}{2}}}.
\end{aligned} \tag{5.8}
$$

Comparing (5.5) with (5.8), we find that if the mean of the jumps' sizes and the jump's sizes squared are set to be 1, i.e. $A = B = 1$, then (5.5) reduces to (5.8), meaning no difference between fair values obtained from the Hull-White model with jumps and the corresponding continuous model.

5.2 Volatility Swaps for Schwartz's One-Factor Model with Jumps

The solution of Schwartz's one-factor model with jumps is given by (3.19) where $\sigma_t^2 = \exp[e^{-\kappa t}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2\kappa})(1-e^{-\kappa t}) + \gamma e^{-\kappa t}\int_0^t e^{\kappa s}dW_s + \int_0^t e^{-\kappa(t-s)}\ln J dN_s]$, which yields the cross product as follows:

$$\begin{aligned}\sigma_t^2\sigma_u^2 &= \exp[(e^{-\kappa t}+e^{-\kappa u})\ln\sigma_0^2 + (\alpha-\frac{\gamma^2}{2\kappa})(2-e^{-\kappa t}-e^{-\kappa u}) \\ &+ \gamma e^{-\kappa t}\int_0^t e^{\kappa s}dW_s + \gamma e^{-\kappa u}\int_0^u e^{\kappa s}dW_s \\ &+ \int_0^t e^{-\kappa(t-s)}\ln J dN_s + \int_0^u e^{-\kappa(u-s)}\ln J dN_s]. \end{aligned} \tag{5.9}$$

Recall that the log of jump sizes $\ln J$ is normally distributed with mean $-\frac{\sigma_J^2}{2}$ and variance σ_J^2. It follows the expectation of (5.9) is given by

$$\begin{aligned}E\sigma_t^2\sigma_u^2 &= \exp[(e^{-\kappa t}+e^{-\kappa u})\ln\sigma_0^2 + (\alpha-\frac{\gamma^2}{2\kappa})(2-e^{-\kappa t}-e^{-\kappa u})] \\ &\quad E[\exp[\gamma e^{-\kappa t}\int_0^t e^{\kappa s}dW_s + \gamma e^{-\kappa u}\int_0^u e^{\kappa s}dW_s]] \\ &\quad E[\exp[\int_0^t e^{-\kappa(t-s)}\ln J dN_s + \int_0^u e^{-\kappa(u-s)}\ln J dN_s]] \\ &= \exp[(e^{-\kappa t}+e^{-\kappa u})\ln\sigma_0^2 + (\alpha-\frac{\gamma^2}{2\kappa})(2-e^{-\kappa t}-e^{-\kappa u})] \\ &\quad \exp[\frac{\gamma^2}{4\kappa}(e^{-\kappa t}+e^{-\kappa u})^2(e^{2\kappa(t\wedge u)}-1) + \frac{\gamma^2}{4\kappa}(1-e^{2\kappa((t\wedge u)-(t\vee u))})] \\ &\quad \exp[\lambda\int_0^{t\wedge u}(e^{\frac{\sigma_J^2}{2}(h(s)-\frac{1}{2})^2-\frac{\sigma_J^2}{8}}-1)ds \\ &\quad +\lambda\int_{t\wedge u}^{t\vee u}(e^{\frac{\sigma_J^2}{2}(h'(s)-\frac{1}{2})^2-\frac{\sigma_J^2}{8}}-1)ds], \end{aligned} \tag{5.10}$$

where $h(s) = e^{-\kappa(t-s)} + e^{-\kappa(u-s)}$ and $h'(s) = e^{-\kappa(t\vee u-s)}$.

Therefore,

$$\begin{aligned}E\sigma_R^4 &= \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_u^2 du dt \\ &= \frac{1}{T^2}\exp(2\alpha-\frac{3\gamma^2}{4\kappa})\int_0^T\int_0^T \exp[(\frac{\gamma^2}{2\kappa}-\alpha+\ln\sigma_0^2)(e^{-\kappa t}+e^{-\kappa u})\end{aligned}$$

$$-\frac{\gamma^2}{4\kappa}e^{2\kappa((t\wedge u)-(t\vee u))}+\frac{\gamma^2}{4\kappa}(e^{-\kappa t}+e^{-\kappa u})^2(e^{2\kappa(t\wedge u)}-1)$$
$$+\lambda\int_0^{t\wedge u}(e^{\frac{\sigma_J^2}{2}(h(s)-\frac{1}{2})^2-\frac{\sigma_J^2}{8}}-1)ds$$
$$+\lambda\int_{t\wedge u}^{t\vee u}(e^{\frac{\sigma_J^2}{2}(h'(s)-\frac{1}{2})^2-\frac{\sigma_J^2}{8}}-1)ds]dudt. \tag{5.11}$$

We don't go further to calculate $Var(\sigma_R^2)$ due to the long expressions that cannot be simplified. As long as we get the mean value of the cross product, it is always possible to approximate the fair value of volatility swaps via the convexity adjustment approximation $K_{vol}=E\sigma_R\approx\sqrt{E\sigma_R^2}-\frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}}$, where $Var(\sigma_R^2)$ can be induced from $E\sigma_R^4$ and $E\sigma_R^2$, which are given by (5.7) and (4.4), respectively.

Theoretically, it is possible to lay out closed-form approximations for volatility swaps' fair values within all SVMs with jumps, as long as explicit solutions of SVMs with jumps are given. However, calculation for expectation of the cross product is so challenging that we only discuss two relatively simple models in this chapter. It would be interesting to see if there are any more efficient approaches.

Chapter 6

Numerical Examples

So far, we have discussed the valuation of variance and volatility swaps for both continuous SVMs and SVMs with jumps. Analytical closed-form solutions are provided within all continuous SVMs and some SVMs with jumps. In this chapter, we lay out numerical examples where we apply results obtained from previous chapters to price variance and volatility swaps and analyze the impacts of parameters, especially of parameters relating to the jump term.

6.1 Calibration of GARCH(1,1) Model

6.1.1 Estimation of Parameters

In the working paper by Théoret, Zabré and Rostan (2002) [40], the fair value of volatility swaps based on the volatility of the S&P 60 Canada index with various maturities was shown within the framework of a continuous GARCH(1,1) model. We will take their results of calibration from five years of daily historical S&P 60 prices (from January 1997 to February 2002) and use them later.

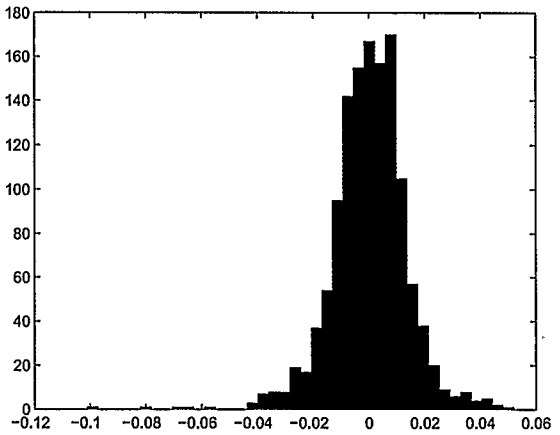
The statistics on log returns of the S&P 60 for five years (1300 observations from January 1997) are presented in Table 6.1 on the following page.

Observing the histogram of log returns of the S&P 60 Canada index (see Figure 6.1 on the following page), we easily observe leptokurtosis. We also note volatility clustering in the return series from Figure 6.2 (on page 62) which shows the log returns of the S&P 60 Canada index on a five year historical period. Therefore, it is reasonable to apply a GARCH(1,1) regression to the series to estimate parameters of a continuous

Table 6.1: Statistics on Log Returns of S&P 60 Canada Index

Series	Log returns S& P Cad
Observations	1300
Mean	0.000235
Median	0.000593
Maximum	0.051983
Minimum	-0.101108
Std. Dev.	0.013567
Skewness	-0.665741
Kurtosis	7.787327

Figure 6.1: Histogram of Log Returns of S&P 60 Canada Index



GARCH(1,1) model.

Recall that the continuous GARCH(1,1) model is described as follows:

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \gamma\sigma_t^2 dW_t. \tag{6.1}$$

According to regression results from [40], we list the estimation of parameters for discrete GARCH(1,1) model summarized in Table 6.2, where we have the following relationships:

$$\begin{aligned} \theta &= \frac{V}{dt} \\ \kappa &= \frac{1-\alpha-\beta}{dt} \end{aligned}$$

Figure 6.2: Time Series of Log Returns of S&P 60 Canada Index

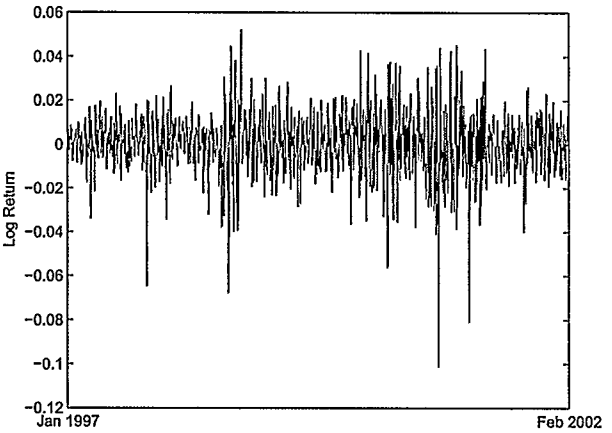


Table 6.2: Estimation of Parameters of GARCH (1,1) Process

ARCH (1,1) coefficient α	0.060445
GARCH (1,1) coefficient β	0.927264
Pearson Kurtosis ξ	7.787327
V	0.00020991
Long-term Volatility θ	0.05289724
κ	3.09733
γ	2.499827486

$$\gamma = \alpha\sqrt{\frac{\xi - 1}{dt}}$$

and choose $dt = \frac{1}{252} = 0.003968354$.

