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Broemeling, L. P. (2015). Methods of Complex Function Theory in Some Problems of Analysis: KMS States and Corona Problem (Master's thesis, University of Calgary, Calgary, Canada). Retrieved from https://prism.ucalgary.ca. doi:10.11575/PRISM/26896

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# Methods of Complex Function Theory in Some Problems of Analysis: KMS States and Corona Problem

by

Luke Paul Broemeling

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

GRADUATE PROGRAM IN MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

May, 2015

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# Abstract


This thesis surveys a pair of topics which both depend on holomorphic functions and Banach algebras. Firstly, the prerequisite background knowledge common to both such as holomorphic functions, Banach spaces and algebras, and module theory is provided.

Secondly, KMS states arising on Cuntz-Krieger algebras are described. A Cuntz-Krieger graph algebra $\mathcal{A}$ is the universal $C^*$ algebra satisfying certain defining relations between its partial isometries $p_v$ which are derived from the directed graph. It becomes a $C^*$ dynamical system when equipped with a gauge action $\alpha_t$ defined on partial isometries by $\alpha_t(p_v) = e^{it}p_v$. Its KMS states can now be studied. (The KMS condition arises in physics in which it is a local equilibrium condition for the states of the operator algebra generated by local observables with the action of conjugation by the (time) evolution operator $U_t = e^{itH}$.) Examples including KMS states on matrix algebras and the generalization of Cuntz-Krieger algebras to Cuntz-Pimsner algebras are provided.

Thirdly, algebras of bounded holomorphic functions are discussed. The maximal ideal space for an algebra $H^\infty(\mathcal{R})$ of bounded holomorphic functions on a Riemann surface $\mathcal{R}$ is described. (In particular, this holds for domains in the complex plane $\mathbb{C}$.) By the correspondence between maximal ideals and algebra homomorphism to $\mathbb{C}$, the maximal ideal space may be equipped with its induced weak* topology which is known as the Gelfand topology. A couple of interesting problems are the corona problem and the complement problem. The corona problem concerns whether the point evaluations (functionals induced by the canonical map $\mathcal{R} \to H^\infty(R)^*$) are weak* are dense in the maximal ideal space and the complement problem concerns whether a nonsquare matrix in the algebra $H^\infty(\mathcal{R})$ can be augmented to become a unipotent matrix of the algebra. Wolff's proof of the corona theorem for $\mathbb{D}$ is given.

# Acknowledgments

In memory of my late grandparents Osborne and Elechia who generously assisted with tuition costs, my great-grandmother Gerry who encouraged me to shoot for the stars, and Paul Bertin, Branden Bell, and Emily Morin who were kind students who died before their time.

To my parents Michael and Rebecca who cultivated my love of learning, and grandparents John and Beverly, for their guidance, encouragement, and prayers.

To my supervisors Alex and Berndt for their financial support, supervision, stimulating classes, and patience.

To Yanmei and Renate for assisting me with paperwork, and drawing my attention to deadlines when needed.

To my siblings, both legal and fictive, for providing me with exercise and recreation, thought-provoking conversation, encouragement to continue on, and healthy perspective due to several emergency room visits and a few hospital stays.

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# Chapter 1

# General Background

## 1.1 Topology

**Definition 1.1.1.** *[56], 76. A* **topology** *on a set $X$ axiomatizes the properties of open subsets of the reals. So an element of a topology $\mathcal{T}$ is called an* **open set***. A topology satisfies the following properties:*

- *Both $\emptyset$ and $X$ are open sets.*
- *Whenever $\mathcal{U}$ is a collection of open sets, then their union $\cup\mathcal{U}$ is also an open set.*
- *Whenever $U_1, \ldots, U_n$ is a finite set of open sets, then their intersection $\cap_i U_i$ is an open set also.*

*A* **topological space** *is a pair $(X, \mathcal{T})$ consisting of a set $X$ equipped with a topology $\mathcal{T}$.*

**Definition 1.1.2.** *A* **neighbourhood** *of a point $x$ is a set containing an open set $U$ having $x$ as an element.*

**Definition 1.1.3.** *[56], 93-94. A set in $X$ is* **closed** *if its complement is* **open***. Consequently, both $\emptyset$ and $X$ are closed, and finite unions or arbitrary intersection of closed sets are closed, by De Morgan's law.*

**Definition 1.1.4.** *A* **base** *for a topology is a set of open sets which generate the topology under the operation of arbitrary unions of collections. A* **basic open set** *is a set in a base. Similarly,* **subbase** *for a topology is a set of open sets whose finite intersections form a base. An element of a subbase is called a* **subbasic open set***. For the set $\mathcal{E}$ to be a base for some topology $\mathcal{T}$ on $X$, it must satisfy the following properties:*

- *it must cover $X$ (that is, $\cup\mathcal{E} = X$),*
- *and for any non-empty intersection of elements $E, F$ there must be a basic open set $H$ such that $H \subseteq E \cap F$.*

*For a set $\mathcal{E}$ to be a subbase for some topology on $X$, it has to be a cover of $X$.*

**Example 1.1.5.** *[56], 77, 79. The powerset $\mathcal{P}(X)$ of a set forms a topology which is known as the* **discrete topology**. *The set of singletons $\{\{x\}|x \in X\}$ forms a base for it.*

**Example 1.1.6.** *The topology generated by taking the open intervals $(a, b)$ in $\mathbb{R}$ as a base is the* **standard topology** *for $\mathbb{R}$. The standard topology for $\mathbb{R}^n$ is similarly that generated by open balls of the form*

$$U(x, \epsilon) = \{y \in \mathbb{R}^n : \|y - x\| < \epsilon\}.$$

**Definition 1.1.7.** *[56], 102. A* **continuous function** *$f$ between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is a function between $X$ and $Y$ having the property that its inverse image $f^{-1}(E)$ for any open set $E$ in $Y$ is open in $X$.*

**Example 1.1.8.** *The continuous functions between $\mathbb{R}^m$ and $\mathbb{R}^n$ match those which are continuous under the $\epsilon - \delta$ definition of continuity.*

**Example 1.1.9.** *For any pair of topological spaces, a constant function $f(x) = c$ is a continuous from $X$ to $Y$, since each inverse image is either $\emptyset$ or $X$.*

It would seem reasonably for a inclusion map from a subspace $Y$ into the space $X$ to be continuous. So then $i^{-1}(E) = E \cap Y$.

**Definition 1.1.10.** *[56], 88. The* **subspace topology** *on a subset $Y$ of a topological space $(X, \mathcal{T})$ is*

$$\mathcal{T}_Y = \{E \cap Y | Y \in \mathcal{T}\}.$$

*Then $Y$ equipped with this topology is called a* **subspace** *of $X$.*

**Example 1.1.11.** *A base for the standard subspace topology on* $[0,1]$ *consists of the open intervals of the form* $(a,b)$ *where* $0 < a < b < 1$*, intervals of the form* $[0,k)$ *for* $0 < k < 1$*, intervals of the form* $(m,1]$ *where* $0 < m < 1$*. The elements of this topology are also known as being* **relatively open in** $[0,1]$.

**Definition 1.1.12.** *Let* $(X,\mathcal{T})$ *be a topological space and* $r : X \to Y$ *a surjective function. The* **quotient topology** *defined by* $r$ *is the topology on a set* $Y$ *consisting of subsets* $E \subset Y$ *such that* $r^{-1}(E) \in \mathcal{T}$.

*Conversely, if* $r$ *is a continous surjective function between the topological spaces* $(X,\mathcal{T})$ *and* $(Y,\mathcal{S})$*, then* $r$ *is called a* **quotient map** *if* $r^{-1}(E)$ *is open in* $X$ *precisely when* $E$ *is open in* $Y$.

**Example 1.1.13.** *Take* $X = [0,1]$ *and let* $Y = S^1 \subseteq \mathbb{C}$*. Then the standard topology on* $S^1$ *is the quotient topology induced from* $[0,1]$ *via the map*

$$r : [0,1] \to S^1$$

*defined as*

$$r(x) = e^{2\pi i x}.$$

**Definition 1.1.14.** *[56], 164. A collection* $\mathcal{A}$ *of subsets of a space* $X$ *is a* **covering** *of* $X$ *if the union* $\cup\mathcal{A} = X$*. An* **open covering** *of* $X$ *is a cover whose elements are open.*

*A* **subcovering** $\mathcal{B}$ *of a covering* $\mathcal{A}$ *is a subset of* $\mathcal{A}$ *which is a covering itself.*

**Definition 1.1.15.** *[56], 164, 182. A topological space* $(X,\mathcal{T})$ *is* **compact** *if any open cover* $\mathcal{U}$ *of* $X$ *has a finite subcover. A set* $K$ *in a topological space is compact if it is compact in its subspace topology. It is* **locally compact** *if each* $x \in X$ *has a compact neighbourhood.*

**Example 1.1.16.** *[56], 173. A set in* $\mathbb{R}^n$ *is compact if and only if it is both closed and bounded.*

**Definition 1.1.17.** *[56], 245. An* **open refinement** $\mathcal{V}$ *of an open covering* $\mathcal{U}$ *is a collection of open sets such that each* $V \in \mathcal{V}$ *is contained in some open set* $U$ *in* $\mathcal{U}$*.*

**Example 1.1.18.** *In* $\mathbb{R}$*, let* $(0,1)$ *be covered by*

$$\mathcal{U} = \{(0, 2/3), (1/3, 1)\}.$$

*Then one open refinement of* $\mathcal{U}$ *would be*

$$\{(0, 1/3), (1/4, 2/3), (1/3, 3/4), (2/3, 1)\}.$$

**Definition 1.1.19.** *[56], 244. A collection of subsets* $\mathcal{A}$ *is locally finite if any point* $x \in X$ *has an open neighbourhood which meets only finitely many elements of* $\mathcal{A}$*.*

**Definition 1.1.20.** *[56], 253. A topological space is* **paracompact** *if any open covering* $\mathcal{U}$ *has a locally finite open refinement* $\mathcal{V}$ *which covers* $X$*.*

**Example 1.1.21.** *For instance, a locally compact space will be paracompact, since a locally finite subcovering is certainly a locally finite open refinement.*

**Definition 1.1.22.** *[56], 98. A topological space is called* **Hausdorff** *if given any two distinct* $x, y \in X$ *there are disjoint open neighbourhoods of* $x$ *and* $y$*.*

**Definition 1.1.23.** *[56], 148, 159. A topological space* $X$ *is* **connected** *if whenever* $X = U \cup V$ *where* $U, V$ *are open disjoint sets, either* $U = \emptyset$ *or* $V = \emptyset$*.*

*Given* $X$*, define an equivalence relation on* $X$ *by setting* $x \sim y$ *if there is a connected subspace of* $X$ *containing both* $x$ *and* $y$*. The equivalence classes are called the* **components** *of* $X$*.*

**Definition 1.1.24.** *[41], 25-26. A* **path** $f$ *in a space* $X$ *is a continuous function from the unit interval* $[0, 1]$ *to* $X$*. The path* $f$ *is a* **loop** *if* $f(0) = f(1)$*.*

*The points $f(0), f(1)$ are called the* **endpoints** *of the path. In the case of a loop, $f(0) = f(1)$ is the* **basepoint**.

