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AN AXIOMATIC APPROACH TO THE HUREWICZ THEOREM FOR SYMMETRIC SPECTRA

by

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Abstract

Two important invariants in classical algebraic topology are the homotopy groups and the homology groups. The first of these are notoriously difficult to compute while the second is reasonably easier. The Hurewicz homomorphism is a natural transformation between these invariants which can make computation of some homotopy groups easier.

In this thesis we will look at the analogue of these invariants, and of the Hurewicz homomorphism, for the category of interest in modern stable homotopy theory.

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Introduction

A classic result in the study of closed orientable surfaces -spaces which locally resemble $\mathbb{R}^2$, have no boundary, and have well-defined notions of interior and exterior - is that these spaces are completely determined, up to isomorphism, by their *genus*, in lay terms that is to say the number of holes passing through them. That is, the complete list of all such surfaces is: the sphere, the torus, the two-handled torus, the three-handled torus, etc. As such there are two classes of functors of interest to this thesis, the homology groups on the category of spaces and homotopy groups on the category of pointed spaces, with the aim of counting "n-dimensional holes" in more general spaces than just closed orientable surfaces. While these functors *do not* provide a classification of spaces or pointed spaces, even up to homotopy, they have proved to be interesting in their own right and are fundamental to the study of classical algebraic topology.

Define, for any integer $n \geq 1$, the n^{th}-homotopy group functor from the category of pointed topological spaces to the category of groups

$$\pi_n(-)\colon \mathbf{Top}_* \longrightarrow \mathbf{Grp}$$

by taking, for any pointed space X, the set of maps from the n-sphere with basepoint chosen on the equator, S^n, to X, denoted $\mathbf{Top}_*(S^n, X)$, and quotienting out by the homotopy equivalence relation. The group operation is given by choosing representatives of each equivalence class, collapsing the equator of the n-sphere to form the one-point-union of two n-spheres, applying the first representative to the first sphere, the second to the second, and returning the homotopy class of the composite. The functor $\pi_0(-)\colon \mathbf{Top}_* \longrightarrow \mathbf{Set}$ can similarly be defined except that the group operation described above does not work for S^0 since it has no equator to collapse so only a set is obtained.

Let $\mathbb{N}$ denote the set of all non-negative integers and let Δ^n denote the n-simplex. Now one can define, for all $n \in \mathbb{N}$, the n^{th}-homology functor from the category of topological spaces to the category of abelian groups

$$\mathrm{H}_n(-)\colon \mathbf{Top} \longrightarrow \mathbf{Ab}$$

by taking, for each space X, the free abelian groups on the sets $\mathbf{Top}(\Delta^k, X))$ for each integer $k \geq -1$[1]. Call these groups C_k and define homomorphisms

$$\partial_{k+1}\colon C_{k+1} \longrightarrow C_k$$

on generators, $f\colon \Delta^{k+1} \longrightarrow X$, by taking f to the alternating sum of its restrictions to the facets of Δ^{k+1}. It then follows that $\operatorname{im}\partial_{k+1} \subseteq \ker\partial_k$ so we may define

$$H_n(X) := \ker\partial_n / \operatorname{im}\partial_{n+1}.$$

One can then define the homology functors on $\mathbf{Top}_*$ by pre-composing with the forgetful functor $\mathrm{U}\colon\ \mathbf{Top}_* \longrightarrow \mathbf{Top}$. We could instead have defined $\mathrm{H}_n(-)$ to be a functor from the category of topological spaces to the category of abelian groups, but we want to define a family of natural transformations from the homotopy groups to the homology groups.

For each $n \in \mathbb{N}$ there is natural transfromation from π_n to H_n called the n^{th} *Hurewicz homomorphism*. The simplest definition of this natural transformation is as follows:

- The homology functors H_n respect the homotopy relation, so for any equivalence class

  $$[f]\colon S^n \longrightarrow X$$

  there is a well-defined $\mathrm{H}_n([f]) := \mathrm{H}_n(f)\colon\ \mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(X)$

- For all integers $n \geq 1$ we have that $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ so, after fixing such an isomorphism for each n, $\mathrm{H}_n([f])\colon \mathbb{Z} \longrightarrow \mathrm{H}_n(X)$

- Define $h_n^X\colon \pi_n(X) \longrightarrow \mathrm{H}_n(X)$ by

  $$[f] \longmapsto \mathrm{H}_n(f) \longmapsto \mathrm{ev}_1(\mathrm{H}_n(f))$$

  where ev_1 is the "evaluate at (a representative of) 1" map.

[1]for this particular definition take $\Delta^{-1} := \emptyset$

The *Hurewicz Theorem* states that if $\pi_k(X) = 0$ for all $k < n$, for some integer $n \geq 2$, then the Hurewicz map $h_n^X : \pi_n(X) \longrightarrow \mathrm{H}_n(X)$ is an isomorphism and $h_{n+1}^X : \pi_{n+1}(X) \longrightarrow \mathrm{H}_{n+1}(X)$ is an epimorphism. This turns out to be a powerful result since to compute $\mathrm{H}_n(X)$ one has all of the tools of homological algebra at one's disposal whereas there are relatively few tools to compute $\pi_n(X)$.

For many years computing homotopy groups was the main goal of homotopy theory [20] but that proved to be untenable, computing all the homotopy groups of any sphere of dimension greater than 1 is still an open problem. The difficulties in computing homotopy groups led homotopy theorists to the study of *stable homotopy* of spaces. It was discovered, via the Freudenthal Suspension Theorem, that one could use the homotopy groups to define the *stable homotopy groups*. It was originally thought these would be easier to compute [20]. As with their unstable predecessors however, the stable homotopy groups likewise proved to be difficult to compute. In fact, computing the stable homotopy groups of the zero sphere is still an open problem. As it turns out the study of the stable homotopy groups proved to be interesting in its own right and, in 1958, Lima defined spectra - sequences of spaces with added *structure maps*. These ultimately proved to be *the* objects to use to study stable homotopy theory. The morphisms took a bit longer. Eventually the *stable homotopy category* was defined [2] and acknowledged to be the appropriate category in which to study stable homotopy theory. This category even came equipped with a symmetric monoidal product. Unfortunately this category was not known to be induced from any kind of homotopic structure on the category of spectra. In particular the symmetric monoidal product of the homotopy category was not induced by a symmetric monoidal product on the category of spectra (see [20] or the introduction to [23]).

Towards the end of the last millennium several categories of spectra where introduced which solved this, namely the category of $\mathscr{S}$-modules of [6], the category of orthogonal spectra of [18], and the category of symmetric spectra of [12]. The last of these will be our category of interest. In particular one can define the category of *symmetric spectra*, in this category it *is* possible to get a symmetric monoidal product which is homotopically well-behaved, further the homotopy category one gets from the category of symmetric spectra is equivalent to the original Bousfield-Friedlander homotopy category of spectra [2].

In this thesis we will provide a list of axioms which, when satisfied by a category $\mathbb{X}$, endow $\mathbb{X}$ with homotopy groups, homology groups, and a Hurewicz homomorphism between them which satisfies a weak version of the Hurewicz Theorem. We will then establish - by appealing to results in an unpublished book project, [23], of Stefan Schwede - that the homotopy category of symmetric spectra satisfies these axioms and, from the weak Hurewicz theorem, we recover the full Hurewicz theorem for symmetric spectra.

This approach has the advantages of (1) portability - in the sense that one could equally well apply this theory to a field ouside of homotopy theory- (2) generality -in our context $\mathbb{X}$ need not be a homotopy category- and (3) the abstraction clarifies how much of the theory one loses if certain key theorems fail, in whole or in part. Additionally it is my hope that this thesis will serve as a good first introduction to stable algebraic topology for future graduate students, being more accessible than [1] and less technical than [23].

This entire thesis is awash in the language and results of category theory and, as such, in Chapter 1 we will provide all of the necessary definitions, notation, results necessary to understand the rest of the document. All of the results in this chapter are standard.

Instead of working over the category of topological spaces, or indeed over a nice subcategory of topological spaces such as CW complexes or compactly generated weak Hausdorff spaces we work instead over simplicial sets. Chapter 2 provides a brief overview of the necessary background. All of the results in this chapter are standard.

The remainder of the background needed to understand the results in this thesis are the introductory definitions and results from the theory of model categories. Chapter 3 provides this background, though much and more of it can be found in any introductory text.

The novel contribution of this thesis is Chapter 4 in which the axioms for a *Hurewicz category* are provided, homotopy and homology groups are constructed, a Hurewicz homomorphism is defined, and a weak version of the Hurewicz Theorem is proved. It should be said that the motivation for the definitions as well as many of the proofs are simple vertical categorifications of the corresponding results in [23] and indeed even some of the elementary methods discussed in [8]. The truly novel contribution is the perspective the abstraction provides as well as the accessibility of the results.

Chapter 5 outlines the construction and basic results of symmetric spectra, including the symmetric monoidal product for which it is so celebrated. Finally, Chapter 6 provides citations to the appropriate results in [23] and [12] which prove that the homotopy category of symmetric spectra is a Hurewicz category. Effectively, Chapters 5 and 6 provide an example for Chapter 4.

Remark. It should be emphasized that, throughout this document, the symbol $\mathbb{N}$ shall denote the set of all non-negative integers; in particular $0 \in \mathbb{N}$.

Remark. The reader should be advised that, throughout this document we write function composition, and more generally composition of morphisms in an arbitrary category, in diagrammatic order. That is, if $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ then the composite is written

$$fg\colon X \longrightarrow Z$$

or, when we need to use more explicit notation

$$f \circ g\colon X \longrightarrow Z.$$

Should ever we need to write function composition in applicative order (such as during a computation) we write

$$g(f(x)) \qquad \text{or} \qquad g(f(-)).$$

Chapter 1

Category Theory

Introduction

Here we present the basic notions of category theory needed to understand the rest of the thesis. The study of algebraic topology in general, and homotopy theory in particular, is enhanced by the heavy use of the language, and results, of category theory throughout the entire subject. As such, this thesis will make prodigious use of category theory on every page and, as such, the following is the background a reader will require, and become intimately familiar with if they aren't already, in reading this document. The theorems and proofs within are all standard as are the definitions, all of which can be found, for instance, in [3], [17], with the exception of the material in Section 1.5 which can be found in [4]. We also set up our notation.

Section 1.1 will start at the very beginning and define categories, functors, as well as special classes of morphisms such as monics and epics. It will also introduce the notion of duality which will be important throughout the thesis, as well as diagrams and the notion of commutativity of a diagram.

Section 1.2 defines functors which will be used throughout, it also introduces natural transformations and functor categories.

Section 1.3 introduces products, coproducts, pushouts, pullbacks, and other important limits and colimits. It also briefly discusses biproducts as these will be relevant to the homotopy category of symmetric spectra. We will then cover adjoint functors and, in particular we state the theorem that left adjoints preserve colimits and right adjoints preserve limits. We conclude this section by introducing ends and coends.

In Section 1.4 we introduce symmetric monoidal products and from this the categorical definitions of monoids and modules; these will play an important part in Chapter 5. We will go on to state some theorems which induce categories of monoids and of modules with limits, colimits, and symmetric monoidal products, from corresponding ones in their parent category.

Finally, in Section 1.5 we discuss enriched categories and we state the Day Convolution. We will appeal to this result later to get the symmetric monoidal product on the category of symmetric spectra, in this case we will be enriching over the category of simplicial sets (see Chapter 2).

1.1 Basics

Definition 1.1.1. A *category*, $\mathbb{X}$, consists of:

- A class of objects $\operatorname{ob}\mathbb{X}$
- For each ordered pair of objects $(X, X') \in \operatorname{ob}\mathbb{X} \times \operatorname{ob}\mathbb{X}$ a set of *morphisms* $\mathbb{X}(X, X')$ called the *hom-set of* X *and* X'
- For each $X \in \operatorname{ob}\mathbb{X}$ an *identity morphism* $1_X \in \mathbb{X}(X, X)$
- For each ordered triple of objects $(X, X', X'') \in \operatorname{ob}\mathbb{X} \times \operatorname{ob}\mathbb{X} \times \operatorname{ob}\mathbb{X}$ a function

  $$\circ : \mathbb{X}(X, X') \times \mathbb{X}(X', X'') \longrightarrow \mathbb{X}(X, X''),$$

  called *composition*, such that, for all $X \in \operatorname{ob}\mathbb{X}$, the functions

  $$1_X \circ - : \mathbb{X}(X, X') \longrightarrow \mathbb{X}(X, X') \qquad \text{and} \qquad - \circ 1_{X'} : \mathbb{X}(X, X') \longrightarrow \mathbb{X}(X, X')$$

  are the identity functions, and such that, for any ordered quadruple (X, X', X'', X''') of objects of $\mathbb{X}$ the composites

  $$\mathbb{X}(X, X') \times \mathbb{X}(X', X'') \times \mathbb{X}(X'', X''') \underset{(-\circ-)\circ-}{\overset{-\circ(-\circ-)}{\rightrightarrows}} \mathbb{X}(X, X''')$$

  are equal. Call this the *associativity condition*. We may write this common composite as $- \circ - \circ -$.

Call a category $\mathbb{X}$ *small* if $\operatorname{ob}\mathbb{X}$ is a set in addition to being a class.

We may also write $f\colon X \longrightarrow X'$ for any morphism $f \in \mathbb{X}(X, X')$, in this case call X the *domain* of f and X' the *codomain* of f.

By abuse of notation, write $X \in \mathbb{X}$ as shorthand for $X \in \operatorname{ob}\mathbb{X}$.

Example 1.1.2. Let **Set** denote the category of sets and functions defined by:

- objects: sets
- morphisms: set functions
- identity morphisms: identity functions, i.e. $1_X\colon X \longrightarrow X$ given by $1_X(x) = x$ for all $x \in X$
- composition: the usual function composition.

That the collection of functions between two sets is itself a set follows from the powerset and comprehension axioms. The associativity condition follows from the usual associativity of set functions. That $1_X \circ -$ and $- \circ 1_{X'}$ are the identity function on $\mathbf{Set}(X, X')$ follows from the definition of function composition.

Notice that since there is no set of all sets by the central argument of Russell's Paradox we get that **Set** is *not* a small category.

We can also define the category of *pointed sets and pointed functions*, $\mathbf{Set}_*$, by:

- objects: pairs (X, x_0) where X is a non-empty set and $x_0 \in X$
- morphisms: a morphism $f\colon (X, x_0) \longrightarrow (Y, y_0)$ is a function $f\colon X \longrightarrow Y$ such that $f(x_0) = y_0$
- identity morphisms: the usual identity functions $1_X\colon (X, x_0) \longrightarrow (X, x_0)$ (since $1_X(x_0) = x_0$ by definition of the identity function)
- composition: the usual function composition.

Example 1.1.3. Let **Top** denote the category of topological spaces and continuous functions given by:

- objects: Topological spaces
- morphisms: Continuous functions
- identity morphisms: the usual identity functions
- composition: the usual function composition.

The identity functions are continuous by basic topology, the argument that this forms a category is then nearly identical to the argument for **Set**.

Again similarly we get the category of *pointed topological spaces and pointed maps*, $\mathbf{Top}_*$

- objects: pairs (X, x_0) where X is a topological space and $x_0 \in X$
- morphisms: continuous functions $f\colon X \longrightarrow Y$ such that $f(x_0) = y_0$
- identity morphisms: the identity functions.
- composition: the usual function composition.

Example 1.1.4. Let **Ab** denote the category of abelian groups and group homomorphisms given by:

- objects: abelian groups
- morphisms: group homomorphisms
- identity morphisms: the identity homomorphisms
- composition: the usual function composition.

Definition 1.1.5. Given a category $\mathbb{X}$ we can define a new category $\mathbb{X}^{\text{op}}$ by reversing the domain and codomain of every morphism, f, of $\mathbb{X}$. Given morphisms $f \in \mathbb{X}^{\text{op}}(X, Y) = \mathbb{X}(Y, X)$ and $g \in \mathbb{X}^{\text{op}}(Y, Z) = \mathbb{X}(Z, Y)$ the composite of f and g is the morphism $gf \in \mathbb{X}(Z, X) = \mathbb{X}^{\text{op}}(X, Z)$.

Remark. For any category $\mathbb{X}$ we have that $(\mathbb{X}^{\text{op}})^{\text{op}} = \mathbb{X}$.

Example 1.1.6. The category $\textbf{Set}^{\text{op}}$ is the category defined by

- objects: sets
- morphisms: a morphism $f^{\text{op}} \colon X \longrightarrow Y$ in $\textbf{Set}^{\text{op}}$ is a function $f \colon Y \longrightarrow X$
- identities: the identity morphism of X in $\textbf{Set}^{\text{op}}$ is the same as the identity morphism of X in $\textbf{Set}^{\text{op}}$.
- composition: Given morphisms $f^{\text{op}} \colon X \longrightarrow Y$ and $g^{\text{op}} \colon Y \longrightarrow Z$ in $\textbf{Set}^{\text{op}}$, the composite $f^{\text{op}} \circ g^{\text{op}}$ is defined to be the morphism $(gf)^{\text{op}}$

The following constitutes an example of an important phenomenon called *duality*:

Definition 1.1.7. Given a category $\mathbb{X}$, call an object $\perp \in \mathbb{X}$ *initial* if, for every object $X \in \mathbb{X}$ the set $\mathbb{X}(\perp, X)$ is a singleton.

Dually, call an object in $\mathbb{X}$ *terminal* if it meets one of the two following, equivalent, conditions

1. if, for every object $X \in \mathbb{X}$ the set $\mathbb{X}(X, \top)$ is a singleton
2. if $\top$ is an initial object in $\mathbb{X}^{\text{op}}$

If an object $\mathbf{0}$ is both initial and terminal in $\mathbb{X}$ then call it a *zero object*.

Remark. In general, given a definition in category theory it is often possible to form a *dual* definition by reversing the direction of all arrows in the diagrams used in the definition. That is, we say that a structure $\mathcal{X}$ in $\mathbb{X}$ satisfies the dual of a definition $\mathcal{D}$ if the structure $\mathcal{X}^{\text{op}}$ in $\mathbb{X}^{\text{op}}$ satisfies the definition $\mathcal{D}$.

In the same vein, if the correct proof of the dual statement, P^{op}, of some property P proceeds exactly the same as the proof of P except with all morphisms reversed, then we can simply state: "the proof is dual". Be advised, just because two statements are dual does *not* mean that the proofs will be dual.

The following serves as a good -if somewhat trivial - example of proof-by-duality:

Proposition 1.1.8. *If $\mathbb{X}$ is a category with initial object $\bot$, then the category $\mathbb{X}^{\mathrm{op}}$ has $\bot$ as a terminal object.*

Proof. For any $X \in \mathbb{X}^{\mathrm{op}}$ the set $\mathbb{X}^{\mathrm{op}}(X, \bot)$ is, by definition of $\mathbb{X}^{\mathrm{op}}$, the set $\mathbb{X}(\bot, X)$ which is a singleton. Thus by Definition 1.1.7 we get that $\bot$ is the terminal object of $\mathbb{X}^{\mathrm{op}}$.

□

The dual to Proposition 1.1.8 is:

Proposition 1.1.9. *If $\mathbb{X}$ is a category with terminal object $\top$, then the category $\mathbb{X}^{\mathrm{op}}$ has $\top$ as an initial object.*

The proof of this statement is obtained by taking the proof of Proposition 1.1.8 and reversing the roles of the domains and codomains:

Proof. For any $X \in \mathbb{X}^{\mathrm{op}}$ the set $\mathbb{X}^{\mathrm{op}}(\top, X)$ is, by definition of $\mathbb{X}^{\mathrm{op}}$, the set $\mathbb{X}(X, \top)$ which is a singleton. Thus by Definition 1.1.7 we get that $\top$ is the terminal object of $\mathbb{X}^{\mathrm{op}}$.

□

In this case we may simply state: The proof of Proposition 1.1.9 is dual to the proof of Proposition 1.1.8.

Definition 1.1.10. Given categories $\mathbb{X}$ and $\mathbb{Y}$, a functor, F, from $\mathbb{X}$ to $\mathbb{Y}$, denoted

$$F \colon \mathbb{X} \longrightarrow \mathbb{Y},$$

is a class function $F \colon \mathrm{ob}\,\mathbb{X} \longrightarrow \mathrm{ob}\,\mathbb{Y}$ together with a collection of set functions $\mathbb{X}(X, X') \longrightarrow \mathbb{Y}(F(X), F(X'))$ such that F *respects identities* - i.e. $1_X \longmapsto 1_{F(X)}$ for each $X \in \mathrm{ob}\,\mathbb{X}$- and such that F *respects composition* - i.e. the functions

$$\mathbb{X}(X, X') \times \mathbb{X}(X', X'') \underset{F(-)\circ F(=)}{\overset{F(-\circ=)}{\rightrightarrows}} \mathbb{Y}\big(F(X), F(X')\big)$$

are equal for all $X, X', X'' \in \mathbb{X}$.

Example 1.1.11. Given any category $\mathbb{X}$ define the *identity functor* $1_{\mathbb{X}}$ by $1_{\mathbb{X}}(X) = X$ for all $X \in \mathbb{X}$ and

$$1_{\mathbb{X}}(f\colon X \longrightarrow X') = f\colon X \longrightarrow X'.$$

Clearly $1_{\mathbb{X}}$ respects identities since

$$1_{\mathbb{X}}(1_X\colon X \longrightarrow X) = 1_X\colon X \longrightarrow X$$

and $1_{\mathbb{X}}$ also respects composition since

$$\begin{aligned} 1_{\mathbb{X}}(- \circ =) &= - \circ = \\ &= 1_{\mathbb{X}}(-) \circ 1_{\mathbb{X}}(=). \end{aligned}$$

Example 1.1.12. For any category $\mathbb{X}$ and any object $X' \in \mathbb{X}$ define the *hom-functor in* X'

$$\mathbb{X}(X', -)\colon \mathbb{X} \longrightarrow \mathbf{Set}$$

on objects to be

$$X \longmapsto \mathbb{X}(X', X)$$

and a morphism $f\colon X \longrightarrow Y'$ is sent to the post-composition function

$$\begin{aligned} \mathbb{X}(X', f)\colon \mathbb{X}(X', X) &\longrightarrow \mathbb{X}(X', Y) \\ (g\colon X' \longrightarrow X) &\longmapsto (\ X' \xrightarrow{g} X \xrightarrow{f} Y\ ). \end{aligned}$$

Example 1.1.13. Define the *free functor* $\mathrm{F}\colon \mathbf{Set} \longrightarrow \mathbf{Ab}$ on objects by

$$\mathrm{F}(X) = \bigoplus_{x \in X} \mathbb{Z}$$

the *free abelian group indexed by* X, and on morphisms, $f\colon X \longrightarrow Y$, by

$$\mathrm{F}(f)\colon\ \mathrm{F}(X) \longrightarrow \mathrm{F}(Y)$$

where $\mathrm{F}(f)$ sends the generator of $\mathrm{F}(X)$ indexed by the element X to the generator of $\mathrm{F}(Y)$ indexed by $f(x)$.

Example 1.1.14. Define the forgetful functor $\mathrm{U}\colon \mathbf{Ab} \longrightarrow \mathbf{Set}$ on objects, G, by taking $\mathrm{U}(G)$ to be the set of elements of G. For a homomorphism $h\colon G \longrightarrow G'$ we define $\mathrm{U}(h)\colon \mathrm{U}(G) \longrightarrow \mathrm{U}(G')$ to be the homomorphism h considered as a set function.

Definition 1.1.15. A functor $F\colon \mathbb{X}^{\mathrm{op}} \longrightarrow \mathbb{Y}$ is often called a *contravariant functor* from $\mathbb{X}$ to $\mathbb{Y}$.

Example 1.1.16. Let $\mathbb{X}$ be any category and let $X' \in \mathbb{X}$ be arbitrary. Define the *contravariant hom-functor in* X'

$$\mathbb{X}(-, X')\colon \mathbb{X}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

on objects by

$$X \longmapsto \mathbb{X}(X, X')$$

and a morphism $f\colon X \longrightarrow Y$ is sent to the pre-composition function

$$\begin{aligned} \mathbb{X}(f, X')\colon \mathbb{X}(Y, X') &\longrightarrow \mathbb{X}(X, X') \\ (g\colon Y \longrightarrow X') &\longmapsto (\ X \xrightarrow{f} Y \xrightarrow{g} X'\ ). \end{aligned}$$

Definition 1.1.17. Given a category $\mathbb{X}$, a *diagram*, D, in $\mathbb{X}$ is a functor $D\colon I \longrightarrow \mathbb{X}$ where I is a small category.

Say that a diagram D *commutes*, or is *commutative*, if, given any two objects $i, j \in I$ where I is the domain of the diagram D and any two paths

$$i \xrightarrow{f_0} i_1 \xrightarrow{f_1} \cdots \longrightarrow i_n \xrightarrow{f_n} j$$

and

$$i \xrightarrow{g_0} j_1 \xrightarrow{g_1} \cdots \longrightarrow j_m \xrightarrow{g_m} j$$

between them, the identity

$$D(f_0) \circ D(f_1) \circ \cdots \circ D(f_n) = D(g_0) \circ D(g_1) \circ \cdots \circ D(g_m)$$

holds. Treat the diagrams

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z \qquad \text{and} \qquad Y \underset{h}{\overset{g}{\rightrightarrows}} Z \xrightarrow{k} W$$

as the diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[h]{} & Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow[k]{} & W \end{array}$$

for the purposes of determining commutativity.

Example 1.1.18. The following diagram commutes for any category $\mathbb{X}$ and any morphisms $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ in $\mathbb{X}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle fg}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

Example 1.1.19. The diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow[f']{} & Y \end{array}$$

commutes if and only if $fg' = gf'$ since those are the only non-trivial composites of the diagram. For future reference, call the composite fg' the *clockwise composite* and gf' the *counterclockwise composite*.

Example 1.1.20. The diagram

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

commutes if $fg = fh$. In this case say f *equalizes* g and h. Notice that, despite the fact that g and h are distinct paths from Y to Z we do not require that $g = h$ in order to say that the diagram commutes.

Dually, say k *coequalizes* g and h if the following diagram commutes

$$Y \underset{h}{\overset{g}{\rightrightarrows}} Z \xrightarrow{k} W\ .$$

Definition 1.1.21. A morphism $f\colon X \longrightarrow Y$ in $\mathbb{X}$ is called *monic* if whenever f coequalizes g and h, we get that $g = h$. That is, a diagram of the form

$$W \overset{g}{\underset{h}{\rightrightarrows}} X \xrightarrow{f} Y$$

commutes if and only if $g = h$.

Dually, a morphism k is *epic* if, whenever k equalizes morphisms g and h, we get that $g = h$.

A morphism which is both monic and epic we will call *bijic*.

A morphism f is a *section* if there is a morphism $g\colon Y \longrightarrow X$ such that $fg = 1_X$. In this case g is called a *retraction*.

A morphism which is both a section and a retraction is called an *isomorphism.* If there is an isomorphism $f\colon X \longrightarrow Y$ then call X and Y *isomorphic*.

If $X, Y \in \mathbb{X}$ and there is a section $s\colon X \longrightarrow Y$ then call X a *retract* of Y.

Example 1.1.22. In **Set** the monic morphisms are exactly the injective functions and the epic morphisms are exactly the surjective functions.

The sections are exactly the injections with non-empty domain. A morphism of **Set** is a retraction exactly when it is epic.

In **Set** the class of bijics, isomorphisms, and bijections coincide.

The following result is an easy exercise and a useful result.

Proposition 1.1.23. *If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are morphisms in a category $\mathbb{X}$ such that*

$$fg\colon X \longrightarrow Z$$

is epic, then g is epic. Dually for monics.

Proof. Let

$$Y \xrightarrow{g} Z \overset{h_0}{\underset{h_1}{\rightrightarrows}} W$$

commute. Then so too does

$$X \xrightarrow{f} Y \xrightarrow{g} Z \overset{h_0}{\underset{h_1}{\rightrightarrows}} W$$

hence $h_0 = h_1$ since fg is epic. Since h_0 and h_1 were arbitrary we get that g must indeed be epic.

□

Definition 1.1.24. Let I be a set and let $\mathbb{X}$ be a category. If $X' \in \mathbb{X}$, call a collection of I-indexed morphisms, $f_i \colon X' \longrightarrow Y_i$, *jointly monic* if whenever we have any pair of morphisms

$$X \overset{g}{\underset{h}{\rightrightarrows}} X'$$

such that $gf_i = hf_i$ for all $i \in I$, then $g = h$.

The dual statement is the definition of *jointly epic.*

Proposition 1.1.25. *If $f \colon X \longrightarrow Y$ is an isomorphism with section $h \colon Y \longrightarrow X$ and retraction $g \colon Y \longrightarrow X$ then $g = h$.*

Proof.

$$g = g1_X = g(fh) = (gf)h = 1_Y h = h.$$

□

In light of this if f is an isomorphism denote the unique morphism which is both its section and retraction by f^{-1}. Call f^{-1} the *inverse* of f.

Remark. We say that an object is *unique up to isomorphism* with respect to some property if, whenever two objects satisfy the given property there is an isomorphism between them which respects the given property. Say an object is unique up to *unique isomorphism* with respect to some property if, whenever two objects satisfy the given property there is a unique isomorphism between the objects which respects the given property. Equivalently we say that an object has a *universal property.*

As an example of this:

Proposition 1.1.26. *If $\mathbb{X}$ has an initial object, then it is unique up to (unique) isomorphism.*

Proof. Let I and J be initial objects of $\mathbb{X}$. By the definition of an initial object we get that $\mathbb{X}(I, J)$ is a singleton, denote this morphism by $f\colon I \longrightarrow J$. Similarly there is a unique morphism $g\colon J \longrightarrow I$. Thus $fg \in \mathbb{X}(I, I) = \{1_I\}$, hence $fg = 1_I$. Similarly $gf = 1_J$ so f is an isomorphism with inverse g.

Notice that f is the only isomorphism from I to J so we indeed get that initial objects are unique up to unique isomorphism.

□

Definition 1.1.27 (see [3] page 34)**.** Let $\mathbb{X}$ be a category, a *congruence* $\sim$ on $\mathbb{X}$ is a collection of equivalence relations on the hom-sets of $\mathbb{X}$ such that, for any morphisms $f\colon X' \longrightarrow X$, $g_0, g_1\colon X \longrightarrow Y$, and $h\colon Y \longrightarrow Y'$, if $g_0 \sim g_1$ then $fg_0h \sim fg_1h$.

Given a congruence $\sim$ on $\mathbb{X}$ we can define the *quotient of* $\mathbb{X}$ *by* $\sim$, $\mathbb{X}/\sim$ to be the category given by

objects: The same as those of $\mathbb{X}$

morphisms: A morphism $\mathbf{f}\colon X \longrightarrow Y$ in $\mathbb{X}/\sim$ is the equivalence class of a morphism $f\colon X \longrightarrow Y$ in $\mathbb{X}$.

identities: The identity of $X \in \mathbb{X}$ is the equivalence class of the identity $1_X\colon X \longrightarrow X$ in $\mathbb{X}$.

composition: Given $\mathbf{f}\colon X \longrightarrow Y$ and $\mathbf{g}\colon Y \longrightarrow Z$, pick representatives $f \in \mathbf{f}$ and $g \in \mathbf{g}$ and let $f \circ g$ be the equivalence class of fg.

Proposition 1.1.28. *If $\mathbb{X}$ is a category and $\sim$ is a congruence on $\mathbb{X}$ then $\mathbb{X}/\sim$ is a category.*

Proof. We need to show that the given composition is well-defined and that the given identities behave appropriately.

First, to show that composition is well-defined choose equivalence classes $[f]\colon X \longrightarrow Y$ and $[g]\colon Y \longrightarrow Z$ and pick representatives $f_0, f_1 \in [f]$ and $g_0, g_1 \in [g]$. Thus $f_0 \sim f_1$ and $g_0 \sim g_1$, we need to show that $f_0g_0 \sim f_1g_1$, but $f_0 \sim f_1$ implies $f_0g_0 = 1_{X'}f_0g_0 \sim 1_{X'}f_1g_0 = f_1g_0$ since $\sim$ is a congruence. Similarly $g_0 \sim g_1$ implies $f_1g_0 = f_1g_01_Y \sim f_1g_11_Y = f_1g_1$. Transitivity then yields $f_0g_0 \sim f_1g_1$ and so composition is well-defined.

To show that the identities are well behaved let $\mathbf{f}\colon X \longrightarrow Y$ in $\mathbb{X}/\sim$ be arbitrary and choose a representative $f \in \mathbf{f}$ so that $[f] = \mathbf{f}$. Then

$$[1_X]\mathbf{f} = [1_X][f] = [1_Xf] = [f] = \mathbf{f}$$

and similarly

$$\mathbf{f}[1_Y] = [f][1_Y] = [f1_Y] = [f].$$

Since $\mathbf{f}$ was arbitary we get that the collection $[1_X]$ are identities for the given composition.

□

Call $\mathbb{X}/\sim$ a *congruence category*.

1.2 More on Functors

Definition 1.2.1. A functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ is called *full* (resp. *faithful*) if the associated functions $\mathbb{X}(X, X') \longrightarrow \mathbb{Y}(F(X), F(X'))$ are all surjections (resp. injections).

Example 1.2.2. For any category $\mathbb{X}$ the identity functor $1_{\mathbb{X}}\colon \mathbb{X} \longrightarrow \mathbb{X}$ is both full and faithful.

Example 1.2.3. Let $\mathbf{1}$ denote the category with one object and one morphism. The identity morphism is the only morphism present, composition is defined trivially, and associativity is trivial. Consider the functor

$$!\colon \mathbf{Set} \longrightarrow \mathbf{1}$$

which sends every set to the unique object of **1** and every morphism to the identity morphism on that object. This is trivially a functor. This functor is also clearly surjective on hom-sets, thus is full.

Example 1.2.4. The forgetful functor $\mathrm{U}\colon \mathbf{Ab} \longrightarrow \mathbf{Set}$ from Example 1.1.14, which takes any abelian group, G, to the set of its elements, is faithful since, if $h, h'\colon G \longrightarrow G'$ are distinct homomorphism then there is some $g \in G$ such that $h(g) \neq h(g')$, but then

$$\mathrm{U}(h)(g) = h(g) \neq h'(g) = \mathrm{U}(h')(g).$$

Example 1.2.5. Let $\mathbb{X}$ and $\mathbb{Y}$ be any categories and let $Y \in \mathbb{Y}$ be arbitrary. Let $c_Y : \mathbb{X} \longrightarrow \mathbb{Y}$ denote the *constant functor* which sends all objects in $\mathbb{X}$ to Y and all morphisms to the identity morphism. This functor will only be faithful if every hom-set in $\mathbb{X}$ has at most one element, and will only be full if $\mathbb{Y}(Y, Y) = \{1_Y\}$.

Definition 1.2.6. Given two functors $F, G\colon \mathbb{X} \longrightarrow \mathbb{Y}$, a natural transformation $\eta\colon F \Longrightarrow G$ is a collection of morphisms $\eta_X\colon F(X) \longrightarrow G(X)$ in $\mathbb{Y}$ indexed by the objects of $\mathbb{X}$ such that, for any $f\colon X \longrightarrow X'$ in $\mathbb{X}$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(X') & \xrightarrow[\eta_{X'}]{} & G(X') \end{array}$$

commutes. We will sometimes denote a natural transformation by

$$\mathbb{X} \underset{G}{\overset{F}{\rightrightarrows}}\!\!{\scriptstyle \Downarrow \eta}\; \mathbb{Y}\ .$$

A natural transformation η such that η_X is an isomorphism for all X is called a *natural isomorphism*.

Example 1.2.7. For any functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ there is the *identity natural transformation* $1_F\colon F \Longrightarrow F$ given by

$$(1_F)_X = 1_{F(X)}\colon F(X) \longrightarrow F(X).$$

Indeed, given any $f\colon X \longrightarrow X'$ the square

$$\begin{array}{ccc} F(X) & \xrightarrow{1_{F(X)}} & F(X) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle F(f)} \\ F(X') & \xrightarrow[1_{F(X')}]{} & F(X') \end{array}$$

commutes since functors preserve the identity morphisms.

