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'NONLINEAR REALIZATIONS OF GROUPS'

by

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ABSTRACT

Recent studies of nonlinear realizations of groups are reviewed. The general methods developed for the construction of nonlinear realizations are demonstrated by a number of detailed calculations. As examples groups are chosen which are of interest for physical applications. Physical consequences arising if these groups are realized nonlinearly are indicated.

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CHAPTER I

INTRODUCTION

In the present thesis we are trying to give a systematic review of some recent studies of nonlinear realizations of groups in physics.

The development of nonlinear realization techniques is due to the search for a better description of elementary particles and their interactions. One of the well-established experimental facts is invariance (at least approximate) of interactions under various groups of symmetry. If we try to fit elementary particles into multiplets corresponding to linear representations of these groups (e.g. SU(3) x SU(3)), we are led to predict too many particles (not all of them seem to exist in nature). Nonlinear realizations provide an elegant means of solving this difficulty.

Historically, the linear realizations (representations) of groups were tried first and the extra particles were gotten rid of by imposing nonlinear constraints on the fields associated with the particles. Such a treatment is already completely equivalent to employing a nonlinear realization which is linear when restricted to a subgroup of the whole group of symmetry (Ref. 19).

Furthermore, nonlinear realizations may be relevant in quantum field theories with so-called spontaneous symmetry breaking, an effect which reduces the symmetry of the physical states (Refs. 14, 19). Spontaneous symmetry breaking is discussed in the literature from various angles and will not be treated here. A few remarks concerning the subject must suffice.

The spontaneous breaking of symmetry means (from a mathematical point of view) choosing a special set of solutions with less symmetry than the equations to which the solutions belong. This way of breaking a symmetry is attractive, for example, as a possibility in which conformal symmetry might be realized in nature. It enables us to introduce Lagrangians invariant under the conformal group which lead to physical solutions with a discrete spectrum of mass (corresponding to the observed particles). If the conformal group were realized linearly, we should expect to observe a continuous mass spectrum of elementary particles (Ref. 23), which clearly is not the case.

When a symmetry is broken spontaneously, the vacuum state cannot be invariant under the whole group of symmetry. It is invariant under some subgroup only (the same subgroup under which the realization becomes linear) (Refs. 18, 19). In such a theory there are many vacuum states, one for each inequivalent representation of the algebra of observables, so that one also speaks of a degenerate vacuum. These various vacuum states differ by different admixtures of zero-mass particles (Ref. 18). It turns out that Lagrangians which are invariant under a nonlinear transformation law of its fields usually also contain some fields which correspond to massless particles (Refs. 17, 19); however, fewer massless particles than would be required in case of a linear transformation law of the fields. By explicitly breaking the symmetry by adding a small mass term to the Lagrangian, we obtain a theory describing massive particles. However, it may perhaps not be necessary to introduce such symmetry breaking mass terms in order to describe massive particles.

The type of ordinary symmetry breaking mentioned in the previous paragraph is discussed by means of explicit examples in Chapter IV. Our main aim is to show what the theory of nonlinear realizations looks like in the form developed and discussed in Refs. 1, 2, 24. This form is motivated not so much by any physical arguments but by a certain mathematical elegance. Our approach is to follow this mathematical line and to demonstrate the theory by working out explicit examples of groups closely related to physics.

In the second chapter we give a review of the general theory of nonlinear realizations of groups, which is complete as far as compact, connected, semisimple groups are concerned (Refs. 1, 2, 24). Some facts concerning linear realizations (representations) of groups are also mentioned here (quoting Refs. 6, 7, 15, 16). In the third chapter we treat as an example the group SU(n) x SU(n) in detail, and an invariant Lagrangian is constructed in a separate section (Ref. 3). Chapter IV shows how the linear and nonlinear realizations are related in the case of the SU(2) x SU(2) group. Also some symmetry breaking terms and weak currents are constructed in the note-section (Ref. 17). The last chapter is a review of an attempt to apply the standard method of the general theory to a noncompact group of physical interest, namely the conformal group (Refs. 20, 5).

The exposition of nonlinear realizations presented in this thesis also leads to some open questions worthy of further investigation (for example, questions arising in connection with noncompact groups, or with the incorporation of gauge invariance into conformally invariant Lagrangians) as pointed out in the text. The thesis intends to serve as a basis for any such research.

CHAPTER II

GENERAL THEORY

a) Representations

First we shall quote some definitions and results concerning Lie groups and their representations (from Refs. 15, 16, 24). We shall pay no attention to details necessary to achieve mathematical rigour (such details can be found in Refs. 6, 7).

An r-parameter Lie group is a group whose elements can be labelled by r independent continuously varying real parameters with one more requirement, namely that the parameters of the product of two elements can be expressed as analytic functions of the parameters of these two elements.

If the parameters labelling group elements vary over a finite range, the group is called compact, if by varying the parameters continuously we can reach the unit element, the group is called connected.

A Lie group which has no proper invariant Abelian subgroup is called semisimple.

When we find a one-to-one correspondence between a set of $(n \times n)$-matrices and elements of a Lie group such that these matrices preserve the group multiplication, we say we have found an n-dimensional faithful representation of the group. When to one matrix there can correspond more elements of the group, we just speak of a representation (not faithful). Obviously, we can think of these matrices as transformations of vectors in an n-dimensional Euclidean space.

A matrix corresponding to an element g of a Lie group can be written (at least for sufficiently small ξ^i) in the form (Refs. 6, 16)

$$R(g) = \exp \{\xi^i X_i\}, \tag{1}$$

where the ξ^i are a special set of r real parameters (called canonical) representing the element g and where the X_i are a set of r (n x n)-matrices representing the so-called generators of the group. They satisfy relations

$$[X_i, X_j] = f_{ij}{}^k X_k, \tag{2}$$

where $f_{ij}{}^k$ are the so-called structure constants of the Lie groups (they are the same in any representation) and where $[X_i, X_j] = X_i X_j - X_j X_i$ is the commutator of the matrices X_i and X_j. The $f_{ij}{}^k$ are antisymmetric in the indices i and j (because of the commutator property) and must satisfy the Jacobi identity (Refs, 6, 16).

$$f_{ij}{}^s f_{ks}{}^m + f_{ki}{}^s f_{js}{}^m + f_{jk}{}^s f_{is}{}^m = 0 \tag{3}$$

Here and everywhere else in the thesis a summation over the same upper and lower index is understood.

The $f_{ij}{}^k$ themselves define a representation of the set of generators, the so-called adjoint representation (we have to take the first of the lower indices to label the matrix and the upper index and the second lower one as the usual row and column indices respectively).

We can define a symmetric tensor g_{ij} by

$$g_{ij} = f_{im}{}^k f_{jk}{}^m \tag{4}$$

It can always be diagonalized by a proper choice of generators of the Lie group (as any real symmetric matrix α can always be diagonalized by a real orthogonal matrix σ, i.e., a σ can be found so that $\sigma \cdot \alpha \cdot \sigma^{-1}$ is diagonal (Ref. 11).

A semisimple Lie group is characterized by det $(g_{ij}) \neq 0$ (Refs. 6, 15, 16), which implies that g has only non-zero diagonal elements when it has been diagonalized. A semisimple connected Lie group is compact if and only if, by a proper choice of parameters of the group, g_{ij} can be made a scalar multiple of Kronecker's symbol δ_{ij} (Ref. 7).

b) Nonlinear Realizations

We would like to describe now the nonlinear realizations in the standard form in which they were given by Coleman *et al* (Refs. 1, 2, 24). We shall again more or less only quote their results since it would be quite difficult to improve on the presentation given in their papers. Coleman *et al* also restrict themselves to compact, connected, semisimple Lie groups, and we shall point out where these assumptions are needed and what changes occur if compactness is removed.

Most of the Lie groups used in physics satisfy all of the assumptions (they are compact, connected and semisimple). One of the exceptions is the Poincaré group, which is neither compact nor semisimple, but for physical applications we are only interested in linear representations of this group anyway. Another, more important exception is the conformal group which violates compactness only. But the standard techniques of nonlinear realizations can be applied even here (see Ref. 20 and Chapter V) with all the appropriate precautions (Ref. 15).

We want to construct a nonlinear realization of a Lie group G which is linear when restricted to some continuous subgroup H of G. It is essential for this construction that the set of generators of G can be split into two parts with certain commutation relations among each other. On the one hand we have the set of generators

V_i ($i = 1, 2, \ldots, s$ where $s \leq r$) of the subgroup H and on the other the remaining generators A_j ($j = 1, 2, \ldots, r-s$) of the group G. While the commutators $[V_i, V_j]$ are, of course, linear combinations of the V's only, the A's can be chosen (if G is compact, semisimple and connected) in such a way that the commutator

$$[V_i, A_j] \tag{5}$$

is a linear combination of A's only (Refs. 1, 24). The conditions in parentheses are sufficient since they guarantee the full reducibility of the adjoint representation of G when restricted to the subgroup H only (Ref. 7). If G were not semisimple and if we chose for H an invariant subgroup, the commutator (5) would be a combination of the generators V only. Why this property of commutator (5) to be a linear combination of the generators A only is essential will become clear in the next paragraph.

Group elements in the neighbourhood of the identity element of G (for compact, semisimple and connected Lie groups it suffices to study such elements -- Ref. 24) can be decomposed uniquely (Refs. 6, 7, 24) into the form

$$e^{\xi \cdot A} e^{u \cdot V}, \tag{6}$$

where ξ and u are sets of $(r - s)$ and s real parameters respectively and where

$$\xi \cdot A = \sum_{i=1}^{r-s} \xi^i A_i$$

and

$$u \cdot V = \sum_{i=1}^{s} u^i V_i.$$

If g_o is an element of the group G, we can define a transformation of ξ under g_o uniquely by

$$g_o e^{\xi\cdot A} = e^{\xi'\cdot A} e^{u'\cdot V} \tag{7}$$

In the case when $g_o = h_o \varepsilon H$, this transformation of the ξ's is indeed linear as can be shown in the following way:

$$h_o e^{\xi\cdot A} = h_o e^{\xi\cdot A} h_o^{-1} h_o = e^{\xi'\cdot A} h_o. \tag{8}$$

This ξ' is a linear function of ξ because if property (5) is used on the right hand side of the known decomposition

$$e^{u\cdot V} A_i e^{-u\cdot V} = A_i + u^j [V_j, A_i] + \frac{u^j u^k}{2!} [V_k, [V_j, A_i]] + \ldots, \tag{9}$$

we get $e^{u\cdot V} A_i e^{-u\cdot V} = R_i{}^j(u) A_j$, where the matrix R is just a collection of all coefficients of A_j from the right hand side of equation (9). It implies clearly $\xi'^j = \xi^i R_i{}^j(u)$, which is linear.

If now R(h) are the matrices of an arbitrary representation of the subgroup H (acting on vectors Ψ) we can show that the mapping $\begin{pmatrix}\xi\\ \Psi\end{pmatrix} \xrightarrow{g_o} \begin{pmatrix}\xi'\\ \Psi'\end{pmatrix}$ defined by (7) and

$$\Psi' = R(e^{u'\cdot V})\Psi \tag{10}$$

is a realization of the group G which becomes linear under H. This follows from the following calculations (for the Ψ-part):

$$g_o e^{\xi\cdot A} = e^{\xi'\cdot A} e^{u'\cdot V} \qquad g_1 e^{\xi'\cdot A} = e^{\xi''\cdot A} e^{u''\cdot V} \tag{11}$$

$$g_1 g_o e^{\xi\cdot A} = e^{\xi''\cdot A} e^{u''\cdot V} e^{u'\cdot V} = e^{\xi''\cdot A} e^{u'''\cdot V},$$

which shows that under the transformation by $g_1 g_0$, Ψ is transformed into $R(e^{u''\circ V})R(e^{u'\circ V})\Psi$. The linearity of mapping (10) under H is clear because Equations (7) and (8) imply $e^{u'\cdot V} = h_0$ so that we get back to the original representation of H.

We note that in the realization (10) the matrix R depends on ξ through u'.

It can be shown (Refs. 1, 2, 4) that manifolds on which a nonlinear realization of a compact, connected, semisimple Lie group is defined (with the additional restriction that the realization becomes linear under a continuous subgroup H of the whole group G) can always be parametrized in such a way that we obtain the standard transformation property (7) for a subset of the coordinates of the manifold and the standard transformation property (10) for the set of remaining coordinates of the manifold (if any). The coordinates which belong to the first set are called the preferred coordinates and they are necessarily a part of any such manifold. Their number is equal to the number of the group generators minus the number of the subgroup generators.

This means that if some coordinates of the manifold are not transformed in the standard way, they can be redefined to create a standard set of coordinates. Here the assumption of compactness enters essentially (Refs. 1, 2, 4).

This redefinition keeps the origin of coordinates fixed (Ref. 24) which implies that if the coordinates of the manifold are interpreted as fields defined on space-time (or their space-time derivatives) of a physical theory,* the described redefinition of coordinates (fields)

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*In this sense the coordinates of the manifold will be called fields in the following, although we do not make any attempt to follow up the consequences implied for quantized fields transforming in this nonlinear manner.

does not change on mass-shell S-matrix elements (neither the exact expressions (proved in Ref. 21), nor their free graph approximations (Refs. 1, 4)). This enables us to restrict our attention to nonlinear realizations in their standard form only.

When the preferred fields are already defined in the standard fashion while there are some other coordinates of the manifold (they might include space-time derivatives of the preferred fields and some other physical fields and their space-time derivatives), jointly denoted by Φ throughout this section, whose transformation properties may still differ from (10), we know from the last paragraph that we can always redefine them into standard form. To give an explicit formula for this redefinition, we shall consider a point P of the manifold with coorinates (ξ,Φ). The group element $e^{-\xi\cdot A}$ will map P into another point P' with coordinates $(0,\Phi')$. Then we define the new coordinates Φ_{New} of the point P under consideration by $\Phi_{New} \equiv \Phi'$. Then (ξ,Φ_{New}) can serve as new coordinates of P where the coordinates Φ_{New} already have the standard transformation property (10).

Let us denote by $T^{\xi}_{g_o}$ the transformation which takes the old coordinates Φ of some point P with preferred coordinates ξ into the old coordinates Φ' of another point P' with preferred coordinates ξ' if P is mapped into P' by applying the group element g_o; in a formula: $\Phi' = T^{\xi}_{g_o}\,\Phi$. With this notation, the definition of the coordinates Φ_{New} of P can be expressed as

$$\Phi_{New} = T^{\xi}_{e^{-\xi\cdot A}}\,\Phi \tag{12}$$

In order to show the standard transformation property of Φ_{New} we write equation (7) as $e^{-\xi'\cdot A}g_o = e^{u'\cdot V}e^{-\xi\cdot A}$. This implies that the redefined Φ coordinates of the point with old coordinates $\left(\xi', T^{\xi}_{g_o}\Phi\right)$, namely $\left(T^{\xi}_{g_o}\Phi\right)_{\text{New}} = T^{\xi'}_{e^{-\xi'\cdot A}}T^{\xi}_{g_o}\,\Phi = T^{\xi}_{e^{-\xi'\cdot A}g_o}\Phi$, are equal to $T^{\xi}_{e^{u'\cdot V}e^{-\xi\cdot A}}\Phi = T^{o}_{e^{u'\cdot V}}T^{\xi}_{e^{-\xi\cdot A}}\Phi = T^{o}_{e^{u'\cdot V}}\Phi_{\text{New}}$.

If the group is not compact, we can have different nonlinear realizations of it which cannot be transformed into each other by a transformation which would leave the physical content of the theory unchanged. The classification of nonlinear realizations of noncompact groups is an open problem. One example of a noncompact group, the conformal group, is treated in Chapter V. For this group, however, it has been shown (Ref, 15) that indeed all its nonlinear realizations are physically equivalent to the nonlinear realization in the standard form. This is briefly discussed in Chapter V.

c) Linearization

If we have any fields $\tilde{\Psi}$ transforming linearly under a group G, they can always be redefined into a set of fields Ψ which have the standard transformation properties (10). One possibility is to set

$$\Psi = R(e^{-\xi\cdot A})\tilde{\Psi} \tag{13}$$

as can be checked easily (Refs. 1. 17, 24).

This construction can be inverted, and any fields which transform linearly under H (such that there exists a linear representation of G which, when restricted to H, becomes equal to the representation of H on the given fields) can be redefined to transform linearly (according to the linear representation of G) under the whole group (Refs. 1, 17, 24).

Furthermore, it has been proved (Ref. 24) that functions of the preferred fields ξ can be constructed so that they also transform according to a linear representation of G, provided this representation reduces under H and has the trivial (unity) representation in its decomposition. This procedure will be demonstrated in Chapter IV, and the connection with the σ-model will be pointed out.

d) Covariant Derivatives

We shall redefine the usual space-time derivatives of fields in order to bring them into a form which transforms in the standard way (10). We know (Refs. 1, 24) that any fields other than the preferred fields ξ (and $\partial_\mu \Psi$ and $\partial_\mu \xi$ certainly are different from ξ) can be redefined (in the sense of the previous section) in such a way. A redefinition is necessary since we have no reason to expect the space-time derivatives of ξ (ξ transform according to equation (7)) to transform according to equation (10) while $\partial_\mu \Psi$ cannot transform according to (10) because of the extra term $R(\partial_\mu e^{u'\cdot V})\Psi$ which arises when we differentiate (10) (for details of the calculation see the next chapter).

For the redefinition of $\partial_\mu \xi$ and $\partial_\mu \Psi$ we use equation (12) (we assume that the preferred fields ξ already transform according to (7) in the form $D_\mu \xi = (\partial_\mu \xi)_{New} = T^\xi_{e^{-\xi\cdot A}}(\partial_\mu \xi)$ and $D_\mu \Psi = (\partial_\mu \Psi)_{New} = T^\xi_{e^{-\xi\cdot A}}(\partial_\mu \Psi)$.

The quantities $D_\mu \xi$ and $D_\mu \Psi$ will be called the covariant derivatives of the fields ξ and Ψ respectively.

In order to carry out the construction of the covariant derivatives explicitly, we use a little trick (Ref. 19). For fields Ψ transforming in the standard way (10) we can define a covariant operation $\Delta_\mu \Psi$ by

$$\Delta_\mu \Psi = R(e^{-\xi\cdot A})\ \partial_\mu (R(e^{\xi\cdot A})\Psi) \tag{14}$$

Using definitions (7) and (10), the covariance is shown by

$$(\Delta_\mu \Psi)' = R(e^{-\xi'\cdot A})\partial_\mu (R(e^{\xi'\cdot A})R(e^{u'\cdot V})\Psi) =$$

$$= R(e^{u'\cdot V})R(e^{-\xi'\cdot A})R(g_0^{-1})\partial_\mu (R(g_0)R(e^{\xi\cdot A})\Psi) = R(e^{u'\cdot V})\Delta_\mu \Psi.$$

Note that g_0 is an x^μ--independent group--element. Thus $\Delta_\mu \Psi$ really transforms according to (10).

Equation (14) can be rewritten as

$$\Delta_\mu \Psi = \partial_\mu \Psi + R(e^{-\xi\cdot A})(\partial_\mu R(e^{\xi\cdot A}))\Psi. \tag{15}$$

The last term of this equation can be rewritten using

$$R(e^{-\xi\cdot A})\partial_\mu R(e^{\xi\cdot A}) = \alpha_\mu{}^i R(A_i) + \beta_\mu{}^j R(V_j), \tag{16}$$

where α and β are some functions of ξ and $\partial_\mu \xi$ and where $R(A_i)$ and $R(V_j)$ are the generators of G in the corresponding representation (see Ref. 19). The functions $\alpha_\mu{}^i$ can be shown to be exactly the $D_\mu \xi^i$ of the previous paragraph (for details see the next chapter) and so they must be transformed according to (10) by themselves. They can be extracted from Equation (15) leaving the rest still covariant. This means that the expressions $\partial_\mu \Psi + \beta_\mu{}^j R(V_j)\Psi$ can serve as the covariant derivatives $D_\mu \Psi$ of the fields Ψ (since they transform according to (10)).