**Definition 1.1.25.** *[41], 28. [56], 160. A topological space $X$ is* **path connected** *if for any arbitrary distinct points $a, b \in X$, there exists a continuous path $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$.*

*Define an equivalence class on $X$: $x \sim y$ whenever there is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$. The equivalence classes are the* **path components** *of $X$.*

## 1.2 Convergence

**Definition 1.2.1.** *[7], I §6.1, §6.4 prop. 5. A* **filter $\mathcal{F}$ on** *$X$ is a subset of $\mathcal{P}X$ satisfying the following properties:*

- *the filter is non-empty; that is, $X \in \mathcal{F}$*
- *if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$ as well.*
- *if the elements $A, B \in \mathcal{F}$, then their intersection $A \cap B \in \mathcal{F}$ also.*

**Ultrafilters** *or* **maximal filters** *satisfy an additional property:*

- *for any $A \subset X$, one of $A$ or $X \setminus A$ is an element in $\mathcal{F}$.*

**Definition 1.2.2.** *[7], I §7.1. A filter $\mathcal{F}$ on a set $X$ converges to a point $x_0$ if each neighbourhood of $x_0$ is an element of $\mathcal{F}$.*

**Example 1.2.3.** *Let $X = [0,1]$. Then the filter*

$$\mathcal{F} = \left\{ E \subseteq [0,1] : E \supseteq (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) \text{ for some } \epsilon < \frac{1}{2} \right\}$$

*converges to the point $1/2$.*

**Definition 1.2.4.** *[7], II §1.1. A* **uniformity** $\mathcal{U}$ *on a set* $X$ *is a filter on* $X \times X$ *satisfying the following properties:*

- *The diagonal* $\Delta = \{(x, x) | x \in X\}$ *is a subset of any element* $E$ *of* $\mathcal{U}$ *(also known as* **an entourage***). In different terms,* $x$ *is always* $E$*-close to itself and* $E$ *forms a relation on* $X$*.*
- *For each entourage* $E \in \mathcal{U}$*, its inverse*

$$E^{-1} = \{(y, x) | (x, y) \in E\}$$

*is also an entourage. That is, if* $x$ *is* $E$*-close to* $y$*, then* $y$ *is* $E^{-1}$*-close to* $x$*.*

- *For any entourage* $E \in \mathcal{U}$*, there is an entourage* $A \in \mathcal{U}$ *such that its (relation) composition with itself,*

$$A \circ A = \{(x, z) : \text{ there exists some } y \text{ such that } (x, y), (y, z) \in A.\}$$

*is a subset of* $E$*.*

*In the specific case of* $X = \emptyset$*, the uniformity is defined to be* $\mathcal{U} = \{\emptyset\}$*. A pair* $(X, \mathcal{U})$ *consisting of a space* $X$ *equipped with a uniformity* $\mathcal{U}$ *is called a* **uniform space***.*

**Definition 1.2.5.** *Adapted from [7], II §1.2. Any uniformity* $\mathcal{U}$ *induces an associated topology. A set* $T$ *is open in this topology if for any* $x \in T$*, there exists some* $W \in \mathcal{U}$ *such that* $W(x) \subseteq T$*.*

Similar to the term $V$-close for $(x, y) \in V$, we can call an subset $E \subset X$, $V$-small if $(x, y) \in V$ for all $x, y \in V$; in set-theoretic notation, $E$ is $V$-small means that $E \times E \subseteq V$.

**Definition 1.2.6.** *[7], II §3.1. A filter* $\mathcal{F}$ *in a uniform space* $(X, \mathcal{U})$ *is called a* **Cauchy filter** *if for any entourage* $E \in \mathcal{U}$*, there exists some* $F \in \mathcal{F}$ *such that* $F \times F \subseteq E$*. That is, it contains an* $E$*-small set* $F$*.*

**Definition 1.2.7.** *[7], II §2.1. A function between uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called* **uniformly continuous** *if for each entourage $V \in \mathcal{V}$, there exists an entourage $E \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x,y) \in E$.*

## 1.3 Metric and Pseudometric Spaces

**Definition 1.3.1.** *[8], IX §1.1, IX §2.1. Let $X$ be a set. A function $\rho : X \times X \to [0,\infty]$ is called a* **pseudometric** *if it satisfies the following properties:*

- *$\rho(x,x) = 0$ for all $x \in X$,*
- *$\rho(x,y) = \rho(y,x)$ for any $x, y \in X$; that is, $\rho$ is symmetric,*
- *$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for any $x, y, z \in X$; that is, $\rho$ satisfies the triangle inequality.*

*It is called a* **metric** *if $\rho(x,y) = 0$ implies that $x = y$ and it is finite.*

**Definition 1.3.2.** *[8], IX §1.2. The topology and uniformity of a pseudometric space $(X, P)$ are generated by basic open sets taking the form*

$$B_\delta(\rho_1, \ldots, \rho_n; x) = \{y \in X | \rho_j(x,y) < \delta \text{ for all } 1 \leq j \leq n\}$$

*and*

$$W_\delta(\rho_1, \ldots, \rho_n) = \{(x,y) \in X \times X | \rho_j(x,y) < \delta \text{ for all } 1 \leq j \leq n\},$$

*respectively, where $\rho_j$ are pseudometrics in $P$.*

## 1.4 Measure Theory

**Definition 1.4.1.** *[29], 111A. A $\sigma$-algebra (whose elements are known as* **measurable sets***) is a subset of $\mathcal{P}(X)$ having the following properties:*

- *both $X$ and $\emptyset$ are measurable,*
- *any complement of a measurable set is measurable,*
- *and a countable union of measurable sets is measurable.*

**Example 1.4.2.** *[29], 111G. The Borel $\sigma$-algebra for a topological measure space is the $\sigma$-algebra generated by the (standard) topology on the space. So any open set or closed set is measurable in that case. In particular, any countable union of closed sets ($F_\sigma$) is measurable and any countable intersection of open sets ($G_\delta$) is measurable.*

**Definition 1.4.3.** *[29], 112D. A $\sigma$-ideal (whose elements are known as* **negligible sets***) is a subset of $\mathcal{P}(X)$ having the properties:*

- *$\emptyset$ is negligible*
- *if $E_i$ is negligible, then $\cup_i E_i$ is negligible*
- *if $F$ is negligible and $E \subseteq F$, then $E$ is also negligible*

**Definition 1.4.4.** *[29], 112A. A* **measure** *on a $\sigma$-algebra is a function $\mu : \Sigma \to [0, \infty]$ such that*

- $\mu(\emptyset) = 0$
- $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ *whenever $A_i$ is a disjoint measurable family.*

**Definition 1.4.5.** *[29], 112A, 112D. [31], 211B-D, 211F. A* **measure space** *is a triple $(X, \Sigma, \mu)$ where $\Sigma$ is a $\sigma$-algebra, and $\mu$ is a measure on $\Sigma$. It has a $\sigma$-ideal $\mathcal{N}$ consisting of subsets of measurable sets whose measure is zero. It is*

- *a* **probability measure** *when $\mu(X) = 1$*
- **finite** *when $\mu(X) < \infty$*
- **$\sigma$-finite** *when $X = \cup_{i=1}^{\infty} E_i$ where $E_i$ are measurable and $\mu(E_i) < \infty$*

- **semi-finite** *when any non-negligible measurable set $E$ has a non-negligible measurable subset $F$ having finite measure*

**Definition 1.4.6.** *[30], 411B. Let $(X, \Sigma, \mu)$ be a measure space and $\mathcal{K}$ a family of subsets of $X$. Then $\mu$ is* **inner regular with respect to** $\mathcal{K}$ *if for any measurable set $E$*

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \in \mathcal{K} \cap \Sigma\}.$$

**Definition 1.4.7.** *[30], 411D. Let $(X, \Sigma, \mu)$ be a measure space and $\mathcal{H}$ a family of subsets of $X$. Then $\mu$ is* **outer regular with respect to** $\mathcal{H}$ *if for any measurable set $E$*

$$\mu(E) = \inf\{\mu(H) : H \supseteq E, H \in \mathcal{H} \cap \Sigma\}.$$

**Example 1.4.8.** *Lebesgue measure is inner regular with respect to the compact sets and outer regular with respect to the open sets.*

**Definition 1.4.9.** *[29], 121Y(c). A* **measurable function** *between measure spaces $(X, \Sigma, \mu)$ and $(Y, T, \nu)$ is a function from $X$ to $Y$ such that $f^{-1}(E) \in \Sigma$ for each $E \in T$.*

**Example 1.4.10.** *[29], 121D. A continuous function between topological measure spaces is a Borel (and Lebesgue) measurable function between them.*

**Remark 1.4.11.** *Note that the Lebesgue measure of a measurable set $E$ will often be denoted $|E|$ from this point.*

## 1.5 Group Theory

**Definition 1.5.1.** *[45], 24. A* **group** *$G$ is a non-empty set equipped with a binary operation $G \times G \to G$ satisfying the following properties:*

- *associativity: $a(bc) = (ab)c$ for any $a, b, c \in G$*

- *equipped with a two-sided identity* $e \in G$*:* $ae = ea = a$ *for all* $a \in G$*,*
- *each element* $a \in G$ *has a (unique) element* $a^{-1} \in G$ *such that* $aa^{-1} = a^{-1}a = e$*.*

*A non-empty set* $M$ *equipped with a binary operation satisfying just the first two properties is called a* **monoid.**

*If* $G$ *further satisfies the property* $ab = ba$ *for all* $a, b \in G$*, it is an* **abelian (or commutative) group**.