Example 1.2.8. There is a power-set functor $\mathcal{P}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ defined on sets by $X \longmapsto \mathcal{P}(X)$, and on functions, $f\colon X \longrightarrow Y$, by

$$\mathcal{P}(f)\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$$

$$X' \subseteq X \longmapsto \{f(x')|x' \in X'\} \subseteq Y$$

For each set X define $\eta_X\colon X \longrightarrow \mathcal{P}(X)$ by $\eta_X(x) = \{x\}$ for all $x \in X$. We claim that this is a natural transformation

$$\mathbf{Set} \underset{\mathcal{P}}{\overset{1_{\mathbf{Set}}}{\Downarrow \eta}} \mathbf{Set}\,.$$

Let $f\colon X \longrightarrow Y$ be any function, we need to prove that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{P}(X) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}(f)} \\ Y & \xrightarrow[\eta_Y]{} & \mathcal{P}(Y) \end{array}$$

By Example 1.1.19 we get that this diagram commutes if and only if its clockwise composite equals its counterclockwise composite. The clockwise composite is, for an arbitrary $x \in X$

$$\begin{aligned} \eta_X\mathcal{P}(f)(x) &= \mathcal{P}(f)(\eta_X(x)) \\ &= \mathcal{P}(f)(\{x\}) \\ &= \{f(x)\} \end{aligned}$$

while the counterclockwise composite, for an arbitrary $x \in X$ is

$$\begin{aligned} f\eta_Y(x) &= \eta_Y(f(x)) \\ &= \{f(x)\} \end{aligned}$$

These agree, hence the diagram commutes. Since f was arbitrary we get that η is a natural transformation.

Definition 1.2.9. If $\mathbb{X}$ and $\mathbb{Y}$ are categories such that, for any pair of functors $F, G\colon \mathbb{X} \longrightarrow \mathbb{Y}$, the class of natural transformations from F to G, denoted $\mathrm{Nat}(F, G)$, is actually a set, then there is a category $\mathbb{Y}^{\mathbb{X}}$

- objects: Functors $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$
- morphisms: Natural transformations $\eta\colon F \Longrightarrow G$
- identities: The identity morphism for the functor F is the identity natural transformation $1_F\colon F \Longrightarrow F$
- composition: Composition between natural transformations $\eta\colon F \Longrightarrow G$ and $\tau\colon G \longrightarrow H$ is given object-wise, i.e. $(\eta\tau)_X = \eta_X \tau_{G(X)}$

In particular, if $\mathbb{X}$ is a small category then $\mathbb{Y}^{\mathbb{X}}$ will always be a category since then, for any functors $F, G\colon \mathbb{X} \longrightarrow \mathbb{Y}$

$$|\,\mathrm{Nat}(F, G)| \leq \left| \prod_{X \in \mathrm{ob}\,\mathbb{X}} \mathbb{Y}(F(X), G(X)) \right|.$$

The right-hand side is a set since $|\,\mathrm{ob}\,\mathbb{X}|$ is and since $\mathbb{Y}(F(X), G(X))$ is for each $X \in \mathrm{ob}\,\mathbb{X}$.

Example 1.2.10. Let Arrow be the category with two objects, call them 0 and 1, and three arrows, the identities on 0 and 1, and a single arrow $0 \longrightarrow 1$. A functor $D\colon$ Arrow $\longrightarrow \mathbb{X}$ is a diagram of the form

$$X_0 \longrightarrow X_1$$

that picks out a morphism of $\mathbb{X}$. Since Arrow is a small category we get, for any other category $\mathbb{X}$, the *arrow category* $\mathrm{Ar}(\mathbb{X}) := \mathbb{X}^{\mathrm{Arrow}}$ on $\mathbb{X}$.

- objects: morphisms of $\mathbb{X}$

- morphisms: if $f, g \in \mathrm{Ar}(\mathbb{X})$ then a morphism $\alpha\colon f \longrightarrow g$ is a commuting diagram of the form

$$\begin{array}{ccc} X_0 & \xrightarrow{\alpha_0} & X_0' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_1 & \xrightarrow[\alpha_1]{} & X_1' \end{array}$$

Definition 1.2.11. Given categories $\mathbb{X}$ and $\mathbb{Y}$ we can form the category $\mathbb{X} \times \mathbb{Y}$ which has, as objects, pairs of the form (X, Y) such that $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$. A morphism $f\colon (X, Y) \longrightarrow (X', Y')$ is a pair of arrows (f_0, f_1) such that $f_0\colon X \longrightarrow X'$ and $f_1\colon Y \longrightarrow Y'$. Composition is done component-wise and associativity follows from the associativity of $\mathbb{X}$ and $\mathbb{Y}$.

In the literature a functor $F\colon \mathbb{X} \times \mathbb{Y} \longrightarrow \mathbb{Z}$ is often referred to as a *bifunctor*.

Example 1.2.12. Given any category $\mathbb{X}$ there is the *hom functor*

$$\mathbb{X}(-,=)\colon \mathbb{X}^{\mathrm{op}} \times \mathbb{X} \longrightarrow \mathbf{Set}$$

defined on objects by

$$(X, X') \mapsto \mathbb{X}(X, X')$$

and on morphisms $f^{\mathrm{op}}\colon X \longrightarrow Y$ in $\mathbb{X}^{\mathrm{op}}$ and $g\colon X' \longrightarrow Y'$ in $\mathbb{X}$ by

$$\mathbb{X}(f^{\mathrm{op}}, g)\colon \mathbb{X}(X, X') \longrightarrow \mathbb{X}(Y, Y')$$
$$(h\colon X \longrightarrow X') \longmapsto (\, Y \xrightarrow{f} X \xrightarrow{h} X' \xrightarrow{g} Y' \,)$$

Definition 1.2.13. Given any group G we may consider G as a category with a single object $*$ and whose morphisms are the elements of G, composition and identity are given by the group operation and the identity element of the group respectively - i.e.

$$\underbrace{g_0 \circ g_1}_{\text{composition}} = \underbrace{g_0 g_1}_{\text{group operation}} .$$

An *action of* G on an object X of a category $\mathbb{X}$ is a functor

$$\cdot : G \longrightarrow \mathbb{X}$$

such that X is the image of $*$. Call such a functor a *G-object of* $\mathbb{X}$. Notice that any object $Y \in \mathbb{X}$ may be made into a G-object by sending all elements of G to the identity morphism $1_Y : Y \longrightarrow Y$, called the *trivial action of G on Y*. A morphism $f : X \longrightarrow Y$ in $\mathbb{X}$ between G-objects is called *G-equivariant* if it is a natural transformation between the actions on X and Y respectively. If X is a G-object and Y is any object of $\mathbb{X}$ then call $f : X \longrightarrow Y$ *G-invariant* if f is G-equivariant when G acts trivially on Y.

If $X \in \mathbb{X}$ is a G-object we can define the quotient of X by G to be the colimit, if it exists, of the diagram

$$X \; \underset{g_0}{\overset{g_\alpha}{\underset{\vdots}{\longrightarrow}}} \; X$$

where $g_0, \ldots, g_\alpha$ are the elements of G. Denote this object by X/G.

Example 1.2.14. In **Set** a G-object, for a fixed group G, is a set X together with a G-indexed set of bijections

$$f_g : X \longrightarrow X$$

such that, for any pair $g, h \in G$ the identity

$$f_g f_h = f_{gh}$$

holds.

The quotient, X/G, of X by G as defined above is the set of equivalence classes of the equivalence relation, $\equiv$, given by $x \equiv y$ if and only if there exists a $g \in G$ such that $g \cdot x = y$.

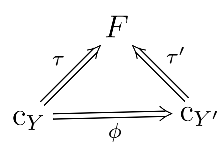
1.3 Limits and Colimits

Recall 1.3.1. Recall from Definition 1.1.7 that an object $\bot \in \mathbb{X}$ is initial if $\mathbb{X}(\bot, X)$ is a singleton for all $X \in \mathbb{X}$. Dually we call an object, $\top$, terminal if $\mathbb{X}(X, \top)$ is a singleton for all $X \in \mathbb{X}$. An object which is both initial and terminal is called a zero object.

Example 1.3.2. In **Set** and **Top** the empty set $\emptyset$ is the initial object and any singleton $\{0\}$ is a terminal object - in the case of **Top** these sets have canonical topologies.

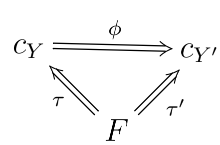
In both $\mathbf{Set}_*$ and $\mathbf{Top}_*$ the pair $(\{0\}, 0)$ is a zero object. To see this, let (X, x_0) be any other pair in either $\mathbf{Set}_*$ or $\mathbf{Top}_*$. Then any map $f\colon (\{0\}, 0) \longrightarrow (X, x_0)$ must satisfy $f(0) = x_0$ by definition, but then that completely determines the map to X. Further, this map exists both in $\mathbf{Set}_*$ and $\mathbf{Top}_*$ by elementary set theory and topology thus there is exactly one map from $(\{0\}, 0)$ to (X, x_0), hence $(\{0\}, 0)$ is a zero object in both categories.

Definition 1.3.3. Given any functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ (in particular any diagram D) define the category $\mathrm{Cones}(F)$, of *cones under* F to be the category whose objects are natural transformations $\tau\colon \mathrm{c}_Y \Longrightarrow F$ for any $Y \in \mathbb{Y}$. A morphism in $\mathrm{Cones}(F)$ is a commuting diagram



Notice that a natural transformation between c_Y and $c_{Y'}$ is just a morphism in $\mathbb{Y}$ from Y to Y', hence the collection of all such diagrams between fixed cones τ and τ' is a set. Composition is given by gluing diagrams together. Objects of $\mathrm{Cones}(F)$ are called *cones*. Call Y the *point* or *vertex* of the cone $\tau\colon \mathrm{c}_Y \longrightarrow F$. The *limit* of F, denoted $\pi\colon \mathrm{c}_{\lim F} \longrightarrow F$, is then, if it exists, any terminal object of $\mathrm{Cones}(F)$.

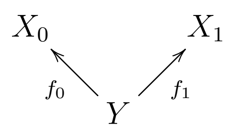
Dually, define $\mathrm{Cocones}(F)$ to be the category whose objects are natural transformations $\tau\colon F \Longrightarrow \mathrm{c}_Y$, called *cones over* F, and where arrows are commuting diagrams



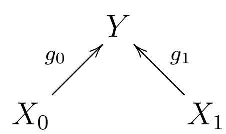
with composition given by gluing diagrams together. Objects of $\mathrm{Cocones}(F)$ are called *cocones*, if $(c_Y, \tau) \in \mathrm{Cocones}(F)$ then call Y the *point* or *vertex* of the cocone (c_Y, τ). A colimit of F, denoted $\iota\colon F \longrightarrow \mathrm{c}_{\operatorname{colim} F}$, if it exists, is then given by any initial object of this category.

Traditionally one speaks of *the* limit or colimit of a functor (or diagram), this is justified by the fact that limits and colimits, if they exist, are unique up to unique isormorphism by Proposition 1.1.26.

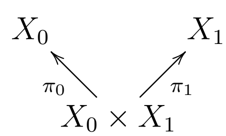
Example 1.3.4. Consider the discrete category on two objects (call it $\mathbf{2}$): i.e. the category with two objects, and the only morphisms are the identity morphisms on those objects. A diagram $D\colon \mathbf{2} \longrightarrow \mathbb{X}$ just picks out two objects $X_0, X_1 \in \mathbb{X}$ and their identity morphisms. A cone on such a diagram (with identity arrows suppressed) then looks like



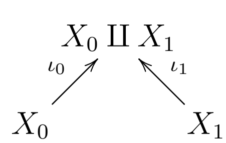
and a cocone looks like



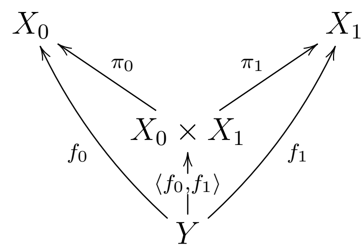
If a terminal cone exists, that is, if the underlying diagram has a limit, call this cone a *product of* X *and* X' and denote it by



If an initial cocone exists, that is, if the underlying diagram has a colimit, call this cocone a *coproduct of* X *and* X' and denote it by



Definition 1.3.5. Given objects $X_0, X_1 \in \mathbb{X}$ with a product $X_0 \times X_1$, if $Y \in \mathbb{X}$ then for any pair of morphisms $f_0 \colon Y \longrightarrow X_0$ and $f_1 \colon Y \longrightarrow X_1$ then there is a unique morphism $\langle f_0, f_1 \rangle \colon Y \longrightarrow X_0 \times X_1$ making the following diagram commute



Dually, define the $\langle g_0 | g_1 \rangle \colon X_0 \amalg X_1 \longrightarrow Z$.

Definition 1.3.6. Given morphisms $f \colon X \longrightarrow X'$ and $g \colon Y \longrightarrow Y'$ there is a morphism $f \times g \colon X \times Y \longrightarrow X' \times Y'$ given by $\langle \pi_0 f, \pi_1 g \rangle$. This, together with an application of the axiom of choice, makes $\times$ into a functor. Dually $\amalg$ is a functor.

Example 1.3.7. In **Set** we can define a functorial product functor. First, for each ordered pair of sets (X, Y) we have the cartesian product $\{(x, y) | x \in X, y \in Y\}$ which we will momentarily denote $X \times_C Y$. We also have projection morphisms $\pi_0 \colon X \times_C Y \longrightarrow X$ and $\pi_1 \colon X \times Y \longrightarrow Y$ which are defined by $(x, y) \longmapsto x$ and $(x, y) \longmapsto y$ respectively. We claim that this is a product in the categorical sense.

Assume we have a pair $f_0 \colon Z \longrightarrow X$ and $f_1 \colon Z \longrightarrow Y$, then define $\langle f_0, f_1 \rangle_C \colon Z \longrightarrow X \times_C Y$ by $z \mapsto (f_0(z), f_1(z))$ for all $z \in Z$. Then $\langle , f_0 \rangle f_{1C}$ is a perfectly well-defined function, further $\langle , f_0 \rangle f_{1C} \pi_0 = f_0$ and $\langle , f_0 \rangle f_{1C} \pi_1 = f_1$. Finally, $\langle , f_0 \rangle f_{1C}$ is clearly the unique morphism with this property. We conclude that $X \times_C Y$ is a product in the categorical sense. Henceforth denote $X \times_C Y$ by $X \times Y$.

Definition 1.3.8. For any set I, take $\mathrm{Disc}(I)$ to be the discrete category on I (i.e. the objects of $\mathrm{Disc}(I)$ are the elements of I and the only morphisms are the necessary identity morphisms.) For any functor $D \colon \mathrm{Disc}(I) \longrightarrow \mathbb{X}$, denote the limit of D by

$$\left(\prod_{i \in I} D(i), \pi \right)$$

- π is a natural transformation with components π_i - and call it the (I*-indexed*) *product*. Dually denote the colimit by

$$\left(\coprod_{i\in I} D(i), \iota\right)$$

- where the components of ι are denoted ι_i for each $i \in I$- and call it the (I*-indexed*) *coproduct*.

A category $\mathbb{X}$ which has all (finite) products is said to be *closed under (finite) products*. Dually, a category which has all (finite) coproducts is said to be *closed under (finite) coproducts*.

We state the following definition for coproducts instead of products because the preservation of coproducts will prove more pertinent in the sequel.

Definition 1.3.9. A functor

$$F\colon \mathbb{X} \longrightarrow \mathbb{Y}$$

preserves coproducts if, given any I-indexed collection $X_i \in \mathbb{X}$ with a coproduct

$$\left(\coprod_{i\in I} X_i, \iota\right)$$

in $\mathbb{X}$, the object

$$\left(F\left(\coprod_{i\in I} X_i\right), F(\iota)\right)$$

is a coproduct of the collection $F(X_i)$ in $\mathbb{Y}$. That is, given any object $Y \in \mathbb{Y}$ and a collection of morphisms

$$f_i\colon F(X_i) \longrightarrow Y$$

there is a unique morphism

$$\phi\colon F\left(\coprod_{i\in I} X_i\right) \longrightarrow Y$$

such that, for all $i \in I$ the diagram

$$\begin{array}{ccc} F\left(\coprod_{i\in I} X_i\right) & \xleftarrow{F(\iota_i)} & F(X_i) \\ & \searrow^{\phi} & \downarrow{\scriptstyle f_i} \\ & & Y \end{array}$$

commutes.

Definition 1.3.10. Let $\mathbb{X}$ be a category with a zero object, $\mathbf{0}$ and let I be a set. Consider any I-indexed set of objects $\{X_i | i \in I\}$ in $\mathbb{X}$. The *I-indexed biproduct* (or just the *biproduct*) of the X_i is an object

$$\bigoplus_{i \in I} X_i$$

together with morphisms

$$\pi_i \colon \bigoplus_{i \in I} X_i \longrightarrow X_i$$

and

$$\iota_i \colon X_i \longrightarrow \bigoplus_{i \in I} X_i$$

for each $i \in I$ such that...

... $\left(\bigoplus_{i \in I} X_i, \pi_j \right)_{j \in I}$ is a product of the X_i.

... $\left(\bigoplus_{i \in I} X_i, \iota_j \right)_{j \in I}$ is a coproduct of the X_i.

... the composite $\iota_i \pi_j = \begin{cases} 1_{X_i}, & i = j \\ 0, & i \neq j \end{cases}$ for all $i, j \in I$.

For $I = \{0, 1\}$ denote the biproduct by $X_0 \oplus X_1$.

Example 1.3.11. The category **Ab** of abelian groups has all finite biproducts. See [13] for details.

Remark. Notice that a biproduct is simultaneously a limit and a colimit. In particular, just because a category $\mathbb{X}$ has all small limits and colimits does not imply that it has any biproducts at all.

We need the following definition for a nice characterization of biproducts:

Definition 1.3.12. If $\mathbb{X}$ is a category with a zero object, then for any I-indexed collection of objects X_i with a product and a coproduct define the *canonical morphism*

$$\kappa\colon \coprod_{i\in I} X_i \longrightarrow \prod_{i\in I} X_i$$

to be the unique morphism satisfying

$$\iota_i \kappa \pi_j = \begin{cases} 1_{X_i}, & i = j \\ 0, & i \neq j \end{cases}$$

Proposition 1.3.13. *Let $\mathbb{X}$ be a category with a zero object and consider an I-indexed collection of objects X_i. The biproduct of the X_i exists if and only if the canonical morphism*

$$\kappa\colon \coprod_{i\in I} X_i \longrightarrow \prod_{i\in I} X_i$$

is an isomorphism.

Proof. First, if the biproduct exists, then choosing $\left(\bigoplus_{i\in I} X_i, \pi_j\right)_{j\in I}$ for a product of the X_i and $\left(\bigoplus_{i\in I} X_i, \iota_j\right)_{j\in I}$ for a coproduct, then the unique morphism κ such that

$$\iota_i \kappa \pi_j = \begin{cases} 1_{X_i}, & i = j \\ 0, & i \neq j \end{cases}$$

is the identity morphism by the definition of the biproduct. The identity morphism is an isomorphism so in this case we're done. That products and coproducts are unique up to isomorphism yields the proof for the general case.

Conversely, if κ is an isomorphism then the triples $\left(\coprod_{i\in I} X_i, \kappa_j \pi_j, \iota_j\right)_{j\in I}$ is a biproduct for the X_i.

∎

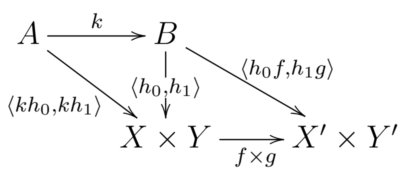
Proposition 1.3.14. *Let $D\colon I \longrightarrow \mathbb{X}$ be a diagram with a limit* $(\lim D, \pi)$*, then the collection* $\{\pi_i\}_{i\in I}$ *is jointly-monic. Dually, if D has a colimit* $(\operatorname{colim} D, \iota)$ *then the collection* $\{\iota_i\}_{i\in I}$ *is jointly-epic.*

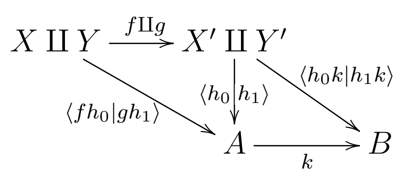
Proof. Assume we have two morphisms $f, g\colon X \longrightarrow \lim D$ such that $f\pi_i = g\pi_i$ for all $i \in I$. Then, by the universal property of the limit we get that there is a unique morphism $\phi\colon X \longrightarrow \lim D$ such that $\phi\pi_i = f\pi_i = g\pi_i$ for all i. In particular $f = \phi = g$ and we're done. The proof for the colimit is dual.

□

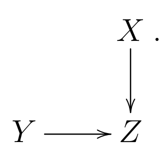
The following is a useful proposition

Proposition 1.3.15. *The following diagrams commute in any category in which they make sense.*

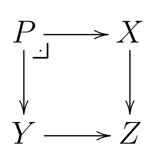




Definition 1.3.16. A *pullback* is the limit of a diagram of the form



and we write



to denote that P is the pullback of the rest of the diagram.

Dually, a *pushout* is a colimit of the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

and we write

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & Q \end{array}$$

to denote that Q is the pushout of the rest of the diagram.

Definition 1.3.17. Similarly an *equalizer* is a limit of a diagram of the form

$$Y \rightrightarrows Z\ .$$

A *coequalizer* is a colimit of a diagram of the same form.

Definition 1.3.18. Let $\mathbb{X}$ be a category with a zero object $\mathbf{0}$. For any pair of objects $X, Y \in \mathbb{X}$ there is a unique morphism

$$\mathbf{0}_{X,Y} \colon X \longrightarrow Y$$

formed by taking the composite

$$X \longrightarrow \mathbf{0} \longrightarrow Y$$

Given any other morphism $f \colon X \longrightarrow Y$ the *kernel* of f is the equalizer of the diagram

$$X \underset{\mathbf{0}_{X,Y}}{\overset{f}{\rightrightarrows}} Y$$

Dually, the coequalizer of this diagram is the *cokernel* of f.

Definition 1.3.19. A category $\mathbb{X}$ is called *complete* if, for every small category I and every functor $D \colon I \longrightarrow \mathbb{X}$ the limit of D exists. Dually, a category is cocomplete if, for every small category I and every functor $D \colon I \longrightarrow \mathbb{X}$ the colimit exists.

Theorem 1.3.20 ([17] p.102)**.** *A category $\mathbb{X}$ is complete if and only if it has all products and equalizers. Dually a category is cocomplete if and only if it has all coproducts and all coequalizers.*

Proposition 1.3.21 (See [17])**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be categories such that the category $\mathbb{Y}^{\mathbb{X}}$ exists. Then limits and colimits in $\mathbb{Y}^{\mathbb{X}}$ are computed pointwise.*

Definition 1.3.22. A functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ *preserves limits* if, given any small category I and any $D\colon I \longrightarrow \mathbb{X}$ such that the limit $(\lim D, \pi)$ exists then $(F(\lim D), F(\pi^D)) \cong (\lim DF, \pi^{DF})$, i.e. F takes an initial object of Cones(D) to an initial object of Cones(DF).

Dually, F preserves colimits if, given any small category I and any $D\colon I \longrightarrow \mathbb{X}$ such that the colimit $(\operatorname{colim} D, \iota)$ exists then. $(F(\operatorname{colim} D), F(\iota^D)) = (\operatorname{colim} DF, \iota^{DF})$

Definition 1.3.23. Given categories $\mathbb{X}$ and $\mathbb{Y}$ and functors

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{Y}$$

we say that F *is left-adjoint to* G (equiv. G *is right-adjoint to* F), written $F \dashv G$ if there is a natural isomorphism

$$\mathbb{X}^{\mathrm{op}} \times \mathbb{Y} \underset{\mathbb{X}(-,G(=))}{\overset{\mathbb{Y}(F(-),=)}{\Downarrow\psi}} \mathbf{Set}$$

If $F \dashv G$ and both F and G are full and faithful then say that F (or G) is an *equivalence.*

The following example of an adjoint pair is given for **Set**, but it occurs and is quite useful in several other categories.

Example 1.3.24. Let Y be any set, then we have functors

$$- \times Y\colon \mathbf{Set} \longrightarrow \mathbf{Set};\quad \begin{array}{ccc} X & \longmapsto & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1_Y} \\ X' & \longmapsto & X' \times Y \end{array}$$

and

$$\mathbf{Set}(Y,-);\quad \begin{array}{ccc} X & \longmapsto & \mathbf{Set}(Y,X) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ X' & \longmapsto & \mathbf{Set}(Y,X') \end{array} \quad \begin{array}{c} {\scriptstyle g} \\ \big\downarrow \\ {\scriptstyle gf} \end{array}$$

from Examples 1.3.7 and 1.2.12 respectively. We claim that these are an adjoint pair. In particular we claim $- \times Y \dashv \mathbf{Set}(Y,-)$.

To prove this we need a natural isomorphism of sets

$$\mathbf{Set}(- \times Y, =) \cong \mathbf{Set}(-, \mathbf{Set}(Y, =)).$$

Let $X, Z \in \mathbf{Set}$ be arbitrary and consider the set $\mathbf{Set}(X \times Y, Z)$. An element $f \in \mathbf{Set}(X \times Y, Z)$ is a function $f\colon X \times Y \longrightarrow Z$, but this can be seen as a function $\widetilde{f}\colon X \longrightarrow \mathbf{Set}(Y, Z)$ by sending each element $x \in X$ to the function $f(x,-)\colon Y \longrightarrow Z$. Conversely, if $g \in \mathbf{Set}(X, \mathbf{Set}(Y, Z))$ then we can define a function $\overline{g}\colon X \times Y \longrightarrow Z$ by sending the pair (x, y) first to the function $g(x)\colon Y \longrightarrow Z$, then to the element $g(x)(y) \in Z$. It is straightforward to see that this pairing is a bijection. That this is natural essentially follows from the fact that all choices made were canonical.

Theorem 1.3.25. *Let $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a functor, if F is a left adjoint then F preserves colimits. Dually, if F is a right adjoint then F preserves limits.*

Proof. This can be found in [17] p. 115.

□

Proposition 1.3.26 ([17] p.111)**.** *Let I be a small category and let $\mathbb{Y}$ be a complete and cocomplete category. Then $\mathbb{Y}^I$ is complete and cocomplete with (co)limits computed pointwise, i.e. given a diagram $D\colon I \longrightarrow \mathbb{Y}^{\mathbb{X}}$ let $D_i := D(i)\colon \mathbb{X} \longrightarrow \mathbb{Y}$, then for any $X \in \mathbb{X}$*

$$(\lim_{i \in I} D)(X) = \lim_{i \in I} D_i(X)$$

and

$$(\operatorname*{colim}_{i \in I} D)(X) = \operatorname*{colim}_{i \in I} D_i(X)$$

Proof. This can be found in [17] p.111.

□

We next define ends and coends, the latter being used in Chapter 5 to define a symmetric monoidal product on the category of Symmetric Spectra. The presentation is derived from the presentation in [14] for enriched categories.

Definition 1.3.27. Let I be a small category and let $F\colon I^{\mathrm{op}} \times I \longrightarrow \mathbf{Set}$ be a functor. Then if we fix an object $i_0 \in I$ we get a new functor $F(i_0, -)\colon I \longrightarrow \mathbf{Set}$, which, for any other object $i_1 \in I$ determines a function

$$F(i_0, -)_{i_0,i_1}\colon I(i_0, i_1) \longrightarrow \mathbf{Set}(F(i_0, i_0), F(i_0, i_1))$$

which takes a morphism $f\colon i_0 \longrightarrow i_1$ and sends it to the function

$$F(i_0, f)\colon F(i_0, i_0) \longrightarrow F(i_0, i_1).$$

Now

$$\begin{aligned} F(i_0, -)_{i_0,i_1} &\in \mathbf{Set}(I(i_0, i_1), \mathbf{Set}(F(i_0, i_0), F(i_0, i_1)) \\ &\cong \mathbf{Set}(I(i_0, i_1) \times F(i_0, i_0), F(i_0, i_1)) \\ &\cong \mathbf{Set}(F(i_0, i_0), \mathbf{Set}(I(i_0, i_1), F(i_0, i_1))) \end{aligned}$$

Denote by $r_{i_0,i_1}\colon F(i_0, i_0) \longrightarrow \mathbf{Set}(I(i_0, i_1), F(i_0, i_1))$ the image of $F(i_0, -)_{i_0,i_1}$ under this bijection. It is straightforward to check that r_{i_0,i_1} sends an element $x \in F(i_0, i_0)$ to the evaluation function

$$\mathrm{ev}_x(f\colon i_0 \longrightarrow i_1) = F(i_0, f)(x).$$

Conversely, by first fixing i_1 and then i_0 we get a function

$$F(-, i_1)_{i_1,i_0}\colon I^{\mathrm{op}}(i_1, i_0) \longrightarrow \mathbf{Set}(F(i_1, i_1), F(i_0, i_1))$$

which takes a morphism $f\colon i_0 \longrightarrow i_1$ in I to the function $F(f, i_1)\colon F(i_1, i_1) \longrightarrow F(i_0, i_1)$. Again

$$\begin{aligned} F(-, i_1)_{i_1, i_0} &\in \mathbf{Set}(I^{\mathrm{op}}(i_1, i_0), \mathbf{Set}(F(i_1, i_1), F(i_0, i_1))) \\ &\cong \mathbf{Set}(I(i_0, i_1), \mathbf{Set}(F(i_1, i_1), F(i_0, i_1))) \\ &\cong \mathbf{Set}(I(i_0, i_1) \times F(i_1, i_1), F(i_0, i_1)) \\ &\cong \mathbf{Set}(F(i_1, i_1), \mathbf{Set}(I(i_0, i_1), F(i_0, i_1))) \end{aligned}$$

Denote the image of $F(-, i_1)_{i_1, i_0}$ under this bijection as l_{i_0, i_1}. Again, this function takes an element $y \in F(i_1, i_1)$ to the evaluation map

$$\mathrm{ev}_y(f\colon i_0 \longrightarrow i_1) = F(f, i_1)(y).$$

In effect, the hom-sets of I *act* on the sets $F(i_0, i_1)$.

Definition 1.3.28. Let I be a small category and let $F\colon I^{\mathrm{op}} \times I \longrightarrow \mathbf{Set}$ be a functor. Define the *end* of F, $\displaystyle\int_{i \in I} F(i, i)$, to be the equalizer

$$\int_{i \in I} F(i, i) \xrightarrow{\pi} \prod_{i \in I} F(i, i) \overset{r}{\underset{l}{\rightrightarrows}} \prod_{(i_0, i_1) \in (\mathrm{ob}\, I)^2} \mathbf{Set}(I(i_0, i_1), F(i_0, i_1))$$

where r is the function whose components are r_{i_0, i_1} and l is the function whose components are the functions l_{i_0, i_1}.

Definition 1.3.29. Dually, define the *coend* of F, $\displaystyle\int^{i \in I} F(i, i)$, to be the coequalizer

$$\coprod_{(i_0, i_1) \in (\mathrm{ob}\, I)^2} I(i_0, i_1) \times F(i_0, i_1) \overset{r'}{\underset{l'}{\rightrightarrows}} \coprod_{i \in (\mathrm{ob}\, I)} F(i, i) \xrightarrow{\iota} \int^{i \in I} F(i, i)\ .$$

Given a functor

$$F\colon \mathbb{X}^{\mathrm{op}} \times \mathbb{X} \times \mathbb{Y}^{\mathrm{op}} \times \mathbb{Y} \longrightarrow \mathbf{Set}$$

one can define the *double coend*

$$\int^{X \in \mathbb{X}, Y \in \mathbb{Y}} F(X, X, Y, Y) := \int^{X \in \mathbb{X}} \int^{Y \in \mathbb{Y}} F(X, X, Y, Y).$$

Dually for ends.

Proposition 1.3.30 (See [17] Section IX.8)**.** *For any functor*

$$F\colon \mathbb{X}^{\mathrm{op}} \times \mathbb{X} \times \mathbb{Y}^{\mathrm{op}} \times \mathbb{Y} \longrightarrow \mathbf{Set}$$

the equation

$$\int^{X\in\mathbb{X}} \int^{Y\in\mathbb{Y}} F(X,X,Y,Y) \cong \int^{Y\in\mathbb{Y}} \int^{X\in\mathbb{X}} F(X,X,Y,Y)$$

holds whenever either coend exists. Dually for ends.

In particular, computing double (co)ends can be done in either order. Clearly this extends to n-fold (co)ends.

Proposition 1.3.31 (See [17] Theorem IX.7.2)**.** *Let*

$$F\colon \mathbb{X} \times \mathbb{Y}^{\mathrm{op}} \times \mathbb{Y} \longrightarrow \mathbf{Set}$$

be a functor such that, for every $X \in \mathbb{X}$ the functor $F(X,-,=)$ has an end. Then

$$\int_{Y\in\mathbb{Y}} F(-,Y,Y)\colon \mathbb{X} \longrightarrow \mathbf{Set}$$

is a functor.

The dual statement also holds.

The reader should be advised that the more common presentation of ends and coends is via a universal property which is more general then our presentation here. The presentation here, I feel, makes developing intuition about what ends and coends are, easier. For a more standard presentation see [17] or [14] for the enriched version.

1.4 Monoids and Modules

In this section we ignore the 'usual' definition of a set theoretic monoid and consider instead the categorified concept. See [17] for more details.

Definition 1.4.1. A *monoidal product* on a category, $\mathbb{X}$, is a functor $\otimes\colon \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ together with an object $E \in \mathbb{X}$ and natural isomorphisms

1. $\lambda \colon E \otimes - \Longrightarrow 1_{\mathbb{X}}$

2. $\rho \colon - \otimes E \Longrightarrow 1_{\mathbb{X}}$

3. $\text{assoc} \colon (- \otimes -) \otimes - \Longrightarrow - \otimes (- \otimes -)$

which, for all objects W, X, Y, and Z, the diagrams

$$
\begin{array}{ccc}
((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{\text{assoc}} & (W \otimes X) \otimes (Y \otimes Z) \\
\downarrow{\scriptstyle \text{assoc}\otimes 1} & & \\
(W \otimes (X \otimes Y)) \otimes Z & & \downarrow{\scriptstyle \text{assoc}} \\
\downarrow{\scriptstyle \text{assoc}} & & \\
W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow[1\otimes\text{assoc}]{} & W \otimes (X \otimes (Y \otimes Z))
\end{array}
\qquad
\begin{array}{ccc}
(X \otimes E) \otimes Y & \xrightarrow{\text{assoc}} & X \otimes (E \otimes Y) \\
{\scriptstyle \rho_X \otimes 1_Y} \searrow & & \swarrow {\scriptstyle 1_X \otimes \lambda_Y} \\
& X \otimes Y &
\end{array}
$$

commute. If, further, there is a natural isomorphism $\text{twist}_{X,X'} \colon X \otimes X' \longrightarrow X' \otimes X$ such that

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\text{twist}_{X,Y}} & Y \otimes X \\
& {\scriptstyle 1_{X\otimes Y}} \searrow & \downarrow{\scriptstyle \text{twist}_{Y,X}} \\
& & X \otimes Y
\end{array}
\qquad
\begin{array}{ccc}
X \otimes E & \xrightarrow{\text{twist}_{X,E}} & E \otimes X \\
& {\scriptstyle \rho_X} \searrow & \downarrow{\scriptstyle \lambda_X} \\
& & X
\end{array}
\qquad
\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{\text{twist}\otimes 1} & (Y \otimes X) \otimes Z \\
\downarrow{\scriptstyle \text{assoc}} & & \downarrow{\scriptstyle \text{assoc}} \\
X \otimes (Y \otimes Z) & & Y \otimes (X \otimes Z) \\
\downarrow{\scriptstyle \text{twist}} & & \downarrow{\scriptstyle 1_Y \otimes \text{twist}} \\
(Y \otimes Z) \otimes X & \xrightarrow[\text{assoc}]{} & Y \otimes (Z \otimes X)
\end{array}
$$

commute then we call $(\otimes, E, \lambda, \rho, \text{assoc}, \text{twist})$ a *symmetric monoidal product*.

Note at this point that, in [17], the diagrams defiining a monoidal category have arrows going opposite the ones presented above. Be advised that these two presentations are equivalent.

If $\mathbb{X}$ has a a monoidal product $(\otimes, E, \lambda, \rho, \text{assoc})$ then call $(\mathbb{X}, \otimes, E, \lambda, \rho, \text{assoc})$ a *monoidal category*. If $(\otimes, E, \lambda, \rho, \text{assoc}, \text{twist})$ is a symmetric monoidal product then call

$$(\mathbb{X}, \otimes, \lambda, \rho, \text{assoc}, \text{twist})$$

a *symmetric monoidal category*. We will reserve the symbols $\lambda, \rho, \text{assoc}$, and twist for monoidal and symmetric monoidal products. As such, we will often refer to the (symmetric) monoidal product $(\otimes, E)$, or even just the (symmetric) monoidal product $\otimes$ and leave the omitted morphisms as understood. Note that the commutativity of the above allows us to refer to the object $X \otimes Y \otimes Z$ without having to parenthesize.