The covariant derivatives will be necessary for constructing invariant Lagrangains. They are generalizations of the ordinary space-time derivatives in the sense that the covariant derivatives are equal to the ordinary ones for $\xi = 0$.

In proving the covariance of Equation (14) we treated the element

g_o as being space-time independent. If this is not the case, we speak of gauge transformations of the second kind. Having elements of G space-time dependent forces us to introduce two additional sets of gauge fields and to redefine the standard derivatives of ξ and Ψ. For details see the next chapter as well as Refs. 3, 19.

CHAPTER III

NONLINEAR REALIZATIONS OF SU(n) x SU(n)

In this chapter we would like to show how the theory of nonlinear realizations applies to the special case of the group $G \equiv SU(n) \times SU(n)$ if the subgroup H is taken to be the diagonal subgroup. To make clear this terminology we shall denote the (n^2-1) generators of the left (right) subgroup of SU(n) x SU(n) by J_i^+ (J_i^-), where $i = 1, 2, \ldots, n^2-1$. In terms of these generators the Lie algebra of SU(n) x SU(n) is described by

$$[J_i^+, J_j^+] = f_{ij}{}^k J_k^+ , [J_i^-, J_j^-] = f_{ij}{}^k J_k^- \text{ and } [J_i^+, J_j^-] = 0, \tag{1}$$

where all indices run from 1 to (n^2-1) and $f_{ij}{}^k$ are the structure coefficients of the SU(n) group. We choose the generators J^+ and J^- because they diagonalize the tensor g_{ij}, which in this case becomes a multiple of Kronecker's delta (see the previous chapter). In this case also, $f_{ij}{}^k$ is totally antisymmetric, because $f_{ijk} = f_{ij}{}^l g_{lk}$ is always antisymmetric (Ref. 16). For more details concerning these standard results see Refs. 1, 3.

If we choose a new set of $(2n^2-2)$ generators of the group G defining $V_i = J_i^+ + J_i^-$ and $A_i = J_i^+ - J_i^-$, we find that in terms of V and A the Lie algebra is described by

$$[V_i, V_j] = f_{ij}{}^k V_k \quad [V_i, A_j] = f_{ij}{}^k A_k \text{ and } [A_i, A_j] = f_{ij}{}^k V_k, \tag{2}$$

where $f_{ij}{}^k$ are the same structure coefficient (Ref. 1).

It follows from the first equation in (2) that the generators V span a subalgebra of the Lie algebra of G. We shall call the corresponding subgroup the diagonal one.

At the right hand side of the second equation in (2) there are no V-terms. This can always be achieved by a proper choice of A's (if the V's are chosen to be the generators of the subgroup H) as we know from the last chapter (all the required conditions are satisfied). It means that the adjoint representation of the Lie algebra of G when restricted to the Lie algebra of H provides (apart from the adjoint representation of H) a representation of the latter on the vector space spanned by the generators A. Correspondingly, the subgroup H is linearly represented on the space G/H of right cosets of H. The non-linear realization of G which we construct using the preferred fields ξ becomes linear and isomorphic to the representation mentioned in the previous sentence, when restricted to the subgroups H (Refs. 1, 3, 12).

The third equation in (2) gives us no A-terms on the right hand side. This is just accidental and we shall say in such a case that the corresponding subgroup H is symmetrical. It enables us to define a parity conjugation which will be used to simplify the form of the nonlinear realization (Refs. 1, 3, 8, 24).

a) Transformation of the Preferred Fields

We introduce some notation first. Any group element $g \varepsilon G$ can be written uniquely in the form $g = e^{A_i \xi^i} e^{V_i u^i} = e^{A \cdot \xi} e^{V \cdot u}$ where ξ^i and u^i are two sets of (n^2-1) real parameters each (Refs. 7, 24).

An operation which takes the group element g with parameters ξ^i and u^i into another group element $\tilde{g}$ with parameters $-\xi^i$ and u^i will be called a parity conjugation (Refs. 1, 3, 8). The element $\tilde{g}$, parity conjugate to g, is then $\tilde{g} = e^{-A \cdot \xi} e^{V \cdot u}$.

In the previous section we have already discussed the linear representation of H on the vector space G/H parametrized by ξ. It can be written explicitly as

$$\xi'^{k} = \exp\,(\theta^{i} f_{ij}{}^{k})\xi^{j} \tag{3}$$

if the transforming element $h_o \varepsilon H$ is equal to $e^{V\cdot\theta}$.

This can also be written in the form

$$e^{A\cdot\xi'} = h_o e^{A\cdot\xi} h_o^{-1} = e^{V\cdot\theta} e^{A\cdot\xi} e^{-V\cdot\theta} \tag{4}$$

as can be seen from the following calculation:

$$e^{V\cdot\theta} A_j e^{-V\cdot\theta} = A_j + \theta^{i}[V_i, A_j] + \frac{\theta^{i}\theta^{l}}{2!}[V_l,[V_i, A_j]] + \ldots =$$

$$= A_j + \theta^{i} f_{ij}{}^{k} A_k + \frac{\theta^{i}\theta^{l}}{2!} f_{lm}{}^{k} f_{ij}{}^{m} A_k + \ldots = \exp\,(\theta^{i} f_{ij}{}^{k}) A_k.$$

It is easy to generalize $h_o e^{A\cdot\xi} = e^{A\cdot\xi'}.h_o$ (equation 4) to define an unique transformation of ξ under an element of G. Replacing h_o on the left-hand side by an arbitrary $g_o \varepsilon G$, we can write

$$g_o e^{A\cdot\xi} = e^{A\cdot\xi'}.e^{V\cdot u'}, \tag{5}$$

which is already the most general nonlinear transformation of ξ and which was described in the previous chapter.

The equation (5) gives actually two mappings, namely $\xi' = \xi'(\xi,g_o)$ which is the transformation of the preferred fields and $u' = u'(\xi,g_o)$ which will be used in the transformation law of any other field.

If we apply the parity conjugation to equation (5) we get

$$\tilde{g}_o e^{-A\cdot\xi} = e^{-A\cdot\xi'} e^{V\cdot u'}.$$

Inverting this equation and multiplying it by equation (5) from the left we find

$$g_o(e^{A\cdot\xi})^2\tilde{g}_o^{-1} = (e^{A\cdot\xi'})^2. \tag{6}$$

When the transformation law of the fields ξ is written in this form one can check that this transformation law really provides a realization of the group G by writing $g_o = g_1 g_2$. This implies

$$(e^{A\cdot\xi'})^2 = g_1 g_2 (e^{A\cdot\xi})^2\, \tilde{g}_2^{-1}\, \tilde{g}_1^{-1},$$

in other words nothing results but a successive application of g_2 and g_1 (Refs. 1, 3, 8).

b) Transformation of Other Fields

Assume that we have other fields Ψ^α which are transformed linearly under H, i.e.,

$$\Psi'^{\alpha} = R^{\alpha}{}_{\beta}(h)\Psi^{\beta}, \tag{7}$$

where $h\varepsilon H$ and where $R^{\alpha}{}_{\beta}$ belong to a representation of H.

Then we can extend this transformation to the whole group G by writing

$$\Psi^{\alpha} \xrightarrow{g_o} R^{\alpha}{}_{\beta}(e^{V\cdot u'})\Psi^{\beta} \tag{8}$$

where $g_o\varepsilon G$ and $u' = u'(\xi, g_o)$ is given by (5).

That this is indeed a realization of G which in the case $g_o = h\varepsilon H$ reduces to the original transformation (7) was already shown in Chapter II.

c) Covariant Derivatives

In this section we are looking for some analog to the usual derivatives $\partial_\mu \xi^j = \frac{\partial \xi^j}{\partial x^\mu}$ and $\partial_\mu \Psi^\alpha = \frac{\partial \Psi^\alpha}{\partial x^\mu}$ of the preferred and the other fields respectively. These analogs can be treated as additional fields and so they can be required to transform according to (8) (Refs. 2, 3, 8, 24).

We know how to transform $\partial_\mu \xi^j$ and $\partial_\mu \Psi^\alpha$ (as implied by equations (5) and (8)), but this transformation does not have the standard form of equation (8). The reason is that not just Ψ^β but also $u'(\xi, g_0)$ is x^μ-dependent through ξ, which (if equation (8) is differentiated with respect to x^μ) creates one additional term and thus spoils the standard behaviour.

In Ref. 2 and in Chapter II it is shown that the covariant derivatives can be defined as follows: If we take all fields and their derivatives together, $(\xi^i, \Psi^\alpha, \partial_\mu \xi^j, \partial_\mu \Psi^\beta)$, and transform them under a group element $g_0 = e^{-\xi \cdot A}$, we get the following result:

$$(\xi, \Psi, \partial_\mu \xi, \partial_\mu \Psi)' = (0, \Psi, D_\mu \xi, D_\mu \Psi).$$

This means that the fields ξ become zero, that the fields Ψ do not change at all and that $D_\mu \xi$ and $D_\mu \Psi$ (just a new notation for $(\partial_\mu \xi)'$ and $(\partial_\mu \Psi)'$) define the desired analogs of $\partial_\mu \xi$ and $\partial_\mu \Psi$ respectively, because they have the standard transformation properties.

Now we shall calculate and check all of this explicitly. To an infinitesimal increase of the space-time coordinates $(x^\mu \to x^\mu + dx^\mu)$ there must correspond increases of ξ and Ψ and, through equation (5), increases of the transformed coordinates ξ', Ψ', and of u'. If we denote these increases by $d\xi$, $d\Psi$, $d\xi'$, $d\Psi'$ and du' respectively, we can write two relations (using equations (5) and (8)),

$$g_o e^{A\cdot(\xi+d\xi)} = e^{A\cdot(\xi'+d\xi')}e^{V\cdot(u'+du')} \quad \text{and}$$

$$(\Psi'+d\Psi')^\alpha = R^\alpha{}_\beta(e^{V\cdot(u'+du')})(\Psi + d\Psi)^\beta.$$

Now we take $g_o = e^{-A\cdot\xi}$ and express the exponentials as power series in $d\xi, d\xi'$ and du' up to the first order. This gives us

$$e^{-A\cdot\xi}(e^{A\cdot\xi} + \frac{\partial e^{A\cdot\xi}}{\partial\xi^i} d\xi^i) = (e^{A\cdot\xi'} + \frac{\partial e^{A\cdot\xi'}}{(\partial\xi')^i}(d\xi')^i)(e^{V\cdot u'} + \frac{\partial e^{V\cdot u'}}{(\partial u')^i}(du')^i)$$

for the first equation and

$$(\Psi' + d\Psi')^\alpha = R^\alpha{}_\beta(e^{V\cdot u'} + \frac{\partial e^{V\cdot u'}}{(\partial u')^i}(du')^i)(\Psi + d\Psi)^\beta$$

for the second one.

Collecting the zero order terms gives us $\xi' = 0$, $u' = 0$ and $\Psi' = \Psi$.

From the first order terms we obtain

$$e^{-A\cdot\xi}\frac{\partial e^{A\cdot\xi}}{\partial\xi^i} d\xi^i = A_i(d\xi')^i + V_i(du')^i \quad \text{and}$$

$$(d\Psi')^\alpha = R^\alpha{}_\beta(V_i)(du')^i\Psi^\beta + \delta^\alpha{}_\beta d\Psi^\beta.$$

Dividing both equations by dx^μ yields

$$e^{-A\cdot\xi}\frac{\partial e^{A\cdot\xi}}{\partial\xi^i}\partial_\mu\xi^i = A_i\frac{(d\xi')^i}{dx^\mu} + V_i\frac{(du')^i}{dx^\mu} \quad \text{and} \tag{9a}$$

$$\frac{(d\Psi')^\alpha}{dx^\mu} = R^\alpha{}_\beta(V_i)\frac{(du')^i}{dx^\mu}\Psi^\beta + \partial_\mu\Psi^\alpha \tag{9b}$$

We have already agreed to call $\frac{(d\xi')^i}{dx^\mu} \equiv D_\mu\xi^i$ a covariant derivative of ξ and $\frac{(d\Psi')^\alpha}{dx^\mu} \equiv D_\mu\Psi^\alpha$ a covariant derivative of Ψ.

Only the difficulty to calculate $e^{-A\cdot\xi}\,\dfrac{de^{A\cdot\xi}}{d\xi^i}$ remains.

This is done in Appendix 3a. There it is shown that

$$e^{-A\cdot\xi}\,\frac{de^{A\cdot\xi}}{d\xi^i} = \left(\delta_i{}^j + \frac{x_i{}^k x_k{}^j}{3!} + \frac{x_i{}^k x_k{}^l x_l{}^m x_m{}^j}{5!} + \ldots\right)A_j - \left(\frac{x_i{}^j}{2!} + \right.$$

$$\left. + \frac{x_i{}^k x_k{}^l x_l{}^j}{4!} + \ldots\right)V_j \equiv \sigma_i{}^j A_j - \rho_i{}^j V_j, \text{ where } x_i{}^j = \xi^k f_{ki}{}^j$$

and where the last step is just a definition of the matrices σ and ρ.

Our two equations now read

$$(\sigma_i{}^j A_j - \rho_i{}^j V_j)\partial_\mu \xi^i = A_j D_\mu \xi^j + V_j \frac{(du')^j}{dx^\mu} \quad \text{and}$$

$$D_\mu \Psi^\alpha = \partial_\mu \Psi^\alpha + R^\alpha{}_\beta(V_j)\frac{(du')^j}{dx^\mu}\,\psi^\beta.$$

Collecting A and V terms in the first equation and replacing $\dfrac{(du')^j}{dx^\mu}$ in the second equation by the explicit expression obtained from the first one yields the final result

$$D_\mu \xi^j = \sigma_i{}^j \partial_\mu \xi^i \quad \text{and} \tag{10a}$$

$$D_\mu \Psi^\alpha = \partial_\mu \Psi^\alpha - R^\alpha{}_\beta(V_j)\rho_i{}^j \partial_\mu \xi^i \psi^\beta, \tag{10b}$$

where the matrices σ and ρ are still functions of ξ.

Let us check that these two expressions transform in the standard way. We can write an X^μ-derivative of equation (5) as

$$g_o \partial_\mu e^{A\cdot\xi} = (\partial_\mu e^{A\cdot\xi'})e^{V\cdot u'} + e^{A\cdot\xi'}(\partial_\mu e^{V\cdot u'}), \tag{11}$$

where g_o has been treated as x^μ-independent.

Since $g_o = e^{A\cdot\xi'}e^{V\cdot u'}e^{-A\cdot\xi}$ (from equation (5)), we get

$$e^{A\cdot\xi'}e^{V\cdot u'}e^{-A\cdot\xi}\partial_\mu e^{A\cdot\xi} = (\partial_\mu e^{A\cdot\xi'})e^{V\cdot u'} + e^{A\cdot\xi'}(\partial_\mu e^{V\cdot u'}).$$

This equation can be rewritten (in two steps) as

$$e^{V\cdot u'}e^{-A\cdot\xi}(\partial_\mu e^{A\cdot\xi})e^{-V\cdot u'} = e^{-A\cdot\xi'}(\partial_\mu e^{A\cdot\xi'}) + (\partial_\mu e^{V\cdot u'})e^{-V\cdot u'} \quad \text{and}$$

$$e^{V\cdot u'}(\sigma_i{}^j(\xi)A_j\partial_\mu\xi^i - \rho_i{}^j(\xi)V_j\partial_\mu\xi^i)e^{-V\cdot u'} = \sigma_i{}^j(\xi')A_j(\partial_\mu\xi')^i -$$

$$- \rho_i{}^j(\xi')V_j(\partial_\mu\xi')^i + (\partial_\mu e^{V\cdot u'})e^{-V\cdot u'}. \tag{12}$$

Separating the A-terms at each side of this equation gives

$$(D_\mu\xi')^j A_j = e^{V\cdot u'}A_j e^{-V\cdot u'}D_\mu\xi^j.$$

Now we can simply repeat the calculation following equation (4) to get the transformation property of $D_\mu\xi$ as

$$(D_\mu\xi')^j = \exp\left((u')^i f_{ik}{}^j\right)D_\mu\xi^k. \tag{13}$$

This has exactly the standard form of equation (8), which was to be proved.

Separating the V-terms in equation (12) gives us the transformation property of $V_j\rho_i{}^j(\xi)\partial_\mu\xi^i$ as follows

$$V_j(\rho_i{}^j\partial_\mu\xi^i)' = e^{V\cdot u'}\rho_i{}^j(\partial_\mu\xi^i)V_j e^{-V\cdot u'} - (\partial_\mu e^{V\cdot u'})e^{-V\cdot u'}.$$

With this relation we can determine the transformation law of the standard derivatives $D_\mu\Psi$.

In this calculation we must, of course, express all generators and group elements in that representation to which the fields Ψ belong (i.e., we must use the $R^{\alpha}{}_{\beta}$ matrices). We get (when replacing Ψ by Ψ' in equation (10b))

$$(D_{\mu}\Psi')^{\alpha} = \partial_{\mu}\ \{R^{\alpha}{}_{\beta}(e^{V\cdot u'})\Psi^{\beta}\} - R^{\alpha}{}_{\beta}(e^{V\cdot u'})\rho_{i}{}^{j}(\partial_{\mu}\xi^{i})R^{\beta}{}_{\gamma}(V_{j})\cdot$$

$$\cdot R^{\gamma}{}_{\delta}(e^{-V\cdot u'}) - R^{\alpha}{}_{\beta}(\partial_{\mu}e^{V\cdot u'})R^{\beta}{}_{\delta}(e^{-V\cdot u'})\}(\Psi')^{\delta}.$$

Using $R^{\beta}{}_{\delta}(e^{-V\cdot u'})(\Psi')^{\delta} = \Psi^{\beta}$ (see equation 8) we finally see that $D_{\mu}\Psi$ is also transformed in the standard form of equation (8):

$$(D_{\mu}\Psi')^{\alpha} = R^{\alpha}{}_{\beta}(e^{V\cdot u'})(\partial_{\mu}\Psi^{\beta} + R^{\beta}{}_{\gamma}(V_{j})\rho_{i}{}^{j}\partial_{\mu}\xi^{i}\Psi^{\alpha}) = R^{\alpha}{}_{\beta}(e^{V\cdot u'})D_{\mu}\Psi^{\beta}. \tag{14}$$

Note: Lagrangians

If we have constructed a Lagrangian as a function of ξ,Ψ, $\partial_{\mu}\xi$ and $\partial_{\mu}\Psi$

$$L = L(\xi,\Psi,\partial_{\mu}\xi,\partial_{\mu}\Psi),$$

we already know (see the third paragraph of Section c) how it transforms under $g_{0} = e^{-A\cdot\xi}$, namely

$$L'(\xi,\Psi,\partial_{\mu}\xi,\partial_{\mu}\Psi) = L(\xi',\Psi',\partial_{\mu}\xi',\partial_{\mu}\Psi') = L(0,\Psi,D_{\mu}\xi,D_{\mu}\Psi).$$

This implies that in order to have a Lagrangian invariant under the group G it has to be a function of $D_{\mu}\xi,\Psi$ and $\partial_{\mu}\Psi$ only and it has to be superficially invariant under the subgroup H (Ref. 3). This already guarantees the invariance under the whole group G due to the standard transformation property of $\Psi,D_{\mu}\xi$ and $D_{\mu}\Psi$. In the case of a system described by the preferred fields only, there is a natural choice for such a Lagrangian, namely (Ref. 3)

$$L = \frac{1}{2} f^2 g_{ij}(D_\mu \xi^i)(D_\mu \xi^j) = -\frac{1}{2} f^2 g_{kl}\sigma_i{}^k \sigma_j{}^l \partial_\mu \xi^i \partial_\mu \xi^j,$$

where f is a numerical constant. Here g_{ij} is the metric tensor defined by equation (4) in Chapter II. In the case of a compact group it will usually be assumed to be in its diagonal form proportional to a Kronecker delta, which can always be achieved by an appropriate parametrization of the group. In what follows we will assume that this has been done. From the invariance of this Lagrangian we can see easily that $g_{kl}\sigma_i{}^k\sigma_j{}^l$ is another invariant metric tensor in the space spanned by ξ (the quadratic form $g_{kl}\sigma_i{}^k\sigma_j{}^l d\xi^i d\xi^j$ is invariant). For a discussion of this geometrical point of view see Refs. 9, 10.

d) Gauge Transformations of the Second Kind

In this section we shall consider the transformations to be x^μ-dependent. This will give us one more additional term $(\partial_\mu g_o)e^{A\cdot\xi}$ on the left-hand side of equation (11). In order to be able to repeat the calculation following this equation and to get the same transformation laws (13) and (14), we must modify the expressions for covariant derivaties of the fields ξ and Ψ by making them dependent on two newly introduced sets of fields (Refs. 1, 3). These fields, which we shall call $v_\mu{}^i$ (a set of (n^2-1) vector fields) and $a_\mu{}^i$ (a set of (n^2-1) axial vector fields), should be transformed under g_o according to

$$\partial_\mu + f(V\cdot v_\mu' + A\cdot a_\mu') = g_o[\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)]g_o^{-1}, \qquad (15)$$

where f is a numerical constant and $V\cdot v_\mu + A\cdot a_\mu$ is the most general element of the Lie algebra of G (Refs. 1, 3)

If we replace ∂_μ by $\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)$ in equation (9a), we obtain

$$e^{-A\cdot\xi}[\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)]e^{A\cdot\xi} = A\cdot\widehat{D_\mu\xi} - V_j\widehat{\rho_i{}^j\partial_\mu\xi^i}, \tag{16}$$

where $\widehat{D_\mu\xi}$ and $\widehat{\rho_i{}^j\partial_\mu\xi^i}$ is an analog of $D_\mu\xi$ and $\rho_i{}^j\partial_\mu\xi^i$ from equations (10). If we use $\widehat{D_\mu\xi}$ and $\widehat{\rho_i{}^j\partial_\mu\xi^i}$ instead of $D_\mu\xi$ and $\rho_i{}^j\partial_\mu\xi^i$ to define covariant derivatives of the fields ξ and Ψ, these new covariant derivatives will have the standard transformation properties (equation (8) or (13), (14)) extended to the gauge transformations of the second kind.