**Definition 1.5.2.** *[45], 30-31. Let* $G, H$ *be monoids. A function* $f : G \to H$ *is a* **homomorphism** *provided that* $f(ab) = f(a)f(b)$ *for all* $a, b \in G$*.*

- *The* **kernel** *of a homomorphism* $f : G \to H$ *is the set* $\{a \in G : f(a) = e\}$*. and the* **image** *of* $f$ *is*
$$f(G) = \{f(a) : a \in G\}.$$
- *If* $f : G \to H$ *is injective, then* $f$ *is called a* **monomorphism***. This is the same as ker*$(f) = \{e\}$*.*
- *If* $f : G \to H$ *is surjective, then* $f$ *is called an* **epimorphism***.*
- *If* $f : G \to H$ *is bijective, then* $f$ *is called an* **isomorphism***. This is the same as the property that there exists an* $f^{-1} : H \to G$ *such that* $ff^{-1} = 1_H$ *and* $f^{-1}f = 1_G$*.*
- *When* $f : G \to G$*, it is called an* **endomorphism***. Finally, an isomorphism* $f : G \to G$ *is called an* **automorphism***.*

**Definition 1.5.3.** *[45], 31. Let* $G$ *be a group, and* $H$ *a non-empty subset of* $G$ *closed under the product operation. If* $H$ *is a group under the product, then* $H$ *is a subgroup of* $G$*,* $H < G$*.*

**Theorem 1.5.4.** *[45], 31. If* $H$ *is a non-empty subset of* $G$*, then* $H$ *is a subgroup precisely when* $ab^{-1} \in H$ *for any* $a, b \in H$*.*

**Definition 1.5.5.** *[45], 32. Let $X$ be a subset of the group $G$. Then the* **subgroup generated by X**, *$\langle X \rangle$ is the intersection of all subgroups of $G$ which contain $X$.*

*A subgroup $H$ of a group $G$ is* **cyclic** *if it can be generated by a single element $g$.*

**Definition 1.5.6.** *A group $G$ equipped with a topology $\mathcal{T}$ is called a* **topological group** *if the operations $(a, b) \mapsto ab$ and $x \mapsto x^{-1}$ are continuous maps (with respect to the topology). A* **discrete group** *is a topological group eqipped with the discrete topology.*

**Example 1.5.7.** *The* **general linear group** *$GL_n(\mathbb{F})$ of a field $\mathbb{F}$ is the group of $n \times n$ matrices $\mathcal{M}_n(\mathbb{F})$ whose determinant is non-zero. It is equipped with the norm topology that it inherits from $\mathcal{M}_n(\mathbb{F})$.*

**Example 1.5.8.** *The* **special linear group** *$SL_n(\mathbb{F})$ is the subgroup of $GL_n(\mathbb{F})$ whose elements have determinants equal to 1. The quotient group obtained by taking its quotient by the subgroup $\{I_n, -I_n\}$ is called the the* **projective special linear group** *$PSL_n(\mathbb{F})$.*

## 1.6 Convex Sets

**Definition 1.6.1.** *[22], I.2.4. Let $\mathcal{X}$ be a vector space and $A \subseteq \mathcal{X}$. $A$ is* **convex** *if for any $x, y \in A$ and $0 \leq t \leq 1$, $tx + (1-t)y \in A$. That is, whenever $A$ contains a pair of points it also contains the line segment joining them.*

**Definition 1.6.2.** *[22], V.7.1. If $K$ is a convex subset of a vector space $\mathcal{X}$, then a point $a$ in $K$ is an* **extreme point** *of $K$ if there is not a proper open line segment containing $a$ and lying entirely in $K$. Denote the set of extreme points of $K$ as $Ext(K)$.*

**Definition 1.6.3.** *[18], 28.1. Let $\mathcal{X}$ be a Hausdorff locally convex space, and $K$ be a compact convex set in it. Let $\overline{K}$ be the cone in$\mathbb{R} \times X$*

$$\overline{K} = \{(\lambda, \lambda k) : k \in K, \lambda > 0\}.$$

*This can be equipped with a partial order. For $x, y \in \overline{K}$, let $x \leq y$ whenever $y - x \in \overline{K}$. We say that $K$ is a* **simplex** *provided that $\overline{K}$ is a lattice under this order.*

**Definition 1.6.4.** *[18], 26.2 Let $\mu$ be a probability measure on a locally convex space $\mathcal{X}$. Then its* **resultant** *$r(\mu)$ is the point of $\mathcal{X}$ such that*

$$\int_X f\,d\mu = f(r(\mu))$$

*for all continuous affine functions $f$ on $\mathcal{X}$.*

**Example 1.6.5.** *If the probability measure $\mu$ is point-supported, then*

$$\int_X f\,d\mu(x) = \sum_{k=1}^{n} c_k f(x_k)$$

*for some $x_k \in \mathcal{X}$ and $c_k \in \mathbb{R}$ such that $\sum_k c_k = 1$. In this case,*

$$r(\mu) = \sum_k c_k x_k.$$

**Definition 1.6.6.** *[18], Pr. 26.6. Let $K$ be a compact convex set in a Hausdorff locally convex space $\mathcal{X}$. A* **face** *$F$ of $K$ is a closed convex set such that every probability measure $\mu$ on $K$ having its resultant $x \in F$ has its* **support**

$$S(\mu) = X \bigcup \{T \text{ is open }, f(x) = 0 \text{ for all } x \in T\}$$

*contained in $K$.*

**Example 1.6.7.** *In particular, if $F$ is a face of a compact convex set $K$ in $\mathcal{X}$ then whenever $\mu$ is a point-supported probability measure on $K$ such that*

$$\int_{\mathcal{X}} f\,d\mu = \sum_{k=1}^{n} c_k f(x_k),$$

*and* $\sum_k c_k x_k \in F$*, then* $x_k \in F$*.*

**Definition 1.6.8.** *[22], IV.1.9. The convex hull of a set* $K$ *in a vector space* $\mathcal{X}$ *is the set consisting of all convex combinations*

$$c_1 x_1 + \cdots + c_n x_n$$

*where* $x_j \in \mathcal{X}$ *and* $c_j \geq 0$, $\sum_j c_j = 1$*.*

**Theorem 1.6.9.** *[22], V.7.4 . If* $K \neq \emptyset$ *is a compact convex subset of a locally convex space* $\mathcal{X}$*, then* $Ext(K) \neq \emptyset$ *and* $K$ *is the closure of the convex hull of* $Ext(K)$*.*

## 1.7 Holomorphic and Harmonic functions

**Definition 1.7.1.** *[15], 14. An open connected subset* $D$ *of* $\mathbb{C}$ *is a* **domain***.*

**Definition 1.7.2.** *[68], 10.2, 10.3. Suppose* $f$ *is a complex-valued function defined in an open set* $\Omega \subseteq \mathbb{C}$*. If* $z_0 \in \Omega$ *and if the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

*exists, it is denoted as* $f'(z_0)$ *and called the* **derivative of** $f$ **at** $z_0$*. If* $f'(z_0)$ *exists for each* $z_0 \in \Omega$*, we say that* $f$ *is* **holomorphic in** $\Omega$ *(also called* **analytic***).*

**Remark 1.7.3.** *The set of holomorphic functions will be denoted* $H(\Omega)$*; by the limit laws, it is a ring. If the domains match and* $f$ *and* $g$ *are holomorphic functions, their composition* $f \circ g$ *is also holomorphic.*

**Definition 1.7.4.** *[68], 11.1. The differential operators* $\partial$ *and* $\overline{\partial}$ *are defined as*

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

**Theorem 1.7.5.** *[68], 11.2. Suppose $f$ is a complex-valued function in $\Omega$ having a differential at each point of $\Omega$. Then $f \in H(\Omega)$ if and only if the Cauchy-Riemann equation $\overline{\partial} f = 0$ holds for each $z \in \Omega$. In that case $f'(z) = (\partial f)(z)$ for each $z \in \Omega$. If $f = u + iv$ for $u, v$ real then $\overline{\partial}(f) = 0$ splits into the* **Cauchy-Riemann equations:** *$u_x = v_y$, $u_y = -v_x$.*

**Definition 1.7.6.** *[68], 11.3, 11.4. The Laplacian $\Delta f$ for a function $f$ having the derivatives $f_{xx}$ and $f_{yy}$ is*

$$\Delta(f) = f_{xx} + f_{yy}.$$

*Note that if $f_{xy} = f_{yx}$ then*

$$\Delta(f) = 4\partial\overline{\partial} f.$$

*A function $f$ continuous in $\Omega$ and having $\Delta f = 0$ in $\Omega$ is a* **harmonic function***. A holomorphic function $f$ is $C^\infty$ and satisfies $\overline{\partial} f = 0$, so is harmonic.*

**Definition 1.7.7.** *[15], 6.1. A* **chain of paths** *$\Gamma$ in a domain $U \subseteq \mathbb{C}$ is a formal sum $c_1\gamma_1 + \ldots + c_n\gamma_n$ of piecewise $C^1$ paths $\gamma_i : [0,1] \to U$ where $c_j \in \mathbb{Z}$. The image $\Gamma^*$ of a chain $\Gamma$ is the union of the images $\gamma_j^*$ of its paths $\gamma_j$.*

*Integration of a function $f \in C(\Gamma^*)$ over a chain $\Gamma$ is defined by linearity:*

$$\int_\Gamma f(z)\, dz = \sum_{k=1}^{n} c_k \int_{\gamma_k} f(z)\, dz$$

*for $f \in C(\gamma_1^* \cup \cdots \cup \gamma_n^*)$.*

**Definition 1.7.8.** *[15], 6.1. Two chains $\Gamma = \sum_j c_j\gamma_k$ and $\Gamma' = \sum_k c_k'\gamma_k'$ are* **equivalent** *if $\int_\Gamma f(z)\, dz = \int_{\Gamma'} f(z)\, dz$ for any function $f \in C(\Gamma \cup \Gamma')$. A* **cycle** *is a chain which is (equivalent to) a sum $\sum_j c_j\gamma_j$ where $\gamma_j$ are closed paths.*

**Definition 1.7.9.** *[15], 6.1. The* **index** *of a point $z$ with respect to the cycle $\Gamma$*

$$Ind_{\Gamma}(z) = \sum_{k=1}^{n} c_k Ind_{\gamma_k}(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} c_k \int_{\gamma_k} \frac{d\zeta}{\zeta - z}.$$

**Theorem 1.7.10.** *Inverse Function Theorem. [36], 233. Let $f(z)$ be a non-constant function which is analytic in the closed disk $|z - z_0| \leq \rho$ such that $f'(z_0) \neq 0$ and $f(z) \neq f(z_0)$ within the punctured closed disc $0 < |z - z_0| \leq \rho$. Choose $\delta > 0$ such that $|f(z) - f(z_0)| \geq \delta$ for each $z$ of the circle $|z - z_0| = \rho$. Then there is a unique $z$ (denoted $f^{-1}(w)$) such that*

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta$$

*for $w$ in the disk $|w - f(z_0)| < \delta$ and $f^{-1}(w)$ is analytic.*

**Theorem 1.7.11.** *Implicit Function Theorem. [36], 235. Let $F(z, w)$ be a continuous function which is analytic in both $z$ and $w$. Let $F(z_0, w_0) = 0$, $\frac{\partial F}{\partial z}(z_0, w_0) \neq 0$. Now choose $\rho$ so that $F(z, w_0) \neq 0$ for $0 < |z - z_0| \leq \rho$. Then there exists a $\delta > 0$ such that for $w$ in the disk $|w - w_0| < \delta$, there is a unique $g(w) = z$ in the disk $|z - z_0| < \rho$ such that $F(z, w) = 0$. Now*

$$g(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta \frac{\partial F}{\partial z}(\zeta, w)}{F(\zeta, w)} d\zeta$$

*is an analytic function defined for $w$ in the open disk $|w - w_0| < \delta$.*

**Definition 1.7.12.** *[67], 3.30. Let $\Omega \subseteq \mathbb{C}$ be open, and $\mathcal{X}$ be a complex topological vector space.*