On the other hand, it seems that the continuous GARCH(1,1) is not good enough to capture the evolution of variance. Simply looking at Figure 6.2 where we find not only the mean-reverting property of log returns, but also lots of spikes, indicating the sharp change in the evolution of variance. Therefore, we consider describing variance by a GARCH(1,1) model with jumps, i.e. adding jump terms to (6.1):

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \gamma\sigma_t^2 dW_t + (J-1)\sigma_{t-}^2 dN_t \tag{6.2}$$

where N_t is a Poisson process with intensity $\lambda > 0$ and independent of W_t, and J is the

jump size with mean $A = EJ$.

To estimate the intensity λ, we first need to define jumps and thus set a criterion: if the absolute value of a log return is greater than three times of the standard deviation of log returns, then we define it as a jump (see [11]).

By this criterion, we find 15 jumps over the five-year period, and therefore, the intensity λ can be described as the frequency of jumps, defined as $\lambda = \frac{N_t}{t}$ (see [1]). In this example, we take $\lambda = \frac{15}{5} = 3$.

As to the jump size, we assume that J follows a log-normal distribution, i.e. $\ln J \sim N(-\frac{\sigma_J^2}{2}, \sigma_J^2)$, with $A = EJ = 1$. In this example, we take $\ln J \sim N(-0.25, 0.5)$.

6.1.2 Simulation of GARCH(1,1) Model

In this section, we use parameters estimated above to simulate a continuous GARCH(1,1) process and a GARCH(1,1) model with jumps, where we take the initial variance $\sigma_0^2 = 0.05$, close to the long-term mean in the continuous case.

By Monte Carlo simulation, we obtain Figure 6.3 and Figure 6.4.

Figure 6.3: Evolution of Variance

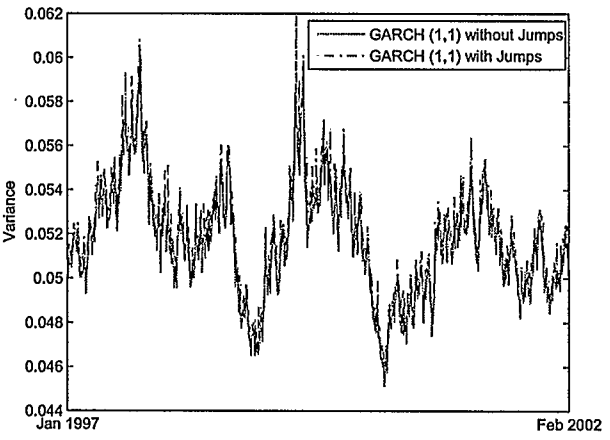
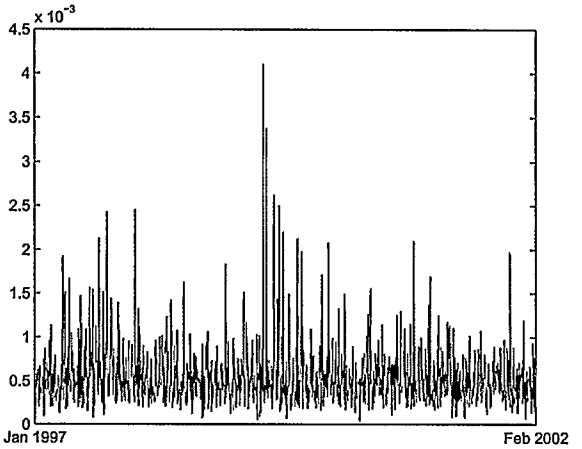


Figure 6.3 shows that the path for the continuous GARCH(1,1) process is close to the

Figure 6.4: Differences between GARCH(1,1) with and without jumps



path for the GARCH(1,1) process with jumps. This is due to our assumption that the mean jump size $A = EJ = 1$. However, we can still figure out when jumps happen by looking at Figure 6.4 which shows the difference between two paths at the same specific times.

6.2 Variance Swaps for the GARCH(1,1) Model

Now we consider the valuation of variance swaps for the GARCH(1,1) model, with the assumptions and estimations of parameters as discussed in Section 6.1.

Recall that the fair value for variance swaps within continuous GARCH (1,1) model is given by

$$K_{var} = E\sigma_R^2 = \frac{\sigma_0^2 - \theta}{\kappa T}(1 - e^{-\kappa T}) + \theta, \tag{6.3}$$

and that the fair value for variance swaps within GARCH (1,1) with jumps is given by

$$\begin{aligned} K_{var} &= E\sigma_R^2 \\ &= \frac{1}{T}\left(\frac{\sigma_0^2}{A\lambda - \lambda - \kappa} + \frac{\kappa\theta}{(A\lambda - \lambda - \kappa)^2}\right)\left(e^{(A\lambda - \lambda - \kappa)T} - 1\right) - \frac{\kappa\theta}{A\lambda - \lambda - \kappa}. \end{aligned} \tag{6.4}$$

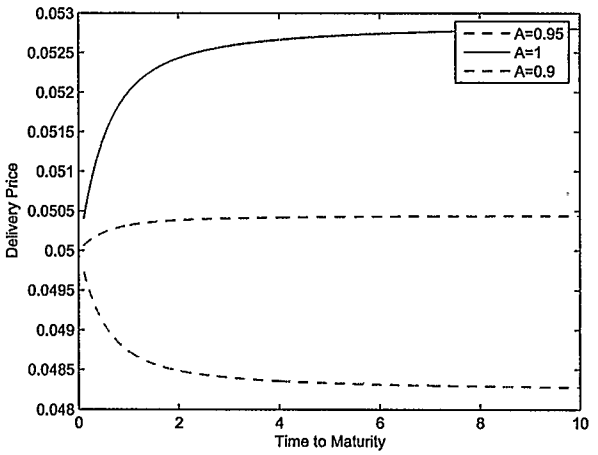
Since if we choose the mean of the jumps' sizes $A = 1$ then (6.4) reduces to (6.3), we will choose a different value for A to see the impact of jumps on the fair value for variance swaps. Values of parameters needed for this valuation are summarized in Table 6.3.

Table 6.3: Parameters for Valuation of Variance Swaps within GARCH (1,1)

θ	0.05289724
κ	3.09733
σ_0^2	0.05
λ	3
A	0.9,1.0,1,1

According to (6.3) and (6.4), we obtain the fair value of variance swaps with maturities up to 10 years shown in Figure 6.5 and Figure 6.6.

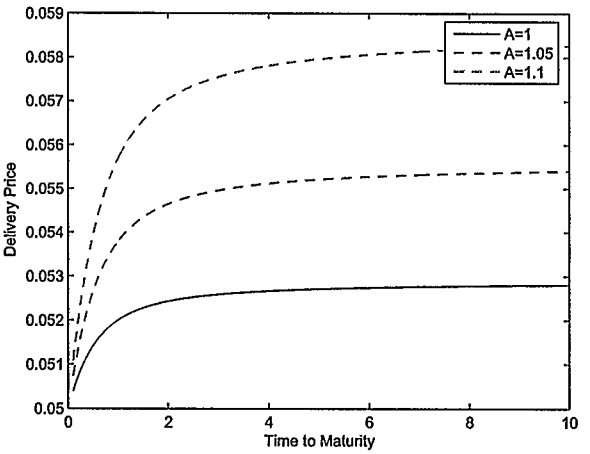
Figure 6.5: Delivery Price for Variance Swaps with $A \leq 1$



From Figure 6.5, we find that if the mean of jumps $A < 1$, then the lower the value of A, the lower the delivery price for variance swaps. As the value for A decreases, the convexity of the curve of delivery price with maturity T changes.

Comparatively, if we look at Figure 6.6, we find that if the mean of jumps $A > 1$, then the larger value of A, the higher delivery price for variance swaps. As the value for

Figure 6.6: Delivery Price for Variance Swaps with $A \geq 1$



A increases, the convexity of the curve of delivery price with maturity T won't change.

6.3 Volatility Swaps for the Continuous Heston Model

In this section, we consider the valuation of volatility swaps for the continuous Heston model.

Recall that the fair value for volatility swaps within the continuous Heston model is given by

$$K_{vol} = E\sigma_R \approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}}, \tag{6.5}$$

where

$$E\sigma_R^2 = \frac{\sigma_0^2 - \theta^2}{\kappa T}(1 - e^{-\kappa T}) + \theta^2, \tag{6.6}$$

$$\begin{aligned} Var(\sigma_R^2) &= \frac{\gamma^2 e^{-2\kappa T}}{2\kappa^3 T^2}[(2e^{2\kappa T} - 4e^{\kappa T}\kappa T - 2)(\sigma_0^2 - \theta^2) \\ &+ (2e^{2\kappa T}\kappa T - 3e^{2\kappa T} + 4e^{\kappa T} - 1)\theta^2], \end{aligned} \tag{6.7}$$

and also note that the last term in (6.5) is called convexity adjustment.

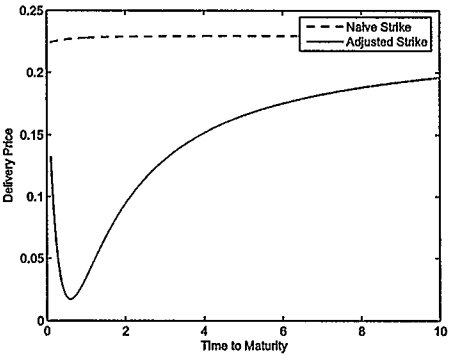
We take values of parameters summarized in Table 6.4 on the following page.