Remark. Let $\mathbb{X}$ be a monoidal category with monoidal product $\otimes$ and let $X, Y, Z \in \mathbb{X}$. If we have morphisms $f \colon X \otimes Y \longrightarrow Y$ and $g \colon Y \otimes Z \longrightarrow Y$ in $\mathbb{X}$ then notice that the diagram

$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{f \otimes Z} & Y \otimes Z \\ {\scriptstyle X \otimes g}\downarrow & & \downarrow{\scriptstyle g} \\ X \otimes Y & \xrightarrow[f]{} & Y \end{array}$$

isn't well-defined since $X \otimes Y \otimes Z$ is usually ambiguous notation. In spite of this, say the above diagram *commutes* if the well-defined diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\mathrm{assoc}^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{f \otimes Z} & Y \otimes Z \\ {\scriptstyle X \otimes g}\downarrow & & & & \downarrow{\scriptstyle g} \\ X \otimes Y & & \xrightarrow[f]{} & & Y \end{array}$$

commutes in the usual sense. Adopt this convention for other diagrams of this type.

Example 1.4.2. Let Σ_n denote the symmetric group on n-elements - in particular Σ_0 is the one-element group. Let Σ denote the category defined by ...

... objects: the objects are the natural numbers $n \in \mathbb{N}$

... morphisms: the morphisms of Σ are

$$\Sigma(n, m) = \begin{cases} \Sigma_n, & m = n \\ \emptyset, & \text{otherwise} \end{cases}$$

... identities: the identities of Σ are the identity elements of Σ_n for each $n \in \mathbb{N}$

... composition: the compostion of Σ is given by the group operation of Σ_n for each $n \in \mathbb{N}$

There is a symmetric monoidal product on Σ given by...

... the functor: $+\colon \Sigma \times \Sigma \longrightarrow \Sigma$ defined by

$$(n, m) \longmapsto n + m$$

$$(\alpha, \beta) \in \Sigma_n \times \Sigma_m \longmapsto \alpha \times \beta \in \Sigma_{n+m}.$$

... the identity object $0 \in \Sigma$.

... For each $n \in \Sigma$ the morphism $\lambda_n := 1_n \colon n + 0 \longrightarrow n$

... For each $n \in \Sigma$ the morphism $\rho_n := 1_n \colon 0 + n \longrightarrow n$

... For every $k, n, m \in \mathbb{N}$ the morphism

$$\mathrm{assoc}_{k,m,n} := 1_{k+m+n} \colon k + (m + n) \longrightarrow (k + m) + n$$

... For every $n, m \in \mathbb{N}$ the morphism $\mathrm{twist}_{m,n} := 1_{m+n} \colon m + n \longrightarrow n + m$

That these morphisms satisfy the necessary diagrams follows easily from the fact that *all* of the morphisms are just the identities, hence all the diagrams are equivalent to the statement "the composite of identity morphisms is an identity morphism," which is true.

Example 1.4.3. The category of abelian groups **Ab** with the following data

- The tensor functor $\otimes\colon \mathbf{Ab} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}$, see [13]
- The identity object $(\mathbb{Z}, +)$
- For each $A \in \mathbf{Ab}$ the morphism $\lambda_A \colon A \otimes \mathbb{Z} \longrightarrow A$ given, on generators, by $a \otimes n \longmapsto n \cdot a := \sum_{k=1}^{n} a$
- For each $A \in \mathbf{Ab}$ the morphism $\rho_A \colon \mathbb{Z} \otimes A \longrightarrow A$ given, on generators, by $n \otimes a \longmapsto n \cdot a$
- For every $A, A', A'' \in \mathbf{Ab}$ the morphism

  $$\mathrm{assoc}_{A,A',A'} \colon (A \otimes A') \otimes A'' \longrightarrow A \otimes (A' \otimes A'')$$

  given, on generators, by

  $$(a \otimes a') \otimes a'' \longmapsto a \otimes (a' \otimes a'').$$

- For every $A, A' \in \mathbf{Ab}$ the morphism twist: $A \otimes A' \longrightarrow A' \otimes A$ given, on generators, by $a \otimes a' \longmapsto a' \otimes a$

Definition 1.4.4. Let $\otimes\colon \mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{X}$ -together with the appropriate natural isomorphisms- be a symmetric monoidal product. If, for all objects $X \in \mathbb{X}$ the functor $X \otimes -\colon \mathbb{X} \longrightarrow \mathbb{X}$ has a right adjoint, denoted $[X, -]$, then call $\mathbb{X}$ a *closed symmetric monoidal category.* Call the object $[X, Y]$ the *internal hom-object for* X *and* Y.

Example 1.4.5. For the category **Ab** the symmetric monoidal product $\otimes$ is closed with $\mathrm{Hom}(A, A')$ the abelian group of homomorphisms from A to A' with operation given by point-wise addition, and the constant function to the identity of A' the identity. To see the proof of the adjunction see [13]

Definition 1.4.6. Let $(\mathbb{X}, \otimes, E)$ be a monoidal category, then a *monoid* in $\mathbb{X}$ is a triple $(M, \mu\colon M \otimes M \longrightarrow M, \eta\colon E \longrightarrow M)$ such that the following diagrams commute

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{1_M \otimes \mu} & M \otimes M \\ {\scriptstyle \mu \otimes 1_m}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \otimes M & \xrightarrow[\mu]{} & M \end{array} \qquad \begin{array}{ccccc} E \otimes M & \xrightarrow{\eta \otimes 1_M} & M \otimes M & \xleftarrow{1_M \otimes \eta} & M \otimes E \\ & {\scriptstyle \lambda_M}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \rho_X} & \\ & & M & & \end{array}$$

Given two monoids (M, μ, η) and (M', μ', η') in $(\mathbb{X}, \otimes, E)$ a *monoid morphism*

$$h\colon (M, \mu, \eta) \longrightarrow (M', \mu', \eta')$$

is a morphism

$$h\colon M \longrightarrow M'$$

such that the following diagrams commute.

$$\begin{array}{ccc} M \otimes M & \xrightarrow{h \otimes h} & M' \otimes M' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ M & \xrightarrow[h]{} & M' \end{array} \qquad \begin{array}{ccc} & M & \\ {\scriptstyle \eta}\nearrow & & \searrow{\scriptstyle h} \\ E & \xrightarrow[\eta']{} & M' \end{array}$$

Example 1.4.7. In the category **Set** the axioms defining a monoid are exactly the same as those defining a monoid in the sense of abstract algebra with μ being the operation and $\eta\colon \{0\} \longrightarrow M$ picking out the identity element.

In the category **Ab** with symmetric monoidal product the tensor product we get that these same axioms are equivalent to the axioms of a unital ring. See [17] page 4.

Once we have monoids in a category, it becomes a natural thing to let these monoids act on other objects.

Definition 1.4.8. Given a monoidal category $(\mathbb{X}, \otimes, E)$ and a monoid (M, μ, η), a left M-module is a morphism $m\colon M \otimes R \longrightarrow R$ such that the following diagrams commute

$$\begin{array}{ccc} M \otimes M \otimes R & \xrightarrow{\mu \otimes 1_R} & M \otimes R \\ {\scriptstyle 1_M \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ M \otimes R & \xrightarrow[m]{} & R \end{array} \qquad \begin{array}{ccc} E \otimes R & \xrightarrow{\eta \otimes 1_R} & M \otimes R \\ & {\scriptstyle \lambda_R}\searrow & \downarrow{\scriptstyle m} \\ & & R \end{array}$$

A morphism $h\colon m \longrightarrow m'$ between modules $m\colon M \otimes R \longrightarrow R$ and $m'\colon M \otimes R' \longrightarrow R'$ is a morphism $h\colon R \longrightarrow R$ in $\mathbb{X}$ such that

$$\begin{array}{ccc} M \otimes R & \xrightarrow{M \otimes h} & M \otimes R' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ R & \xrightarrow[h]{} & R' \end{array}$$

commutes.

Right M-modules are defined analogously.

Example 1.4.9. In the category **Set** a monoid R is a monoid in the usual sense, and the axioms above describe a left action of a monoid R on a set M.

In the category **Ab**, we got above that a monoid is exactly a unital ring, so if R is a monoid in **Ab**, then a left R-module in **Ab**, in the categorical sense, is precisely a left R-module in the sense of ring theory. See [17].

Definition 1.4.10. Let $(\mathbb{X}, \otimes, E)$ be a monoidal category and let (M, μ, η) be a monoid in $(\mathbb{X}, \otimes, E)$. Then there is a category ${}^M\mathbb{X}$ whose objects are left modules $m\colon M \times R \longrightarrow R$ and whose morphisms are morphisms of left modules as per Definition 1.4.8.

The identity morphisms and composition are inherited from $\mathbb{X}$. There is a forgetful functor $\mathrm{U}\colon {}^M\mathbb{X} \longrightarrow \mathbb{X}$ which takes a module $m\colon M \otimes R \longrightarrow R$ to R and which is the identity function on morphisms.

We say that ${}^M\mathbb{X}$ inherits (co)limits from $\mathbb{X}$ if U preserves (co)limits.

Proposition 1.4.11. *Let $(\mathbb{X}, \otimes, E)$ be a complete and cocomplete, closed symmetric monoidal category. If (M, μ, η) is a commutative monoid in $\mathbb{X}$ and ${}^M\mathbb{X}$ is the category of M-modules then ${}^M\mathbb{X}$ inherits limits and colimits from $\mathbb{X}$ and is itself closed symmetric monoidal with monoidal product*

$$\otimes_M\colon {}^M\mathbb{X} \times {}^M\mathbb{X} \longrightarrow {}^M\mathbb{X}$$

defined on modules $\alpha\colon M \otimes A \longrightarrow A$ and $\beta\colon M \otimes B \longrightarrow B$ by

$$A \otimes_M B := \text{coequalizer}(\ A \otimes M \otimes B \underset{\beta}{\overset{\text{twist}^{\otimes} \circ \alpha}{\rightrightarrows}} A \otimes B\ )$$

Given modules homomorphisms $f\colon \alpha \longrightarrow \alpha'$ and $g\colon \beta \longrightarrow \beta'$ the morphism $f \otimes_M g$ is given by the universal property of the coequalizer, namely:

$$\begin{array}{ccccc} A \otimes M \otimes B & \underset{\beta}{\overset{\text{twist} \circ \alpha}{\rightrightarrows}} & A \otimes B & \xrightarrow{\iota_{A,B}} & A \otimes_M B \\ \downarrow{\scriptstyle f \otimes 1_m \otimes g} & & \downarrow{\scriptstyle f \otimes g} & & \downarrow{\scriptstyle f \otimes_M g} \\ A' \otimes M \otimes B' & \underset{\beta'}{\overset{\text{twist} \circ \alpha'}{\rightrightarrows}} & A' \otimes B' & \xrightarrow{\iota_{A',B'}} & A' \otimes_M B' \end{array} .$$

For a proof of this Proposition see [19]. Proposition 1.2.14 there proves that ${}^M\mathbb{X}$ is complete and cocomplete, Proposition 1.2.15 proves that ${}^M\mathbb{X}$ is symmetric monoidal with the given monoidal product, and Proposition 1.2.17 gives the closed structure.

1.5 The Day Convolution

Definition 1.5.1 ([14])**.** Let $\mathcal{V} = (\mathbb{V}, \otimes, E)$ be a monoidal category. A $\mathcal{V}$-*category* $\mathbb{A}$, also called a $\mathcal{V}$-enriched category, consists of

1. a class $\operatorname{ob}\mathbb{A}$ of objects
2. for each ordered pair $(A, A') \in \operatorname{ob}\mathbb{A} \times \operatorname{ob}\mathbb{A}$ a *hom-object* $\mathbb{A}(A, A') \in \mathbb{V}$
3. for each $A \in \operatorname{ob}\mathbb{A}$ an *identity object* $1_A \colon E \longrightarrow \mathbb{A}(A, A)$
4. for each ordered triple $(A, A', A'') \in \operatorname{ob}\mathbb{A}\times\operatorname{ob}\mathbb{A}\times\operatorname{ob}\mathbb{A}$ a *composition morphism*

$$c_{A,A',A''} \colon \mathbb{A}(A, A') \otimes \mathbb{A}(A', A'') \longrightarrow \mathbb{A}(A, A'')$$

making the following diagrams commute

$$\begin{array}{ccc}
\mathbb{A}(A, A') \otimes \mathbb{A}(A', A'') \otimes \mathbb{A}(A'', A''') & \xrightarrow{1\otimes c_{A',A'',A'''}} & \mathbb{A}(A, A') \otimes \mathbb{A}(A', A''') \\
{\scriptstyle c_{A,A',A''}\otimes 1}\downarrow & & \downarrow{\scriptstyle c_{A,A',A'''}} \\
\mathbb{A}(A, A'') \otimes \mathbb{A}(A'', A''') & \xrightarrow[c_{A,A'',A'''}]{} & \mathbb{A}(A, A''')
\end{array}$$

$$\begin{array}{ccc}
E \otimes \mathbb{A}(A, A') & \xrightarrow{1_A\otimes 1_{\mathbb{A}(A,A')}} & \mathbb{A}(A, A) \otimes \mathbb{A}(A, A') \\
& {\scriptstyle \lambda_{\mathbb{A}(A,A')}}\searrow & \downarrow{\scriptstyle c_{A,A,A'}} \\
& & \mathbb{A}(A, A')
\end{array}$$

$$\begin{array}{ccc}
\mathbb{A}(A, A') \otimes \mathbb{A}(A', A') & \xleftarrow{1_{\mathbb{A}(A,A')}\otimes 1_{A'}} & \mathbb{A}(A, A') \otimes E \\
{\scriptstyle c_{A,A',A'}}\downarrow & \swarrow{\scriptstyle \rho_{\mathbb{A}(A,A')}} & \\
\mathbb{A}(A, A') & &
\end{array}$$

Example 1.5.2. Every category, as defined above, is a $(\mathbf{Set}, \times, \top)$-category.

Example 1.5.3. Given any ($\mathbf{Set}$-enriched) category $\mathbb{X}$ we may enrich $\mathbb{X}$ over $(\mathbf{Top}, \times, \top)$ by giving the hom-sets $\mathbb{X}(X, Y)$ the discrete topology. Since $\top$ has the descrete topology in $\mathbf{Top}$, since the finite product of discrete spaces is discrete, and since every function from a discrete space is continuous, we may use the composition and identity morphisms from $\mathbf{Set}$. The diagrams above clearly commute in this case.

Definition 1.5.4 ([14])**.** A $\mathcal{V}$-functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{V}$-categories consists of

1. A function on objects $\operatorname{ob} F\colon\ \operatorname{ob}\mathbb{A} \longrightarrow \operatorname{ob}\mathbb{B}$
2. A collection of morphisms in $\mathbb{V}$: $F_{A,A'}\colon \mathbb{A}(A,A') \longrightarrow \mathbb{B}(F(A),F(A'))$ such that the following diagrams commute for all $A, A' \in \mathbb{A}$

$$
\begin{array}{ccc}
\mathbb{A}(A,A')\otimes\mathbb{A}(A',A'') & \xrightarrow{\ c\ } & \mathbb{A}(A,A'') \\
\Big\downarrow {\scriptstyle F_{A,A'}\otimes F_{A',A''}} & & \Big\downarrow {\scriptstyle F_{A,A''}} \\
\mathbb{B}(F(A),F(A'))\otimes\mathbb{B}(F(A'),F(A'')) & \longrightarrow & \mathbb{B}(F(A),F(A''))
\end{array}
\qquad
\begin{array}{ccc}
 & & \mathbb{A}(A,A) \\
 & \nearrow^{1_A} & \Big\downarrow {\scriptstyle F_{A,A}} \\
I & \searrow_{1_{F(A)}} & \\
 & & \mathbb{B}(F(A),F(A))
\end{array}
$$

Example 1.5.5. Notice that, in the case of $(\mathbf{Set},\times,\top)$ the right-hand Diagram of 2 is equivalent to the statement that F respects composition, and the right-hand diagram is equivalent to the statement that F respects identities. Hence every functor, as defined previously, is a $(\mathbf{Set},\times,\{0\})$-functor in this sense.

Definition 1.5.6 ([14])**.** Given $\mathcal{V}$-functors $F,G\colon \mathbb{A} \longrightarrow \mathbb{B}$ we can define a $\mathcal{V}$-natural transformations $\alpha\colon F \Longrightarrow G$ to be an $\operatorname{ob}\mathbb{A}$-indexed familiy of morphisms $\alpha_A\colon E \longrightarrow \mathbb{B}(F(A),G(A))$ in $\mathbb{A}$ such that the following diagram commutes.

$$
\begin{array}{ccccc}
 & & E\otimes\mathbb{A}(A,A') & \xrightarrow{\ \alpha_A\otimes G_{A,A'}\ } & \mathbb{B}(F(A),G(A))\otimes\mathbb{B}(G(A),G(A')) \\
 & \nearrow^{\lambda^{-1}} & & & \Big\downarrow {\scriptstyle c} \\
\mathbb{A}(A,A') & & & & \mathbb{B}(F(A),G(A')) \\
 & \searrow_{\rho^{-1}} & & & \Big\uparrow {\scriptstyle c} \\
 & & \mathbb{A}(A,A')\otimes E & \xrightarrow[\ F_{A,A'}\otimes\alpha_{A'}\ ]{} & \mathbb{B}(F(A),F(A'))\otimes\mathbb{B}(F(A'),G(A'))
\end{array}
$$

Example 1.5.7. As before in $(\mathbf{Set},\times,\top)$, this diagram is a diagrammatization of the axioms for a natural transformation in the usual sense, hence the $(\mathbf{Set},\times,\top)$-natural transformations are exactly the usual natural transformations.

Definition 1.5.8. If $\mathcal{V}$ is a symmetric monoidal category and $\mathbb{X}$ is a $\mathcal{V}$-category, and

$$F\colon \mathbb{X}^{\mathrm{op}}\times\mathbb{X} \longrightarrow \mathcal{V}$$

is a $\mathcal{V}$-functor, then we may define the $\mathcal{V}$-*coend* as in Definition 1.3.29 except with **Set** replaced with $\mathcal{V}$. If $\mathcal{V}$ is also closed then we may likewise define the $\mathcal{V}$-*end*.

Theorem 1.5.9 (Day convolution, [4])**.** *Let $\mathcal{V}$ be a complete, cocomplete, closed symmetric monoidal category and let $(\mathbb{X}, \Box, U)$ be a monoidal $\mathcal{V}$-category. The functor category $\mathcal{V}^{\mathbb{X}}$ is a closed symmetric monoidal $\mathcal{V}$-category with monoidal product given by the double coend*

$$F \star G(-) := \int^{Y,Z} F(Y) \otimes G(Z) \otimes \mathbb{X}(Y \Box Z, -).$$

This monoidal product is called the Day convolution*.*

Chapter 2

Simplicial Sets

Introduction

2.1 Basics

Let Δ denote the category defined as follows:

- objects: the ordered sets $[n] := \{0 < 1 < \cdots < n\}$ for each natural number $n \in \mathbb{N}$
- morphisms: a morphism $f\colon [n] \longrightarrow [m]$ is a weakly order preserving function, i.e. if $i \leq j$ then $f(i) \leq f(j)$.
- composition: the usual composition of functions
- identities: the usual identity functions

Call this the *simplicial category.*

Definition 2.1.1. Let Δ^n denote the *n-simplex*

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_{k=0}^{n} x_k = 1, \forall i \in [n]\, x_i \geq 0\}$$

given the subspace topology. For each $0 \leq k \leq n$ define the kth facet of Δ^n to be the subspace

$$\Delta^n_k := \{(x_0, \ldots, x_n) \in \Delta^n | x_k = 0\}$$

Remark. Notice that the kth facet of Δ^n is linearly homeomorphic to Δ^{n-1}, so we can consider the facets of Δ^n_k to get $(n-2)$-dimensional faces of Δ^n and so on.

Definition 2.1.2. The *simplex functor*

$$\Delta^{-} \colon \Delta \longrightarrow \mathbf{Top}$$

is given by

$$[n] \longmapsto \Delta^n$$

and

$$\begin{aligned}(f\colon [n] \longrightarrow [m]) \longmapsto (\Delta^f \colon \Delta^n \longrightarrow \Delta^m) \\ (x_0, \ldots, x_n) \mapsto \left(\sum_{i \in f^{-1}(0)} x_i, \sum_{i \in f^{-1}(1)} x_i, \ldots, \sum_{i \in f^{-1}(m)} x_i \right)\end{aligned}$$

Intuitively, Δ^f affinely sends the facet of Δ^n indexed by k to the face of Δ^m spanned by the elementary vectors $\{e_i | i \in \operatorname{im} f \setminus \{f(k)\}\}$.

That Δ^{-} sends identities to identities follows from the fact $\sum_{i \in 1_{[n]}^{-1}(j)} x_i = x_j$. That Δ^{-} respects composition is straightforwards but tedious.

Definition 2.1.3. Define a *simplicial set* to be a functor

$$X \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

Denote the set $X([n])$ by X_n and the image of the function $f\colon [n] \longrightarrow [m]$ by $X(f)\colon X_m \longrightarrow X_n$. Since Δ is small there is a functor category $\mathbf{Set}^{\Delta^{\mathrm{op}}}$. Denote this category by **sSet**, called the *category of simplicial sets*.

Example 2.1.4. Denote by $\Delta[n] := \Delta(-, [n])\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. Call this simplicial set the *nth simplicial simplex*.

Example 2.1.5. Given a set X define the *discrete simplicial set* on X to be the constant functor $\mathrm{c}_X \colon \Delta^{\mathrm{op}} \longrightarrow \mathbb{X}$.

Definition 2.1.6. Let G be a group, a *G-simplicial set* is a G-object in the category **sSet**, equivalently a G-simplicial set, X, is a functor

$$X \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{G\text{-}Set}$$

Intuitively the sets X_n of a simplicial set $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ should be thought of as a set of n-simplexes, functions $X(f)\colon X_n \longrightarrow X_m$ then describe how to collapse and include the n-simplexes of X_n into the m-simplexes X_m. This intuition will made formal by the *geometric realization functor* of Definition 2.1.7 below, but first we need the following.

Recall from Example 1.5.3 that any **Set**-enriched category, $\mathbb{X}$, is also **Top**-enriched by taking the hom-sets of $\mathbb{X}$ to be discrete topological spaces. In paritcular, the category Δ is **Top**-enriched, thus for a fixed simplicial set

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

we may post-compose with the functor

$$\mathrm{Disc}\colon \mathbf{Set} \longrightarrow \mathbf{Top}$$

which sends a set to the discrete space on that set. This gives us a simplicial space

$$X \circ \mathrm{Disc}\colon \mathbf{Set} \longrightarrow \mathbf{Top}$$

which we may cross with the simplex functor to obtain a functor

$$(X \circ \mathrm{Disc}) \times \Delta^{-}\colon \Delta^{\mathrm{op}} \times \Delta \longrightarrow \mathbf{Top}$$

where Δ is considered as **Top**-enriched. Thus we may define the coend

$$\int^{[n]\in\Delta} X_n \times \Delta^n$$

as per Definition 1.5.8. Given that, we can make the following definition.

Definition 2.1.7. Define the *geometric realization* functor $|-|\colon \mathbf{sSet} \longrightarrow \mathbf{Top}$ by

$$(X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}) \longmapsto \int^{[n]\in\Delta} X_n \times \Delta^n$$

on objects, and to send a morphism of **sSet** to map given by the universal property of coequalizers as usual.

Example 2.1.8. There is an isomorphism

$$|\Delta[n]| \cong \Delta^n$$

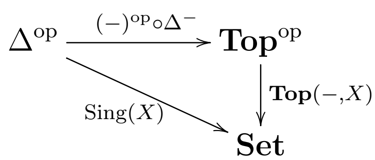
for each $n \in \mathbb{N}$.

Example 2.1.9. For any set X the geometric realization of the discrete simplicial set on X is the discrete space on X:

$$|\,c_X\,| \cong \operatorname{Disc} X.$$

There is also a functor in the other direction:

Definition 2.1.10. Define the functor $\operatorname{Sing}\colon \mathbf{Top} \longrightarrow \mathbf{sSet}$, called *singularization*, by setting $\operatorname{Sing}(X)$ to be the composite:



and similarly for morphisms. Call the simplicial set $\operatorname{Sing}(X)$ the *singular complex of* X.

Example 2.1.11.

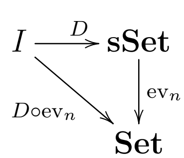
$$\operatorname{Sing}(\{0\}) \cong \Delta[0]$$

Theorem 2.1.12 (See [7] Proposition I.2.2)**.**

$$|-| \dashv \operatorname{Sing}$$

Definition 2.1.13. For each $n \in \mathbb{N}$ let $\mathrm{ev}_n\colon \mathbf{sSet} \longrightarrow \mathbf{Set}$ be the *evaluation at degree n* functor, defined by sending the simplicial set X to the set X_n and the natural transformation $\eta\colon X \longrightarrow Y$ to its nth component $\eta_n\colon X_n \longrightarrow Y_n$.

For any functor $D\colon I \longrightarrow \mathbf{sSet}$ then, for each $n \in \mathbb{N}$, let D_n be the composite in the following diagram



Theorem 2.1.14. *The category* **sSet** *is complete and cocomplete, further limits and colimits are computed pointwise.*

Proof. This is a direct consequence of Proposition 1.3.21 and the completeness and cocompleteness of **Set**.

□

Definition 2.1.15. We can also define the category of *pointed simplicial sets* to be the functor category

$$\mathbf{sSet}_* := \mathbf{Set}_*^{\Delta^{\mathrm{op}}}.$$

Example 2.1.16. Define the n^{th} pointed simplicial simplex, $\Delta_*[n]$, to be given by

$$\Delta_*[n]_k := (\Delta([k],[m]), \mathrm{c}_m \colon [k] \longrightarrow [m])$$

where $\mathrm{c}_m(i) = m$ for all $i \in [k]$.

Example 2.1.17. Let

$$S^0 = (\{-1, 1\}, 1)$$

be an object of $\mathbf{Set}_*$, denote by $\mathcal{S}^0$ the constant functor

$$\Delta \longrightarrow \mathbf{Set}_*$$

which picks out S^0, i.e. $\mathcal{S}^0_n = S^0$ and

$$\mathcal{S}^0(\alpha \colon [n] \longrightarrow [m]) = 1_{S^0} \colon S^0 \longrightarrow S^0.$$

Call this pointed simplicial set the *simplicial* 0*-sphere.*

Definition 2.1.18. Recall that $\Delta_*[1]$ is the simplicial set

$$\Delta(-,[1]) = \big\{\text{All order preserving functions from the input to the ordinal } \{0 \leq 1\}\big\}$$

together with the constant functions

$$\mathrm{c}_1 \colon [n] \longrightarrow [1]$$

as the basepoints of the various levels. Consider the graded-function

$$i\colon \mathcal{S}^0 \longrightarrow \Delta[1]$$

given pointwise by taking

$$i_n\colon \mathcal{S}^0_n \longrightarrow \Delta[1]$$

to be the function sending the element

$$-1 \in \mathcal{S}^0 = (\{-1, 1\}, 1)$$

to the constant function

$$\mathrm{c}_0\colon [n] \longrightarrow [1]$$

and the basepoint

$$1 \in \mathcal{S}^0$$

to the basepoint

$$\mathrm{c}_1\colon [n] \longrightarrow [1]$$

of $\Delta[1]$. This is a morphism of simplicial sets since pre- and post- composing a constant function by any function is again a constant function. Now define the simplicial set

$$\mathcal{S}^1 := \operatorname{coker}(i\colon \mathcal{S}^0 \longrightarrow \Delta[1]_*)$$

called the *simplicial* 1-*sphere*.

As with the category of simplicial sets we have the geometric realization and singularization functors

$$\mathbf{sSet}_* \underset{\text{Sing}}{\overset{|-|}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Top}_*$$

as well as completeness, cocompleteness, pointwise evaluation of limits, etc. Further, since $\mathbf{Set}_*$ has a zero object, so does $\mathbf{sSet}_*$. Thus there is the canonical map, denoted τ

$$X \amalg Y \xrightarrow{\langle\langle 1_X|0\rangle, \langle 0|1_Y\rangle\rangle} X \times Y$$

Example 2.1.19. The geometric realization of the simplicial 0-sphere is homeomorphic to the usual 0-sphere

$$|\mathcal{S}^0| \cong S^0.$$

Definition 2.1.20. Given pointed simplicial sets $X, Y \in \mathbf{sSet}_*$ define

$$X \wedge Y := \operatorname{coker}\left(X \amalg Y \xrightarrow{\tau} X \times X \right) = X \times Y / X \amalg Y.$$

Call $X \wedge Y$ the *smash product of* X *and* Y. Denote the canonical map

$$X \times Y \longrightarrow X \wedge Y$$

by $q_{X,Y}$.

Given pointed simplicial sets X and Y let $x \in X_n$ and $y \in Y_n$ for some $n \in \mathbb{N}$. Denote the image of

$$(x, y) \in X_n \times Y_n = (X \times Y)_n$$

under the quotient map $X \times Y \longrightarrow X \wedge Y$ by $x \wedge y$.

Definition 2.1.21. Let G be a group, a *G-simplicial set*, X, is a G-object in the category of simplicial sets. Equivalently, a G-simplicial set is a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{G\text{-}Set}$$

where **G-Set** is the category of G-sets and G-equivariant functions.

Definition 2.1.22. Let X and Y be G-simplicial sets, then the smash product $X \wedge Y$ has a G-action given by $g \cdot (x \wedge y) = (g \cdot x) \wedge (g \cdot y)$. Call this action the *diagonal action of* G *on* $X \wedge Y$.

Denote by $X \wedge_G Y$ the simplicial set formed by quotienting $X \wedge Y$ by the diagonal action of G.

Theorem 2.1.23. *The smash product* $\wedge$ *is a symmetric monoidal product on* **sSet** *with unit* $\mathcal{S}^0$.

Proof Sketch. First we show that

$$X \wedge (Y \wedge Z) \cong \frac{X \times Y \times Z}{(X \times Y) \amalg (X \times Z) \amalg (Y \times Z)}$$

naturally and similarly

$$(X \wedge Y) \wedge Z \cong \frac{X \times Y \times Z}{(X \times Y) \amalg (X \times Z) \amalg (Y \times Z)}$$

naturally, hence $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ naturally.

Then define...

... $\text{twist}^{\wedge}_{X,Y}$ via the diagram

$$\begin{array}{ccccc}
X \amalg Y & \xrightarrow{\tau_{X,Y}} & X \times Y & \longrightarrow & X \wedge Y \\
\downarrow{\scriptstyle \text{flip}} & & \downarrow{\scriptstyle \text{twist}^{\times}} & & \vdots{\scriptstyle \exists!} \\
Y \amalg X & \xrightarrow[\tau_{Y,X}]{} & Y \times X & \longrightarrow & Y \wedge X
\end{array}$$

... $\text{assoc}^{\wedge}_{X,Y,Z}$ via the pair of natural isomorphisms above

... $\lambda^{\wedge}_{X}$ via the computation (in **Set**):

$$\begin{aligned}
(X \wedge \mathcal{S}^0)_n &= (X_n \times \mathcal{S}^0_n)/(X_n \amalg \mathcal{S}^0_n) \\
&= (X \times \{-1, 1\})/(X \sqcup \{-1\}) \\
&\cong X \times \{1\} \\
&\cong X
\end{aligned}$$

we can then check that the appropriate diagrams commute.

□

Remark. It will often be convenient to choose the quotient

$$\left(\prod_{i \in I} X_i\right) / \coprod_{i \in I} (\amalg_{j \neq i} X_j)$$

as *the* model for $\bigwedge_{i \in I} X_i$.

Definition 2.1.24. For each pair of pointed simplicial sets X and Y define the *mapping complex* from X to Y to be the simplicial set

$$\mathrm{maps}(X,Y) := \mathbf{sSet}_*(X \wedge \Delta[-], Y)\colon \Delta \longrightarrow \mathbf{Set}_*$$

Theorem 2.1.25 (See [7])**.** *For every pointed simplicial set X there is an adjunction*

$$- \wedge X \dashv \mathrm{maps}(X, -)$$

In particular, the smash product is a closed symmetric monoidal product on the category of simplicial sets.

Definition 2.1.26. For integers $n \geq 2$ define the *simplicial n-sphere* by

$$\mathcal{S}^n := \bigwedge_{i=1}^{n} \mathcal{S}^1.$$

Example 2.1.27.

$$|\mathcal{S}^n| \cong S^n$$

Definition 2.1.28. For any permutation $\sigma \in \Sigma_n$ define the map $\sigma\cdot\colon \mathcal{S}^n \longrightarrow \mathcal{S}^n$ in level k by

$$\sigma_k\colon \mathcal{S}^n_k \longrightarrow \mathcal{S}^n_k$$
$$s_1 \wedge \cdots \wedge s_n \longmapsto s_{\sigma(1)} \wedge \cdots \wedge s_{\sigma(n)}$$

Together these maps assemble into a Σ_n action on $\mathcal{S}^n$ for each $n \in \mathbb{N}$. Call this the *standard action of Σ_n on $\mathcal{S}^n$*. To see that this action is well-defined notice that, for a fixed $\sigma \in \Sigma_n$ there is a morphism

$$\prod_{1 \leq i \leq n} \mathcal{S}^1 \longrightarrow \prod_{1 \leq i \leq n} \mathcal{S}^1$$

given by

$$(s_1, \ldots, s_n) \longmapsto (s_{\sigma(1)}, \ldots, s_{\sigma(n)})$$

induced by the universal property of the product. Notice further that, in general, $s_k = *$ for some $1 \leq k \leq n$ if and only if

$$(s_1, \ldots, s_n) \in \operatorname{im}\left(\kappa\colon \prod_{1\leq i\leq n} \prod_{j\neq i} \mathcal{S}^1 \longrightarrow \prod_{1\leq i\leq n} \mathcal{S}^1\right)$$

holds. Thus, $(s_1, \ldots, s_n) \in \operatorname{im}\kappa$ if and only if $(s_{\sigma(1)}, \ldots, s_{\sigma(n)}) \in \operatorname{im}\kappa$ so, by the universal property of the cokernel, we get our desired morphism

$$\mathcal{S}^n \longrightarrow \mathcal{S}^n.$$

Definition 2.1.29. Call the functor $\mathcal{S}^1 \wedge -\colon \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$ the *suspension* functor. Call its adjoint $\Omega := \operatorname{maps}(\mathcal{S}^1, -)\colon \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$ the *loops* functor.

Given a map $f\colon \mathcal{S}^1 \wedge X \longrightarrow Y$ let $\widetilde{f}\colon X \longrightarrow \Omega(Y)$ denote its adjoint.

Chapter 3

Model Categories

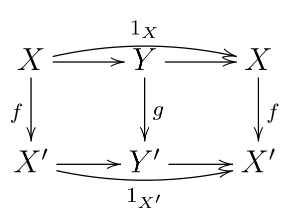
Introduction

This presentation is condensed from the presentations given in [9] and [5]. All the results and proofs presented in this chapter are standard and can be found in any introductory reference such as [[9], [5], [11]].

3.1 Basics

Definition 3.1.1. A *model category* is a category $\mathbb{X}$ together with three -composition closed- distinguished class of morphisms of $\mathbb{X}$ called *fibrations, cofibrations*, and *weak equivalences* satisfying the following axioms

1. $\mathbb{X}$ is complete and cocomplete
2. (2-out-of-3 Axiom)If f and g are composable maps in $\mathbb{X}$ such that two of f, g, and fg are weak equivalences then so is the third.
3. (Retract Axiom) If f and g are maps in $\mathbb{X}$ and g belongs to one of the three distinguished classes and f is a retract of g in $\mathrm{Ar}(\mathbb{X})$- in the sense of Definition 1.1.21- then f belongs to the same distinguished class(es) as g. Equivalently f is a retract of g if there is a commuting diagram



4. (Lifting Axiom) Let f be a cofibration, g a fibration, and let one of them also be a weak equivalence. Whenever there is a commuting diagram of the following form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X' & \longrightarrow & Y' \end{array}$$

then there is a morphism $h\colon X' \longrightarrow Y$ such that

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle g} \\ X' & \longrightarrow & Y' \end{array}$$

commutes. Note that h need not be unique.

5. (Factorization Axiom) Every map $f \in \mathbb{X}$ has (at least) two factorizations, $f = iq$ where i is a cofibration and weak equivalence (a trivial cofibration) and q is a fibration, and $f = jp$ where j is a cofibration and p is a fibration and a weak equivalence (a trivial fibration)

If $\perp$ is the initial object of $\mathbb{X}$ and $\top$ is the terminal object - the existence of these is guaranteed by Axiom 1- then an object $X \in \mathbb{X}$ is called *fibrant* if $X \to \top$ is a fibration, X is called *cofibrant* if $\perp \to X$ is a cofibration, and X is *fibrant-cofibrant* if it is both fibrant and cofibrant.

