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FIRST-ORDER DECOUPLED FINITE ELEMENT METHOD OF THE THREE-DIMENSIONAL PRIMITIVE EQUATIONS OF THE OCEAN*

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Abstract. This paper is concerned with a first-order fully discrete decoupled method for solving the three-dimensional (3D) primitive equations of the ocean with the Dirichlet boundary conditions on the side, where a decoupled semi-implicit scheme is used for the time discretization, and the $P_1(P_1) - P_1 - P_1(P_1)$ finite element for velocity, pressure, and density is used for the spatial discretization of these equations. The $H^1 - L^2 - H^1$ optimal error estimates for the numerical solution $(u_h^n, p_h^n, \theta_h^n)$ and the L^2 optimal error estimate for (u_h^n, θ_h^n) are established under the restriction of $0 < h \le \beta_1$ and $0 < \tau \le \beta_2$ for some positive constants β_1 and β_2. Moreover, numerical investigations are provided to show that the first-order decoupled method is of almost unconditional convergence with accuracy $\mathcal{O}(h + \tau)$ in the H^1-norm and $\mathcal{O}(h^2 + \tau)$ in the L^2-norm for solving the 3D primitive equations of the ocean. Numerical results are given to verify the theoretical analysis.

Key words. primitive equations of ocean, optimal error estimate, $P_1(P_1) - P_1 - P_1(P_1)$ finite element, second-order decoupled implicit/explicit method

AMS subject classifications. 65N30, 76M10

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1. Introduction.

Given a smooth bounded domain $\omega \subset R^2$ and the cylindrical domain $\Omega = \omega \times (-d, 0) \subset R^3$, consider the following three-dimensional (3D) viscous primitive equations (PEs) of the ocean in Ω:

$$u_t + L_1 u + (u \cdot \nabla)u + w\partial_z u + \nabla P + f\vec{k} \times u = F_1, \tag{1.1}$$

$$\theta_t + L_2\theta + (u \cdot \nabla)\theta + w\partial_z\theta - \sigma w = F_2, \tag{1.2}$$

$$\nabla \cdot u + \partial_z w = 0, \tag{1.3}$$

$$\partial_z P + \gamma\theta = 0. \tag{1.4}$$

The unknowns for these 3D viscous PEs are the fluid velocity field $(u, w) = (u_1, u_2, w) \in R^3$ with $u = (u_1, u_2)$ being horizontal, the density θ, and the pressure P. $f = f_0(\beta + y)$ is the given Coriolis rotation frequency with a β-plane approximation, F_1 and F_2 are two given functions, $\vec{k}$ is a vertical unit vector, and $\sigma > 0$ and $\gamma > 0$ are given constants. The elliptic operators L_1 and L_2 are given, respectively, as follows:

$$L_i = -\nu_i\Delta - \mu_i\partial_z^2, \quad i = 1,\ 2,$$

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where the positive constants ν_1 and μ_1 are the horizontal and vertical viscosity coefficients, and the positive constants ν_2 and μ_2 are the horizontal and vertical thermal diffusivity coefficients, respectively. The following notation is introduced for convenience:

$$u_t = \frac{\partial u}{\partial t}, \quad \theta_t = \frac{\partial \theta}{\partial t}, \quad \nabla = (\partial_x, \partial_y), \ \Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad \partial_{x_i x_i} = \partial^2_{x_i},$$

with $i = 1, 2, 3$ and $(x_1, x_2, x_3) = (x, y, z)$.

We partition the boundary of Ω into the following three parts:

$$\Gamma_u = \{(x, y, z) \in \bar{\Omega}; z = 0\},$$

$$\Gamma_b = \{(x, y, z) \in \bar{\Omega}; z = -d\},$$

$$\Gamma_s = \{(x, y, z) \in \bar{\Omega}; (x, y) \in \partial\omega, \ -d \le z \le 0\}.$$

Next, we provide system (1.1)–(1.4) with the following boundary conditions with the wind-driven on the top surface and nonslip and nonheat fluxes on the side walls and the bottom (see, e.g., p. 246 in [5], p. 160 in [18], and p. 1037 in [25]):

$$\begin{array}{rl}
\text{on} & \Gamma_u, \ \partial_z u = d\,\tau^*, \ w = 0, \ \ \partial_z \theta = -\alpha(\theta - \theta^*), \\
\text{on} & \Gamma_b, \ \partial_z u = 0, \ w = 0, \partial_z \theta = 0, \\
\text{on} & \Gamma_s, \ u \cdot n = 0, \ \ \frac{\partial u}{\partial n} \times n = 0, \ \ \frac{\partial \theta}{\partial n} = 0, \\
\text{or on} & \Gamma_s, \ u = 0, \ \ \frac{\partial \theta}{\partial n} = 0,
\end{array}$$

where $\tau^* = \tau^*(x, y)$ is the wind stress on the ocean surface, α is a positive constant, n is the normal vector of Γ_s, and $\theta^* = \theta^*(x, y)$ is the typical density distribution of the top surface of the ocean. Based on the above conditions, it is natural to assume that $\tau^*(x, y)$ and $\theta^*(x, y)$ satisfy

$$\tau^* \cdot n = 0, \ \frac{\partial \tau^*}{\partial n} \times n = 0, \quad \text{or } \tau^* = 0, \text{ and } \frac{\partial \theta^*}{\partial n} = 0 \text{ on } \partial\omega.$$

Due to this condition, we can convert the previous boundary conditions into the homogeneous ones by replacing (u, θ) by $(u + \frac{1}{2}([(z + d)^2 - \frac{1}{3}h^3]\tau^*, \theta + \theta^*)$ (refer to p. 248 in [5]).

Hence, we consider the following boundary conditions for the 3D viscous PEs:

$$w|_{\Gamma_u \cup \Gamma_b} = 0, \tag{1.5}$$

$$\partial_z u|_{\Gamma_u \cup \Gamma_b} = 0, \quad u \cdot n|_{\Gamma_s} = 0, \quad \frac{\partial u}{\partial n} \times n|_{\Gamma_s} = 0, \tag{1.6a}$$

$$\text{or } \ \partial_z u|_{\Gamma_u \cup \Gamma_b} = 0, \quad u|_{\Gamma_s} = 0, \tag{1.6b}$$

$$\partial_z \theta|_{\Gamma_b} = (\partial_z \theta + \alpha\theta)|_{\Gamma_u} = 0, \quad \frac{\partial \theta}{\partial n}|_{\Gamma_s} = 0. \tag{1.7}$$

Refer to (28) and (29) on p. 248 in [5], (1.3) and (1.4) on p. 160 in [18], and Remark 2.1 on p. 1038 in [25] for the boundary conditions (1.5)–(1.7).

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The initial conditions of $u(x, y, z, t)$ and $\theta(x, y, z, t)$ are given by

$$u(x, y, z, 0) = u_0(x, y, z), \quad \theta(x, y, z, 0) = \theta_0(x, y, z). \tag{1.8}$$

Using the Dirichlet boundary condition (1.5) of w on $\Gamma_u \cap \Gamma_b$, (1.3), and (1.4), we have

$$w(x, y, z, t) = -\int_{-d}^{z} \nabla \cdot u(x, y, \xi, t) d\xi, \quad \int_{-d}^{0} \nabla \cdot u(x, y, \xi, t) d\xi = 0,$$

$$P(x, y, z, t) = p(x, y, t) - \gamma \int_{-d}^{z} \theta(x, y, \xi, t) d\xi.$$

With the above statements, one obtains the initial boundary value problem of the 3D viscous PEs:

$$\begin{aligned} u_t + L_1 u + \nabla p(x, y, t) - \gamma \int_{-d}^{z} \nabla\theta(x, y, \xi, t) d\xi + f\vec{k} \times u + (u \cdot \nabla) u \\ - \left(\int_{-d}^{z} \nabla \cdot u(x, y, \xi, t) d\xi \right) \partial_z u = F_1, \end{aligned} \tag{1.9}$$

$$\theta_t + L_2\theta + \sigma \int_{-d}^{z} \nabla \cdot u(x, y, \xi, t) d\xi + (u \cdot \nabla)\theta - \left(\int_{-d}^{z} \nabla \cdot u(x, y, \xi, t) d\xi \right) \partial_z \theta = F_2, \tag{1.10}$$

$$\nabla \cdot \bar{u} = 0, \tag{1.11}$$

together with the boundary conditions (1.6) and (1.7) and the initial condition (1.8), where

$$\bar{\phi}(x, y) = \frac{1}{d} \int_{-d}^{0} \phi(x, y, z) dz, \quad \tilde{\phi} = \phi - \bar{\phi},$$

for any function $\phi(x, y, z)$ in Ω.

Remark 1.1. Recall from [5, 18] that $F_1 = 0$ and $\gamma = 1$ in (1.1) and (1.4) and $\sigma = 0$ in (1.2) and (1.10).

The 3D viscous PEs are very important research subjects in the field of geophysical fluid from both the theoretical and numerical points of view. There are some well-known difficulties associated with these fundamental equations of the 3D oceanic model due to their strong nonlinearity and coupling.

A mathematical study of the PEs was originated in a series of articles by Lions, Temam, and Wang in the early 1990s [22, 23, 24], where a mathematical formulation for the PEs, which resembles that of the Navier–Stokes equations, was established. Also, an asymptotic analysis and the finite-dimensional behavior of the 3D viscous PEs in a thin domain as the depth of the domain goes to zero were studied in [16, 17]. For a more extensive discussion and review on this subject, the reader is referred to the recent articles [4, 5, 12, 18, 27, 28].

A numerical simulation study of a first-order coupled method based on a two-grid finite difference method for the PEs of the ocean was proposed by Medjo and Temam in [25]. However, an error analysis was not given in [25]. To the best of our knowledge,

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it was the first time that the first-order decoupled method for the 3D viscous PEs of the ocean was represented. The derivation of error estimates for the numerical solutions of these equations has been challenging. In a recent paper [13], He analyzed only time discretization of problem (1.9) and (1.11) in the finite time interval $[0,T]$ with the boundary conditions (1.6) and (1.7) and the initial condition (1.8); the spatial variables remain continuous. Setting τ to be a time step size, $t_n = n\tau$ and $T = N\tau$, we consider the time discrete approximation (u^n, p^n, θ^n) of $(u(t_n), p(t_n), \theta(t_n))$ by (2.17) and (2.18). He also discussed the stability of the first-order decoupled semi-implicit scheme and the optimal error estimates of the time discrete solution (u^n, p^n, θ^n) with $n = 1, 2, \ldots, N$ [13].

In the first part of this paper we will propose the first-order fully discrete decoupled method for solving the 3D PEs of the ocean in the case of the Dirichlet boundary conditions on the side, where the decoupled semi-implicit scheme is used for the time discretization and the $P_1(P_1) - P_1 - P_1(P_1)$ finite element for the velocity, pressure, and density is used for the spatial discretization of these equations. The $H^1 - L^2 - H^1$ optimal error estimates of the numerical solution $(u_h^n, p_h^n, \theta_h^n)$ are provided by the induction method and the Gronwall lemma, and the L^2 optimal error estimate of (u_h^n, θ_h^n) is provided by using the negative norm technique and the Gronwall lemma under the restriction of $0 < h \le \beta_1$ and $0 < \tau \le \beta_2$ for some positive constants β_1 and β_2. The main results of this paper are stated as follows.

THEOREM 1.1. *Under the assumptions* (A1)–(A3) *below, if* $0 < h \le \beta_1$ *and* $0 < \tau \le \beta_2$ *for some positive constants* β_1 *and* β_2*, then* $(u_h^n, p_h^n, \theta_h^n)$ *satisfies the following error estimates:*

$$\sigma(t_m)[\|u(t_m) - u_h^m\|_{L^2}^2 + \|\theta(t_m) - \theta_h^m\|_{L^2}^2] \le \kappa(h^4 + \tau^2), \tag{1.12}$$

$$\begin{aligned}\sigma(t_m)\Big[&\nu_2\|\nabla(\theta(t_m) - \theta_h^m)\|_{L^2}^2 + \mu_2\|\partial_z(\theta(t_m) - \theta_h^m)\|_{L^2}^2 \\ &+\mu_2\alpha\|(\theta(t_m) - \theta_h^m)(z=0)\|_{L^2(\omega)}^2\Big] \\ &+ \sigma(t_m)[\nu_1\|\nabla(u(t_m) - u_h^m)\|_{L^2}^2 + \mu_1\|\partial_z(u(t_m) - u_h^m)\|_{L^2}^2] \le \kappa(h^2 + \tau^2),\end{aligned} \tag{1.13}$$

$$\tau\sum_{n=1}^{m}\sigma(t_n)\|p(t_n) - p_h^n\|_{L^2(\omega)}^2 \le \kappa(h^2 + \tau^2), \tag{1.14}$$

where $\sigma(t) = \min\{1, t\}$ *and* κ *is a general positive constant depending on the data* $(\Omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2, f, F_1, F_2, u_0, \theta_0, T)$*, which can have a different value at its different occurrence.*

The second part of this paper is mainly focused on validating stability and numerical accuracy of the first-order decoupled method for the 3D PEs of the ocean. In the numerical investigation part, we first construct a test case with analytical solutions and validate the theoretical L^2 and H^1 error estimates. The results demonstrate that the theoretical results coincide with the numerical results very well. Second, the first-order decoupled method is used to simulate a classical practical problem of double gyres. The correct double gyres phenomena are simulated. By the simulation of the double gyres problem, the reliability of the proposed first-order decoupled method is validated.

This paper is outlined as follows. In section 2, some basic mathematical notation and important inequalities are recalled and basic lemmas and estimates of the

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nonlinear terms are provided. In section 3, the $P_1(P_1) - P_1 - P_1(P_1)$ finite element discretization based on the first-order decoupled scheme is introduced. In section 4, we use the induction method and the Gronwall lemma to obtain the $H^1 - L^2 - H^1$ optimal error estimates of $(u_h^n, p_h^n, \theta_h^n)$ to the exact solution $(u(t_n), p(t_n), \theta(t_n))$ under the restriction of $0 < h \le \beta_1$ and $0 < \tau \le \beta_2$. In section 5, the negative norm technique is used to provide the L^2 optimal error estimate of (u_h^n, θ_h^n) to $(u(t_n), \theta(t_n))$. In section 6, numerical investigations are provided to show the efficiency of the first-order decoupled method for solving the 3D PEs of the ocean. Finally, in the appendix, we give a detailed proof of an important and complex H^1 error estimate.

2. Preliminaries. For the 3D domain Ω, 2D domain ω, $m \ge 0$ and $p \ge 1$, we introduce the standard Sobolev spaces $H^m(\Omega)$ and $H^m(\omega)$ or $H^m(\Omega)^2$ and $H^m(\omega)^2$ with the norms $\|\cdot\|_{H^m}$ and $\|\cdot\|_{H^m(\omega)}$ and seminorms $|\cdot|_{H^m}$ and $|\cdot|_{H^m(\omega)}$, respectively. For details of the Sobolev spaces, refer to Adams [1]. Here, we introduce the following spaces:

$$H_1 = \{v \in L^2(\Omega)^2; \mathrm{div}\ \bar{v} = 0, \quad v \cdot n|_{\Gamma_s} = 0\}, \quad H_2 = L^2(\Omega), \quad X_2 = V_2 = H^1(\Omega),$$

$X_1 = \{v \in H^1(\Omega)^2;\ \ v \cdot n|_{\Gamma_s} = 0\}$ in the case of (1.6a), $X_1 = \{v \in H^1(\Omega)^2;\ \ v|_{\Gamma_s} = 0\}$ in the case of (1.6b), and $V_1 = X_1 \cap H_1$. Next, we introduce the Sobolev spaces X_0, M_0, and V_0 in the 2D domain ω by

$$X_0 = \bar{X}_1 = \left\{ \bar{v} \in H^1(\omega)^2; \bar{v} = d^{-1} \int_{-d}^{0} v(x,y,z)dz, \quad v \in X_1 \right\},$$

$$M_0 = L_0^2(\omega) = \left\{ q \in L^2(\omega); \int_\omega q(x,y)dxdy = 0 \right\}, \quad V_0 = \{\bar{v} \in X_0; \nabla \cdot \bar{v} = 0\}.$$

Furthermore, $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\omega$ will be used to denote the inner product in $L^2(\Omega)$ and $L^2(\omega)$ or $L^2(\Omega)^2$ and $L^2(\omega)^2$ or $L^2(\Omega)^4$ and $L^2(\omega)^4$, respectively.