According to (6.5), (6.6) and (6.7), we obtain the fair value of volatility swaps with maturities up to 10 years, shown in Figure 6.7, as well as the convexity adjustment, shown in Figure 6.8.

Table 6.4: Parameters for Valuation of Volatility Swaps within Continuous Heston Model

θ^2	0.05289724
κ	3.09733
σ_0^2	0.05
γ	2.499827486

Figure 6.7: Delivery Price for Volatility Swaps

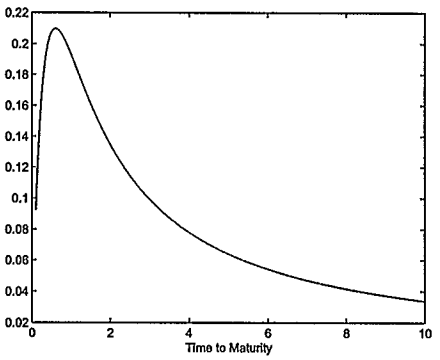


If we disregard the left tail of the adjusted strike curve in Figure 6.7, we find that as maturity time increases, the adjusted delivery price approaches the naive strike, which indicates that the impact of the convexity adjustment decreases with maturity. We can confirm this by looking at Figure 6.8 below, where the convexity adjustment decreases with maturity.

6.4 Variance and Volatility Swaps for Hull-White Model

In this section, we will take a look at fair values of variance and volatility swaps within both the continuous Hull-White model and the Hull-White model with jumps.

Figure 6.8: Convexity Adjustment



Recall that the fair value of variance swaps for the continuous Hull-White model and the Hull-White model with jumps are given by, respectively,

$$K_{var-nojump} = \frac{\sigma_0^2}{\alpha T}(e^{\alpha T} - 1), \tag{6.8}$$

$$K_{var-withjump} = \frac{\sigma_0^2}{(A\lambda - \lambda + \alpha)T}(e^{(A\lambda-\lambda+\alpha)T} - 1). \tag{6.9}$$

The fair value of volatility swaps for the continuous Hull-White model and the Hull-White model with jumps are given by, respectively,

$$K_{vol-nojump} \approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}}, \tag{6.10}$$

where

$$E\sigma_R^2 = \frac{\sigma_0^2}{\alpha T}(e^{\alpha T} - 1), \tag{6.11}$$

$$\begin{aligned} Var(\sigma_R^2) &= \frac{\sigma_0^4}{\alpha T^2(\alpha+\gamma^2)(2\alpha+\gamma^2)}[2\alpha e^{(2\alpha+\gamma^2)T} - \frac{(\alpha+\gamma^2)(2\alpha+\gamma^2)}{\alpha}e^{2\alpha T} \\ &+ \frac{2\gamma^2}{\alpha}(2\alpha+\gamma^2)e^{\alpha T} - \frac{\gamma^2}{\alpha}(\alpha+\gamma^2)], \end{aligned} \tag{6.12}$$

and

$$K_{vol-withjump} \approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8(E\sigma_R^2)^{\frac{3}{2}}}, \tag{6.13}$$

where

$$E\sigma_R^2 = \frac{\sigma_0^2}{(A\lambda-\lambda+\alpha)T}(e^{(A\lambda-\lambda+\alpha)T}-1), \tag{6.14}$$

$$Var(\sigma_R^2) = \frac{\sigma_0^4}{T^2}[\frac{2}{m(m+n)}e^{(m+n)T}-\frac{1}{n^2}e^{2nT}+2(\frac{1}{n^2}-\frac{1}{mn})e^{nT} + \frac{2}{n(m+n)}-\frac{1}{n^2}], \tag{6.15}$$

where $m=\alpha+\gamma^2+\lambda(B-A)$ and $n=\alpha+\lambda(A-1)$ with $A=EJ$ and $B=EJ^2$.

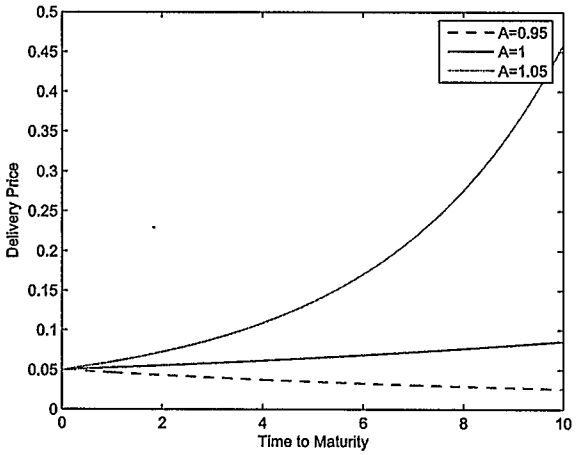
For simplicity, we take values of parameters shown in Table 6.5.

Table 6.5: Parameters for Valuation within Hull-White Model

α	0.1
σ_0^2	0.05
λ	5
γ	0.2

In order to analyze the impact of jumps, we choose different values for A and B. Using (6.8) and (6.9), we obtain the fair value of variance swaps with maturities up to 10 years in Figure 6.9.

Figure 6.9: Delivery Price for Variance Swaps

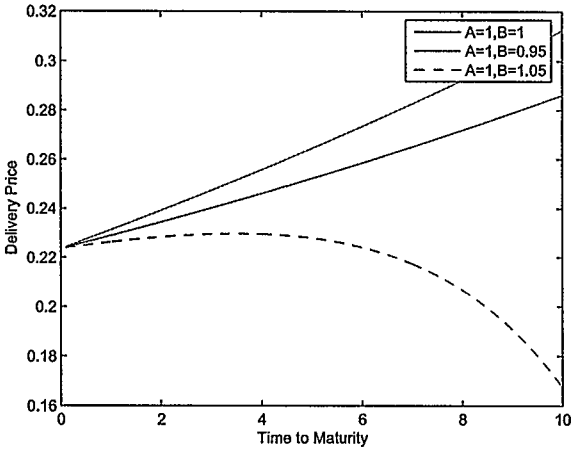


We can see from Figure 6.9 that the delivery price of variance swaps is negatively related to the mean of jumps A. In other words, the larger the value of A, the lower the

delivery price of variance swaps. If A is large enough, the delivery price will not increase any more as time increases.

Using (6.10) and (6.13), we get the fair value of volatility swaps with maturities up to 10 years in Figure 6.10.

Figure 6.10: Delivery Price for Volatility Swaps



As we can see, the mean of jump size squared will greatly effect the delivery price of volatility swaps. The smaller of value B, the higher delivery price it will be. If B is large enough, delivery price won't increase with time series and will decrease with an acceleration.

To summarize these models, the impact of jumps on delivery prices of variance and volatility swaps varies. One thing is clear, however, as we can see from the examples above, jumps play an important role in the pricing of variance and volatility swaps, and we should pay attention to it.

Chapter 7

Conclusions

Throughout this thesis, we have been concerned with the valuation of variance and volatility swaps for stochastic volatility models (SVMs) with and without jumps. The history for variance and volatility swaps is not long, but they are now more widely used than ever before, making it urgent to derive explicit formulae to price swaps.

In the thesis, we examined seven popular continuous SVMs and provided analytical closed-form solutions for fair values of both variance and volatility swaps. It is not hard to note that almost all the models we considered show the property of mean-reversion, which satisfies expectations from historical data. The only exception we considered is the Hull-White model, which is actually a geometric Brownian process.

However, the mean-reversion property cannot capture the evolution of volatility completely, so we extended continuous SVMs to SVMs with jumps. Under SVMs with jumps, we derived closed-form formulae for delivery price of variance swaps. As to the valuation of volatility swaps, we worked this out for only some of these models, even though explicit formulae for variance were shown for all SVMs with jumps we considered, except the Heston model. Note that we obtained closed-form formulae of variance and volatility swaps for both the continuous Hull-White model and the Hull-White model with jumps, and for both the continuous Schwartz's one-factor model and Schwartz's one-factor model with jumps.

We also compared results from continuous SVMs with those from SVMs with jumps. We found that in most cases, just as we had anticipated, if the mean value of jump sizes is set to be 1, meaning no jumps in average, then there is no difference between results from

continuous SVMs and those from the corresponding SVMs with jumps. The exception we considered is Schwartz's one-factor model.

The last part of this thesis is devoted to numerical examples, where we see the impact of jumps on swaps' fair values for different types of SVMs. We cannot simply summarize how the behavior would affect swaps' fair values, and we can only say with certainty that the behavior of jumps plays an important role in the valuation of variance and volatility swaps.

Besides what we have done in this thesis, there is still a long way to go for the pricing of variance and volatility swaps. The valuation of volatility swaps is still difficult within the framework of SVMs. As mentioned in Chapter 5, even though Brockhaus and Long (2000) introduced the convexity adjustment to approximate volatility swaps' fair values, the calculation of the convexity adjustment is quite complicated. It would be interesting to see any efficient approaches for this problem. Moreover, another difficult point lies in the calibration of jump-diffusion SVMs, which we didn't discuss much in this thesis.