To prove that equations (15) and (16) together with

$$\widehat{D_\mu\Psi}^\alpha = \partial_\mu\Psi^\alpha - R^\alpha{}_\beta(V_j)\widehat{\rho_i{}^j\partial_\mu\xi^i}\Psi^\beta \tag{17}$$

really define such covariant derivatives of the fields ξ and Ψ (given by $\widehat{D_\mu\xi}$ and $\widehat{D_\mu\Psi}$ respectively) we start by calculating how the left-hand side of equation (16) transforms under $g_0\epsilon G$. Since

$$e^{A\cdot\xi'} = g_0 e^{A\cdot\xi}e^{-V\cdot u'}$$

we get

$$e^{-A\cdot\xi'}[\partial_\mu + f(V\cdot v'_\mu + A\cdot a'_\mu)]e^{A\cdot\xi'} = e^{V\cdot u'}e^{-A\cdot\xi}g_0^{-1}g_0[\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)]\cdot$$

$$\cdot g_0^{-1}g_0 e^{A\cdot\xi}e^{-V\cdot u'} = e^{V\cdot u'}e^{-A\cdot\xi}[\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)]e^{A\cdot\xi}e^{-V\cdot u'},$$

where we have used equation (15) essentially.

This calculation shows that all we have to do to transform the left-hand side of equation (16) is to multiply it by $e^{V\cdot u'}$ from the left and by $e^{-V\cdot u'}$ from the right. If the same operation is applied to the right-hand side of (16) it will give us the transformation properties of $\widehat{D_\mu \xi}$ and $\widehat{D_\mu \Psi}$ (through $\widehat{\rho_i{}^j \partial_\mu \xi^i}$) which turn out to be of the standard form. To show that, we just note that equation (9a) is transformed by applying the same operation (when g_o is x^μ-independent). This implies transformations (13) and (14).

Everthing which has been said so far in this section about nonlinear realizations applies to all compact, connected, semisimple Lie groups, but now we are going to use the specific Lie algebra of the SU(n) x SU(n) group.

To calculate the covariant derivatives $D_\mu \xi$ and $D_\mu \Psi$ as defined by equations (16) and (17) we need the following results, which are proved in Appendix 3b):

$$e^{-A\cdot\xi} V_i e^{A\cdot\xi} = (\delta_i{}^j + \frac{x_i{}^k x_k{}^j}{2!} + \frac{x_i{}^k x_k{}^l x_l{}^m x_m{}^j}{4!} + \ldots)V_j$$

$$- (x_i{}^j + \frac{x_i{}^k x_k{}^l x_l{}^j}{3!} + \ldots)A_j = \cosh x_i{}^j V_j - \sinh x_i{}^j A_j \quad \text{and}$$

similarly $e^{-A\cdot\xi} A_i e^{A\cdot\xi} = \cosh x_i{}^j A_j - \sinh x_i{}^j V_j$, where again $x_i{}^j = \xi^k f_{ki}{}^j$. Comparing A and V terms of both sides of equations (16) (and remembering $e^{-A\cdot\xi}\partial_\mu e^{A\cdot\xi} = A\cdot D_\mu \xi - V_j \rho_i{}^j \partial_\mu \xi^i$) gives us

$$\widehat{D_\mu \xi^j} = D_\mu \xi^j + f(\cosh x_i{}^j a_\mu{}^i - \sinh x_i{}^j v_\mu{}^i) \text{ and} \tag{18a}$$

$$\widehat{\rho_i{}^j \partial_\mu \xi^i} = \rho_i{}^j \partial_\mu \xi^i - f(\cosh x_i{}^j v_\mu{}^i - \sinh x_i{}^j a_\mu{}^i) \text{ respectively.} \tag{18b}$$

e) The Gauge Fields

In the previous section we replaced ∂_μ by $\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)$

to get the new covariant derivatives. Now it is obvious how to construct the covariant generalization of the curl operation. We just calculate

$$[\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)][\partial_\nu + f(V\cdot v_\nu + A\cdot a_\nu)] - [\partial_\nu + f(V\cdot v_\nu + A\cdot a_\nu)]\cdot$$

$$\cdot[\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)]$$

and separate the coefficients of fV_i and fA_i which we call $F_{\mu\nu}{}^i$ and $\overline{F}_{\mu\nu}{}^i$ respectively.

We find (see Appendix 3c)

$$F_{\mu\nu}{}^i = \partial_\mu v_\nu{}^i - \partial_\nu v_\mu{}^i + ff_{jk}{}^i(v_\mu{}^j v_\nu{}^k - a_\mu{}^j a_\nu{}^k) \text{ and}$$

$$\overline{F}_{\mu\nu}{}^i = \partial_\mu a_\nu{}^i - \partial_\nu a_\mu{}^i + ff_{jk}{}^i(a_\mu{}^j v_\nu{}^k - v_\mu{}^j a_\nu{}^k)$$

From this definition follow immediately (see equation (16)) the transformation properties of the tensors F, namely

$$(F_{\mu\nu}{}^i A_i + \overline{F}_{\mu\nu}{}^i V_i)' = F_{\mu\nu}{}^i g_o A_i g_o^{-1} + \overline{F}_{\mu\nu}{}^i g_o V_i g_o^{-1}, \tag{19}$$

where $g_o \varepsilon G$ represents the transformation.

Now we can construct a Lagrangian of the gauge fields v_μ and a_μ in the generalized Yang-Mills form as (see Ref. 3)

$$L_{va} = -\frac{1}{4}\delta_{ij}(F_{\mu\nu}{}^i F^{\mu\nu j} + \overline{F}_{\mu\nu}{}^i \overline{F}^{\mu\nu j})$$

(not mixing F with $\overline{F}$ because of parity conservation).

To show the invariance of such a Lagrangian would be easy if the tensors F had the standard transformation properties. Since that is not the case, we want to redefine F and rewrite L_{va} in order to make both manifestly covariant.

To do so we redefine first the gauge fields v and a which also are not transformed in the standard way of equation (8). Using the method described in the Introduction and Section c) of this chapter we set $g_0 = e^{-A\cdot\xi}$ in equation (15). Thus we can define new fields $\overset{\circ}{v}_\mu$ and $\overset{\circ}{a}_\mu$ by

$$\overset{\circ}{v}_\mu\cdot V + \overset{\circ}{a}_\mu\cdot A = e^{-A\cdot\xi}(v_\mu\cdot V + a_\mu\cdot A)e^{A\cdot\xi}, \tag{20}$$

where $\overset{\circ}{v}_\mu$ and $\overset{\circ}{a}_\mu$ must have the standard transformation properties. We can rewrite equation (20) in an explicit form (using the identities of Appendix 3b) as

$$\overset{\circ}{v}_\mu{}^j = \cosh\,(x_i{}^j)v_\mu{}^i - \sinh\,(x_i{}^j)a_\mu{}^i \quad \text{and}$$

$$\overset{\circ}{a}_\mu{}^j = \cosh\,(x_i{}^j)a_\mu{}^i - \sinh\,(x_i{}^j)v_\mu{}^i.$$

(This demonstrates the standard behaviour of $\widehat{D_\mu\xi}$ and $\widehat{D_\mu\Psi}$ (defined by equations (18)) under x^μ-independent transformations).

By using the same method we can redefine the tensors F to give us new tensors E having the standard transformation property. Setting $g_0 = e^{-A\cdot\xi}$, equation (19) becomes

$$E_{\mu\nu}{}^i A_i + \overline{E}_{\mu\nu}{}^i V_i = F_{\mu\nu}{}^i(\cosh\,(x_i{}^j)A_j - \sinh\,(x_i{}^j)V_j) +$$

$$+ \overline{F}_{\mu\nu}{}^i(\cosh\,(x_i{}^j)V_j - \sinh\,(x_i{}^j)A_j) \quad \text{or, separately,}$$

$$E_{\mu\nu}{}^i = F_{\mu\nu}{}^i \cosh x_i{}^j - \overline{F}_{\mu\nu}{}^i \sinh x_i{}^j \quad \text{and} \tag{21a}$$

$$\overline{E}_{\mu\nu}{}^i = \overline{F}_{\mu\nu}{}^i \cosh x_i{}^j - F_{\mu\nu}{}^i \sinh x_i{}^j \tag{21b}$$

These relations can be used to check that $L_{va} = -\frac{1}{4}\delta_{ij}(E_{\mu\nu}{}^{i}E^{\mu\nu j} + \overline{E}_{\mu\nu}{}^{i}\overline{E}^{\mu\nu j})$ is evidently invariant under the whole group G because of the standard transformation properties of $E_{\mu\nu}$ and $\overline{E}_{\mu\nu}$ and because of the orthonormality of the matrices $R^{\alpha}{}_{\beta}$ in equation (8).

The standard transformation property of E can be checked by using its definition, the relation $e^{-A\cdot\xi'}g_o = e^{V\cdot u'}e^{-A\cdot\xi}$ and the fact that the F's are transformed linearly under H. We can also demonstrate it explicitly by rewriting E in terms of fields and covariant derivatives which all transform in the standard way under the second gauge transformations as follows:

$$E_{\mu\nu}{}^{i} = \widehat{D_{\mu}\mathring{v}_{\nu}}{}^{i} - \widehat{D_{\nu}\mathring{v}_{\mu}}{}^{i} + f_{lm}{}^{i}[\widehat{D_{\mu}\xi}{}^{l}\mathring{a}_{\nu}{}^{m} - \widehat{D_{\nu}\xi}{}^{l}\mathring{a}_{\mu}{}^{m} - f(\mathring{v}_{\mu}{}^{l}\mathring{v}_{\nu}{}^{m} + \mathring{a}_{\mu}{}^{l}\mathring{a}_{\nu}{}^{m})] \quad \text{and} \tag{22a}$$

$$\overline{E}_{\mu\nu}{}^{i} = \widehat{D_{\mu}\mathring{a}_{\nu}}{}^{i} - \widehat{D_{\nu}\mathring{a}_{\mu}}{}^{i} + f_{lm}{}^{i}[\widehat{D_{\mu}\xi}{}^{l}\mathring{v}_{\nu}{}^{m} - \widehat{D_{\nu}\xi}{}^{l}\mathring{v}_{\mu}{}^{m} - f(\mathring{v}_{\mu}{}^{l}\mathring{a}_{\nu}{}^{m} + \mathring{a}_{\mu}{}^{l}\mathring{v}_{\nu}{}^{m})] \tag{22b}$$

This agrees with the original definition of E given by (21a) and (21b) as shown in Appendix 3d).

CHAPTER IV

LINEARIZING THE NONLINEAR REALIZATION OF SU(2) x SU(2)

In this chapter we shall carry out explicitly the construction of linear realizations out of a nonlinear one which was mentioned in Chapter II. The basic idea consists in redefining the fields on which the nonlinear realization is defined. The construction is quite complicated and that is why we shall treat the case where the group G is just the SU(2) x SU(2) group. The construction has practical importance because in linear representations, as opposed to nonlinear ones, the usual space-time derivatives are already covariant under transformations of G.

When linearizing the preferred fields, we have to obtain a representation of G on the redefined fields which when restricted to H, has the trivial (unity) representation in its decomposition into a direct sum of irreducible representations of H (Refs. 1, 17, 24 and Chapter II).

We know all linear representations of the SU(2) x SU(2) group. They are usually denoted by (j^+, j^-), where $2j^+ + 1$ and $2j^- + 1$ are the dimensions of the representations of the left and right subgroup, which are combined in a direct product to give a representation of the whole group.

The representations with $j^+ = j^- = j$ decompose under the diagonal subgroup H into irreducible representations with I = 0, 1, 2, ..., 2j (the label I is called the isospin of the corresponding (2I + 1)-dimensional representation) and thus contain the trivial one, which has $I = 0$. These are actually all such representations, because for $j^+ \neq j^-$ we get the following possible isospins: $I = |j^+ - j^-|, \ldots, j^+ + j^-$, which means I is always different from zero (Ref. 17).

a) Linearization of the Preferred Fields

The algebra of generators of the SU(2) x SU(2) group is (see the previous chapter)

$$[V_a, V_b] = [A_a, A_b] = i\varepsilon_{abc}V_c \text{ and } [V_a, A_b] = i\varepsilon_{abc}A_c, \tag{1}$$

where the indices a, b and c run from 1 to 3 (summation over a double index is understood throughout this chapter), where ε_{abc} is Kronecker's totally antisymmetric tensor and the imaginary unit i stays here for our convenience only (it makes the generators A and V hermitian - they were antihermitian in the previous chapter).

We can construct two Casimir operators for this algebra,

$$C_1 = (V \cdot V) + (A \cdot A) = 2(J^+ \cdot J^+) + 2(J^- \cdot J^-) \quad \text{and} \tag{2a}$$

$$C_2 = (V \cdot A) = (A \cdot V) \tag{2b}$$

where J^+ and J^- have been defined in the first paragraph of Chapter III. If we apply them to a vector of the (j^+, j^-) representation space, we get $2j^+(j^+ + 1) + 2j^-(j^- + 1)$ and $(j^+ - j^-)(j^+ + j^- + 1)$ as eigenvalues of this vector under C_1 and C_2 respectively.

In a representation eligible for our purpose ($j^+ = j^- = j$) we have

$$C_1 = 4j(j + 1) \quad \text{and } C_2 \equiv 0. \tag{3}$$

We need to explain how we define the operation of a generator on a field. The previous chapter enabled us to calculate a transformation property of a field under a group element g_0. We now write $g_0 = e^{-\omega X}$ (where X is any one of the group generators and ω is an infinitesimal parameter) and calculate an infinitesimal change of the field under such an element. This infinitesimal change divided by ω is equal to what results if X operates on the field. In this sense generators will be called operators in the following.

According to this definition we may say that

$$V_a \xi_b = i \varepsilon_{abc} \xi_c \tag{4}$$

(ξ_c is a set of three preferred fields), which is in complete correspondence with equation (3) of the previous chapter. If $f(\xi_i)$ is a function of the preferred field, our definition also gives us for a generator X operating on f

$$Xf(\xi_i) = \frac{\partial f}{\partial \xi_i} X \xi_i \, ,$$

which will be used extensively in the calculations.

To construct the (j, j) representation of the group G out of the preferred fields we first need a function of these fields (let us call it $S(\xi)$)which is an isoscalar (meaning $V_a S = 0$) under the diagonal subgroup H. Such a function will enable us to construct all (j, j) representations as we can see from the following paragraph, where we present results of Ref. 17, but using slightly different arguments.

If we apply the operator $A_+ = A_1 + iA_2$ to S n times, we can prove easily (using commutation relations (1)) that

$$V_3(A_+)^n S = n(A_+)^n S \tag{5}$$

$$(V \cdot V)(A_+)^n S = n(n + 1)(A_+)^n S \tag{6}$$

This shows that the functions $(A_+)^n S$ are the highest eigenfunctions (having the maximal V_3-eigenvalue) of the isospin = n representation of H. To get all the (2n + 1) eigen functions of this representation we just apply the lowering operators $V_- = V_1 - iV_2$ to $(A_+)^n S$ successively 2n times.

To complete the (j, j) representation of the whole group G we have to have all the subgroup representations with isospin equal to 0, 1, 2, ..., 2j.
This induces one more condition upon the function S

$$(A_+)^{2j+1} S = 0 \tag{7}$$

All we have to do now is to find an isoscalar function S of the preferred fields satisfying condition (7).

In order to accomplish this, we need the complete transformation properties of the preferred fields. The transformation law under the subgroup H has already been mentioned in equation (4). The most general form of the transformation law under the remaining generators of the group G is (Refs. 4, 12, 17, 22) in the case of real fields the following:

$$A_a \xi_b = -i(\delta_{ab} f(\xi^2) + \xi_a \xi_b g(\xi^2)), \tag{8}$$

where $\xi^2 = \sum_{a=1}^{3} \xi_a \xi_a$, δ_{ab} is Kronecker's delta, $f(\xi^2)$ is an arbitrary function of ξ^2, and where $g(\xi^2)$ is related to $f(\xi^2)$ as shown below.

Here it is worth mentioning that the generality of (8) implies the subgroup is symmetric (Ref. 12). Also, (8) is a more general transformation law than the standard transformation law of the preferred fields described in the previous chapter (but they are physically equivalent in the sense of that chapter).

From the condition $[A_c, A_a]\xi_b = i\varepsilon_{cad}V_d\xi_b$, which follows from the Lie algebra of generators, we can calculate the function $g(\xi^2)$ as

$$g = \frac{1 + 2ff'}{f - 2\xi^2 f'} \tag{9}$$

where $f' = \dfrac{df(\xi^2)}{d\xi^2}$ (for the proof see Appendix 4a).

It is not easy to construct an isoscalar function $S(\xi)$ satisfying condition (7) and we shall devote to this task the rest of this section. We could try the obvious isoscalar ξ^2, but it does not terminate as condition (7) requires.

