1 Introduction

In this paper we present an algorithm for a geometric optimization problem in 3-dimensional space: Given a convex polyhedron, find the maximum volume inscribed tetrahedron which has one face coinciding with a face of the polyhedron. This problem is a generalization of the problem of finding the maximum area inscribed triangle in a polygon where one of its edges coincides with an edge of the polygon. Polyhedral problems are more difficult to deal with in general than the corresponding polygonal problems as is for example seen in the case of intersection of convex objects in two vs. three dimensions [illegible] and less work has therefore been published on such problems.

Our problem is closely related to the problem of finding the maximum area inscribed triangle in a planar convex polygon. This problem is discussed by a number of authors [2, 6]. Here we show that the maximum volume tetrahedron inscribed in a three dimensional polyhedron can be found in $O(n \log n)$ time.

We first explain the relationship between our problem and the diameter and the width problems. These problems and some problems connected with them are basic problems in computational geometry, and a number of interesting results exist for these problems. For example, Shamos [11] uses the rotating caliper method to show that the diameter problem in plane can be solved in $O(n \log n)$ time using $O(n)$ space. For the special case of the diameter problem for a convex polygon in the plane it has been shown that it can be solved in $O(n)$ time and $O(n)$ space. (The same result was obtained by [illegible] [illegible] using the geometric transform method.) In higher dimensions [illegible] obtained the result that the diameter can be found in $O(n^{2-a(d)}(\log n)^{1-a(d)})$ time where d is the number of dimensions and $a(d) = 2^{-d-1}$ [13]. Aggarwal improved the result when $n = 3$ to get an algorithm which requires $O(n^{3/2}(\log n)^{3/2})$ time [1]. The width problem has been considered by Houle and Toussaint [8]. In two dimensions they showed that the width of n given points can be found in $O(n \log n)$ time using $O(n)$ space and for the special case of the width of a planar convex n-gon it can be found in $O(n)$ time and $O(n)$ space. In three dimension the running time for finding the width of n given points is $O(n \log n + I)$, where I is the number of pairs of edges of the convex hull of the n points that admit parallel planes of support. In the worst case $I = O(n)$.

In Section 2 we discuss the two-dimensional case. It is clear that both the width problem and our problem is almost the same in two dimensions. In order to solve our problem in three dimensions we need to discuss a particular class of search problems known as reciprocal search problems in Section 3. We then give an algorithm for our problem in three dimensions in Section 4.

2 Two dimensions

Given a set S of n points we restate the problems in two dimensions as follows.

Problem 1 *(The diameter problem):*

Find two vertices x, y of S such that

$$d(x,y) = \max\{d(x',y') : x',y' \in S\}. \tag{1}$$

Problem 2 *(The width problem) Find the minimum distance between parallel lines of support of S. (Two parallel lines are parallel lines of support of S if there is a point of S on each of the two lines and where all points of S are in between the two lines.)*

Problem 3 *Find the maximum area inscribed triangle in a convex hull where one edge coincides with an edge of the convex hull.*

For convenience we suppose that S is the set of vertices of the convex hull of S and let P denote the convex polygon with the vertex set S. We also introduce some definitions.

DEFINITION 2.1 *Two edges e_1 and e_2 are called an antipodal e-e pair if they are contained in two parallel lines l_1, l_2, $l_1 \neq l_2$, such that $e_1 \in l_1$ and $e_2 \in l_2$. Such an edge pair is denoted by $\{e_1, e_2\}$.*

DEFINITION 2.2 *Two vertices v_1 and v_2 are called an antipodal v-v pair if they are contained in two parallel lines l_1, l_2, $l_1 \neq l_2$, such that $v_1 \in l_1$ and $v_2 \in l_2$. Such a vertex pair is denoted by $\{v_1, v_2\}$.*

DEFINITION 2.3 *An edge e and a vertex v is called an antipodal v-e pair if they are contained in two parallel lines l_1, l_2, $l_1 \neq l_2$, such that $v \in l_1$ and $e \in l_2$. Such a vertex-edge combination is denoted by $\{v, e\}$.*

Using the above definitions the first two problems can be restated in the following form [8].