- *a function $f : \Omega \to \mathcal{X}$ is* **weakly holomorphic in** $\Omega$ *if $\alpha \circ f$ is holomorphic in $\Omega$ for each $\alpha \in \mathcal{X}^*$.*
- *a function $f : \Omega \to \mathcal{X}$ is* **strongly holomorphic in** $\Omega$ *if*

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

*exists (in the topology of $\mathcal{X}$) for every $z \in \Omega$.*

**Theorem 1.7.13.** *[67], 3.31. Let $\Omega \subseteq \mathbb{C}$ be open, and $f$ a weakly holomorphic function $f$ from $\Omega$ to a complex topological vector space $\mathcal{X}$. The following conclusions hold:*

- *$f$ is strongly continuous in $\Omega$*
- *Cauchy's Theorem and Cauchy's Integral Formula hold: if $\gamma$ is a closed path in $\Omega$ such that $Ind_\gamma(w) = 0$ for every $w \notin \Omega$, then*

$$\int_\gamma f(\zeta)\, d\zeta = 0$$

*and*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{z - \zeta}\, d\zeta$$

*if $z \in \Omega$ and $Ind_\gamma(z) = 1$. If $\gamma_1, \gamma_2$ are closed paths in $\Omega$ such that $Ind_{\gamma_1}(z) = Ind_{\gamma_2}(z)$ for every $z \notin \Omega$, then*

$$\int_{\gamma_1} f(\zeta)\, d\zeta = \int_{\gamma_2} f(\zeta)\, d\zeta.$$

- *$f$ is strongly holomorphic in $\Omega$.*

**Theorem 1.7.14.** *[68], Thm. 10.16, Cor. to Thm.10.6, Thm 10.7. Let $\Omega$ be an open set. Every $f \in H(\Omega)$ is representable by power series in $\Omega$. That is, there is a power series expansion for $f$ about any point of $\Omega$. Furthermore, this expansion is unique.*

*Proof.* Let $a$ be an arbitrary point of $\Omega$. Consider $R > 0$ such that $B_R(a) \subseteq \Omega$. For any $z \in B_r(a)$ where $0 < r < R$

$$f(z) = f(z)\mathrm{Ind}_{\delta B_r(a)} = \frac{1}{2\pi i} \int_{\delta B_r(a)} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

since the winding number of a point inside a circle with respect to its positive-oriented boundary is 1. Now the uniformly converging sum

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta-a)^{n+1}} = \frac{1}{\zeta - z}$$

with the help of Fubini's theorem (applies to the line integral since the circle is rectifiable and its integrand is continuous, to the sum because of uniform convergence) gives

$$f(z) = \frac{1}{2\pi i}\int_{\delta B_r(a)} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta-a)^{n+1}} f(\zeta)\, d\zeta = f(z) = \sum_{n=0}^{\infty}\left(\int_{\delta B_r(a)} \frac{f(\zeta)\, d\zeta}{(\zeta-a)^{n+1}}\right)(z-a)^n = \sum_k c_k\,(z-a)^n$$

So we have a power series expansion. Now differentiation under the integral shows that

$$f^k(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n\,(z-a)^{n-k}\,;$$

in particular, for each $k \in \mathbb{N}$ it follows that $f^k(a) = n!c_k$. Consequently, the power series is uniquely determined. ■

**Theorem 1.7.15.** *[68], Thm. 10.18. Let $\Omega$ be a domain in $\mathbb{C}$, $f \in H(\Omega)$, and*

$$\mathcal{Z}(f) = \{a \in \Omega : f(a) = 0\}\,.$$

*Then either $\mathcal{Z}(f) = \Omega$ or $\mathcal{Z}(f)$ has no limit point in $\Omega$. In the latter case there corresponds to each $a \in \mathcal{Z}(f)$ a positive integer $m = m(a)$ (the* **order of the zero***) such that*

$$f(z) = (z-a)^m g(z)$$

*for all $z \in \Omega$, where $g \in H(\Omega)$, $g(a) \neq 0$. Furthermore, $\mathcal{Z}(f)$ is countable.*

**Corollary 1.7.16.** *The Identity Principle. [68], Cor. to Thm. 10.18. If $f, g \in H(\Omega)$ where $\Omega$ is a domain and if $f(z) = g(z)$ for all $z$ in a set having a limit point in $\Omega$, then $f(z) = g(z)$*

*for all* $z \in \Omega$.

**Theorem 1.7.17.** *[68], Thm. 10.35. Suppose* $f \in H(\Omega)$, *where* $\Omega \subseteq \mathbb{C}$ *is an arbitrary open set. If* $\Gamma$ *is a cycle in* $\Omega$ *satisfying*

$$Ind_{\Gamma}(z) = 0$$

*for each* $z \notin \Omega$, *then*

$$f(z) Ind_{\Gamma}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)\, dw}{w - z}$$

*for each* $z \in \Omega$ *not on a path of* $\Gamma$, *and*

$$\int_{\Gamma} f(z)\, dz = 0.$$

*If* $\Gamma_0$ *and* $\Gamma_1$ *are cycles in* $\Omega$ *such that* $Ind_{\Gamma_0}(z) = Ind_{\Gamma_1}(z)$ *for each* $z \notin \Omega$, *then*

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz.$$

**Corollary 1.7.18.** *By the equivalence of weak and strong holomorphicity, this result applies to vector-valued holomorphic functions.*

**Definition 1.7.19.** *A continuous real valued function* $f$ *on an open set* $\Omega$ *is* **subharmonic** *if for any* $B_r(a) \subseteq \Omega$,

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

*That is, its value at a point* $a$ *is bounded above by its averages along circles about* $a$.

**Example 1.7.20.** *The modulus* $|f|$ *of holomorphic function is subharmonic.*

*Proof.* Note that for $B_r(a) \subseteq \Omega$ (using the Cauchy integral formula on $\gamma(t) = a + re^{i\theta}$)

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta$$

$$|f(z)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z+re^{i\theta})\,d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z+re^{i\theta})\right|\,d\theta.$$

∎

**Theorem 1.7.21.** *[36], 395. Let $u(z)$ be a continuous subharmonic real-valued function on a domain $\Omega$. If $u(z)$ is bounded above by $M$ on $\Omega$, then $u(z)$ attains $M$ on $\Omega$ only if it is constant.*

*Proof.* Let $u(z)$ attains its maximum $M$ at $z$, $B_R(z) \subseteq \Omega$, and let $r$ be an arbitrary real value such that $0 < r < R$. Now as

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta,$$

it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta}) - u(z)\,d\theta \geq 0.$$

Since $u(z) \leq M$ on $\Omega$, it follows that $\frac{1}{2\pi}\left(u(z) - u(z+re^{i\theta})\right)\,d\theta \geq 0$, so

$$\int_0^{2\pi} (u(z+re^{i\theta}) - M)\,d\theta = 0.$$

This means that $u(z+re^{i\theta}) = M$ almost everywhere on $\{z||z-a| = r\}$, so by continuity $u(z) = M$ on the circle. So $\{z \in \Omega|u(z) = M\}$ is open.

Now let $z_k$ be a convergent net of points (to $z'$) in $\{z \in \Omega|u(z) = M\}$. Then by continuity of $u(z)$,

$$u(z') = \lim_k u(z_k) = M.$$

Consequently, $\{z \in \Omega|u(z) = M$ is relatively closed. Consequently, $\Omega$ is closed and open in $\Omega$, so $\{z \in \Omega|u(z) = M\} = \emptyset$ or $\{z \in \Omega|u(z) = M\} = \Omega\}$. That is, $u(z)$ doesn't attain the upper bound in $\Omega$, or $u(z)$ is a constant. ∎

## 1.8 Fundamental Group

**Definition 1.8.1.** *[41], 25. A* **homotopy of paths (with fixed endpoints)** *in a topological space $X$ is family $f_t$ of paths $f_t : [0,1] \to X$ such that*

- $f_t(0) = x_0$ *and* $f_t(1) = x_1$ *for any* $t \in [0,1]$,
- *The function* $F : [0,1] \times [0,1] \to X$ *defined by* $F(s,t) = f_t(s)$ *is continuous.*

*If $f_0$ and $f_1$ are connected by a homotopy, they are* **homotopic***; $f_0 \simeq f_1$.*

**Definition 1.8.2.** *[41], 26. If $f, g : [0,1] \to X$ are paths in a topological space $X$ such that $f(1) = g(0)$, then their* **product path** *$f \cdot g$ is defined*

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq 1/2, \\ g(2s-1) & 1/2 \leq s \leq 1 \end{cases}.$$

**Remark 1.8.3.** *[41], 26. Note that if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ that $f_0 \cdot g_0 \simeq g_0 \cdot g_1$. This means that the product path operation is compatible with the equivalence relation of path homotopy (with fixed endpoints).*

**Definition 1.8.4.** *[41], 26-27. The* **fundamental group** *$\pi_1(X, x_0)$ is the set of all homotopy classes $[f]$ of loops $f : [0,1] \to X$ (at the basepoint $x_0$). It is a group with respect to the operation $[f][g] = [f \cdot g]$.*

*The element $[f]^{-1}$ is the homotopy class of the* **inverse path** *$\overline{f}$ of $f$; that is,*

$$\overline{f}(t) = f(1-t).$$

*The identity element 1 is the homotopy class of the constant path $c(t) = x_0$.*

**Remark 1.8.5.** *[41], Prop. 1.5, 28. A group isomorphism $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ can be defined as*

$$\beta_h([f]) = [h \cdot f \cdot \overline{h}],$$

*where $h$ is any path joining $x_0$ to $x_1$. This shows that the fundamental group (denoted as $\pi_1(X)$) is unique up to isomorphism, provided that $X$ is path-connected.*

**Definition 1.8.6.** *[41], 28. A path connected topological space $X$ is* **simply connected** *if it has a trivial fundamental group. Otherwise, it is* **multiply connected**.

**Example 1.8.7.** *[15], prop. 6.4, 210. If $U$ is a domain of $\mathbb{C}$, then it is simply connected if and only if its complement in the Riemann sphere $S^2$ has no bounded components.*

## 1.9 Banach Spaces

**Definition 1.9.1.** *[47], 2.1. An* **inner product** *on a complex vector space $\mathcal{X}$ is a mapping $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that*

$$\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle \tag{1.1}$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}, \tag{1.2}$$

$$\langle x, x \rangle \geq 0. \tag{1.3}$$

*whenever $x, y, z \in \mathcal{X}$.*

*If $\langle x, x \rangle = 0$ only when $x = 0$, we say $\langle \cdot, \cdot \rangle$ is a* **definite inner product**.

**Definition 1.9.2.** *[47], 7. A* **norm** *$\|\cdot\|$ on a vector space $\mathcal{X}$ is a function $\mathcal{X} \to [0, \infty)$ which satisfies the properties for any $x, y \in \mathcal{X}$, $\lambda \in \mathbb{C}$:*

$$\|x\| = 0 \text{ if and only if } x = 0 \tag{1.4}$$

$$\|x + y\| \leq \|x\| + \|y\| \text{ (triangle inequality)} \tag{1.5}$$

$$\|\lambda x\| = |\lambda| \, \|x\| \tag{1.6}$$

*If (1.4) is dropped, the other conditions define a* **semi-norm**.