Denote by A_1 the Stokes-type operator associated with the PEs (see [5, 18]); that is, $A_1 = PL_1$, where P is the L^2-orthogonal projection from $L^2(\Omega)^2$ to H_1. Also, we denote $A_2 = L_2$. Then we define the bilinear forms $a_i : X_i \times X_i \to R,\ i = 1, 2$, as follows:

$$a_1(u, v) = \nu_1(\nabla u, \nabla v)_\Omega + \mu_1(u_z, v_z)_\Omega = (L_1^{\frac{1}{2}} u, L_1^{\frac{1}{2}} v)_\Omega,$$

$$a_2(\theta, \phi) = \nu_2(\nabla\theta, \nabla\phi)_\Omega + \mu_2(\theta_z, \phi_z)_\Omega + \mu_2\alpha(\theta(z=0), \phi(z=0))_\omega = (L_2^{\frac{1}{2}}\theta, L_2^{\frac{1}{2}}\phi)_\Omega.$$

Define $D(A_i) = \{\phi \in H_i; A_i\phi \in H_i\}, i = 1, 2$, with the norm $\|A_i \cdot\|_{L^2}$. Then $V_i = D(A_i^{\frac{1}{2}})$ and $a_i(\phi, \psi) = (A_i^{\frac{1}{2}}\phi, A_i^{\frac{1}{2}}\psi)_\Omega = (L_i^{\frac{1}{2}}\phi, L_i^{\frac{1}{2}}\psi)_\Omega$ for $\phi,\ \psi \in V_i$.

We have the following Poincaré inequalities [1, 5]:

$$4\gamma_0\|u\|_{L^2}^2 \le \nu_1\|\nabla u\|_{L^2}^2 \quad \forall u \in X_1, \tag{2.1}$$

$$4\gamma_0\|\theta\|_{L^2}^2 \le \mu_2(\|\partial_z\theta\|_{L^2}^2 + \alpha\|\theta(z=0)\|_{L^2(\omega)}^2) \quad \forall\theta \in X_2, \tag{2.2}$$

for some positive constant γ_0 depending on Ω or (ω, d). Here and after, we shall use the letters c and C (with or without subscripts) to denote a general positive constant depending on the data $(\Omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2)$ and the data $(\Omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2, F_1, F_2, u_0, \theta_0, T)$, respectively, which can have a different value at their different occurrences.

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Generally, we need a further assumption on the regularity results of the solution of the Stokes-type system associated with the PEs of the ocean and the modified Poisson equation when the domain ω is sufficient smooth.

(A1) For a given $g_1 \in H^k(\Omega)^2$, the steady modified Stokes-type system

$$-\nu_1 \Delta v - \mu_1 \partial_{zz} v + \nabla q(x,y) = g_1 \text{ in } \Omega, \quad \operatorname{div} \bar{v}(x,y) = 0 \text{ in } \omega, \tag{2.3}$$

admits a unique solution $(v,q) \in (H^2(\Omega)^2 \cap V_1) \times (L_0^2(\omega) \cap H^1(\omega))$ for the boundary conditions $\partial_z v|_{\Gamma_u \cup \Gamma_b} = 0$ and $v \cdot n|_{\Gamma_s} = 0$, $\frac{\partial v}{\partial n} \times n|_{\Gamma_s} = 0$ or $v|_{\Gamma_s} = 0$ such that

$$\|v\|_{2,\Omega}^2 + \|q\|_{1,\omega}^2 \le c\|g_1\|_{0,\Omega}^2, \tag{2.4}$$

and for a given $g_2 \in L^2(\Omega)$, the elliptic equation

$$-\nu_2 \Delta \phi - \mu_2 \partial_{zz} \phi = g_2 \text{ in } \Omega \tag{2.5}$$

admits a unique solution $\phi \in H^2(\Omega)$ for the boundary condition

$$\partial_z \psi|_{\Gamma_b} = (\partial_z \psi + \alpha\psi)|_{\Gamma_u} = 0, \ \left.\frac{\partial \psi}{\partial n}\right|_{\Gamma_s} = 0, \tag{2.6}$$

such that

$$\|\phi\|_{2,\Omega} \le c\|g_2\|_{0,\Omega}^2. \tag{2.7}$$

The second part in assumption (A1) is a classical result. Some details of the first part in assumption (A1) can be found on pp. 2740–2741 in [19], pp. 56–57 in [29], and pp. 308 and 311 in [30] in the case of the boundary conditions (1.5), (1.6b), and (1.7). In the case of the boundary conditions (1.5), (1.6a), and (1.7), some results in assumption (A1) can be proved in a similar manner as in [19, 29, 30].

We also make the following assumption about the prescribed data for problem (1.9)–(1.11):

(A2) The initial data $(u_0, \theta_0) \in D(A_1) \times D(A_2)$ and (F_1, F_2), $(\partial_z F_1, \partial_z F_2)$, (F_{1t}, F_{2t}), $(F_{1tt}, F_{2tt}) \in L^\infty([0,T]; L^2(\Omega)^2) \times L^\infty([0,T]; L^2(\Omega))$ such that, for some positive constant C,

$$\begin{aligned}\|A_1 u_0\|_{L^2}^2 + \|A_2\theta_0\|_{L^2}^2 + \sup_{0\le t\le T}\{\|F_1(t)\|_{L^2}^2 + \|F_2(t)\|_{L^2}^2 + \|\partial_z F_1(t)\|_{L^2}^2 + \|\partial_z F_2(t)\|_{L^2}^2\} \\ + \sup_{0\le t\le T}\{\|F_{1t}(t)\|_{L^2}^2 + \|F_{2t}(t)\|_{L^2}^2 + \|F_{1tt}(t)\|_{L^2}^2 + \|F_{2tt}(t)\|_{L^2}^2\} \le C.\end{aligned}$$

Also, we recall the following important inequality (see [5, 12]):

$$\int_\omega \int_{-d}^0 |\nabla u(x,y,\xi)| d\xi \int_{-d}^0 |\partial_z \phi||w| dz dx dy \le c_0 \|\nabla u\|_{L^2}^{\frac12} \|\Delta u\|_{L^2}^{\frac12} \|\partial_z \phi\|_{L^2}^{\frac12} \|\nabla \partial_z \phi\|_{L^2}^{\frac12} \|w\|_{L^2} \tag{2.8}$$

for $u \in D(A_1)$, $(\phi, w) \in D(A_1) \times L^2(\Omega)^2$ or $(\phi, w) \in D(A_2) \times L^2(\Omega)$, and the following Sobolev and Ladyzhenskaya inequalities [1, 5, 8, 10, 20]:

$$\begin{aligned}&\|\phi\|_{L^4(\omega)} \le c_0 \|\phi\|_{L^2(\omega)}^{\frac12} \|\phi\|_{H^1}^{\frac12} \quad \forall \phi \in H^1(\omega), \\ &\|\phi\|_{L^\infty(\omega)} \le c_0 \|\phi\|_{L^2(\omega)}^{\frac12} \|\phi\|_{H^2(\omega)}^{\frac12} \quad \forall \phi \in H^2(\omega),\end{aligned} \tag{2.9}$$

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$$\|\phi\|_{L^3} \leq c_0 \|\phi\|_{L^2}^{\frac{1}{2}} \|\phi\|_{H^1}^{\frac{1}{2}}, \quad \|\phi\|_{L^4(\partial\Omega)} + \|\phi\|_{L^6} \leq c_0 \|\phi\|_{H^1} \tag{2.10}$$

for all $\phi \in X_1$ or X_2, where the norm $\|\cdot\|_{L^q}$ denotes $\|\cdot\|_{L^q(\Omega)^2}$ or $\|\cdot\|_{L^q(\Omega)}$ and

$$\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^3} + \|\partial_z \phi\|_{L^3} \leq c_0 \|A_i^{\frac{1}{2}}\phi\|_{L^2}^{\frac{1}{2}} \|A_i \phi\|_{L^2}^{\frac{1}{2}} \tag{2.11}$$

for all $\phi \in D(A_i)$, with $i = 1,\ 2$.

It is easy to see that for $v \in X_1$ and $\phi \in X_2$, there hold [5]

$$\begin{aligned} & h\left(\int_{-d}^{z} \nabla \cdot v(x,y,\xi)d\xi, \phi\right)_\Omega - \left(\int_{-d}^{z} \nabla\phi(x,y,\xi)d\xi, v\right)_\Omega \\ & = \left(\int_{-d}^{0} \nabla \cdot v(x,y,\xi)d\xi, \int_{-d}^{0} \phi(x,y,\xi)d\xi\right)_\omega, \end{aligned} \tag{2.12}$$

$$f\vec{k} \times v \cdot v = 0 \quad \forall v \in L^2(\Omega)^2. \tag{2.13}$$

Moreover, we define the trilinear form

$$b(v,\phi,\psi) = ((v\cdot\nabla)\phi,\psi)_\Omega - \left(\left(\int_{-d}^{z} \nabla\cdot v d\xi\right)\partial_z\phi, \psi\right)_\Omega$$

for all $v \in X_1, (\phi,\psi) \in H^1(\Omega)^2 \times H^1(\Omega)^2$ or $(\phi,\psi) \in H^1(\Omega) \times H^1(\Omega)$. It can be checked that

$$b(v,\phi,\psi) + b(v,\psi,\phi) = -\left(\int_{-d}^{0} \nabla\cdot v d\xi, \phi(z=0)\cdot\psi(z=0)\right)_\omega \tag{2.14}$$

for all $v \in X_1,\ \phi,\ \psi \in H^1(\Omega)$ or $H^1(\Omega)^2$.

With the above statements, the weak form of (1.9)–(1.11) with the boundary conditions (1.5)–(1.7) and the initial condition (1.8) is as follows: Find $(u,p,\theta)(t) \in X_1 \times M_0 \times X_2$ with $0 \leq t \leq T$ such that, for $(v,q,\phi) \in X_1 \times M_0 \times X_2$,

$$\begin{aligned} (u_t,v)_\Omega + a_1(u,v) + b(u,u,v) + (f\vec{k}\times u, v)_\Omega - \gamma\left(\int_{-d}^{z} \nabla\theta(x,y,\xi,t)d\xi, \phi\right)_\Omega & \\ - (\nabla\cdot v, p)_\Omega + (\nabla\cdot u, q)_\Omega = (F_1, v)_\Omega, & \end{aligned} \tag{2.15}$$

$$(\theta_t,\phi)_\Omega + a_2(\theta,\phi) + b(u;\theta,\phi) + \sigma\left(\int_{-d}^{z} \nabla\cdot u(x,y,\xi,t)d\xi, \phi\right)_\Omega = (F_2,\phi)_\Omega. \tag{2.16}$$

Refer to [12] for more details.

Letting τ be a time step size, $t_n = n\tau,\ T = N\tau,\ u^0 = u_0$, and $\theta^0 = \theta_0$, then we recall the time discrete decoupled semi-implicit scheme of (2.15)–(2.16) in [13] as follows: For each n, find (θ^n, u^n, p^n) satisfying

(2.17)

$$\begin{aligned} & (d_t u^n, v)_\Omega + a_1(u^n, v) - (\nabla\cdot v, p^n)_\Omega + (\nabla\cdot u^n, q)_\Omega - \gamma(\textstyle\int_{-d}^{z} \nabla\theta^{n-1}(x,y,\xi)d\xi, v)_\Omega \\ & \quad + (f\vec{k}\times u^n, v)_\Omega + b(u^{n-1}, u^{n-1}, \bar{v}) + b(u^{n-1}, u^n, \tilde{v}) = (F_1^n, v)_\Omega, \end{aligned}$$

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(2.18)
$$(d_t\theta^n,\phi)_\Omega + a_2(\theta^n,\phi) + \sigma\left(\int_{-d}^{z}\nabla\cdot u^{n-1}(x,y,\xi)d\xi,\phi\right)_\Omega + b(u^{n-1},\theta^n,\phi) = (F_2^n,\phi)_\Omega$$

for all $(v,q,\phi)\in X_1\times M_0\times X_2$, $\tilde{v}=v-\bar{v}$, and

$$d_tu^n=\frac{1}{\tau}(u^n-u^{n-1}),\ d_t\theta^n=\frac{1}{\tau}(\theta^n-\theta^{n-1}),\ F_1^n=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}F_1(t)dt,\ F_2^n=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}F_2(t)dt.$$

Here, θ^n can be solved by the linearized elliptic equation (32) without (u^n,p^n), and (31) can be rewritten into the Stokes equations on $(\bar{u}^n,p^n)$ in the 2D domain ω without θ^n and the linearized elliptic equations on $\tilde{u}^n$ in the 3D domain Ω without θ^n and $(\bar{u}^n,p^n)$.

From [13], there hold the following. stability and convergence results.

THEOREM 2.1. *Suppose that the assumptions* (A1) *and* (A2) *hold and* $0<\tau<1$ *satisfies the stability condition* $C_1\tau\le 1$ *for some positive constant* C_1. *Then there hold*

(2.19)
$$\|A_1^{\frac12}u^m\|_{L^2}^2+\|A_2^{\frac12}\theta^m\|_{L^2}^2+\tau\sum_{n=1}^{m}[\|A_1u^n\|_{L^2}^2+\|d_tu^n\|_{L^2}^2+\|A_2\theta^n\|_{L^2}^2+\|d_t\theta^n\|_{L^2}^2]\le\kappa_1,$$

(2.20)
$$\|d_tu^m\|_{L^2}^2+\|d_t\theta^m\|_{L^2}^2+\|A_1u^m\|_{L^2}^2+\|A_2\theta^m\|_{L^2}^2+\tau\sum_{n=1}^{m}[\|A_1^{\frac12}d_tu^n\|_{L^2}^2+\|A_2^{\frac12}d_t\theta^n\|_{L^2}^2]\le\kappa_2,$$

$$\|p^m\|_{H^1(\omega)}^2+\tau\sum_{n=1}^{m}[\|u^n\|_{H^3}^2+\|\theta^n\|_{H^3}^2+\|p^n\|_{H^2(\omega)}^2]\le\kappa_3, \tag{2.21}$$

$$\begin{aligned}\|u(t_m)-u^m\|_{L^2}^2+\|\theta(t_m)-\theta^m\|_{L^2}^2&+\sigma(t_m)(\|A_1^{\frac12}(u(t_m)-u^m)\|_{L^2}^2\\&+\|A_2^{\frac12}(\theta(t_m)-\theta^m)\|_{L^2}^2)\\&+\tau\sum_{n=1}^{m}\sigma(t_n)\|p(t_n)-p^n\|_{L^2(\omega)}^2\\&+\sigma^2(t_m)\|p(t_m)-p^m\|_{L^2(\omega)}^2\le\kappa\tau^2\end{aligned} \tag{2.22}$$

for $0\le m\le N$, *where* κ_i *with* $i=1,2,3$ *are some positive constants depending on the data* $(\Omega,\alpha,\sigma,\gamma,d,\nu_1,\mu_1,\nu_2,\mu_2,\ f,F_1,F_2,u_0,\theta_0,T)$.

In order to analyze the convergence of the numerical solution, we need the following Gronwall lemma [26].