Denote a weak equivalence from X to Y by $X \xrightarrow{\simeq} Y$, a fibration by $X \twoheadrightarrow Y$, and a cofibration by $X \rightarrowtail Y$.

Definition 3.1.2. Let $\mathbb{X}$ be a model category, a *fibrant replacement* on $\mathbb{X}$ is

- an object (class) function $\mathrm{fib}\colon \mathrm{ob}\,\mathbb{X} \longrightarrow \mathrm{ob}\,\mathbb{X}$ such that $\mathrm{fib}(X)$ is fibrant for all $X \in \mathbb{X}$
- a trivial cofibration $X \underset{\simeq}{\overset{q_X}{\rightarrowtail}} \mathrm{fib}(X)$ for each $X \in \mathbb{X}$

If fib is the object function of a functor, also called fib, and $q\colon 1_{\mathbb{X}} \Longrightarrow \mathrm{fib}$ is natural then call the pair (fib, q) a *functorial fibrant replacement.*

Dually, a *cofibrant replacement* on $\mathbb{X}$ is

- a class function $\mathrm{cofib}\colon \mathrm{ob}\,\mathbb{X} \longrightarrow \mathrm{ob}\,\mathbb{X}$ such that $\mathrm{cofib}(Y)$ is cofibrant for all $Y \in \mathbb{X}$
- a trivial fibration $\mathrm{cofib}(Y) \xrightarrow[\simeq]{p_Y} Y$ for each $Y \in \mathbb{X}$

Similarly, if cofib is a functor and the p_Y are the components of a natural transformation then call (cofib, p) a *functorial cofibrant replacement.*

Notice that Axiom 5 guarantees the existence of a fibrant replacement by factoring the unique map to the terminal object. Dually, factoring the unique map from the initial object gives a cofibrant replacement. Further, if the factorizations are functorial then so are the replacements. One of the reasons for forcing q_X and p_Y to be a cofibration and a fibration respectively is this:

Proposition 3.1.3. *Let $\mathbb{X}$ be a model category with fibrant replacement* (fib, q) *and* (cofib, p)*, then for all $X \in \mathbb{X}$*

$\mathrm{fib} \circ \mathrm{cofib}(X)$ *is fibrant-cofibrant*

$\mathrm{cofib} \circ \mathrm{fib}(X)$ *is fibrant-cofibrant*

Proof. We prove $\mathrm{fib} \circ \mathrm{cofib}(X)$ is fibrant-cofibrant. Clearly $\mathrm{fib} \circ \mathrm{cofib}(X)$ is cofibrant since cofib has its image in cofibrant objects. Now, the following diagram commutes

$$\begin{array}{ccc} \mathrm{fib} \circ \mathrm{cofib}(X) & \xrightarrow{p_{\mathrm{fib}(X)}} & \mathrm{fib}(X) \\ & \searrow & \downarrow \\ & & \top \end{array}$$

Further, $p_{\mathrm{fib}(X)}$ is a fibration by the definition of cofibrant replacement and $\mathrm{fib}(X) \longrightarrow \top$ is a fibration since $\mathrm{fib}(X)$ is fibrant. Since the class of fibrations are composition closed we then get that $\mathrm{fib} \circ \mathrm{cofib}(X) \longrightarrow \top$ is a fibration, hence $\mathrm{fib} \circ \mathrm{cofib}(X)$ is fibrant.

The proof that $\mathrm{cofib} \circ \mathrm{fib}(X)$ is fibrant-cofibrant is dual.

□

The objective of giving a category $\mathbb{X}$ a model structure is to add new morphisms to $\mathbb{X}$ in such a way that each weak equivalence becomes an isomorphism. Further, we want the resulting category to be as similar to $\mathbb{X}$ as possible. This problem, called *localization*, of inverting a class of morphisms in a category, isn't unique to homotopy theory and a model structure is *not* the only way to solve it. The nice thing about the model category approach is that the process of inverting the weak equivalences in a model category is ultimately a matter of quotienting out by compatible equivalence relations on the hom-sets. That is, we will be able to define the hom-sets of the new category as equivalence classes of maps in the model category. Be advised, a morphism in the new category, called the *homotopy category*, need not be an equivalence class of morphisms with the same domain or codomain.

Definition 3.1.4. Consider any solid arrow diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ X' & \longrightarrow & Y' \end{array}$$

If, for any choice of horizontal arrows the dotted morphism exists and makes the entire diagram commute then we say that f *has the left-lifting property (LLP) with respect to* g or, equivalently, that g *has the right-lifting property (RLP) with respect to* f. Given sets of morphisms L and R in a category $\mathbb{X}$, if every element of L has the left-lifting property with respect to every morphism in R (or vice-versa) then we say that L *has the left-lifting property with respect to* R and that R *has the right-lifting property with respect to* L. Notice that we do not require that the lift be unique.

Proposition 3.1.5. *Let $\mathbb{X}$ be a model category and let g be a morphism of $\mathbb{X}$. Then g is a fibration if and only if it has the RLP with respect to every trivial cofibration. If g has the RLP with respect to every cofibration then g is a trivial fibration.*

Dually, a morphism f of $\mathbb{X}$ is a cofibration exactly when it has the LLP with respect to every trivial fibration, and a trivial cofibration whenever it has the LLP with respect to every fibration.

Proof. We shall prove the statement for $g\colon X \longrightarrow Y$ a fibration. The proof for f a cofibration is then dual.

First, if g is a fibration then Axiom 4 says exactly that g has the RLP with respect to all trivial cofibrations.

Now assume g is any morphism which has the RLP with respect to all trivial cofibrations. By condition 5 we may write $g = ip$ where i is a trivial cofibration and p is a fibration. Notice then that the following solid arrow diagram commutes

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle i}\downarrow{\scriptstyle \simeq} & \overset{\widetilde{g}}{\nearrow} & \downarrow{\scriptstyle g} \\ Z & \underset{p}{\twoheadrightarrow} & Y \end{array}$$

so $\widetilde{g}$ exists since g has the RLP with respect to, among other morphisms, i. Notice now that the following diagram commutes

$$\begin{array}{ccccc} & & \overset{1_X}{\frown} & & \\ X & \xrightarrow{i} & Z & \xrightarrow{\widetilde{g}} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ Y & = & Y & = & Y \end{array}$$

so, by Axiom 3, since p is a fibration, so is g.

If instead, g has the RLP with respect to every cofibration then we can have instead factor g so that i is a cofibration and p is the trivial fibration. Our lift, $\widetilde{g}$, still exists and the retract axiom now implies that g, like p, is a trivial fibration.

□

Corollary 3.1.6. *Every isomorphism has the left and right lifting properties with respect to all other morphisms, hence all isomorphisms of a model category are both trivial fibrations and trivial cofibrations.*

Given that the plan is to invert all weak equivalences it is most definitely comforting to know that the isomorphisms must be weak equivalences.

Proposition 3.1.7. *If $f\colon X \longrightarrow Y$ is a cofibration and $g\colon X \longrightarrow Z$ is any morphism, then the pushout of f along g is a cofibration. If f is a trivial cofibration then so is the pushout of f along g.*

Proof. Take the pushout

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ \downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[g']{} & W \end{array} \tag{3.1}$$

and let $h\colon A \longrightarrow B$ be a trivial fibration. Consider any commuting diagram of the form

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & A \\ {\scriptstyle f'}\downarrow & & \simeq\downarrow{\scriptstyle h} \\ W & \xrightarrow[\beta]{} & B \end{array} \tag{3.2}$$

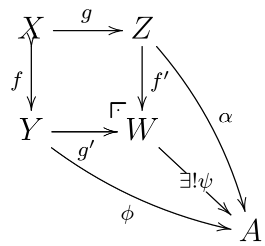
then by combining Diagrams 3.1 and 3.2 to get that

$$\begin{array}{ccccc} X & \xrightarrow{g} & Z & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & & & & \simeq\downarrow{\scriptstyle h} \\ Y & \xrightarrow[g']{} & W & \xrightarrow[\beta]{} & B \end{array}$$

commutes, we hence get a lift.

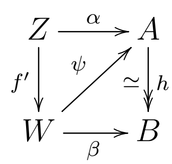
$$\begin{array}{ccccc} X & \xrightarrow{g} & Z & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & & \overset{\phi}{\nearrow} & & \simeq\downarrow{\scriptstyle h} \\ Y & \xrightarrow[g']{} & W & \xrightarrow[\beta]{} & B \end{array} \tag{3.3}$$

We then get the following commuting diagram



(3.4)

by the universal property of the pushout. We then claim that



commutes. Compute:

$$f'\psi = \alpha \qquad \text{by Diagram 3.4}$$

while

$$\begin{aligned} f'\psi h &= \alpha h \\ &= f'\beta \qquad \text{by Diagram 3.3} \end{aligned}$$

and

$$\begin{aligned} g'\psi h &= \phi h \qquad \text{by Diagram 3.4} \\ &= g'\beta \qquad \text{by Diagram 3.3} \end{aligned}$$

So $\psi h = \beta$ since the pair (f', g') is bi-epic. We conclude then that ψ is the necessary lift. Since g was an arbitrary trivial fibration we get that f' has the LLP with respect to all trivial fibrations and is thus a cofibration.

If instead f had been a trivial cofibration then, applying the same argument but taking g to be any fibration, we get that f' has the LLP with respect to all fibrations and is thus a trivial cofibration.

The dual argument proves that if g is a (trivial) fibration, then its pullback along any morphism f is a (trivial) fibration.

□

Example 3.1.8. The category of all topological spaces $\mathbf{Top}_*$ with one of two model structures, the first due to Quillen [21] and is called the *Quillen model structure*:

- weak equivalences: the weak homotopy equivalences (maps $f\colon X \longrightarrow Y$ such that $\pi_n(f)\colon \pi_n(X) \longrightarrow \pi_n(Y)$ is an isomorphism for all $n \in \mathbb{N}$).
- fibrations: maps with the RLP with respect to all inclusions $\iota_0\colon D^n \longrightarrow D^n \times I$ of the n-disk into the cylinder on the n-disk. These maps are called *Serre fibrations.*
- cofibrations: maps with the LLP with respect to all trivial fibrations.

The second, called the *Hurewicz model structure*, is due to Strøm in [24]

- weak equivalences: homotopy equivalences (maps $f\colon X \longrightarrow Y$ such that there is a $g\colon Y \longrightarrow X$ such that $fg \sim 1_X$ and $gf \sim 1_Y$)
- fibrations: maps with the RLP with respect to all inclusions $\iota_0\colon A \longrightarrow A \times I$ of any topological space A into the cylinder with base A. These maps are called *Hurewicz fibrations.*
- cofibrations: all maps with the LLP with respect to all trivial fibrations.

The Quillen model structure is the *usual* one.

Example 3.1.9 ([21])**.** The category of simplicial sets $\mathbf{sSet}_*$ with

- weak equivalences: A morphism $f\colon X \longrightarrow Y$ in $\mathbf{sSet}_*$ is a weak equivalence if its geometric realization $|f|$ is a weak homotopy equivalence.
- cofibrations: all (pointwise) monomorphisms.

- fibrations: a morphism $f\colon X \longrightarrow Y$ is a fibration if f has the RLP with respect to every acyclic cofibration.

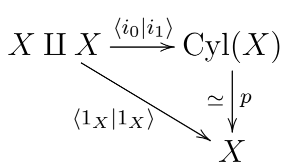
Example 3.1.10. The category of non-negatively graded chain complexes of R-modules for any ring R, $\mathbf{Ch_{\geq 0}}(R)$. Good references for this are [5] and [25].

- weak equivalences: chain maps $f\colon X \longrightarrow Y$ such that $\mathrm{H}_n(f)\colon \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(Y)$ is an isomorphism for each $n \in \mathbb{N}$.
- fibrations: a chain map $g\colon X \longrightarrow Y$ is a fibration if $g_n\colon X_n \longrightarrow Y_n$ is epic in $R-\mathrm{mod}$ for all $n > 0$.
- cofibrations: a chain map $f\colon X \longrightarrow Y$ is a cofibration if $f_n\colon X_n \longrightarrow Y_n$ is monic in $R-\mathrm{mod}$ and satisfies the condition that $\operatorname{coker} f_n$ is a projective R-module.

3.2 Homotopy in a Model Category

Recall from Definition 1.3.5 that $\langle f|g\rangle\colon X \amalg Y \longrightarrow Z$ is the unique map satisfying $\iota_0\langle f|g\rangle = f\colon X \longrightarrow Z$ and $\iota_1\langle f|g\rangle = g\colon Y \longrightarrow Z$.

Definition 3.2.1. Let $\mathbb{X}$ be a model category and let $X \in \mathbb{X}$, then *any* diagram of the form



such that p is a weak equivalence is called a *cylinder object*. Notice that we do not mean for the object $\mathrm{Cyl}(X)$ to be unique, nor is $\mathrm{Cyl}(X)$ supposed to be of the form $X \times I$ for some object $I \in \mathbb{X}$. For convenience we will suppress notation and simply refer to a cylinder object $\mathrm{Cyl}(X)$ when, in fact, we mean the entire diagram.

Call a cylinder object such that $\langle i_0|i_1\rangle$ is a cofibration a *good cylinder object.* Call a good cylinder object such that p is a trivial fibration a *very good cylinder object.*

Given two morphisms $f, g\colon X \longrightarrow Y$ in $\mathbb{X}$ call f and g *left homotopic*, denoted $f \sim_l g$, if there is a commuting diagram of the form

$$\begin{array}{ccc} & & Y \\ & \nearrow^{\langle f|g\rangle} & \uparrow{\scriptstyle H} \\ X \amalg X & \xrightarrow{\langle i_0|i_1\rangle} & \mathrm{Cyl}(X) \\ & \searrow_{\langle 1_X|1_X\rangle} & \simeq \downarrow{\scriptstyle p} \\ & & X \end{array}$$

such that the lower triangle is a cylinder object. In this context call H a *left homotopy*. Call H *good* if the cylinder object used in its definition is good, similarly define *very good* left homotopies to be those defined on very good cylinder objects.

Example 3.2.2. In $\mathbf{Top}_*$ with the Quillen model structure the diagram

$$X \amalg X \xrightarrow{\langle i_0|i_1\rangle} X \times [0,1] \xrightarrow{\pi_0} X$$

where $i_j\colon X \longrightarrow X \times [0,1]$ is the identification of X with $X \times \{j\}$ in $X \times [0,1]$, and π_0 is the usual projection map of a product, is a cylinder object.

Definition 3.2.3. If $\mathbb{X}$ is a model category, then for any object $Y \in \mathbb{X}$ call any diagram of the form

$$\begin{array}{ccc} Y & \underset{\simeq}{\xrightarrow{s}} & \mathrm{Path}(Y) \\ & \searrow_{\langle 1_Y,1_Y\rangle} & \downarrow{\scriptstyle \langle p_0,p_1\rangle} \\ & & Y \times Y \end{array}$$

such that s is a weak equivalence a *path object.* As with cylinder objects we identify the object $\mathrm{Path}(Y)$ with the diagram defining it and call a path object *good* if $\langle p_0, p_1\rangle$ is a fibration, call a good path object *very good* if s is a trivial cofibration. Given any maps $f, g\colon X \longrightarrow Y$ in $\mathbb{X}$ call f and g *right homotopic*, denoted $f \sim_r g$, if there is a commuting diagram of the form

$$\begin{array}{ccccc} Y & \underset{\simeq}{\xrightarrow{s}} & \mathrm{Path}(Y) & \xleftarrow{F} & X \\ & \searrow_{\langle 1_Y,1_Y\rangle} & \downarrow{\scriptstyle \langle p_0,p_1\rangle} & \swarrow_{\langle f,g\rangle} & \\ & & Y \times Y & & \end{array}$$

In this context call the map F a *right homotopy*. Call F *good* if the path object defining it is good. Similarly define a *very good* right homotopy.

Example 3.2.4. Again in **Top** with the Quillen model structure let $\mathrm{maps}([0,1],Y) := \mathbf{Top}([0,1],Y)$ endowed with the compact-open topology. Then the diagram

$$Y \xrightarrow[\simeq]{\mathrm{c}_Y} \mathrm{maps}([0,1],Y) \xrightarrow{\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle} Y \times Y$$

where $\mathrm{c}_Y \colon Y \longrightarrow \mathrm{maps}([0,1],Y)$ sends each element $y \in Y$ to the constant map $\mathrm{c}_y \colon I \longrightarrow Y$, and $\mathrm{ev}_i \colon \mathrm{maps}([0,1],Y)$ sends the function $f \colon [0,1] \longrightarrow Y$ to $f(i), i = 0,1$.

Notice that Axiom 5 applied to $\langle 1_X | 1_X \rangle$ and $\langle 1_X, 1_X \rangle$ ensure the existance of good and very good cylinder and path objects for each $X \in \mathbb{X}$ respectively.

Lemma 3.2.5. *If $X \in \mathbb{X}$ is cofibrant then $X \amalg X$ is also. Dually, if Y is fibrant then so is $Y \times Y$.*

Proof. We prove the case when X is cofibrant. The case when Y is fibrant is by duality.

Since the X is cofibrant we know that $\perp \rightarrowtail X$ is a cofibration, further the following is a pushout diagram

$$\begin{array}{ccc} \perp & \rightarrowtail & X \\ \downarrow & & \downarrow \iota_0 \\ X & \xrightarrow[\iota_1]{} & X \amalg X \end{array}$$

so, by Proposition 3.1.7 we get that ι_0 is a cofibration, hence $\perp \longrightarrow X \amalg X$ is a cofibration since the composition of cofibrations is a cofibration.

□

Lemma 3.2.6. *If X is cofibrant and we have a good cylinder object*

$$X \amalg X \xrightarrow{\langle i_0 | i_1 \rangle} \mathrm{Cyl}(X) \xrightarrow{p} X \ ,$$

then i_k is a trivial cofibration for $k \in \{0, 1\}$. Dually if Y is fibrant and we have a good path object

$$Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{\langle p_0, p_1 \rangle} Y \times Y$$

then p_k is a trivial fibration for $k \in \{0, 1\}$.

Proof. We prove the statement for cofibrant X, the proof for fibrant Y is by duality.

First, by the definition of a cylinder object we get that p is a weak equivalence. Also, 1_X is a weak equivalence by Prop 3.1.6, so since the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{i_k} & \mathrm{Cyl}(X) \\ & {\scriptstyle 1_X} \searrow & \simeq \downarrow {\scriptstyle p} \\ & & X \end{array}$$

i_k must be a weak equivalence by the 2-out-of-3 Axiom.

Since pushouts preserve cofibrations, and since

$$\begin{array}{ccc} \bot & \longrightarrow & X \\ \downarrow & & \downarrow {\scriptstyle \iota_0} \\ X & \xrightarrow[\iota_1]{} & X \amalg X \end{array}$$

is a pushout diagram, X is cofibrant if and only if $\bot \longrightarrow X$ is a cofibration, which in turn implies that ι_k is a cofibration for $k \in \{0, 1\}$. Finally

$$i_k = \iota_k \langle i_0 | i_1 \rangle$$

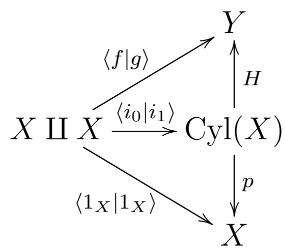
by definition, so i_k is a cofibration being the composite of cofibrations.

□

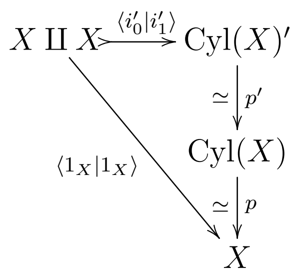
Lemma 3.2.7. *If $f \sim_l g \colon X \longrightarrow Y$ then there is a good homotopy $H \colon \mathrm{Cyl}(X) \longrightarrow Y$ such that $i_0 H = f$ and $i_1 H = g$. If Y is fibrant then there is a very good homotopy.*

The dual statements for right homotopy also holds.

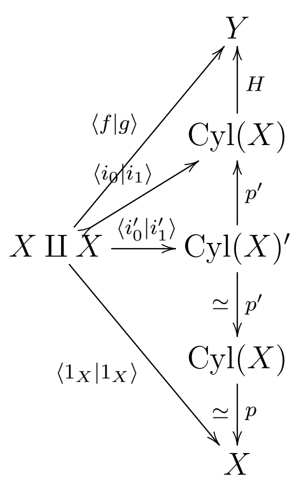
Proof. Assume $f \sim_l g$ and pick a witness



Now, if we apply Axiom 5 to the map $\langle i_0|i_1\rangle\colon X \amalg X \longrightarrow \mathrm{Cyl}(X)$ to get $\langle i_0|i_1\rangle = \langle i_0'|i_1'\rangle p'$ where $\langle i_0'|i_1'\rangle$ is a cofibration and p' is a trivial fibration, then $p'p$ is a weak equivalence so

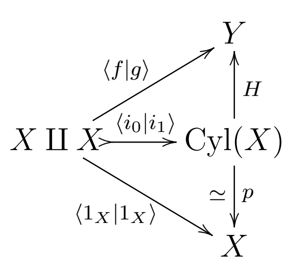


is a good cylinder object. In addition, the following diagram commutes



hence there is a good homotopy $H' := p'H$ witnessing that $f \sim_l g$.

If Y is fibrant then take a good homotopy



and factor

$$\begin{array}{ccc} \mathrm{Cyl}(X) & \xrightarrow[\simeq]{p_0} & \mathrm{Cyl}(X)' \\ & \underset{p}{\overset{\simeq}{\searrow}} & \simeq\big\downarrow{\scriptstyle p_1} \\ & & X \end{array}$$

where p_0 is a trivial cofibration and p_1 is a trivial fibration. This is possible by Axioms 5 and 2. Then the following diagram commutes and is a very good cylinder object

$$\begin{array}{ccc} X \amalg X & \xrightarrow{\langle i_0p_0 | i_1p_0\rangle} & \mathrm{Cyl}(X)' \\ & \underset{\langle 1_X | 1_X\rangle}{\searrow} & \simeq\big\downarrow{\scriptstyle p_1} \\ & & X \end{array}$$

Further, the solid arrow diagram

$$\begin{array}{ccc} \mathrm{Cyl}(X) & \xrightarrow{H} & Y \\ {\scriptstyle p_0}\big\downarrow\simeq & \overset{H'}{\nearrow} & \big\downarrow \\ \mathrm{Cyl}(X)' & \longrightarrow & \top \end{array}$$

commutes, thus providing the dotted lift $H' \colon \mathrm{Cyl}(X) \longrightarrow Y$. Compute

$$\begin{aligned} \langle i_0p_0 | i_1p_0\rangle H' &= \langle i_0p_0H' | i_1p_0H'\rangle \\ &= \langle i_0H | i_1H\rangle \\ &= \langle f | g\rangle, \end{aligned}$$

thus we have our very good homotopy.

□

The next result will allow us to conclude that, on the full subcategory of $\mathbb{X}$ consisting of fibrant-cofibrant the relations $\sim_l$ and $\sim_r$ coincide, further they form a congruence on this subcategory and thus we may form the congruence category. This congruence category will in fact be equivalent to the homotopy category we wish to construct.

Proposition 3.2.8. *Let $\mathbb{X}$ be a model category and let $X \in \mathbb{X}$ cofibrant. The following statements are true*

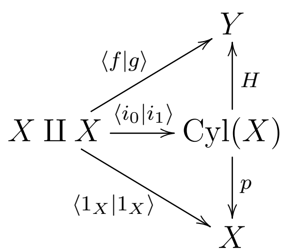
1. *If $f, g \colon X \longrightarrow Y$ are morphisms such that $f \sim_l g$ then $f \sim_r g$*

2. *The relation* $\sim_l$ *is an equivalence relation on* $\mathbb{X}(X, Y)$ *for all* $Y \in \mathbb{X}$

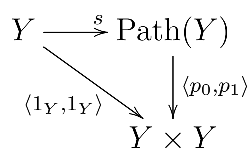
The dual statements for fibrant objects Y*, are also true.*

Proof. 1. First, we will prove that if $\mathbb{X}$ is cofibrant and $f \sim_l g \colon X \longrightarrow Y$ then $f \sim_r g$.

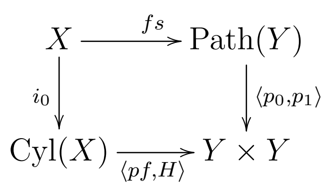
Let $f \sim_l g$ and let



witness that fact. Further, assume that H was chosen to be a good homotopy (i.e. so that $\mathrm{Cyl}(X)$ is also good). Now pick any good path object for Y



And consider the (not-obviously-commuting) diagram



Notice that, composing clockwise around the square yields

$$fs\langle p_0, p_1\rangle = \langle fsp_0, fsp_1\rangle \qquad \text{by Proposition 1.3.15}$$

$$= \langle f, f\rangle \qquad \text{since } sp_k = 1_Y \text{ by the definition of a path object}$$

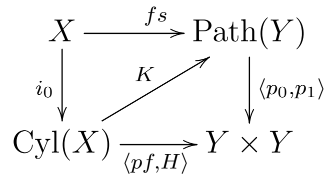
while compsing counterclockwise yields

$$i_0\langle pf, H\rangle = \langle i_0pf, i_0H\rangle \qquad \text{by Propostion 1.3.15}$$

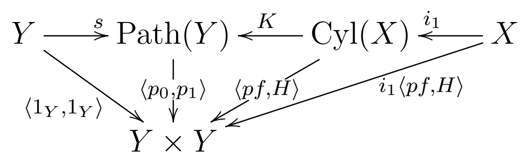
$$= \langle 1_X f, f\rangle \qquad \text{by the definition of a cylinder object and of a homotopy}$$

$$= \langle f, f\rangle$$

so our diagram commutes. Since $\mathrm{Path}(Y)$ is good we know that $\langle p_0, p_1\rangle$ is a fibration, we also know that i_0 is a trivial cofibration by Lemma 3.2.6 so there is a lift



Assemble the diagram



This diagram commutes by the definition of K, further

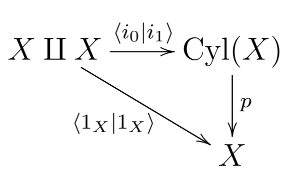
$$
\begin{aligned}
i_1\langle pf, H\rangle &= \langle i_1 pf, i_1 H\rangle && \text{by Proposition 1.3.15} \\
&= \langle f, g\rangle && \text{by the defintion of a cylinder object and of } H,
\end{aligned}
$$

hence i_1K is a right homotopy from f to g.

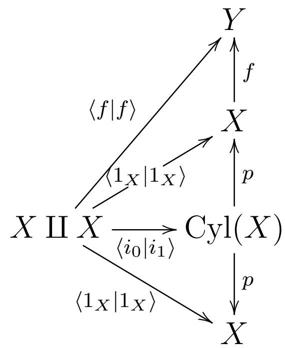
> **Aside:** In the topological setting the map i_0 is the inclusion, $x \longmapsto (x, 0)$, into the honest-to-goodness cylinder $X \times I$ and p is the projection map from $X \times I$ onto X. Going the other way the composite fs first sends each point of x to $f(x) \in Y$, then s sends it to the constant path at $f(x)$. The maps p_0 and p_1 are the endpoint projections. The lift K then becomes the map which takes a pair $(x, t) \in X \times I$ and sends it to the path $s \mapsto H(x, ts)$.

2. Now, we will prove that $\sim_l$ is an equivalence relation on $\mathbb{X}(X, Y)$ when X is cofibrant.

First, if $f\colon X \longrightarrow Y$ is any map, then choose any cylinder object

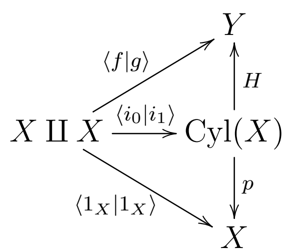


and expand it to the diagram

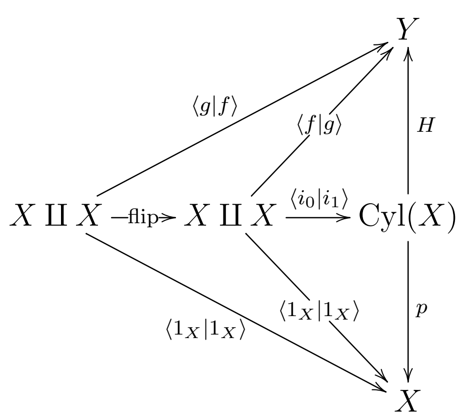


This diagram commutes by Proposition 1.3.15.

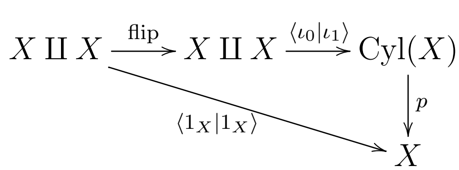
For symmetry assume $f \sim_l g \colon X \longrightarrow Y$ and let



witness that relation. Recall that there is a unique map flip: $X \amalg X \longrightarrow X \amalg X$ such that for any $\langle f|g\rangle \colon X \amalg X \longrightarrow Y$ we have flip$\langle f|g\rangle = \langle g|f\rangle$. Add this to the diagram and get the commuting diagram

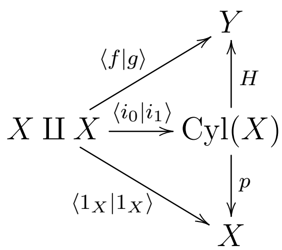


Since p is a weak equivalence we get that

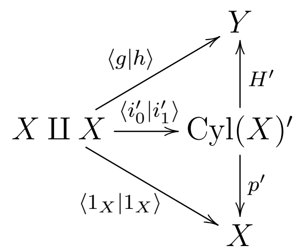


is another cylinder object, hence $g \sim_l f$.

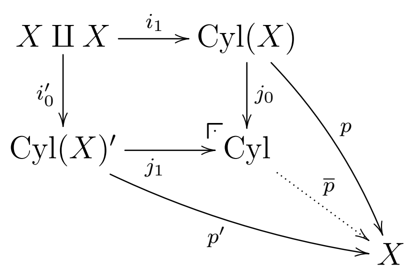
Finally, assume $f \sim_l g$ and $g \sim_l h$ and choose good homotopies



and

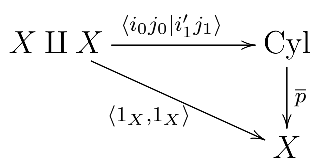


and consider the pushout diagram together with the external commuting diagram



(3.5)

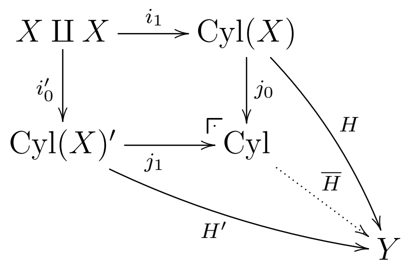
This assembles into the diagram



which commutes since

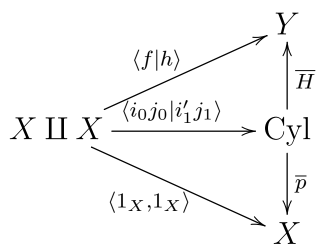
$$\begin{aligned} \langle i_0 j_0|i_1' j_1\rangle\overline{p} &= \langle i_0 j_1\overline{p}|i_1' j_0\overline{p}\rangle \\ &= \langle i_0 p|i_1' p'\rangle && \text{by Diagram 3.5} \\ &= \langle 1_X|1_X\rangle \end{aligned}$$

Since i_1 is a trivial cofibration Proposition 3.1.7 gives that j_1 is a trivial cofibration. Further, since $j_1\overline{p} = p$, and since j_1 and p are weak equivalences by the last sentence and by definition respectively, the 2-out-of-3 Axiom tells us that $\overline{p}$ is a weak equivalence. In particular we get that Cyl and its accompanying diagram is a cylinder object. We can also generate



(3.6)

which allows us to construct



which commutes since

$$\langle i_0 j_0 | i'_1 j_1\rangle \overline{H} = \langle i_0 j_0 \overline{H} | i'_1 j_1 \overline{H}\rangle$$

$$= \langle i_0 H | i'_1 H'\rangle \qquad \text{by Diagram 3.6}$$

$$= \langle f | h\rangle$$

Thus $f \sim_l g$ via Cyl and so $\sim_l$ is indeed an equivalence relation.

□

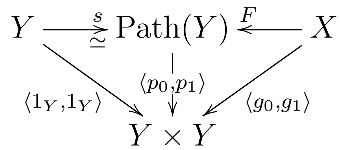
Corollary 3.2.9. *If X is cofibrant and Y is fibrant then the relations $\sim_l$ and $\sim_r$ on $\mathbb{X}(X,Y)$ coincide. Call this common relation* homotopy *and denote it by the symbol $\sim$.*

Proposition 3.2.10. *Let $\mathbb{X}_{fc}$ denote the full subcategory of $\mathbb{X}$ consisting of fibrant-cofibrant objects. The homotopy relation $\sim$ is then a congruence on $\mathbb{X}_{fc}$.*

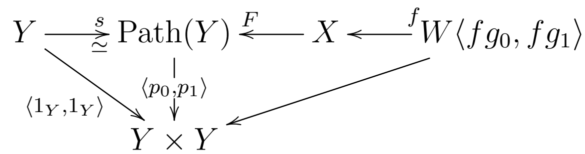
Proof. By Corollary 3.2.9 we have that $\sim$ is an equivalence relation on all hom-sets of $\mathbb{X}_{fc}$.

Let $f\colon W \longrightarrow X$, $g_0, g_1\colon X \longrightarrow Y$, and $h\colon Y \longrightarrow Z$ be morphisms in $\mathbb{X}_{fc}$ such that $g_0 \sim g_1$. We need to show that $fg_0h \sim fg_1h$.

Since X and Y are fibrant-cofibrant we get that there is a right homotopy

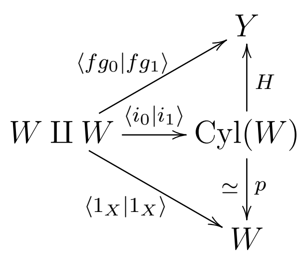


Pre-composing F with the morphism f then yields the commuting diagram

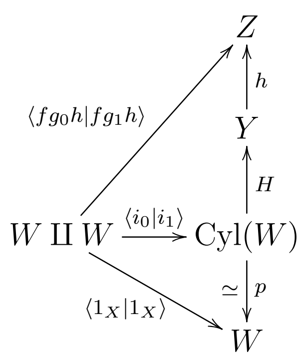


thus $F' = fF$ is a right homotopy witnessing $fg_0 \sim fg_1$.

Since left homotopy and right homotopy agree on fibrant-cofibrant objects choose a left-homotopy



and post compostion of H by h yields a commuting diagram



Thus $H' := Hh$ yields a left homotopy witnessing $fg_0h \sim fg_1h$.

We conclude that $\sim$ is a congruence on $\mathbb{X}_{fc}$.