**Definition 1.9.3.** *[47], 1.5.5. The norm of an operator $T : \mathcal{X} \to \mathcal{Y}$ between normed spaces is defined as*

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0, x \in \mathcal{X}\right\}.$$

**Example 1.9.4.** *In a Hilbert space $X$ equipped with an inner product $\langle \cdot, \cdot \rangle$), we can define an associated* **semi-norm** *by $\|x\| = \sqrt{\langle x, x \rangle}$. If $\langle , \rangle$ is definite, this will be a norm. For instance, the usual norm on $\mathbb{R}^n$ comes from the dot product on $\mathbb{R}^n$.*

**Proposition 1.9.5.** *[22], 3. Cauchy-Bunyakowsky-Schwarz: Let $\mathcal{X}$ be a vector space equipped with an inner product $\langle , \rangle$ and its associated norm. For any $x, y \in \mathcal{X}$:*

$$|\langle x, y \rangle| \leq \|x\| \; \|y\| \tag{1.7}$$

*Proof.* Given any $\alpha \in \mathbb{C}$ and $x, y \in \mathcal{X}$,

$$0 \leq \langle x - \alpha y, x - \alpha y \rangle$$

$$0 \leq \langle x, x \rangle - \overline{\alpha} \langle y, x \rangle - \alpha \langle x, y \rangle + |\alpha|^2 \langle y, y \rangle .$$

Now let $\langle y, x \rangle = be^{i\theta}$ and define $\alpha = e^{-i\theta}$. Then

$$0 \leq \langle x, x \rangle - 2bt + t^2 \langle y, y \rangle$$

which means that the discriminant $\Delta$ of this quadratic

$$4b^2 - 4 \langle x, x \rangle \langle y, y \rangle \leq 0$$

$$b^2 \leq \langle x, x \rangle \langle y, y \rangle .$$

This means that

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \ \langle y, y\rangle$$

so that

$$|\langle x, y\rangle| \leq \|x\| \ \|y\| ,$$

as required. ∎

**Definition 1.9.6.** *[47], §1.2.6. Let $X$ be a vector space equipped with a family of semi-norms $P$. The topology defined by a family of semi-norms $P$ has a subbase generated by sets of the form*

$$U(x;p;\epsilon) = \{y \in \mathcal{X} : p(x-y) < \epsilon\}$$

*where $p \in P$; that is, it has a base generated by basic open neighbourhoods of the form*

$$U(x;p_1,\ldots,p_n;\epsilon) = \{y \in \mathcal{X} : p_i(x-y) < \epsilon \text{ for each } 1 \leq i \leq n\} .$$

*The uniformity defined by this family is that generated by the ensembles*

$$W(p_1,\ldots,p_n;\epsilon) = \{(x,y) \in \mathcal{X} \times \mathcal{X} : p_i(x-y) < \epsilon \text{ for each } 1 \leq i \leq n\} .$$

*The topology defined by a norm $\|\cdot\|$ is that generated by the basic open sets*

$$U(x;\epsilon) = \{y \in \mathcal{X} : \|x-y\| < \epsilon\} .$$

*The uniformity defined by a norm is that generated by the ensembles*

$$W(\epsilon) = \{(x,y) \in \mathcal{X} \times \mathcal{X} : \|x-y\| < \epsilon\} .$$

*If the semi-norms form a separating family (e.g., there is no point $z$ in the topological* *vector space which each semi-norm vanishes), then the associated topology is a (Hausdorff) locally convex topology.*

**Definition 1.9.7.** *[47], 1.5. A* **Banach space** *is a normed vector space which is complete relative to the uniformity defined by its norm* $\|\cdot\|$.

**Example 1.9.8.** *[29], 133A-E. [31], 242-244A,242-244D. Let* $(X, \Sigma, \mu)$ *be a measure space, and let* $L^0(\mu)$ *be the space of functions* $f : X \to \mathbb{C}$ *defined and measurable almost everywhere. Then for* $1 \leq p < \infty$

$$\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}$$

*and*

$$\|f\|_\infty = ess\ sup\, |f(x)| = \inf \{t \in \mathbb{R} : \mu (\{x : |f(x)| > t\}) = 0\}$$

*define norms on* $L^0(\mu)$*. Then define the* **function spaces** $L^p(\mu)$ *as*

$$L^p(\mu) = \left\{ f \in L^0(\mu) : \|f\|_p < \infty \right\}.$$

**Definition 1.9.9.** *[31], 283A, 284A. Let* $h$ *a function from* $\mathbb{R}$ *to* $\mathbb{C}$*. If* $h$ *is differentiable of every order and*

$$\sup_{x \in \mathbb{R}} |x|^k \left| h^{(m)}(x) \right| < \infty$$

*for any positive integers* $k, m$*, then* $h$ *is a* **(rapidly decreasing) test function***. The* **Fourier transform** *and* **inverse Fourier transform** *are defined on them as follows:*

$$\hat{h}(y) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{-ixy} h(x) \, dx,$$

$$\check{h}(y) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{ixy} h(x) \, dx = \hat{h}(-y).$$

**Proposition 1.9.10.** *[31], 284H, 284P. For* $f, g \in L^2(\mathbb{R})$*,* $g$ *is the Fourier transform of* $f$

*provided that*

$$\int_{x=-\infty}^{\infty} g(x)h(x)\,dx = \int_{x=-\infty}^{\infty} f(x)\hat{h(x)}\,dx$$

*for any test function* $h : \mathbb{R} \to \mathbb{C}$*. Fourier transformation* $f \mapsto \hat{f}$ *is a linear isometry on* $L^2(\mathbb{R})$*. In particular,*

$$\|f\|_2 = \left\|\hat{f}\right\|_2$$

*for any* $f \in L^2(\mathbb{R})$*.*

**Definition 1.9.11.** *[37], 48. The Hardy spaces on the disk* $\mathbb{D} = \left\{re^{i\theta} \in \mathbb{C} : 0 \le r < 1\right\}$ *come from the norms*

$$\|f\|_{H^p}^p = \sup_{0\le r<1} \frac{1}{2\pi} \int \left|f(re^{i\theta})\right|^p \, d\theta$$

*for* $1 \le p < \infty$ *and the supremum norm*

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|\,.$$

*Then the Hardy spaces* $1 \le p \le \infty$ *consist of holomorphic functions having finite norms*

$$H^p(\mathbb{D}) = \{f \in H(\mathbb{D}) : \|f\|_{H^p} < \infty\}\,.$$

**Definition 1.9.12.** *[37], 49. The Hardy spaces on the upper halfplane* $\mathcal{H} = \{x + iy \in \mathbb{C} : y > 0\}$ *are defined from the norms*

$$\|f\|_{H^p}^p = \sup_{y>0} \int |f(x+iy)|^p \, dx$$

*for* $0 < p < \infty$ *and the supremum norm*

$$\|f\|_{H^\infty} = \sup_{z\in\mathcal{H}} |f(z)|\,.$$

*These Hardy spaces consist of holomorphic functions having finite norms.*

$$H^p(\mathcal{H}) = \{f \in H(\mathcal{H}) : \|f\|_{H^p} < \infty\}.$$

**Definition 1.9.13.** *[42], 39. The Hardy spaces $H^p(d\theta)$ for $1 \le p \le \infty$ on the circle consist of functions $f \in L^p(d\theta)$ with vanishing negative Fourier coefficients; that is,*

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} f(\theta)\, d\theta = 0,$$

*for $n \in \mathbb{Z}, n > 0$.*

**Definition 1.9.14.** *The Hardy spaces $H^p(dt)$ for $1 \le p \le \infty$ on the real line consist of functions $f \in L^p(dt)$ such that*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{i-t}{i+t}\right)^n \frac{f(t)}{1+t^2}\, dt = 0,$$

*for $n \in \mathbb{Z}, n > 0$.*

**Proposition 1.9.15.** *[42], 38-39.*

*The Poisson integral*

$$f(re^{i\phi}) = \int_0^{2\pi} g(e^{i\theta}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, d\theta$$

*of a function $g \in H^p(d\theta)$ for $1 \le p \le \infty$ is in the corresponding Hardy space $H^p(\mathbb{D})$. On the other hand, any function $f \in H^p(\mathbb{D})$ has non-tangential limits almost everywhere on the unit circle, and these define a function $g \in H^p(d\theta)$ whose Poisson integral agrees with $f$ almost everywhere.*

**Proposition 1.9.16.** *[42], 128. (After a conformal transformation to carry the right half-plane to the upper halfplane since the original theorem is stated in terms of the right half-*

*plane.)*

*Let $f$ be a function in the Hardy space $H^p(\mathcal{H})$ where $1 \le p \le \infty$. Then its nontangential limits form a boundary function $g \in H^p(dt)$ and $f$ is the Poisson integral of $g$; that is,*

$$f(x+iy) = \frac{1}{\pi}\int_{t=-\infty}^{\infty} f(t)\frac{y\,dt}{(x-t)^2+y^2}.$$

**Proposition 1.9.17.** *To construct a bounded harmonic function from its boundary values on a more general domain $D$ having a Green's function $G(z,w)$ the harmonic measure may be used. Define the harmonic measure (with respect to $w \in D$) for a region whose boundary is the union of a finite number of analytic arcs as*

$$d\mu_w = -\frac{1}{2\pi}\frac{\partial g}{\partial n}\,ds$$

*where $g$ is the Green's function for $D$ having a pole at $w$. Then*

$$u(w) = \oint_{\partial D} u(z)\,d\mu_w(z)$$

*for any bounded harmonic function $u$ in $D$.*

**Remark 1.9.18.** *Note that in the case of $H^p$ spaces, that there is not a direct correspondence between $H^p$ of the unit disk and of the upper halfplane. This occurs because of convention: $H^p$ of the unit disc means $H^p(d\theta/2\pi)$, and $H^p$ of the upper halfplane means $H^p(dt)$, the harmonic functions they represent. Of course $d\theta/2\pi$ corresponds to $dt/(1+t^2)$ and $dt$ corresponds to $\frac{1}{2}\sec^2(\theta/2)\,d\theta$, since $x = \tan(\theta/2)$. One consequence of this is that $H^p(d\theta)$ contains non-zero constants, while $H^p(dt) \cong H^p(\sec^2(\theta/2)\,d\theta)$ does not.*

**Definition 1.9.19.** *[22], V.1.1. Let $\mathcal{X}$ be a Banach space. The* **weak$^*$ topology** *on the* **dual space** *$\mathcal{X}^* = \mathcal{B}(\mathcal{X},\mathbb{C})$ a Banach space $\mathcal{X}$ is the topology defined by the family of* *semi-norms*

$$p_x(\alpha) = |\alpha(x)| = |\langle x, \alpha \rangle|$$

*where the notation* $\langle x, \alpha \rangle$ *for* $x \in \mathcal{X}, \alpha \in \mathcal{X}^*$ *means* $\alpha(x)$.