LEMMA 2.2. *Let* C_0 *be a positive constant and* $a_n,\ b_n$, *and* d_n *be three positive series satisfying*

$$a_m+\tau\sum_{n=1}^{m}b_n\le\tau\sum_{n=0}^{m-1}d_na_n+C_0,\quad m\ge1.$$

Then

$$a_m+\tau\sum_{n=1}^{m}b_n\le C_0\exp\left(\tau\sum_{n=0}^{m-1}d_n\right)\quad\forall m\ge1.$$

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Finally, we recall the integral version of the Minkowsky inequality for the L^p spaces [5].

LEMMA 2.3. *Let* $\Omega_1 \subset R^{m_1}$ *and* $\Omega_2 \subset R^{m_2}$ *be two measurable sets, where* m_1 *and* m_2 *are positive integers, and let* $f(\xi,\eta)$ *be measurable over* $\Omega_1 \times \Omega_2$. *Then*

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2}|f(\xi,\eta)|d\eta\right)^p d\xi\right]^{\frac{1}{p}} \le \int_{\Omega_2}\left(\int_{\Omega_1}|f(\xi,\eta)|^p d\xi\right)^{\frac{1}{p}} d\eta.$$

3. First-order decoupled semi-implicit finite element method.

From now on, h is a real positive parameter approaching 0. To avoid the lengthy and technical treatments of the smooth curved boundary of ω, we assume from now on that ω is a planar polygonal domain. For the cases with curved boundaries we can combine some standard techniques as developed in [7, 9, 21] with the analysis in this work to establish the error estimates for the corresponding finite element solutions.

Let $\tau_{2h} = \{\tilde{K}\}$ be quasi-uniformly regular partitions of ω made of triangles with diameters bounded by $2h$, and let $-d = z_0 < z_1 < \cdots < z_L = 0$ be quasi-uniformly regular partitions of $[-d, 0]$ with $I_j = z_j - z_{j-1} \le h$ for $j = 1, 2, \ldots, L$. Also, we define $\tau_h = \{K\}$ by subdividing each triangle $\tilde{K} \in \tau_{2h}$ into four triangles K_1, K_2 , K_3, and K_4 by the midpoints of the sides. Then we obtain prismatic grids $\{K \times I_j\}$ on the 3D domain $\bar{\Omega}$ that are used by Rebollo and Gonzalez [6] for the hydrostatic approximation of the Navier–Stokes equations. First, we will establish the finite element space pair (X_{0h}, M_h) based on the $P_1 - P_1$ elements on $\bar{\omega}$ as follows:

$$X_{0h} = S_h^2 \cap X_0, \quad S_h = \{\phi_h \in C^0(\bar{\omega}) \cap H^1(\omega) : \quad \phi_h|_K \in P_1(K) \ \ \forall K \in \tau_h\},$$

$$M_{0h} = \{q_h \in C^0(\bar{\omega}) \cap M_0 : \quad q_h|_K \in P_1(K) \ \ \forall K \in \tau_{2h}\}.$$

Next, we will establish the finite element spaces $X_{1h} \subset X_1$ based on the $P_1(P_1)$ element and $X_{2h} \subset X_2$ based on the $P_1(P_1)$ element on the prismatic grids $\{K \times I_j\}$ on the 3D domain $\bar{\Omega}$ as follows:

$$J_h = \mathrm{span}\{\psi_h \in H^1([0,d]) : \quad \psi_h|_{I_j} \in P_1(I_j) \ \forall j = 1, 2, \ldots, L\}, \ \ X_{2h} = S_h \times J_h = V_{2h},$$

$$X_{1h} = \{v_h \in (S_h \times J_h)^2; v_h \cdot n|_{\Gamma_s} = 0\}$$

in the case of (1.6a) and

$$X_{1h} = \{v_h \in (S_h \times J_h)^2; v_h|_{\Gamma_s} = 0\}$$

in the case of (1.6b).

Remark 3.1. From [2], the finite element space pair (X_{0h}, M_{0h}) satisfies the discrete inf-sup condition. Moreover, the other finite element pair (X_{0h}, M_{0h}) can be found in [3, 11]. Also, another finite element space pair of the 3D horizontal velocity and the 2D pressure satisfying the discrete inf-sup condition is constructed by Rebollo and Gonzalez in [6].

Here, we need to define the subspace V_{1h} of X_{1h} as follows:

$$V_{1h} = \{v_h \in X_{1h}; (\nabla \cdot \bar{v}_h, q_h)_\omega = 0 \ \ \forall q_h \in M_{0h}\}.$$

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Furthermore, we define the L^2 projection operator $P_{1h} : L^2(\Omega)^2 \to V_{1h}$ and $P_{2h} : L^2(\Omega) \to V_{2h}$ by

$$(P_{1h}u, v_h)_\Omega = (u, v_h)_\Omega \quad \forall u \in L^2(\Omega)^2, \quad v_h \in V_{1h}, \quad (P_{2h}\theta, \phi_h)_\Omega = (\theta, \phi_h)_\Omega,$$
$$\forall \theta \in L^2(\Omega)^2, \quad \phi_h \in V_{2h}.$$

We easily deduce that the above finite element spaces (X_{1h}, M_{0h}, X_{2h}) based on the $P_1(P_1) - P_1 - P_1(P_1)$ element satisfy the usual approximation assumption (A3):

- There exist the mappings $I_h \in \mathcal{L}(V_1; V_{1h})$ and $\pi_h \in \mathcal{L}(V_2; X_{1h})$ such that

$$\|I_h u - u\|_{L^2} + h\|L_1^{\frac{1}{2}}(I_h u - u)\|_{L^2} \le ch^l\|v\|_{H^l}, \quad l = 1, 2, \tag{3.1}$$
$$\|\pi_h\theta - \theta\|_{L^2} + h\|L_2^{\frac{1}{2}}(\pi_h\theta - \theta)\|_{L^2} \le ch^l\|\theta\|_{H^l}, \quad l = 1, 2. \tag{3.2}$$

- The L^2-orthogonal projection operator $\rho_h : M_0 \to M_{0h}$ satisfies

$$\|p - \rho_h p\|_{L^2(\omega)} \le ch^l\|p\|_{H^l(\omega)}, \quad \forall p \in H^l(\omega) \cap M_0, \quad l = 1,\ 2. \tag{3.3}$$

- There exists a constant $\beta_0 > 0$ such that

$$\sup_{\bar v_h \in X_{0h}} \frac{(\nabla \cdot \bar v_h, q_h)_\Omega}{\|\nabla \bar v_h\|_{L^2(\omega)}} \ge \beta_0\|q_h\|_{L^2(\omega)}. \tag{3.4}$$

- The following inverse inequality holds:

$$\|\phi_h\|_{L^\infty(\omega)} \le ch^{-1}\|\phi_h\|_{L^2(\omega)}, \quad \|L_i^{\frac{1}{2}}\phi_h\|_{L^2} \le ch^{-1}\|\phi_h\|_{L^2} \ \ \forall \phi_h \in X_{ih},\ i = 1,\ 2. \tag{3.5}$$

The following estimates, which are the consequences of properties (3.1)–(3.3), will be very useful:

$$\|u - P_{1h}u\|_{L^2} + h\|L_1^{\frac{1}{2}}(u - P_{1h}u)\|_{L^2} \le ch^l\|u\|_{H^l} \quad \forall u \in H^l(\Omega)^2 \cap V_1, \tag{3.6}$$
$$\|\theta - P_{2h}\theta\|_{L^2} + h\|L_2^{\frac{1}{2}}(\theta - P_{2h}\theta)\|_{L^2} \le ch^l\|\theta\|_{H^l} \quad \forall \theta \in H^l(\Omega) \tag{3.7}$$

for $l = 1, 2$.

Now, we define the discrete forms of the operators A_1 and A_2:

$$(A_{ih}\phi_h, \psi_h)_\Omega = (A_i^{\frac{1}{2}}\phi_h, A_i^{\frac{1}{2}}\psi_h)_\Omega = (L_i^{\frac{1}{2}}\phi_h, L_i^{\frac{1}{2}}\psi_h)_\Omega \quad \forall \phi_h, \psi_h \in V_{ih}$$

with $i = 1,\ 2$. Here, we introduce the discrete Sobolev norm $\|\phi_h\|_r = \|A_{ih}^{\frac{r}{2}}\phi_h\|_{L^2}$ for $\phi_h \in V_{ih}$ and $r \in R$, where

$$\|\phi_h\|_0 = \|\phi_h\|_{L^2}, \quad \|\phi_h\|_{-1} = \|A_{ih}^{-\frac{1}{2}}\phi_h\|_{L^2} \quad \forall \phi_h \in V_{ih}.$$

With the above statements, the standard finite element approximation of (2.17) and (2.18) based on $X_{1h} \times M_{0h} \times X_{2h}$ reads as follows: Find $(u_h^n, p_h^n, \theta_h^n) \in X_{1h} \times M_{0h} \times X_{2h}$ such that

$$(d_t u_h^n, v_h)_\Omega + a_1(u_h^n, v_h) - (\nabla \cdot v_h, p_h^n)_\Omega + (\nabla \cdot u_h^n, q_h)_\Omega - \gamma\left(\int_{-d}^{z} \nabla\theta_h^{n-1}(x, y, \xi)d\xi, v_h\right)_\Omega$$
$$+ (f\vec{k} \times u_h^n, v_h)_\Omega + b(u_h^{n-1}, u_h^{n-1}, \bar v_h) + b(u_h^{n-1}, u_h^n, \tilde v_h) = (F_1^n, v_h)_\Omega, \tag{3.8}$$

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(3.9)
$$(d_t\theta_h^n, \phi_h)_\Omega + a_2(\theta_h^n, \phi_h) + \sigma\left(\int_{-d}^{z} \nabla\cdot u_h^{n-1}(x,y,\xi)d\xi, \phi_h\right)_\Omega + b(u_h^{n-1}, \theta_h^n, \phi_h) = (F_2^n, \phi_h)_\Omega$$

for all $(v_h, q_h, \phi_h) \in X_{1h} \times M_{0h} \times X_{2h}$, $\tilde{v}_h = v_h - \bar{v}_h$. Here, (u_h^n, p_h^n) can be solved by (3.8) without θ_h^n and θ_h^n can be solved by (3.9) without (u_h^n, p_h^n), respectively. Set

$$u_h^n = \bar{u}_h^n + \tilde{u}_h^n, \quad \bar{u}_h^n = d^{-1}\int_{-d}^{0} u_h^n(x,y,z)dz, \quad \tilde{u}_h^n = u_h^n - \bar{u}_h^n.$$

Then (3.8) can be rewritten as follows: For each n and $(u_h^{n-1}, \theta_h^{n-1})$, find $(\bar{u}_h^n, p_h^n, \tilde{u}_h^n)$ such that

(3.10)
$$(d_t\bar{u}_h^n, \bar{v}_h)_\Omega + a_1(\bar{u}_h^n, \bar{v}_h) - (\nabla\cdot\bar{v}_h, p_h^n)_\Omega + (\nabla\cdot\bar{u}_h^n, q_h)_\Omega - \gamma(\textstyle\int_{-d}^{z}\nabla\theta_h^{n-1}d\xi, \bar{v}_h)_\Omega$$
$$+ (f\vec{k}\times\bar{u}_h^n, \bar{v}_h)_\Omega + b(u_h^{n-1}, u_h^{n-1}, \bar{v}_h) = (\bar{F}_1^n, \bar{v}_h)_\Omega,$$

$$(d_t\tilde{u}_h^n, \tilde{v}_h)_\Omega + a_1(\tilde{u}_h^n, \tilde{v}_h) - \gamma\left(\int_{-d}^{z}\nabla\theta_h^{n-1}d\xi, \tilde{v}_h\right)_\Omega$$
$$+ (f\vec{k}\times\tilde{u}_h^n, \tilde{v}_h)_\Omega + b(u_h^{n-1}, u_h^n, \tilde{v}_h) = (\tilde{F}_1^n, \tilde{v}_h)_\Omega \tag{3.11}$$

for all $v_h \in X_{1h}$ and $q \in M_{0h}$, together with the initial condition: $\bar{u}_h^0 = \bar{u}_{0h}(x,y)$ and $\tilde{u}_h^0 = \tilde{u}_{0h}(x,y,z)$.

$(\bar{u}_h^n, p_h^n)$ can be solved by the Stokes equations (3.10) in the 2D domain ω without $\tilde{u}_h^n$ and $\tilde{u}_h^n$ can be solved by the linearized equations (3.11) in the 3D domain Ω. Hence, the scheme (3.9)–(3.11) is a first-order decoupled semi-implicit scheme with respect to θ_h^n, $(\bar{u}_h^n, p_h^n)$, and $\tilde{u}_h^n$.

4. $H^1 - L^2 - H^1$ error estimates. In this section, we shall provide the optimal $H^1 - L^2 - H^1$ error estimates of the numerical solution $(u_h^n, p_h^n, \theta_h^n)$ to (u^n, p^n, θ^n) by the induction method and the Gronwall lemma.

First, using (2.1)–(2.14), (3.6)–(3.7), and Lemma 2.3, we deduce the following important inequalities:

(4.1)
$$\begin{aligned}
|b(u,\phi,\bar{w}_h)| &\le c\|L_1^{\frac12}u\|_{L^2}\|L_i^{\frac12}\phi\|_{L^2}\|A_{ih}^{\frac12}\bar{w}_h\|_{L^2} \quad \forall u\in X_1, v\in X_i, w_h\in V_{ih},\\
|b(u,v-P_{ih}v,w_h)| &\le ch\|L_1^{\frac12}u\|_{L^2}\|A_i v\|_{L^2}\|A_{ih}^{\frac12}w_h\|_{L^2} \ \forall u\in X_1,\ v\in D(A_i), w_h\in V_{ih},\\
|b(u,v,\bar{w}_h)| &\le ch^{-\frac12}\|L_1^{\frac12}u\|_{L^2}\|L_i^{\frac12}v\|_{L^2}\|\bar{w}_h\|_{L^2} \quad \forall u\in X_1,\ v\in X_i,\ w_h\in V_{ih},\\
|b(u,v,w_h)| &\le ch^{-1}\|L_1^{\frac12}u\|_{L^2}\|L_i^{\frac12}v\|_{L^2}\|w_h\|_{L^2} \quad \forall u\in X_1,\ v\in X_i,\ w_h\in V_{ih},\\
|b(u,v_h,w_h)| &\le ch^{-\frac12}\|L_1^{\frac12}u\|_{L^2}\|A_{ih}^{\frac12}v_h\|_{L^2}^{\frac12}\|A_{ih}v_h\|_{L^2}^{\frac12}\|w_h\|_{L^2} \ \forall u\in X_1, v_h, w_h\in V_{ih},\\
|b(u,v,v)| &= 0 \quad \forall u\in V_1,\ v\in X_i,\\
|b(u,v,v)| &\le c\|L_1^{\frac12}u\|_{L^2}\|L_i^{\frac12}v\|_{L^2}^2 \quad \forall u\in X_1,\ v\in X_i,\\
|b(u,v,w_h)| &\le c\|u\|_{L^2}\|v\|_{H^3}\|A_{ih}^{\frac12}w_h\|_{L^2} \ \forall u\in H_1, v\in H^3(\Omega)^i\cap X_i, w_h\in V_{ih},\\
|b(u,v,w_h)| &\le c\|u\|_{H^3}\|L_i^{\frac12}v\|_{L^2}\|w_h\|_{L^2} \quad \forall u\in H^3(\Omega)^2\cap V_1,\ v\in X_i,\ w\in H_i,
\end{aligned}$$

where $i = 1, 2$.