We have to take some general function of ξ^2. Let us call it $h_o(\xi^2)$. When we apply the operator A_+ (multiplied by i for our convenience) to such a function we obtain

$$(iA_+)h_o(\xi^2) = h_o'(\xi^2)(iA_+)\xi^2 = h_o'\, 2\sum_{a=1}^{3} (\delta_{1a}f + \xi_1\xi_a g + i\delta_{2a}f +$$

$$+ i\xi_2\xi_a g)\xi_a = 2h_o'(f + \xi^2 g)(\xi_1 + i\xi_2) = h_1(\xi^2)\xi_+, \tag{10}$$

where a prime always means a first derivative of a function with respect to ξ^2 and where the last step is just a definition of

$$h_1(\xi^2) = 2(f(\xi^2) + \xi^2 g(\xi^2))h_o'(\xi^2) \text{ and of } \xi_+ = \xi_1 + i\xi_2.$$

Similarly we can define $h_2(\xi^2)$ through $(iA_+)^2 h_o = h_2(\xi_+)^2$ (the common argument ξ^2 of the functions h is assumed, and most generally

$$(iA_+)^n h_o = h_n(\xi_+)^n. \tag{11}$$

Such a relation defines the functions h_n properly, as can be checked using equation (8) repeatedly. Furthermore, this definition gives us a recurrence relation for the functions h_n in the form

$$h_{n+1} = ngh_n + 2h_n'(f + \xi^2 g). \tag{12}$$

For n = 0 this has been derived in (10), and we can prove similarly that it holds for (n+1) if it holds for n.

To guarantee a proper termination of the functions h (equation 7), we have to impose the condition

$$h_{2j+1} = 0 \tag{13}$$

Each function h_n must be an eigenfunction of both Casimir operators with the required eigenvalues (see equation 3). Since the operator A_+ commutes with the Casimir operators we can require this for the h_o-function only and we shall get the desired eigenvalues for all the other functions h automatically.

Since h_o is a scalar under H and thus obeys $V_a h_o = 0$ (a = 1, 2, 3), it is an eigenvector of $C_2 = A \cdot V$ with eigenvalue zero. This together with the requirement that h_o be an eigenvector of C_1 (see equation (3)) then implies that

$$C_1 h_o = \sum_{a=1}^{3} A_a A_a h_o = 4j(j + 1)h_o \tag{14}$$

To rewrite this equation in an explicit form we calculate $A_a h_o = ih_1 \xi_a$ and then apply the operator A_a once more and sum over a. The result is

$$\sum_{a=1}^{3} A_a A_a h_o = -(2h_1'(f + \xi^2 g)\xi^2 + h_1(3f + \xi^2 g)) = -(h_2\xi^2 + 3h_1 f).$$

Equation (14) now reads

$$h_2\xi^2 + 3h_1 f + 4j(j + 1) = 0 \tag{15}$$

Equations (12), (13) and (15) define the functions h_n completely.

To solve them in this form would be still quite difficult. The greatest contribution of Ref. 17 consists in introducing a new set of functions ν_n of a new variable u given by

$$u = -f/\sigma, \quad \sigma = (f^2 + \xi^2)^{\frac{1}{2}}, \text{ and } \nu_n(u) = \sigma^n h_n(\xi^2). \tag{16}$$

This redefinition enables us to rewrite the recurrence relation (12) in a nice form,

$$\nu_{n+1}(u) = \frac{d}{du}\,\nu_n(u), \tag{17}$$

as can be checked easily using $\frac{d\sigma}{d\xi^2} = \frac{2ff'+1}{2\sigma}$, $\frac{du}{d\xi^2} = \frac{f - 2\xi^2 f'}{2\sigma^3}$

and relation (9).

Equation (13) now reads

$$\nu_{2j+1}(u) = \left(\frac{d}{du}\right)^{2j+1} \nu_o(u) = 0 \tag{18}$$

and rewriting equation (15) yields

$$(1 - u^2)\nu_2(u) - 3u\nu_1(u) + 4j(j + 1) = 0. \tag{19}$$

Equation (18) tells us that $\nu_o(u)$ must be a polynomial of degree 2j or less in the variable u. Equation (19) can be solved by a power series expansion, $\nu_o(u) = \Sigma b_n u^n$, which yields a recurrence relation for the coefficients b_n,

$$(2r + 2)(2r + 3)b_{2r+2} = (4r(r + 1) - 4j(j + 1)b_{2r} \tag{20}$$

where $r = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Because the recurrence relation (20) together with equation (18) implies that b_{2j+1} must vanish while $b_{2j} \neq 0$, we have to take $b_o \neq 0$ and $b_1 = 0$ for j being an integer and $b_o = 0$ and $b_1 \neq 0$ for j being a half-integer number.

The function $h_o(\xi^2) = \nu_o(u)$ is the isoscalar function of the preferred fields which we were trying to find and which has all the required properties.

The way to generate the remaining functions of the (j, j)-representation from h_o has been discussed at the beginning of this section.

In the case of $j = \frac{1}{2}$ the $(\frac{1}{2}, \frac{1}{2})$ representation decomposes (when restricted to H) into representations with I = 0 and 1. The eigenvalues correspond to the σ-particle and three pions known from the so-called σ-model (Ref. 4).

<u>Note: Redefinition of the Preferred Fields</u>

We can define a set of new fields by the equation

$$\xi_a^* = \xi_a H(\xi^2) \tag{21}$$

where $H(\xi^2)$ is any function of ξ^2 for which $H(0) \neq 0$. (As discussed in Chapter II this will give us the same physical results). The previous section can be rewritten in terms of these new fields if we replace the functions f and g by functions F and G of the new variable

ξ^{*2}. These functions, in order to satisfy equations (4), (8) and (9) must be of the form

$$F(\xi^{*2}) = f(\xi^2)H(\xi^2) \text{ and } G(\xi^{*2}) = (gH + 2H'(f + \xi^2 g))/H^2 \qquad (22)$$

as can be seen by writing $A_a\xi_b^* = A_a(\xi_b H(\xi^2)) = (\delta_{bc}H + 2\xi_b\xi_c H')A_a\xi_c =$ $= -i(\delta_{ab}fH + \xi_a\xi_b(gH + 2H'f + 2H'\xi^2 g))$ and $V_a\xi_b^* = V_a(\xi_b H(\xi^2)) =$ $= (\delta_{bc}H + 2\xi_b\xi_c H')i\varepsilon_{acd}\xi_d = i\varepsilon_{abc}\xi_c^*$.

Furthermore, we have to redefine $\sigma^* = H\sigma$ and $u^* = u$ (see definition 16). We now notice that $h_n(\xi^2)(\xi_+)^n = h_n(\xi^{*2})(\xi_+^*)^n$ (see the third of equations 16), which means that the linearized fields do not change under such a redefinition of the preferred fields.

b) Linearization of Other Fields

We already know from the last chapter how the other fields Ψ transform under an element h of the diagonal subgroup H, namely according to a linear representation of H, in a formula: $(\psi')^\alpha = R^\alpha{}_\beta(h)\Psi^\beta$. The generators then operate on the fields ψ as follows:

$$V_a\Psi = t_a\Psi \qquad (23)$$

where by t_a we understand the matrix $R(V_a)$. For the generators A the most general form of their action on the Ψ's can be derived to be (Refs. 12, 17, 22)

$$A_a\Psi = v(\xi^2)\varepsilon_{abc}\xi_c t_b\Psi, \qquad (24)$$

where we sum over indices b and c.

From the Lie algebra we obtain $[A_a, A_b]\Psi = i\varepsilon_{abc}V_c\Psi$, which yields the following expression for the function v:

$$v(\xi^2) = \frac{1}{f(\xi^2)\pm(f^2 + \xi^2)^{\frac{1}{2}}} = \frac{1}{\sigma(1 \pm u)} .$$

Here f is the arbitrary function of the last section. (For a proof see Appendix 4b).

We want to extend the linear transformation property (23) over the whole group SU(2) x SU(2).

The simplest case occurs when equation (23) defines an irreducible representation of H. We can expand it to either of the following representations of the whole group G, (t, 0) or (0, t).

In the extended representation there will correspond to each A_a a matrix which we shall denote by X_a. The set of X and t-matirces must satisfy the commutation relations (1),

$$[t_a, t_b] = [X_a, X_b] = i\varepsilon_{abc}t_c \text{ and } [t_a, X_b] = i\varepsilon_{abc}X_c \qquad (25)$$

Linearization of the fields Ψ will be done by finding a matrix $M(\xi)$ (a function of the preferred fields) which will multiply Ψ to give a result transforming linearly.

This means we are looking for a matrix $M(\xi)$ satisfying

$$V_a(M(\xi)\Psi) = -t_aM(\xi)\Psi \text{ and } A_a(M(\xi)\Psi) = -X_aM(\xi)\Psi. \qquad (26)$$

It can be checked (using equations 23 and 24) that (26) is equivalent to

$$V_aM(\xi) = -[t_a, M] \text{ and } A_aM(\xi) = -X_aM - \varepsilon_{abc}\xi_c vMt_b. \qquad (27)$$

The second of these equations can be written more explicitly as

$$\frac{\partial M}{\partial \xi_b}(-i)(\delta_{ab}f + \xi_a\xi_b g) = -X_a M - \epsilon_{abc}\xi_c vMt_b \tag{28}$$

Multiplying this by ξ_a and summing over a we obtain

$$(-i)(f + \xi^2 g)\xi_a \frac{\partial M}{\partial \xi_a} = -(X\cdot\xi)M, \tag{29}$$

where

$$(X\cdot\xi) = \sum_{a=1}^{3} X_a\xi_a.$$

To solve this equation, we shall make an ansatz for M as a power series of $(iX\cdot\xi)$ with ξ^2-dependent coefficients

$$M(\xi) = \sum_{n=1}^{\infty} a_n(\xi^2)(iX\cdot\xi)^n.$$

Such a choice already satisfies the first of the equations (27). Inserting this trial solution into equation (29) gives us

$$(f + \xi^2 g)(2\xi^2 a_n' + na_n) = -a_{n-1}. \tag{30}$$

If we assume $a_n(\xi^2) = \frac{1}{n!}\left(\frac{\lambda}{(\xi^2)^{1/2}}\right)^n$, where λ is a function of u only, equation (30) simplifies to

$$\frac{d\lambda}{du} = -\frac{1}{(1-u^2)^{1/2}} \tag{31}$$

This has a solution $\lambda = \arccos(-u) \leq 0$.

With this solution for $\lambda(u)$ we find that

$$M(\xi) = \exp\,(i\lambda(X\cdot\xi)/(\xi^2)^{1/2}). \tag{32}$$

This M is obviously also invariant under the redefinition (21) of the preferred fields.

M can be calculated explicitly for the representations ($\frac{1}{2}$,0) and (0,$\frac{1}{2}$) (Appendix 4c). We obtain the following result:

$$M = \left(\frac{1 - u}{2}\right)^{\frac{1}{2}} \left(1 \mp \frac{i}{\sigma(1 - u)} (S\cdot\xi)\right), \tag{33}$$

where the S are the Pauli matrices.

Similarly we get for the representations (1,0) and (0,1)

$$M = \left(1 \mp \frac{i}{\sigma} (X\cdot\xi) - \frac{1 + u}{\xi^2} (X\cdot\xi)^2\right), \tag{34}$$

where X_a is the matrix with coefficients $(X_a)_{bc} = -i\varepsilon_{abc}$.

c) Relation Between M and $(V-)^k h_n(\xi+)^n$

If the linearized version of the fields Ψ is transformed according to the (j, j)-representation of G, then there exists a relation between the matrix M and these functions which were constructed in section a) from the preferred fields to represent the group G linearly. By $|n,m\rangle$ we shall denote the base vectors of the (j, j)-representation defined by the following properties:

$$t_3|n,m\rangle = m|n,m\rangle \text{ and } (t\cdot t)|n,m\rangle = n(n + 1)|n,m\rangle.$$

As before, the t are matrices representing the operators V in the (j, j)-representation, n may be any number from the series 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, ..., 2j and m satisfies $|m| \leq n$.

We can prove (Ref. 17) that the matrix elements $\langle 0,0|M(\xi)|n,m\rangle$ are exactly (up to a constant factor) these functions constructed in section a) to form the (j, j)-representation which are characterized by the same quantum numbers n and m.

For example in the case of the $(\frac{1}{2},\frac{1}{2})$ representation we have

$$M(\xi) = \exp\left(\frac{i\lambda}{2(\xi^2)^{\frac{1}{2}}}(S\cdot\xi)\right) \otimes \exp\left(-\frac{i\lambda}{2(\xi^2)^{\frac{1}{2}}}(S\cdot\xi)\right) \tag{35}$$

(compare with equation (32), $S = 2X$ are Pauli matrices), which yields (see Appendix 4d) $\langle 0,0|M|0,0\rangle = -u$ and $\langle 0,0|M|1,+1\rangle = -i\xi_+/\sigma$ which are the functions of section a) obtained from $\nu_0(u) = -u$ (up to the constant factors appearing in the last three expressions) as can be seen from equations (11), (16) and (17) and from

$(V_1 - iV_2)\,\frac{\xi+}{\sigma} = -2\frac{\xi_3}{\sigma} = -2\frac{\xi_0}{\sigma}$ (by definition of ξ_0) and

$(V_1 - iV_2)\,\frac{\xi_3}{\sigma} = \frac{\xi-}{\sigma}$.

Note: The Standard Form of $f(\xi^2)$

In the previous chapter a particular kind of nonlinear realization was constructed. The results of the present chapter are more general (but restricted to SU(2) x SU(2)) because the function $f(\xi^2)$ describing the transformation laws is completely arbitrary. (Physically no generality is gained by this freedom to choose $f(\xi^2)$ as was discussed in Chapter II).

It would be interesting to know what the function $f(\xi^2)$ would have to be to give the special case of Chapter II. After some calculation the answer turns out to be (Ref. 17)

$$f(\xi^2) = -(\xi^2)^{\frac{1}{2}} \cot (\xi^2)^{\frac{1}{2}}.$$

The special kind of transformation implied by this special function f can always be obtained from a general function f if we redefine the preferred fields as

$\xi_a^* = \xi_a \frac{\lambda}{(\xi^2)^{\frac{1}{2}}}$ with λ defined in (31).

This equation implies that $\lambda = (\xi^{*2})^{\frac{1}{2}}$, and since λ is invariant under such a redefinition of the preferred fields, we can rewrite equation (32) in terms of the new fields ξ^* as

$$M(\xi^*) = \exp(-iX\cdot\xi^*)$$

d) Lagrangians.

In this section we shall more or less quote Ref. 17 to indicate possible physical applications of the theory of nonlinear realizations.

In the Introduction we mentioned briefly the symmetry breaking terms in Lagrangians and now we are going to construct them explicitly.

For a term which remains invariant under the diagonal subgroup H but which breaks the symmetry of a Lagrangian under the whole group G we can naturally take the isoscalar function h_0 (from section a). We can expand it as a power series in ξ^2 and it gives after some easy manipulations.

$$h_0 = \frac{3m^2}{8j(j+1)u_0'} - \frac{1}{2}m^2\xi^2 + \frac{m^2}{4u_0'}\left(\frac{4j(j+1)-3}{5}(u_0')^2 - u_0''\right)(\xi^2)^2 + \ldots \tag{36}$$

where the coefficient of ξ^2 has been called $-\frac{1}{2}m^2$ and where u is the function of ξ^2 defined in (16) and where u_0' and u_0'' mean the first and second derivative of u at the point $\xi^2 = 0$. The first term of (36) is a constant and has no physical significance. The second term is called a mass term because it indicates the mass of the particles ξ. It gives us the reason for adding such a symmetry breaker to the Lagrangian, which would otherwise describe massless particles only (see expression 39). The third term of (36) will contribute to the interaction term of the whole Lagrangian.

The next term of the Lagrangian should be invariant under the whole group (it is the basic term not breaking the SU(2) x SU(2) symmetry) and it should contain the space-time derivatives of ξ

(if we consider the preferred fields only). Since we know that

$$V_a\partial_\mu M|0,0\rangle = -t_a\partial_\mu M|0,0\rangle \quad \text{and} \quad A_a\partial_\mu M|0,0\rangle = -X_a\partial_\mu M|0,0\rangle \tag{37}$$

(from equation 26), we can conclude that $\langle 0,0|\partial_\mu M^+\partial^\mu M|0,0\rangle$ provides an expression with the desired properties.* It can be expanded as a power series again

$$g\langle 0,0|\partial_\mu M^+\partial^\mu M|0,0\rangle = -\frac{1}{2}(\partial_\mu\xi\cdot\partial_\mu\xi) - \frac{u_0'' - (u_0')^2}{4u_0'}\,\xi^2(\partial_\mu\xi\cdot\partial^\mu\xi) - \\ - \frac{u_0'' + (u_0')^2}{2u_0'}\,(\xi\cdot\partial_\mu\xi)^2 + \ldots, \tag{38}$$

where g is a number which makes the coefficient of $(\partial_\mu\xi\cdot\partial^\mu\xi)$ equal to $-\frac{1}{2}$ (to get the so called kinematic term $-\frac{1}{2}(\partial_\mu\xi\cdot\partial^\mu\xi)$), where u_0' and u_0'' have the same meaning as before and where $(\xi\cdot\partial_\mu\xi)$ and $(\partial_\mu\xi\cdot\partial^\mu\xi)$ are the usual scalar products. Equation (38) (as well as 36) is correct up to the fourth order of ξ.

Collecting the right hand sides of equations (36) and (38), we get for the total Lagrangian

$$\mathcal{L} = -\frac{1}{2}m^2\xi^2 - \frac{1}{2}(\partial_\mu\xi\cdot\partial^\mu\xi) + \frac{1}{4}u_0'\Big(\frac{1}{5}(4j(j+1)-3)m^2(\xi^2)^2 + \\ + \xi^2(\partial_\mu\xi\cdot\partial^\mu\xi) - 2(\xi\cdot\partial_\mu\xi)^2\Big) = \frac{1}{2}m^2\xi^2 - \frac{1}{2}(\partial_\mu\xi\cdot\partial^\mu\xi) + \mathcal{L}_{\xi-\xi}, \tag{39}$$

where $\mathcal{L}_{\xi-\xi}$ is the interaction part. In equation (39) we omitted the constant term of equation (36) as well as another term proportional to u_0'' (because it is also proportional to the four-divergence of

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* $\partial_\mu M^+$ means the hermitian conjugate of $\partial_\mu M$.

$\xi^2(\xi\cdot\partial_\mu\xi)$ and therefore does not contribute to the space-time integral of $\mathcal{L}$).

We shall add one more note concerning the weak currents.

Experimental evidence indicates that the weak strangeness conserving current changes the third component of the isospin by 1. This, together with our sign conventions for the currents, implies that the weak strangeness current is transformed according to the (1,0)-representation of the group G. We can check that the following expression transforms that way; furthermore, it is obviously a four vector in space-time

$$J_\mu = G\langle 0,0|M^+(t_a + X_a)\partial_\mu M|0,0\rangle = G\,\frac{4j(j+1)}{3}\,\frac{i}{f^2+\xi^2}\,((\vec{\xi}\times\partial_\mu\vec{\xi})_a +$$

$$+ f\partial_\mu\xi_a - \xi_a\partial_\mu f).$$

So it can serve as an expression for the weak strangeness conserving current (here G is a numerical constant and $\vec{\xi}\times\partial_\mu\xi$ is the usual vector product).

Similarly we know from experiment that the weak strangeness violating current changes the third component of the isospin by $\frac{1}{2}$ and must transform according to the $(\frac{1}{2},0)$-representation of the group G. To construct such an expression we take some additional fields Ψ associated with the K-mesons) which transform under the diagonal subgroup H according to the usual 2-dimensional representation (isospin $= \frac{1}{2}$). Then we need the matrix $M(\xi)$ in the $(\frac{1}{2},0)$-, $(0,\frac{1}{2})$- and $(\frac{1}{2},\frac{1}{2})$-representation of the group G (we shall denote these by $M(\frac{1}{2},0)$, $M(0,\frac{1}{2})$ and $M(\frac{1}{2},\frac{1}{2})$ respectively). Now Ref. 17 suggests two expressions for the weak strangeness violating current, which can be shown to have the desired properties and which can be expanded in a power series.