Problem 1 *Find an antipodal $v-v$ pair $\{v_1, v_2\}$ such that*

$$d(v_1, v_2) = \max\{d(v_1', v_2') : \{v_1', v_2'\}. \textit{ is an antipodal } v\text{-}v \textit{ pair}\} \tag{2}$$

Problem 2 *Find an antipodal $v-e$ pair $\{v, e\}$ such that*

$$d(v, e) = \min\{d(v', e') : \{v', e'\} \textit{ is an antipodal } v\text{-}e \textit{ pair}\}. \tag{3}$$

Let $\Delta(v, e)$ denote the triangle defined by an edge e and vertex v and let $|\Delta(v, e)|$ denote the area of the triangle $\Delta(v, e)$. We then have the following theorem.

THEOREM 2.1 *Let e be an edge and v be a vertex. Then $\Delta(v, e)$ is the largest area inscribed triangle rooted in P iff*

$$(i) \quad \{v, e\} \textit{ is an antipodal } v\text{-}e \textit{ pair} \tag{4}$$

$$(ii) \quad |\Delta(v, e)| = \max\{|\Delta(v', e') : \{v', e'\} \textit{ is an antipodal } v\text{-}e \textit{ pair}\}. \tag{5}$$

Proof: ($\Rightarrow$) (ii) is obvious.

We construct a line lv through v such that l is parallel to e and we suppose that le is the line through e. It is easy to show that lv and le are lines of support of P. Therefore $\{v, e\}$ is an antipodal v-e pair.

($\Leftarrow$) Suppose that $\{v, e\}$ is an antipodal v-e pair and that

$$|\Delta(v, e)| = \max\{|\Delta(v', e')| : \{v', e'\} \textit{is an antipodal } v\text{-}e \textit{ pair}\}. \tag{6}$$

We fix any edge e'. Then there is at least one vertex v' such that $\{v', e'\}$ is an antipodal v-e pair. Obviously

$$|\Delta(v', e')| = \max\{|\Delta(v'', e')| : v'' \textit{is a vertex of } P\}. \tag{7}$$

Thus $\Delta(v, e)$ is the largest area inscribed triangle rooted in P. □

M. I. Shamos [11] presented a very elegant method called the rotating caliper method which has been used to solve a number of problems [12]. Houle and Toussaint [8] used the method to generate all antipodal v-e pairs and then to get the width of P. The running time is $O(n)$. Obviously Problem 3 can be solved by almost the same algorithm.

Algorithm 2.2

1. *Find an initial antipodal v-e pair in $O(\log n)$ time.*

2. *Use the rotating caliper method to generate all antipodal v-e pairs in $O(n)$ time.*

3. *Calculate the areas of the triangles defined by the antipodal v-e pairs in $O(n)$ time.*

4. *Report the maximal area inscribed triangle rooted on an edge in P.*

3 A class of reciprocal search problem

Within the general class of search problems Guibas and Seidel [7] gives a definition of a subclass which they call reciprocal search problems.

A reciprocal search problem can informally be defined as follows: Let $\mathcal{B}$ be a set of blue objects and $\mathcal{G}$ be a set of green objects. A relation ρ is defined between elements of $\mathcal{D}$ and $\mathcal{G}$ respectively. The resiprocal search problem is then to find all pairs of differently coloured, related objects, i.e. find all $\{b, g\}$ with $b \in \mathcal{B}$ and $g \in \mathcal{G}$, such that $b\rho g$.

Reciprocal search problems arise frequently in computational geometry. Typically, the objects involved are points, line segments, rectangles or other figures with simple descriptions, and the relation ρ might be something like "having non-empty intersection."

In this section we discuss a special class of reciprocal search problems.

A convex subdivision of the plane is a partition of the plane into a finite number of open convex sets. It is not hard to see that convex sets can be only the following three types: a set can be a 2-dimensional region, i.e. it is an open convex polygon (possibly unbounded); a set can be a 1-dimensional edge, i.e. a (possibly unbounded) interval on some straight line; and a set can be a 0-dimensional vertex.