□

Proposition 3.2.11. *Let* $\mathbb{X}$ *is be model category with fibrant and cofibrant replacements* (fib, p) *and* (cofib, q) *respectively. For all* $f\colon X \longrightarrow Y$ *in* $\mathbb{X}$ *there are morphisms*

$$\mathrm{fib}(f)\colon\ \mathrm{fib}(X) \longrightarrow \mathrm{fib}(Y)$$

and $\mathrm{cofib}(f)\colon\ \mathrm{cofib}(X) \longrightarrow \mathrm{cofib}(Y)$*, unique up to right and left homotopy respectively, making the following diagrams commute.*

$$\begin{array}{ccc} \mathrm{cofib}(X) & \xrightarrow{\mathrm{cofib}(f)} & \mathrm{cofib}(Y) \\ \simeq \big\downarrow q_X & & q_Y \big\downarrow \simeq \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ p_X \big\downarrow \simeq & & p_Y \big\downarrow \simeq \\ \mathrm{fib}(X) & \xrightarrow[\mathrm{fib}(f)]{} & \mathrm{fib}(Y) \end{array} \tag{3.7}$$

Further, if any of f*,* $\mathrm{fib}(f)$*, or* $\mathrm{cofib}(f)$ *are weak equivalences then they all are.*

Proof. Given a morphism $f\colon X \longrightarrow Y$ in $\mathbb{X}$ consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & Y & \overset{p_Y}{\underset{\simeq}{\rightarrowtail}} & \mathrm{fib}(Y) \\ p_X \big\downarrow \simeq & & & & \big\downarrow \\ \mathrm{fib}(X) & & \longrightarrow & & \top \end{array}.$$

Since $\top$ is the terminal object of $\mathbb{X}$ we get that this diagram commutes, hence we get a lift, $\mathrm{fib}(f)\colon\ \mathrm{fib}(X) \longrightarrow \mathrm{fib}(Y)$, by Axiom 4. Notice that the upper triangle in the diagram above is exactly the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ p_X \big\downarrow \simeq & & p_Y \big\downarrow \simeq \\ \mathrm{fib}(X) & \xrightarrow[\mathrm{fib}(f)]{} & \mathrm{fib}(Y) \end{array} \tag{3.8}$$

Now let $\mathrm{fib}(f)_0, \mathrm{fib}(f)_1\colon\ \mathrm{fib}(X) \longrightarrow \mathrm{fib}(Y)$ are any two morphisms making Diagram 3.8 commute. Then choose a good path object for $\mathrm{fib}(Y)$ and consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\ f\ } & Y & \overset{p_Y}{\underset{\simeq}{\rightarrowtail}} & \mathrm{fib}(Y) & \overset{s}{\underset{\simeq}{\longrightarrow}} & \mathrm{Path}(\mathrm{fib}(Y)) \\ p_X \big\downarrow \simeq & & & & & & \big\downarrow \langle p_0, p_1 \rangle \\ \mathrm{fib}(X) & & & \xrightarrow[\langle \mathrm{fib}(f)_0, \mathrm{fib}(f)_1 \rangle]{} & & & \mathrm{fib}(Y) \times \mathrm{fib}(Y) \end{array}$$

Compute clockwise

$$\begin{aligned} fp_Y s\langle p_0, p_1\rangle &= fp_Y\langle 1_{\mathrm{fib}(Y)}, 1_{\mathrm{fib}(Y)}\rangle \\ &= \langle fp_Y, fp_Y\rangle \end{aligned}$$

and counterclockwise

$$\begin{aligned} p_X\langle \mathrm{fib}(f)_0, \mathrm{fib}(f)_1\rangle &= \langle p_X\mathrm{fib}(f)_0, p_X\mathrm{fib}(f)_1\rangle \\ &= \langle fp_Y, fp_Y\rangle && \text{by 3.8 and our choices of } \mathrm{fib}(f)_i \end{aligned}$$

hence the diagram commutes, thus Axiom 4 provides a lift $K\colon\ \mathrm{fib}(X) \longrightarrow \mathrm{Path}(\mathrm{fib}(Y))$ such that

$$\begin{array}{ccccc} \mathrm{fib}(Y) & \xrightarrow[\simeq]{s} & \mathrm{Path}(\mathrm{fib}(Y)) & \xleftarrow{K} & \mathrm{fib}(X) \\ & \searrow^{\langle 1_{\mathrm{fib}(Y)}, 1_{\mathrm{fib}(Y)}\rangle} & \downarrow{\scriptstyle \langle p_0, p_1\rangle} & \swarrow_{\langle \mathrm{fib}(f)_0, \langle\rangle\, \mathrm{fib}(f)_1)} & \\ & & \mathrm{fib}(Y) \times \mathrm{fib}(Y) & & \end{array}$$

commutes, hence $\mathrm{fib}(f)_0 \sim_r \mathrm{fib}(f)_1$.

Since p_X and p_Y are weak equivalences the 2-out-of-3 axiom implies that f is a weak equivalence if and only if fp_Y is, which is a weak equivalence if and only if $p_X\mathrm{fib}(f)$ is, and $p_X\mathrm{fib}(f)$ is a weak equivalence if and only if $\mathrm{fib}(f)$ is. Hence f is a weak equivalence if and only if $\mathrm{fib}(f)$ is.

If f is a weak equivalence then the composite fp_Y is a weak equivalence by the 2-out-of-3 Axiom, thus $p_X\mathrm{fib}(f)$ is also a weak equivalence by the 2-out-of-3 Axiom, hence $\mathrm{fib}(f)$ is a weak equivalence.

The dual argument provides the rest of the proof.

□

This immediately implies

Corollary 3.2.12. *For each morphism $f\colon X \longrightarrow Y$ in $\mathbb{X}$ there is a morphism,*

$$\mathrm{fibcofib}(f)\colon\ \mathrm{fibcofib}(X) \longrightarrow \mathrm{fibcofib}(Y),$$

unique up to homotopy such that

$$\begin{array}{ccc}
\mathrm{fibcofib}(X) & \xrightarrow{\mathrm{fibcofib}(f)} & \mathrm{fibcofib}(Y) \\
{\scriptstyle q_{\mathrm{fib}(X)}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle q_{\mathrm{fibcofib}(Y)}} \\
\mathrm{fib}(X) & \xrightarrow{\mathrm{fib}(f)} & \mathrm{fib}(Y) \\
\simeq\uparrow{\scriptstyle p_X} & & {\scriptstyle p_Y}\uparrow\simeq \\
X & \xrightarrow[f]{} & Y
\end{array}$$

commutes. Further, $\mathrm{fibcofib}(f)$ *is a weak-equivalence if and only if* f *is also.*

Proposition 3.2.13. *Given a morphism* $f\colon X \longrightarrow Y$ *where* X *is fibrant, then, if* g *is a trivial cofibration, there is a morphism* $f'\colon Y \longrightarrow X$ *such that* $ff' = 1_Y$ *and* $f'f \sim_r 1_Y$*. If* Y *is also fibrant then the converse also holds.*

Dually, given a morphism $g\colon X \longrightarrow Y$ *where* Y *is cofibrant, if* g *is a trivial fibration then there is a morphism* $g'\colon Y \longrightarrow X$ *such that* $f'f = 1'_X$ *and* $gg' \sim_l 1_X$*. If* X *is also cofibrant then the converse holds.*

Proof. Assume $f\colon X \longrightarrow Y$ is a trivial cofibration and X is fibrant, then the following diagram commutes

$$\begin{array}{ccc}
X & = & X \\
{\scriptstyle f}\downarrow\simeq & & \downarrow \\
Y & \longrightarrow & \top
\end{array}$$

thus we get a lift $f'\colon X \longrightarrow Y$ such that $ff' = 1_X$. Choose a good path object for Y and consider the diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow[\simeq]{s} & \mathrm{Path}(Y) \\
{\scriptstyle f}\downarrow\simeq & & & & \downarrow{\scriptstyle \langle p_0,p_1\rangle} \\
Y & & \xrightarrow[\langle f'f,1_Y\rangle]{} & & Y \times Y
\end{array}$$

Clockwise:

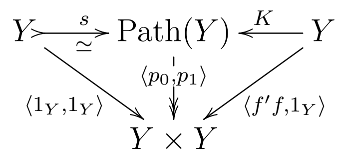
$$\begin{aligned}
fs\langle p_0, p_1\rangle &= f\langle 1_Y, 1_Y\rangle \\
&= \langle f, f\rangle
\end{aligned}$$

and counterclockwise:

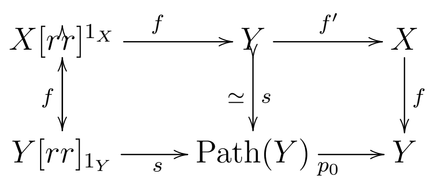
$$
\begin{aligned}
f\langle f'f, 1_Y\rangle &= \langle ff'f, f\rangle \\
&= \langle 1_X f, f\rangle \\
&= \langle f, f\rangle
\end{aligned}
$$

so we get a lift $K\colon Y \longrightarrow \mathrm{Path}(Y)$ such that $K\langle p_0, p_1\rangle = \langle f'f, 1_Y\rangle$ hence $f'f \sim_r 1_Y$.

Conversely, if for $f\colon X \longrightarrow Y$ with Y fibrant, and there is a morphism $f'\colon Y \longrightarrow X$ such that $ff' = 1_X$ and $f'f \sim_r 1_Y$, then there is a very good right homotopy witnessing the latter by Lemma 3.2.7. Let



be such a homotopy, then the following diagram commutes



so f is a retract of s, hence f is a trivial cofibration by the Retract Axiom.

The proof of the dual is dual.

□

Corollary 3.2.14. *Given fibrant cofibrant objects $X, Y \in \mathbb{X}$, a morphism $w\colon X \longrightarrow Y$ is a weak equivalence if and only if there is a morphism $w'\colon Y \longrightarrow X$ such that $ww' \sim 1_X$ and $w'w \sim 1_Y$.*

Proof. Factor $w = w_0 w_1$ where $w_0\colon X \longrightarrow Z$ is a trivial cofibration and $w_1\colon Z \longrightarrow Y$ is a trivial fibration, this factorization exists by Axiom 5 and the 2-out-of-3 Axiom. Since X is fibrant there is a morphism $w_0'\colon Z \longrightarrow X$ such that $w_0 w_0' \sim_l 1_X$ and $w_0' w_0 = 1_Z$. Since Y is cofibrant there is a morphism $w_1'\colon Y \longrightarrow Z$ such that $w_1 w_1' = 1_Z$ and $w_1' w_1 \sim_r 1_Y$, hence

$$\begin{aligned} w w_1' w_0' &= w_0 w_1 w_1' w_0' \\ &\sim_r w_0 w_0' \\ &= 1_X \end{aligned}$$

and

$$\begin{aligned} w_1' w_0' w &= w_1' w_0' w_0 w_1 \\ &\sim_l w_1' w_1 \\ &= 1_Y. \end{aligned}$$

So, since left homotopy and right homotopy agree on fibrant-cofibrant objects we get that $w w_1' w_0' \sim 1_X$ and $w_1' w_0' w \sim 1_Y$.

Conversely assume $f\colon X \longrightarrow Y$ is a morphism between fibrant-cofibrant objects such that there is a morphism $g\colon Y \longrightarrow X$ such that $fg \sim 1_X$ and $gf \sim 1_Y$. We will show that f is a weak equivalence.

Factor f as a trivial cofibration followed by a fibration

$$\begin{array}{ccc} X & \overset{f_0}{\underset{\simeq}{\rightarrowtail}} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_1} \\ & & Y \end{array}$$

Notice that since X is cofibrant and f_0 is a cofibration we have that Z is cofibrant. Also, since f_1 is a fibration and Y is fibrant we get that Z is fibrant. Hence Z is fibrant-cofibrant. Now choose a good homotopy

$$\begin{array}{ccc} & & X \\ & {\scriptstyle \langle gf|1_Y\rangle}\nearrow & \uparrow{\scriptstyle H} \\ Y \amalg Y & \overset{\langle i_0|i_1\rangle}{\rightarrowtail} & \mathrm{Cyl}(Y) \\ & {\scriptstyle \langle 1_Y|1_Y\rangle}\searrow & \simeq\downarrow{\scriptstyle p} \\ & & Y \end{array},$$

since Y is cofibrant i_0 is a trivial cofibration by Lemma 3.2.6 and assemble the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{g} & X & \xrightarrow{f_0} & Z \\ {\scriptstyle i_0}\downarrow\simeq & & & & \downarrow{\scriptstyle f_1} \\ \mathrm{Cyl}(Y) & & \xrightarrow[H]{} & & Y \end{array} .$$

Compute

$$\begin{aligned} i_0 H &= gf && \text{by the definition of } H \\ &= gf_0f_1 \end{aligned}$$

so the diagram commutes, Axiom 4 gives us a lift $H' \colon \mathrm{Cyl}(X) \longrightarrow Z$ such that $i_0H' = gf_0$ and $H'f_1 = H$. H' then provides a homotopy from gf_0 to $h := i_1H'$. Since f_0 is a trivial cofibration whose domain is fibrant, by Proposition 3.2.13 we get that there is a map $f_0' \colon Z \longrightarrow X$ such that $f_0f_0' = 1_X$ and $f_0'f_0 \sim_r 1_Y$. Now we compute

$$\begin{aligned} hf_1 &= i_1H'f_1 \\ &= i_1H \\ &= 1_Y \end{aligned}$$

and

$$\begin{aligned} f_1h &\sim_l f_1gf_0 \\ &\sim_r f_0'f_0f_1gf_0 \\ &= f_0'fgf_0 \\ &\sim f_0'f_0 \\ &\sim 1_Z. \end{aligned}$$

Since left and right homotopy agree between fibrant-cofibrant objects we get that $f_1h \sim 1_Y$. In summary, $f_1 \colon Z \longrightarrow Y$ is a morphism between cofibrant objects such that there exists an $h \colon Y \longrightarrow Z$ such that $f_1h \sim_l 1_Z$ and $hf_1 = 1_Y$, thus f_1 is a trivial fibration by Proposition

3.2.13. In particular then, f is the composite of two weak equivalences and thus is also a weak equivalence.

□

Definition 3.2.15. Let $\mathbb{X}_{fc}/\sim$ be the congruence category defined in Definition 1.1.27 and let $\operatorname{Ho}\mathbb{X}$ be the category with

objects: those of $\mathbb{X}$.

morphisms: $\operatorname{Ho}\mathbb{X}(X,Y) := \pi(\operatorname{fibcofib}(X), \operatorname{fibcofib}(Y))$.

identities: In $\operatorname{Ho}\mathbb{X}$ let the identity of X be $[1_{\operatorname{fibcofib}(X)}]\colon \operatorname{fibcofib}(X) \longrightarrow \operatorname{fibcofib}(Y)$.

composition: If $[f] \in \operatorname{Ho}\mathbb{X}(X,Y)$ and $[g] \in \operatorname{Ho}\mathbb{X}(Y,Z)$ define the composite $[f] \circ [g]$ to be the equivalence class $[fg]$.

Define $\gamma^{\mathbb{X}}\colon \mathbb{X} \longrightarrow \operatorname{Ho}(\mathbb{X})$ by $\gamma^{\mathbb{X}}(X) = X$ and $\gamma^{\mathbb{X}}(f\colon X \longrightarrow X')$ to be the homotopy class of $\operatorname{fibcofib}(f)\colon \operatorname{fibcofib}(X) \longrightarrow \operatorname{fibcofib}(X')$ which, by Corollary 3.2.12 is unique.

When context allows we will drop the superscript and simply refer to the functor γ when it is clear which model category this functor is associated to.

Proposition 3.2.16. *The above definition actually make* $\operatorname{Ho}(\mathbb{X})$ *into a category and* γ *into a functor.*

Remark. If we restrict our attention, in $\operatorname{Ho}\mathbb{X}$, to the objects which were fibrant-cofibrant in $\mathbb{X}$ then we get the congruence category $\mathbb{X}_{cf}/\sim$ from Definition 1.1.27, and $\operatorname{Ho}\mathbb{X}$ is then obtained by enlarging the isomorphisms classes of $\mathbb{X}_{cf}/\sim$ to provide a one-to-one correspondence with $\operatorname{ob}\mathbb{X}$.

Proof. The composition in $\operatorname{Ho}\mathbb{X}$ is well-defined since it is in $\mathbb{X}_{fc}/\sim$. The provided morphisms in $\operatorname{Ho}\mathbb{X}$ are identities for the given composition since they are in $\mathbb{X}_{fc}/\sim$.

To show that γ is a functor we need to show that it respects domains, codomains, identities, and composition.

First, let $f\colon X \longrightarrow Y$ in $\mathbb{X}$ be arbitrary, then

$$\gamma(f) = [\mathrm{fibcofib}(f)] \in \pi(\mathrm{fibcofib}(X), \mathrm{fibcofib}(Y)) = \mathrm{Ho}\,\mathbb{X}(X, Y)$$

so γ respects domains and codomains. Notice that the following diagram commutes:

$$\begin{CD}
\mathrm{fibcofib}(X) @>{1_{\mathrm{fibcofib}(X)}}>> \mathrm{fibcofib}(X)\\
@V{q_{\mathrm{fib}(X)}}V{\simeq}V @V{\simeq}V{q_{\mathrm{fibcofib}(X)}}V\\
\mathrm{fib}(X) @>{1_{\mathrm{fib}(X)}}>> \mathrm{fib}(X)\\
@V{p_X}V{\simeq}V @A{p_X}A{\simeq}A\\
X @>>{1_X}> X
\end{CD}$$

so, by Corollary 3.2.12 we get that $\gamma(1_X) = [1_{\mathrm{fibcofib}(X)}]$ so γ respects identities.

Now let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ in $\mathbb{X}$, then the following diagram commutes

$$\begin{CD}
\mathrm{fibcofib}(X) @>{\mathrm{fibcofib}(f)\mathrm{fibcofib}(g)}>> \mathrm{fibcofib}(Z)\\
@V{q_{\mathrm{fib}(X)}}V{\simeq}V @V{\simeq}V{q_{\mathrm{fibcofib}(Z)}}V\\
\mathrm{fib}(X) @>{\mathrm{fib}(f)\mathrm{fib}(g)}>> \mathrm{fib}(Z)\\
@V{p_X}V{\simeq}V @A{p_Z}A{\simeq}A\\
X @>>{fg}> Z
\end{CD}$$

so $\gamma(fg) = [fg] = [f][g] = \gamma\, f \gamma\, g$ and thus γ respects composition. We conclude that γ is a functor.

□

Proposition 3.2.17. *For any map $f\colon X \longrightarrow Y$ in $\mathbb{X}$ we have that $\gamma(f)$ is an isomorphism if and only if f is a weak equivalence.*

Proof. By definition $\gamma(f) = [\mathrm{fibcofib}(f)]$ and $\mathrm{fibcofib}(f)$ is a weak equivalence if and only if f is by Corollary 3.2.12. Now if $[\mathrm{fibcofib}(f)]$ is an isomorphism then there is a morphism

$g\colon \operatorname{fibcofib} Y \longrightarrow \operatorname{fibcofib} X$ such that

$$\operatorname{fibcofib}(f)g \sim 1_{\operatorname{fibcofib}(X)}$$

$$g\operatorname{fibcofib}(f) \sim 1_{\operatorname{fibcofib}(Y)}$$

but this is true if and only if $\operatorname{fibcofib}(f)$ is a weak equivalence by Proposition 3.2.14.

□

Proposition 3.2.18. *Let $\mathbb{X}$ be a model category and let $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a functor such that, whenever $w\colon X \longrightarrow X'$ in $\mathbb{X}$ is a weak equivalence we get that $F(w)$ is an isomorphism. Then there is a unique functor $\widetilde{F}\colon \operatorname{Ho}\mathbb{X} \longrightarrow \mathbb{Y}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{F} & \mathbb{Y} \\ {\scriptstyle\gamma}\downarrow & \nearrow_{\widetilde{F}} & \\ \operatorname{Ho}\mathbb{X} & & \end{array}$$

Proof. Let $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be such a functor. First, if

$$\begin{array}{ccc} X \amalg X & \xrightarrow{\langle i_0 | i_1 \rangle} & \operatorname{Cyl}(X) \\ & {\scriptstyle\langle 1_X | 1_X \rangle}\searrow & \simeq\downarrow{\scriptstyle p} \\ & & X \end{array}$$

is a cylinder object then, since p is a weak equivalence we get that $F(p)$ is an isomorphism. Futher

$$F(i_0)F(p) = F(i_0 p) = F(1_X) = F(i_1 p) = F(i_1)F(p)$$

so $F(i_0) = F(i_1)$. So if $f \sim_l f'\colon X \longrightarrow X'$ in $\mathbb{X}$ let H be a homotopy from f to f', then

$$\begin{aligned} F(f) &= F(i_0 H) \\ &= F(i_0)F(H) \\ &= F(i_1)F(H) \\ &= F(i_1 H) \\ &= F(f'). \end{aligned}$$

Dually $F(p_0) = F(p_1)$ whenever $\langle p_0, p_1 \rangle$ fits into a path object diagram, so $g \sim_r g'$ implies $F(g) = F(g')$. In particular then define $\widetilde{F}\colon \operatorname{Ho}\mathbb{X} \longrightarrow \mathbb{Y}$ by $\widetilde{F}(X) = F(X)$ (on objects), and for $[f] \in \mathbb{X}(X, Y)$ define $\widetilde{F}([f])$ to be the composite

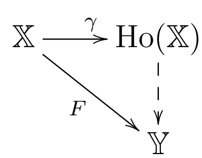
$$\widetilde{F}([f]) = F(p_X) \circ F(q_{\mathrm{fib}(X)})^{-1} \circ \gamma(f) \circ F(q_{\mathrm{fib}(Y)}) \circ F(p_Y)^{-1}$$

This morphism function clearly respects domains and codomains. That $\widetilde{F}$ respects identities and composition follows from the use of conjugation by the morphism $F(p)F(q)^{-1}$ in the definition of $\widetilde{F}$. To see that $F = \gamma\widetilde{F}$ one need only apply F to the commuting diagram in the statement of Corollary 3.2.12.

□

3.3 Derived Functors

If we have a model category $\mathbb{X}$ and a functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ then it would be nice if we could factor F through γ as in the following diagram



If we can find an honest-to-goodness factorization then clearly that's fantastic. However, as with all things functorial it is beneficial to look for solutions "up to natural transformation". With this in mind, we define the left and right derived functors as follows:

Definition 3.3.1 ([5]). Given a model category $\mathbb{X}$, any category $\mathbb{Y}$, and any functor

$$F\colon \mathbb{X} \longrightarrow \mathbb{Y}$$

define the *left derived functor of* F, denoted L_F to be, if it exists, a functor

$$\operatorname{Ho}(\mathbb{X}) \longrightarrow \mathbb{Y}$$

and a natural transformation α

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\gamma} & \mathrm{Ho}(\mathbb{X}) \\ & {\scriptstyle F}\searrow \;\Swarrow \alpha & \downarrow L_F \\ & & \mathbb{Y} \end{array}$$

such that, for any other pair consisting of a functor $G\colon \mathrm{Ho}(\mathbb{X}) \longrightarrow \mathbb{Y}$ and a natural transformation

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\gamma} & \mathrm{Ho}(\mathbb{X}) \\ & {\scriptstyle F}\searrow \;\Swarrow \beta & \downarrow G \\ & & \mathbb{Y} \end{array}$$

there is a unique $\beta'\colon G \Longrightarrow L_F$ such that the diagram

$$\begin{array}{ccc} \gamma G & \overset{\beta}{\Longrightarrow} & F \\ {\scriptstyle \gamma(\beta')}\Downarrow & \nearrow\!\!\!\nearrow {\scriptstyle \alpha} & \\ \gamma L_F & & \end{array}$$

commutes.

Dually, the *right derived functor of F* is a functor $R_F\colon \mathrm{Ho}\,\mathbb{X} \longrightarrow \mathbb{Y}$ and a natural transformation $\omega\colon F \Longrightarrow \gamma R_F$ which uniquely factors any natural transformation

$$\eta\colon F \Longrightarrow \gamma G.$$

We then get the following nice theorem.

Theorem 3.3.2 (see [5] Proposition 9.3)**.** *Let $\mathbb{X}$ be a model category and let $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a functor. If for every weak equivalence $f\colon X \longrightarrow X'$ between cofibrant objects in $\mathbb{X}$ we have that $F(f)$ is an isomorphism then (L_F, α) exists. Furthermore, α is a natural isomorphism when restricted to the full subcategory of cofibrant objects of $\mathbb{X}$.*

Dually, if $G\colon \mathbb{X} \longrightarrow \mathbb{Y}$ sends weak equivalences between fibrant objects to isomorphisms then (R_G, ω) exists and again ω is a natural isomorphism when restricted to the full subcategory of fibrant objects of $\mathbb{X}$.

Proof Sketch. The idea is this: by a construction similar to 3.2.12 we get that cofibrant replacement system $(\mathrm{cofib}, \mathrm{q})$ can be extended to a functor $\mathbb{X} \longrightarrow \pi_{\mathrm{r}}\,\mathbb{X}_c$ which sends each weak equivalence to the equivalence class of a weak equivalence.

It can then be shown that $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ respects right homotopy and thus induces a functor $F'\colon \pi_{\mathrm{r}} \mathbb{X}_c \longrightarrow \mathbb{Y}$ by $F'(C) = F(C)$ for all objects $C \in \pi_{\mathrm{r}} \mathbb{X}_c$ and $F'([f]) = [F(f)]$, hence if an equivalence class of $\pi_{\mathrm{r}} \mathbb{X}_c$ can be represented by a weak equivalence, then F' sends that equivalence class to an isomorphism.

It then follows that the composite $F'(\mathrm{cofib}(-))\colon \mathbb{X} \longrightarrow \mathbb{Y}$ takes weak equivalences to isomorphisms, hence by Proposition 3.2.18 there is a unique functor, suggestively denoted by $L_F\colon \operatorname{Ho}\mathbb{X} \longrightarrow \mathbb{Y}$, such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\text{cofib}} & \pi_{\mathrm{r}} \mathbb{X}_c \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle F'} \\ \operatorname{Ho}\mathbb{X} & \xrightarrow[L_F]{} & \mathbb{Y} \end{array} .$$

The natural transformation $\alpha\colon L_F \Longrightarrow F$ is obtained by setting

$$\alpha_X := F(\mathrm{q}_X)\colon F(\mathrm{cofib}(X)) \longrightarrow F(X).$$

That α_X is an isomorphism when X is cofibrant follows from F sending weak equivalences between cofibrant objects to isomorphisms.

The pair (L_F, α) is then the left derived functor of F.

□

In particular, if F is a functor which behaves nicely with respect to weak equivalences on the subcategory of cofibrant objects, then it can be tweaked to behave nicely on the entire homotopy category by precomposing with the cofibrant replacement functor and invoking the universal property of γ.

The next definition will be important in Chapter 6 where we will use it to build a symmetric monoidal product on $\operatorname{Ho}\mathbb{X}$ from a symmetric monoidal product on $\mathbb{X}$.

Definition 3.3.3. Given a functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ where $\mathbb{X}$ and $\mathbb{Y}$ are model categories, then we have the composite $F\gamma^{\mathbb{Y}}\colon \mathbb{X} \longrightarrow \operatorname{Ho}\mathbb{Y}$. Let $\mathbf{L}_F$ and $\mathbf{R}_F$ denote the left and right derived functor of this composite when they exist. Call $\mathbf{L}_F$ the *total left derived functor of F* and call $\mathbf{R}_F$ the *total right derived functor of F.*

Theorem 3.3.4. *Let $\mathbb{X}$ be a model category and let $\wedge$ be a closed symmetric monoidal product on $\mathbb{X}$ with unit S^0. If $X \wedge - \colon \mathbb{X} \longrightarrow \mathbb{X}$ has a total left-derived functor, T^l_X, for each $X \in \mathbb{X}$ then $\operatorname{Ho}\mathbb{X}$ has a symmetric monoidal product, $\wedge^{\mathrm{L}}$, given by $X \wedge^{\mathrm{L}} Y = T^l_X(Y)$ on objects with unit $\gamma\, S^0$*

Definition 3.3.5. Let $\mathbb{X}$ and $\mathbb{Y}$ be model categories and consider functors $F \dashv G \colon \mathbb{X} \longrightarrow \mathbb{Y}$ satisfying the following additional properties:

1. F sends cofibrations in $\mathbb{X}$ to cofibrations in $\mathbb{Y}$
2. G sends fibrations in $\mathbb{Y}$ to fibrations in $\mathbb{X}$

Call such a pair a *Quillen adjunction*, call F a *left Quillen functor* and call G a *right Quillen functor.*

If in addition we have that, for every cofibrant $X \in \mathbb{X}$ and every fibrant $Y \in \mathbb{Y}$ any morphism $X \longrightarrow G(Y)$ is a weak equivalence if and only if $F(X) \longrightarrow Y$ is a weak equivalence, then call the pair (F, G) a *Quillen equivalence.*

Example 3.3.6. Between the categories of simplicial sets and topological spaces there is the geometric realization - singularization adjunction:

$$|-| \dashv \operatorname{Sing} \colon \mathbf{sSet} \longrightarrow \mathbf{Top}$$

This is a Quillen equivalence by the definition of the model structure on **sSet**.

Theorem 3.3.7 (See [9] Theorems 8.5.18 and 8.5.23)**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be model categories and let $F \dashv G \colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a Quillen adjunction, then*

1. $\mathbf{L}_F \dashv \mathbf{R}_G \colon \operatorname{Ho}(\mathbb{X}) \longrightarrow \operatorname{Ho}(\mathbb{Y})$
2. *$F \dashv G$ is a Quillen equivalence if and only if $\mathbf{L}_F \dashv \mathbf{R}_G$ is an equivalence of categories.*

Provided one ignores the fact that the 'category' of all Model Categories is not locally small, a Quillen adjunction is the "correct" notion of a morphism between model categories, further Quillen equivalences are the correct notion of equivalence between model categories.

Chapter 4

Hurewicz, Categorically

Introduction

The objective of this section is to define a category $\mathbb{X}$, ultimately called a *Hurewicz* category, with enough structure so that the Hurewicz theorem both makes sense - that is, there are homotopy groups, homology groups, and a natural transformation between them - and is true - for n-connected objects of $\mathbb{X}$ the $(n+1)^{\text{st}}$ Hurewicz homomorphism is an isomorphism and the $(n+2)^{\text{nd}}$ is an epimorphism.

The reason for this is threefold, first is to be able to use the theory to outline the proof of the Hurewicz theorem in the case of symmetric spectra, second is to highlight the similarities and differences between the proofs for the case of spectra and the case of simplicial sets, and third because abstraction has a wonderful way of shedding light on phenomenae.

The reader should be advised that Chapters **??** and 6 are devoted to providing the example of a Hurewicz category.

4.1 Sphere Monoidal Categories

In this subsection, we define spheres in $\mathbb{X}$ and establish their important properties. Spheres will be at the heart of defining π_n and the subcategory of $\mathbb{X}$ analogous to the n-connected objects.

We begin with the algebraic structure of spheres. In particular we give a definition of a *cogroup object*, note that more general definitions exist. In particular I do *not* claim to be the first to define nor to study these objects however, I was unable to find a reference to the first appearance of these objects in the literature. We restrict ourselves to the following definition for simplicity.

Definition 4.1.1. A *cogroup object* in a category $\mathbb{X}$, with initial object $\perp$, which is closed under finite coproducts is an object $G \in \mathbb{X}$ together with morphisms

$$\delta\colon G \longrightarrow G \amalg G \qquad \varepsilon\colon G \longrightarrow \perp \quad \upsilon\colon G \longrightarrow G$$

such that (G, δ, ε) is an *internal comonoid* - i.e. a comonoid in $(\mathbb{X}, \amalg, \perp)$ and such that

$$\begin{array}{ccc} G & \xrightarrow{\ \delta\ } & G \amalg G \\ {\scriptstyle\delta}\downarrow & \searrow^{1_G} & \downarrow{\scriptstyle\langle \upsilon | 1_G \rangle} \\ G \amalg G & \xrightarrow[\langle 1_G | \upsilon \rangle]{} & G \end{array}$$

commutes. If δ is cocommutative, then G is called an *abelian cogroup object.*

Example 4.1.2. In Ho $\mathbf{Top}_*$, each $\mathcal{S}^n$ for $n \geq 1$ is a cogroup. The morphism δ is the homotopy class of the map which collapses the $\mathcal{S}^{n-1}$ equator, the morphism ε is the map to the singleton, and υ is the homotopy class of the map which reflects $\mathcal{S}^n$ about the equator.

Motivated by this example we make the following definition. For what follows recall the definition of the *arrow category* from Definition 1.2.10.

Definition 4.1.3. Let $\mathbb{X}$ be a symmetric monoidal category $\mathbb{X}$ with all finite coproducts and a zero object $\mathbf{0}$.

A *sphere monoidal structure* on $\mathbb{X}$ consists of the following data.

1. For each $n \in \mathbb{Z}$ a cogroup object $\mathscr{S}^n$ such that the object $\mathscr{S}^0$ is the unit of the monoidal product on $\mathbb{X}$. Let the comultiplication morphism of $\mathscr{S}^n$ be denoted by

$$\delta_n\colon \mathscr{S}^n \longrightarrow \mathscr{S}^n \amalg \mathscr{S}^n$$

and be called the *collapse morphism* for $\mathscr{S}^n$.

2. A family of isomorphisms

$$\alpha_{n,m}\colon\ \mathscr{S}^n \wedge \mathscr{S}^m \longrightarrow \mathscr{S}^{n+m}$$

such that the diagrams

$$\begin{array}{ccc}
\mathscr{S}^n \wedge \mathscr{S}^m & \xrightarrow{\delta_n \wedge 1_{\mathscr{S}^m}} & (\mathscr{S}^n \amalg \mathscr{S}^n) \wedge \mathscr{S}^m \\
{\scriptstyle \alpha_{n,m}}\uparrow & & \\
\mathscr{S}^{n+m} & & \uparrow{\scriptstyle \langle \iota_0 \wedge \mathscr{S}^m \,|\, \iota_1 \wedge \mathscr{S}^m \rangle} \\
{\scriptstyle \delta_{n+m}}\downarrow & & \\
\mathscr{S}^{n+m} \amalg \mathscr{S}^{n+m} & \xrightarrow[\alpha_{n,m} \amalg \alpha_{n,m}]{} & (\mathscr{S}^n \wedge \mathscr{S}^m) \amalg (\mathscr{S}^n \wedge \mathscr{S}^m)
\end{array} \tag{4.1}$$

along with the evident pair of the diagram above, and

$$\begin{array}{ccc}
\mathscr{S}^{n+m+k} & \xrightarrow{\alpha_{n,m+k}} & \mathscr{S}^n \wedge \mathscr{S}^{m+k} \\
{\scriptstyle \alpha_{n+m,k}}\downarrow & & \downarrow{\scriptstyle \mathscr{S}^n \wedge \alpha_{m,k}} \\
\mathscr{S}^{n+m} \wedge \mathscr{S}^k & \xrightarrow[\alpha_{n,m} \wedge \mathscr{S}^k]{} & \mathscr{S}^n \wedge \mathscr{S}^m \wedge \mathscr{S}^k
\end{array} \tag{4.2}$$

commute for all $n, m, k \in \mathbb{Z}$.

3. A functor

$$\text{Cone}:\ \text{Ar}\,\mathbb{X} \longrightarrow \mathbb{X}$$

called the *mapping cone* and a natural transformation

$$\mathrm{i}_f\colon\ \text{cod}(f) \Longrightarrow \text{Cone}\, f.$$

If, in addition, $\delta_n\colon\ \mathscr{S}^n \longrightarrow \mathscr{S}^n \amalg \mathscr{S}^n$ is cocommutative for each $n \in \mathbb{Z}$, we say that $\mathbb{X}$ is an cocommutative sphere monoidal category.

Given a sphere monoidal category $\mathbb{X}$ and a subcategory $\mathbb{Y}$, say that Cone *restricts to* $\mathbb{Y}$ if there is a functor $C\colon\ \text{Ar}\,\mathbb{Y} \longrightarrow \mathbb{Y}$ such that the following diagram

$$\begin{array}{ccc}
\text{Ar}\,\mathbb{Y} & \xrightarrow{C} & \mathbb{Y} \\
\downarrow & & \downarrow \\
\text{Ar}\,\mathbb{X} & \xrightarrow[\text{Cone}]{} & \mathbb{X}
\end{array}$$

commutes.

Remark. Diagram 4.1 relates the cogroup structures of the various cogroups. In cases where $\wedge$ preserves coproducts this diagram actually completely determines the cogroup structure of $\mathscr{S}^n$ from those of $\mathscr{S}^1$ and $\mathscr{S}^{-1}$.

The following two propositions, and the outlines of their proofs, are standard.

Proposition 4.1.4. *In any category $\mathbb{X}$ with finite coproducts, G has a cogroup structure if and only if $\mathbb{X}(G, -)$ may be considered as a functor from $\mathbb{X}$ to* **Grp**.

Proof Sketch. If $(G, \delta, \varepsilon, \upsilon)$ is a cogroup then the fact that $(\mathbb{X}(G, X), \mathbb{X}(\delta, X), \mathbb{X}(\varepsilon, X), \mathbb{X}(\upsilon, X))$ is a group in **Set** follows from the fact that

$$\mathbb{X}(-, X) \colon \mathbb{X} \longrightarrow \mathbf{Set}$$

takes coproducts to products, and commuting diagrams to opposite commuting diagrams. It is also true that the opposite diagrams of those given are exactly those defining a group in **Set**. That

$$\mathbb{X}(G, h) \colon \mathbb{X}(G, X) \longrightarrow \mathbb{X}(G, Y)$$

is a homomorphism follows from the identity

$$\langle f | g \rangle \circ h = \langle f \circ h | g \circ h \rangle$$

from Proposition 1.3.15.

Conversely, if $\mathbb{X}(G, -)$ may be considered as a functor to the category of groups then let...

... $\delta := \iota_0 \cdot \iota_1 \colon G \longrightarrow G \amalg G$

... $\varepsilon := \mathrm{id}_{\mathbb{X}(G,\perp)}$

... υ be the inverse of $1_G \colon G \longrightarrow G$ under the group operation of $\mathbb{X}(G, G)$.

and chase the appropriate diagrams.

$\square$

Proposition 4.1.5. *In a category $\mathbb{X}$ with finite coproducts, G is a cocommutative cogroup if and only if $\mathbb{X}(G,-)$ may be considered as a functor to* **Ab**.