These expressions are

$$J_\mu^{(1)} = \partial_\mu(M(\tfrac{1}{2},0)\Psi) = -\partial_\mu\Psi + \frac{i}{2f_o}\,\partial_\mu((\xi\cdot S)\Psi) + \frac{1}{4f_o^{\,2}}\,(\xi\cdot\partial_\mu\xi)^2\Psi +$$

$$+ \frac{\xi^2}{8f_o^{\,2}}\;\partial_\mu\Psi + \ldots \qquad \text{and}$$

$$J_\mu^{(2)} = i\Psi^T M^T(0,\tfrac{1}{2})S_2\partial_\mu M(\tfrac{1}{2},\tfrac{1}{2})|0,0\rangle = \frac{i}{(2)^{\frac{1}{2}}f_o}\,(S\cdot\partial_\mu\xi)\Psi +$$

$$+ \frac{1}{2(2)^{\frac{1}{2}}f_o}\,(S\cdot\xi)(S\cdot\partial_\mu\xi)\Psi + \ldots$$

where the S's are the Pauli matrices, f_o is the value of the function f at $\xi^2 = 0$ and Ψ^T and M^T are the transposed matrices Ψ and M respectively.

CHAPTER V

THE CONFORMAL GROUP

a) Definition and notation

The conformal group can be characterized as the set of all point transformations of Minkowski space which map a space-time vector x_μ into another space-time vector x'_μ such that infinitesimal null-vectors dx_μ remain null vectors.* In other words, $dx_\mu dx^\mu = 0$ implies $dx'_\mu (dx')^\mu = 0$, where the metric is the Lorentz metric $g^{\mu\nu} = (+1, -1, -1, -1)$ (Ref. 23).

These transformations consist of the Poincaré group transformations plus the so-called special conformal transformations depending on four parameters β_μ and defined by

$$\frac{x_\mu}{x^2} \longrightarrow \frac{x_\mu}{x^2} + \beta_\mu = \frac{x'_\mu}{(x')^2} \tag{1}$$

plus the dilatations depending on one parameter σ and defined by

$$x_\mu \longrightarrow x_\mu e^{-\sigma} = x'_\mu \tag{2}$$

where x^2 and $(x')^2$ denote the Lorentz-invariant scalar products of x_μ and x'_μ with themselves, both formed, of course, with the same metric $g^{\mu\nu}$ as above (Ref. 23).

We shall show in this section that the conformal group is equivalent to a group of linear transformations in a six-dimensional space preserving the following metric:

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* We shall express most relations in terms of the quantities $x_\mu \equiv (t, -\vec{x})$ instead of $x^\mu \equiv (t, \vec{x})$ for our convenience only. All relevant equations can be rewritten in terms of x^μ easily (by raising indices).

$$g^{AB} = (+1,\ -1,\ -1,\ -1,\ -1,\ +1), \tag{3}$$

where the indices A and B run through the numbers 0, 1, 2, 3, 5 and 6 (Ref. 20). Such a group will be called O(4,2). To be more precise, if Λ is an element of O(4,2) (Λ is a (six by six)-dimensional matrix)* which maps a six-vector η_A into η'_A via $\eta'_A = \Lambda_A{}^B \eta_B$, we must have

$$\Lambda_A{}^B g^{AC} \Lambda_C{}^D = g^{BD}, \tag{4}$$

where summation over indices B and C is understood. Equation (4) can also be written shortly as $\Lambda^T g \Lambda = g$, where Λ^T is the transpose of the matrix Λ.

Suppose we have a representation of the group O(4,2) given by matrices R. A matrix representing an element $\Lambda_A{}^B = \delta_A{}^B + \varepsilon_A{}^B$ (where $\varepsilon_A{}^B$ is a set of 36 infinitesimally small numbers) can be written in the form

$$R(\Lambda) = \mathbb{1} + \frac{i}{2}\, \varepsilon_A{}^B J_B{}^A, \tag{5}$$

where $\mathbb{1}$ is a unit matrix and where $J_B{}^A$ is a set of thirty-six matrices generating the representation (they are of the same dimension as $\mathbb{1}$ and just fifteen of them are linearly independent). We get a similar expression for another element $\Lambda' = \mathbb{1} + \varepsilon'$ (in general ε' is different from ε and $\mathbb{1}$ is here the (six by six)-unit matrix) of the group O(4,2). Using equation (5) gives us (up to the first order of ε and ε')

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* It is difficult to keep the previously introduced notation throughout this chapter because of the existing (and overlapping) standard notations. A general element of the group O(4,2) (the conformal group) will be called Λ (instead of the old g) and the letter g will be saved for the metric tensor.

$$R(\Lambda)(\mathbb{1}+\frac{i}{2}\varepsilon'_{AB}J^{BA})R^{-1}(\Lambda) = R(\Lambda)R(\Lambda')R(\Lambda^{-1}) = R(\Lambda(\mathbb{1}+\varepsilon')\Lambda^{-1}) =$$

$$= \mathbb{1}+\frac{i}{2}(\Lambda\varepsilon'\Lambda^{-1})_{CD}J^{DC} = \mathbb{1}+\frac{i}{2}\Lambda_C{}^A\varepsilon'_A{}^B g_{BE}\Lambda_D{}^E J^{DC} = \tag{6}$$

$$= \mathbb{1}+\frac{i}{2}\Lambda_C{}^A\varepsilon'_{AB}\Lambda_D{}^B J^{DC},$$

where we used the relation $\Lambda^{-1} = g\Lambda^T g$ and the usual practice of raising and lowering indices.

Comparing the first and the last expression of (6) we can cancel the unit matrix and the factor $\frac{i}{2}$. In order to cancel out the common factor ε'_{AB} (which is antisymmetric in A and B as follows from equation (4)) we have to antisymmetrize its coefficients on both sides. We can do this without loss of generality by choosing J_{AB} to be antisymmetric in the indices A and B. Then equation (6) reads

$$R(\Lambda)J^{BA}R^{-1}(\Lambda) = \Lambda_C{}^A\Lambda_D{}^B J^{DC} \tag{7}$$

This can be rewritten once more as

$$(\mathbb{1}+\frac{i}{2}\varepsilon_{CD}J^{CD})J^{BA}(\mathbb{1}-\frac{i}{2}\varepsilon_{CD}J^{DC}) = (\delta_C{}^A+\varepsilon_C{}^A)(\delta_D{}^B+\varepsilon_D{}^B)J^{DC},$$

which (after cancelling some terms) yields the result

$$\frac{i}{2}\varepsilon_{CD}[J^{DC}, J^{BA}] = \varepsilon_C{}^A J^{BC} + \varepsilon_D{}^B J^{DA} = \varepsilon_{CD}g^{DA}J^{BC} + \varepsilon_{DC}g^{CB}J^{DA}.$$

In order to cancel ε_{CD} in this equation we have to write its coefficients as a sum of symmetric and antisymmetric terms in C, D. The symmetric terms do not contribute, because ε_{CD} is antisymmetric, and we are left with the antisymmetric coefficients

$$\frac{1}{i}[J^{AB}, J^{CD}] = g^{AD}J^{BC} - g^{AC}J^{BD} + g^{BC}J^{AD} - g^{BD}J^{AC}, \tag{8}$$

which is the final result of this calculation and defines the Lie algebra of the O(4,2) group.

It can be checked easily, that the following matrices (Ref. 20) provide a representation of this algebra:

$$(J^{AB})_C{}^D = i(\delta_C{}^A g^{BD} - \delta_C{}^B g^{AD}). \tag{9}$$

These matrices represent the generators in the six-dimensional self-representation of the group O(4,2). (Here A,B are indices labelling the matrix and C,D are the usual row and column indices).

Similarly we get (lowering the indices A,B)

$$(J_{AB})_C{}^D = i(g_{AC}\delta_B{}^D - g_{BC}\delta_A{}^D). \tag{10}$$

In Appendix 5a) there is an explicit list of these matrices, which is convenient to have for any practical calculation.

To make the connection with the conformal group we redefine these generators into a set

$$P_\mu = J_{5\mu} + J_{6\mu} \qquad K_\mu = J_{5\mu} - J_{6\mu} \quad \text{and} \quad D = J_{56}, \tag{11}$$

where μ runs from 0 to 3. For $A \leq 3$ and $B \leq 3$ we keep J_{AB} unchanged except for denoting these indices by Greek letters.

The structure equations of the Lie algebra of O(4,2) in terms of these new generators become (using equation (8))

$$[J_{\alpha\beta}, J_{\mu\nu}] = i(g_{\alpha\nu}J_{\beta\mu} - g_{\beta\nu}J_{\alpha\mu} + g_{\beta\mu}J_{\alpha\nu} - g_{\alpha\mu}J_{\beta\nu}) \tag{12a}$$

$$[P_\mu, P_\nu] = 0 \quad [K_\mu, K_\nu] = 0 \quad \text{and} \quad [J_{\mu\nu}, D] = 0 \tag{12b}$$

$$[P_\mu, D] = -iP_\mu \;,\; [K_\mu, D] = iK_\mu \quad \text{and} \quad [K_\mu, P_\nu] = 2i(J_{\mu\nu} - g_{\mu\nu}D) \qquad (12c)$$

$$[P_\alpha, J_{\mu\nu}] = i(g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu) \text{ and } [K_\alpha, J_{\mu\nu}] = i(g_{\alpha\mu}K_\nu - g_{\alpha\nu}K_\mu). \qquad (12d)$$

This is just the Lie algebra of the conformal group (Ref. 23), where $J_{\mu\nu}$ are the generators of the Lorentz transformations, P_μ are the generators of the translations, K_μ are the generators of the special conformal transformations and D is the generator of the dilatations.

Now we can show quite easily (Appendix 5b) using the generators in the self-representation that

$$e^{i(\alpha\cdot P)}\eta = \begin{pmatrix} \eta_\mu + \alpha_\mu(\eta_5 + \eta_6) \\ \eta_5 + (\alpha\cdot\eta) + \frac{1}{2}\alpha^2(\eta_5 + \eta_6) \\ \eta_6 - (\alpha\cdot\eta) - \frac{1}{2}\alpha^2(\eta_5 + \eta_6) \end{pmatrix} \qquad (13a)$$

$$e^{i(\beta\cdot K)}\eta = \begin{pmatrix} \eta_\mu + \beta_\mu(\eta_5 - \eta_6) \\ \eta_5 + (\beta\cdot\eta) + \frac{1}{2}\beta^2(\eta_5 - \eta_6) \\ \eta_6 + (\beta\cdot\eta) + \frac{1}{2}\beta^2(\eta_5 - \eta_6) \end{pmatrix} \qquad (13b)$$

$$e^{(i\sigma D)}\eta = \begin{pmatrix} \eta_\mu \\ \eta_5\cosh\sigma + \eta_6\sinh\sigma \\ \eta_5\sinh\sigma + \eta_6\cosh\sigma \end{pmatrix}, \qquad (13c)$$

where η is a six-vector, α_μ and β_μ are two sets of four real parameters each, $(\alpha\cdot P)$, $(\beta\cdot K)$, $(\alpha\cdot\eta)$ and $(\beta\cdot\eta)$ mean the usual scalar product in four dimensions (for example $(\alpha \cdot \eta) = \alpha_0\eta_0 - \alpha_1\eta_1 - \alpha_2\eta_2 - \alpha_3\eta_3$) and σ is a real parameter.

Because of the isometrism between O(4,2) and the conformal group it must be possible to redefine the six-vector η to get four components which would transform like a space-time vector, and two additional components say k and λ. One possibility is the following (Ref. 20):

$$x_\mu = \frac{\eta_\mu}{\eta_5 + \eta_6} \qquad k = \eta_5 + \eta_6 \quad \text{and } \lambda = \eta_5 - \eta_6 \tag{14}$$

It can be shown easily (using equations (13)) that the x_μ really have the correct transformation properties under the Poincaré subgroup and under dilatations. To obtain also agreement with the transformation property (1), we have to restrict the η's to lie on the hyperquadratic

$$\eta^2 = \eta_o\eta_o - \eta_1\eta_1 - \eta_2\eta_2 - \eta_3\eta_3 - \eta_5\eta_5 + \eta_6\eta_6 = 0 \tag{15}$$

This only means we loose one of our independent parameters, say λ, which is equal to kx^2.

With the restriction (15), the expression (14) for x^μ can be interpreted as the definition of homogeneous coordinates for space-time. In another connection these homogeneous coordinates serve the usual purpose of distinguishing between various infinite points of space-time. This is important because the conformal group maps finite points into infinity and vice versa. Since it is also a one-to-one map, we must distinguish between different points at infinity.

We can now write

$$\frac{x_\mu}{x^2} = \frac{\eta_\mu}{\eta_5 - \eta_6}$$

and, using equation (13b), verify the transformation law (1).

For a later purpose we define an operator ∂_{AB}. If $f(\eta)$ is a scalar function of η, we can calculate an infinitesimal change of this function under an element Λ of the group O(4,2). With $\Lambda_A{}^B = \partial_A{}^B + \varepsilon_A{}^B$ the change of f can be expressed as

$$\delta f(\eta) = f(\eta') - f(\eta) = \frac{\partial f}{\partial \eta_A} (\eta' - \eta)_A = \frac{\partial f}{\partial \eta_A} \varepsilon_A{}^B \eta_B = \varepsilon^{AB} \eta_B \partial_A f,$$

where the last step is a definition of the operator ∂_A.
Since ε^{AB} is antisymmetric in the indices A,B, we antisymmetrize its coefficients by writing

$$\delta f(\eta) = \frac{1}{2}\varepsilon^{AB}(\eta_B \partial_A - \eta_A \partial_B) f(\eta) = -\frac{1}{2}\varepsilon^{AB} \partial_{AB} f(\eta). \tag{16}$$

The last step is a definition of the operator ∂_{AB}.

This operator can easily be rewritten in terms of the new set of independent variables x_μ, k and λ:

$$\begin{aligned}
\partial_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu \\
\partial_{5\mu} + \partial_{6\mu} &= \partial_\mu + 2x_\mu k \frac{\partial}{\partial \lambda} \\
\partial_{5\mu} - \partial_{6\mu} &= \frac{\lambda}{k} \partial_\mu - 2x_\mu ((x \cdot \partial) - k \frac{\partial}{\partial k}) \\
\partial_{56} &= -(x \cdot \partial) + k \frac{\partial}{\partial k} - \lambda \frac{\partial}{\partial \lambda} .
\end{aligned} \tag{17}$$

If we use here condition (15) for the six-vector η (this condition is invariant under the O(4,2) group) and if we redefine the function $f(\eta)$ as follows:

$$f(\eta) \longrightarrow f(\eta) - (\lambda - kx^2) \frac{\partial f}{\partial \lambda}$$

(such a redefinition does not change the function for any η satisfying condition (15) but makes its first derivative with respect to λ vanish there for these η's), we can rewrite equation (17) (Ref. 20) as

$$\begin{aligned}
\partial_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu \\
\partial_{5\mu} + \partial_{6\mu} &= \partial_\mu \\
\partial_{5\mu} - \partial_{6\mu} &= x^2 \partial_\mu - 2x_\mu ((x \cdot \partial) - k \frac{\partial}{\partial k}) \\
\partial_{56} &= -(x \cdot \partial) + k \frac{\partial}{\partial k} .
\end{aligned} \qquad (18)$$

The standard way of dealing with $k \frac{\partial}{\partial k}$ is to assume that $f(\eta)$ is a homogeneous function of k of degree ℓ, which implies that it satisfies Euler's equation,

$$k \frac{\partial}{\partial k} f(\eta) = \ell f(\eta).$$

A general function $f(\eta)$ can be written as a sum of such homogeneous functions of different degrees ℓ.

b) Transformation of the Preferred Fields

For constructing a nonlinear realization of the preferred fields we use the standard method of the second chapter. It does not guarantee generality of our results, i.e., there might be different physically inequivalent nonlinear realizations of this group since the conformal group is noncompact (see Chapters I and II), but it can be shown (Ref. 5) that in the case of the conformal group this method does provide the most general results.* Some attempts to generalize the results to all noncompact groups have been made in Refs. 12, 13, but the arguments are not quite clear. Further work remains to be done to give an exhausting answer to this question.

In the case of the conformal group we choose as subgroup H the homogeneous Lorentz subgroup because then the commutation relations are of the required form discussed in connection with expression (5) in Chapter II. Thus we are left with nine preferred fields. Since four of them turn out to transform just like the four coordinates of the space-time vector we will interpret them as x_μ. Then there are five preferred fields left.

Because any group element Λ can be written uniquely in the form

$$\Lambda = e^{i(x\cdot P)}\, e^{i(\phi\cdot K)}\, e^{-i\sigma D}\, e^{i\varepsilon_{\mu\nu}J^{\mu\nu}}$$

(Ref. 7) (splitting the preferred fields into three parts is unessential since the decomposition is still unique), we can define a transformation of the preferred fields under a group element Λ_0 by

$$\Lambda_0 e^{i(x\cdot P)}\, e^{i(\phi\cdot K)}\, e^{-i\sigma D} = e^{ix'_\mu P^\mu}\, e^{i\phi'_\mu K^\mu}\, e^{-i\sigma' D} h, \tag{19}$$

where h is an element of H uniquely determined by (19).

To calculate the transformation laws of the preferred fields explicitly is rather tricky. Since the 6-dimensional self-representation

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*In Ref. 5 the author starts "by defining the most general commutators between the preferred fields and the generators of the group. The Jacobi identities are then used to derive differential equations for the functions introduced in the commutators. Explicit field transformations are shown to define transformation laws for these functions and these laws are then used to show that the commutators are equivalent to the very simple form ... of Ref 20."

of the conformal group (discussed above) is well-known and since a number of relations are probably simplest in this representation, we shall express all group elements and generators in the following in this representation. Then we notice that

$$e^{ix'_\mu P^\mu} e^{i\phi'_\mu K^\mu} e^{-i\sigma' D} = \Lambda_o e^{ix_\mu P^\mu} e^{i\phi_\mu K^\mu} e^{-i\sigma D} h^{-1} \tag{20}$$

implies that the last two columns of the (6x6)-matrix

$$M = e^{ix_\mu P^\mu} e^{i\phi_\mu K^\mu} e^{-i\sigma D} \tag{21}$$

transform as two ordinary six-vectors. This is so because in our representation the matrix h^{-1} is obviously in fully reduced block form containing a four-dimensional part, effecting the homogeneous Lorentz transformation on the first four components of a 6-vector, and a two-dimensional unit matrix in the indices 5,6. This implies that the matrix Mh^{-1} has the same last two columns as M, and therefore $\Lambda_o Mh^{-1}$ has as its last two columns the image of the last two columns under Λ_o.

Furthermore, the transformation of the last two columns of the matrix M must determine the transformation laws of the preferred fields uniquely (because to any transformation of the preferred fields which transforms the last two columns of M properly we can always find a matrix h satisfying relation (19), but we also know that such a transformation of the preferred fields (satisfying relation (19)) must be unique).

Second we assume that the quantities x_μ transform like components of the space-time vector; and using this trial transformation law of x_μ, we calculate the transformation laws for ϕ_μ and σ from equation (19) as we know from the last paragraph. If this problem has a solution, it must be the very unique solution of the whole task, and then we know that the assumption is the correct one.

Thus, asusming the transformation properties of x_μ as known, we have reduced our problem to calculating only ϕ_μ' and σ' by using just the last two columns of the matrix equation (20) (which are equal to the last two columns of equation (19)).