Let $\mathcal{B}$ be a set of n points of the plane, let $\mathcal{C}$ be a convex subdivision of the plane and let $\mathcal{G} = \{\overline{\mathcal{P}} : P \text{ is an open convex polygon of } \mathcal{C}\}$, where $\overline{P}$ is the closure of P and $|\mathcal{C}| = n$.

For each $v \in \mathcal{B}$ we want to find a $P \in \mathcal{G}$ such that $v \in P$. This problem is different from finding the set $\{\{v, P\} : v \in \mathcal{B}, P \in \mathcal{G}, v \in P\}$.

Algorithm 3.1

1. *Define a large triangle $T(ABC)$ such that*
   - *(a) For each $P \in \mathcal{G}$ if P is bounded then $P \subset T$;*
   - *(b) Every vertex of T must belong to some unbounded 1-dimensional edge of $\mathcal{C}$.*

     (See Figure 1)

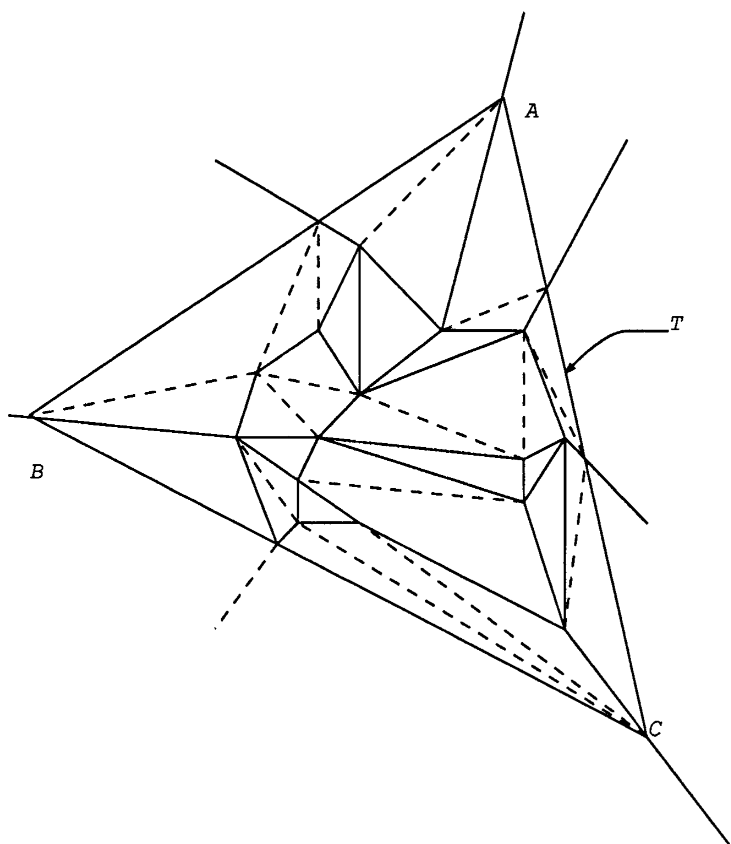

Figure 1: The triangulation

Thus we get two subdivisions, one is $\mathcal{G}_T = \{P \cap T : P \in \mathcal{G}\}$ *which is subdivision of* T *and one which is* $\mathcal{G}_{T^c} = \{P \cap T^c : P \in \mathcal{G}\}$ *which is a subdivision of* T^c *where* T^c *is the complement of* T*. Furthermore,* $\mathcal{G}_{T^c}$ *is divided into three parts* $\mathcal{G}_{T^c}(AB)$, $\mathcal{G}_{T^c}(BC)$, $\mathcal{G}_{T^c}(AC)$ *where the element of* $\mathcal{G}_{T^c}(AB)$ *is an element of* $\mathcal{G}_{T^c}$ *which intersect the line segment* AB*.*

2. *For a fixed point* z *of* B *check whether* Z *is in* T *or not.*

3. *If* z *is not in* T*, then*

   (a) *Check which one of* $\cup\mathcal{G}_{T^c}(AB), \cup\mathcal{G}_{T^c}(AC)$ *and* $\cup\mathcal{G}_{T^c}(BC)$ *contains the point* z*.*