LEMMA 4.1. *Suppose that assumptions* (A1)–(A3) *hold and* $0 < \tau < 1$ *satisfies the stability condition:* $C_1\tau \le 1$. *Then* $(u_h^n, p_h^n, \theta_h^n)$ *satisfies the following error estimate:*

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$$
\begin{aligned}
&[\|e^n\|_{L^2}^2+\|\varepsilon^n\|_{L^2}^2]-[\|e^{n-1}\|_{L^2}^2+\|\varepsilon^{n-1}\|_{L^2}^2]+\|L_1^{\frac12}(u^n-u_h^n)\|_{L^2}^2\tau+\|L_2^{\frac12}(\theta^n-\theta_h^n)\|_{L^2}^2\tau\\
&+\frac12(1-c_1\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2})(\|A_{1h}^{\frac12}e^n\|_{L^2}^2+\|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2)\tau\\
&+[1-c_2\|A_1u^{n-1}\|_{L^2}^2\tau]\|e^n-e^{n-1}\|_{L^2}^2-c_2\|A_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^4\tau\\
&-c_2[h^2\|A_1u^n\|_{L^2}^2+h^2\|A_2\theta^n\|_{L^2}^2+\|A_1^{\frac12}(u^n-u^{n-1})\|_{L^2}^2]\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2\tau\\
&\le d_{n-1}(\|e^{n-1}\|_{L^2}^2+\|\varepsilon^{n-1}\|_{L^2}^2)\tau+ch^2(\|A_1u^n\|_{L^2}^2+\|A_2\theta^n\|_{L^2}^2)\tau\\
&+ch^4(\|A_1^{\frac12}u^{n-1}\|_{L^2}^2+\|A_1^{\frac12}u^n\|_{L^2}^2+\|A_2\theta^n\|_{L^2}^2)\|u^{n-1}\|_{H^3}^2\tau\\
&+ch^4(\|u^n\|_{H^3}^2+\|\theta^n\|_{H^3}^2)\|A_1u^{n-1}\|_{L^2}^2\tau\\
&+ch^4(\|p^n\|_{H^2(\omega)}^2+\|A_1u^n\|_{L^2}^2+\|A_1u^{n-1}\|_{L^2}^2+\|A_2\theta^n\|_{L^2}^2)\tau
\end{aligned}
$$

for some positive constants c_1 and c_2, where $(e^n,r^n,\varepsilon^n)=(P_{1h}u^n-u_h^n,\rho_hp^n-p_h^n,P_{2h}\theta^n-\theta_h^n)$ and

$$d_{n-1}=c+c\|u^n\|_{H^3}^2+c\|\theta^n\|_{H^3}^2.$$

Proof. It follows from (2.17), (2.18), (3.8), and (3.9) that

(4.2)
$$
\begin{aligned}
&(d_t(u^n-u_h^n),v_h)_\Omega+a_1(u^n-u_h^n,v_h)-\gamma(\textstyle\int_{-d}^z\nabla(\theta^{n-1}-\theta_h^{n-1})d\xi,v_h)_\Omega\\
&+b(u^{n-1}-u_h^{n-1},u^{n-1}-u^n,\bar v_h)+(f\vec k\times(u^n-u_h^n),v_h)_\Omega\\
&+b(u_h^{n-1}-u^{n-1},u^{n-1}-u_h^{n-1}-(u^n-u_h^n),\bar v_h)+b(u^{n-1},u^{n-1}-u_h^{n-1}-(u^n-u_h^n),\bar v_h)\\
&+b(u^{n-1}-u_h^{n-1},u^n,v_h)+b(u_h^{n-1}-u^{n-1},u^n-u_h^n,v_h)+b(u_h^{n-1}-u^{n-1},u^n-u_h^n,v_h)\\
&-(\nabla\cdot v_h,p^n-p_h^n)_\Omega+(\nabla\cdot(u^n-u_h^n),q_h)_\Omega=0\quad\forall(v_h,q_h)\in X_{1h}\times M_{0h},
\end{aligned}
$$

$$
\begin{aligned}
&(d_t(\theta^n-\theta_h^n),\phi_h)_\Omega+a_2(\theta^n-\theta_h^n,\phi_h)+b(u^{n-1}-u_h^{n-1},\theta^n,\phi_h)\\
&+b(u_h^{n-1}-u^{n-1},\theta^n-\theta_h^n,\phi_h)\\
&+b(u^{n-1},\theta^n-\theta_h^n,\phi_h)+\sigma(\textstyle\int_{-d}^z\nabla\cdot(u^{n-1}-u_h^{n-1})d\xi,\phi_h)_\Omega=0\quad\forall\phi_h\in X_{2h}
\end{aligned}\tag{4.3}
$$

with the initial condition $e^0=0$ and $\varepsilon^0=0$.

Taking $(v_h,q_h)=(e^n,r^n)$ in (4.2) and $\phi_h=\varepsilon^n$ in (4.3), adding these two relations, and using (2.13)–(2.14), we find

(4.4)
$$
\begin{aligned}
&\tfrac{1}{2\tau}[\|e^n\|_{L^2}^2-\|e^{n-1}\|_{L^2}^2+\|e^n-e^{n-1}\|_{L^2}^2]+\tfrac{1}{2\tau}[\|\varepsilon^n\|_{L^2}^2-\|\varepsilon^{n-1}\|_{L^2}^2+\|\varepsilon^n-\varepsilon^{n-1}\|_{L^2}^2]\\
&+\tfrac12[\|A_{1h}^{\frac12}e^n\|_{L^2}^2+\|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2+\|L_1^{\frac12}(u^n-u_h^n)\|_{L^2}^2+\|L_2^{\frac12}(\theta^n-\theta_h^n)\|_{L^2}^2]\\
&+b(u^{n-1}-u_h^{n-1},u^{n-1}-u^n,\bar e^n)+b(u_h^{n-1}-u^{n-1},u^{n-1}-u_h^{n-1}-(u^n-u_h^n),\bar e^n)\\
&+b(u^{n-1}-u_h^{n-1},u^n,e^n)+b(u_h^{n-1}-u^{n-1},u^n-u_h^n,e^n)+(f\vec k\times(u^n-P_{1h}u^n),e^n)_\Omega\\
&+b(u^{n-1}-u_h^{n-1},\theta^n,\varepsilon^n)+b(u_h^{n-1}-u^{n-1},\theta^n-\theta_h^n,\varepsilon^n)-(\nabla\cdot e^n,p^n-\rho_hp^n)_\Omega\\
&+b(u^{n-1},u^{n-1}-u_h^{n-1}-(u^n-u_h^n),\bar e^n)+b(u^{n-1},u^n-P_{1h}u^n,e^n)\\
&+b(u^{n-1},\theta^n-P_{2h}\theta^n,\varepsilon^n)+\sigma(\textstyle\int_{-d}^z\nabla\cdot(u^{n-1}-u_h^{n-1})d\xi,\varepsilon^n)_\Omega\\
&-\gamma(\textstyle\int_{-d}^z\nabla(\theta^{n-1}-\theta_h^{n-1})d\xi,e^n)_\Omega\\
&=\tfrac12[\|L_1^{\frac12}(u^n-P_{1h}u^n)\|_{L^2}^2+\|L_2^{\frac12}(\theta^n-P_{2h}\theta^n)\|_{L^2}^2].
\end{aligned}
$$

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Due to (3.6), (3.7), (4.1), and the Young inequality, we have

$$\begin{aligned}|b(u^{n-1}-u_h^{n-1},u^n-u^{n-1},\bar{e}^n)| &\le \frac{1}{32}\|A_{1h}^{\frac12}\bar{e}^n\|_{L^2}^2\\ &+c\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2\|A_1^{\frac12}(u^n-u^{n-1})\|_{L^2}^2,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&|b(u_h^{n-1}-u^{n-1},u^{n-1}-u_h^{n-1}-(u^n-u_h^n),\bar{e}^n)| \le c\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}\|A_{1h}^{\frac12}e^n\|_{L^2}^2\\ &+\tfrac{1}{32}\|A_{1h}^{\frac12}e^n\|_{L^2}^2+c(h^2\|A_1u^n\|_{L^2}^2+\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2)\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2,\end{aligned} \tag{4.6}$$

$$|b(u_h^{n-1}-u^{n-1},u^n-P_{1h}u^n,e^n)| \le \frac{1}{32}\|A_{1h}^{\frac12}e^n\|_{L^2}^2+ch^2\|A_1u^n\|_{L^2}^2\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2, \tag{4.7}$$

$$|b(u_h^{n-1}-u^{n-1},e^n,e^n)| \le c\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}\|A_{1h}^{\frac12}e^n\|_{L^2}^2, \tag{4.8}$$

$$|b(u_h^{n-1}-u^{n-1},\theta^n-P_{2h}\theta^n,\varepsilon^n)| \le \frac{1}{32}\|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2+ch^2\|A_2\theta^n\|_{L^2}^2\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2, \tag{4.9}$$

$$|b(u_h^{n-1}-u^{n-1},\varepsilon^n,\varepsilon^n)| \le c\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}\|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2, \tag{4.10}$$

$$\begin{aligned}|b(u^{n-1}-u_h^{n-1},u^n,e^n)|+|b(u^{n-1}-u_h^{n-1},\theta^n,\varepsilon^n)| &\le \tfrac{1}{32}(\|A_{1h}^{\frac12}e^n\|_{L^2}^2+\|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2)\\ &+c(\|u^n\|_{H^3}^2+\|\theta^n\|_{H^3}^2)\|u^{n-1}-u_h^{n-1}\|_{L^2}^2,\end{aligned} \tag{4.11}$$

$$\begin{aligned}&|b(u^{n-1},u^{n-1}-u_h^{n-1}-(u^n-u_h^n),\bar{e}^n)| \le \tfrac{1}{32}\|A_{1h}\bar{e}^n\|_{L^2}^2\\ &+ch^4\|A_1u^{n-1}\|^2(\|A_1u^{n-1}\|_{L^2}^2+\|A_1u^n\|_{L^2}^2)+c\|A_1u^{n-1}\|_{L^2}^2\|e^n-e^{n-1}\|_{L^2}^2,\end{aligned} \tag{4.12}$$

$$|b(u^{n-1},u^n-P_{1h}u^n,e^n)| \le \frac{1}{32}\|A_{1h}^{\frac12}e^n\|_{L^2}^2+ch^4\|A_1u^n\|_{L^2}^2\|u^{n-1}\|_{H^3}^2, \tag{4.13}$$

$$|b(u^{n-1},\theta^n-P_{2h}\theta^n,\varepsilon^n)| \le \frac{1}{32}\|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2+ch^4\|A_2\theta^n\|_{L^2}^2\|u^{n-1}\|_{H^3}^2, \tag{4.14}$$

$$\begin{aligned}&\sigma|(\textstyle\int_{-d}^z\nabla\cdot(u^{n-1}-u_h^{n-1})d\xi,\varepsilon^n)_\Omega|+\gamma|(\int_{-d}^z\nabla(\theta^{n-1}-\theta_h^{n-1})d\xi,e^n)_\Omega|\\ &\le \tfrac{1}{32}\|A_{1h}^{\frac12}e^n\|_{L^2}^2+\tfrac{1}{32}\|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2+c(\|u^{n-1}-u_h^{n-1}\|_{L^2}^2+\|\theta^{n-1}-\theta_h^{n-1}\|_{L^2}^2),\end{aligned} \tag{4.15}$$

$$\begin{aligned}&|(f\vec{k}\times(u^n-P_{1h}u^n),e^n)_\Omega|+|(\nabla\cdot e^n,p^n-\rho_hp^n)_\Omega|\\ &\le \tfrac{1}{32}\|A_{1h}e^n\|_{L^2}^2+ch^4(\|A_1u^n\|_{L^2}^2+\|p^n\|_{H^2(\omega)}^2).\end{aligned} \tag{4.16}$$

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Combining the above estimates with (4.4) and using (3.6)–(3.7), we have completed the proof of Lemma 4.1. ∎

Now, we need to define the following H^1-projections: $R_{ih} : V_i \to V_{ih}$ such that

$$a_i(R_{ih}\phi, \psi_h) = a_i(\phi, \psi_h) \quad \forall \phi \in V_i, \quad \psi_h \in V_{ih}.$$

By the standard error estimate technique, we can prove the following error estimate results:

$$\|R_{1h}u - u\|_{L^2} + h\|L_1^{\frac{1}{2}}(R_{1h}u - u)\|_{L^2} \le ch^l\|u\|_{H^l} \quad \forall u \in H^l(\Omega)^2 \cap V_1, \tag{4.17}$$

$$\|R_{2h}\theta - \theta\|_{L^2} + h\|L_2^{\frac{1}{2}}(R_{2h}\theta - \theta)\|_{L^2} \le ch^l\|\theta\|_{H^l} \quad \forall \theta \in H^l(\Omega) \cap V_2 \tag{4.18}$$

for $l = 1, 2$.

Lemma 4.2. *Under the assumptions of Lemma* 4.1, $(u_h^n, p_h^n, \theta_h^n)$ *satisfies the following error estimate:*

$$\begin{aligned}
&\|L_1^{\frac{1}{2}}(u^m - u_h^m)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^m - \theta_h^m)\|_{L^2}^2 + \tfrac{\tau}{2}\textstyle\sum_{n=1}^m[\|d_te^n\|_{L^2}^2 \\
&+\|A_{1h}e^n\|_{L^2}^2 + \|d_t\varepsilon^n\|_{L^2}^2 + \|A_{2h}\varepsilon^n\|_{L^2}^2] \\
&\le Ch^2 + \tau Ch^{-1}\tau\textstyle\sum_{n=1}^m \|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2 \\
&+\tau C\textstyle\sum_{n=1}^m[\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2] \\
&+\tfrac{\tau c}{h^2}\textstyle\sum_{n=1}^m(\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^4 \\
&+\|L_2^{\frac{1}{2}}(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^4)(\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2) \\
&+\tau\textstyle\sum_{n=0}^{m-1} d_n(\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2),
\end{aligned} \tag{4.19}$$

where $(e^n, r^n, \varepsilon^n) = (R_{1h}u^n - u_h^n, \rho_h p^n - p_h^n, R_{2h}\theta^n - \theta_h^n)$ *and*

$$\begin{aligned}
d_{n-1} &= c + c\|u^n\|_{H^3}^2 + c\|\theta^n\|_{H^3}^2 + c\|u^{n-1}\|_{H^3}^2 \\
&\quad + \frac{c}{h}[\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2 + \|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2] \\
&\quad + ch^{-1}\|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2 + c\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2(\|A_1u^n\|_{L^2}^2 + \|A_2\theta^n\|_{L^2}^2).
\end{aligned}$$

Proof. This proof will be provided in Appendix A. ∎

Theorem 4.3. *Suppose that assumptions* (A1)–(A3) *hold and* $0 < h < 1$ *and* $0 < \tau < 1$ *satisfies the following convergence condition:*

$$C_1\tau \le 1, \quad 2c_1\sqrt{\kappa_5}(h + \tau) \le 1, \quad 8c_2(\kappa_2 + \kappa_3)\tau \le 1, \quad 8c_2\kappa_2h^2 \le 1, \quad \kappa_5(h + \tau) \le 1. \tag{4.20}$$

Then $(u_h^n, p_h^n, \theta_h^n)$ *satisfies the following stability and error estimates:*

$$\|u^m - u_h^m\|_{L^2}^2 + \|\theta^m - \theta_h^m\|_{L^2}^2 + \tau\textstyle\sum_{n=1}^m[\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2] \le \kappa_4h^2, \tag{4.21}$$

$$\begin{aligned}
&\|L_1^{\frac{1}{2}}(u^m - u_h^m)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^m - \theta_h^m)\|_{L^2}^2 + \tau\textstyle\sum_{n=1}^m[\|A_{1h}(R_{1h}u^n - u_h^n)\|_{L^2}^2 \\
&+ \|A_{2h}(R_{2h}\theta^n - \theta_h^n)\|_{L^2}^2] + \tau\textstyle\sum_{n=1}^m[\|d_t(u^n - u_h^n)\|_{L^2}^2 + \|d_t(\theta^n - \theta_h^n)\|_{L^2}^2] \le \kappa_5(h^2 + h\tau).
\end{aligned} \tag{4.22}$$

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Here, the constants κ_2 and κ_3 are from Theorem 2.1*, which are the regularity constant bounds of the solution (u^n, p^n, θ^n) in some norms, and κ_4 and κ_5 are the error constant bounds of $(u_h^n, p_h^n \theta_h^n)$ to (u^n, p^n, θ^n) in some norms.*

Proof. We will prove Theorem 4.3 by the induction method. It follows from (4.1), (4.19), and (4.20) that (4.21) and (4.22) hold for $m = 0, 1$. Assuming that (4.21) and (4.22) hold for $m = 0,\ 1, \dots, J$, we need to prove (4.21) and (4.22) for $m = J + 1$.