Proof Sketch. If $(G,\delta,\varepsilon,\upsilon)$ is cocommutative then

$$\delta \circ \text{flip} = \delta$$

by definition, hence

$$\begin{aligned}\mathbb{X}(\delta, X) &= \mathbb{X}(\delta), X \circ \mathbb{X}(\text{flip}, X)\\ &= \mathbb{X}(\delta, X) \circ \text{twist}_\times\end{aligned}$$

holds, thus the operation $\mathbb{X}(\delta, X)$ is commutative on $\mathbb{X}(G,X)$. This holds for arbitrary $X \in \mathbb{X}$.

Conversely, if $\mathbb{X}(G,-)$ may be considered as a functor to the category of abelian groups, then, by the proof sketch of Proposition 4.1.4 we get that

$$\delta = \iota_0 + \iota_1 = \iota_1 + \iota_0 = \delta \circ \text{flip}.$$

□

Definition 4.1.6. Let $\mathbb{X}$ be an cocommutative sphere monoidal category and define

$$\pi_n \colon \mathbb{X} \longrightarrow \mathbf{Ab}$$

by

$$\pi_n(f\colon X \longrightarrow Y) := \mathbb{X}(\mathscr{S}^n, f)\colon \mathbb{X}(\mathscr{S}^n, X) \longrightarrow \mathbb{X}(\mathscr{S}^n, Y)$$

where the group structure on $\mathbb{X}(\mathscr{S}^n, X)$ is as in Proposition 4.1.4. Call π_n the n^{th}-*homotopy group* of $\mathbb{X}$.

Definition 4.1.7. For a fixed $n \in \mathbb{Z}$ consider the class of subcategories, $\mathcal{X}$, of $\mathbb{X}$ such that

1. $\mathscr{S}^k \in \mathcal{X}$ for all $k \geq n$.

2. $\mathcal{X}$ has all finite coproducts.

3. the mapping cone $\mathrm{Cone}: \ \mathrm{Ar}\,\mathbb{X} \longrightarrow \mathbb{X}$ restricts to $\mathcal{X}$.

The intersection of all such subcategories is denoted $\mathcal{X}_n$ and an object of $\mathcal{X}_n$ is called n-*generated.*

Definition 4.1.8. Call an cocommutative sphere monoidal category *good* if the following hold

1. $\pi_n(\mathscr{S}^n) \cong \mathbb{Z}$ for each $n \in \mathbb{Z}$.

2. $\pi_n(\mathscr{S}^{n+k}) \cong 0$ for all $n \in \mathbb{Z}$ and every integer $k \geq 1$.

3. π_n preserves coproducts for all $n \in \mathbb{Z}$.

4. for any $n \in \mathbb{Z}$ and any morphism f of $\mathbb{X}$ we have that the equation

$$\operatorname{coker} \pi_n(f) = \ker \pi_n(\mathrm{i}_f)$$

holds.

5. The homomorphisms $\pi_n(\mathrm{i}_f)\colon \pi_n(\operatorname{cod} f) \longrightarrow \pi_n(\operatorname{Cone} f)$ are epimorphisms whenever f is a morphism between n-generated objects.

4.2 Freudenthal Categories

In the previous section we established the necessary definitions to provide a symmetric monoidal category, $\mathbb{X}$, with $\mathbb{Z}$-indexed collections of spheres, homotopy groups, and distinguished classes of objects inspired by the spheres, homotopy groups, and n-connected objects of $\mathrm{Ho}\,\mathbf{Top}_*$. We will now turn our attention towards giving $\mathbb{X}$ a Freudenthal Suspension Theorem-like property.

Theorem 4.2.1. *Let $\mathbb{X}$ be a cocommutative sphere monoidal category with monoidal product $\wedge$ such that*

$$\mathbf{0} \wedge - \colon \mathbb{X} \longrightarrow \mathbb{X}$$

sends every object to the zero object of $\mathbb{X}$*. Then for any objects* $X, Y \in \mathbb{X}$*, and any* $n, m \in \mathbb{Z}$ *there is a natural morphism*

$$\varphi_{n,m} \colon \pi_n(X) \otimes \pi_m(Y) \longrightarrow \pi_{n+m}(X \wedge Y)$$

defined on generators by sending $f \otimes g$ *to the composite*

$$\mathscr{S}^{n+m} \xrightarrow{\alpha_{n,m}} \mathscr{S}^n \wedge \mathscr{S}^m \xrightarrow{f \wedge g} X \wedge Y$$

for any pair of maps $f \colon \mathscr{S}^n \longrightarrow X$ *and* $g \colon \mathscr{S}^m \longrightarrow Y$*. Call this homomorphism the* splicing homomorphism*.*

Proof. Define a function

$$\pi_n X \times \pi_m Y \longrightarrow \pi_{n+m} X \wedge Y \tag{4.3}$$

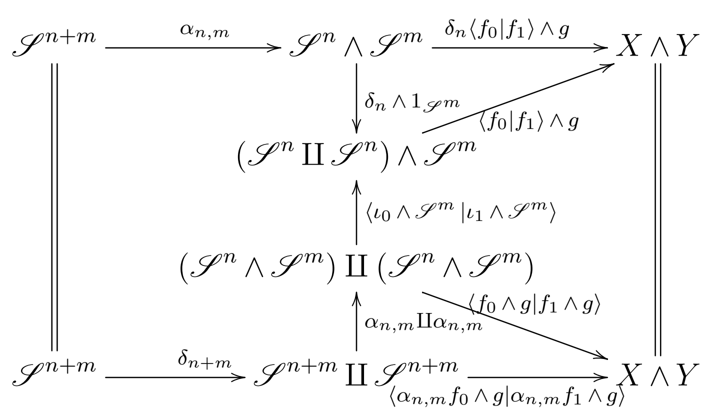
$$(f, g) \longmapsto (\ \mathscr{S}^{n+m} \xrightarrow{\alpha_{n,m}} \mathscr{S}^n \wedge \mathscr{S}^m \xrightarrow{f \wedge g} X \wedge Y\ ) \tag{4.4}$$

Since $\mathbf{0} \wedge -$ is the constant morphism on $\mathbf{0}$ we get that, for any $g \in \pi_m Y$, any $n \in \mathbb{Z}$, and any $X \in \mathbb{X}$, the pair $(\mathbf{0}_{\mathscr{S}^n X}, g)$ is sent, via 4.3 to the composite

$$\mathscr{S}^{n+m} \xrightarrow{\alpha_{n,m}} \mathscr{S}^n \wedge \mathscr{S}^m \xrightarrow{\mathbf{0}_{\mathscr{S}^n} \wedge g} \mathbf{0} \wedge Y \xrightarrow{\cong} \mathbf{0} \longrightarrow X \wedge Y$$

which is just the zero morphism, hence the identity element of $\pi_{n+m}(X \wedge Y)$.

Let $f_0, f_1 \in \pi_n X$ and $g \in \pi_m(Y)$. We have the following diagram



which commutes since the left-hand rectangle commutes by assumption, the upper-right triangle commutes by the functoriality of $\wedge$, the lower-right triangle commutes by Proposition 1.3.15, and the remaining square can be seen to commute by precomposing with the morphisms

$$\iota_0, \iota_1 \colon \mathscr{S}^n \wedge \mathscr{S}^m \longrightarrow (\mathscr{S}^n \wedge \mathscr{S}^m) \amalg (\mathscr{S}^n \wedge \mathscr{S}^m)$$

which are bi-epic.

Notice that the top horizontal composite is the image of $(f_0 + f_1, g)$ under the function (4.3), and the lower composite is the sum of the images of (f_0, g) and (f_1, g). Hence our morphism is linear in the first variable.

By symmetry the function (4.3) is also linear in the second variable, hence bilinear. By the universal property of the tensor product we then get a map

$$\varphi_{n,m} \colon \pi_n(X) \otimes \pi_m(Y) \longrightarrow \pi_{n+m}(X \wedge Y)$$

defined exactly the way we wanted. This morphism is clearly natural.

□

Definition 4.2.2. Let $(\mathbb{X}, \wedge, \mathscr{S}^0)$ be a good cocommutative sphere monoidal category. Recall then that

$$\pi_1(\mathscr{S}^1) \cong \mathbb{Z}$$

by Definition 4.1.8. If, for every $n \in \mathbb{Z}$ the splicing morphism

$$\pi_1 \mathscr{S}^1 \otimes \pi_n(X) \longrightarrow \pi_{1+n}(\mathscr{S}^1 \wedge X)$$

is a natural isomorphism then call $(\mathbb{X}, \wedge, \mathscr{S}^0)$ a *Freudenthal category.*

Lemma 4.2.3. *In any sphere monoidal category, $\mathbb{X}$, for any $n, m \in \mathbb{Z}$ and any $X \in \mathbb{X}$ the diagram*

$$\begin{array}{ccc} \pi_k(X) \otimes \pi_m(Y) \otimes \pi_n(Z) & \xrightarrow{\pi_k(X)\otimes\varphi_{m,n}} & \pi_k(X) \otimes \pi_{m+n}(Y \wedge Z) \\ \downarrow{\scriptstyle \varphi_{k,m}\otimes\pi_n(Z)} & & \downarrow{\scriptstyle \varphi_{k,m+n}} \\ \pi_{k+m}(X \wedge Y) \otimes \pi_n(Z) & \xrightarrow[\varphi_{k+m,n}]{} & \pi_{k+m+n}(X \wedge Y \wedge Z) \end{array} \tag{4.5}$$

commutes.

Proof. Given morphisms $f\colon \mathscr{S}^k \longrightarrow X$, $g\colon \mathscr{S}^m \longrightarrow Y$, and $h\colon \mathscr{S}^n \longrightarrow Z$ compute clockwise

$$
\begin{aligned}
f \otimes g \otimes h &\longmapsto f \otimes (\ \mathscr{S}^{m+n} \xrightarrow{\alpha_m n} \mathscr{S}^m \amalg \mathscr{S}^n \xrightarrow{g \wedge h} Y \wedge Z\ ) \\
&= f \otimes (\alpha_{m,n}(g \wedge h)) \\
&\longmapsto (\ \mathscr{S}^{k+m+n} \xrightarrow{\alpha_{k,m+n}} \mathscr{S}^k \wedge \mathscr{S}^{m+n} \xrightarrow{f \wedge (\alpha_{m,n} \circ g \wedge h)} X \wedge Y \wedge Z\ ) \\
&= \alpha_{k,m+n} \circ f \wedge (\alpha_{m,n}(g \wedge h))
\end{aligned}
$$

and counterclockwise

$$
\begin{aligned}
f \otimes g \otimes h &\longmapsto (\ \mathscr{S}^{k+m} \xrightarrow{\alpha_{k,m}} \mathscr{S}^k \wedge \mathscr{S}^m \xrightarrow{f \wedge g} X \wedge Y\ ) \otimes h \\
&= (\alpha_{k,m} \circ (f \wedge g)) \otimes h \\
&\longmapsto\ \mathscr{S}^{k+m+n} \xrightarrow{\alpha_{k+m,n}} \mathscr{S}^{k+m} \wedge \mathscr{S}^n \xrightarrow{(\alpha_{k,m} \circ (f \wedge g)) \wedge h} X \wedge Y \wedge Z \\
&= \alpha_{k+m,n} \circ (\alpha_{k,m} \circ (f \wedge g)) \wedge h
\end{aligned}
$$

The following diagram commutes

$$
\begin{array}{ccccc}
\mathscr{S}^{k+m+n} & \xrightarrow{\alpha_{k+m,n}} & \mathscr{S}^{k+m} \wedge \mathscr{S}^n & \xrightarrow{f \wedge (\alpha_{m,n}(g \wedge h))} & X \wedge Y \wedge Z \\
{\scriptstyle \alpha_{k,m+n}} \big\downarrow & (1) & \big\downarrow {\scriptstyle \alpha_{k,m} \wedge 1_{\mathscr{S}^n}} & (2) & \big\| \\
\mathscr{S}^k \wedge \mathscr{S}^{m+n} & \xrightarrow[1_{\mathscr{S}^k} \wedge \alpha_{m,n}]{} & \mathscr{S}^k \wedge \mathscr{S}^m \wedge \mathscr{S}^n & \xrightarrow[f \wedge g \wedge h]{} & X \wedge Y \wedge Z \\
{\scriptstyle (\alpha_{k,m} \circ (f \wedge g)) \wedge h} \big\downarrow & (3) & \big\downarrow {\scriptstyle f \wedge g \wedge h} & (4) & \big\| \\
X \wedge Y \wedge Z & = & X \wedge Y \wedge Z & = & X \wedge Y \wedge Z
\end{array}
$$

The square labeled 1 commutes by Diagram 4.2, 2 and 3 commute since functors preserve composition, and 4 commutes clearly. This implies that the clockwise and counterclockwise computations above agree, hence Diagram 4.5 commutes. We conclude that the lemma is true.

□

Corollary 4.2.4. *In any good cocommutative sphere monoidal category $\mathbb{X}$, for any $n, m \in \mathbb{Z}$ and any $X \in \mathbb{X}$ the following diagram commutes*

$$\begin{array}{ccc}
\pi_{m-1}(\mathscr{S}^{m-1}) \otimes \pi_n(X) & \xrightarrow{\varphi_{m-1,n}} & \pi_{m-1+n}(\mathscr{S}^{m-1} \wedge X) \\
\Big\downarrow {\scriptstyle \lambda_{\otimes}^{-1}} & & \Big\downarrow {\scriptstyle \lambda_{\otimes}^{-1}} \\
\pi_1(\mathscr{S}^1) \otimes \pi_{m-1}(\mathscr{S}^{m-1}) \otimes \pi_n(X) & & \pi_1(\mathscr{S}^1) \otimes \pi_{m-1+n}(\mathscr{S}^{m-1} \wedge X) \\
\Big\downarrow {\scriptstyle \varphi_{1,m-1} \otimes 1_{\pi_n(X)}} & & \Big\downarrow {\scriptstyle \varphi_{1,m-1+n}} \\
\pi_{1+(m-1)}(\mathscr{S}^1 \wedge \mathscr{S}^{m-1}) \otimes \pi_n(X) & & \pi_{m+n}(\mathscr{S}^1 \wedge \mathscr{S}^{m-1} \wedge X) \\
\Big\downarrow {\scriptstyle \pi_m(\alpha^{-1}_{1,m-1}) \otimes 1_{\pi_n(X)}} & & \Big\downarrow {\scriptstyle \pi_{m+n}(\alpha^{-1}_{1,m-1} \wedge 1_X)} \\
\pi_m(\mathscr{S}^m) \otimes \pi_n(X) & \xrightarrow[\varphi_{m,n}]{} & \pi_{n+m}(\mathscr{S}^m \wedge X)
\end{array} \tag{4.6}$$

Proof. Notice that the diagram

$$\begin{array}{ccc}
\pi_{m-1}(\mathscr{S}^{m-1}) \otimes \pi_n(X) & \xrightarrow{\varphi_{m-1,n}} & \pi_{m-1+n}(\mathscr{S}^{m-1} \wedge X) \\
\Big\downarrow {\scriptstyle \lambda_{\otimes}^{-1}} & & \Big\downarrow {\scriptstyle \lambda_{\otimes}^{-1}} \\
\pi_1(\mathscr{S}^1) \otimes \pi_{m-1}(\mathscr{S}^{m-1}) \otimes \pi_n(X) & \xrightarrow{1_{\pi_1(\mathscr{S}^1)} \otimes \varphi_{m-1,n}} & \pi_1(\mathscr{S}^1) \otimes \pi_{m-1+n}(\mathscr{S}^{m-1} \wedge X) \\
\Big\downarrow {\scriptstyle \varphi_{1,m-1} \otimes 1_{\pi_n(X)}} & & \Big\downarrow {\scriptstyle \varphi_{1,m-1+n}} \\
\pi_{1+(m-1)}(\mathscr{S}^1 \wedge \mathscr{S}^{m-1}) \otimes \pi_n(X) & \xrightarrow{\varphi_{m,n}} & \pi_{m+n}(\mathscr{S}^1 \wedge \mathscr{S}^{m-1} \wedge X) \\
\Big\downarrow {\scriptstyle \pi_m(\alpha^{-1}_{1,m-1}) \otimes 1_{\pi_n(X)}} & & \Big\downarrow {\scriptstyle \pi_{m+n}(\alpha^{-1}_{1,m-1} \wedge 1_X)} \\
\pi_m(\mathscr{S}^m) \otimes \pi_n(X) & \xrightarrow[\varphi_{m,n}]{} & \pi_{n+m}(\mathscr{S}^m \wedge X)
\end{array}$$

commutes since each internal square commutes - the top one by the naturality of λ, the bottom by the naturality of $\varphi_{n,m}$, and the middle one by Lemma 4.2.3. But the exterior rectangle of this diagram is just Diagram 4.2.4.

□

Proposition 4.2.5. *In a Freudenthal category the splicing map*

$$\varphi_{m,n} \colon \pi_m(\mathscr{S}^m) \otimes \pi_n(X) \longrightarrow \pi_{m+n}(\mathscr{S}^m \wedge X)$$

is an isomorphism for all $m, n \in \mathbb{Z}$ and every $X \in \mathbb{X}$.

Proof. First notice that, in a Freudenthal category, the vertical composites of the diagram of Corollary 4.2.4 are isomorphisms. Hence, for any $n, m \in \mathbb{Z}$ and any $X \in \mathbb{X}$

$$\varphi_{m-1,n,:}\, \pi_{m-1}(\mathscr{S}^{m-1}) \otimes \pi_n(X) \longrightarrow \pi_{m-1+n}(\mathscr{S}^{m-1} \wedge X)$$

is an isomorphism if and only if

$$\varphi_{m,n,:}\, \pi_m(\mathscr{S}^m) \otimes \pi_n(X) \longrightarrow \pi_{m+n}(\mathscr{S}^m \wedge X)$$

is an isomorphism.

We proceed by induction. Let $P(m)$ be the statement: $\forall n \in \mathbb{Z}, \forall X \in \mathbb{X}$ the morphisms

$$\varphi_{m,n} \colon \pi_m(\mathscr{S}^m) \otimes \pi_n(X) \longrightarrow \pi_{m+n}(\mathscr{S}^m \wedge X)$$

and

$$\varphi_{-m,n} \colon \pi_{-m}(\mathscr{S}^{-m}) \otimes \pi_n(X) \longrightarrow \pi_{-m+n}(\mathscr{S}^m \wedge X)$$

are isomorphisms.

Our base case is $P(0)$: we need to show that

$$\varphi_{0,n} \colon \pi_0(\mathscr{S}^0) \otimes \pi_n(X) \longrightarrow \pi_n(\mathscr{S}^0 \wedge X)$$

is an isomorphism, but by our note above we have that this is an isomorphism if and only if

$$\varphi_{1,n} \colon \pi_1(\mathscr{S}^1) \otimes \pi_n(X) \longrightarrow \pi_{1+n}(\mathscr{S}^1 \wedge X)$$

is, which is true by the definition of a Freudenthal category. This concludes our base case.

Now assume $P(k)$ is true for some $k \in \mathbb{Z}$, that is, assume

$$\varphi_{k,n} \colon \pi_k(\mathscr{S}^k) \otimes \pi_n(X) \longrightarrow \pi_{k+n}(\mathscr{S}^k \wedge X) \quad \text{and} \quad \varphi_{-k,n} \colon \pi_{-k}(\mathscr{S}^{-k}) \otimes \pi_n(X) \longrightarrow \pi_{-k+n}(\mathscr{S}^{-k} \wedge X)$$

are isomorphisms, we want to prove $P(k+1)$. This follows from our note above taking m to k and $-k+1$ respectively.

We conclude that $P(m)$ holds for all m.

□

In summary we now have that

$$\pi_{n+k}(\mathscr{S}^n \wedge X) \cong \pi_k(X)$$

for any $n, k \in \mathbb{Z}$ and any $X \in \mathbb{X}$. This is our Freudenthal Suspension Theorem-like property.

4.3 Hurewicz Categories

Finally we define the homology functors for a Freudenthal category $\mathbb{X}$ and the associated Hurewicz homomorphism. From this we prove a slightly weaker form of the Hurewicz Theorem. In particular, we prove the theorem for n-generated objects, which is a (possibly) weaker notion than that of $(n-1)$-connected objects.

Definition 4.3.1. Let $(\mathbb{X}, \wedge, \mathscr{S}^0)$ be a Freudenthal category. Call an object, $\mathrm{H}\,\mathbb{Z}$, such that

1. $\pi_k(\mathscr{S}^0) \cong \pi_k(\mathrm{H}\,\mathbb{Z})$ for all integers $k \leq 0$
2. If $\eta_{\mathrm{H}\,\mathbb{Z}}$ is a generator of $\pi_0(\mathrm{H}\,\mathbb{Z}) \cong \mathbb{Z}$ then

   $$\pi_1(\eta_{\mathrm{H}\,\mathbb{Z}})\colon \pi_1(\mathscr{S}^0) \longrightarrow \pi_1(\mathrm{H}\,\mathbb{Z})$$

   is an epimorphism.
3. there is a natural isomorphism $\mathrm{Cone}(1_{\mathrm{H}\,\mathbb{Z}} \wedge -) \cong \mathrm{H}\,\mathbb{Z} \wedge \mathrm{Cone}(-)$
4. the functor $\mathrm{H}\,\mathbb{Z} \wedge -$ preserves coproducts.

an *EM object* (EM for Eilenberg-MacLane). Call a tuple $(\mathbb{X}, \wedge, \mathrm{H}\,\mathbb{Z})$ a *Hurewicz category.*

Definition 4.3.2. If $(\mathbb{X}, \wedge, \mathrm{H}\,\mathbb{Z})$ is a Hurewicz category, then define the *homology functor*

$$\mathrm{H}_n(-) := \pi_n(\mathrm{H}\,\mathbb{Z} \wedge -)\colon \mathbb{X} \longrightarrow \mathbf{Ab}$$

Definition 4.3.3. Since $\pi_0(\mathrm{H}\,\mathbb{Z}) \cong \mathbb{Z}$ by hypothesis let $\eta_{\mathrm{H}\,\mathbb{Z}}\colon\ \mathscr{S}^0 \longrightarrow \mathrm{H}\,\mathbb{Z}$ be a fixed generator of $\pi_0(\mathrm{H}\,\mathbb{Z})$. Now for each $n \in \mathbb{Z}$ and each $X \in \mathbb{X}$ define the morphism

$$h_n^X\colon \pi_n(X) \to \mathrm{H}_n(X)$$

to be the composite

$$\pi_n(X) \xrightarrow{\pi_n(\lambda^{-1})} \pi_n(\mathscr{S}^0 \wedge X) \xrightarrow{\pi_n(\eta_{\mathrm{H}\,\mathbb{Z}} \wedge X)} \pi_n(\mathrm{H}\,\mathbb{Z} \wedge X) = \mathrm{H}_n(X)$$

Proposition 4.3.4. *The hurwicz homomorphism h is a natural transformation from $\pi_n(-)$ to $\mathrm{H}_n(-)$.*

Proof. Given any morphism $f\colon X \longrightarrow Y$ in $\mathbb{X}$ we get that the diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\lambda^{-1}} & \mathcal{S}^0 \wedge X & \xrightarrow{\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_X} & \mathrm{H}\,\mathbb{Z} \wedge X \\
{\scriptstyle f}\downarrow & & {\scriptstyle \mathcal{S}^0 \wedge f}\downarrow & & \downarrow{\scriptstyle \mathrm{H}\,\mathbb{Z} \wedge f} \\
Y & \xrightarrow{\lambda^{-1}} & \mathcal{S}^0 \wedge Y & \xrightarrow{\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_Y} & \mathrm{H}\,\mathbb{Z} \wedge Y
\end{array}$$

commutes since the left hand square commutes by the naturality of λ^{-1} and the right-hand square commutes since both paths compose to $\eta_{\mathrm{H}\,\mathbb{Z}} \wedge f$ by the functoriality of $\wedge$. Since the top composite is sent to h_X by π_n and the lower composite is sent to h_Y, applying π_n to this commuting diagram gives us the naturality diagram of h. □

4.4 The Weak Hurewicz Theorem

Throughout this section assume that $\mathbb{X}$ is a Hurewicz cateogry.

Lemma 4.4.1. *For any $X \in \mathbb{X}$ and any $n \in \mathbb{Z}$ the Hurewicz morphism*

$$h_n^X \colon \pi_n(X) \longrightarrow \mathrm{H}_n(X)$$

is an epimorphism if and only if the map

$$\varphi_{0,n}\colon \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n(X) \longrightarrow \pi_n(\mathrm{H}\,\mathbb{Z} \wedge X)$$

is an epimorphism. In addition, if one is an isomorphism, then so is the other.

Proof. We claim that the following diagram commutes

$$\begin{array}{ccc}
\pi_n(X) & \xrightarrow{\eta_{\mathrm{H}\,\mathbb{Z}} \otimes -} & \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n(X) \\
{\scriptstyle \pi_n(\lambda_X^{-1})}\downarrow & & \downarrow{\scriptstyle \varphi_{0,n}} \\
\pi_n(\mathcal{S}^1 \wedge X) & \xrightarrow[\pi_n(\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_X)]{} & \pi_n(\mathrm{H}\,\mathbb{Z} \wedge X)
\end{array}$$

In particular, let $f\colon \mathscr{S}^n \longrightarrow X$ be an element of $\pi_n(X)$, then following the diagram clockwise yields

$$\begin{aligned}(f\colon \mathscr{S}^n \longrightarrow X) &\longmapsto \eta_{\mathrm{H}\,\mathbb{Z}} \otimes f \\ &\longmapsto (\ \mathscr{S}^n \xrightarrow{\alpha_{0,n}} \mathscr{S}^0 \wedge \mathscr{S}^n \xrightarrow{\eta_{\mathrm{H}\,\mathbb{Z}} \wedge f} \mathrm{H}\,\mathbb{Z} \wedge X\ ) \\ &= (\ \mathscr{S}^n \xrightarrow{\alpha_{0,n}} \mathscr{S}^0 \wedge \mathscr{S}^n \xrightarrow{\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_{\mathscr{S}^n}} \mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^n \xrightarrow{1_{\mathrm{H}\,\mathbb{Z}} \wedge f} \mathrm{H}\,\mathbb{Z} \wedge X\ )\end{aligned}$$

and counterclockwise:

$$\begin{aligned}(f\colon \mathscr{S}^n \longrightarrow X) &\longmapsto (\ \mathscr{S}^n \xrightarrow{f} X \xrightarrow{\lambda_X^{-1}} \mathcal{S}^0 \wedge X\ ) \\ &\longmapsto (\ \mathscr{S}^n \xrightarrow{f} X \xrightarrow{\lambda_X^{-1}} \mathcal{S}^0 \wedge X \xrightarrow{\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_X} \mathrm{H}\,\mathbb{Z} \wedge X\ )\end{aligned}$$

These assemble into the following diagram

$$\begin{array}{ccccc} \mathscr{S}^n & \xrightarrow{\alpha_{0,n} = \lambda_{\mathscr{S}^n}^{-1}} & \mathscr{S}^0 \wedge \mathscr{S}^n & \xrightarrow{\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_{\mathscr{S}^n}} & \mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^n \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{S}^0 \wedge f} & & \big\downarrow{\scriptstyle \mathrm{H}\,\mathbb{Z} \wedge f} \\ X & \xrightarrow[\lambda_X{}^{-1}]{} & \mathscr{S}^0 \wedge X & \xrightarrow[\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_X]{} & \mathrm{H}\,\mathbb{Z} \wedge X \end{array}$$

which commutes by the naturality of λ and $\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1$. Thus f has the same image under both composites in the diagram for any $f \in \pi_n(X)$, hence our diagram commutes.

Since $\eta_{\mathrm{H}\,\mathbb{Z}}$ is a generator of $\pi_0(\mathrm{H}\,\mathbb{Z}) \cong \mathbb{Z}$ we get that $\eta_{\mathrm{H}\,\mathbb{Z}} \otimes -$ is an isomorphism, hence

$$\varphi_{0,n} = (\eta_{\mathrm{H}\,\mathbb{Z}} \otimes -)^{-1} \circ \pi_n(\lambda_X^{-1}) \circ \pi_n(\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_X)$$

implies that, if $\mathcal{E}$ is any class of morphisms in $\mathbb{X}$ which is closed under precomposition by isomorphisms, we get that the statement

$$\varphi_{0,n} \in \mathcal{E} \iff \pi_n(\eta_{\mathrm{H}\,\mathbb{Z}} \wedge 1_X) \in \mathcal{E}$$

holds. The Lemma then follows from the fact that the classes of epimorphisms and of isomorphisms respectively are closed under precomposition by isomorphisms.

□

For the following theorem we need the following result from homological algebra. The proof is an easy consequence of the famous Five Lemma (see [25] Exercise 1.3.3).

Lemma 4.4.2. *Given a diagram of abelian groups of the form*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

with exact rows, then if α and β are isomorphisms, so is γ.

Theorem 4.4.3 (Weak Hurewicz Theorem)**.** *Let $(\mathbb{X}, \wedge, \mathrm{H}\,\mathbb{Z})$ be a Hurewicz category then for each $n \in \mathbb{Z}$ the n^{th} and $(n+1)^{st}$ Hurewicz homomorphisms, h_n and h_{n+1}, are respectively a natural isomorphism and a pointwise epimorphism on the full subcategory of n-generated objects, $\mathcal{X}_n$.*

Proof. For an arbitrary $n \in \mathbb{Z}$ we will let $\mathcal{I}_n$ denote the full subcategory of $\mathcal{X}_n$ on which h_n restricts to a natural isomorphism and let $\mathcal{E}_n$ denote the full subcategory of $\mathbb{X}_n$ on which h_{n+1} restricts to a pointwise epimorphism. We will prove that $\mathscr{S}^k \in \mathcal{I}_n$ and $\mathscr{S}^k \in \mathcal{E}_n$ for all $k \geq n$, then we will show that both $\mathcal{I}_n$ and $\mathcal{E}_n$ are closed under coproducts, and that Cone restricts to both $\mathcal{I}_n$ and $\mathcal{E}_n$. This will prove that $\mathcal{I}_n = \mathcal{X}_n = \mathcal{E}_n$.

First, by Lemma 4.4.1 we get that, for any object $X \in \mathbb{X}$, the morphisms h_n^X and h_{n+1}^X are, respectively, an isomorphism and an epimorphism if and only if

$$\varphi_{0,n}\colon \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n(X) \longrightarrow \pi_n(\mathrm{H}\,\mathbb{Z} \wedge X) \quad \text{and} \quad \varphi_{0,n+1}\colon \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_{n+1}(X) \longrightarrow \pi_{n+1}(\mathrm{H}\,\mathbb{Z} \wedge X)$$

are an isomorphism and an epimorphism respectively.

Claim 1. *For all integers $k \geq 0$ the sphere $\mathscr{S}^{n+k}$ is a object of $\mathcal{I}_n$.*

Proof of Claim 1. Since every Hurewicz category is a Freudenthal category we have, by Proposition 4.2.5 that

$$\varphi_{0,n}\colon \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n(\mathscr{S}^n) \longrightarrow \pi_n(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^n)$$

is an isomorphism by recalling that $\otimes$ and $\wedge$ are symmetric, and letting $X = \mathrm{H}\,\mathbb{Z}$ in the statement of the proposition. In particular, $\mathscr{S}^n \in \mathcal{I}_n$. Then for any integer $k \geq 1$ we claim that the following diagram commutes:

$$\begin{array}{ccc} \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n(\mathscr{S}^{n+k}) & \xrightarrow{\varphi_{0,n}} & \pi_n(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^{n+k}) \\ \Big\downarrow \cong & & \cong \Big\uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

This follows by recalling that $\pi_n(\mathscr{S}^{n+k}) \cong 0$ when $k \geq 1$ by the definition of a good cocommutative sphere monoidal category and that

$$\begin{aligned} \pi_n(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^{n+k}) &\cong \pi_{-k}(\mathrm{H}\,\mathbb{Z}) \otimes \pi_{n+k}(\mathscr{S}^{n+k}) \\ &\cong 0 \otimes \mathbb{Z} \\ &\cong 0 \qquad\qquad \text{when } k \geq 1 \end{aligned}$$

by our conditions on $\mathrm{H}\,\mathbb{Z}$. Thus we get that

$$\varphi_{0,n} \colon \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n(\mathscr{S}^{n+k}) \longrightarrow \pi_n(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^{n+k})$$

is a morphism between zero groups, hence an isomorphism, for any integer $k \geq 0$, hence $\mathscr{S}^{n+k} \in \mathcal{I}_n$ for all integers $k \geq 0$.

■

Claim 2. *For all integers $k \geq 0$ the sphere $\mathscr{S}^{n+k}$ is an object of $\mathcal{E}_n$.*

Proof of Claim 2. The following diagram commutes

$$\begin{array}{ccc} \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_{n+1}(\mathscr{S}^n) & \xrightarrow{\varphi_{0,n+1}} & \pi_{n+1}(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^n) \\ {\scriptstyle \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes {\varphi_{1,n}}^{-1}} \Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle {\varphi_{1,n}}^{-1}} \\ \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_1(\mathscr{S}^0) \otimes \pi_n(\mathscr{S}^n) & \xrightarrow[\varphi_{0,1} \otimes \pi_n(\mathscr{S}^n)]{} & \pi_1(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^0) \otimes \pi_n(\mathscr{S}^n) \\ {\scriptstyle \rho} \Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle \rho} \\ \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_1(\mathscr{S}^0) & \xrightarrow[\varphi_{0,1}]{} & \pi_1(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^0) \end{array} \tag{4.7}$$

since the top square commutes by Lemma 4.2.3, and the bottom square commutes by the naturality of ρ. It follows from the definition of the Hurewicz homomorphism, and from Condition 2 of Definition 4.3.1 that

$$h_1^{\mathscr{S}^0} : \pi_1(\mathscr{S}^0) \longrightarrow \mathrm{H}_1(\mathscr{S}^0)$$

is an epimorphism, and hence that

$$\varphi_{0,1} : \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_1(\mathscr{S}^0) \longrightarrow \pi_1(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^0)$$

is an epimorphism. Thus, the commutativity of 4.7 allows us to write

$$\varphi_{0,n+1} : \pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_{n+1}(\mathscr{S}^n) \longrightarrow \pi_{n+1}(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^n)$$

as a composite of epimorphisms, thus making it an epimorphism. We get that $\mathscr{S}^n \in \mathcal{E}_n$. The argument preceding this claim, with n everywhere replaced by $(n+1)$ shows that

$$\varphi_{0,n+1} : \pi_0\mathrm{H}\,\mathbb{Z} \otimes \pi_{n+1}(\mathscr{S}^{n+k}) \longrightarrow \pi_{n+1}(\mathrm{H}\,\mathbb{Z} \wedge \mathscr{S}^{n+k})$$

is an isomorphism, thus an epimorphism, when $k \geq 1$ and so $\mathscr{S}^{n+k} \in \mathcal{E}_n$ for all integers $k \geq 1$. This concludes the proof of the claim. ■

Claim 3. *The classes $\mathcal{I}_n$ and $\mathcal{E}_n$ are closed under coproducts.*

Proof of Claim 3. Now assume we have a collection $X_i \in \mathcal{I}_n$ indexed by a set I, we want to show that

$$\pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n\left(\textstyle\coprod_{i\in I} X_i\right) \xrightarrow{\varphi_{0,n}} \pi_n\left(\mathrm{H}\,\mathbb{Z} \wedge \textstyle\coprod_{i\in I} X_i\right)$$

is an isomorphism. We get that

$$\begin{array}{ccc}
\bigoplus_{i\in I}(\pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n(X_i)) & \xrightarrow[\cong]{\bigoplus_{i\in I}\varphi_{0,n}^{X_i}} & \bigoplus_{i\in I}\pi_n(\mathrm{H}\,\mathbb{Z} \wedge X_i) \\
{\scriptstyle \langle \pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\iota_i)\rangle^{\oplus}_{i\in I}}\downarrow\cong & & \cong\downarrow{\scriptstyle \langle \pi_n(\mathrm{H}\,\mathbb{Z}\wedge\iota_i)\rangle^{\oplus}_{i\in I}} \\
\pi_0(\mathrm{H}\,\mathbb{Z}) \otimes \pi_n\left(\coprod_{i\in I} X_i\right) & \xrightarrow[\varphi_{0,n}^{I}]{} & \pi_n\left(\mathrm{H}\,\mathbb{Z} \wedge \coprod_{i\in I} X_i\right)
\end{array}$$

commutes by precomposing the diagram with the morphisms

$$\iota_j\colon \pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(X_j)\longrightarrow\bigoplus_{i\in I}(\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(X_i))$$

for each $j\in I$ and invoking the naturality of $\varphi_{0,n}$. The vertical morphism are isomorphisms since π_n, $\mathrm{H}\,\mathbb{Z}\wedge -$, and $\otimes$ each preserve coproducts by the definition of a good cocommutative sphere monoidal category, by our definition of $\mathrm{H}\,\mathbb{Z}$, and since $\otimes$ is a left adjoint (see [13] Theorem IV.5.10) respectively. The top morphism is a corpoduct of isomorphisms by hypothesis and is thus an isomorphism. This implies that $\varphi^I_{0,n}$ can be written as the composite of isomorphisms and is thus itself an isomorphism. We conclude that $\mathcal{I}_n$ is closed under coproducts.