Appendix A

Proof of Lemmata in Chapter Two

A.1 Lemma 2.2-1

Proof for Lemma 2.2-1 as follows:

Assume that $t > u$, then

$$\begin{aligned} E(\int_0^t e^{\alpha s} dW_s \int_0^u e^{\alpha s} dW_s) &= E[(\int_0^u e^{\alpha s} dW_s + \int_u^t e^{\alpha s} dW_s) \int_0^u e^{\alpha s} dW_s] \\ &= E[(\int_0^u e^{\alpha s} dW_s)^2] + E(\int_0^u e^{\alpha s} dW_s \int_u^t e^{\alpha s} dW_s) \\ &= \int_0^u e^{2\alpha s} ds + 0 \\ &= \frac{1}{2\alpha}(e^{2\alpha u} - 1), \end{aligned}$$

where we apply the Itô isometry $E[(\int_0^u f(s)dW_s)^2] = E\int_0^u f^2(s)ds$ and the property of independence of Wiener process which yields $E(\int_0^u e^{\alpha s} dW_s \int_u^t e^{\alpha s} dW_s) = 0$.

Similarly, if we assume $t < u$, then

$$\begin{aligned} E(\int_0^t e^{\alpha s} dW_s \int_0^u e^{\alpha s} dW_s) &= E[(\int_0^t e^{\alpha s} dW_s + \int_t^u e^{\alpha s} dW_s) \int_0^t e^{\alpha s} dW_s] \\ &= E[(\int_0^t e^{\alpha s} dW_s)^2] + E(\int_0^t e^{\alpha s} dW_s \int_t^u e^{\alpha s} dW_s) \\ &= \int_0^t e^{2\alpha s} ds + 0 \\ &= \frac{1}{2\alpha}(e^{2\alpha t} - 1). \end{aligned}$$

Therefore, we have

$$E(\int_0^t e^{\alpha s} dW_s \int_0^u e^{\alpha s} dW_s) = \frac{1}{2\alpha}[e^{2\alpha(t \wedge u)} - 1].$$

A.2 Lemma 2.2-2

Proof for Lemma 2.2-2 as follows:

Assume that $t > u$, then

$$\begin{aligned}
& E[\exp(\int_0^t e^{-\kappa(t-s)}dW_s + \int_0^u e^{-\kappa(u-s)}dW_s)] \\
= \; & E[\exp(e^{-\kappa t}\int_0^t e^{\kappa s}dW_s + e^{-\kappa u}\int_0^u e^{\kappa s}dW_s)] \\
= \; & E[\exp((e^{-\kappa t}+e^{-\kappa u})\int_0^u e^{\kappa s}dW_s + e^{-\kappa t}\int_u^t e^{\kappa s}dW_s)] \\
= \; & E[\exp((e^{-\kappa t}+e^{-\kappa u})\int_0^u e^{\kappa s}dW_s)]E[\exp(e^{-\kappa t}\int_u^t e^{\kappa s}dW_s)] \\
= \; & \exp[\frac{1}{2}(e^{-\kappa t}+e^{-\kappa u})^2\frac{1}{2\kappa}(e^{2\kappa u}-1)]\exp[\frac{1}{2}e^{-2\kappa t}\frac{1}{2\kappa}(e^{2\kappa t}-e^{2\kappa u})] \\
= \; & \exp[\frac{1}{4\kappa}(e^{-\kappa t}+e^{-\kappa u})^2(e^{2\kappa u}-1)+\frac{1}{4\kappa}(1-e^{2\kappa(u-t)})],
\end{aligned}$$

where we apply the property $E[\exp(\gamma\int_0^t f(s)dW_s)] = \exp[\frac{\gamma^2}{2}\int_0^t f^2(s)ds]$.

Similarly, if we assume $t < u$, then

$$\begin{aligned}
& E[\exp(\int_0^t e^{-\kappa(t-s)}dW_s + \int_0^u e^{-\kappa(u-s)}dW_s)] \\
= \; & E[\exp(e^{-\kappa t}\int_0^t e^{\kappa s}dW_s + e^{-\kappa u}\int_0^u e^{\kappa s}dW_s)] \\
= \; & E[\exp((e^{-\kappa t}+e^{-\kappa u})\int_0^t e^{\kappa s}dW_s + e^{-\kappa u}\int_t^u e^{\kappa s}dW_s)] \\
= \; & E[\exp((e^{-\kappa t}+e^{-\kappa u})\int_0^t e^{\kappa s}dW_s)]E[\exp(e^{-\kappa u}\int_t^u e^{\kappa s}dW_s)] \\
= \; & \exp[\frac{1}{2}(e^{-\kappa t}+e^{-\kappa u})^2\frac{1}{2\kappa}(e^{2\kappa t}-1)]\exp[\frac{1}{2}e^{-2\kappa u}\frac{1}{2\kappa}(e^{2\kappa u}-e^{2\kappa t})] \\
= \; & \exp[\frac{1}{4\kappa}(e^{-\kappa t}+e^{-\kappa u})^2(e^{2\kappa t}-1)+\frac{1}{4\kappa}(1-e^{2\kappa(t-u)})].
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& E[\exp(\int_0^t e^{-\kappa(t-s)}dW_s + \int_0^u e^{-\kappa(u-s)}dW_s)] \\
= \; & \exp[\frac{1}{4\kappa}(1-e^{2\kappa(t\wedge u - t\vee u)})+\frac{1}{4\kappa}(e^{-\kappa t}+e^{-\kappa u})^2(e^{2\kappa(t\wedge u)}-1)].
\end{aligned}$$

A.3 Lemma 2.2-3

Proof for Lemma 2.2-3 as follows:

Assume that $t > u$, then

$$\begin{aligned}
& E[\int_0^t e^{(\kappa+\frac{\gamma^2}{2})s+\gamma(W_t+W_u-W_s)}ds] \\
= & \int_0^u e^{(\kappa+\frac{\gamma^2}{2})s}E[e^{\gamma(W_t-W_u+2(W_u-W_s)+W_s)}]ds \\
& + \int_u^t e^{(\kappa+\frac{\gamma^2}{2})s}E[e^{\gamma(W_t-W_s+W_u)}]ds \\
= & \int_0^u e^{(\kappa+\frac{\gamma^2}{2})s}e^{\frac{\gamma^2}{2}(t-u)+2\gamma^2(u-s)+\frac{\gamma^2}{2}s}ds \\
& + \int_u^t e^{(\kappa+\frac{\gamma^2}{2})s}e^{\frac{\gamma^2}{2}(t-s+u)}ds \\
= & e^{\frac{\gamma^2}{2}t+\frac{3\gamma^2}{2}u}\int_0^u e^{(\kappa-\gamma^2)s}ds \\
& + e^{\frac{\gamma^2}{2}(t+u)}\int_u^t e^{\kappa s}ds \\
= & \frac{1}{\kappa-\gamma^2}(e^{(\kappa-\gamma^2)u}-1)e^{\frac{\gamma^2}{2}t+\frac{3\gamma^2}{2}u}+\frac{1}{\kappa}(e^{\kappa t}-e^{\kappa u})e^{\frac{\gamma^2}{2}(t+u)},
\end{aligned}$$

where we apply the property $E[\exp(\gamma\int_0^t f(s)dW_s)] = \exp[\frac{\gamma^2}{2}\int_0^t f^2(s)ds]$.

Similarly, if we assume $t < u$, then

$$\begin{aligned}
& E[\int_0^t e^{(\kappa+\frac{\gamma^2}{2})s+\gamma(W_t+W_u-W_s)}ds] \\
= & \int_0^t e^{(\kappa+\frac{\gamma^2}{2})s}E[e^{\gamma(W_u-W_t+2(W_t-W_s)+W_s)}]ds \\
= & \int_0^t e^{(\kappa+\frac{\gamma^2}{2})s}e^{\frac{\gamma^2}{2}(u-t)+2\gamma^2(t-s)+\frac{\gamma^2}{2}s}ds \\
= & e^{\frac{\gamma^2}{2}u+\frac{3\gamma^2}{2}t}\int_0^t e^{(\kappa-\gamma^2)s}ds \\
= & \frac{1}{\kappa-\gamma^2}(e^{(\kappa-\gamma^2)t}-1)e^{\frac{\gamma^2}{2}u+\frac{3\gamma^2}{2}t}.
\end{aligned}$$

Therefore, we obtain

$$E[\int_0^t e^{(\kappa+\frac{\gamma^2}{2})s+\gamma(W_t+W_u-W_s)}ds]$$

$$= \frac{1}{\kappa-\gamma^2}(e^{(\kappa-\gamma^2)(t\wedge u)}-1)e^{\frac{\gamma^2}{2}(t\vee u)+\frac{3\gamma^2}{2}(t\wedge u)} + I_{\{t>u\}}\frac{1}{\kappa}(e^{\kappa t}-e^{\kappa u})e^{\frac{\gamma^2}{2}(t+u)},$$

where $I_{\{t>u\}}$ is an indicate function taking value 1 if $t > u$ and 0 if $t \leq u$.