From Lemma 4.1, (4.20), and the induction assumption, there holds

(4.23)
$$[\|e^n\|_{L^2}^2 + \|\varepsilon^n\|_{L^2}^2] - [\|e^{n-1}\|_{L^2}^2 + \|\varepsilon^{n-1}\|_{L^2}^2] + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2\tau + \|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2\tau$$
$$- \tfrac{1}{2}\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2\tau \le d_{n-1}(\|e^{n-1}\|_{L^2}^2 + \|\varepsilon^{n-1}\|_{L^2}^2)\tau$$
$$+ ch^2(\|A_1 u^n\|_{L^2}^2 + \|A_2\theta^n\|_{L^2}^2)\tau$$
$$+ ch^4(\|A_1^{\frac{1}{2}}u^{n-1}\|_{L^2}^2 + \|A_1^{\frac{1}{2}}u^n\|_{L^2}^2 + \|A_2\theta^n\|_{L^2}^2)\|u^{n-1}\|_{H^3}^2\tau$$
$$+ ch^4(\|u^n\|_{H^3}^2 + \|\theta^n\|_{H^3}^2)\|A_1 u^{n-1}\|_{L^2}^2\tau$$
$$+ ch^4(\|p^n\|_{H^2(\omega)}^2 + \|A_1 u^n\|_{L^2}^2 + \|A_1 u^{n-1}\|_{L^2}^2 + \|A_2\theta^n\|_{L^2}^2)\tau,$$

where $(e^n, r^n, \varepsilon^n) = (P_{1h}u^n - u_h^n, \rho_h p^n - p_h^n, P_{2h}\theta^n - \theta_h^n)$ and $d_{n-1} = c + c\|u^n\|_{H^3}^2 + c\|\theta^n\|_{H^3}^2$. Summing (4.23) from $n = 1$ to $n = J+1$ and using Theorem 2.1, we deduce

$$\begin{aligned}[\|e^{J+1}\|_{L^2}^2 + \|\varepsilon^{J+1}\|_{L^2}^2 + \tau\textstyle\sum_{n=1}^{J+1}[\frac{1}{2}\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2] \\ \le \tau\textstyle\sum_{n=0}^{J} d_n(\|e^n\|_{L^2}^2 + \|\varepsilon^n\|_{L^2}^2) + Ch^2.\end{aligned} \tag{4.24}$$

Applying Lemma 2.2 to (4.24) and using (3.6), (3.7), and Theorem 2.1, we have showed that (4.21) holds for $m = J+1$. Finally, using (4.20), (4.21), and the induction assumption in (4.19) with $m = J + 1$, we obtain

(4.25)
$$\|L_1^{\frac{1}{2}}(u^{J+1} - u_h^{J+1})\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^{J+1} - \theta_h^{J+1})\|_{L^2}^2 + \tau\textstyle\sum_{n=1}^{J+1}[\|d_t e^n\|_{L^2}^2 + \|A_{1h}e^n\|_{L^2}^2$$
$$+\|d_t\varepsilon^n\|_{L^2}^2 + \|A_{2h}\varepsilon^n\|_{L^2}^2]$$
$$\le C(h^2 + h\tau) + \tau\textstyle\sum_{n=0}^{J} d_n\|(\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2),$$

where C and $\tau\sum_{n=0}^{J} d_n$ are not dependent of κ_5. Applying Lemma 2.2 to (4.25) and using (4.20) and Theorem 2.1, we have proven that (4.22) holds for $m = J + 1$. The proof is completed. ▯

LEMMA 4.4. *Under the assumptions of Theorem* 4.3*, $(u_h^n, p_h^n, \theta_h^n)$ satisfies the following error estimate:*

$$\tau\sum_{n=1}^{m}\|p^n - p_h^n\|_{L^2(\omega)}^2 \le \kappa(h^2 + h\tau). \tag{4.26}$$

Proof. It follows from (4.2) that

$$\begin{aligned}&d(\nabla\cdot\bar{v}_h, \rho_h p^n - p_h^n)_\omega = (d_t(u^n - u_h^n), \bar{v}_h)_\Omega + a_1(u^n - u_h^n, \bar{v}_h)\\ &+ b(u^{n-1} - u_h^{n-1}, u^{n-1}, \bar{v}_h)\\ &+ b(u_h^{n-1}, u^{n-1} - u_h^{n-1}, \bar{v}_h) + (f\vec{k}\times(u^n - u_h^n), \bar{v}_h)_\Omega\\ &- \gamma(\textstyle\int_{-d}^{z}\nabla(\theta^{n-1} - \theta_h^{n-1})d\xi, \bar{v}_h)_\Omega - (\nabla\cdot\bar{v}_h, p^n - \rho_h p^n)_\Omega, \quad \forall\bar{v}_h \in X_{0h}.\end{aligned} \tag{4.27}$$

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It follows from (4.27) and assumption (A3) that

(4.28)
$$\|\rho_h p^n - p_h^n\|_{L^2(\omega)} \le c\|d_t(R_{1h}u^n - u_h^n)\|_{L^2} + c\|d_t u^n - R_{1h}d_t u^n\|_{L^2} + c\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}$$
$$+ c\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}\|A_1^{\frac{1}{2}}u^{n-1}\|_{L^2} + c\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}$$
$$+ c\|u^n - u_h^n\|_{L^2} + c\|L_2^{\frac{1}{2}}(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + c\|p^n - \rho_h p^n\|_{L^2}.$$

Hence, by using (4.28), (4.17), Theorem 2.1, Theorem 4.3, and assumption (A3), we deduce (4.26). ∎

5. L^2 error estimates.

In this section, we shall provide the L^2 error estimates of the numerical solution (u_h^n, θ_h^n) to (u^n, θ^n) by using the negative norm technique without using the standard duality argument in [15].

First, we need modified versions of the discrete Sobolev inequalities in [14].

LEMMA 5.1. *Suppose that assumptions* (A1)–(A3) *hold. Then, for each $\phi_h \in V_{ih}$ with $i = 1, 2$, there hold*

$$\|A_{ih}^{\frac{1}{2}}\phi_h\|_{L^6} \le c\|A_{ih}\phi_h\|_{L^2}, \quad \|\phi_h\|_{L^\infty} + \|A_{ih}^{\frac{1}{2}}\phi_h\|_{L^3} \le c\|A_{ih}^{\frac{1}{2}}\phi_h\|_{L^2}^{\frac{1}{2}}\|A_{ih}\phi_h\|_{L^2}^{\frac{1}{2}}.$$

Second, we also need the following extended versions of the stability results of $(d_t u^n, d_t\theta^n)$ in Theorem 2.1.

LEMMA 5.2. *Under the assumptions of Theorem* 2.1, *for each $1 \le m \le N$ there holds*

(5.1)
$$\sigma(t_m)[\|A_1^{\frac{1}{2}}d_t u^m\|_{L^2}^2 + \|A_2^{\frac{1}{2}}d_t\theta^m\|_{L^2}^2] + \tau\textstyle\sum_{n=1}^m \sigma(t_n)[\|A_1 d_t u^n\|_{L^2}^2 + \|A_2 d_t\theta^n\|_{L^2}^2] \le \kappa,$$
$$m = 1, \dots, N.$$

Proof. This lemma can be proven by (2.17), (2.18), (4.1), Theorem 2.1, and the standard energy estimate technique. Hence we omit the proof. ∎

LEMMA 5.3. *Under the assumptions of Theorem* 4.3, *if $0 < \tau < 1$ satisfies*

$$c_3\tau + 2c_4\kappa_2^2\tau \le 1 \tag{5.2}$$

for some positive constants c_3 and c_4, then (u_h^n, θ_h^n) satisfies the following error estimate:

$$\|e^m\|_{-1}^2 + \|\varepsilon^m\|_{-1}^2 + \tau\sum_{n=1}^m(\|u^n - u_h^n\|_{L^2}^2 + \|\theta^n - \theta_h^n\|_{L^2}^2) \le \kappa(h^4 + \tau h^2), \tag{5.3}$$

where $(e^n, \varepsilon^n) = (P_{1h}u^n - u_h^n, P_{2h}\theta^n - \theta_h^n)$.

Proof. Taking $(v_h, q_h) = (A_{1h}^{-1}e^n, 0)$ in (4.2) and $\phi_h = A_{2h}^{-1}\varepsilon^n$ in (4.3) and adding these two relations, we find

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(5.4)
$$\frac{1}{2\tau}[\|e^n\|_{-1}^2 - \|e^{n-1}\|_{-1}^2 + \|e^n - e^{n-1}\|_{-1}^2] + \frac{1}{2\tau}[\|\varepsilon^n\|_{-1}^2 - \|\varepsilon^{n-1}\|_{-1}^2 + \|\varepsilon^n - \varepsilon^{n-1}\|_{-1}^2]$$
$$+ (R_{1h}u^n - u^n, e^n)_\Omega + \|e^n\|_0^2 + (R_{2h}\theta^n - \theta^n, \varepsilon^n)_\Omega + \|\varepsilon^n\|_0^2$$
$$+ b(u^{n-1} - u_h^{n-1}, u^{n-1} - u^n, \overline{A_{1h}^{-1}e}^n) + b(u^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \overline{A_{1h}^{-1}e}^n)$$
$$+ b(u_h^{n-1} - u^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \overline{A_{1h}^{-1}e}^n)$$
$$+ b(u_h^{n-1} - u^{n-1}, u^n - u_h^n, A_{1h}^{-1}e^n) + b(u^{n-1} - u_h^{n-1}, u^n, A_{1h}^{-1}e^n)$$
$$+ b(u^{n-1}, u^n - u_h^n, A_{1h}^{-1}e^n) + (f\vec{k} \times (u^n - P_{1h}u^n), A_{1h}^{-1}e^n)_\Omega$$
$$+ b(u^{n-1} - u_h^{n-1}, \theta^n, A_{2h}^{-1}\varepsilon^n) + b(u^{n-1}, \theta^n - \theta_h^n, A_{2h}^{-1}\varepsilon^n)$$
$$+ b(u_h^{n-1} - u^{n-1}, \theta^n - \theta_h^n, A_{2h}^{-1}\varepsilon^n) - (\nabla \cdot A_{1h}^{-1}e^n, p^n - \rho_h p^n)_\Omega$$
$$+ \sigma\left(\int_{-d}^z \nabla \cdot (u^{n-1} - u_h^{n-1})d\xi, A_{2h}^{-1}\varepsilon^n\right)_\Omega - \gamma\left(\int_{-d}^z \nabla(\theta^{n-1} - \theta_h^{n-1})d\xi, A_{1h}^{-1}e^n\right)_\Omega = 0,$$

where $(e^n, r^n, \varepsilon^n) = (P_{1h}u^n - u_h^n, \rho_h p^n - p_h^n, P_{2h}\theta^n - \theta_h^n)$, $e^0 = 0$ and $\varepsilon^0 = 0$.

Due to (4.1), (3.5)–(3.7), and Lemma 5.1, we have

$$|b(u^{n-1} - u_h^{n-1}, u^n - u^{n-1}, \overline{A_{1h}^{-1}e}^n)| \le \frac{1}{32}\|e^n\|_0^2$$
(5.5)
$$+ c\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2 \|A_1^{\frac{1}{2}}(u^n - u^{n-1})\|_{L^2}^2,$$

(5.6)
$$|b(u^{n-1} - u_h^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \overline{A_{1h}^{-1}e}^n)| \le \frac{1}{32}\|e^n\|_0^2$$

(5.7)
$$+ c\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2 \|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2 + c\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 \|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2,$$

(5.8)
$$|b(u^{n-1} - u_h^{n-1}, u^n - u_h^n, A_{1h}^{-1}e^n)| \le \frac{1}{32}\|e^n\|_0^2 + c\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 \|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2,$$

(5.9)
$$|b(u^{n-1} - u_h^{n-1}, \theta^n - \theta_h^n, A_{2h}^{-1}\varepsilon^n)| \le \frac{1}{32}\|\varepsilon^n\|_0^2 + c\|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2 \|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2,$$

(5.10)
$$|b(u^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \overline{A_{1h}^{-1}e}^n)| \le \tfrac{1}{32}\|e^n\|_0^2 + \tfrac{1}{32}\|e^{n-1}\|_0^2$$
$$+ ch^4(\|A_1 u^{n-1}\|_{L^2}^2 \|A_1 u^n\|_{L^2}^2 + \|A_1 u^{n-1}\|_{L^2}^4) + c\|A_1 u^{n-1}\|_{L^2}^4 \|e^n\|_{-1}^2,$$

(5.11)
$$|b(u^{n-1}, u^n - u_h^n, A_{1h}^{-1}e^n)| \le \frac{1}{32}\|e^n\|_0^2 + ch^4\|A_1 u^{n-1}\|_{L^2}^2 |A_1 u^n\|_{L^2}^2 + c\|A_1 u^{n-1}\|_{L^2}^4 \|e^n\|_{-1}^2,$$

(5.12)
$$|b(u^{n-1}, \theta^n - \theta_h^n, A_{2h}^{-1}\varepsilon^n)| \le \frac{1}{32}\|\varepsilon^n\|_0^2 + ch^4\|A_1 u^{n-1}\|_{L^2}^2 |A_2\theta^n\|_{L^2}^2 + c\|A_1 u^{n-1}\|_{L^2}^4 \|\varepsilon^n\|_{-1}^2,$$

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$$|b(u^{n-1}-u_h^{n-1},u^n,A_{1h}^{-1}e^n)| \le \frac{1}{32}\|e^n\|_0^2+\frac{1}{32}\|e^{n-1}\|_0^2+ch^4\|A_1u^{n-1}\|_{L^2}^2|A_1u^n\|_{L^2}^2 + c\|A_1u^{n-1}\|_{L^2}^4\|e^n\|_{-1}^2, \tag{5.13}$$

$$|b(u^{n-1}-u_h^{n-1},\theta^n,A_{2h}^{-1}\varepsilon^n)| \le \frac{1}{32}\|\varepsilon^n\|_0^2+\frac{1}{32}\|e^{n-1}\|_0^2+ch^4\|A_1u^{n-1}\|_{L^2}^2|A_2\theta^n\|_{L^2}^2 + c\|A_2\theta^n\|_{L^2}^4\|\varepsilon^n\|_{-1}^2, \tag{5.14}$$

$$\gamma\Big|\left(\int_{-d}^z \nabla\cdot(\theta^{n-1}-\theta_h^{n-1})d\xi, A_{1h}^{-1}e^n\right)_\Omega\Big| \le \frac{1}{32}\|e^n\|_0^2+\frac{1}{32}\|\varepsilon^{n-1}\|_0^2+c\|e^n\|_{-1}^2 + ch^4\|A_2\theta^{n-1}\|_{L^2}^2, \tag{5.15}$$

$$\sigma\Big|\left(\int_{-d}^z \nabla\cdot(u^{n-1}-u_h^{n-1})d\xi, A_{2h}^{-1}\varepsilon^n\right)_\Omega\Big| \le \frac{1}{32}\|e^{n-1}\|_0^2+\frac{1}{32}\|\varepsilon^n\|_0^2+c\|\varepsilon^n\|_{-1}^2 + \sigma ch^4\|A_1u^{n-1}\|_{L^2}^2, \tag{5.16}$$

$$|(f\vec{k}\times(u^n-u_h^n),A_{1h}^{-1}e^n)_\Omega| \le \frac{1}{32}\|e^n\|_0^2+ch^4\|A_1u^n\|_{L^2}^2+c\|e^n\|_{-1}^2, \tag{5.17}$$

$$|(\nabla\cdot A_{1h}^{-1}e^n, p^n-\rho_hp^n)_\Omega| \le \frac{1}{32}\|e^n\|_0^2+ch^4\|p^n\|_{H^2(\omega)}^2, \tag{5.18}$$

$$|(R_{1h}u^n-u^n,e^n)_\Omega|+|(R_{2h}\theta^n-\theta^n,\varepsilon^n)_\Omega| \le \frac{1}{32}(\|e^n\|_0^2+\|\varepsilon^n\|_0^2) + ch^4(\|A_1u^n\|_{L^2}^2+\|A_2\theta^n\|_{L^2}^2). \tag{5.19}$$