In the calculation that follows, it is more convenient (while completely equivalent) to use the difference and the sum of the last two columns of M instead of using the last two columns themselves. The difference and the sum are

$$B = \begin{pmatrix} 2(\phi_\mu + x_\mu\phi^2)e^\sigma \\ \\ (1 + 2(x\cdot\phi) + x^2\phi^2 + \phi^2)e^\sigma \\ \\ -(1 + 2(x\cdot\phi) + x^2\phi^2 - \phi^2)e^\sigma \end{pmatrix} \text{ and } S = \begin{pmatrix} 2x_\mu e^{-\sigma} \\ \\ (1 + x^2)e^{-\sigma} \\ \\ (1 - x^2)e^{-\sigma} \end{pmatrix} \qquad (22)$$

respectively, as can be seen in Appendix 5c).

We shall consider the transformation of the preferred fields in three separate cases:

I. under the Poincaré subgroup,

II. under the special conformal subgroup,

III. under the dilatatation subgroup.

I. Here we consider transformations under the Poincaré subgroup. Let

$$\Lambda_o = e^{i\alpha_\mu P^\mu}\Lambda ,$$

where Λ is a homogeneous Lorentz transformation.

According to the previous notes we let x_μ transform like the space-time vector and B_A and S_A as six-vectors. This implies that $\frac{B_\mu}{B_5+B_6}$ and $\frac{S_\mu}{S_5+S_6}$ also transform like space-time vectors while B_5+B_6 and S_5+S_6 are conserved (check with equation 13a). All we have to show now is the consistency of these transformations. We can reformulate them once more by saying that

$$B_\mu - x_\mu(B_5+B_6) \equiv 2\phi_\mu e^\sigma \quad \text{and} \quad S_\mu - x_\mu(S_5+S_6) \equiv 0$$

transform like four-vectors while

$$B_5+B_6 \equiv \phi^2 e^\sigma \quad \text{and} \quad S_5+S_6 \equiv e^{-\sigma}$$

transform like scalars under an element of the Poincaré subgroup.

This can be satisfied easily by letting ϕ_μ transform as a four-vector and σ as a scalar.

We summarize these transformation properties in the following:

$$\phi'_\mu = \Lambda_\mu{}^\nu \phi_\nu, \ \sigma' = \sigma \quad \text{and} \quad x'_\mu = \Lambda_\mu{}^\nu x_\nu + \alpha_\mu, \tag{23}$$

where $\Lambda_\mu{}^\nu$ is the space-time part of the matrix Λ.

Last we calculate

$$h = (M')^{-1}\Lambda_o M = e^{i\sigma D}\, e^{-i\Lambda_\mu{}^\nu \phi_\nu K^\mu}\, e^{-i\Lambda_\mu{}^\nu x_\nu P^\mu}\, e^{-i\alpha_\mu P^\mu}\, e^{i\alpha_\mu P^\mu}\Lambda \cdot$$

$$\cdot e^{ix_\mu P^\mu}\, e^{i\phi_\mu K^\mu}\, e^{-i\sigma D} = \Lambda,$$

where we used

$$\Lambda e^{ix_\mu P^\mu} = e^{i\Lambda_\mu{}^\nu x_\nu P^\mu}\Lambda, \ \Lambda e^{i\phi_\mu K^\mu} = e^{i\Lambda_\mu{}^\nu \phi_\nu K^\mu}\Lambda \ \text{and} \ \Lambda e^{-i\sigma D} = e^{-i\sigma D}\Lambda.$$

This calculation can be repeated step by step even if ϕ_μ and σ are functions of x_μ. We only have to modify (23) to

$$\phi'_\mu(x') = \Lambda_\mu{}^\nu \phi_\nu(x), \quad \sigma'(x') = \sigma(x) \quad \text{and} \quad x'_\mu = \Lambda_\mu{}^\nu x_\nu + \alpha_\mu. \tag{24}$$

We notice that the transformations are linear if $\Lambda_o \varepsilon H$ as we would have expected.

II. Now, let $\Lambda_o = e^{i\beta_\mu K^\mu}$ be a special conformal transformation. We want

$$\frac{B_\mu}{B_5 - B_6}, \quad \frac{S_\mu}{S_5 - S_6} \quad \text{and} \quad \frac{x_\mu}{x^2}$$

to transform in the fashion of equation (1) (the requirement $B^2 = S^2 = 0$ is satisfied) and $B_5 - B_6$ and $S_5 - S_6$ to be conserved under such a transformation (check with equation 13b). This implies that the following quantities are conserved:

$$B_\mu - \frac{x_\mu}{x^2}(B_5 - B_6) \equiv \phi_\mu e^\sigma - \frac{x_\mu}{x^2}(1 + 2(x\cdot\phi))e^\sigma$$

$$B_5 - B_6 \equiv (1 + 2(x\cdot\phi) + x^2\phi^2)e^\sigma$$

$$S_\mu - \frac{x_\mu}{x^2}(S_5 - S_6) \equiv 0$$

and $\quad S_5 - S_6 \equiv x^2 e^{-\sigma}$

These conditions are consistent with each other (the second expression divided by the first one squared gives us the last one). They can be satisfied by the following transformations:

$$x'_\mu = \frac{x_\mu + x^2\beta_\mu}{1 + 2(x\cdot\beta) + x^2\beta^2}, \tag{25a}$$

which is equivalent to equation (1),

$$\sigma' = \sigma - \ln(1 + 2(x\cdot\beta) + x^2\beta^2), \tag{25b}$$

which follows from the last condition, and

$$\phi'_\mu = (1 + 2(x\cdot\beta) + \beta^2 x^2)\phi_\mu + ((1 + 2(x\cdot\phi))(1 + 2(x\cdot\beta)) -$$

$$- 2x^2(\beta\cdot\phi))\beta_\mu - (2(\beta\cdot\phi) + \beta^2(1 + 2(x\cdot\phi)))x_\mu,$$

which can be checked after some tedious calculations.

For the matrix h we obtain (up to the first order in β)

$$h = \mathbb{1} + 2ix^\mu\beta^\nu J_{\mu\nu} + \ldots$$

as is shown in Appendix 5d).

III. If $\Lambda_o = e^{i\lambda D}$ is a dilatation we must require that $\beta_\mu \equiv (\phi_\mu + x_\mu\phi^2)e^\sigma$ and $S_\mu \equiv x_\mu e^{-\sigma}$ are conserved under such a transformation while the quantities $B_5+B_6 \equiv \phi^2 e^\sigma$ and $S_5+S_6 \equiv e^{-\sigma}$ are multiplied by e^σ (check with equation 13c). This can be satisfied simultaneously by

$$x'_\mu = x_\mu e^{-\lambda}, \quad \phi'_\mu = \phi_\mu e^\lambda \quad \text{and} \quad \sigma' = \sigma - \lambda$$

as can be seen easily. These relations imply that

$$h = (M')^{-1}\Lambda_o M = e^{i(\sigma-\lambda)D}\, e^{-i\phi_\mu K^\mu e^\lambda}\, e^{-ix_\mu P^\mu e^{-\lambda}}\, e^{i\lambda D}\, e^{ix_\mu P^\mu}\, \cdot$$

$$\cdot\, e^{i\phi_\mu K^\mu}\, e^{-i\sigma D} = \mathbb{1}$$

because $e^{i\lambda D}\, e^{ix_\mu P^\mu} = e^{ix_\mu P^\mu e^{\lambda}}\, e^{i\lambda D}$ and $e^{i\lambda D}\, e^{i\phi_\mu K^\mu} = e^{i\phi_\mu K^\mu e^{-\lambda}}\, e^{i\lambda D}$

as follows by interpreting these expressions as elements of the conformal group acting on the space-time vector.

c) Transformation of Other Fields

Let us assume we have a set of fields $\tilde{\Psi}$ which transform linearly under the conformal group

$$\tilde{\Psi}'(\eta') = R(\Lambda)\tilde{\Psi}(\eta), \tag{26}$$

where Λ is an element of the conformal group

and R is a matrix representing it.

In order to turn $\tilde{\Psi}$ into a set of fields which transform in the standard nonlinear fashion, we just redefine it as follows:

$$\Psi(\eta) = R(e^{i\sigma D}\, e^{-i\phi_\mu K^\mu}\, e^{-ix_\mu P^\mu})\tilde{\Psi}(\eta) = R(M^{-1})\tilde{\Psi}(\eta) \tag{27}$$

transforming $\Psi(\eta)$ by Λ gives (as already shown in Chapter II)

$$\Psi'(\eta') = R(h)R(M^{-1})R(\Lambda^{-1})R(\Lambda)\tilde{\Psi}(\eta) = R(h)\Psi(\eta), \tag{28}$$

which follows from equations (20) and (26). Here $h = h(\Lambda,M)$ is the element of the homogeneous Lorentz subgroup which has been calculated in the previous section.

If $\Psi(\eta)$ is a homogeneous function of η of degree ℓ (i.e., $\Psi(x) = \Psi(\frac{\eta}{k}) = k^{-\ell}\Psi(\eta)$, we can interpret equation (28) as the transformation law of fields defined on space-time. We shall work this out explicitly for transformations of the form $\Lambda = \delta_A{}^B + \varepsilon_A{}^B$, where $\varepsilon_A{}^B$ is a set of infinitesimally small numbers.

In that case we can write equation (28) as

$$\Psi'(\eta') = (\mathbb{1} + \frac{i}{2}\,\omega_{\mu\nu}R(J^{\mu\nu}))\Psi(\eta), \tag{29}$$

where $\omega_{\mu\nu}$ is a set of numbers corresponding to $h_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu$.

Now we can perform the following manipulations:

$$\Psi'(x') = \Psi'(\frac{\eta'}{k'}) = (k')^{-\ell}\Psi'(\eta') = (\frac{k'}{k})^{-\ell}(\mathbb{1} + \frac{i}{2}\,\omega_{\mu\nu}R(J^{\mu\nu}))\Psi(x) =$$

$$= (\mathbb{1} - \ell\,\frac{\delta k}{k}\mathbb{1} + \frac{i}{2}\,\omega_{\mu\nu}R(J^{\mu\nu}))\Psi(x), \tag{30}$$

where δk is an infinitesimal increase of k.

This finally yields (since $\Psi'(x') = \Psi'(x)+\delta x_\mu\partial^\mu\Psi'(x) =$ $= \Psi'(x)+\delta x_\mu\partial^\mu\Psi(x)$ up to the first order in infinitesimals),

$$\delta\Psi(x) = \Psi'(x) - \Psi(x) = \Psi'(x') - \delta x_\mu\partial^\mu\Psi(x) - \Psi(x) =$$

$$= -(\delta x_\mu\partial^\mu + \ell\,\frac{\delta k}{k} - \frac{i}{2}\,\omega_{\mu\nu}R(J^{\mu\nu}))\Psi(x) \tag{31}$$

Having a quick look into the previous section gives us immediately

I. for the Poincaré subgroup, where $\delta k = 0$, $\delta x_\mu = \varepsilon_\mu{}^\nu x_\nu + \alpha_\mu$ and

$h_\mu{}^\nu = \delta_\mu{}^\nu + \varepsilon_\mu{}^\nu$,

$$\delta\Psi(x) = -\,(\varepsilon_\mu{}^\nu x_\nu + \alpha_\mu)\partial^\mu\Psi(x) + \frac{i}{2}\,\varepsilon_{\mu\nu}R(J^{\mu\nu})\Psi(x). \tag{32}$$

II. for the special conformal subgroup, where $\delta k = 2(\beta\cdot\eta)$ from equation (13b), $\delta x_\mu = x^2\beta_\mu - 2x_\mu(x\cdot\beta)$ and $h = \mathbb{1} + \frac{i}{2}x^\mu\beta^\nu J_{\mu\nu}$,

$$\delta\Psi(x) = (2(x\cdot\beta)x_\mu - x^2\beta_\mu)\partial^\mu\Psi(x) - 2\ell(\beta\cdot x)\Psi(x) + 2ix^\mu\beta^\nu R(J_{\mu\nu})\Psi(x) \tag{33}$$

III. and for the dilatation subgroup, where $\delta k = \lambda k$ (from equation 13c), $\delta x_\mu = -\lambda x_\mu$ and $h_\mu{}^\nu = \delta_\mu{}^\nu$,

$$\delta\Psi(x) = \lambda(x_\mu\partial^\mu - 1)\Psi(x). \tag{34}$$

These formulas are handy when calculating currents from a Lagrangian density L for the preferred and other fields.

The general expression for a current associated with a symmetry transformation of a Lagrangian according to Gell-Mann and Le'vy is given by

$$j_\mu(x) = \frac{\partial L}{\partial(\partial^\mu\Psi)}\,\delta\Psi + \frac{\partial L}{\partial(\partial^\mu\phi_\lambda)}\,\delta\phi_\lambda + \frac{\partial L}{(\partial\,\partial^\mu\sigma)}\,\partial\sigma + \delta x_\mu L. \tag{35}$$

For inhomogeneous Lorentz transformations the currents obtained from expression (35) are the energy-momentum and angular momentum and boost tensors.

Special conformal transformations and dilatations give us new currents, namely

$$j_\mu{}^{(K^\nu)} = \frac{\partial L}{\partial(\partial^\mu\Psi)}(2x_\nu(x\cdot\partial) - x^2\partial_\nu - 2\ell x_\nu + 2ix^\alpha R(J_{\alpha\nu}))\Psi(x) +$$

$$+ \frac{\partial L}{\partial(\partial^\mu\phi\lambda)}((2x_\nu(x\cdot\partial) - x^2\partial_\nu + 2x_\nu)\phi_\lambda - 2x_\lambda\phi_\nu + g_{\lambda\nu} + 2(x\cdot\phi)g_{\lambda\nu}) +$$

$$\frac{\partial L}{\partial(\partial^\mu\sigma)}\;((2x_\nu(x\cdot\partial) - x^2\partial_\nu)\sigma - 2x_\nu) + (x^2 g_{\mu\nu} - 2x_\mu x_\nu)L$$

and

$$j_\mu{}^{(D)} = \frac{\partial L}{\partial(\partial^\mu\Psi)}((x\cdot\partial) - \ell)\Psi(x) + \frac{\partial L}{\partial(\partial^\mu\phi_\lambda)}((x\cdot\partial)+1)\phi_\lambda + \frac{\partial L}{\partial(\partial^\mu\sigma)}\,\cdot$$

$$\cdot\;((x\cdot\partial)\sigma - 1) - x_\mu L \text{ respectively.}$$

d) Covariant Derivatives

To construct covariant derivaties of the preferred fields and other fields we start with the adjoint representation of the conformal group. It is the representation according to which the operator ∂_{AB} defined in Section a) is transformed (Ref. 20). If we apply the operator ∂_{AB} to arbitrary linearly transformed fields $\tilde{\Psi}$ (see equation 26) we get an expression with complicated transformation properties.

Using the general ideas of Chapter II we can define a new operator by

$$D_{AB}\tilde{\Psi} = (M^{-1})_A{}^C (M^{-1})_B{}^D R(M^{-1}) \partial_{CD}\tilde{\Psi}, \tag{36}$$

where the matrices R are defined by equation (26) and the matrix M by equation (21).

That $D_{AB}\tilde{\Psi}$ indeed has the standard transformation properties (for each of its indices separately) under an element Λ can be shown in the following way:

$$(D_{AB}\tilde{\Psi})' = (hM^{-1}\Lambda^{-1})_A{}^C (hM^{-1}\Lambda^{-1})_B{}^D R(hM^{-1}\Lambda^{-1}) \Lambda_C{}^E \Lambda_D{}^F \partial_{EF}(R(\Lambda)\Psi) =$$

$$= h_A{}^C h_B{}^D R(h) D_{CD}\tilde{\Psi}, \tag{37}$$

where we have used equations (20) and (21), the known transformation properties of ∂_{AB} (according to the adjoint representation) and the assumption that the parameters of Λ are space-time independent.

Equation (36) can be written in a more explicit form by noticing that (Ref. 20)

$$\partial_{AB} = \frac{i}{2}(e^{ix_\mu P^\mu})_A{}^C (e^{ix_\mu P^\mu})_B{}^D (K_\mu \partial^\mu + 2\ell D)_{CD}. \tag{38}$$

This is proved in Appendix 5e).

With the help of this relation we obtain

$$D_{AB}\tilde{\Psi} = \frac{i}{2} R(M^{-1})(M^{-1}e^{ix_\mu P^\mu})_A{}^C (M^{-1}e^{ix_\mu P^\mu})_B{}^D (K_\mu \partial^\mu + 2\ell D)_{CD} \cdot \tag{39}$$

$$\cdot \tilde{\Psi} = \frac{i}{2} R(M^{-1})(e^{i\sigma D}e^{i\phi_\mu K^\mu})_A{}^C (e^{i\phi_\mu K^\mu}e^{-i\sigma D})_{DB} \cdot$$

$$\cdot (K_\mu \partial^\mu + 2\ell D)_C{}^D \tilde{\Psi} = \frac{i}{2} R(M^{-1})(K_\mu e^\sigma(\partial_\mu + 2\ell\phi_\mu) + 2\ell D)_{AB}\tilde{\Psi},$$

where we have used the general relation $\Lambda_B{}^A = (\Lambda^{-1})^A{}_B$, (valid for any element of O(4,2)), equation (21) and the formulas $e^{i\sigma D}K_\mu e^{-i\sigma D} = K_\mu e^\sigma$ and $e^{-i\phi_\mu K^\mu} D e^{i\phi_\mu K_\mu} = D + \phi_\mu K^\mu$.

The expression $(K_\mu e^\sigma(\partial_\mu + 2\ell\phi_\mu) + 2\ell D)_{AB}$ can be evaluated using the explicit matrices of Appendix 5a) with the final result for $D_{AB}\tilde{\Psi}$

$$\begin{aligned} &D_{\mu\nu}\tilde{\Psi} = 0 \\ &(D_{5\mu} + D_{6\mu})\tilde{\Psi} = R(M^{-1})e^\sigma(\partial_\mu + 2\ell\phi_\mu)\tilde{\Psi} \\ &(D_{5\mu} - D_{6\mu})\tilde{\Psi} = 0 \\ &D_{56}\tilde{\Psi} = \ell\tilde{\Psi}. \end{aligned} \tag{40}$$

The covariant derivaties we are looking for must be contained in the expression for $(D_{5\mu} + D_{6\mu})\tilde{\Psi}$, which transforms (using equation 37) according to

$$(D_{5\mu} + D_{6\mu})\tilde{\Psi} \longrightarrow h_\mu{}^\nu R(h)(D_{5\mu} + D_{6\mu})\tilde{\Psi} \; . \tag{41}$$

The expression for $(D_{5\mu} + D_{6\mu})\tilde{\Psi}$ can be rewritten using the standard fields Ψ (defined by equation 28) as

$$(D_{5\mu} + D_{6\mu})\tilde{\Psi} = R(M^{-1})e^{\sigma}(\partial_{\mu} + 2\ell\phi_{\mu})R(M)\Psi = e^{\sigma}(\partial_{\mu} + 2\ell\phi_{\mu})\Psi + \\ + R(M^{-1})e^{\sigma}R(\partial_{\mu}M)\Psi. \tag{42}$$

The matrix $R(M^{-1})R(\partial_{\mu}M)$ can be expressed as a linear combination of the generators of the conformal group (in the corresponding representation) and then separated into two parts, the first part being a linear combination of the generators of the subgroup H only and the second part being a linear combination of the remaining generators.

We know from the general discussion of Chapter II that the second part (multiplied by e^{σ}) must be covaraint on its own and that it gives us the covariant derivatives of the preferred fields, while the first part (multiplied by e^{σ}) must be added to the term $e^{\sigma}(\partial_{\mu} + 2\ell\phi_{\mu})\Psi$ (of expression (42)) to complete the covariant derivative of Ψ.