To see that $\mathcal{E}_n$ is closed under coproducts use the same argument as above with $\mathcal{I}_n$ replaced with $\mathcal{E}_n$ and recall that all isomorphisms are epimorphisms.

■

Claim 4. *The mapping cone* Cone *restricts to both* $\mathcal{I}_n$ *and* $\mathcal{E}_n$*.*

Proof of Claim 4. Let $\mathcal{A}$ be a subcategory of $\mathcal{X}_n$ and let $f\colon X\longrightarrow Y$ be a morphism in $\mathcal{A}$. We need to show

$$\varphi_{0,n}\colon \pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{Cone}\,f)\longrightarrow\pi_n(\mathrm{H}\,\mathbb{Z}\wedge\mathrm{Cone}(f))$$

is an isomorphism if $\mathcal{A}=\mathcal{A}$ or an epimorphism if $\mathcal{A}=\mathcal{E}_n$. Since $\mathcal{A}$ is subcategory of $\mathcal{X}_n$ we get that our conditions on Cone and on n-generated objects imply that the following

$$\pi_n(X)\xrightarrow{\pi_n(f)}\pi_n(Y)\xrightarrow{\pi_n(\mathrm{i}_f)}\pi_n(\mathrm{Cone}(f))\longrightarrow 0$$

is an exact sequence whenever f is a morphism of $\mathcal{A}$. Since tensoring is right exact (see [13]), and since Cone commutes with $-\wedge\mathrm{H}\,\mathbb{Z}$ by Definition 4.3.1 we get two more exact sequences

$$\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(X)\xrightarrow{\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(f)}\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(Y)\xrightarrow{\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{i}_f)}\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{Cone}(f))\longrightarrow 0$$

and

$$\pi_n(\mathrm{H}\,\mathbb{Z}\wedge X) \xrightarrow{\pi_n(\mathrm{H}\,\mathbb{Z}\wedge f)} \pi_n(\mathrm{H}\,\mathbb{Z}\wedge Y) \xrightarrow{\pi_n(\mathrm{H}\,\mathbb{Z}\wedge \mathrm{i}_f)} \pi_n(\mathrm{H}\,\mathbb{Z}\wedge \mathrm{Cone}(f)) \longrightarrow 0$$

we then have the diagram:

$$\begin{array}{ccccccc}
\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(X) & \xrightarrow{\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(f)} & \pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(Y) & \xrightarrow{\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{i}_f)} & \pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{Cone}(f)) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi_{0,n}} & & \downarrow{\scriptstyle \varphi_{0,n}} & & \downarrow{\scriptstyle \varphi_{0,n}} & & \downarrow{\scriptstyle 0} \\
\pi_n(\mathrm{H}\,\mathbb{Z}\wedge X) & \xrightarrow[\pi_n(\mathrm{H}\,\mathbb{Z}\wedge f)]{} & \pi_n(\mathrm{H}\,\mathbb{Z}\wedge Y) & \xrightarrow[\pi_n(\mathrm{H}\,\mathbb{Z}\wedge \mathrm{i}_f)]{} & \pi_n(\mathrm{H}\,\mathbb{Z}\wedge \mathrm{Cone}(f)) & \longrightarrow & 0
\end{array} \tag{4.8}$$

which commute by the naturality of the splicing homomorphisms.

If $\mathcal{A}=\mathcal{I}_n$ then since $X,Y\in\mathcal{A}$ we get that $\varphi_{0,n}^{\mathrm{H}\,\mathbb{Z},X}$ and $\varphi_{0,n}^{\mathrm{H}\,\mathbb{Z},Y}$ are isomorphisms thus Lemma 4.4.2 implies that $\varphi_{0,n}^{\mathrm{H}\,\mathbb{Z},\mathrm{Cone}\,f}$ is an isomorphism. Since $X,Y\in\mathcal{X}_n$ we get that $\mathrm{Cone}\,f\in\mathcal{X}_n$ so the previous conclusion imples that $\mathrm{Cone}(f)\in\mathcal{I}_n$. Since f was arbitrary we get that $\mathcal{I}_n$ is closed under mapping cones.

Similarly, if $\mathcal{A}=\mathcal{E}_n$ then since $X,Y\in\mathcal{A}$ we get that $\varphi_{0,n}^{\mathrm{H}\,\mathbb{Z},X}$ and $\varphi_{0,n}^{\mathrm{H}\,\mathbb{Z},Y}$ are epimorphisms. In particular, from the Diagram 4.8 we get that

$$\begin{array}{ccc}
\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(Y) & \xrightarrow{\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{i}_f)} & \pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{Cone}(f)) \\
\downarrow{\scriptstyle \varphi_{0,n}} & & \downarrow{\scriptstyle \varphi_{0,n}} \\
\pi_n(\mathrm{H}\,\mathbb{Z}\wedge Y) & \xrightarrow[\pi_n(\mathrm{H}\,\mathbb{Z}\wedge \mathrm{i}_f)]{} & \pi_n(\mathrm{H}\,\mathbb{Z}\wedge \mathrm{Cone}(f))
\end{array}$$

commutes. Thus $\pi_0(\mathrm{H}\,\mathbb{Z})\otimes\pi_n(\mathrm{i}_f)\varphi_{0,n}^{\mathrm{H}\,\mathbb{Z},\mathrm{Cone}\,f}$ can be written as a composite of epimorphisms, hence $\varphi_{0,n}^{\mathrm{H}\,\mathbb{Z},\mathrm{Cone}\,f}$ is an epimorphism by Proposition 1.1.23. Again since $\mathcal{E}_n$ is a subcategory of $\mathcal{X}_n$ we get that $\mathrm{Cone}\,f\in\mathcal{X}_n$ so the previous conclusion allows us to deduce that $\mathrm{Cone}\,f\in\mathcal{E}_n$.

■

Finally, since both $\mathcal{I}_n$ and $\mathcal{E}_n$ are subcategories of $\mathcal{X}_n$ which, in turn, is a subcategory of $\mathbb{X}$ we get that Claims 1 through 4 above imply that $\mathcal{I}_n$ and $\mathcal{E}_n$ contain $\mathscr{S}^{n+k}$ for all $k\geq 0$, are closed under coproducts, and Cone restricts to both, hence $\mathcal{X}_n$ must be a subcategory of each, thus $\mathcal{I}_n=\mathcal{X}_n=\mathcal{E}_n$.

□

Chapter 5

Symmetric Spectra

Introduction

The category of spectra has long been the natural category in which to study stable homotopy. Unfortunately the original category of spectra, here called *sequential spectra*, suffers from the following defect: the homotopy category of sequential spectra has a symmetric monoidal structure which is not -indeed cannot be- inherited from the original category. The category of symmetric spectra is a subcategory of the category of sequential spectra, discovered by Jeff Smith (see the introduction to [12]), which is equipped with a symmetric monoidal product which descends to its own homotopy category, which is happily equivalent to the original.

5.1 Basics

The original definition of a spectrum (what we call here a sequential spectrum) is from the PhD thesis of Elon Lima [15]. A more modern reference is [1].

Definition 5.1.1 ([15])**.** Define a *sequential spectrum*, X, to be the following collection of data:

- A sequence, X_n, of pionted simplicial sets for each $n \in \mathbb{N}$.
- For each $n \in \mathbb{N}$ a basepoint-preserving *structure map* $\sigma_n \colon \mathcal{S}^1 \wedge X_n \to X_{n+1}$

A *morphism of sequential spectra* $f \colon X \to Y$ is a collection of maps $f_n \colon X_n \to Y_n$ such that

$$\begin{array}{ccc} \mathcal{S}^1 \wedge X_n & \xrightarrow{\sigma_n^X} & X_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \mathcal{S}^1 \wedge Y_n & \xrightarrow[\sigma_n^Y]{} & Y_{n+1} \end{array}$$

commutes.

Recall here the definition of a G-object.

Definition 5.1.2 ([12])**.** A *symmetric spectrum* (or just a *spectrum*) X is a sequential spectrum with the following additional data

- Every simplicial set X_n comes equipped with a basepoint-preserving action of Σ_n
- The structure maps σ_n are $\Sigma_1 \times \Sigma_n$-equivariant for all $n \in \mathbb{N}$, further the composite

$$\mathcal{S}^m \wedge X_n \overset{\mathcal{S}^{m-1} \wedge \sigma_n}{\longrightarrow} \mathcal{S}^{m-1} \wedge X_{n+1} \overset{\mathcal{S}^{m-2} \wedge \sigma_{n+1}}{\longrightarrow} \cdots \overset{\sigma_{n+m}}{\longrightarrow} X_{n+m}$$

  is $\Sigma_m \times \Sigma_n$ invariant. The action of Σ_m on $\mathcal{S}^m$ is the standard one from Definition 2.1.28.

Example 5.1.3. Define the *sphere spectrum* to be the symmetric spectrum $\mathbb{S}^0$ by...

... $\mathbb{S}^0_n = \mathcal{S}^n$.

... the Σ_n action on $\mathbb{S}^0_n = \mathcal{S}^n$ is the standard one.

... $\sigma_n \colon \mathcal{S}^1 \wedge \mathcal{S}^n \longrightarrow \mathcal{S}^{n+1}$ is the isomorphism of Definition 2.1.26.

Definition 5.1.4. Given any simplicial set X define the n^{th} *free symmetric spectrum on* X, $\mathrm{F}_n(X)$, to be the spectrum defined by...

... $\mathrm{F}_n(X)_m := \begin{cases} *, & m < n \\ \mathcal{S}^n \wedge X & m \geq n \end{cases}$

... the Σ_m action on $\mathrm{F}_n(X)_m$ is (by necessity) trivial for $m < n$, and is given by the standard action on $\mathcal{S}^m$ and the trivial action on X when $m \geq n$.

... $\sigma_m \colon \mathcal{S}^1 \wedge \mathrm{F}_n(X)_m \longrightarrow \mathrm{F}_n(X)_{m+1}$ is the isomorphism of Definition 2.1.26 smashed with the identity morphism on X

Example 5.1.5. For each integer $n \geq 1$ define $\mathbb{S}^n := \mathrm{F}_0\, \mathcal{S}^n$ and $\mathbb{S}^{-n} := \mathrm{F}_n\, \mathcal{S}^0$.

Example 5.1.6 ([17] [8])**.** For any $X \in \mathbf{sSet}_*$ and each $n \in \mathbb{N}$ define the *nth symmetric product* of X by

$$\mathrm{SP}_n(X) = \underbrace{X \times \cdots \times X}_{n-\text{times}} / \Sigma_n$$

where Σ_n acts on $\prod_{i=1}^{n} X$ by permuting the entries. Since X is pointed we may define an inclusion

$$\mathrm{SP}_n(X) \hookrightarrow \mathrm{SP}_{n+1}(X) \tag{5.1}$$

given by taking the map $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n, *)$ on $\prod_{i=1}^{n} X$ and noticing that if we take Σ_n to be a subgroup of Σ_{n+1} via $\Sigma_n \cong \Sigma_n \times \Sigma_1 \hookrightarrow \Sigma_{n+1}$ then this map is Σ_n equivariant and so descends to a map on the quotients, namely the inclusions from Equation 5.1.

Define the *symmetric product of* X by

$$\mathrm{SP}(X) := \mathrm{colim}(\, \mathrm{SP}_1(X) \hookrightarrow \mathrm{SP}_2(X) \hookrightarrow \cdots).$$

For each $n \in \mathbb{N}$ define

$$\widetilde{\sigma}_n \colon \mathcal{S}^1 \wedge \prod_{i=1}^{n} X \longrightarrow \prod_{i=1}^{n} \mathcal{S}^1 \wedge X$$

to be the map with i^{th} component

$$\mathcal{S}^1 \wedge \pi_i \colon \mathcal{S}^1 \wedge \prod_{i=1}^{n} X \longrightarrow \mathcal{S}^1 \wedge X.$$

Clearly σ_n is $\Sigma_1 \times \Sigma_n$ equivariant, thus, since the symmetric product is a cokernel and $\mathcal{S}^1 \wedge -$ commutes with colimits, it induces a morphism

$$\sigma_n \colon \mathcal{S}^1 \wedge \mathrm{SP}_n(X) \longrightarrow \mathrm{SP}_n\left(\mathcal{S}^1 \wedge X\right).$$

which is given by

$$s \wedge [x_1, \ldots, x_n] \mapsto [s \wedge x_1, \ldots s \wedge x_n]$$

for any $s \in \mathcal{S}^1$ and any $[x_1, \ldots, x_n] \in \mathrm{SP}_n(X)$.

The collection

$$\left\{ \mathcal{S}^1 \wedge \mathrm{SP}_n(X) \xrightarrow{\sigma_n} \mathrm{SP}_n(\mathcal{S}^1 \wedge X) \longrightarrow \mathrm{SP}(\mathcal{S}^1 \wedge X) \,|n \in \mathbb{N}\right\}$$

forms a cocone which yields a morphism

$$\sigma\colon \mathcal{S}^1 \wedge \mathrm{SP}(X) \longrightarrow \mathrm{SP}\left(\mathcal{S}^1 \wedge X\right) \tag{5.2}$$

Definition 5.1.7 (See [12] for an alternate definition)**.** The *Eilenberg-MacLane* spectrum, $\mathrm{H}\,\mathbb{Z}$, is the spectrum defined by $\mathrm{H}\,\mathbb{Z}_n := \mathrm{SP}(\mathcal{S}^n)$ and with structure maps given by (5.2) above.

This definition is enough to give a sequential spectrum, but in fact more is true:

Proposition 5.1.8. *The Eilenberg-MacLane spectrum,* $\mathrm{H}\,\mathbb{Z}$*, is a symmetric spectrum when we give* $\mathrm{H}\,\mathbb{Z}_n = \mathrm{SP}(\mathcal{S}^n)$ *the diagonal action of* Σ_n *induced by the standard action of* Σ_n *on* $\mathcal{S}^n$*.*

The proof of this proposition is a routine verification.

5.2 Spectra Categorically

Definition 5.2.1. Let Σ denote the category whose objects are the natural numbers, the set of endomorphisms on n is Σ_n, and no other morphisms exist. Composition is given by the group operations of the symmetric groups. Consider Σ as enriched over $\mathbf{sSet}_*$ by setting

$$\Sigma(n, n) := \coprod_{\gamma \in \Sigma_n} \mathcal{S}^0.$$

Define

$$+\colon \Sigma \wedge \Sigma \longrightarrow \Sigma$$

by $(n, m) \mapsto n + m$ and $(\sigma, \tau) \in \Sigma_n \times \Sigma_m$ is sent to $\sigma \times \tau \in \Sigma_{n+m}$.

Proposition 5.2.2.

$$+\colon \Sigma \wedge \Sigma \longrightarrow \Sigma$$

is a symmetric monoidal product on Σ *which respects the enrichment over* $\mathbf{sSet}_*$.

Again, the proof of this is routine.

Definition 5.2.3 ([12])**.** A *symmetric sequence*, A, is a functor from Σ to $\mathbf{sSet}_*$.

Example 5.2.4 ([12])**.** Define $S\colon \Sigma \longrightarrow \mathbf{sSet}_*$ by $S(n) = \mathcal{S}^n$, the n-sphere, and S sends Σ_n to the automorphisms of $\mathcal{S}^n$ representing the standard action of Σ_n on $\mathcal{S}^n$ and the basepoint $*$ of $\Sigma(n,n)$ to the basepoint of the simplicial set $\mathrm{maps}_*(\mathcal{S}^n, \mathcal{S}^n)$. Call S the *sphere sequence.*

Theorem 5.2.5. *Since* $+$ *respects the enrichment over* $\mathbf{sSet}_*$*, and since* $\mathbf{sSet}_*$ *is equipped with a symmetric monoidal product of its own (namely* $\wedge$*), the Day convolution provides a closed symmetric monoidal product for the functor category* $\mathbf{sSet}_*^{\Sigma}$.

Remark. By Day's theorem we get that this symmetric monoidal product takes the form, in level $n \in \mathbb{N}$:

$$F \otimes G(n) := \int^{p,q \in \mathbb{N}} \Sigma(n, p+q) \wedge F(p) \wedge G(q) \cong \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} F_p \wedge G_q$$

this isomorphism follows by unravelling the definition of a coend.

The Day construction gives us that the unit for $\otimes$ is the functor $U := \Sigma(0, -)$ which can be described as

$$\Sigma(0,-)_n := \Sigma(0,n) = \left\{ \begin{array}{ll} \{1_{\Sigma_0}\}^+, & n = 0 \\ *, & \text{otherwise} \end{array} \right\} \cong \left\{ \begin{array}{ll} \mathcal{S}^0, & n = 0 \\ *, & \text{otherwise} \end{array} \right.$$

It also provides a natural isomorphism $\lambda\colon -\otimes U \Longrightarrow 1_{\mathbf{sSet}^{\Sigma}}$.

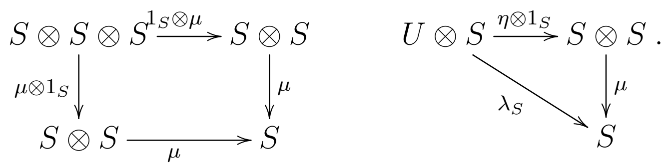
Definition 5.2.6 ([12])**.** Now define $\mu\colon S \otimes S \to S$ to be given, in level n and on each summand, by

$$\mu_{p,q}\colon \Sigma_{p+q}^+ \wedge_{\Sigma_p \times \Sigma_q} \mathcal{S}^p \wedge \mathcal{S}^q \longrightarrow \mathcal{S}^{p+q} \qquad \alpha \wedge_{\Sigma_p \times \Sigma_q} s_0 \wedge s_1 \mapsto \alpha \cdot (s_0 \wedge s_1)$$

and the map $\eta\colon U \to S$ to be the identity on level 0 and the basepoint-inclusion elsewhere (by necessity). The second of these is clearly well-defined. The first is well defined since the basepoint of Σ^+_{p+q} is disjoint from the rest of the simplicial set .

Proposition 5.2.7 ([12])**.** *(S, μ, η) is a monoid over $(\mathbf{sSet}^\Sigma, \otimes, U)$.*

Proof. We need to prove that the following diagrams commute



For the square diagram, let $n \in \mathbb{N}$ and pick any arbitrary $s \in (S \otimes S \otimes S)_n$. Then

$$s \in \Sigma_{p+q+r} \wedge_{\Sigma_p \times \Sigma_q \times \Sigma_r} \mathcal{S}^p \wedge \mathcal{S}^q \wedge \mathcal{S}^r$$

and we can write

$$s = (\alpha_p \times \alpha_q \times \alpha_r) \wedge_{\Sigma_p \times \Sigma_q \times \Sigma_r} s_p \wedge s_q \wedge s_r.$$

We calculate:

$$\begin{aligned}
(\mu \otimes 1_S) \circ \mu(s) &= \mu((\mu \otimes 1_S)(s)) \\
&= \mu((\mu \otimes 1_S)((\alpha_p \times \alpha_q \times \alpha_r) \wedge_{\Sigma_p \times \Sigma_q \times \Sigma_r} s_p \wedge s_q \wedge s_r)) \\
&= \mu(1 \times \alpha_r \wedge_{\Sigma_{p+q} \times \Sigma_r} ((\alpha_p \times \alpha_q) \cdot s_p \wedge s_q) \wedge s_r) \\
&= (\alpha_p \times \alpha_q \times \alpha_r) \cdot (s_p \wedge s_q \wedge s_r).
\end{aligned}$$

Also

$$\begin{aligned}
(1_S \otimes \mu) \circ \mu(s) &= \mu((1_S \otimes \mu)(s)) \\
&= \mu((1_S \otimes \mu)((\alpha_p \times \alpha_q \times \alpha_r) \wedge_{\Sigma_p \times \Sigma_q \times \Sigma_r} (s_p \wedge s_q \wedge s_r)) \\
&= \mu((\alpha_p \times 1) \wedge_{\Sigma_p \times \Sigma_{q+r}} s_p \wedge (\alpha_p \times \alpha_r) \cdot (s_p \wedge s_r)) \\
&= (\alpha_p \times \alpha_q \times \alpha_r) \cdot (s_p \wedge s_q \wedge s_r).
\end{aligned}$$

So the square diagram commutes.

∎

We now have a monoid in a symmetric monoidal category. For any module, (X, m), over (S, μ, η) the diagrams

$$
\begin{array}{ccc}
S \otimes S \otimes X & \xrightarrow{\mu \otimes 1_X} & S \otimes X \\
{\scriptstyle 1_S \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\
S \otimes X & \xrightarrow[m]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
U \otimes X & \xrightarrow{\eta \otimes 1_X} & S \otimes X \\
 & \searrow{\scriptstyle \lambda_X} & \downarrow{\scriptstyle m} \\
 & & X
\end{array}
$$

commute.

Theorem 5.2.8 ([12])**.** *Spectra are exactly the modules over the nice commutative monoid, S, in a functor category which has been given a universal closed symmetric monoidal product.*

Finally, since $\otimes$ is a closed symmetric monoidal product we get that $X \otimes -$ preserves coequalizers for all $X \in \mathbf{sSet}^{\Sigma}$ hence Proposition 1.4.11 tells us that ${}^{S}\mathbf{sSet}^{\Sigma}$ is complete and cocomplete with a closed symmetric monoidal product defined by:

$$X \wedge Y := \text{coequalizer}(\ X \otimes S \otimes Y \underset{1_X \otimes m_Y}{\overset{(\text{twist} \circ m_X) \otimes 1_Y}{\rightrightarrows}} X \otimes Y\ )$$

Definition 5.2.9. Notice that, by Proposition 1.4.11 the unit of $\wedge$ is the monoid S which becomes the symmetric spectrum $\mathbb{S}^0$ of Example 5.1.3.

5.3 Some Useful Constructions

It would be nice to be able to generate endofunctors on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ "for free" by taking endofunctors on **sSet** and applying them level-wise to each symmetric spectrum. The following notion, *prolongation*, is a condition on endofunctors of **sSet** for doing exactly that. We will use mostly the presentation given in [12]. The only difference is that what we call prolongable pairs here, they call *pointed S_*-functors*.

Definition 5.3.1 (See [12] Definition 1.2.8)**.** Let $F\colon \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$ be a functor and let $h_{X,Y}\colon F(X) \wedge Y \longrightarrow F(X \wedge Y)$ be a natural transformation betwen functors from

$$\mathbf{sSet}_* \times \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*.$$

Call the pair (F, h) *prolongable* if it satisfies the following conditions:

(0) $F(\mathbf{0}) = \mathbf{0}$ where $\mathbf{0}$ denotes the zero object of $\mathbf{sSet}$.

(1) Recall that ρ is the right unit transformation for $\wedge$, i.e. $\rho_X\colon X \wedge \mathbb{S}^0 \xrightarrow{\cong} X$. For all $X \in \mathbf{sSet}$ we get that

$$\begin{array}{ccc} F(X) \wedge \mathcal{S}^0 & \xrightarrow{h_{X,\mathcal{S}^0}} & F(X \wedge \mathcal{S}^0) \\ & \searrow^{\rho_{F(X)}} & \downarrow{\scriptstyle F(\rho_X)} \\ & & F(X) \end{array}$$

commutes.

(2) For all $X, Y, Z \in \mathbf{sSet}$ we get that

$$\begin{array}{ccc} (F(X) \wedge Y) \wedge Z & \xrightarrow{h_{X,Y} \wedge 1_Z} & F(X \wedge Y) \wedge Z \\ {\scriptstyle \text{assoc}}\downarrow & & \downarrow{\scriptstyle h_{X \wedge Y, Z}} \\ F(X) \wedge (Y \wedge Z) & \xrightarrow[h_{X, Y \wedge Z}]{} & F(X \wedge Y \wedge Z) \end{array}$$

commutes.

A natural transformation $\tau\colon F \to F'$ between prolongable pairs (F, h) and (F', h') is *prolongable* if, for all $X, Y \in \mathbf{sSet}$ the diagram

$$\begin{array}{ccc} F(X) \wedge Y & \xrightarrow{h_{X,Y}} & F(X \wedge Y) \\ {\scriptstyle \tau_X \wedge Y}\downarrow & & \downarrow{\scriptstyle \tau_{X \wedge Y}} \\ F'(X) \wedge Y & \xrightarrow[h'_{X,Y}]{} & F'(X \wedge Y) \end{array}$$

commutes.

The next theorem is given as a definition in [12] with some of the details left to the reader. The presentation here differs from that of [12] most in a change of terminology: what we call *prolongable* is called a *$\mathcal{S}_*$-functor or $\mathcal{S}_*$-natural transformation* in [12]. Further, the presentation given here is slightly more formal than that given in [12] as might be inferred from the change-of-status from ‘Definition’ to ‘Theorem’. The reader is strongly encouraged to compare the two presentations.

Theorem 5.3.2 ([12] Definition 1.2.9)**.** *If (F, h) is prolongable then for any symmetric spectrum, X, considered as a functor $X\colon \Sigma \longrightarrow \mathbf{sSet}$, then*

1. *The composite XF is again an $S-$module (i.e. a symmetric spectrum).*
2. *Post-composition by F extends to a functor, called the* prolongation of (F, h) *on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ by acting pointwise on maps via F. By abuse of notation we let F denote the prolongation of (F, h).*
3. *If $\tau\colon F \Longrightarrow F'$ is prolongable, then τ induces a natural transformation between the prolongations of F and F'*

Let $K, L, M \in \mathbf{sSet}$ and define the map $h\colon L^K \wedge M \longrightarrow (L \wedge M)^K$ to be the adjoint of the map $\widetilde{h}\colon L^K \longrightarrow ((L \wedge M)^K)^M$ which sends each map $f\colon K \wedge \Delta[m] \longrightarrow L$ to the composite

$$K \wedge \Delta[m] \xrightarrow{\text{diag}} (K \wedge \Delta[m]) \wedge (K \wedge \Delta[m]) \xrightarrow{f \wedge c_-} L \wedge M \ ,$$

where c_- is the constant map on the input element of M.

Let $\mathrm{assoc}_{-,=,\equiv}\colon (- \wedge =) \wedge \equiv \longrightarrow - \wedge (= \wedge \equiv)$ denote the associativity morphism of $\wedge$.

Proposition 5.3.3 (See [12])**.** *For any $K \in \mathbf{sSet}$ the pairs $(K \wedge -, \mathrm{assoc}_{K,-,=})$ and $(-^K, h)$ are prolongable. Furthermore, $K \wedge - \dashv -^K$ via the prolongation of the usual natural transformations.*

Example 5.3.4 ([12])**.** Let I be $\Delta[1]$, with the vertex $\{1\}$ as the basepoint. Define I^+ be $\Delta[1]^+$, the 1-simplex with an added basepoint. Then write $- \wedge \mathrm{I}$, $- \wedge \mathrm{I}^+$, $-^{\mathrm{I}}$, and $-^{\mathrm{I}^+}$ for the respective prolongations of the apparent functors which exist by Proposition 5.3.3.

We will now turn our attention to a pair of useful constructions on symmetric spectra, inspired by -and nearly identical to- the constructions in the categories of spaces and simplicial sets.

For the next definition note that, for any simplicial set X, the map $i_0\colon X \longrightarrow X \wedge \mathrm{I}$ given level-wise by letting $(i_0)_n\colon X_n \longrightarrow X_n \wedge \mathrm{I}$ be the map sending $x \mapsto x \wedge 1$, is the prolongation of the -prolongable- natural transformation $- \wedge 1\colon\ - \wedge \mathcal{S}^0 \longrightarrow - \wedge \mathrm{I}$ and is thus a morphism of symmetric spectra.

Definition 5.3.5 (Mapping Cone, [23])**.** Given a morphism $f\colon X \longrightarrow Y$ of symmetric spectra define the *mapping cone* $\mathrm{Cone}(f)$ to be the pushout

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \iota_0}\downarrow & & \downarrow{\scriptstyle i_f} \\ X \wedge \mathrm{I} & \xrightarrow[\;f'\;]{} & \mathrm{Cone}(f) \end{array}$$

Since $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ is cocomplete, and since the underlying pushout diagram is a diagram in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$, the mapping cone of any morphism $f\colon X \longrightarrow Y$ of symmetric spectra exists and is itself a symmetric spectrum.

Proposition 5.3.6 ([22] Proposition I.4.7)**.** *For every morphism $f\colon X \longrightarrow Y$ in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ there is a long exact sequence*

$$\cdots \xrightarrow{\delta_{n+1}} \pi_n(X) \xrightarrow{\pi_n(f)} \pi_n(Y) \xrightarrow{\pi_n(\mathrm{i}_f)} \pi_n(\mathrm{Cone}(f)) \xrightarrow{\delta_n} \pi_{n-1}(X) \xrightarrow{\pi_{n-1}(f)} \cdots .$$

Again for the next definition the maps $\mathrm{ev}_0, \mathrm{ev}_1\colon Z^{\mathrm{I}} \longrightarrow Z$ are components of the prolongations of natural transformaitons and thus are morphisms of symmetric spectra.

Definition 5.3.7 (Homotopy fiber [23])**.** Given a morphism $g\colon Y \to Z$ of symmetric spectra define the *homotopy fiber*, $\mathrm{hofiber}(g)$, to be the pullback

$$\begin{array}{ccc} \mathrm{hofiber}(g) & \xrightarrow{\;p_g\;} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle \langle *, g\rangle} \\ Z^{\mathrm{I}} & \xrightarrow[\;\langle \mathrm{ev}_0, \mathrm{ev}_1\rangle\;]{} & Z \times Z \end{array}$$

Again, the homotopy fibre of any morphism $g\colon Y \longrightarrow Z$ is a symmetric spectrum, by the completeness of $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

5.4 Some Homotopy-Oriented Results/Definitions

Spectra were originally born from the following theorem of Freudenthal.

Recall the suspension functor of Definition 2.1.29

$$\mathcal{S}^1 \wedge -\colon \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$$

which induces, for each $k \in \mathbb{N}$, a map

$$\mathcal{S}^1 \wedge -\colon \mathbf{sSet}_*(\mathcal{S}^k, X) \longrightarrow \mathbf{sSet}_*(\mathcal{S}^{k+1}, \mathcal{S}^1 \wedge X)$$

for every X and k. This map descends to the suspension homomorphism

$$s_{k,X}\colon \pi_k(X) \longrightarrow \pi_{k+1}(\mathcal{S}^1 \wedge X).$$

Theorem 5.4.1 (Freundenthal Suspension Theorem)**.** *Let X be an $(n-1)$ connected space, then whenever $k < 2n-1$ we have that*

$$s_{k,X}\colon \pi_k(X) \longrightarrow \pi_{k+1}(\mathcal{S}^1 \wedge X)$$

is an isomorphism and $s_{k+1,X}$ is surjective.

In particular, as a consequence of this theorem, given a space X we can define the sequence of groups

$$\pi_n(X) \longrightarrow \pi_{n+1}(\mathcal{S}^1 \wedge X) \longrightarrow \cdots \longrightarrow \pi_{n+m}(\mathcal{S}^m \wedge X) \longrightarrow \cdots \longrightarrow$$

and we know that eventually this sequence will stabilize since every space is, at worst, (-1)-connected, and whenever X is m-connected $\mathcal{S}^1 \wedge X$ is $(m+1)$-connected.

From this we can define the *nth stable homotopy group of* X, $\pi_n^s(X)$ by

$$\pi_n^s(X) := \operatorname*{colim}_k \pi_{n+k}(\mathcal{S}^k \wedge X)$$

and be assured that these new groups are, at least in principle, computable. The study of these stable homotopy groups then lead to the discovery of sequential spectra, $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}_*}$, and the subsequent definition of the n^{th} homotopy group of a spectrum X to be

$$\pi_n(X) := \operatorname*{colim}_k \pi_{n+k}(X_k) \tag{5.3}$$

where the morphisms making up this system look like

$$\pi_{n+k}(X_k) \xrightarrow{s_{k,X_k}} \pi_{n+k+1}(\mathcal{S}^1 \wedge X_k) \xrightarrow{\pi_{n+k+1}(\sigma_k)} \pi_{n+k+1}(X_{k+1})$$

Recall 5.4.2. Recall from Definition 2.1.29 that the suspension functor

$$\mathcal{S}^1 \wedge -\colon \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$$

has a right adjoint $\Omega\colon \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$ where Ω takes a simplicial set X to the simplicial set $\operatorname{maps}(\mathcal{S}^1, X)$. Given a map $f\colon \mathcal{S}^1 \wedge X \longrightarrow Y$, let

$$\widetilde{f}\colon X \longrightarrow \operatorname{maps}(\mathcal{S}^1, Y)$$

denote its right adjoint.

Recall also that a map $f\colon X \longrightarrow Y$ in $\mathbf{sSet}_*$ is a weak equivalence if and only if f induces isomorphisms on homotopy groups.

Definition 5.4.3 ([1])**.** Call a symmetric spectrum X an Ω-*symmetric spectrum* if, for every $n \in \mathbb{N}$ the map $\widetilde{\sigma_n}\colon X_n \longrightarrow \Omega\, X_{n+1}$ is a weak equivalence.

Definition 5.4.4 ([12])**.** Call a symmetric spectrum A *injective* if, for every $f\colon X \longrightarrow Y$ such that f_n is a trivial cofibration for all $n \in \mathbb{N}$, and any $g\colon X \longrightarrow A$, there is a map $h\colon Y \longrightarrow A$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ A & & \end{array}$$

commute. Notice that we make no claim about the uniqueness of h.

Note on Terminology. To save on syllables, if X is a symmetric spectrum which is both injective and an Ω-symmetric spectrum we will simply call it an *injective* Ω *spectrum* and drop the adjective 'symmetric'. Note that we do *not* intend this to mean that we consider X as a sequential spectrum, only that *injective* Ω*-symmetric spectrum* sounds ridiculous in the author's humble opinion.

Remark. The reason for the name *injective* is this: in the case of symmetric spectra of simplicial sets, a map f is level-wise a trivial cofibration if and only if it is monic and a weak equivalence. Compare this to the category of R-modules where a module M is injective if the same property above holds whenever f is monic.

In Chapter 3 we defined what it means for two morphisms in a model category to be homotopic (see Theorem 3.2.9). In order to define the homotopy equivalence relation in a model category however we first need weak equivalences. To define the weak equivalences on the category of symmetric spectra that we want we can proceed in one of two ways. The usual way (as found in [12] and [23]) requires first that we define a homotopy relation, use that to define the weak equivalences, and then note that the defined homotopy relation and the one given to us by the model category structure coincide on fibrant-cofibrant objects. Since fibrant cofibrant objects are the the only ones we care about we ignore the differences between the two. The alternate way involves first constructing a fibrant-cofibrant functor and then saying that a morphism of spectra is a weak equivalence if its fibrant-cofibrant replacement is level-wise a weak equivalence. We adopt the former approach, see [10] for the latter approach in a more general setting.

Definition 5.4.5 ([12])**.** For any symmetric spectrum X we have the symmetric spectrum $X \wedge \mathrm{I}^+$ from Example 5.3.4. If Y is an injective symmetric spectrum and X is any symmetric spectrum then define a *homotopy* to be any morphism $H\colon X \wedge \mathrm{I}^+ \longrightarrow Y$. We say that H is a homotopy from $f := H(- \wedge 0)\colon X \longrightarrow Y$ to $g := H(- \wedge 1)\colon X \longrightarrow Y$. Define a relation $\sim$ on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}(X, Y)$ by $f \simeq g$ if and only if there is a homotopy H from f to g.

Remark. As one might expect from the use of language, the spectrum $X \wedge \mathrm{I}^+$ is going to play the role of the cylinder object in a model category structure on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ for morphisms between symmetric injective Ω-spectra X and Y.

Proposition 5.4.6 (See [23]). *If Y is an injective symmetric spectrum then, for any symmetric spectrum X, $\sim$ is an equivalence relation on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}(X, Y)$.*

Definition 5.4.7 ([12]). Call a morphism $f\colon X \longrightarrow Y$ of symmetric spectra a *stable equivalence* if, for every injective symmetric spectrum A, the map

$$[Y, A] \xrightarrow{[f,Y]} [X, A]$$

is a bijection.