   (b) *If* $z \in \cup\mathcal{G}_{T^c}(AB)$ *(or* $\cup\mathcal{G}_{T^c}(AC)$*, or* $\cup\mathcal{G}_{T^c}(BC)$*), then check which element* P_{T^c} *of* $\mathcal{G}_{T^c}(AB)$ *contains the point* z*.*

   (c) *Check which element* P *of* $\mathcal{G}$ *contains the above element* P_{T^c}*.*

4. *If* z *is in* T*, then*

   (a) *Define a refinement of* $\mathcal{G}$ *to get a triangulation* Π *(see Figure 3.1).*

   (b) *Use the triangulation refinement method [9] to find which element* tz *of* Π *contains* z*.*

   (c) *Find the element* P *of* $\mathcal{G}$ *which contains* tz*.*

5. *For each point* z *repeat the above procedure to find an element of* $\mathcal{G}$ *which contains the point* z*.*

Analysis of the algorithm

From $|\mathcal{B}| = n$ and $|\mathcal{C}| = n$, it follows that step 1) can be done in $O(n)$ time. Obviously steps 2), 3.a), 3.c) and 4.c) can be finished in constant time. Step 4.a) can be done in $O(n)$ time because $|\mathcal{C}| = n$ and every polygon is convex. Step 3.b) can be found by using binary search method, so the running time is $O(\log n)$. By the result of Kirkpatrick [9] the running time of the step 4.b) is $O(\log n)$. Thus using a divide-and-conquer analysis the running time of the algorithm is $O(n \log n)$.

THEOREM 3.2 *Let $\mathcal{B}$ be a set of n points in the plane and let $\mathcal{C}$ be a convex subdivision of the plane and $\mathcal{G} = \{\overline{P} : P$ is an open convex polygon of $\mathcal{C}\}$. The problem of determining a $v \in \mathcal{B}$ for each $P \in \mathcal{G}$ such that $v \in P$ can be solved in $O(n \log n)$ time using $O(n)$ space.*

4 Problem 3 in Three Dimensions

In 1979 K. Q. Brown [3] used several geometric transforms to obtain fast geometric algorithms. As an example he devised an $O(n)$ algorithm for solving the diameter problem in the plane. Houle and Toussaint [8] used his idea to discuss the width problem in two and three dimensions.

Let P be a convex polyhedron. Similarly to the planar case we define the antipodal v-v pair, the antipodal v-e pair, the antipodal v-f pair, the antipodal e-e pair, the antipodal e-f pair, the antipodal e-f pair and the antipodal e-f pair. Before we point out the difference between Problem 2 and Problem 3 in the three dimensions, we prove the following results.

THEOREM 4.1 *(i) For a facet f (2-dimensional face) of P there is at least one vertex v such that the pair $\{v, f\}$ is the antipodal v-f pair.*

(ii) For a facet f of P if $\{v_1, f\}$ and $\{v_2, f\}$ are antipodal v-f pairs then the line segment $\overline{v_1, v_2}$ is parallel to f.

Proof: (i) Take a line l which is perpendicular to f. We project all vertices into the line l. There is at least one element v such that its projection is maximum. We construct a plane π_v through v and perpendicular to l. Obviously π_v and π_f which is a plane through f are parallel planes of support of P. So $\{v, f\}$ is the antipodal v-f pair.

(ii) It is easy to show that (ii) follows from (i). □

THEOREM 4.2 *Let f be a facet of P and v be a vertex of P. Then $T(v, f)$ is the largest volume inscribed tetrahedron rooted in P if and only if*

(i) $\{v, f\}$ *is an antipodal* $v - f$ *pair;*

(ii) $|T(v, f)| = \max\{|T(v', f')| : \{v', f'\}$ *is an antipodal v-f pair of P*$\}$,

where $T(v, f)$ denotes the tetrahedron defined by $\{v, f\}$ and $|T(v, f)|$ denotes the volume of $T(c, f)$.