Combining the above estimates with (5.4) and using (3.6) and (3.7) yields

$$\begin{aligned}
&[\|e^n\|_{-1}^2+\|\varepsilon^n\|_{-1}^2]-[\|e^{n-1}\|_{-1}^2+\|\varepsilon^{n-1}\|_{-1}^2]+\frac{5}{4}(\|e^n\|_0^2+\|\varepsilon^n\|_0^2)\tau\\
&-\frac{1}{4}(\|e^{n-1}\|_0^2+\|\varepsilon^{n-1}\|_0^2)\tau\\
&+[1-c_3\tau-c_4(\|A_1u^{n-1}\|_{L^2}^4+\|A_1u^n\|_{L^2}^4+\|A_2\theta^n\|_{L^2}^4)\tau](\|e^n-e^{n-1}\|_{-1}^2\\
&+\|\varepsilon^n-\varepsilon^{n-1}\|_{-1}^2)\\
&\le c(\|L_1^{\frac12}(u^n-u^{n-1})\|_{L^2}^2+\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{l^2}^2)\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2\tau\\
&+c(\|L_1^{\frac12}(u^n-u_h^n)\|_{L^2}^2+\|L_2^{\frac12}(\theta^n-\theta_h^n)\|_{L^2}^2)\|L_1^{\frac12}(u^{n-1}-u_h^{n-1})\|_{L^2}^2\\
&+ch^4[\|A_1u^{n-1}\|_{L^2}^2+\|A_1u^n\|_{L^2}^2+\|A_2\theta^n\|_{L^2}^2+\|A_2\theta^{n-1}\|_{L^2}^2+\|p^n\|_{H^2(\omega)}^2]\tau\\
&+ch^4[\|A_1u^{n-1}\|_{L^2}^4+\|A_1u^n\|_{L^2}^4+\|A_2\theta^n\|_{L^2}^4]\tau+d_{n-1}(\|e^{n-1}\|_{-1}^2+\|\varepsilon^{n-1}\|_{-1}^2)\tau
\end{aligned} \tag{5.20}$$

for some positive constants c_3 and c_4, where $d_{n-1}=c+c(\|A_1u^{n-1}\|_{L^2}^4+\|A_1u^n\|_{L^2}^4+\|A_2\theta^n\|_{L^2}^4)$. Summing (5.20) from $n=1$ to $n=m$ and using (3.6), (3.7), Theorems 2.1 and 4.3, and assumption (A3), we deduce

$$\|e^m\|_{-1}^2+\|\varepsilon^m\|_{-1}^2+\tau\sum_{n=1}^m(\|u^n-u_h^n\|_0^2+\|\theta^n-\theta_h^n\|_0^2)\le Ch^4+\tau\sum_{n=0}^{m-1}d_n(\|e^n\|_{-1}^2+\|\varepsilon^n\|_{-1}^2). \tag{5.21}$$

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Applying Lemma 2.2 to (5.21), we arrive at (5.3). ∎

THEOREM 5.4. *Under the assumptions of Lemma* 5.3, (u_h^n, θ_h^n) *satisfies the following error estimate:*

$$\sigma(t_m)[\|u^m - u_h^m\|_{L^2}^2 + \|\theta^m - \theta_h^m\|_{L^2}^2] \le \kappa(h^4 + \tau^2). \tag{5.22}$$

Proof. Setting $(e^n, r^n, \varepsilon^n) = (R_{1h}u^n - u_h^n, \rho_h p^n - p_h^n, R_{2h}\theta^n - \theta_h^n)$ and taking $(v_h, q_h) = (e^n, r^n)$ in (4.2) and $\phi_h = \varepsilon^n$ in (4.3), adding these two relations, and using (2.13) and (2.14), we find

$$\begin{aligned}
&\frac{1}{2\tau}[\|e^n\|_{L^2}^2 - \|e^{n-1}\|_{L^2}^2 + \|e^n - e^{n-1}\|_{L^2}^2] + \frac{1}{2\tau}[\|\varepsilon^n\|_{L^2}^2 - \|\varepsilon^{n-1}\|_{L^2}^2 + \|\varepsilon^n - \varepsilon^{n-1}\|_{L^2}^2]\\
&+ \|A_{1h}^{\frac12}e^n\|_{L^2}^2 + \|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2 + (d_tu^n - R_{1h}d_tu^n, e^n)_\Omega + (d_t\theta^n - R_{2h}d_t\theta^n, \varepsilon^n)_\Omega\\
&+ b(u^{n-1} - u_h^{n-1}, u^{n-1} - u^n, \bar{e}^n) + b(u^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \bar{e}^n)\\
&+ b(u^{n-1} - u_h^{n-1}, u^n, e^n) + b(u^{n-1}, u^n - R_{1h}u^n, e^n) + (f\vec{k} \times (u^n - R_{1h}u^n), e^n)_\Omega\\
&+ b(u_h^{n-1} - u^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \bar{e}^n) + b(u_h^{n-1} - u^{n-1}, u^n - u_h^n, e^n)\\
&+ b(u^{n-1} - u_h^{n-1}, \theta^n, \varepsilon^n) + b(u_h^{n-1} - u^{n-1}, \theta^n - \theta_h^n, \varepsilon^n) + \gamma b(u^{n-1}, \theta^n - R_{2h}\theta^n, \varepsilon^n)\\
&- (\nabla\cdot e^n, p^n - \rho_h p^n)_\Omega + \sigma\left(\int_{-d}^z \nabla\cdot(u^{n-1} - u_h^{n-1})d\xi, \varepsilon^n\right)_\Omega\\
&- \gamma\left(\int_{-d}^z \nabla(\theta^{n-1} - \theta_h^{n-1})d\xi, e^n\right)_\Omega = 0.
\end{aligned}$$

In the calculations of (4.10)–(4.16), by replacing (P_{1h}, P_{2h}) by (R_{1h}, R_{2h}), we deduce exactly the same estimates as (4.10)–(4.16). Combining these estimates with the above relation yields

(5.23)

$$\begin{aligned}
&[\|e^n\|_{L^2}^2 + \|\varepsilon^n\|_{L^2}^2 - [\|e^{n-1}\|_{L^2}^2 + \|\varepsilon^{n-1}\|_{L^2}^2]\\
&+ \frac12(1 - c_1\|L_1^{\frac12}(u^{n-1} - u_h^{n-1})\|_{L^2})(\|A_{1h}^{\frac12}e^n\|_{L^2}^2 + \|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2)\tau\\
&\le 2|(d_tu^n - R_{1h}d_tu^n, e^n)_\Omega|\tau + 2|(d_t\theta^n - R_{2h}d_t\theta^n, \varepsilon^n)_\Omega|\tau\\
&+ c(\|A_1^{\frac12}(u^n - u^{n-1})\|_{L^2}^2 + \|L_1^{\frac12}(u^{n-1} - u_h^{n-1})\|_{l^2}^2)\|L_1^{\frac12}(u^{n-1} - u_h^{n-1})\|_{L^2}^2\tau\\
&+ c(h^2\|L_1^{\frac12}(u^n - u_h^n)\|_{L^2}^2 + ch^2\|A_1u^n\|_{L^2}^2 + +h^2\|A_2\theta^n\|_{L^2}^2)\|L_1^{\frac12}(u^{n-1} - u_h^{n-1})\|_{L^2}^2\tau\\
&+ ch^4\|u^{n-1}\|_{H^3}^2[\|A_1u^n\|_{L^2}^2 + \|A_2\theta^n\|_{L^2}^2]\tau + ch^4[\|A_1u^n\|_{L^2}^2 + \|A_2\theta^n\|_{L^2}^2 + \|p^n\|_{H^2(\omega)}^2]\tau\\
&+ c\|A_1u^{n-1}\|_{L^2}^2\|u^n - u_h^n\|_{L^2}^2\tau + d_{n-1}(\|u^{n-1} - u_h^{n-1}\|_{L^2}^2 + \|\theta^{n-1} - \theta_h^{n-1}\|_{L^2}^2)\tau,
\end{aligned}$$

where $d_{n-1} = c + c(\|u^n\|_{H^3}^2 + \|\theta^n\|_{H^3}^2)$.

Hence, multiplying (5.23) by $\sigma(t_n)$, summing from $n = 1$ to $n = m$, and using (4.17), (4.18), Theorem 2.1, Theorem 4.3, Lemma 5.2, and Lemma 5.3, we deduce

(5.24)

$$\begin{aligned}
&\sigma(t_m)[\|u^m - u_h^m\|_{L^2}^2 + \|\theta^m - \theta_h^m\|_{L^2}^2] + \tfrac14\tau\textstyle\sum_{n=1}^m \sigma(t_n)(\|A_{1h}^{\frac12}e^n\|_{L^2}^2 + \|A_{2h}^{\frac12}\varepsilon^n\|_{L^2}^2)\\
&\le C(h^4 + h^2\tau) + 2\tau\textstyle\sum_{n=0}^{m-1} d_n\sigma(t_n)(\|u^n - u_h^n\|_{L^2}^2 + \|\theta^n - \theta_h^n\|_{L^2}^2).
\end{aligned}$$

Applying Lemma 2.2 to (5.24), we arrive at (5.22).

Finally, by combining Theorem 4.3, Theorem 5.4, and Lemma 5.3 with Theorem 2.1, we deduce Theorem 1.1. ∎

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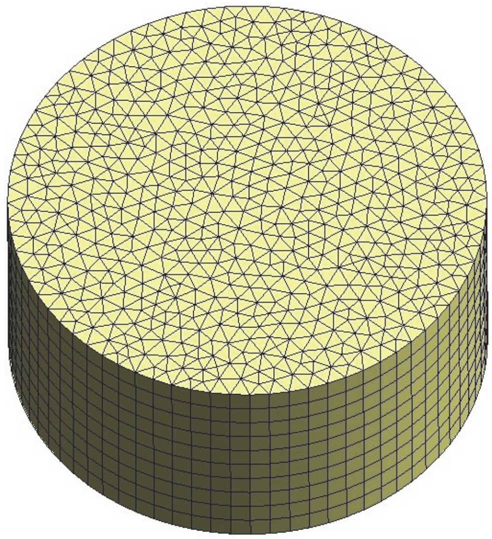

FIG. 1. *Schematic mesh partition for the cylindrical computational domain* Ω.

6. Numerical investigations. In this part, numerical investigations are implemented to validate the theoretical error analysis in the previous sections. Meanwhile, the proposed numerical method is used to simulate a double gyres problem for further validation of reliability of this numerical method.

6.1. Numerical accuracy. In this part, we check the numerical accuracy of the numerical scheme by flows with analytic solutions. The computational domain is $\Omega = \omega \times [-1, 0]$, where $\omega = \{(x, y) : x^2 + y^2 <= r^2\}$ and $r = 0.5$. The exact velocity $u = (u_1, u_2)$ and p are set by the following formulations:

$$u = \phi_y(x, y)\psi_z(z),\ v = \phi_x(x, y)\psi_z(z),\ p = x + y, \tag{6.1}$$

where $\phi(x, y) = (x^2 + y^2)(x^2 + y^2 - r^2)\exp(-(\nu_1 + \mu_1)t)$ and $\psi(z) = z^2(z + 1)^2$. According to (1.3), w and the external force F_1 can be determined explicitly. θ is set by

$$\theta = \psi(x, y)\psi(z)\exp(-(\nu_1 + \mu_1)t). \tag{6.2}$$

The right-hand side F_2 of (1.2) can be determined by (6.2). $\nu_1 = \mu_1 = \nu_2 = \mu_2 = 0.1$, $f_0 = \beta = 10^{-5}$, $\sigma = 0.001$, $\gamma = 1$, and $\alpha = 0$.

In order to validate the theoretical results in the error analysis, we choose a series of mesh scales: $h = \{1/20, 1/25, 1/30, 1/40, 1/45, 1/50, 1/55, 1/60, 1/65, 1/70\}$. In simulations, we set the discrete time step $\tau = 0.0001 < h^2$. The schematic mesh partition is illustrated in Figure 1. In Figures 2 and 3, the L^2 and H^1 relative errors of u and θ are shown. According to Theorem 1.1, for $\tau < h^2$, we have

$$\|u(t_m) - u_h^m\|_{L^2}/\|u(t_m)\|_{L^2} \sim \|\theta(t_m) - \theta_h^m\|_{L^2}/\|\theta(t_m)\|_{L^2} \sim h^2$$

and

$$|u(t_m) - u_h^m|_1/|u(t_m)|_1 \sim |\theta(t_m) - \theta_h^m|_1/|\theta(t_m)|_1 \sim h.$$

The above relations can be observed clearly in Figures 2 and 3. In Figures 4 and 5, the 3D plots of u, v, w, p, and θ are illustrated at $t = 0.2$.

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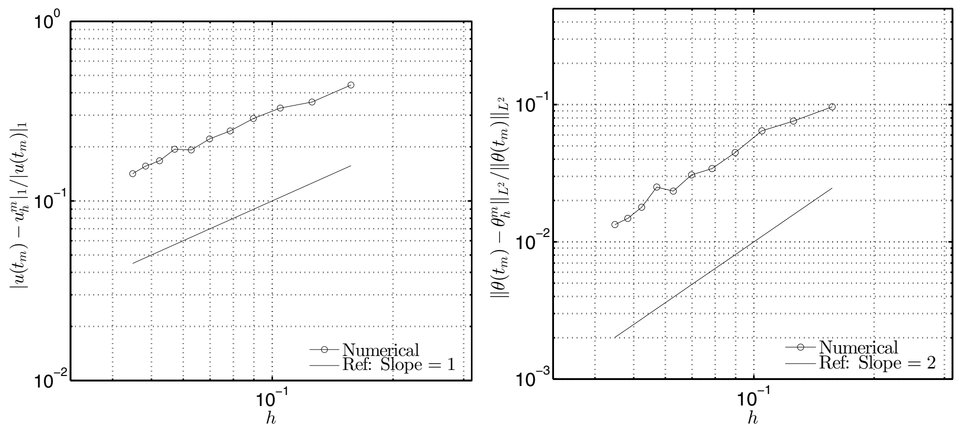
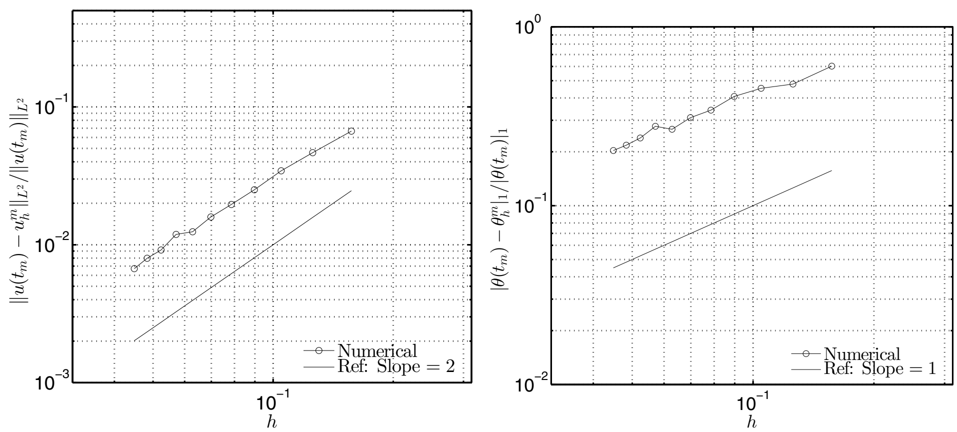