A.4 Lemma 2.2-5

Proof for Lemma 2.2-5 as follows:

Assume that $t > u$, then

$$\begin{aligned}
& E[\int_0^t\int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)+\gamma(W_t+W_u-W_x-W_y)}dydx] \\
= & \int_0^u\int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t+W_u-W_x-W_y)}]dydx \\
& + \int_u^t\int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t+W_u-W_x-W_y)}]dydx \\
= & \int_0^u\int_0^x e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t-W_u+2(W_u-W_x)+W_x-W_y)}]dydx \\
& + \int_0^u\int_x^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t-W_u+2(W_u-W_y)+W_y-W_x)}]dydx \\
& + \int_u^t\int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t-W_x+W_u-W_y)}]dydx \\
= & \int_0^u\int_0^x e^{(\kappa+\frac{\gamma^2}{2})(x+y)}e^{\frac{\gamma^2}{2}(t-u)+2\gamma^2(u-x)+\frac{\gamma^2}{2}(x-y)}dydx \\
& + \int_0^u\int_x^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)}e^{\frac{\gamma^2}{2}(t-u)+2\gamma^2(u-y)+\frac{\gamma^2}{2}(y-x)}dydx \\
& + \int_u^t\int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)}e^{\frac{\gamma^2}{2}(t-x+u-y)}dydx \\
= & e^{\frac{\gamma^2}{2}t+\frac{3\gamma^2}{2}u}\int_0^u\int_0^x e^{(\kappa-\gamma^2)x}e^{\kappa y}dydx \\
& + e^{\frac{\gamma^2}{2}t+\frac{3\gamma^2}{2}u}\int_0^u\int_x^u e^{(\kappa-\gamma^2)y}e^{\kappa x}dydx \\
& + e^{\frac{\gamma^2}{2}t+\frac{\gamma^2}{2}u}\int_u^t\int_0^u e^{\kappa y}e^{\kappa x}dydx \\
= & e^{\frac{\gamma^2}{2}t+\frac{\gamma^2}{2}u}[e^{2\kappa u}(\frac{1}{\kappa(2\kappa-\gamma^2)}+\frac{1}{\kappa(\kappa-\gamma^2)}-\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa^2})
\end{aligned}$$

$$
\begin{aligned}
&-e^{\kappa u}\left(\frac{2}{\kappa(\kappa-\gamma^2)}-\frac{1}{\kappa^2}\right)\\
&+e^{\gamma^2 u}\left(\frac{1}{\kappa(\kappa-\gamma^2)}+\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa(2\kappa-\gamma^2)}\right)\\
&+\frac{1}{\kappa^2}(e^{\kappa(t+u)}-e^{\kappa t})],
\end{aligned}
$$

where we apply the property $E[\exp(\gamma\int_0^t f(s)dW_s)]=\exp[\frac{\gamma^2}{2}\int_0^t f^2(s)ds]$.

Similarly, if we assume $t<u$, then

$$
\begin{aligned}
& E[\int_0^t\int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)+\gamma(W_t+W_u-W_x-W_y)}dydx]\\
=\ & E[\int_0^u\int_0^t e^{(\kappa+\frac{\gamma^2}{2})(x+y)+\gamma(W_t+W_u-W_x-W_y)}dxdy]\\
=\ & \int_0^t\int_0^t e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t+W_u-W_x-W_y)}]dxdy\\
& +\int_t^u\int_0^t e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t+W_u-W_x-W_y)}]dxdy\\
=\ & \int_0^t\int_0^y e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_u-W_t+2(W_t-W_y)+W_y-W_x)}]dxdy\\
& +\int_0^t\int_y^t e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_u-W_t+2(W_t-W_x)+W_x-W_y)}]dxdy\\
& +\int_t^u\int_0^t e^{(\kappa+\frac{\gamma^2}{2})(x+y)}E[e^{\gamma(W_t-W_x+W_u-W_y)}]dxdy\\
=\ & \int_0^t\int_0^y e^{(\kappa+\frac{\gamma^2}{2})(x+y)}e^{\frac{\gamma^2}{2}(u-t)+2\gamma^2(t-y)+\frac{\gamma^2}{2}(y-x)}dxdy\\
& +\int_0^t\int_y^t e^{(\kappa+\frac{\gamma^2}{2})(x+y)}e^{\frac{\gamma^2}{2}(u-t)+2\gamma^2(t-x)+\frac{\gamma^2}{2}(x-y)}dxdy\\
& +\int_t^u\int_0^t e^{(\kappa+\frac{\gamma^2}{2})(x+y)}e^{\frac{\gamma^2}{2}(t-x+u-y)}dxdy\\
=\ & e^{\frac{\gamma^2}{2}u+\frac{3\gamma^2}{2}t}\int_0^t\int_0^y e^{(\kappa-\gamma^2)y}e^{\kappa x}dxdy\\
& +e^{\frac{\gamma^2}{2}u+\frac{3\gamma^2}{2}t}\int_0^t\int_y^t e^{(\kappa-\gamma^2)x}e^{\kappa y}dxdy\\
& +e^{\frac{\gamma^2}{2}t+\frac{\gamma^2}{2}u}\int_t^u\int_0^t e^{\kappa y}e^{\kappa x}dxdy\\
=\ & e^{\frac{\gamma^2}{2}t+\frac{\gamma^2}{2}u}[e^{2\kappa t}\left(\frac{1}{\kappa(2\kappa-\gamma^2)}+\frac{1}{\kappa(\kappa-\gamma^2)}-\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-e^{\kappa t}\left(\frac{2}{\kappa(\kappa-\gamma^2)}-\frac{1}{\kappa^2}\right)\\
&+e^{\gamma^2 t}\left(\frac{1}{\kappa(\kappa-\gamma^2)}+\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa(2\kappa-\gamma^2)}\right)\\
&+\frac{1}{\kappa^2}\left(e^{\kappa(t+u)}-e^{\kappa u}\right)].
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&E[\int_0^t\int_0^u e^{(\kappa+\frac{\gamma^2}{2})(x+y)+\gamma(W_t+W_u-W_x-W_y)}dydx]\\
=\;& e^{\frac{\gamma^2}{2}t+\frac{\gamma^2}{2}u}[e^{2\kappa(t\wedge u)}\left(\frac{1}{\kappa(2\kappa-\gamma^2)}+\frac{1}{\kappa(\kappa-\gamma^2)}-\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa^2}\right)\\
&-e^{\kappa(t\wedge u)}\left(\frac{2}{\kappa(\kappa-\gamma^2)}-\frac{1}{\kappa^2}\right)\\
&+e^{\gamma^2(t\wedge u)}\left(\frac{1}{\kappa(\kappa-\gamma^2)}+\frac{1}{(\kappa-\gamma^2)(2\kappa-\gamma^2)}-\frac{1}{\kappa(2\kappa-\gamma^2)}\right)\\
&+\frac{1}{\kappa^2}\left(e^{\kappa(t+u)}-e^{\kappa(t\vee u)}\right)].
\end{aligned}
$$

Appendix B

Proof of Propositions in Chapter Three

B.1 Proposition 3.2-1

Proof for Proposition 3.2-1 as follows:

Assume that for $t \in [\tau_k, \tau_{k+1})$, the following is true:

$$\begin{aligned}\sigma_t^2 &= (J_1 J_2 \dots J_k)\sigma_0^2 e^{-\alpha t} + \sum_{n=0}^{k-1}(J_{n+1}J_2 \dots J_k)\gamma e^{-\alpha(t-\tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\ &+ \gamma e^{-\alpha(t-\tau_k)} \int_{\tau_k}^{t} e^{\alpha s} dW_s.\end{aligned}$$

In particular,

$$\begin{aligned}\sigma_{\tau_{k+1}-}^2 &= (J_1 J_2 \dots J_k)\sigma_0^2 e^{-\alpha \tau_{k+1}-} + \sum_{n=0}^{k-1}(J_{n+1}J_2 \dots J_k)\gamma e^{-\alpha(\tau_{k+1}--\tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\ &+ \gamma e^{-\alpha(\tau_{k+1}--\tau_k)} \int_{\tau_k}^{\tau_{k+1}-} e^{\alpha s} dW_s.\end{aligned}$$

Then, for $t = \tau_{k+1}$, we get

$$\begin{aligned}\sigma_{\tau_{k+1}}^2 &= (J_1 J_2 \dots J_k J_{k+1})\sigma_0^2 e^{-\alpha \tau_{k+1}-} \\ &+ J_{k+1}\sum_{n=0}^{k-1}(J_{n+1}J_2 \dots J_k)\gamma e^{-\alpha(\tau_{k+1}--\tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\ &+ J_{k+1}\gamma e^{-\alpha(\tau_{k+1}--\tau_k)} \int_{\tau_k}^{\tau_{k+1}-} e^{\alpha s} dW_s.\end{aligned}$$

Further, for $t \in [\tau_{k+1}, \tau_{k+2})$, we obtain

$$\begin{aligned}\sigma_t^2 &= \sigma_{\tau_{k+1}}^2 e^{-\alpha(t-\tau_{k+1})} + \gamma e^{-\alpha(t-\tau_{k+1})} \int_{\tau_{k+1}}^{t} e^{\alpha s} dW_s \\ &= (J_1 J_2 \dots J_k J_{k+1})\sigma_0^2 e^{-\alpha t} + J_{k+1}\sum_{n=0}^{k-1}(J_{n+1}J_2 \dots J_k)\gamma e^{-\alpha(t-\tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s\end{aligned}$$

$$
\begin{aligned}
&+ \quad J_{k+1}\gamma e^{-\alpha(t-\tau_k)}\int_{\tau_k}^{\tau_{k+1}} e^{\alpha s}dW_s + \gamma e^{-\alpha(t-\tau_{k+1})}\int_{\tau_{k+1}}^{t} e^{\alpha s}dW_s \\
&= \quad (J_1J_2\ldots J_kJ_{k+1})\sigma_0^2e^{-\alpha t} + \sum_{n=0}^{k}(J_{n+1}J_2\ldots J_{k+1})\gamma e^{-\alpha(t-\tau_n)}\int_{\tau_n}^{\tau_{n+1}} e^{\alpha s}dW_s \\
&+ \quad \gamma e^{-\alpha(t-\tau_{k+1})}\int_{\tau_{k+1}}^{t} e^{\alpha s}dW_s.
\end{aligned}
$$