Remark. The class of stable equivalences will form the class of weak equivalences for a model category structure on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

Theorem 5.4.8 ([23]). *For each symmetric spectrum X there is an injective Ω-symmetric spectrum $\omega\, X$ and a weak equivalence $\mathrm{p}_X\colon X \longrightarrow \omega(X)$. By the axiom of choice we get a function $\omega\colon \mathbb{X} \longrightarrow \mathbb{X}$ such that $\omega(X) = X$ whenever X is itself an injective Ω-symmetric spectrum.*

Remark. The injective Ω-spectra are the class of fibrant objects in one of the model structures on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

Theorem 5.4.9 ([23] see Example II.1.15). *For every symmetric spectrum X and each $n \in \mathbb{Z}$ we have*

$$\pi_n(X) \cong [\mathcal{S}^n, \omega\, X] \cong [\omega\, \mathcal{S}^n, \omega\, X]$$

naturally.

Definition 5.4.10 ([23]). Call a symmetric spectrum X *flat* if the functor

$$X \wedge -\colon \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*} \longrightarrow \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$$

sends level cofibrations to level cofibrations.

Remark. For simplicial sets, a level cofibration of $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ is a categorical monomorphism. In algebra, a module, M, is flat if $M \otimes -$ sends monomorphisms to monomorphisms, hence the term.

Remark. The flat spectra will form the class of cofibrant objects for another model structure on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$, in light of this the following will be important.

Theorem 5.4.11 ([23])**.** *If X and Y are flat, then $X \wedge Y$ is also flat. In paticular, $\wedge$ restricts to a symmetric monoidal product on the full subcategory of flat spectra.*

Proof. Let $f\colon A \longrightarrow B$ be a level cofibration in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$, then

$$Y \wedge f \colon Y \wedge A \longrightarrow Y \wedge B$$

is a level cofibration since Y is flat, so

$$X \wedge (Y \wedge f)\colon X \wedge (Y \wedge A) \longrightarrow X \wedge (Y \wedge B)$$

is a level cofibration since X is flat, so

$$(X \wedge Y) \wedge f \colon (X \wedge Y) \wedge A \longrightarrow (X \wedge Y) \wedge B$$

is a level cofibration by isomorphism closure of cofibrations under composition. Hence $X \wedge Y$ is flat.

□

Chapter 6

Hurewicz for Symmetric Spectra

Introduction

In this chapter we will define the homotopy category of $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ and prove that it satisfies the axioms of Chapter 4. Note that proving that a given category is a model category is typically quite diffucult; the category of Symmetric Spectra is no exception and, as such I leave all proofs regarding the model category structure of $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ to [23], [22], and [12].

6.1 The Homotopy Category of Symmetric Spectra

We start by defining the category which will become our example of a Hurewicz category.

Recall 6.1.1. The stable equivalences of spectra, as per Definition 5.4.7, are the morphisms $f\colon X \longrightarrow Y$ such that, for every injective spectrum A, the map

$$[Y, A] \xrightarrow{[f,A]} [X, A]$$

is a bijection.

Theorem 6.1.2 ([12] Definition 5.1.1 and Theorem 5.1.2)**.** *The following determines the* absolute injective model structure *on* $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$*:*

- *weak equivalences: the stable equivalences*
- *cofibrations: morphisms $f\colon X \longrightarrow Y$ such that $f_n\colon X_n \longrightarrow Y_n$ is a cofibration for all $n \in \mathbb{N}$*
- *fibrations: maps with the RLP with respect to the trivial cofibrations.*

In this model structure every object is cofibrant so the identity functor and identity natural transformation serve as a cofibrant replacement.

The fibrant objects of the absolute injective model structure are the injective Ω spectra of the note following 5.4.4. The pair (ω, p) from Theorem 5.4.8 serve as a fibrant-replacement.

Theorem 6.1.3 ([23] Theorem II.3.1)**.** *The smash product of symmetric spectra, $\wedge$, descends to a closed symmetric monoidal product, called the* derived smash product*, in* $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$*, denoted $\wedge^{\mathrm{L}}$, with unit $\gamma(S^0)$.*

Proof Sketch. The proof proceeds as follows: while the absolute injective model structure on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ doesn't lend itself well to interacting with the smash product of $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$, we can find a new model category structure on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ -called the *absolute flat model structure*, [23] Theorem III.4.11(iii) - which has the same weak equivalences as the absolute injective model structure, and thus induces equivalent homotopy categories.

The cofibrant objects of the absolute flat model structure are the flat symmetric spectra of Definition 5.4.10. Proposition I.5.50 of [23] gives us that, if X is flat, then $X \wedge -$ preserves stable equivalences between flat object, i.e. $X \wedge -$ preserves weak equivalences between cofibrant objects of the absolute flat model structure on $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$. Thus, by Theorem 3.3.2 we get that

$$X \wedge - \colon \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*} \longrightarrow \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$$

has a total left derived functor $\mathbf{L}_{X \wedge -}$. By Theorem 3.3.4 we get a closed symmetric monoidal product

$$(\wedge^{\mathrm{L}}, \mathrm{assoc} \circ \gamma, \lambda \circ \gamma, \rho \circ \gamma, \mathrm{twist} \circ \gamma, \gamma(S^0)).$$

□

Corollary 6.1.4. *Since $\wedge^{\mathrm{L}}$ is a closed monoidal product $X \wedge^{\mathrm{L}} -$ is a left adjoint, and thus preserves coproducts, for every $X \in \mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.*

Proposition 6.1.5 ([23] Proposition II.1.10)**.** *The following are true:*

- $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *has all coproducts and* γ *preserves coproducts*

- $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *has all products and* γ *preserves all finite products*

- *For any finite family of symmetric spectra the canonical morphism, in* $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$*, from the coproduct to the product is a stable equivalence.*

Corollary 6.1.6. *The category* $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *has all finite biproducts - a zero object in particular. By necessity the zero object of* $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *is the image of the zero object of* $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *under* γ*.*

Proof. Since γ preserves finite products and coproducts the image of the cannonical morphism from the coproduct to the product in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ is the cannonical morphism in $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$, but since the former is a stable equivalence the latter is an isomorphism by Proposition 3.2.17. Hence by Proposition 1.3.13 we get that $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ has all finite biproducts.

Since a zero object is by definition the biproduct of the empty functor we get that $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ has a zero object.

□

Proposition 6.1.7 ([23])**.** *For any* $X \in \mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *we have* $X \wedge^{\mathrm{L}} \mathbf{0} \cong \mathbf{0} \wedge^{\mathrm{L}} X \cong \mathbf{0}$.

Proof. A zero object is an iniital object, which in turn is the colimit of the empty diagram,, hence $\mathbb{X} \wedge^{\mathrm{L}} \mathbf{0} \cong \mathbf{0}$ since $X \wedge^{\mathrm{L}} -$ is a left adjoint by Theorem 6.1.3.

□

In sum, $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ is a symmetric monoidal category which is closed under finite coproducts and has a zero object. As per Definition 4.1.3, $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ is now eligible to be given the structure of an cocommutative sphere monoidal category.

6.2 Hurewicz Category Structure

Our objective now is to give $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ the structure of a Freudenthal category, then to add to that the structure of a Hurewicz category. The details of the proofs are left to [23] and

[22].

Definition 6.2.1. For each $n \in \mathbb{Z}$ let the n^{th} sphere object of $\operatorname{Ho}\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ be

$$\mathbf{S}^n := \gamma(\mathbb{S}^n).$$

Recall that, by Definition 4.1.3, the sphere object indexed by 0, in this case $\mathbf{S}^0$, must be the unit of the symmetric monoidal product. In our case, $\mathbf{S}^0$ satisfies this requirement by Theorem 6.1.3.

Proposition 6.2.2. *For each $n \in \mathbb{Z}$ the object $\mathbf{S}^n$ together with the diagonal morphism and the zero morphism*

$$\Delta\colon \mathbf{S}^n \longrightarrow \mathbf{S}^n \oplus \mathbf{S}^n \qquad \mathbf{0}_{\mathbf{S}^n}\colon \mathbf{S}^n \longrightarrow \mathbf{0}$$

determine an abelian cogroup structure.

Proof Sketch. In any category, $\mathbb{X}$, with all finite biproducts it is routine to verify that all objects $X \in \mathbb{X}$ may be given the structure of an internal cocommutative comonoid using the diagonal and zero morphisms as the comultiplication and counit morphisms respectively. By Corollary 6.1.6 we know that $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ has all finite biproducts, hence the n-sphere $\mathbf{S}^n$ is an internal cocommutative comonoid in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$. That $\mathbf{S}^n$ is a cogroup follows from Example II.1.15 of [23] and Proposition 4.1.5.

□

Proposition 6.2.3 ([23])**.** *There are isomorphisms*

$$\alpha_{-,=}\colon S^{-+=} \longrightarrow S^{-} \wedge^{\mathrm{L}} S^{=}$$

satisfying the relevant properties from Definition 4.1.3 in $\operatorname{Ho}\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

Proof. These isomorphisms $\alpha_{n,m}$ are given by [23] Proposition II.4.4.

□

Recall the definition of the functor

$$\mathrm{Cone}\colon \mathrm{Ar}(\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}) \longrightarrow \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$$

and the natural transformation

$$\mathrm{i}\colon \mathrm{cod} \Longrightarrow \mathrm{Cone},$$

from Definition 5.3.5.

Definition 6.2.4. For any $\mathbf{f} \in \mathrm{Ho}\,\mathbb{X}(X, Y)$ choose a representative $f\colon \omega X \longrightarrow \omega Y$ and let $\mathrm{Cone}(\mathbf{f}) := \mathrm{Cone}(f)$ and

$$\mathrm{i}_{\mathbf{f}} := \gamma(\mathrm{p}_Y \mathrm{i}_f).$$

This mapping cone then inherits the long exact sequence property from the spectra case.

Proposition 6.2.5 ([22])**.** *The functor*

$$\mathrm{Cone}: \ \mathrm{Ar}\,\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*} \longrightarrow \mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$$

and natural transformation

$$\mathrm{i}_{\mathbf{f}} := \gamma(\mathrm{p}_Y \mathrm{i}_f).$$

of Defintion 6.2.4 are well-defined.

Proof. This follows from the fact that Cone is defined to be homotopy-invariant.

□

The previous propositions yield:

Proposition 6.2.6. *The tuple*

$$(\wedge^{\mathrm{L}}, \mathbf{S}^n, \alpha_{n,m}, \mathrm{Cone}\,, \mathrm{i}\,)_{n,m\in\mathbb{Z}}$$

provides an cocommutative sphere monoidal structure for $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

Hence we also have our splicing homomorphism

$$\varphi^{X,Y}_{n,m} \colon \pi_n(X) \otimes \pi_m(Y) \longrightarrow \pi_{n+m}(X \wedge^{\mathrm{L}} Y)$$

natural in $X, Y \in \mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

Proposition 6.2.7. *The splicing morphism*

$$\varphi^{\mathbf{S}^0,X}_{1,n} \colon \pi_1(\mathbf{S}^1) \otimes \pi_n(X) \longrightarrow \pi_{1+n}(\mathbf{S}^1 \wedge^{\mathrm{L}} X)$$

of $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *is an isomorphism for any* $X \in \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *and any* $n \in \mathbb{Z}$*.*

Proof Sketch. The proof of this follows from the discusion following Proposition I.5.13, together with Proposition I.6.7 of [23].

□

Theorem 6.2.8. *For all* $n \in \mathbb{Z}$ *the following are true*

$$\pi_m(\mathbf{S}^n) = \begin{cases} \mathbb{Z}, & m = n \\ 0, & m < n \end{cases} \qquad \pi_m(\mathrm{H}\,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & m = 0 \\ 0, m \neq 0 \end{cases}.$$

Proof Sketch. There is a class of symmetric spectra, called *semistable* (see Example I.3.11 and Definition I.3.14 of [23]) which, by Proposition I.6.3 of [23], satisfy the property that the homotopy groups of any semistable spectrum are computed as in (5.3) using the homotopy groups of the simplicial sets of its levels.

Since $\mathbb{S}^0$ has isomorphisms for all its structure maps, Definition I.3.14 of [23] implies that $\mathbb{S}^0$ is semistable. Then for all $m < 0$ we get

$$\begin{aligned} \pi_m(\mathbb{S}^0) &\cong \operatorname*{colim}_{k>m} \pi_{m+k}(\mathbb{S}^0_k) \\ &= \operatorname*{colim}_{k>m} \pi_{m+k}(\mathcal{S}^k) \\ &\cong \operatorname*{colim}_{k>m} 0 \\ &\cong 0 \end{aligned}$$

since the unstable homotopy groups of the n-sphere is trivial below degree n.

For $m = 0$ we get

$$\begin{aligned}\pi_0(\mathbb{S}^0) &\cong \operatorname*{colim}_{k>0} \pi_k(\mathbb{S}^0_k)\\ &= \operatorname*{colim}_{k>0} \pi_k(\mathcal{S}^k)\\ &\cong \operatorname*{colim}_{k>0} \mathbb{Z}\\ &\cong \mathbb{Z}\end{aligned}$$

since $\pi_k(S^k) \cong \mathbb{Z}$ for all integers $k \geq 1$.

By Proposition 6.2.7 we get that

$$\pi_m(\mathbf{S}^n) \cong \pi_{m-n}(\mathbf{S}^0) \cong \begin{cases} \mathbb{Z}, & m-n=0 \\ 0, & m-n<0 \end{cases}$$

Finally, Proposition I.3.16(vi) of [23] together with Dold-Thom Theorem (see [8] Theorem 4.K.6) imply that $\mathrm{H}\,\mathbb{Z}$ is semistable. This allows us to compute

$$\begin{aligned}\pi_0(\mathrm{H}\,\mathbb{Z}) &\cong \operatorname*{colim}_{k>0} \pi_k(\mathrm{H}\,\mathbb{Z}_k)\\ &= \operatorname*{colim}_{k>0} \pi_k(\mathrm{SP}\,(()\,\mathcal{S}^k))\\ &\cong \operatorname*{colim}_{k>0} \mathrm{H}_k(\mathcal{S}^k)\\ &\cong \operatorname*{colim}_{k>0} \mathbb{Z} & \cong \mathbb{Z}\end{aligned}$$

and, for $m \neq 0$

$$\begin{aligned}\pi_0(\mathrm{H}\,\mathbb{Z}) &\cong \operatorname*{colim}_{k>0} \pi_k(\mathrm{H}\,\mathbb{Z}_k)\\ &= \operatorname*{colim}_{k>0} \pi_k(\mathrm{SP}\,(()\,\mathcal{S}^k))\\ &\cong \operatorname*{colim}_{k>0} \mathrm{H}_k(\mathcal{S}^k)\\ &\cong \operatorname*{colim}_{k>0} 0 & \cong 0\end{aligned}$$

which completes the proof.

□

Lemma 6.2.9. *The class of n-generated objects of the sphere monoidal category* $(\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}, \mathbf{S}, \mathrm{Cone}\,)$ *is contained in the class of $(n+1)$-connected objects. That is, if X is n-generated, then $\pi_k(X) \cong 0$ for all $k < n$.*

Proof. Let $\mathcal{C}_n$ denote the full subcategory of $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ consisting of of $(n-1)$-connected spectra, i.e. for any symmetric spectrum, X, we have $X \in \mathcal{C}_n$ if and only if $\pi_k(X) \cong 0$ for all $k < n$. By Theorem 6.2.8 we have that $\mathbf{S}^m \in \mathcal{C}_n$ for all $m \geq n$.

Since the homotopy groups preserve coproducts, and since the coproduct of any collection of zero groups is again the zero group, we get that $\mathcal{C}_n$ is closed under coproducts.

Finally, given a morphism $f \colon X \longrightarrow Y$ in $\mathcal{C}_n$, Proposition 5.3.6 an exact sequence

$$\pi_k(X) \xrightarrow{\pi_k(f)} \pi_k(Y) \xrightarrow{\pi_k(\mathrm{i}_f)} \pi_k(\mathrm{Cone}\, f) \xrightarrow{\partial} \pi_{k-1}(X)$$

for any $k < n$. Since $X, Y \in \mathcal{C}_n$ we get that

$$\pi_k(X) \cong \pi_k(Y) \cong \pi_{k-1}(X) \cong 0$$

therefore, by exactness, $\pi_k(\mathrm{Cone}\, f) \cong 0$. We conclude that Cone restricts to $\mathcal{C}_n$, hence the class of n-generated objects is contained in the class of $(n+1)$-connected objects of $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

□

Corollary 6.2.10. *For any morphism $f \colon X \longrightarrow Y$ in the category of n-generated objects of $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ the sequence*

$$\pi_n(X) \xrightarrow{\pi_n(f)} \pi_n(Y) \xrightarrow{\pi_n(\mathrm{i}_f)} \pi_n(\mathrm{Cone}\, f) \longrightarrow 0$$

is exact.

Proof. This follows from the long exact sequence of Proposition 5.3.6 and the isomorphism

$$\pi_{n-1}(X) \cong 0$$

given by Lemma 6.2.9 when X is n-generated.

□

Proposition 6.2.11 ([23] Proposition I.6.12(i)). *For any* $\coprod_{i\in I} X_i \in \mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *we have that*

$$\pi_n\left(\coprod_{i\in I} X_i\right) \cong \bigoplus_{i\in I} \pi_n(X_i)$$

that is, sphere groups commute with coproducts.

Proposition 6.2.12. *The cocommutative sphere monoidal structure of Proposition 6.2.6 is good.*

Proof. Recall from Definition 4.1.8 that an cocommutative sphere monoidal structure is good if the following are true.

1. $\pi_n(\mathscr{S}^n) \cong \mathbb{Z}$ for each $n \in \mathbb{Z}$.
2. $\pi_n(\mathscr{S}^{n+k}) \cong 0$ for all $n \in \mathbb{Z}$ and every integer $k \geq 1$.
3. π_n preserves coproducts for all $n \in \mathbb{Z}$.
4. for any $n \in \mathbb{Z}$ and any morphism f of $\mathbb{X}$ we have that the equation

   $$\operatorname{coker} \pi_n(f) = \ker \pi_n(\mathrm{i}_f)$$

   holds.
5. The homomorphisms $\pi_n(\mathrm{i}_f)\colon \pi_n(\operatorname{cod} f) \longrightarrow \pi_n(\operatorname{Cone} f)$ are epimorphisms whenever f is a morphism between n-generated objects.

Conditions (1) and (2) are true by Theorem 6.2.8, condition (3) by Proposition 6.2.11, condition (4) by Proposition 5.3.6, and condition (5) by Corollary 6.2.10.

□

Now Proposition 6.2.7 can be re-written as:

Proposition 6.2.13. *The triple* $(\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}, \wedge^{\mathrm{L}}, \alpha)$ *forms a Freudenthal category.*

Now to give the Hurewicz category structure:

Proposition 6.2.14. *In* $(\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}, \wedge^{\mathrm{L}}, \alpha, \mathrm{Cone})$ *the spectrum* $\mathrm{H}\,\mathbb{Z}$ *from Definition 5.1.7 is an EM object, thus making* $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *into a Hurewicz category.*

Proof Sketch. Since $\wedge$ is a closed symmetric monoidal product, the functor $\mathrm{H}\,\mathbb{Z}\wedge -$ is a left adjoint and thus preserves colimits. Since Cone is defined as a colimit in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$, the required natural isomorphism is the one given by the universal property of the colimit. That $\mathrm{H}\,\mathbb{Z}$ has the appropriate homotopy groups for non-positive integers is Theorem 6.2.8. Finally, since

$$\pi_1(\eta_{\mathrm{H}\,\mathbb{Z}})\colon \pi_1(\mathbf{S}^0) \longrightarrow \pi_1(\mathrm{H}\,\mathbb{Z})$$

is surjective follows from the fact that $\pi_1(\mathrm{H}\,\mathbb{Z}) \cong 0$ again by Theorem 6.2.8.

□

What follows is important for completing the class of n-generated objects to the class of $(n-1)$-connected objects.

Theorem 6.2.15 ([23] Proposition I.6.12)**.** *Given a sequence of symmetric spectra*

$$X^0 \xrightarrow{f_0} X^1 \xrightarrow{f_1} X_2 \longrightarrow \cdots$$

we get that, for any $k \in \mathbb{Z}$*,*

$$\pi_k(\operatorname*{colim}_n X^n) \cong \operatorname*{colim}_n \pi_k(X^n)$$

and

$$\mathrm{H}_k(\operatorname*{colim}_{nX^n}) \cong \operatorname*{colim}_n \mathrm{H}_k(X^n).$$

Proposition 6.2.16. *Given a sequence of symmetric spectra*

$$X^0 \xrightarrow{f_0} X^1 \xrightarrow{f_1} X_2 \longrightarrow \cdots$$

if the Hurewicz homomorphisms $h_k^{X^n}$ *are all isomorphisms or are all epimorphsms then*

$$h^{\operatorname{colim}_n X^n}\colon \pi_k(\operatorname*{colim}_n X^n) \longrightarrow \mathrm{H}_k(\operatorname*{colim}_n X^n)$$

is, respectively, an isomorphism or an epimorphism whenever both colimits exist.

Proof. In general if I is a small category and we have two diagrams

$$D, D' \colon I \longrightarrow \mathbb{X}$$

and a natural transformation

$$\beta \colon D \Longrightarrow D'$$

such that the components of β are all isomorphisms, then clearly $\operatorname{colim} D \cong \operatorname{colim} D'$ whenever either colimit exists.

If, instead, each component of β is an epimorphism then choose an initial cocone $(\operatorname{colim} D', \iota^{D'})$ over D'. Precomposing $\iota^{D'}$ with the natural transformation β yields the cocone $(\operatorname{colim} D', \beta\iota^{D'})$ over D. If an initial colimit $(\operatorname{colim} D, \iota^{D})$ exists let

$$\Phi \colon \operatorname{colim} D \longrightarrow \operatorname{colim} D'$$

denote the unique morphism such that

$$\iota_i^D \Phi = \beta_i \iota_i^{D'}$$

holds for all $i \in I$. We want to show that Φ is epic. Let

$$f, g \colon \operatorname{colim} D' \longrightarrow X$$

be any morphisms such that

$$\Phi f = \Phi g$$

holds. Then for each $i \in I$ we have

$$\begin{aligned} \beta_i \iota_i^{D'} f &= \iota_i^D \Phi f \\ &= \iota_i^D \Phi g \\ &= \beta_i \iota_i^{D'} g \end{aligned}$$

hence

$$\iota_i^{D'} f = \iota_i^{D'} g$$

for each $i \in I$ since the β_i are epic. Now since the $\iota_i^{D'}$ are multi-epic we get that $f = g$. In conclusion Φ must be epic.

Now, if we take I to be the underlying category of the diagram

$$X^0 \xrightarrow{f_0} X^1 \xrightarrow{f_1} X_2 \longrightarrow \cdots$$

and we take D and D' to be the diagrams obtained by applying the k^{th} homotopy and homology functors to the above diagram then our work above tells us that the morphism

$$\pi_k(\operatorname*{colim}_n X^n) \longrightarrow \mathrm{H}_k(\operatorname*{colim}_n X^n)$$

given by the universal property is an isomorphism or an epimorphism as appropriate. That this morhpism coincides with the Hurewicz homomorphism then follows from the commutativity of

$$\begin{array}{ccc} \mathrm{H}_k(X^i) & \longrightarrow & \mathrm{H}_k(\operatorname{colim} X^n) \\ {\scriptstyle h_k^{X^i}} \uparrow & & \uparrow {\scriptstyle h_k^{\operatorname{colim} X^n}} \\ \pi_k(X^i) & \longrightarrow & \pi_k(\operatorname{colim} X^n) \end{array} ,$$

which is by the naturality of the Hurewicz homomorphism, and the universal property of the colimit.

□

6.3 The Hurewicz Theorem for Symmetric Spectra

Now we have homotopy groups, we have homology groups, we have the Hurewicz homomorphism, and we have that, for each $n \in \mathbb{Z}$ the n^{th} Hurewicz homomorphism h_n is a natural isomorphism on the subcategory of $\operatorname{Ho}\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ of n-generated objects. To get the more usual Hurewicz homomorphism - the one which states that h_n and h_{n+1} are a natural isomorphism and a pointwise epimorphism respectively on the full subcategory of $(n-1)$-connected objects. We will show that every $(n-1)$-connected object in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ is weakly equivalent to the union of a chain of inclusions of symmetric spectra which are n-generated. We will then show that the Hurewicz homomorphism respects these unions.

Recall the following lemma from group theory, a proof of which can be found in [13] as Corollary I.9.3.

Lemma 6.3.1 ([13]). *Let A and B be abelian groups. There are free abelian groups F_0 and F_1, and a homomorphism $\phi\colon F_0 \to F_1 \oplus A$ such that $B \cong \operatorname{coker}\phi$.*

The following proof is inspired from the case of spaces, see [8].

Lemma 6.3.2. *Let X be any symmetric spectrum, Y be an injective Ω-symmetric spectrum, and let $\phi\colon X \to Y$ be a morphism of symmetric spectra such that, for some integer $n \in \mathbb{Z}$, $\pi_k(\phi)$ is an isomorphism for all $k \leq n$. Then there is an injective Ω-symmetric spectrum X' and a morphism $\phi'\colon X' \to Y$ such that $\pi_k(\phi')$ is an isomorphism for all $k \leq n+1$. If X is $(n+1)$-generated then X' is too.*

Proof. Let X, Y, and ϕ be as in the statement of the lemma. Consider the morphism

$$\pi_{n+1}(\phi)\colon \pi_{n+1}(X) \longrightarrow \pi_{n+1}(Y)$$

By Lemma 6.3.1 there are free abelian groups F_0 and F_1 and a homomorphism $\Phi\colon F_0 \to F_1 \oplus \pi_{n+1}(X)$ such that $\operatorname{coker}\Phi \cong \pi_{n+1}(Y)$.

Let $F_0 \cong \bigoplus_{i\in I}\mathbb{Z}$ and $F_1 \cong \bigoplus_{j\in J}\mathbb{Z}$. Further since $\pi_{n+1}(\mathbf{S}^{n+1}) \cong \mathbb{Z}$ and since $\pi_{n+1}(-)$ commute with coproducts we may change the (co)domain of Φ, namely write

$$\pi_{n+1}\left(\bigvee_{i\in I}\mathbf{S}^{n+1}\right) \xrightarrow{\Phi} \pi_{n+1}(\bigvee_{j\in J}\mathbf{S}^{n+1} \vee X)\,.$$

Let $\widetilde{\Phi}\colon I \to \mathrm{U}(\pi_{n+1}(\vee_{j\in J}\mathbf{S}^{n+1} \vee X)$ denote the function of sets which, for each $i \in I$ picks out the image of the generator of the i^{th} $\pi_{n+1}(\mathbf{S}^{n+1})$ - i.e. $\widetilde{\Phi}$ is the right adjoint of Φ under the free-forgetful adjunction - and for each $i \in I$ choose a representative

$$f_i\colon\, \omega(\mathbf{S}^{n+1}) \longrightarrow \omega\left(\bigvee_{j\in J}\mathbf{S}^{n+1} \vee X\right)$$

of the homotopy class $\widetilde{\Phi}(i)$. Define

$$f\colon \bigvee_{i\in I} \omega(\mathbf{S}^{n+1}) \longrightarrow \omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)$$

to be the morphism with components f_i, then

$$\pi_{n+1}(f)\colon \pi_{n+1}\left(\bigvee_{i\in I} \omega(\mathbf{S}^{n+1})\right) \longrightarrow \pi_{n+1}\left(\omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)\right),$$

but

$$\begin{aligned}\pi_{n+1}\left(\omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)\right) &= \pi\,\mathbb{X}\left(\omega(\mathbf{S}^{n+1}), \omega\left(\omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)\right)\right)\\ &= \pi\,\mathbb{X}\left(\omega\,\mathbf{S}^{n+1}, \omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)\right)\\ &= \pi_{n+1}\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)\end{aligned}$$

where the second equality holds since $\omega^2 = \omega$ by Theorem 5.4.8. Identify the codomain of $\pi_{n+1}(f)$ with $\pi_{n+1}\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)$. Under this identification we get that $\pi_{n+1}(f) = \Phi$.

Further, since π_{n+1} preserves coproducts we have the following composite

$$F_1 = \pi_{n+1}\left(\bigvee_{j\in J} \omega(\mathbf{S}^{n+1})\right) \hookrightarrow \pi_{n+1}\left(\left(\bigvee_{j\in J} \mathbf{S}^{n+1}\right) \vee X\right) \xrightarrow{\ q\ } \operatorname{coker}\phi = \pi_{n+1}(Y)$$

So as before pick out representatives $g_j\colon\ \omega(\mathbf{S}^{n+1}) \longrightarrow \omega\, Y = Y$ of the images of the generators of $\pi_{n+1}\left(\bigvee_{j\in J} \mathbf{S}^{n+1}\right)$, then let

$$g\colon \bigvee_{j\in J} \omega(\mathbf{S}^{n+1}) \longrightarrow Y$$

denote the universal morphism generated by the collection g_j. By construction then

$$\pi_{n+1}(g \vee \phi) \cong q\colon \pi_{n+1}\left(\left(\bigvee_{j\in J} \mathbf{S}^{n+1}\right) \vee X\right) \longrightarrow \operatorname{coker}\Phi.$$

Now, for each $i \in I$ we have that

$$\begin{aligned}\pi_{n+1}(g\vee\phi)\left(f_i\colon\ \omega(\mathbf{S}^{n+1}) \longrightarrow \omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)\right) &= f_i g \vee \phi\\ &= \iota_i f g \vee \phi\\ &= \pi_{n+1}(fg\vee\phi)(\iota_i)\\ &= 0\end{aligned}$$

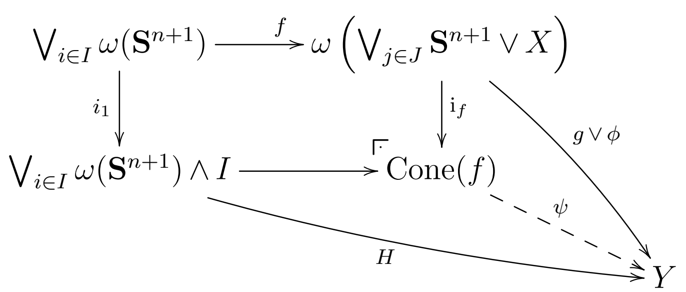
so there is a null-homotopy $H_i\colon\ \omega(\mathbf{S}^{n+1}) \wedge \mathrm{I} \longrightarrow \omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)$. Since

$$\left(\bigvee_{i\in I} \omega(\mathbf{S}^{n+1})\right) \wedge \mathrm{I} \cong \bigvee_{i\in I} (\omega(\mathbf{S}^{n+1}) \wedge \mathrm{I})$$

we can assemble the collection H_i into a null-homotopy H. This is a legitimate null-homotopy since the codomain is fibrant,

$$H\colon\ \left(\bigvee_{i\in I} \omega(\mathbf{S}^{n+1})\right) \wedge \mathrm{I} \longrightarrow \omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right).$$

By the definition of the mapping cone we have



The exterior diagram commutes by the definition of a homotopy so the dashed map, ψ, exists.

Now we want to show that $\pi_{n+1}(\psi)$ is an isomorphism. Since functors take commuting diagrams to commuting diagrams, and since $\pi_{n+1}(g \vee \phi)$ is surjective, so to is $\pi_{n+1}(\psi)$. To prove injectivity pick any $x \in \ker \pi_{n+1}(\psi)$, now since

$$\pi_{n+1}\left(\omega\left(\bigvee_{j\in J} \mathbf{S}^{n+1} \vee X\right)\right) \longrightarrow \pi_{n+1}(\mathrm{Cone}(f))$$

is surjective (by the induced long-exact sequence generated by the mapping cone) pick a preimage

$$y \in \pi_{n+1}\left(\left(\bigvee_{j\in J} \mathbf{S}^{n+1}\right) \vee X\right)$$

of x. Then y is in the kernel of $\pi_{n+1}(g \vee \phi)$ by commutativity so y is in the image of $\pi_{n+1}(f)$. However, because $\pi_{n+1}(\mathrm{Cone}(f)) \cong \operatorname{coker} \pi_{n+1}(f)$ we get that $x = 0$, hence $\pi_{n+1}(\psi)$ is injective. Thus $\pi_{n+1}(\psi)$ is an isomorphism.

Finally, for $k < n+1$ we know that $\pi_k(\mathbf{S}^{n+1}) \cong 0$ so Proposition 5.3.6 gives us an exact sequence

$$0 \xrightarrow{\pi_k(f)} \pi_k\left(\omega\left(\bigvee_{j\in J}\mathbf{S}^{n+1}\vee X\right)\right) \xrightarrow{\pi_k(\mathrm{i}_f)} \pi_k(\mathrm{Cone}\, f) \longrightarrow 0$$

and exactness forces $\pi_k(\mathrm{i}_f)$ to be an isomorphism, hence $\pi_k(\psi)$ is an isomorphim.

The theorem is then complete if we take $X' := \mathrm{Cone}(f)$ and $\phi' := \psi$. Notice that if X is n-generated, then X' was obtained from X by taking a coproduct of $(n+1)$-spheres, taking a coproduct of X with another coproduct of $(n+1)$-spheres, and then taking the mapping cone of a morphism between these two, all of which are either $(n+1)$-generated objects or restrict to the category of $(n+1)$-generated objects, hence X' is $(n+1)$-generated if X is.

□

Theorem 6.3.3. *A symmetric spectrum* $X \in \mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *has* $\pi_k(X) = 0$ *for all integers* $k < n$ *if and only if either* X *is stably equivalent to an* n*-generated spectrum in the sense of Definition 4.1.7 or there exists a sequence* $\{X_i\}_{k\in\mathbb{N}}$ *of* n*-generated spectra together with inclusions* $\iota_k\colon X_k \longrightarrow X_{k+1}$ *in* $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *such that*

$$X \simeq \operatorname*{colim}_{k\in\mathbb{N}} X_k.$$

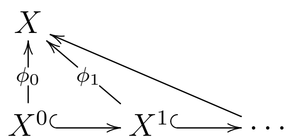
Call such a spectrum $(n-1)$-connected*.*

Proof. We already have by Proposition 6.2.9 that an n-generated symmetric spectrum, X, satisfies $\pi_k(X) = 0$ for all $k < n$. By Theorem 6.2.15 $\pi_k(\mathrm{colim}\, X^j) = \mathrm{colim}(\pi_k(X^j)) = \mathrm{colim}\, 0 = 0$ for all integers $k < n$ whenever the X^j are n-generated. Since we get that if $Y \sim X$ and X satisfies $\pi_k(X) = 0$ for all integers $k < n$ then the same is true of Y since stable equivalences in $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ descend to isomorphisms in $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$.

First, since every symmetric spectrum is stably equivalent to a fibrant-cofibrant one it is enough to prove the converse for those objects and then again appeal to the fact that stable equivalences descend to isomorphisms in $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ by the construction of model categories. As such, pick a fibrant-cofibrant symmetric spectrum X which is $(n-1)$-connected. By definition we have that

$$\pi_k(S^n) \cong \pi_k(X)$$

via the zero morphism for all integers $k \leq n-1$. By $\aleph_0$ many applications of the lemma above we get an ascending sequence of spectra (taking $X^0 := S^n$), and a cocone on that sequence



Notice that since the proof of Lemma 6.3.2 only makes use of X, spheres of the appropriate dimension, and mapping cones we get each of the X_i are n-generated. Now define

$$X' := \operatorname*{colim}_{n \in \mathbb{N}} X^n$$

and consider the unique morphism $\phi \colon X' \longrightarrow X$, we want to show that this morphism induces isomorphisms on all homotopy groups. Now by Theorem 6.2.15 we get that, for any $k \in \mathbb{Z}$ the morphism

$$\pi_k(\phi) \colon \pi_k(X') \longrightarrow \pi_k(X)$$

is the unique map

$$\operatorname*{colim}_{j,l} \pi_{k+l}(X^j_l) \longrightarrow \pi_k(X)$$

but by construction, for sufficiently large values of j the maps $\pi_{k+l}(X^j_l) \longrightarrow \pi_{k+l}(X_l)$ are isomorphisms, so ϕ must be also.

In particular then there is a weak equivalence $X' \longrightarrow \omega X$. Thus, in the homotopy category we get that the n-connected objects are exactly what they're supposed to be.

□

So now we can finally obtain the *Hurewicz Theorem:*

Theorem 6.3.4. *On the subcategory of* $\operatorname{Ho} \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}_*}$ *consisting of n-connected objects, the* $(n+1)^{st}$ *Hurewicz homomorphism,* h_{n+1}*, is a natural isomorphsim and the* $(n+2)^{nd}$ *Hurewicz homomorphism,* h_{n+2}*, is a pointwise epimorphism.*

Proof. We proved in the weak Hurewicz theorem that on the $(n+1)$-generated objects h_{n+1} is a natural isomorphism. So we need only show that if $X \simeq \operatorname{colim}_{k\in\mathbb{N}} X_k$ then h^X_{n+1} is an isomorphism. Since weak equivalences induce isomorphisms on homotopy it is enough to check this when we have actual equality. This follows from Proposition 6.2.16.

$\square$

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