(a) $\|u(t_m) - u_h^m\|_{L^2}/\|u(t_m)\|_{L^2} \sim h^2$ (b) $\|\theta(t_m) - \theta_h^m\|_{L^2}/\|\theta(t_m)\|_{L^2} \sim h^2$

FIG. 2. *L^2 relative errors of u and θ at $t = 0.02$.*



(a) $\|u(t_m) - u_h^m\|_{H^1}/\|u(t_m)\|_{H^1} \sim h$ (b) $\|\theta(t_m) - \theta_h^m\|_{H^1}/\|\theta(t_m)\|_{H^1} \sim h$

FIG. 3. *H^1 relative errors of u and θ at $t = 0.02$.*

6.2. Double gyres simulations. In order to carry out practical investigations conveniently, the nondimensional form of the PEs is introduced. Let U and θ_0 be the reference values of the horizontal velocity and density, respectively. Meanwhile, we also consider L and H as the reference scales of the horizontal and vertical lengths. Let $\delta = H/L$ be the aspect ratio; we introduce the following rescaled quantities:

$$u = Uu', w = \delta U w', \theta = \theta_0 \theta', t = t'L/U, x = Lx', y = Ly', z = Hz' = \delta L z' \tag{6.3}$$

and

$$F_1 = F_1' U^2/L, F_2 = F_2' U \theta_0/L. \tag{6.4}$$

The corresponding parameters Re_{ν_1}, Re_{μ_1}, Re_{ν_2}, Re_{μ_2}, R_1, R_2, R_γ and R_θ are defined

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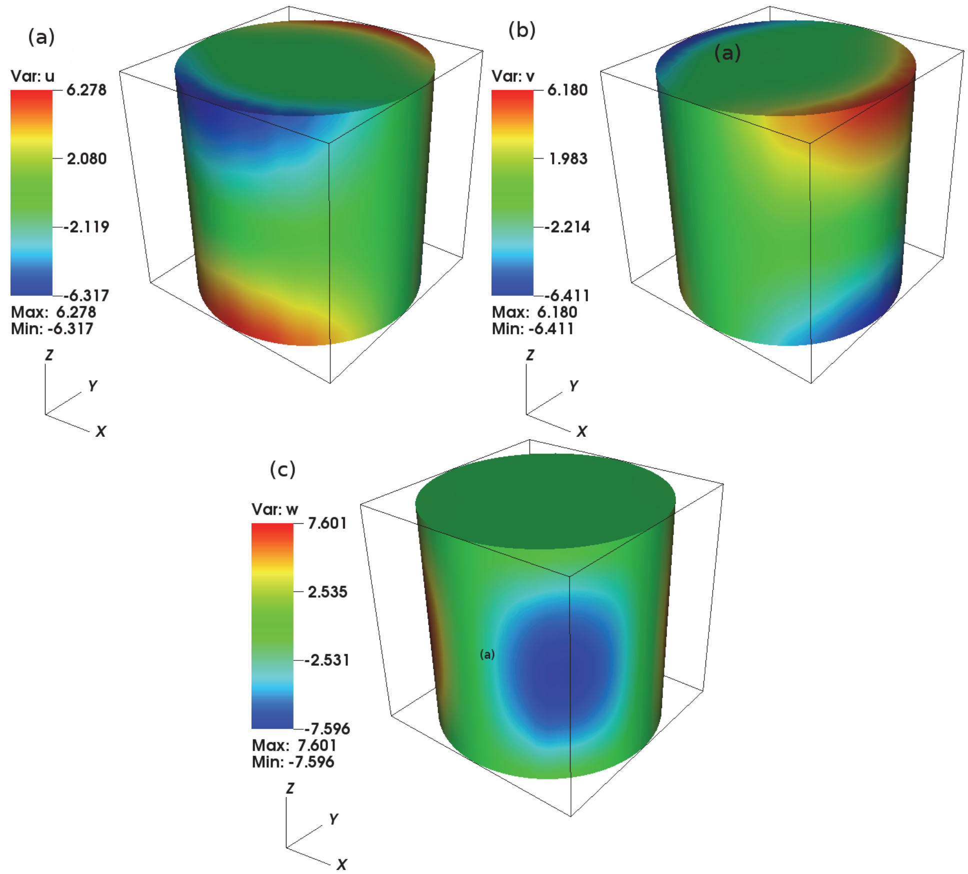

FIG. 4. *Velocity* $u(x,y,z,t)$ (a), $v(x,y,z,t)$ (b), *and* $w(x,y,z,t)$ (c) *at* $t = 0.2$.

as follows:

(6.5)
$$1/Re_{\nu_1} = \nu_1/LU, 1/Re_{\mu_1} = \mu_1 L/H^2 U = \mu_1/\delta^2 LU, 1/R_1 = f_0 L/U, 1/R_\gamma = \gamma U^2/H,$$

(6.6)
$$1/Re_{\nu_2} = \nu_2/LU, 1/Re_{\mu_2} = \mu_2 L/H^2 U = \mu_2/\delta^2 LU, 1/R_2 = f_0/\beta L, 1/R_\theta = \sigma L/U\theta_0.$$

Substituting (6.5) and (6.6) into (1.1)–(1.4) and dropping the super index primes, the nondimensional form of the PEs reads as follows:

$$u_t - \frac{1}{Re_{\nu_1}}\Delta u - \frac{1}{Re_{\mu_1}}\partial_z w + (u\cdot\nabla)u + w\partial_z u + \nabla P + \frac{1}{R_1}(1+R_2 y)\vec{k}\times u = F_1, \tag{6.7}$$

$$\theta_t - \frac{1}{Re_{\nu_2}}\Delta\theta - \frac{1}{Re_{\mu_2}}\partial_z\theta + (u\cdot\nabla)\theta + w\partial_z\theta - 1/R_\theta w = F_2, \tag{6.8}$$

$$\nabla\cdot u + \partial_z w = 0, \tag{6.9}$$

$$\partial_z P + \frac{1}{R_\gamma}\theta = 0. \tag{6.10}$$

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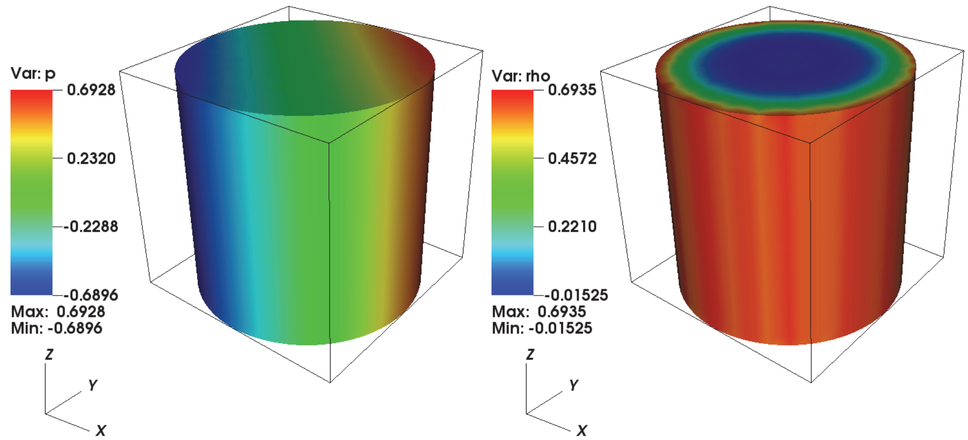

FIG. 5. $p(x, y, t)$ *(left) and* $\theta(x, y, z, t)$ *(right) at* $t = 0.2$.

The basic parameters used in the simulation are listed below:

Coriolis parameter	$f_0 = 5 \times 10^{-5}\mathrm{s}^{-1}$
$f_0 + \beta y$	$\beta = 2 \times 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$
Wind stress amplitude	$\tau_0 = 0.1\,\mathrm{Nm}^{-2}$
Density scale	$\rho_0 = 1022\,\mathrm{kgm}^{-3}$
Acceleration of gravity	$g = 9.806\,\mathrm{ms}^{-2}$
Reference velocity	$U = 1\,\mathrm{ms}^{-1}$

The surface of the ocean is forced by a steady wind stress $\tau^* = (\tau_x^*, \tau_y^*) = (-\tau_0 \cos(2\pi y/L), 0)$, where τ_0 is the amplitude. The boundary conditions for the velocity and density on Γ_u are given as follows:

$$\tau_v^* = (\tau_x^*, 0), \alpha = 0.001. \tag{6.11}$$

The physical domain is $\Omega_p = \omega_p \times [-500\mathrm{m}, 0]$, and $\omega_p = [0, 2\times 10^6\mathrm{m}] \times [0, 2\times 10^6\mathrm{m}]$. The computational domain is $\Omega = \omega \times [-1, 0]$, where ω is a square with the edge length of unity.

First, we consider a low Reynolds number problem with $Re_{\nu_1} = Re_{\nu_2} = 10$ and $Re_{\mu_1} = Re_{\mu_2} = 20$. The time step is $\tau = 0.0001$, the horizontal mesh scale h_h of ω is $1/20$, and the vertical mesh scale h_v is $1/15$. The schematic domain with a mesh partition is shown in Figure 6. For these values of the Reynolds number, the flow reaches a steady state characterized by two gyres (cyclonic and anticyclonic) in Figure 7. Also, in Figure 8, the pressure $P(x, y, z)$ and the averaged pressure $p(x, y)$ are shown at the slice $x = 0.5$. We observe that along the vertical direction, at each horizontal plane, the pressure $p(x, y)$ has the same profile and the pressure $P(x, y, z)$ has different profiles with respect to different depths.

In order to validate the reliability of the proposed method, we choose larger Reynolds numbers $Re_{\nu_1} = Re_{\nu_2} = 20$ and $Re_{\mu_1} = Re_{\mu_2} = 200$ and implement a long-time simulation. In Figures 9 and 10, the stream tracers are given at different dimensionless time t. It is clear to observe that the flows are characterized by two gyres. Furthermore, the approximate steady state is reached for the long-time evolution.

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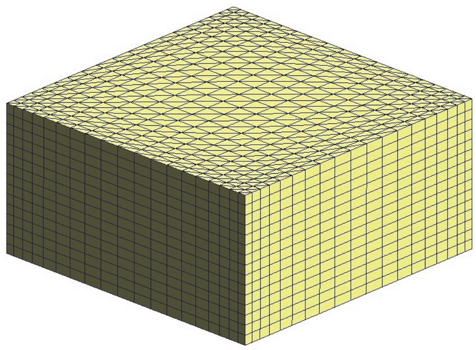

FIG. 6. *Schematic mesh partition for the cube computational domain* Ω.

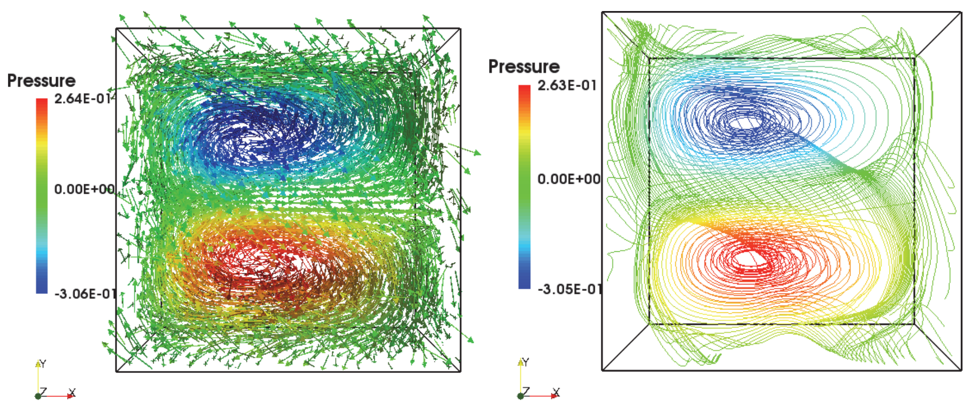

FIG. 7. 3*D Velocity vectors (left) and stream tracers (right).*

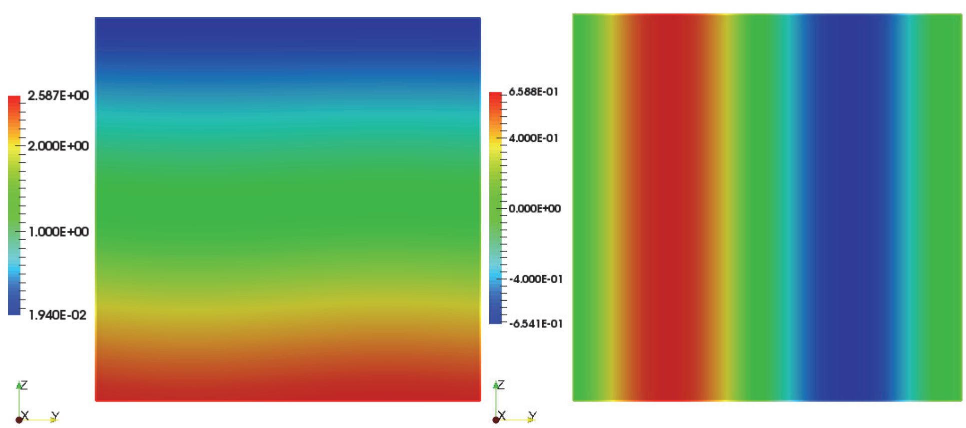

FIG. 8. *Slices of pressure* $P(x, y, z)$ *and surface pressure* $p(x, y)$ *at* $x = 0.5$*: (left)* $P(x, y, z)$*; (right)* $p(x, y)$.