Therefore, we conclude with

$$
\begin{aligned}
\sigma_t^2 &= (J_1J_2\ldots J_{N_t})\sigma_0^2e^{-\alpha t} + \sum_{n=0}^{N_t-1}(J_{n+1}J_2\ldots J_{N_t})\gamma e^{-\alpha(t-\tau_n)}\int_{\tau_n}^{\tau_{n+1}} e^{\alpha s}dW_s \\
&+ \gamma e^{-\alpha(t-\tau_{N_t})}\int_{\tau_{N_t}}^{t} e^{\alpha s}dW_s.
\end{aligned}
$$

B.2 Proposition 3.3-1

Proof for Proposition 3.3-1 as follows:

Assume that for $t \in [\tau_k, \tau_{k+1})$, the following is true:

$$
\begin{aligned}
\sigma_t^2 &= (J_1J_2\ldots J_k)(\sigma_0^2-\beta)e^{-\alpha t} + \beta\sum_{n=0}^{k-1}(J_{n+2}J_2\ldots J_k)(J_{n+1}-1)e^{-\alpha(t-\tau_{n+1})} \\
&+ \sum_{n=0}^{k-1}(J_{n+1}J_2\ldots J_k)\gamma e^{-\alpha(t-\tau_n)}\int_{\tau_n}^{\tau_{n+1}} e^{\alpha s}dW_s \\
&+ \gamma e^{-\alpha(t-\tau_k)}\int_{\tau_k}^{t} e^{\alpha s}dW_s + \beta.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\sigma_{\tau_{k+1}-}^2 &= (J_1J_2\ldots J_k)(\sigma_0^2-\beta)e^{-\alpha\tau_{k+1}-} \\
&+ \beta\sum_{n=0}^{k-1}(J_{n+2}J_2\ldots J_k)(J_{n+1}-1)e^{-\alpha(\tau_{k+1}--\tau_{n+1})} \\
&+ \sum_{n=0}^{k-1}(J_{n+1}J_2\ldots J_k)\gamma e^{-\alpha(\tau_{k+1}--\tau_n)}\int_{\tau_n}^{\tau_{n+1}} e^{\alpha s}dW_s \\
&+ \gamma e^{-\alpha(\tau_{k+1}--\tau_k)}\int_{\tau_k}^{\tau_{k+1}-} e^{\alpha s}dW_s + \beta.
\end{aligned}
$$

Then, for $t = \tau_{k+1}$, we get

$$\begin{aligned}
\sigma^2_{\tau_{k+1}} &= (J_1 J_2 \dots J_k J_{k+1})(\sigma_0^2 - \beta) e^{-\alpha \tau_{k+1}-} \\
&+ J_{k+1}\beta \sum_{n=0}^{k-1} (J_{n+2} J_2 \dots J_k)(J_{n+1} - 1) e^{-\alpha(\tau_{k+1} - - \tau_{n+1})} \\
&+ J_{k+1} \sum_{n=0}^{k-1} (J_{n+1} J_2 \dots J_k) \gamma e^{-\alpha(\tau_{k+1} - - \tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\
&+ J_{k+1} \gamma e^{-\alpha(\tau_{k+1} - - \tau_k)} \int_{\tau_k}^{\tau_{k+1}-} e^{\alpha s} dW_s + J_{k+1}\beta.
\end{aligned}$$

Further, for $t \in [\tau_{k+1}, \tau_{k+2})$, we obtain

$$\begin{aligned}
\sigma_t^2 &= (\sigma^2_{\tau_{k+1}} - \beta) e^{-\alpha(t - \tau_{k+1})} + \beta + \gamma \int_{\tau_{k+1}}^{t} e^{-\alpha(t-s)} dW_s \\
&= (J_1 J_2 \dots J_k J_{k+1})(\sigma_0^2 - \beta) e^{-\alpha t} \\
&+ J_{k+1}\beta \sum_{n=0}^{k-1} (J_{n+2} J_2 \dots J_k)(J_{n+1} - 1) e^{-\alpha(t - \tau_{n+1})} \\
&+ J_{k+1} \sum_{n=0}^{k-1} (J_{n+1} J_2 \dots J_k) \gamma e^{-\alpha(t - \tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\
&+ J_{k+1} \gamma e^{-\alpha(t - \tau_k)} \int_{\tau_k}^{\tau_{k+1}-} e^{\alpha s} dW_s + (J_{k+1} - 1)\beta e^{-\alpha(t - \tau_{k+1})} \\
&+ \beta + \gamma \int_{\tau_{k+1}}^{t} e^{-\alpha(t-s)} dW_s \\
&= (J_1 J_2 \dots J_k J_{k+1})(\sigma_0^2 - \beta) e^{-\alpha t} \\
&+ \beta \sum_{n=0}^{k} (J_{n+2} J_2 \dots J_k J_{k+1})(J_{n+1} - 1) e^{-\alpha(t - \tau_{n+1})} \\
&+ \sum_{n=0}^{k} (J_{n+1} J_2 \dots J_k J_{k+1}) \gamma e^{-\alpha(t - \tau_n)} \int_{\tau_n}^{\tau_{n+1}} e^{\alpha s} dW_s \\
&+ \gamma e^{-\alpha(t - \tau_{k+1})} \int_{\tau_{k+1}}^{t} e^{\alpha s} dW_s + \beta.
\end{aligned}$$

Therefore, we conclude with

$$\sigma_t^2 = (J_1 J_2 \dots J_{N_t})(\sigma_0^2 - \beta) e^{-\alpha t} + \beta \sum_{n=0}^{N_t - 1} (J_{n+2} J_2 \dots J_{N_t})(J_{n+1} - 1) e^{-\alpha(t - \tau_{n+1})}$$

$$+ \sum_{n=0}^{N_t-1}(J_{n+1}J_2\ldots J_{N_t})\gamma e^{-\alpha(t-\tau_n)}\int_{\tau_n}^{\tau_{n+1}} e^{\alpha s}dW_s$$
$$+ \gamma e^{-\alpha(t-\tau_{N_t})}\int_{\tau_{N_t}}^{t} e^{\alpha s}dW_s + \beta.$$

B.3 Proposition 3.4-1

Proof for Proposition 3.4-1 as follows:

Assume that for $t \in [\tau_k, \tau_{k+1})$, the following is true:

$$\sigma_t^2 = (J_1J_2\ldots J_k)\sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})t + \gamma W_t].$$

In particular,

$$\sigma^2_{\tau_{k+1}-} = (J_1J_2\ldots J_k)\sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})\tau_{k+1-} + \gamma W_{\tau_{k+1}-}].$$

Then, for $t = \tau_{k+1}$, we get

$$\sigma^2_{\tau_{k+1}} = (J_1J_2\ldots J_kJ_{k+1})\sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})\tau_{k+1} + \gamma W_{\tau_{k+1}}].$$

Further, for $t \in [\tau_{k+1}, \tau_{k+2})$, we obtain

$$\begin{aligned}\sigma_t^2 &= \sigma^2_{\tau_{k+1}} \exp[(\alpha - \frac{\gamma^2}{2})(t-\tau_{k+1}) + \gamma(W_t - W_{\tau_{k+1}})] \\ &= (J_1J_2\ldots J_kJ_{k+1})\sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})t + \gamma W_t].\end{aligned}$$

Therefore, we conclude with

$$\sigma_t^2 = (J_1J_2\ldots J_{N_t})\sigma_0^2 \exp[(\alpha - \frac{\gamma^2}{2})t + \gamma W_t].$$

B.4 Proposition 3.6-1

Proof for Proposition 3.6-1 as follows:

Assume that for $t \in [\tau_k, \tau_{k+1})$, the following is true:

$$\begin{aligned}
\sigma_t^2 &= (J_1 J_2 \dots J_k)\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\
&+ \kappa\alpha_0 \sum_{n=0}^{k-1} [(J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)} e^{\gamma(W_t^1 - W_{\tau_n}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_n + \sigma W_{\tau_n}^2} \int_{\tau_n}^{\tau_{n+1}} e^{(\mu-\frac{\sigma^2}{2})s + \sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s - \gamma W_s^1} ds] \\
&+ \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_k)} e^{\gamma(W_t^1 - W_{\tau_k}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_k + \sigma W_{\tau_k}^2} \int_{\tau_k}^{t} e^{(\mu-\frac{\sigma^2}{2})s + \sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s - \gamma W_s^1} ds.
\end{aligned}$$

In particular,

$$\begin{aligned}
\sigma_{\tau_{k+1}-}^2 &= (J_1 J_2 \dots J_k)\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})\tau_{k+1}-} e^{\gamma W_{\tau_{k+1}-}^1} \\
&+ \kappa\alpha_0 \sum_{n=0}^{k-1} [(J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_n)} e^{\gamma(W_{\tau_{k+1}-}^1 - W_{\tau_n}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_n + \sigma W_{\tau_n}^2} \int_{\tau_n}^{\tau_{n+1}} e^{(\mu-\frac{\sigma^2}{2})s + \sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s - \gamma W_s^1} ds] \\
&+ \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_k)} e^{\gamma(W_{\tau_{k+1}-}^1 - W_{\tau_k}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_k + \sigma W_{\tau_k}^2} \int_{\tau_k}^{\tau_{k+1}-} e^{(\mu-\frac{\sigma^2}{2})s + \sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s - \gamma W_s^1} ds.
\end{aligned}$$