*Similarly, the* **weak topology** *on* $\mathcal{X}$ *is the topology defined by the family of semi-norms*

$$p_\alpha(x) = |\langle x, \alpha \rangle| \, .$$

**Definition 1.9.20.** *[67], 4.6. Let* $\mathcal{X}$ *be a Banach space and* $K$ *be a subspace of* $\mathcal{X}$*. Then the* **annihilator** *of* $K$ *is*

$$K^\perp = \{\alpha \in \mathcal{X}^* : \langle k, \alpha \rangle = 0 \text{ for all } k \in K\} \, .$$

**Example 1.9.21.** *Let* $\mathcal{X}$ *be the Banach space* $L^1([0,1])$*, and* $K$ *be its subspace consisting of the odd functions (that is,* $f(-x) = -f(x)$*). Its annihilator is the functionals*

$$f \mapsto \int_0^1 fg \, dt$$

*where* $g \in L^\infty([0,1])$ *and* $g$ *is even.*

**Definition 1.9.22.** *Let* $\mathcal{X}$ *be a Banach space and let* $K$ *be a closed linear subspace of* $\mathcal{X}$*. Then the* **quotient norm** *on* $\mathcal{X}/K$ *is*

$$\|a + K\| = \inf \{\|a + k\| : k \in K\} \, .$$

*The* $Y^\perp$ **annihilator norm** *of an element* $x \in \mathcal{X}$ *is*

$$\|x\|_{Y^\perp} = \sup \left\{ |\langle x, \alpha \rangle| : \alpha \in Y^\perp, \|\alpha\| \leq 1 \right\} .$$

**Proposition 1.9.23.** *[37], I.4.1. Let* $\mathcal{X}$ *be a Banach space and let* $Y$ *be a closed subspace.*

*Then the dual space $Y^*$ is isometrically isomorphic to $\mathcal{X}/Y^\perp$. Also, the dual space $(\mathcal{X}/Y)^*$ is isometrically isomorphic to the annihilator $Y^\perp$.*

*In particular,*

$$\sup\{|\langle y,\alpha\rangle| : y\in Y, \|y\|\le 1\} = \inf\left\{\|\alpha-\beta\| : \beta\in Y^\perp\right\}$$

*since $\|\alpha\|_{Y^*} = \left\|\alpha + Y^\perp\right\|$, and*

$$\inf\{\|x-y\| : y\in Y\} = \sup\left\{|\langle x,\alpha\rangle| : \alpha\in Y^\perp, \|\alpha\|\le 1\right\}$$

*since $\|x+Y\| = \|x\|_{Y^\perp}$.*

## 1.10 Banach Algebras

**Definition 1.10.1.** *[47], 3.1.1. An algebra $\mathcal{A}$ is a* **normed algebra** *when $\mathcal{A}$ is a normed space such that $\|AB\|\le\|A\|\,\|B\|$ for all $A,B\in\mathcal{A}$. If it has a unit $I$, then $\|I\|=1$. If $\mathcal{A}$ is a Banach space relative to $\|\cdot\|$, it is a* **Banach algebra**. *(Here, every algebra is taken to be over $\mathbb{C}$.)*

When the algebra is a subalgebra of $B(\mathcal{H})$ for a Hilbert space it will also be equipped with its strong-operator and weak-operator topologies which are defined by pseudo-metrics of the form

$$p_x(T) = \|Tx\|$$

and

$$\omega_{x,y}(T) = |\langle Tx|y\rangle|\,,$$

respectively with $x,y\in\mathcal{H}$.

**Definition 1.10.2.** *[47], 3.2.1. The spectrum $sp(a)$ of an element $a$ of a unital Banach* *algebra $\mathcal{A}$ consists of those complex numbers $\lambda$ such that $a - \lambda 1$ is non-invertible. For a non-unital Banach algebra, take the spectrum of $a$ in its algebra's unitization.*

**Example 1.10.3.** *In $M_n(\mathbb{C})$, the spectrum of an element $a$ consists exactly of its eigenvalues. In $C(X)$, on the other hand, the spectrum of an element $f$ consists of its range $f(X)$.*

## 1.11 Module Theory

**Definition 1.11.1.** *[45], IV, 1.1. A* **left $R$-module** *over a ring $R$ is an additive abelian group $M$ together with a function $R \times M \to M$ (where $(r, a) \mapsto ra$) such that for all $a, b \in M$, $r, s \in R$*

1. $r(a + b) = ra + rb$
2. $(r + s)a = ra + sa$
3. $r(sa) = (rs)a$

*If $R$ has an identity $1_R$ and*

$$1_R a = a$$

*for all $a \in M$ then $M$ is called a* **unitary left $R$-module**. *If $R$ is a division ring, then a unital left $R$-module is called a* **left vector space**. *If $R$ is a field, it is simply called a* **vector space**.

*The properties for a* **right $R$-module** *are analogous.*

**Definition 1.11.2.** *[45], IV, 1.2 Let $A, B$ be left $R$-modules. A function $f : A \to B$ is an* **$R$-module homomorphism** *provided that $f(a + b) = f(a) + f(b)$ and $f(ra) = rf(a)$. If $R$ is a division ring, then $f$ is called a* **linear transformation**.

**Definition 1.11.3.** *[45], IV, 1.3 Let $A$ be a left $R$-module. If $B$ is a non-empty subset of $A$, then $B$ is a* **$R$-submodule** *of $A$ if $B$ is an additive subgroup of $A$ and $rb \in B$ for all $r \in R, b \in B$.*

*A submodule of a vector space is called a* **subspace***.*

**Example 1.11.4.** *The intersection of all the submodules of an R-module A containing the set X is a submodule of A known as the* **(sub)module generated by the set X***. X is a* **generating set** *for A if this submodule is A itself. A is* **finitely generated** *if it has a finite generating set.*

**Definition 1.11.5.** *[45], IV, 1.6 Let B be an R-submodule of an R-module A. Then the* **quotient module** *A/B is the quotient group equipped with the module structure*

$$r(a + B) = ra + B.$$

*The map $\pi : a \mapsto a + B$ is called the* **canonical epimorphism***.*

**Definition 1.11.6.** *[45], IV, 1.16. A sequence of module homomorphisms*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

*is* **exact at** *B if ker(g) = im(f).*

*A finite sequence of homomorphisms*

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$$

*is exact when $ker(f_{k+1}) = im(f_k)$ for each $0 \le k < n$.*

*Similarly, an infinite sequence of homomorphisms*

$$\cdots \xrightarrow{f_{k-1}} A_{k-1} \xrightarrow{f_k} A_k \xrightarrow{f_{k+1}} A_{k+1} \xrightarrow{f_{k+2}} \cdots$$

*is exact when $ker(f_{k+1}) = im(f_k)$ for each integer $k \in \mathbb{Z}$.*

*An exact sequence of the form*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*is called a* **short exact sequence***.*

**Definition 1.11.7.** *[52], III, §4. An R-module A is* **free** *if it admits a non-empty basis or is the zero module. A* **basis** *for a module is a linearly independent generating set.*

**Definition 1.11.8.** *[45], IV, 3.1. An R-module P is projective if given any diagram of R-module homomorphisms*

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \Big\downarrow f & & \\ A & \xrightarrow{\;g\;} & & \longrightarrow & 0 \end{array}$$

*there exists a homomorphism $h : P \to A$ making the diagram commute.*

**Proposition 1.11.9.** *[45], IV, 3.4 The following conditions on an R-module P are equivalent:*

- *P is projective;*
- *Every short exact sequence*

  $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$$

  *is split exact (that is, $B \cong A \oplus P$);*
- *there is a free module F and an R-module K such that $F \cong K \oplus P$ (that is, P is a direct summand of a free module).*

**Definition 1.11.10.** *[52], XXI, §2. An R-module A is* **stably free** *if there exists a finite free module B such that $A \oplus B$ is finite free (so is isomorphic to $R^n$ for some positive integer n). In particular, such an A is projective and finitely generated.*

# Chapter 2

# Part I: KMS States on Graph Algebras

## 2.1 Introduction: KMS States

This part of the thesis concerns KMS states of graph algebras equipped with the gauge action. These states satisfy a special condition on the strip

$$E_\beta = \{z \in \mathbb{C} | 0 \leq \mathrm{Im}(z) \leq \beta\}$$

in the complex plane. This condition is

$$\omega(B\alpha_t(A)) = f_{AB}(t)$$

$$\omega(\alpha_t(A)B) = f_{AB}(t + i\beta)$$

where $f_{AB}$ is an analytic function on the strip depending on $A, B \in \mathcal{A}$ and $\alpha_t$ is the gauge action of the graph algebra. These states currently find application in quantum statistical mechanics and number theory.

## 2.2 Convexity and Matrix Functions

**Definition 2.2.1.** *A function $f : D \to \mathbb{R}$ where $D$ is a convex subset of $\mathbb{R}$ is* **convex** *if given any $x, y$ in its domain*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

*whenever* $0 \leq t \leq 1$. *If this inequality is actually strict for all* $x \neq y$, *the function* $f$ *is* **strictly convex**.

*If* $f$ *is differentiable and satisfies the condition*

$$f(x) - f(y) \leq f'(x)(x-y) - \frac{\alpha}{2}|x-y|^2$$

*for some* $\alpha > 0$, *then* $f$ *is* **strongly convex**.

**Proposition 2.2.2.** *A differentiable function* $f : D \to \mathbb{R}$ *defined on a convex set* $D$ *is convex if and only if given any* $x, y$ *in its domain*

$$f(x) - f(y) \geq f'(y)(x-y).$$

*If this equality is strict for* $x \neq y$, $f$ *is also strictly convex.*

*Proof.* This is shown by observing that the defining condition can be rewritten as

$$\frac{f(tx + (1-t)y) - f(y)}{t} \leq f(x) - f(y).$$

Taking the limit this becomes

$$\lim_{t \to 0^+} \frac{f(tx + (1-t)y) - f(y)}{t} \leq f(x) - f(y)$$

$$\frac{d}{dt}\left(f(tx + (1-t)y)\right|_{t=0} \leq f(x) - f(y)$$

$$f'(y)(x-y) \leq f(x) - f(y)$$

∎

**Remark 2.2.3.** *Note that if $f$ is strongly convex then*

$$f(x) - f(y) \geq f'(y)(x-y) + \frac{\alpha}{2}|x-y|^2 > f'(y)(x-y)$$

*so $f$ is strictly convex.*

**Example 2.2.4.** *The function*

$$f : [0, \infty) \to [0, \infty)$$

$$f(t) = \begin{cases} t \ln t & t > 0 \\ 0 & t = 0 \end{cases}$$

*is strictly convex and, in fact, satisfies the condition*

$$f(x) - f(y) > f'(y)(x-y) \tag{2.1}$$

*whenever $x \neq y$.*

*Proof.* It suffices to show that for any $x, y \in \mathbb{R}$ where $x, y > 0$ and $x \neq y$ that

$$x \ln x - y \ln y > (1 + \ln y)(x - y);$$

in other words,

$$x \ln x - x + y - x \ln y > 0$$

$$\ln \frac{x}{y} + \frac{y}{x} > 1$$

$$\frac{y}{x} - \ln \frac{y}{x} > 1.$$

So I need to show that the function $g(p) = p - \ln p > 1$ for $p > 0$ such that $p \neq 1$. Now its derivative is

$$g'(p) = 1 - \frac{1}{p} = \frac{p-1}{p}$$

which is negative whenever $p < 1$ and positive whenever $p > 1$. Now $g(1) = 1$, so by the first derivative test, $g(p) > 1$ for all positive $p \neq 1$. ■