Proof: ($\Rightarrow$) (ii) is obvious. So we only need to show (i). We construct a plane Π_v which passes through v and is parallel to f. All of the vertices of P must be between Π_v and Π_f which is a plane through f from the conditions given. So $\{v, f\}$ is an antipodal pair.

($\Leftarrow$) For each facet f of P by Theorem 4.1 there is a vertex v such that $\{v, f\}$ is the antipodal v-f pair and $|T(v, f)| = \max\{|T(v', f)| : v'$ is vertex of $P\}$. So if $\{v, f\}$ satisfy the condition (ii) then $T(v, f)$ is the largest volume inscribed tetrahedron rooted in P.

□

Thus we get the following theorem directly from Theorem 3.1 and 3.2.

THEOREM 4.3 *Let f be a facet of P and v be a vertex of P. Then $T(v, f)$ is the largest volume inscribed tetrahedron rooted in P if and only if*

(i) $\{v, f\}$ *is an antipodal pair;*

(ii) $|T(v, f)| = \max\{|T(v_f, f)| : f$ *is a facet of* P *and* $\{v_f, f\}$ *is an antipodal* v-f *pair*$\}$.

We suppose that there is no facet of P which is normal to $v - y$ plane. The slope of the non-vertical plane $z = ax + by + c$ is really a two-dimensional vector, so every such plane maps into a point in $\mathcal{R}^2$:

$$z = ax + by + c \longrightarrow (a, b) \in \mathcal{R}^2. \tag{8}$$

Thus we get the following algorithm for the problem in three dimensions.

Algorithm 4.4

1. *Divide the polyhedron into an upper half hull and a lower half hull.*
2. *Transform each half hull into a planar convex subdivisions $\mathcal{C}_u$ and $\mathcal{C}_d$ respectively. For each of the subdivisions:*
   - *A face on the half hull maps into a vertex of the subdivision.*

- *An edge of the half hull maps into an edge of the subdivision.*
- *A vertex on the half hull maps into an open convex polygon of the subdivision.*

Let V_u *denote the set of all vertices of* $\mathcal{C}_u$ *and* V_d *denote the set of all vertices of* $\mathcal{C}_d$*. Let* P_u *denote the set of all open convex polygons of* $\mathcal{C}_u$ *and* P_d *denote the set of all open convex polygons of* $\mathcal{C}_d$*.*

3. *Let* $\mathcal{B}_u = V_u$ *(or* $\mathcal{B}_d = V_d$*) and* $\mathcal{G}_u = \{\overline{P} : P \in P_u\}$ *(or* $G_d = \{\overline{P} : P \in P_d\}$*).*

   Use the reciprocal search method of section 3 to find the set

$$\begin{aligned} \mathcal{A} &= \{\{b, P_b\} : \forall b \in \mathcal{B}_u, P_b \text{ is some element of } \mathcal{G}_u \text{ and } b \in P_b\} \\ &\cup \{\{b, P_B\} : \forall b \in \mathcal{B}_d, P_b \text{ is some element of} \mathcal{G}_d \text{ and } b \in P_b\}. \quad (9) \end{aligned}$$

4. *Define a set* $\mathcal{A}^*$ *of antipodal v-f pairs*

$$\mathcal{A}^* = \{\{v_b, f_b\} : \{b, P_b\} \in \mathcal{A}\}. \tag{10}$$

5. *Calculate the volume of* $T(v_b, f_b)$ *for each* $\{v_b, f_b\} \in \mathcal{A}^*$
6. *Report the largest volume inscribed tetrahedron rooted in* P*.*

Analysis of the algorithm 4.4

Step 1) is completed in linear time. Brown outlined the method for computing the transforms in Step 2) in linear time [3]. By the reciprocal research method of section 3 $\mathcal{A}$ can be found in $O(n \log n)$ time. Steps 4) and 5) can be done in linear time. Thus using a divide-and-conquer analysis the running time of the algorithm is $O(n \log n)$.

THEOREM 4.5 *Let P be a convex polyhedron. Then the largest volume inscribed tetrahedron rooted in P can be found in $O(n \log n)$ time and $O(n)$ space.*

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