Then, for $t = \tau_{k+1}$, we get

$$\begin{aligned}
\sigma_{\tau_{k+1}}^2 &= (J_1 J_2 \dots J_k J_{k+1})\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})\tau_{k+1}-} e^{\gamma W_{\tau_{k+1}-}^1} \\
&+ J_{k+1}\kappa\alpha_0 \sum_{n=0}^{k-1} [(J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_n)} e^{\gamma(W_{\tau_{k+1}-}^1 - W_{\tau_n}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_n + \sigma W_{\tau_n}^2} \int_{\tau_n}^{\tau_{n+1}} e^{(\mu-\frac{\sigma^2}{2})s + \sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s - \gamma W_s^1} ds] \\
&+ J_{k+1}\kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_k)} e^{\gamma(W_{\tau_{k+1}-}^1 - W_{\tau_k}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_k + \sigma W_{\tau_k}^2} \int_{\tau_k}^{\tau_{k+1}-} e^{(\mu-\frac{\sigma^2}{2})s + \sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s - \gamma W_s^1} ds.
\end{aligned}$$

Further, for $t \in [\tau_{k+1}, \tau_{k+2})$, we obtain

$$\sigma_t^2 = \sigma_{\tau_{k+1}}^2 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})} e^{\gamma(W_t^1 - W_{\tau_{k+1}}^1)}$$

$$
\begin{aligned}
&+ \quad \kappa\alpha_{\tau_{k+1}} e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})} e^{\gamma(W_t^1 - W_{\tau_{k+1}}^1)} \int_{\tau_{k+1}}^{t} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds \\
&= \quad (J_1 J_2 \dots J_k J_{k+1}) \sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\
&+ \quad J_{k+1}\kappa\alpha_0 \sum_{n=0}^{k-1} [(J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)} e^{\gamma(W_t^1 - W_{\tau_n}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_n+\sigma W_{\tau_n}^2} \int_{\tau_n}^{\tau_{n+1}} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds] \\
&+ \quad J_{k+1}\kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_k)} e^{\gamma(W_t^1 - W_{\tau_k}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_k+\sigma W_{\tau_k}^2} \int_{\tau_k}^{\tau_{k+1}} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds \\
&+ \quad \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})} e^{\gamma(W_t^1 - W_{\tau_{k+1}}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_{k+1}+\sigma W_{\tau_{k+1}}^2} \int_{\tau_{k+1}}^{t} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds \\
&= \quad (J_1 J_2 \dots J_k J_{k+1}) \sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\
&+ \quad \kappa\alpha_0 \sum_{n=0}^{k} [(J_{n+1} J_2 \dots J_k J_{k+1}) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)} e^{\gamma(W_t^1 - W_{\tau_n}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_n+\sigma W_{\tau_n}^2} \int_{\tau_n}^{\tau_{n+1}} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds] \\
&+ \quad \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})} e^{\gamma(W_t^1 - W_{\tau_{k+1}}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_{k+1}+\sigma W_{\tau_{k+1}}^2} \int_{\tau_{k+1}}^{t} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds.
\end{aligned}
$$

Therefore, we conclude with

$$
\begin{aligned}
\sigma_t^2 \quad &= \quad (J_1 J_2 \dots J_{N_t}) \sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t} e^{\gamma W_t^1} \\
&+ \quad \kappa\alpha_0 \sum_{n=0}^{N_t-1} [(J_{n+1} J_2 \dots J_{N_t}) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)} e^{\gamma(W_t^1 - W_{\tau_n}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_n+\sigma W_{\tau_n}^2} \int_{\tau_n}^{\tau_{n+1}} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds] \\
&+ \quad \kappa\alpha_0 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{N_t})} e^{\gamma(W_t^1 - W_{\tau_{N_t}}^1)} \\
&\quad e^{(\mu-\frac{\sigma^2}{2})\tau_{N_t}+\sigma W_{\tau_{N_t}}^2} \int_{\tau_{N_t}}^{t} e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s^2} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s^1} ds.
\end{aligned}
$$

B.5 Proposition 3.7-1

Proof for Proposition 3.7-1 as follows:

Assume that for $t \in [\tau_k, \tau_{k+1})$, the following is true:

$$\begin{aligned}
\sigma_t^2 &= (J_1 J_2 \dots J_k)\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t+\gamma W_t} \\
&+ \kappa\theta \sum_{n=0}^{k-1} (J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)+\gamma(W_t - W_{\tau_n})} \int_{\tau_n}^{\tau_{n+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds \\
&+ \kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_k)+\gamma(W_t - W_{\tau_k})} \int_{\tau_k}^{t} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds.
\end{aligned}$$

In particular,

$$\begin{aligned}
\sigma_{\tau_{k+1}-}^2 &= (J_1 J_2 \dots J_k)\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})\tau_{k+1}-+\gamma W_{\tau_{k+1}-}} \\
&+ \kappa\theta \sum_{n=0}^{k-1} (J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_n)+\gamma(W_{\tau_{k+1}-} - W_{\tau_n})} \int_{\tau_n}^{\tau_{n+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds \\
&+ \kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_k)+\gamma(W_{\tau_{k+1}-} - W_{\tau_k})} \int_{\tau_k}^{\tau_{k+1}-} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds.
\end{aligned}$$

Then, for $t = \tau_{k+1}$, we get

$$\begin{aligned}
\sigma_{\tau_{k+1}}^2 &= (J_1 J_2 \dots J_k J_{k+1})\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})\tau_{k+1}-+\gamma W_{\tau_{k+1}-}} \\
&+ J_{k+1}\kappa\theta \sum_{n=0}^{k-1} (J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_n)+\gamma(W_{\tau_{k+1}-} - W_{\tau_n})} \int_{\tau_n}^{\tau_{n+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds \\
&+ J_{k+1}\kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(\tau_{k+1}- - \tau_k)+\gamma(W_{\tau_{k+1}-} - W_{\tau_k})} \int_{\tau_k}^{\tau_{k+1}-} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds.
\end{aligned}$$

Further, for $t \in [\tau_{k+1}, \tau_{k+2})$, we obtain

$$\begin{aligned}
\sigma_t^2 &= \sigma_{\tau_{k+2}}^2 e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})+\gamma(W_t - W_{\tau_{k+1}})} \\
&+ \kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})+\gamma(W_t - W_{\tau_{k+1}})} \int_{\tau_{k+1}}^{t} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds \\
&= (J_1 J_2 \dots J_k J_{k+1})\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t+\gamma W_t} \\
&+ J_{k+1}\kappa\theta \sum_{n=0}^{k-1} (J_{n+1} J_2 \dots J_k) e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)+\gamma(W_t - W_{\tau_n})} \int_{\tau_n}^{\tau_{n+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s} ds
\end{aligned}$$

$$
\begin{aligned}
&+ \quad J_{k+1}\kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_k)+\gamma(W_t-W_{\tau_k})}\int_{\tau_k}^{\tau_{k+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s}ds \\
&+ \quad \kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})+\gamma(W_t-W_{\tau_{k+1}})}\int_{\tau_{k+1}}^{t} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s}ds \\
&= \quad (J_1J_2\ldots J_kJ_{k+1})\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t+\gamma W_t} \\
&+ \quad \kappa\theta\sum_{n=0}^{k}(J_{n+1}J_2\ldots J_kJ_{k+1})e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)+\gamma(W_t-W_{\tau_n})}\int_{\tau_n}^{\tau_{n+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s}ds \\
&+ \quad \kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{k+1})+\gamma(W_t-W_{\tau_{k+1}})}\int_{\tau_{k+1}}^{t} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s}ds.
\end{aligned}
$$

Therefore, we conclude with

$$
\begin{aligned}
\sigma_t^2 \quad &= \quad (J_1J_2\ldots J_{N_t})\sigma_0^2 e^{-(\kappa+\frac{\gamma^2}{2})t+\gamma W_t} \\
&+ \quad \kappa\theta\sum_{n=0}^{N_t-1}(J_{n+1}J_2\ldots J_{N_t})e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_n)+\gamma(W_t-W_{\tau_n})}\int_{\tau_n}^{\tau_{n+1}} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s}ds \\
&+ \quad \kappa\theta e^{-(\kappa+\frac{\gamma^2}{2})(t-\tau_{N_t})+\gamma(W_t-W_{\tau_{N_t}})}\int_{\tau_{N_t}}^{t} e^{(\kappa+\frac{\gamma^2}{2})s-\gamma W_s}ds.
\end{aligned}
$$

Appendix C

Proof of Theorems in Chapter Five

C.1 Theorem 5.1-1

Proof for Theorem 5.1-1 as follows:

Assume that $t > u$ so that $N_t \geq N_u$, then

$$\begin{aligned}
E\sigma_t^2\sigma_u^2 &= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)} E[J_1^2 J_2^2 \dots J_{N_u}^2 J_{N_u+1} \dots J_{N_t} e^{\gamma(W_t+W_u)}] \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)} E[J_1^2 J_2^2 \dots J_{N_u}^2 J_{N_u+1} \dots J_{N_t}] E[e^{\gamma(W_t-W_u+2W_u)}] \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(t-u)+2\gamma^2 u} \\
&\quad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P(N_u = n) P(N_t - N_u = k) B^n A^k \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(t-u)+2\gamma^2 u} \\
&\quad \sum_{n=0}^{\infty} e^{-\lambda u}\frac{(B\lambda u)^n}{n!} \sum_{k=0}^{\infty} e^{-\lambda(t-u)}\frac{(A\lambda(t-u))^k}{k!} \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(t-u)+2\gamma^2 u+\lambda(B-1)u+\lambda(A-1)(t-u)} \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}t+\frac{3\gamma^2}{2}u+\lambda(A-1)t+\lambda(B-A)u},
\end{aligned}$$

where we apply the property of independent increment of Poisson process.