**Corollary 2.2.5.** *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be hermitian matrices, Tr the trace on $\mathcal{M}_n(\mathbb{C})$, and $f : D \to \mathbb{R}$ a convex differentiable function defined on a convex set $D \supseteq \sigma(A) \cup \sigma(B)$. Then*

$$Tr(f(A) - f(B) - f'(B)(A - B)) \geq 0$$

*If $f$ is strongly convex or otherwise satisfies the condition (2.1) then this expression equals zero only when $A = B$.*

*Proof.* Choose $a_j, \phi_j$ to be the eigenvalues and orthonormal eigenvectors of $A$, and $b_k, \psi_k$ to be those of $B$. These are real values since $A$ and $B$ are hermitian. Now

$$\mathrm{Tr}(f(A) - f(B) - f'(B)(A - B)) = \sum_{j=1}^{n} (\phi_j, f(A) - f(B) - f'(B)(A - B)\phi_j)$$

$$= \sum_{j=1}^{n} (\phi_j, f(a_j)I - f(B) - f'(B)(a_j I - B)\phi_j)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} (\phi_j, \psi_k)(\psi_l, \phi_j)(\psi_k, f(a_j)I - f(B) - f'(B)(a_j I - B)\psi_l)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} (\phi_j, \psi_k)(\psi_l, \phi_j)(\psi_k, (f(a_j) - f(b_l) - f'(b_l)(a_j - b_l))\psi_l).$$

By orthogonality, the trace $\mathrm{Tr}(f(A) - f(B) - f'(B)(A - B))$ is

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |(\phi_j, \psi_k)|^2 (f(a_j) - f(b_k) - f'(b_k)(a_j - b_k)).$$

Since $f$ is convex, the term in parentheses is non-negative, so the sum is non-negative. If $f$ is strongly convex or otherwise satisfies (2.1), then this sum equals zero only if $a_j = b_k$ for each $j, k$, which means that $A$ and $B$ are equal to some multiple of $I_n$, so $A = B$. ■

**Proposition 2.2.6.** *If $A$ and $B$ are positive matrices, then*

$$Tr(A \log A - A \log B) \geq Tr(A - B).$$

*Proof.* Take $f$ to be the strictly convex function $t \log t$. In that case,

$$\mathrm{Tr}(A \log A - B \log B - (1 + \log B)(A - B)) \geq 0$$

$$\mathrm{Tr}(A \log A - B \log B - A \log B + B \log B - (A - B)) \geq 0$$

$$\mathrm{Tr}(A \log A - A \log B) \geq \mathrm{Tr}(A - B)$$

■

## 2.3 $C^*$ Algebras

**Definition 2.3.1.** *A $C^*$ algebra is a Banach algebra with an involution $*$ which satisfies the following conditions:*

- $*$ *is conjugate linear,*
- $a^{**} = a$,
- $(ab)^* = b^* a^*$,
- $\|a^* a\| = \|a\|^2$.

**Example 2.3.2.** *The algebras $C(X)$ of continuous functions on a compact set $X$, $C_0(X)$ of continuous functions vanishing at infinity on a locally compact set, and the algebra $M_n(\mathbb{C})$ of complex $n$ by $n$ matrices provide examples of $C^*$ algebras. With the exception of $C_0$, these are unital, and with the exception of $M_n$, these are commutative.*

**Definition 2.3.3.** *An element $a$ in a $C^*$ algebra is called*

- **self-adjoint** *if* $a = a^*$*,*
- **positive** *if* $a = b^*b$ *for some* $b \in \mathcal{A}$*,*
- **unitary** *if* $a^*a = aa^* = 1$*,*
- **a projection** *if* $a = a^2 = a^*$*.*

*The sets of self-adjoints, unitaries, and projections in a* $C^*$*-algebra* $\mathcal{A}$ *are denoted Re* $\mathcal{A}$*,* $\mathcal{U}(\mathcal{A})$*, and* $\mathcal{P}(\mathcal{A})$*, respectively.*

**Definition 2.3.4.** *A linear transformation between* $C^*$ *algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *is*

- **hermitian** *if it carries self-adjoints to self-adjoints,*
- **positive** *if it carries positive elements to positive elements.*

**Example 2.3.5.** *Properties of an element* $a$ *often restrict its spectrum* $sp(a)$*:*

- *for a self-adjoint* $a$*,* $sp(a)$ *is a subset of the real line* $\mathbb{R}$*,*
- *for a positive* $a$*,* $sp(a)$ *is a subset of the positive reals* $\mathbb{R}^+$*,*
- *for a unitary* $u$*,* $sp(u)$ *is a subset of the unit circle* $S^1$*,*
- *for a projection* $p$*,* $sp(p)$ *is a subset of the two element set* $\{0, 1\}$*. The spectrum consists of both elements unless* $p = 0$ *or* $p = 1$*.*

**Definition 2.3.6.** *[66], 1.1.6, ex. 1.3 in 1.4.*

*The* **unitization** $\mathfrak{A}$ *of a* $C^*$*-algebra* $\mathcal{A}$ *is defined as follows:*

$$\mathfrak{A} = \{a + \alpha 1_{\mathfrak{A}} : a \in \mathcal{A}, \alpha \in \mathbb{C}\}.$$

*This underlying set (*$\mathcal{A} \oplus \mathbb{C}$*) is equipped with the multiplication*

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta),$$

*the involution*

$$(a, \alpha)^* = (a^*, \overline{\alpha}).$$

*Define the maps* $\iota : \mathcal{A} \to \mathfrak{A}$ *as* $a \mapsto (a, 0)$ *and* $\pi : \mathfrak{A} \to \mathbb{C}$ *as* $(a, \alpha) \mapsto \alpha$*; we now get the short exact sequence* $0 \longrightarrow \mathcal{A} \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$

*The unique* $C^*$ *norm for* $\mathfrak{A}$ *is*

$$\|x\|_{\mathfrak{A}} = \max\left\{|\pi(x)|, \sup\left\{\|ax\|_{\mathcal{A}} : a \in \mathcal{A}, \|a\|_{\mathcal{A}} \leq 1\right\}\right\}.$$

*Given a morphism of* $C^*$ *algebras* $\phi : \mathcal{A} \to \mathcal{B}$*, the corresponding morphism* $\overline{\phi} : \mathfrak{A} \to \mathfrak{B}$ *is*

$$\phi(a + \alpha 1_{\mathfrak{A}}) = \phi(a) + \alpha 1_{\mathfrak{B}};$$

*it has to take this form because the following diagram must commute.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} & & \| & & \\ 0 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathfrak{B} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \end{array}$$

**Definition 2.3.7.** *A state* $\rho$ *on a* $C^*$*-algebra is a positive linear functional having its norm* $\|\rho\| = 1$*. An extreme point of the set of states is called a* **pure state***.*

**Example 2.3.8.** *On the* $C_{\mathbb{C}}[0, 1]$*, any regular probability measure* $\mu$ *on* $[0, 1]$ *generates a state* $\rho_\mu$ *where*

$$\rho_\mu(f) = \int_{[0,1]} f(t)\, \mu(dt).$$

*The extreme points of the space of probability measures consists of those supported by a single point. This means that the pure states can be constructed as* $\rho(f) = f(x_0)$ *where* $x_0$ *is a point in the interval* $[0, 1]$*. (Of course, topological probability measures on metrizable compact spaces are regular.) Note that these states are multiplicative, which holds in general for commutative* $C^*$ *algebras.*

**Example 2.3.9.** *Consider the algebra $\mathcal{M}_n(\mathbb{C})$ of complex $n \times n$ matrices. This is a finite-dimensional Hilbert space when equipped with the inner product $\langle A, B\rangle = Tr(AB^*)$. A basis consisting of the* **standard matrix units** *is defined as*

$$[E_{jk}]_{lm} = [\delta_{jl}\delta_{km}].$$

*The functionals $A \mapsto \langle A, E_{jk}\rangle$ are a basis for $\mathcal{M}_n(\mathbb{C})^*$, so an arbitrary functional takes the form*

$$A \mapsto \sum_{jk} q_{kj} \langle A, E_{jk}\rangle$$

*for some $q_{jk} \in \mathbb{C}$ for $1 \leq j, k \leq n$. This is*

$$A \mapsto \left\langle A, \sum_{jk} \overline{q}_{kj} E_{jk} \right\rangle$$

$$A \mapsto \langle A, Q^*\rangle = Tr(QA)$$

*for the matrix $Q = [q_{jk}]$.*

*A state $\rho$ takes the form $\rho_Q(E) = Tr(QE)$ where $Q$ is a self-adjoint, positive (definite), matrix having a trace of 1. (See (2.3.10).) For pure states, $Q$ is a (minimal) projection.*

*Unlike in the previous example, these pure states are not multiplicative. For instance, take*

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

*Then $\rho_Q(A^2) = 1$, but $\rho_Q(A) = 0$ and $\rho_Q(A^2) \neq \rho_Q(A)^2$.*

**Remark 2.3.10.** *In a matrix algebra $\mathcal{M}_n(\mathbb{C})$ there is a one-to-one correspondence between states $\phi$ and density matrices $Q$; that is, positive self-adjoint matrices having a trace of 1.*

*Proof.* The dual space of $\mathcal{M}_n(\mathbb{C})$ exactly consists of the functionals $A \mapsto \mathrm{Tr}(AQ)$ where $Q \in \mathcal{M}_n(\mathbb{C})$ (see 2.3.9). A functional $\phi$ is hermitian if and only if its matrix $Q_\phi$ is self-adjoint. Consider

$$\phi(E^*) = \mathrm{Tr}(E^* Q_\phi) = \overline{\mathrm{Tr}(EQ_\phi)} = \overline{\phi(E)}$$

to see that $\phi$ is hermitian when $Q_\phi$ is self-adjoint and the argument reverses to show the converse.