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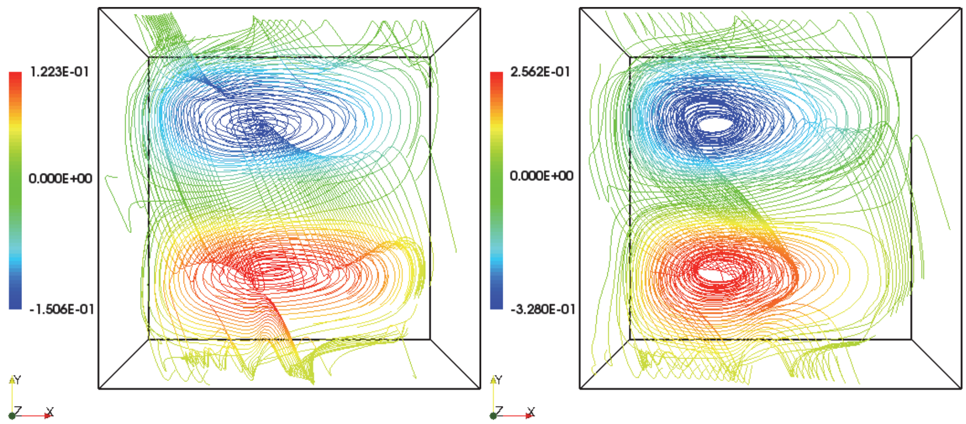
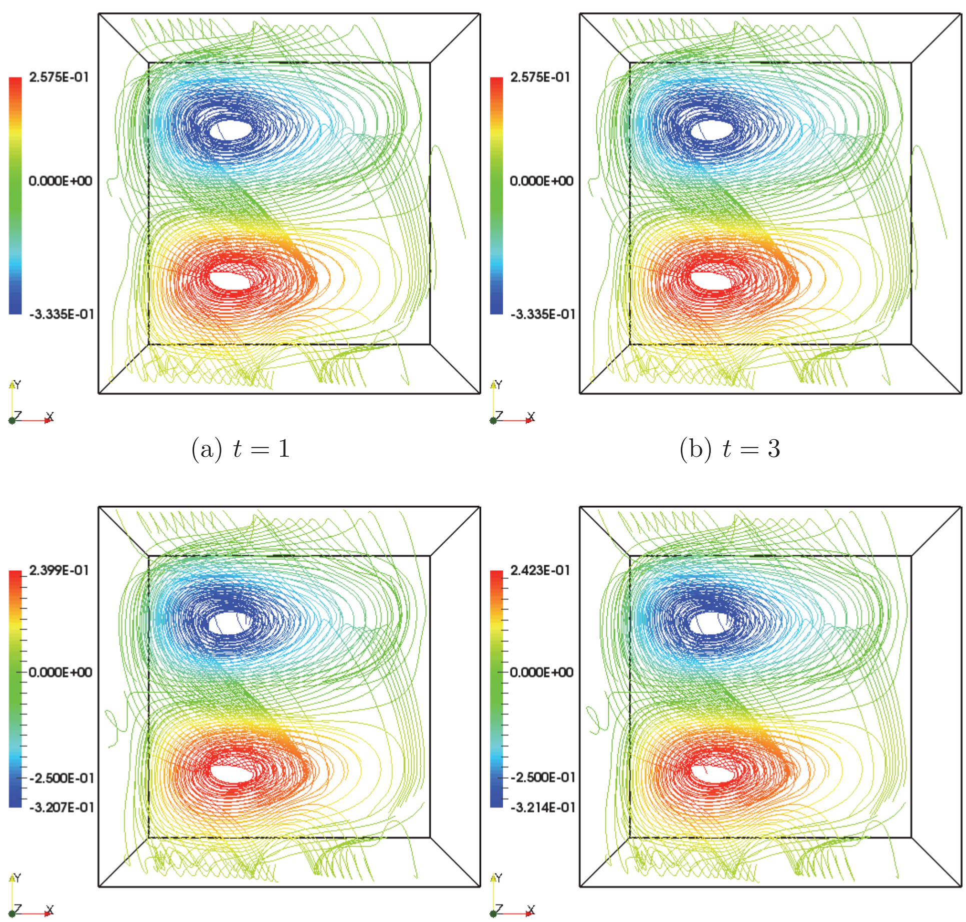


(a) $t = 0.1$ (b) $t = 0.5$

FIG. 9. *Stream tracers at early stages. Colorbar is set by pressure (t denotes dimensionless time unit).*



(a) $t = 1$ (b) $t = 3$

(c) $t = 6$ (d) $t = 9$

FIG. 10. *Stream tracers at long-time stages. Colorbar is set by pressure.*

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Appendix A. The proof of Lemma 4.2.

Proof. Taking $(v_h, q_h) = (A_{1h}e^n + d_te^n, 0)$ in (4.2) and $\phi_h = A_{2h}\varepsilon^n + d_t\varepsilon^n$ in (4.3) and adding these two relations, we find

(A.1)

$$
\begin{aligned}
&(d_tu^n - d_tR_{1h}u^n, A_{1h}e^n + d_te^n)_\Omega + \frac{1}{\tau}[\|A_{1h}^{\frac{1}{2}}e^n\|_{L^2}^2 - \|A_{1h}^{\frac{1}{2}}e^{n-1}\|_{L^2}^2\\
&+ \|A_{1h}^{\frac{1}{2}}(e^n - e^{n-1})\|_{L^2}^2]\\
&+ (d_t\theta^n - d_tR_{2h}\theta^n, A_{2h}\varepsilon^n + d_t\varepsilon^n)_\Omega + \frac{1}{\tau}[\|A_{2h}^{\frac{1}{2}}\varepsilon^n\|_{L^2}^2 - \|A_{2h}^{\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2\\
&+ \|A_{2h}^{\frac{1}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2]\\
&+ \|d_te^n\|_{L^2}^2 + \|d_t\varepsilon^n\|_{L^2}^2 + \|A_{1h}e^n\|_{L^2}^2 + \|A_{2h}\varepsilon^n\|_{L^2}^2 + (A_{1h}e^n + d_te^n, \nabla(p^n - \rho_hp^n))_\Omega\\
&+ b(u^{n-1} - u_h^{n-1}, u^{n-1} - u^n, \overline{A_{1h}e^n + d_te^n}) + (f\vec{k} \times (u^n - u_h^n), A_{1h}e^n + d_te^n)_\Omega\\
&+ b(u_h^{n-1} - u^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \overline{A_{1h}e^n + d_te^n})\\
&+ b(u^{n-1} - u_h^{n-1}, u^n, A_{1h}e^n + d_te^n) + b(u_h^{n-1} - u^{n-1}, u^n - u_h^n, A_{1h}e^n + d_te^n)\\
&+ b(u^{n-1} - u_h^{n-1}, \theta^n, \varepsilon^n) + b(u_h^{n-1} - u^{n-1}, \theta^n - \theta_h^n, \varepsilon^n)\\
&- (\nabla \cdot (A_{1h}e^n + d_te^n), p^n - \rho_hp^n)_\Omega\\
&+ b(u^{n-1}, u^{n-1} - u_h^{n-1} - (u^n - u_h^n), \overline{A_{1h}e^n + d_te^n}) + b(u^{n-1}, u^n - u_h^n, A_{1h}e^n + d_te^n)\\
&+ b(u^{n-1}, \theta^n - \theta_h^n, A_{2h}\varepsilon^n + d_t\varepsilon^n) + \sigma\left(\int_{-d}^z \nabla \cdot (u^{n-1} - u_h^{n-1})d\xi, A_{2h}\varepsilon^n + d_t\varepsilon^n\right)_\Omega\\
&- \gamma\left(\int_{-d}^z \nabla(\theta^{n-1} - \theta_h^{n-1})d\xi, A_{1h}e^n + d_te^n\right)_\Omega = 0.
\end{aligned}
$$

Due to (4.1), (4.17), (4.18), and the Young inequality, we have the following estimates:

(A.2)

$$
\begin{aligned}
&|(d_tu^n - R_{1h}d_tu^n, A_{1h}e^n + d_te^n)_\Omega| \le \tfrac{1}{32}(\|A_{1h}e^n\|_{L^2}^2 + \|d_te^n\|_{L^2}^2) + ch^2\|A_1^{\frac{1}{2}}d_tu^n\|_{L^2}^2,\\
&|(d_t\theta^n - R_{2h}d_t\theta^n, A_{2h}\varepsilon^n + d_t\varepsilon^n)_\Omega| \le \tfrac{1}{32}(\|A_{2h}\varepsilon^n\|_{L^2}^2 + \|d_t\varepsilon^n\|_{L^2}^2) + ch^2\|A_2^{\frac{1}{2}}d_t\theta^n\|_{L^2}^2,\\
&|(A_{1h}e^n + d_te^n, \nabla(p^n - \rho_hp^n))_\Omega| \le \tfrac{1}{32}(\|A_{1h}e^n\|_{L^2}^2 + \|d_te^n\|_{L^2}^2) + ch^2\|p\|_{H^2(\omega)}^2,\\
&|(f\vec{k} \times (u^n - u_h^n), A_{1h}e^n + d_te^n)_\Omega| \le \tfrac{1}{32}(\|A_{1h}e^n\|_{L^2}^2 + \|d_te^n\|_{L^2}^2) + c\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
\gamma\Big|\left(\int_{-d}^z \nabla(\theta^{n-1} - \theta_h^{n-1})d\xi, A_{1h}e^n + d_te^n\right)_\Omega\Big| &\le \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2 + \|d_te^n\|_{L^2}^2)\\
&+ c\|L_2^{\frac{1}{2}}(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^2
\end{aligned}\tag{A.3}
$$

$$
\begin{aligned}
\sigma\Big|\left(\int_{-d}^z \nabla \cdot (u^{n-1} - u_h^{n-1})d\xi, A_{2h}\varepsilon^n + d_t\varepsilon^n\right)_\Omega\Big| &\le \frac{1}{32}(\|A_{2h}\varepsilon^n\|_{L^2}^2 + \|d_t\varepsilon^n\|_{L^2}^2)\\
&+ c\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2,
\end{aligned}\tag{A.4}
$$

$$
\begin{aligned}
|b(u^{n-1} - u_h^{n-1}, u^{n-1} - u^n, \overline{A_{1h}e^n + d_te^n})| &\le \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2 + \|d_te^n\|_{L^2}^2)\\
&+ ch^{-1}\|A_1^{\frac{1}{2}}(u^n - u^{n-1})\|_{L^2}^2\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2,
\end{aligned}\tag{A.5}
$$

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$$|b(u_h^{n-1}-u^{n-1},u^{n-1}-u_h^{n-1},\overline{A_{1h}e^n+d_te^n})| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2+\|d_te^n\|_{L^2}^2)$$
$$+ch^{-1}\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^2\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^2, \tag{A.6}$$

$$|b(u_h^{n-1}-u^{n-1},u^n-u_h^n,\overline{A_{1h}e^n+d_te^n})| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2+\|d_te^n\|_{L^2}^2)$$
$$+ch^{-1}\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^2\|L_1^{\frac{1}{2}}(u^n-u_h^n)\|_{L^2}^2, \tag{A.7}$$

$$|b(u_h^{n-1}-u^{n-1},u^n-R_{1h}u^n,A_{1h}e^n+d_te^n)| \leq c\|A_1u^n\|_{L^2}\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})$$
$$\|_{L^2}\|A_{1h}e^n+d_te^n\|_{L^2} \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2+\|d_te^n\|_{L^2}^2)$$
$$+c\|A_1u^n\|_{L^2}^2\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^2, \tag{A.8}$$

$$|b(u_h^{n-1}-u^{n-1},e^n,A_{1h}e^n+d_te^n)| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2+\|d_te^n\|_{L^2}^2)$$
$$+ch^{-2}\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^4\|A_{1h}^{\frac{1}{2}}e^n\|_{L^2}^2, \tag{A.9}$$

$$|b(u_h^{n-1}-u^{n-1},\theta^n-R_{2h}\theta^n,A_{2h}\varepsilon^n+d_t\varepsilon^n)| \leq \frac{1}{32}(\|A_{2h}\varepsilon^n\|_{L^2}^2+\|d_t\varepsilon^n\|_{L^2}^2)$$
$$+c\|A_2\theta^n\|_{L^2}^2\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^2, \tag{A.10}$$

$$|b(u_h^{n-1}-u^{n-1},\varepsilon^n,A_{2h}\varepsilon^n+d_t\varepsilon^n)| \leq \frac{1}{32}(\|A_{2h}\varepsilon^n\|_{L^2}^2+\|d_t\varepsilon^n\|_{L^2}^2)$$
$$+ch^{-2}\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^4\|A_1^{\frac{1}{2}}\varepsilon^n\|_{L^2}^2, \tag{A.11}$$

$$|b(u^{n-1}-u_h^{n-1},u^n,A_{1h}e^n+d_te^n)| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2+\|d_te^n\|_{L^2}^2)$$
$$+c\|u^n\|_{H^3}^2\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^2, \tag{A.12}$$

$$|b(u^{n-1}-u_h^{n-1},\theta^n,A_{2h}\varepsilon^n+d_t\varepsilon^n)| \leq \frac{1}{32}(\|A_{2h}\varepsilon^n\|_{L^2}^2+\|d_t\varepsilon^n\|_{L^2}^2)$$
$$+c\|\theta^n\|_{H^3}^2\|L_1^{\frac{1}{2}}(u^{n-1}-u_h^{n-1})\|_{L^2}^2, \tag{A.13}$$

$$|b(u^{n-1},u^n-P_{1h}u^n,A_{1h}e^n+d_te^n)| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2+\|d_te^n\|_{L^2}^2)$$
$$+ch^2\|u^{n-1}\|_{H^3}^2\|A_1u^n\|_{L^2}^2, \tag{A.14}$$

$$|b(u^{n-1},e^n,A_{1h}e^n+d_te^n)| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2+\|d_te^n\|_{L^2}^2)$$
$$+c\|A_1u^{n-1}\|_{L^2}^4\|L_1^{\frac{1}{2}}(u^n-u_h^n)\|_{L^2}^2$$
$$+ch^2\|A_1u^{n-1}\|_{L^2}^4\|A_1u^n\|_{L^2}^2, \tag{A.15}$$

$$|b(u^{n-1},\theta^n-P_{2h}\theta^n,A_{2h}\varepsilon^n+d_t\varepsilon^n)| \leq \frac{1}{32}(\|A_{2h}\varepsilon^n\|_{L^2}^2+\|d_t\varepsilon^n\|_{L^2}^2)$$
$$+ch^2\|u^{n-1}\|_{H^3}^2\|A_2\theta^n\|_{L^2}^2, \tag{A.16}$$

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$$|b(u^{n-1}, \varepsilon^n, A_{2h}\varepsilon^n + d_t\varepsilon^n)| \leq \frac{1}{32}(\|A_{2h}\varepsilon^n\|_{L^2}^2 + \|d_t\varepsilon^n\|_{L^2}^2)$$
$$+ c\|A_1 u^{n-1}\|_{L^2}^4 \|A_2\theta^n\|_{L^2}^2$$
$$+ c\|A_1 u^{n-1}\|_{L^2}^4 \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2, \tag{A.17}$$

$$|b(u^{n-1}, u^{n-1} - u_h^{n-1}, \overline{A_{1h}e^n + d_t e^n})| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2 + \|d_t e^n\|_{L^2}^2)$$
$$+ c\|A_1 u^{n-1}\|_{L^2}^2 \|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2, \tag{A.18}$$

$$|b(u^{n-1}, u^n - u_h^n, \overline{A_{1h}e^n + d_t e^n})| \leq \frac{1}{32}(\|A_{1h}e^n\|_{L^2}^2 + \|d_t e^n\|_{L^2}^2)$$
$$+ c\|A_1 u^{n-1}\|_{L^2}^2 \|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2. \tag{A.19}$$

Combining (A.1) with these inequalities and using Theorem 2.1, we obtain

(A.20)
$$[\|A_{1h}^{\frac{1}{2}}e^n\|_{L^2}^2 - \|A_{1h}^{\frac{1}{2}}e^{n-1}\|_{L^2}^2] + [\|A_{2h}^{\frac{1}{2}}\varepsilon^n\|_{L^2}^2 - \|A_{2h}^{\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2]$$
$$+ \tfrac{1}{2}[\|d_t e^n\|_{L^2}^2 + \|d_t\varepsilon^n\|_{L^2}^2]\tau + \tfrac{1}{2}[\|A_{1h}e^n\|_{L^2}^2 + \|A_{2h}\varepsilon^n\|_{L^2}^2]\tau$$
$$\leq \tau c h^2(\|A_1^{\frac{1}{2}}d_t u^n\|_{L^2}^2 + \|A_2^{\frac{1}{2}}d_t\theta^n\|_{L^2}^2 + \|p^n\|_{H^2(\omega)}^2)$$
$$+ \tau c h^{-1}\tau^2\|A_1^{\frac{1}{2}}d_t u^n\|_{L^2}^2 \|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2$$
$$+ \tau c h^2(\|A_1 u^n\|_{L^2}^2 + \|A_2\theta^n\|_{L^2}^2)(\|u^{n-1}\|_{H^3}^2 + \|A_1 u^{n-1}\|_{L^2}^4)]$$
$$+ \tau c(1 + \|A_1 u^{n-1}\|_{L^2}^2 + \|A_1 u^{n-1}\|_{L^4}^4)(\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2)$$
$$+ \tau c h^{-2}(\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^4 + \|L_2^{\frac{1}{2}}(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^4)(\|L_1^{\frac{1}{2}}(u^n - u_h^n)\|_{L^2}^2$$
$$+ \|L_2^{\frac{1}{2}}(\theta^n - \theta_h^n)\|_{L^2}^2)$$
$$+ \tau d_{n-1}\|(\|L_1^{\frac{1}{2}}(u^{n-1} - u_h^{n-1})\|_{L^2}^2 + \|L_2^{\frac{1}{2}}(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^2).$$

Summing (A.20) from $n = 1$ to m and using (4.17), (4.18), and Theorem 2.1, we have completed the proof of Lemma 4.2. ∎

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