Similarly, if $t \leq u$ so that $N_t \leq N_u$, then

$$\begin{aligned}
E\sigma_t^2\sigma_u^2 &= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)} E[J_1^2 J_2^2 \dots J_{N_t}^2 J_{N_t+1} \dots J_{N_u} e^{\gamma(W_t+W_u)}] \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)} E[J_1^2 J_2^2 \dots J_{N_t}^2 J_{N_t+1} \dots J_{N_u}] E[e^{\gamma(W_u-W_t+2W_t)}] \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(u-t)+2\gamma^2 t} \\
&\quad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P(N_t = n) P(N_u - N_t = k) B^n A^k
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(u-t)+2\gamma^2 t} \\
&\quad \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(B\lambda t)^n}{n!}\sum_{k=0}^{\infty} e^{-\lambda(u-t)}\frac{(A\lambda(u-t))^k}{k!} \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(u-t)+2\gamma^2 t+\lambda(B-1)t+\lambda(A-1)(u-t)} \\
&= \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}u+\frac{3\gamma^2}{2}t+\lambda(A-1)u+\lambda(B-A)t}.
\end{aligned}
$$

Therefore, we conclude with

$$
E\sigma_t^2\sigma_u^2 = \sigma_0^4 e^{(\alpha-\frac{\gamma^2}{2})(t+u)+\frac{\gamma^2}{2}(t\vee u)+\frac{3\gamma^2}{2}(t\wedge u)+\lambda(A-1)(t\vee u)+\lambda(B-A)(t\wedge u)}.
$$

Appendix D

Stochastic Background

D.1 Wiener Process

A Wiener process is a continuous time stochastic process named in honor of Norbert Wiener. The definition of a standard Wiener process is given as follows ([35]):

Definition D.1-1. A standard Wiener process (also known as a standard Brownian motion) is a stochastic process $\{W_t\}_{t\in\mathbb{R}^+}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ which satisfies the following properties:

1) $W_0 = 0$;

2) W_t is a continuous function of t with probability 1;

3) The process $\{W_t\}_{t\in\mathbb{R}^+}$ has stationary and independent increment;

4) For any $t, s \in \mathbb{R}^+$, $W_{t+s} - W_s \sim N(0, t)$.

From the definition above, we can induce some other properties of a standard Wiener process:

1) $EW_t = 0$ and $EW_t^2 = Var(W_t) = t$ for any $t \in \mathbb{R}^+$;

2) $P(W_t \in [a, b]) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}} dx$;

3) $EW_tW_s = EW_{t\wedge s}(W_{t\vee s} - W_{t\wedge s} + W_{t\wedge s}) = EW_{t\wedge s}(W_{t\vee s} - W_{t\wedge s}) + EW_{t\wedge s}^2 = t \wedge s$;

4) For any fixed $s \in \mathbb{R}^+$, the process $\{W_{t+s} - W_s\}_{t\in\mathbb{R}^+}$ is a standard Wiener process;

5) $Ee^{\gamma W_t} = e^{\frac{\gamma^2}{2}t}$.

D.2 Poisson Process

A Poisson process, named after the French mathematician Siméon-Denis Poisson, is a stochastic process defined in terms of the occurrence of events. The definition of Poisson process is given as follows ([26]):

Definition D.2-1. Let $(T_i)_{i\geq 1}$ be a sequence of identical independent exponentially distributed random variables with parameter $\lambda > 0$, i.e. with density $I_{\{x>0\}}\lambda e^{-\lambda x}$. Set $\tau_n := \sum_{i=1}^{n} T_i$. We call Poisson process with intensity λ the process N_t defined by

$$N_t := \sum_{n\geq 1} I_{\{\tau_n \leq t\}} = \sum_{n\geq 1} I_{\{\tau_n \leq t \leq \tau_{n+1}\}}$$

where I is an indicate function.

If N_t is a Poisson process with intensity λ, then for any $t > 0$ the random variable N_t follows a Poisson law with parameter $\lambda > 0$

$$P(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Some other properties are listed below:

1) $EN_t = \lambda t$ and $Var(N_t) = \lambda t$;

2) $Es^{N_t} = e^{\lambda t(s-1)}$;

3) Poisson process is a process with independent increments, i.e. for any $t, s \geq 0, n \geq 0$, $P(N_{t+s} - N_s = n) = P(N_t = n)$;

4) Poisson distribution is a good approximation of the binomial distribution $B(p, n.x)$ for small p;

5) Let $(U_i)_{i\geq 1}$ be identical independent distribution random variables taking values in $(-1, +\infty)$ and N_t be a Poisson process with intensity $\lambda > 0$ which is independent of $(U_i)_{i\geq 1}$. Then the compound Poisson process is defined in the following way:

$$L_t := \prod_{i=1}^{N_t}(1 + U_i).$$

It can be shown that

$$EL_t = e^{\lambda t E U_1}.$$

D.3 Itô Integral

Itô calculus, named after Kiyoshi Itô, treats mathematical operations on stochastic processes. Its most important concept is the Itô integral. The definition of Itô integral is given as follows ([31]):

Definition D.3-1. For a given probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ and function $f(t,\omega)$ where $\omega \in \Omega$, and given W_t a standard Wiener process. On interval $[0,t]$, we split $[0,t]$ to n partitions by $0 = u_0 \leq u_1 \leq u_2 \leq \ldots \leq u_n = t$, and then define

$$\int_0^t f(u,\omega)dW_u = \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i^*,\omega)[W_{u_{i+1}} - W_{u_i}](\omega)$$

where $u_i^* = u_i$. If the limit of right-hand side exists, then we say $\int_0^t f(u,\omega)dW_u$ is an Itô integral.

Note that in the definition above, we choose the left end-point u_i for u_i^* as the integrating point, which is quite different from the Riemann integral where we can choose any point on interval $[u_i, u_{i+1}]$ for u_i^*.

Define the Itô integral $I(t) := \int_0^t f(u)dW_u$, and below are some properties for $I(t)$:

1) (Linearity) If $I_i(t) := \int_0^t f_i(u)dW_u$, $i = 1,2$, then $aI_1(t) \pm bI_2(t) = \int_0^t (af_1(u) \pm bf_2(u))dW_u$;

2) $EI(t) = 0$;

3) (Itô Isometry) $EI^2(t) = E\int_0^t f^2(u)du$;

4) $Ee^{\int_0^t f(u)dW_u} = e^{\frac{1}{2}\int_0^t f^2(u)du}$.

From Itô integral, we have the definition of stochastic differential equation as follows:

Definition D.3-2. A stochastic differential equation is given by

$$dx(t) = a(t, x(t))dt + \sigma(t, x(t))dW_t$$

where $a(t, x)$ and $\sigma(t, x)$ are measurable bounded functions. More formal definition is the following:

$$x(t) = x(0) + \int_0^t a(s, x(s))ds + \int_0^t \sigma(s, x(s))dW_s$$

where the last term is an Itô integral.

The most powerful tool to solve stochastic differential equation is Itô's lemma, described as below ([30]):

Theorem D.3-3. *Given $f(X_t, t)$ a twice-differentiable function of t and X_t, where X_t is a stochastic process following the stochastic differential equation*

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

with well behaved drift and volatility functions, $\mu(X_t, t)$ and $\sigma(X_t, t)$, then we have

$$df(X_t, t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2(X_t, t)dt$$

or, after substituting dX_t,

$$df(X_t, t) = (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}\mu(X_t, dt) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2(X_t, t))dt + \frac{\partial f}{\partial X_t}\sigma(X_t, dt)dW_t.$$

Now we consider a stochastic differential equation with jumps,defined as follows:

Definition D.3-4. A stochastic differential equation with jumps has the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t + \nu(X_t, t)dN_t$$

where the last term is the jump component with the jump amplitude $\nu(X_t, t)$ and N_t is the distribution of the number of jumps until time t. More formal definition is the

following:

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s + \sum_{i=1}^{N_t} \nu(X_{\tau_i}, \tau_i)$$

where τ_i are times when jumps occur.

The Itô's lemma for stochastic differential equation with jumps is described as follows ([2]):

Theorem D.3-5. *Given $f(X_t, t)$ a twice-differentiable function of t and X_t, where X_t is a stochastic process following the stochastic differential equation*

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t + \nu(X_t, t)dN_t$$

then we have

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{u-})dX_u + \frac{1}{2}\int_0^t f''(X_{u-})\sigma^2(X_u, u)du \\ &\quad - \sum_{i=1}^{N_t} f'(X_{\tau_i-})(X_{\tau_i} - X_{\tau_i-}) + \sum_{i=1}^{N_t}(f(X_{\tau_i}) - f(X_{\tau_i-})) \end{aligned}$$

or in stochastic differential equation form

$$\begin{aligned} df(X_t) &= f'(X_{t-})dX_t + \frac{1}{2}f''(X_{t-})\sigma^2(X_t, t)dt \\ &\quad - f'(X_{\tau_i-})(X_{\tau_i} - X_{\tau_i-})dN_t + (f(X_{\tau_i}) - f(X_{\tau_i-}))dN_t \end{aligned}$$

where τ_i are times when jumps occur.

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