It is easy to separate the matrix $R(M^{-1})R(\partial_{\mu}M)$ into these two parts in the self representation. In this representation we can separate the part corresponding to H from the rest simply by separating the space-time part of the matrix $M^{-1}\partial_{\mu}M$ from the last two columns of $M^{-1}\partial_{\mu}M$. This yields (after multiplying by e^{σ})

$$(M^{-1}e^{\sigma}\partial_{\mu}M)_{\lambda\rho} = 2(g_{\mu\lambda}\phi_{\rho} - g_{\mu\rho}\phi_{\lambda}) = -2ie^{\sigma}\phi^{\nu}(J_{\mu\nu})_{\lambda\rho},$$

$$M^{-1}e^{\sigma}\partial_{\mu}B = \begin{pmatrix} 2e^{2\sigma}(\partial_{\mu}\phi_{\nu} + g_{\mu\nu}\phi^{2} - 2\phi_{\mu}\phi_{\nu}) \\ e^{\sigma}(\partial_{\mu}\sigma + 2\phi_{\mu}) \\ -e^{\sigma}(\partial_{\mu}\sigma + 2\phi_{\mu}) \end{pmatrix} \tag{43}$$

$$M^{-1}e^{\sigma}\partial_\mu S = \begin{pmatrix} 2g_{\mu\nu} \\ \\ -e^{\sigma}(\partial_\mu\sigma + 2\phi_\mu) \\ \\ -e^{\sigma}(\partial_\mu\sigma + 2\phi_\mu) \end{pmatrix}.$$

Details are provided in Appendix 5f). (B and S were defined in 22).

The final expressions for the covariant derivatives are then

$$D_\mu\sigma = e^{\sigma}(\partial_\mu\sigma + 2\phi_\mu),\ D_\mu\phi_\nu = e^{2\sigma}(\partial_\mu\phi_\nu + g_{\mu\nu}\phi^2 - 2\phi_\mu\phi_\nu),$$

and $D_\mu\Psi = e^{\sigma}(\partial_\mu + 2\ell\phi_\mu - 2i\phi^\nu R(J_{\mu\nu}))\Psi.$ (44)

These have (by construction) the desired transformation properties

$$D_\mu\sigma \longrightarrow h_\mu{}^\nu D_\nu\sigma\ , \qquad D_\mu\phi_\lambda \longrightarrow h_\mu{}^\nu h_\lambda{}^\rho D_\nu\phi_\rho$$

and $D_\mu\Psi \longrightarrow h_\mu{}^\nu R(h) D_\mu\Psi.$ (45)

If we want to construct Lagrangians which in addition to being conformally invariant are also invariant under the usual gauge transformations of the second kind, the covariant derivatives will have to be further modified. How this is to be done must be left for a future investigation. This is perhaps an interesting problem because both, the gauge transformations of the second kind and the special conformal transformations, are x^μ-dependent transformations, so that there may be an interplay between the preferred fields ϕ_μ and the vector field v_μ which must be introduced to achieve gauge invariance.

APPENDIX 3a

We would like to evaluate

$$e^{-A\cdot\xi}\,\frac{\partial e^{A\cdot\xi}}{\partial\xi^i} = e^{-A\cdot\xi}\Bigg(A_i + \frac{A_i(A\cdot\xi)+(A\cdot\xi)A_i}{2!} + $$
$$+ \frac{A_i(A\cdot\xi)^2 + (A\cdot\xi)A_i(A\cdot\xi) + (A\cdot\xi)^2A_i}{3!} + \ldots\Bigg) \qquad (1)$$

Each fraction in the parentheses can be written as a sum of terms with a decreasing number of commutators and a corresponding numerical coefficient equal to

$$\frac{1}{(k+1)!(n-k-1)!}\,,$$

where $n!$ is the denomination of the fraction and $k<n$ is the number of commutators in that term. To give an example we consider the fourth fraction.

$$\frac{1}{4!}\,(A_i(A\cdot\xi)^3 + (A\cdot\xi)A_i(A\cdot\xi)^2 + (A\cdot\xi)^2A_i(A\cdot\xi) + (A\cdot\xi)^3A_i) =$$

$$= \frac{1}{4!}\,([A_i,(A\cdot\xi)](A\cdot\xi)^2 + 2(A\cdot\xi)[A_i,(A\cdot\xi)](A\cdot\xi) + 3(A\cdot\xi)^2[A_i,(A\cdot\xi)] +$$

$$+ (A\cdot\xi)^3A_i) = \frac{1}{4!}\,([[A_i,(A\cdot\xi)],(A\cdot\xi)](A\cdot\xi) + 3(A\cdot\xi)[[A_i,(A\cdot\xi)],(A\cdot\xi)] +$$

$$+ 6(A\cdot\xi)^2[A_i,(A\cdot\xi)] + 4(A\cdot\xi)^3A_i) = \frac{1}{4!}\,([[[A_i,(A\cdot\xi)],(A\cdot\xi)],(A\cdot\xi)] +$$

$$+ 4(A\cdot\xi)[[A_i,(A\cdot\xi)](A\cdot\xi)] + 6(A\cdot\xi)^2[A_i,(A\cdot\xi)] + 4(A\cdot\xi)^3A_i) =$$

$$= \frac{1}{4!}\,[[[A_i,(A\cdot\xi)],(A\cdot\xi)],(A\cdot\xi)] + \frac{1}{3!}\,(A\cdot\xi)[[A_i,(A\cdot\xi)],(A\cdot\xi)] +$$

$$+ \frac{1}{2!2!} (A\cdot\xi)^2[A_i,(A\cdot\xi)] + \frac{1}{3!} (A\cdot\xi)^3 A_i .$$

When all terms on the right hand side of equation (1) have been expanded as in this example, we collect all terms with the same number k of commutators and sum them up. The resulting coefficients of such multicommutator terms are clearly

$$\frac{e^{A\cdot\xi}}{(k+1)!}$$

which yields for the explicit result

$$e^{-A\cdot\xi} \frac{\partial e^{A\cdot\xi}}{\partial\xi^i} = A_i + \frac{[A_i,(A\cdot\xi)]}{2!} + \frac{[[A_i,(A\cdot\xi)],(A\cdot\xi)]}{3!} +$$

$$+ \frac{[[[A_i,(A\cdot\xi)],(A\cdot\xi)],(A\cdot\xi)]}{4!} + \ldots] = A_i - \frac{\xi^k f_{ki}{}^j V_j}{2!} +$$

$$+ \frac{\xi^k f_{ki}{}^j \xi^l f_{lj}{}^m A_m}{3!} - \frac{\xi^k f_{ki}{}^j \xi^l f_{lj}{}^m \xi^n f_{nm}{}^r A_r}{4!} + \ldots =$$

$$= (\delta_i{}^j + \frac{x_i{}^k x_k{}^j}{3!} + \frac{x_i{}^k x_k{}^l x_l{}^m x_m{}^j}{5!} + \ldots)A_j - \frac{x_i{}^j}{2!} + \frac{x_i{}^l x_l{}^k x_k{}^j}{4!} + \ldots)V_j =$$

$$= \sigma_i{}^j A_j - \rho_i{}^j V_j ,$$

where $f_{ij}{}^k$ are the structure coefficients,

$x_i{}^j = \xi^k f_{ki}{}^j$ and where the last step is just a definition of the matrices σ and ρ.

APPENDIX 3b

We can show

$$e^{-A\cdot\xi}V_i e^{A\cdot\xi} = V_i - \xi^k[A_k,V_i] + \frac{\xi^k\xi^l[A_l,[A_k,V_i]]}{2!} -$$

$$- \frac{\xi^k\xi^l\xi^m[A_m,[A_l,[A_k,V_i]]]}{3!} + \ldots = V_i - \xi^k f_{ki}{}^j A_j +$$

$$+ \frac{\xi^k f_{ki}{}^j \xi^l f_{lj}{}^m A_m}{2!} - \frac{\xi^k f_{ki}{}^j \xi^l f_{lj}{}^r \xi^m f_{mr}{}^s A_s}{3!} + \ldots =$$

$$= (\delta_i{}^j + \frac{x_i{}^k x_k{}^j}{2!} + \frac{x_i{}^k x_k{}^l x_l{}^m x_m{}^j}{4!} + \ldots)V_j - (x_i{}^j +$$

$$+ (x_i{}^j + \frac{x_i{}^k x_k{}^l x_l{}^j}{3!} + \ldots)A_j$$

and similarly (by exchanging A_i and V_i in the result) we obtain

$$e^{-A\cdot\xi}A_i e^{A\cdot\xi} = A_i - \xi^k[A_k,A_i] + \frac{\xi^k\xi^l[A_l,[A_k,A_i]]}{3!} + \ldots =$$

$$= (\delta_i{}^j + \frac{x_i{}^k x_k{}^j}{2!} + \ldots)A_j - (x_i{}^j + \frac{x_i{}^k x_k{}^l x_l{}^j}{3!} + \ldots)V_j,$$

where $x_i{}^j = \xi^k f_{ki}{}^j$.

APPENDIX 3c

We calculate

$$(\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu))(\partial_\mu + f(V\cdot v_\nu + A\cdot a_\nu)) - (\partial_\nu + f(V\cdot v_\nu + A\cdot a_\nu))\cdot$$

$$\cdot(\partial_\mu + f(V\cdot v_\mu + A\cdot a_\mu)) = f(V\cdot\partial_\mu v_\nu) + f(A\cdot\partial_\mu a_\nu) - f(V\cdot\partial_\nu v_\mu) -$$

$$- f(A\cdot\partial_\nu a_\mu) + f^2 v_\mu{}^i v_\nu{}^j[V_i,V_j] + f^2 v_\mu{}^i a_\nu{}^j[V_i,A_j] + f^2 a_\mu{}^i a_\nu{}^i[A_i,A_j] +$$

$$+ f^2 a_\mu{}^i v_\nu{}^j[A_i,V_j] = fV_i(\partial_\mu v_\nu{}^i - \partial_\nu v_\mu{}^i + ff_{kj}{}^i(v_\mu{}^k v_\nu{}^j + a_\mu{}^k a_\nu{}^j)) +$$

$$+ fA_i(\partial_\mu a_\nu{}^i - \partial_\nu a_\mu{}^i + ff_{kj}{}^i(v_\mu{}^k a_\nu{}^j + a_\mu{}^k v_\nu{}^j))$$

which is equal to

$$fV_i F_{\mu\nu}{}^i + fA_i \overline{F}_{\mu\nu}{}^i$$

by a definition of the tensors $F_{\mu\nu}$ and $\overline{F}_{\mu\nu}$.

APPENDIX 3d

Proving the equivalence of formulas (21) and (22) of Chapter III requires a rather lengthy calculation and we can give just a few steps to guide an interested reader. The covariant derivaties of the fields $\overset{\circ}{v}$ and $\overset{\circ}{a}$ are

$$D_\mu \overset{\circ}{v}_\nu{}^i = \partial_\mu \overset{\circ}{v}_\nu{}^i - f_{kj}{}^i(\rho_l{}^k \partial_\mu \xi^l - f\overset{\circ}{v}_\mu{}^k)\overset{\circ}{v}_\nu{}^j$$

and $$D_\mu \overset{\circ}{a}_\nu{}^i = \partial_\mu \overset{\circ}{a}_\mu{}^i - f_{kj}{}^i(\rho_l{}^k \partial_\mu \xi^l - f\overset{\circ}{v}_\mu{}^k)\overset{\circ}{a}_\nu{}^j$$

respectively.

These expressions follow from equations (17) and (18b) of Chapter III, where we use

$$R^i{}_j(V_k) = f_{kj}{}^i,$$

(the adjoint representation of the subgroup H).

For the covariant derivatives of the preferred fields we have (using equation (18a) and the definition of $\overset{\circ}{a}_\mu$)

$$D_\mu \xi^l = \sigma_i{}^l \partial_\mu \xi^i + f\overset{\circ}{a}_\mu{}^l.$$

We insert these derivatives into equations (21) and then replace $\overset{\circ}{v}_\mu{}^j$ by $(v_\mu{}^i \cosh x_i{}^j - a_\mu{}^i \sinh x_i{}^j)$ and $\overset{\circ}{a}_\mu{}^j$ by $(a_\mu{}^i \cosh x_i{}^j - v_\mu{}^i \sinh x_i{}^j)$. The biggest difficulty is in calculating the following terms:

$$\partial_\mu(\sinh x_i{}^j) \quad \text{and} \quad \partial_\mu(\cosh x_i{}^j).$$

They can be evaluated by a method of Appendix 3a). We shall quote the final result only

$$\partial_\mu(\sinh x_i{}^j) = f_{lk}{}^j(\cosh(x_i{}^k)\sigma_m{}^l\partial_\mu\xi^m - \sinh(x_i{}^k)\rho_m{}^l\partial_\mu\xi^m)$$

and $$\partial_\mu(\cosh x_i{}^j) = f_{lk}{}^j(\sinh(x_i{}^k)\sigma_m{}^l\partial_\mu\xi^m - \cosh(x_i{}^k)\rho_m{}^l\partial_\mu\xi^m),$$

with the matrices σ and ρ defined in Appendix 3a).

Collecting all terms in the expression (22a), we obtain finally

$$G_{\mu\nu}{}^j = (\partial_\mu v_\mu{}^i - \partial_\nu v_\mu{}^i + f_{jk}{}^i(v_\mu{}^j v_\gamma{}^k + a_\mu{}^j a_\nu{}^k)) \cosh x_i{}^j -$$

$$- (\partial_\mu a_\nu{}^i - \partial_\nu a_\mu{}^i + ff_{jk}{}^i(a_\mu{}^j v_\nu{}^k + v_\mu{}^j a_\nu{}^k) \sinh x_i{}^j =$$

$$= F_{\mu\nu}{}^i \cosh x_i{}^j - \overline{F}_{\mu\nu}{}^i \sinh x_i{}^j,$$

which is equation (21a).

$\overline{G}_{\mu\nu}$ can be evaluated similarly.

APPENDIX 4a

To prove equation (9) of Chapter IV we apply the operator A_c to equation (8). We get:

$$A_c A_a \xi_b = (-i)(\delta_{ab} f' 2\xi_d + \delta_{ad}\xi_b g + \xi_a \delta_{bd} g + \xi_a \xi_b g' 2\xi_d)\cdot$$

$$\cdot(-i)(\delta_{cd} f + \xi_c \xi_d g) = -(2ff'\xi_c \delta_{ab} + \delta_{ac}\,{}_b fg + \delta_{bc}\,{}_a fg + 2\xi_a \xi_b \xi_c\cdot$$

$$\cdot fg' + 2\delta_{ab}\xi_c \xi^2 gf' + 2\xi_a \xi_b \xi_c g^2 + 2\xi_a \xi_b \xi_c \xi^2 gg')$$

as well as $A_a A_c \xi_b$ simply by exchanging the indices a and c. Now we can calculate

$$[A_c, A_a]\xi_b = -(2ff'(\xi_c \delta_{ab} - \xi_a \delta_{cb}) + fg(\xi_a \delta_{cb} - \xi_c \delta_{ab}) +$$

$$+ 2\xi^2 gf'(\xi_c \delta_{ab} - \xi_a \delta_{cb})),$$

which should be equal to

$$i\varepsilon_{cad} V_d \xi_b = \xi_c \delta_{ab} - \xi_a \delta_{cb}.$$

Comparing both results gives us

$$2ff' + 2f'g\xi^2 - fg + 1 = 0$$

which is equation (9).

APPENDIX 4b

To calculate the function v of equation (24) we use a similar method. First we calculate

$$A_aA_b\Psi = A_a(v\varepsilon_{bde}\xi_e t_d\Psi) = (2v'\xi_c\varepsilon_{bde}\xi_e t_d\Psi + v\varepsilon_{bde}\delta_{ec}t_d\Psi)\cdot$$

$$\cdot(-i)(\delta_{ac}f + \xi_a\xi_c g) + v\varepsilon_{bde}\xi_e t_d v\varepsilon_{akm}\xi_m t_k\Psi = (-i)(2v'f\xi_a\cdot$$

$$\cdot\varepsilon_{bde}\xi_e t_d\Psi + 2v'g\xi^2\xi_a\varepsilon_{bde}\xi_e t_d\Psi + vf\varepsilon_{bda}t_d\Psi + vg\xi_a\cdot$$

$$\cdot\varepsilon_{bde}t_d\Psi) + v^2\varepsilon_{bde}\xi_e t_d\varepsilon_{akm}\xi_m t_k\Psi = (-i)(2v'f + 2v'g\xi^2 +$$

$$+ vg)\xi_a\varepsilon_{bde}\xi_e t_d\Psi + (-i)vf\varepsilon_{abd}t_d\Psi + v^2\varepsilon_{bde}\xi_e t_d\varepsilon_{akm}\xi_m t_k\Psi.$$

We find $A_bA_a\Psi$ by exchanging indices a and b, and combining these two expressions gives us

$$[A_a,A_b]\Psi = (-i)(2v'f + 2v'g\xi^2 + vg)(\xi_a\varepsilon_{bde} - \xi_b\varepsilon_{ade})\xi_e t_d\Psi +$$

$$+ (-i)2vf\varepsilon_{abd}t_d\Psi + iv^2\varepsilon_{bde}\xi_e\varepsilon_{akm}\xi_m\varepsilon_{dkn}t_n\Psi.$$

The last term of the right-hand side can be written as

$$-iv^2\varepsilon_{abc}\xi_c(\xi\cdot t)\Psi.$$

$[A_a,A_b]\Psi$ should be equal to $i\varepsilon_{abc}V_c\Psi = -i\varepsilon_{abc}t_c\Psi$. This equality (where for $[A_a,A_b]\Psi$ we use the expression derived above) can be multiplied by $i\varepsilon_{abj}$ and summed over a and b. This yields

$$(2v'f + 2v'g\xi^2 + vg)(\xi^2 t_j - \xi_j(\xi\cdot t))\Psi + 2vft_j\Psi + v^2\xi_j\cdot$$

$$\cdot(\xi\cdot t)\Psi = t_j\Psi.$$

Comparing coefficients of $t_j\Psi$ and $\xi_j(\xi\cdot t)$ gives us

$$(2v'f + 2v'g\xi^2 + vg)\xi^2 + 2vf = 1$$

and $(2v'f + 2v'g\xi^2 + vg) = v^2$

Combining these two equations we get

$$v^2\xi^2 + 2vf - 1 = 0$$

which has the solution

$$v = \frac{1}{f \pm (f^2 + \xi^2)^{\frac{1}{2}}}.$$

This solution can be shown to satisfy both of the original equations as well.

APPENDIX 4c

We want to calculate the matrix

$$\exp\left(\frac{i\lambda}{2(\xi^2)^{\frac{1}{2}}}\ (S\cdot\xi)\right),$$

where S' are the Pauli's matrices.

From the algebra of these matrices we know that

$$(S\cdot\xi)^{2k} = (\xi^2)^k \quad\text{and}\quad (S\cdot\xi)^{2k+1} = (\xi^2)^k(S\cdot\xi),$$

where k is an integer number.

This enables us to calculate

$$\exp\left(\frac{i\lambda}{2(\xi^2)^{\frac{1}{2}}}(S\cdot\xi)\right) = \left(1 + \frac{i\lambda}{2(\lambda^2)^{\frac{1}{2}}}(S\cdot\xi) + \frac{1}{2!}\left(\frac{i\lambda}{2(\xi^2)^{\frac{1}{2}}}\right)^2 \xi^2 + \frac{1}{3!}\left(\frac{i\lambda}{2(\xi^2)^{\frac{1}{2}}}\right)^3\cdot\right.$$

$$\left.\cdot\ \xi^2(S\cdot\xi) + \ldots\right) = \left(1 + \frac{1}{2!}\left(\frac{i\lambda}{2}\right)^2 + \frac{1}{4!}\left(\frac{i\lambda}{2}\right)^4 + \ldots\right) + \frac{1}{(\xi^2)^{\frac{1}{2}}}(S\cdot\xi)\left(\frac{i\lambda}{2} + \right.$$

$$\left. + \frac{1}{3!}\left(\frac{i\lambda}{2}\right)^3 + \frac{1}{5!}\left(\frac{i\lambda}{2}\right)^5 + \ldots\right) = \cos\frac{\lambda}{2} + \frac{i}{(\xi^2)^{\frac{1}{2}}}(S\cdot\xi)\sin\frac{\lambda}{2} = \left(\frac{1-u}{2}\right)^{\frac{1}{2}} -$$

$$- \frac{i}{\sigma(1-u^2)^{\frac{1}{2}}}\left(\frac{1+u}{2}\right)^{\frac{1}{2}}(S\cdot\xi) = \left(\frac{1-u}{2}\right)^{\frac{1}{2}}\left(1 - \frac{i}{\sigma(1-u)}(S\cdot\xi)\right),$$

where λ is defined by equation (31).

APPENDIX 4d

We know from equation (35)

$$M = \frac{1-u}{2}\left(\mathbb{1} - \frac{i}{\sigma(1-u)}(S\cdot\xi)\right)\otimes\left(\mathbb{1} + \frac{i}{\sigma(1-u)}(S\cdot\xi)\right) =$$

$$= \frac{1-u}{2}\begin{pmatrix} 1 - \frac{i\xi_2}{\sigma(1-u)} & -\frac{i(\xi_1-i\xi_2)}{\sigma(1-u)} \\ -\frac{i(\xi_1+i\xi_2}{\sigma(1-u)} & 1 + \frac{i\xi_3}{\sigma(1-u)} \end{pmatrix}\otimes\begin{pmatrix} 1 + \frac{i\xi_3}{\sigma(1-u)} & \frac{i(\xi_1-i\xi_2)}{\sigma(1-u)} \\ \frac{i(\xi_1+i\xi_2)}{\sigma(1-u)} & 1 - \frac{i\xi_3}{\sigma(1-u)} \end{pmatrix}$$

and from the Clebsh-Gordan coefficients for spin = ½

$$|1,0\rangle = \frac{1}{(2)^{1/2}}\left(\binom{1}{0}\otimes\binom{0}{1} - \binom{0}{1}\otimes\binom{1}{0}\right),$$

$$|1,-1\rangle = \binom{0}{1}\otimes\binom{0}{1}, \quad |1,1\rangle = \binom{1}{0}\otimes\binom{1}{0}$$

and $$|1,0\rangle = \frac{1}{(2)^{1/2}}\left(\binom{1}{0}\otimes\binom{0}{1} + \binom{0}{1}\otimes\binom{1}{0}\right).$$

We can write immediately

$$\langle 0,0|M|0,0\rangle = \frac{1-u}{4}\left(1 - \left(\frac{i\xi_3}{\sigma(1-u)}\right)^2 + \left(1 + \frac{i\xi_3}{\sigma(1-u)}\right)^2 - 2\,\frac{\xi_1^2+\xi_2^2}{\sigma^2(1-u)^2}\right) =$$

$$= \frac{1-u}{2}\left(1 - \frac{\xi^2}{\sigma^2(1-u)^2}\right) = \frac{1-u}{2}\left(1 - \frac{\sigma^2(1-u^2)}{\sigma^2(1-u)^2}\right) = \frac{1-u}{2}\left(1 - \frac{1+u}{1-u}\right) = -u,$$

$$<0,0|M|1,1> = \frac{1-u}{2(2)^{\frac{1}{2}}}\left(\left(1 - \frac{i\xi_3}{\sigma(1-u)}\right)(-i)\,\frac{\xi_1+i\xi_2}{\sigma(1-u)} - \left(1 + \frac{i\xi_3}{\sigma(1-u)}\right) i\,\frac{\xi_1+i\xi_2}{\sigma(1-u)}\right) =$$

$$= (-i)\,\frac{\xi^+}{\sigma}\,,$$

$$<0,0|M|1,-1> = \frac{1-u}{2(2)^{\frac{1}{2}}}\left(\left(1 - \frac{i\xi_3}{\sigma(1-u)}\right)(-i)\frac{\xi_1-i\xi_2}{\sigma(1-u)} - \left(1 + \frac{i\xi_3}{\sigma(1-u)}\right) i\,\frac{\xi_1-i\xi_2}{\sigma(1-u)}\right) =$$

$$= (-i)\,\frac{\xi^-}{\sigma}$$

and finally

$$<0,0|M|1,0> = \frac{1-u}{4}\left(\left(1 - \frac{i\xi_3}{\sigma(1-u)}\right)^2 - \left(1 + \frac{i\xi_3}{\sigma(1-u)}\right)^2\right) =$$

$$= (-i)\,\frac{\xi_3}{\sigma}\,.$$

APPENDIX 5a

Now we shall list the generators of the self-representation of the conformal group. They can be derived from formula (10) of Chapter V.

$$(J_{AB})_C{}^D \quad = \quad i(g_{AB}\delta_B{}^D - g_{BC}\delta_A{}^D).$$

$$J_{50} = i\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad J_{60} = i\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$J_{51} = i\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad J_{61} = i\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$J_{52} = i\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad J_{62} = i\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$J_{53} = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad J_{63} = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot \end{pmatrix}$$

$$J_{65} = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{pmatrix}$$

$$J_{10} = i \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad J_{20} = i \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$J_{30} = i \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad J_{21} = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$J_{31} = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad J_{32} = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

APPENDIX 5b

Using the explicit matrices of Appendix 5a we can calculate the following matrices:

$$i(\alpha\cdot P) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & +\alpha_0 & +_{0} \\ \cdot & \cdot & \cdot & \cdot & +\alpha_1 & +\alpha_1 \\ \cdot & \cdot & \cdot & \cdot & +\alpha_2 & +\alpha_2 \\ \cdot & \cdot & \cdot & \cdot & +\alpha_3 & +\alpha_3 \\ +\alpha_0 & -\alpha_0 & -\alpha_2 & -\alpha_3 & \cdot & \cdot \\ -\alpha_0 & +\alpha_1 & \alpha_2 & \alpha_2 & \cdot & \cdot \end{pmatrix},$$

$$i(\beta\cdot K) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & +\beta_0 & -\beta_0 \\ \cdot & \cdot & \cdot & \cdot & +\beta_1 & -\beta_1 \\ \cdot & \cdot & \cdot & \cdot & +\beta_2 & -\beta_2 \\ \cdot & \cdot & \cdot & \cdot & +\beta_3 & -\beta_3 \\ +\beta_0 & -\beta_1 & -\beta_2 & -\beta_3 & \cdot & \cdot \\ +\beta_0 & -\beta_1 & -\beta_2 & -\beta_3 & \cdot & \cdot \end{pmatrix}$$

and

$$i\sigma D = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +\sigma \\ \cdot & \cdot & \cdot & \cdot & +\sigma & \cdot \end{pmatrix},$$

where

$$\alpha \cdot P = \alpha_o P_o - \alpha_1 P_1 - \alpha_2 P_2 - \alpha_3 P_3$$

and where

$$\beta \cdot K = \beta_o K_o - \beta_1 K_1 - \beta_2 K_2 - \beta_3 K_3.$$

From here on it is easy to calculate any power of these matrices (the third power of the first two matrices is already the zero matrix and the third matrix squared consists of two blocks, one is the four-dimensional zero matrix and the other is the unit matrix multiplied by σ^2). It means we can easily calculate any power series expansion consisting of these matrices. Expanding the exponentials of equations (13) of Chapter V gives us the final results.

APPENDIX 5c

We have already calculated the matrices $e^{i(x\cdot P)}$, $e^{i(\phi\cdot K)}$ and $e^{-i\sigma D}$. Then we find their product to be

$$M=e^{i(x\cdot P)}e^{i(\phi\cdot K)}e^{-i\sigma D} = \begin{bmatrix} \partial_\mu{}^\nu+2x_\mu\phi^\nu & \phi_\mu e^\sigma+x_\mu\phi^2e^\sigma+ & -\phi_\mu e^\sigma-x_\mu\phi^2e^\sigma+ \\ & +x_\mu e^{-\sigma} & +x_\mu e^{-\sigma} \\ x^\nu+\phi^\nu(1+x^2) & (x\cdot\phi)e^\sigma+\cosh\sigma+ & -(x\cdot\phi)e^\sigma-\sinh\sigma- \\ & +\frac{e^\sigma}{2}\phi^2(1+x^2)+ & -\frac{e^\sigma}{2}\phi^2(1+x^2)+ \\ & +\frac{x^2}{2}e^{-\sigma} & +\frac{x^2}{2}e^{-\sigma} \\ -x^\nu+\phi^\nu(1+x^2) & -(x\cdot\phi)e^\sigma-\sinh\sigma+ & (x\cdot\phi)e^\sigma+\cosh\sigma- \\ & +\frac{e^\sigma}{2}\phi^2(1-x^2)- & -\frac{e^\sigma}{2}\phi^2(1-x^2)- \\ & -\frac{x^2}{2}e^{-\sigma} & -\frac{x^2}{2}e^{-\sigma} \end{bmatrix}$$

Summing the last two columns yields

$$S = \begin{pmatrix} 2x_\mu e^{-\sigma} \\ (1+x^2)e^{-\sigma} \\ (1+x^2)e^{-\sigma} \end{pmatrix},$$

while subtracting gives us

$$B = \begin{pmatrix} 2(\phi_\mu + x_\mu \phi^2)e^\sigma \\ (1 + 2(x\cdot\phi) + x^2\phi^2 + \phi^2)e^\sigma \\ -(1 + 2(x\cdot\phi) + x^2\phi^2 - \phi^2)e^\sigma \end{pmatrix} .$$

APPENDIX 5d

Here we want to evaluate the expression

$$e^{i\sigma D}(1 - 2i(\beta\cdot x)D)e^{-i(\phi\cdot K)}(1 - 2i(x\cdot\beta)(\phi\cdot K) - i(\beta\cdot K) - 2i(x\cdot\phi)(\beta\cdot K) +$$

$$+ 2i(\beta\cdot\phi)(x\cdot K))e^{-i(x\cdot P)}(1 - ix^2(\beta\cdot P) + 2i(x\cdot\beta)(x\cdot P)(1 + i(\beta\cdot K))\cdot$$

$$\cdot\; e^{i(x\cdot P)}e^{i(\phi\cdot K)}e^{-i\sigma D}$$

up to the first order in β.

First we use the relation

$$e^{-i(x\cdot P)}K_\nu e^{i(x\cdot P)} = K_\nu - ix^\mu[P_\mu,K_\nu] - \frac{x^\mu x^\tau}{2!}[P_\tau,[P_\mu K_\nu]] = K_\nu + 2x^\mu J_{\mu\nu} +$$

$$+ 2x_\nu D + x^2 P_\nu - 2(x\cdot P)x_\nu\ ,$$

which enables us to write

$$e^{-i(x\ P)}(\beta\cdot K)e^{i(x\cdot P)} = (K\cdot\beta) + 2x^\mu{}_\beta{}^\nu J_{\mu\nu} + 2(x\cdot\beta)D + x^2(\beta\cdot P) -$$

$$- 2(x\cdot P)(x\cdot\beta).$$

Our original expression now reads

$$e^{i\sigma D}(1 - 2i(\beta\cdot x)D)e^{-i(\phi\cdot K)}(1 - 2i(x\cdot\beta)(\phi\cdot K) - 2i(x\cdot\phi)(\beta\cdot K) +$$

$$+ 2i(\beta\cdot\phi)(x\cdot K) + 2ix^\mu{}_\beta{}^\nu + 2i(x\cdot\beta)D)e^{i(\phi\cdot K)}e^{-i\sigma D}.$$

Exploying finally $e^{-i(\phi\cdot K)}J_{\mu\nu}e^{i(\phi\cdot K)} = J_{\mu\nu} + \phi_\mu K_\nu - \phi_\nu K_\mu$ and

$e^{-i(\phi\cdot k)}De^{i(\phi\cdot K)} = D + (\phi\cdot K)$ leaves us with

$$e^{i\sigma D}(1 - 2i(\beta\cdot x)D)(1 + 2ix^\mu{}_\beta{}^\nu J_{\mu\nu} + 2i(x\cdot\beta)D)e^{-i\sigma D} = 1 + 2ix^\mu{}_\beta{}^\nu J_{\mu\nu}.$$

APPENDIX 5e

To prove relation (38) of Chapter 5 we write (similarly as in Appendix 5b)

$$i(K_\mu \partial^\mu + 2\ell D)_A{}^B = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & +\partial_0 & -\partial_0 \\ \cdot & \cdot & \cdot & \cdot & +\partial_1 & -\partial_1 \\ \cdot & \cdot & \cdot & \cdot & +\partial_2 & -\partial_2 \\ \cdot & \cdot & \cdot & \cdot & +\partial_3 & -\partial_3 \\ +\partial_0 & -\partial_1 & -\partial_2 & -\partial_3 & \cdot & +2\ell \\ +\partial_0 & -\partial_1 & -\partial_2 & -\partial_3 & +2\ell & \cdot \end{pmatrix}$$

or, multiplying this by g_{BE},

$$i(K_\mu \partial^\mu + 2\ell D)_{AE} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -\partial_0 & -\partial_0 \\ \cdot & \cdot & \cdot & \cdot & -\partial_1 & -\partial_1 \\ \cdot & \cdot & \cdot & \cdot & -\partial_2 & -\partial_2 \\ \cdot & \cdot & \cdot & \cdot & -\partial_3 & -\partial_3 \\ +\partial_0 & +\partial_1 & +\partial_2 & +\partial_3 & \cdot & +2\ell \\ +\partial_0 & +\partial_1 & +\partial_2 & +\partial_3 & -2\ell & \cdot \end{pmatrix}.$$

Multiplying again by the matrix

$$e^{i(x\cdot P)} = \begin{pmatrix} +1 & \cdot & \cdot & \cdot & x_0 & x_0 \\ \cdot & +1 & \cdot & \cdot & x_1 & x_1 \\ \cdot & \cdot & +1 & \cdot & x_2 & x_2 \\ \cdot & \cdot & \cdot & +1 & x_3 & x_3 \\ +x_0 & -x_1 & -x_2 & -x_3 & 1+\frac{x^2}{2} & \frac{x^2}{2} \\ -x_0 & +x_1 & +x_2 & +x_3 & -\frac{x^2}{2} & 1-\frac{x^2}{2} \end{pmatrix}$$

from the left and by the matrix

$$
(e^{i(x\cdot P)})^T = \begin{pmatrix} +1 & \cdot & \cdot & \cdot & -x_0 & -x_0 \\ \cdot & +1 & \cdot & \cdot & -x_1 & +x_1 \\ \cdot & \cdot & +1 & \cdot & -x_2 & +x_2 \\ \cdot & \cdot & \cdot & +1 & -x_3 & +x_3 \\ x_0 & x_1 & x_2 & x_3 & 1+\frac{x^2}{2} & -\frac{x^2}{2} \\ x_0 & x_1 & x_2 & x_3 & \frac{x^2}{2} & 1-\frac{x^2}{2} \end{pmatrix}
$$

from the right (the operator ∂_μ is not to be applied to this matrix) we obtain

$$
2\cdot \begin{bmatrix} x_\mu\partial_\nu - x_\nu\partial_\mu & -\frac{1+x^2}{2}\partial_\mu + ((x\cdot\partial)-\ell)x_\mu & \frac{1-x^2}{2}\partial_\mu - ((x\cdot\partial)-\ell)x_\mu \\ \frac{1+x^2}{2}\partial_\nu - ((x\cdot\partial)-\ell)x_\nu & 0 & -(x\cdot\partial)+\ell \\ \frac{1-x^2}{2}\partial_\nu + ((x\cdot\partial)-\ell)x_\nu & (x\cdot\partial)-\ell & 0 \end{bmatrix}
$$

where the matrix can be seen equal to operator ∂_{AB} (see equations (18)).

APPENDIX 5f

Finally we shall evaluate the expressions

$M^{-1}e^{\sigma}\partial_{\mu}M$, $M^{-1}e^{\sigma}\partial_{\mu}B$ and $M^{-}e^{\sigma}\partial_{\mu}S$ needed to calculate covariant derivatives in Chapter V (see equations (43) of this Chapter).

M^{-1} can be calculated easily by taking the matrix M from Appendix 5c) and transposing it with the proper change of signs (elements having one and only one of its row or column indices equal to 0 or 6 change sign, the other do not) gives us

$$M^{-1} = \begin{pmatrix} \partial_{\mu}{}^{\nu}+2\phi_{\mu}x^{\nu} & -x_{\mu}-\phi_{\mu}(1+x^2) & -x_{\mu}+\phi_{\mu}(1-x^2) \\ -\phi^{\nu}e^{\sigma}-x^{\nu}\phi^2e^{\sigma}- & (x\cdot\phi)e^{\sigma}+\cosh\sigma+ & (x\cdot\phi)e^{\sigma}+\sinh\sigma- \\ -x^{\nu}e^{-\sigma} & +\frac{e^{\sigma}}{2}\phi^2(1+x^2)+\frac{x^2}{2}e^{-\sigma} & -\frac{e^{\sigma}}{2}\phi^2(1-x^2)+\frac{x^2}{2}e^{-\sigma} \\ -\phi^{\nu}e^{\sigma}-x^{\nu}\phi^2e^{\sigma}+ & (x\cdot\phi)e^{\sigma}+\sinh\sigma+ & (x\cdot\phi)e^{\sigma}+\cosh\sigma- \\ +x^{\nu}e^{-\sigma} & +\frac{e^{\sigma}}{2}\phi^2(1+x^2)-\frac{x^2}{2}e^{-\sigma} & -\frac{e^{\sigma}}{2}\phi^2(1+x^2)-\frac{x^2}{2}e^{-\sigma} \end{pmatrix}$$

By applying the operator ∂_{λ} to the matrix M we obtain

$$(\partial_{\lambda}M)_A{}^{\nu} = \begin{pmatrix} 2g_{\mu\lambda}\phi^{\nu} + 2x_{\mu}\partial_{\lambda}\phi^{\nu} \\ \delta_{\lambda}{}^{\nu} + \partial_{\lambda}\phi^{\nu}(1+x^2) + 2x_{\lambda}\phi^{\nu} \\ -\delta_{\lambda}{}^{\nu} + \partial_{\lambda}\phi^{\nu}(1-x^2) - 2x_{\lambda}\phi^{\nu} \end{pmatrix}.$$

for the first four columns of M and

$$\partial_\lambda B = \begin{pmatrix} 2(\partial_\lambda \phi_\nu + g_{\lambda\nu}\phi^2 + 2x_\nu(\phi\cdot\partial_\lambda\phi)e^\sigma + 2\partial_\lambda\sigma(\phi_\lambda + x_\nu\phi^2)e^\sigma \\ (2\phi_\lambda+2(x\cdot\partial_\lambda\phi)+2x_\lambda\phi^2+2x^2(\phi\cdot\partial_\lambda\phi)+2(\phi\cdot\partial_\lambda\phi))e^\sigma+\partial_\lambda\sigma(1+2(x\cdot\phi)+x^2\phi^2+\phi^2)e^\sigma \\ -(2\phi_\lambda+2(x\cdot\partial_\lambda\phi)+2x_\lambda\phi^2+2x^2(\phi\cdot\partial_\lambda\phi)-2(\phi\cdot\partial_\lambda\phi))e^\sigma-\partial_\lambda\sigma(1+2(x\cdot\phi)+x^2\phi^2-\phi^2)e^\sigma \end{pmatrix}$$

and

$$\partial_\lambda S = \begin{pmatrix} 2g_{\lambda\nu}e^{-\sigma}-2\partial_\lambda\sigma x_\nu e^{-\sigma} \\ 2x_\lambda e^{-\sigma}-\partial_\lambda\sigma(1+x^2)e^{-\sigma} \\ -2x_\lambda e^{-\sigma}-\partial_\lambda\sigma(1+x^2)e^{-\sigma} \end{pmatrix}$$

for the difference and the sum of the last two columns of M respectively.

Multiplying these last three expressions by the matrix M^{-1} from the left we get expressions (43) of Chapter V.

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