Next, $\phi$ is positive exactly when $Q_\phi$ is positive. If $Q_\phi$ is positive, then $Q = TT^*$ for some matrix $T \in \mathcal{M}_n(\mathbb{C})$, so that for a positive element $E^*E$

$$\phi(E^*E) = \mathrm{Tr}(E^*ETT^*) = \mathrm{Tr}(T^*E^*ET) = \mathrm{Tr}((TE)^*TE) \geq 0.$$

On the other hand, if $\phi$ is positive, then as it is hermitian $Q_\phi$ is self-adjoint. Consequently, $Q_\phi$ is orthogonally diagonalizable as $U\Lambda U^*$. Then as the projections $UE_{jj}U^*$ are positive (where $E_{jk}$ is the standard system of matrix units),

$$\phi(UE_{jj}U^*) = \mathrm{Tr}(U\Lambda U^*UE_{jj}U^*) = \mathrm{Tr}(U\Lambda E_{jj}U^*) = \mathrm{Tr}(\Lambda E_{jj}) = \lambda_j \geq 0,$$

which shows that $Q_\phi$ is positive.

Finally as

$$\phi(I) = \mathrm{Tr}(Q_\phi)$$

$\phi$ is a state exactly when $Q_\phi$ is a positive matrix whose trace is 1; that is, a density matrix. ■

## 2.4 von Neumann algebras

**Definition 2.4.1.** *Let $\mathcal{E} \subseteq B(\mathcal{H})$. Then the commutant $\mathcal{E}'$ is*

$$\mathcal{E}' = \{A \in B(\mathcal{H}) | AE = EA \text{ for all } E \in \mathcal{E}\}.$$

**Definition 2.4.2.** *A Von Neumann algebra $\mathcal{R}$ is a $C^*$ subalgebra of $B(\mathcal{H})$ which is weak-*

*operator closed. Equivalently,* $\mathcal{R} = \mathcal{R}''$*. The* **central carrier** *of an element in a it is the smallest projection p in it which commutes with each element in* $\mathcal{R}$ *and satisfies* $pa = a$*.*

**Example 2.4.3.** *The algebra of essentially bounded functions of a measure space* $L^\infty(X, \mu)$ *equipped with its pointwise product and its supremum norm is a von Neumann algebra.*

**Definition 2.4.4.** *[48], 7.1.11. A state* $\omega$ *in a von Neumann algebra* $\mathcal{R}$ *is* **normal** *when* $\omega(H_i) \to \omega(H)$ *whenever* $H_a$ *is an monotone increasing net of operators with least upper bound* $H$*.*

**Proposition 2.4.5.** *[48], 7.3.5. If* $\mathcal{A}$ *is a self-adjoint algebra of operators (containing the identity* $I$*) on the Hilbert space* $\mathcal{H}$ *and* $\omega$ *is a positive linear functional* $\omega \leq \omega_x|_{\mathcal{A}}$ *for some vector* $x \in \mathcal{H}$*, then there is a positive operator* $H'$ *in the unit ball of* $\mathcal{A}'$ *such that*

$$\omega(A) = \omega_x(H'A)$$

*for all* $A \in \mathcal{A}$*.*

**Proposition 2.4.6.** *[10], 5.3.29. Let* $\mathcal{M}$ *be a von Neumann algebra with faithful normal state* $\omega$*,* $\sigma_t$ *its corresponding modular group and* $\phi$ *be another normal state on* $\mathcal{M}$*. Then the following are equivalent:*

- $\phi$ *is a* $\sigma$*-KMS state, and*
- *There exists a positive (possibly unbounded) operator* $T$ *affiliated with* $\mathcal{M} \cap \mathcal{M}'$ *such that*

  $$\phi(A) = \omega(T^{1/2} A T^{1/2})$$

  *for all* $A \in \mathcal{M}$*.*

## 2.5 The GNS Construction

**Lemma 2.5.1.** *[47], 2.1.1. Let $\mathcal{X}$ be a complex vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. The subset of $\mathcal{X}$*

$$\mathcal{L} = \{z \in \mathcal{X} : \langle z, z \rangle = 0\}$$

*is a linear subspace. Furthermore,*

$$\langle x + \mathcal{L}, y + \mathcal{L} \rangle_0 = \langle x, y \rangle$$

*defines a definite inner product on $\mathcal{X}/\mathcal{L}$.*

*Proof.* Let

$$\mathcal{L}_1 = \{x \in \mathcal{X} | \langle x, y \rangle = 0 \text{ for each } y \in \mathcal{X}\}$$

which is a subspace of $\mathcal{X}$ containing $\mathcal{L}$. But they are equal because

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle} = 0$$

whenever $x \in \mathcal{L}$. Now the quotient inner product is well-defined since given $w, v \in \mathcal{L}$,

$$\langle x + w, y + v \rangle = \langle x, x \rangle + \langle x, v \rangle + \langle w, y \rangle + \langle w, v \rangle = \langle x, y \rangle .$$

This inner product is definite since if

$$\langle x + \mathcal{L}, x + \mathcal{L} \rangle_0 = \langle x, x \rangle = 0,$$

then $x \in \mathcal{L}$, and $x = 0_{\mathcal{X}/\mathcal{L}}$. ∎

**Proposition 2.5.2.** *[47], 4.5.1. If $\rho$ is a state of a $C^*$-algebra $\mathcal{A}$, the* **left kernel**

$$\mathcal{L}_\rho = \{A \in \mathcal{A} | \rho(A^*A) = 0\},$$

*is a closed left ideal in $\mathcal{A}$, and $\rho(B^*A) = 0$ whenever $A \in \mathcal{L}_\rho$ and $B \in \mathcal{A}$. The equation*

$$\langle A + \mathcal{L}_\rho, B + \mathcal{L}_\rho \rangle = \rho(B^*A)$$

*defines a definite inner product on the the quotient algebra $\mathcal{A}/\mathcal{L}$ (not necessarily complete).*

*Proof.* Define

$$\langle A, B \rangle_0 = \rho(B^*A);$$

then this pairing is linear in the first variable (composition of $X \mapsto XA$ and $\rho$) and conjugate-linear in the second variable because $\rho$ being positive, is immediately hermitian. (Note that an arbitrary bounded self-adjoint element $A$ takes the form $E - F$, for $E, F$ positive; these elements may be constructed with the use of the continuous functional calculus. The element $A$ is normal since it is self-adjoint, and the continuous functions $t \mapsto \max(t, 0)$ and $t \mapsto \max(0, -t)$ now generate the desired $E$ and $F$. Then $\rho(E - F) = \rho(E) - \rho(F)$ which is clearly real.) Next note that $\langle A, B \rangle_0 = \overline{\langle B, A \rangle_0}$ for the same reason: $\rho(B^*A) = \overline{\rho(A^*B)}$. Also, as $\rho$ is positive, $\langle a, a \rangle_0 \geq 0$. This shows that $\langle \cdot, \cdot \rangle_0$ is an inner product.

Next construct the left kernel of $\rho$:

$$\mathcal{L}_\rho = \{A \in \mathcal{A} | \langle A, A \rangle_0 = 0\}.$$

It follows from 2.5.1 that $\mathcal{L}_\rho$ is a subspace of $\mathcal{A}$ and

$$\langle A + \mathcal{L}_\rho, B + \mathcal{L}_\rho \rangle = \langle A, B \rangle_0 = \rho(B^*A)$$

defines a definite inner product. Then by Cauchy-Schwarz (1.9.5),

$$|\langle A, B\rangle_0|^2 \leq \langle A, A\rangle_0 \langle B, B\rangle_0$$

$$|\rho(B^*A)^2| \leq \rho(A^*A)\rho(B^*B) = 0$$

whenever $A \in \mathcal{L}_\rho$ and $B \in \mathcal{A}$.

Now to show that $\mathcal{L}_\rho$ is a left ideal, let $A, B$ be arbitrary elements of $\mathcal{L}_\rho$, $\mathcal{A}$, respectively. Then

$$|\langle BA, BA\rangle_0| = |\langle A, B^*BA\rangle_0| \leq \langle A, A\rangle_0 \, \langle B^*BA, B^*BA\rangle_0 = 0$$

so $\rho((BA)^*BA) = 0$. Finally, $\mathcal{L}_\rho$ is closed since $\rho$ is continuous. ∎

**Definition 2.5.3.** *[47], 275-276. A* **representation** *of a $C^*$ algebra $\mathcal{A}$ is a *-homomorphism $\rho$ into the bounded operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. A vector $x_\rho$ is* **cyclic for** *$\rho$ if*

$$\pi_\rho(\mathcal{A})x_\rho = \{\pi(A)x_\rho : A \in \mathcal{A}\}$$

*is dense in $\mathcal{H}$. The representation is* **faithful** *if $\pi(A) > 0$ for any positive $A$ in $\mathcal{A}$.*

**Proposition 2.5.4.** *Gelfand-Neumark-Segal Construction: [47], 4.5.2. Suppose that $\rho$ is a state of a unital $C^*$ algebra $\mathcal{A}$. Then there is a cyclic representation $\pi_\rho$ on a Hilbert space $\mathcal{H}_\rho$, and a unit cyclic vector $x_\rho$ for $\pi_\rho$, such that $\rho = \omega_{x_\rho} \circ \pi_\rho$; that is,*

$$\rho(A) = \langle \pi_\rho(A)x_\rho, x_\rho\rangle$$

*for all $A \in \mathcal{A}$.*

*Proof.* Take $\mathcal{H}_\rho$ to be the completion of the pre-Hilbert quotient space $\mathcal{A}/\mathcal{L}_\rho$ in the inner product

$$\langle A + \mathcal{L}_\rho, B + \mathcal{L}_\rho\rangle = \rho(B^*A).$$

To show that

$$\pi_\rho(A)(B + \mathcal{L}_\rho) = AB + \mathcal{L}_\rho$$

is well-defined, assume that $B - B' \in \mathcal{L}_\rho$. Then

$$\pi_\rho(A)(B + \mathcal{L}_\rho) = AB + \mathcal{L}_\rho = AB' + A(B - B') + \mathcal{L}_\rho = AB' + \mathcal{L}_\rho = \pi_\rho(A)(B' + \mathcal{L}_\rho).$$

Using the fact that $\|A^*A\|\, I - A^*A$ is positive, and that $B^*\left(\|A^*A\|\, I - A^*A\right) B$ is also, consider

$$\|A\|^2 \, \|B + \mathcal{L}_\rho\|^2 - \|\pi(A)(B + \mathcal{L}_\rho)\|^2$$

$$\|A\|^2 \, \|B + \mathcal{L}_\rho\|^2 - \|AB + \mathcal{L}_\rho\|^2$$

$$\|A\|^2 \, \langle B + \mathcal{L}_\rho, B + \mathcal{L}_\rho\rangle - \langle AB + \mathcal{L}_\rho, AB + \mathcal{L}_\rho\rangle$$

$$\|A\|^2 \rho(B^*B) - \rho(B^*A^*AB)$$

$$\rho(B^*(\|A\|^2 I - A^*A)B) \geq 0.$$

This shows that $\|\pi(A)\| \leq \|A\|$, so $\pi(A)$ is bounded. It then extends to a bounded operator (also denoted as) $\pi_\rho(A)$ on $\mathcal{H}_\rho$.

As $\pi(I)$ is the identity on $\mathcal{A}/\mathcal{L}_\rho$, $\pi_\rho(I)$ is the identity on $\mathcal{H}_\rho$. Furthermore for $A, B, C \in \mathcal{A}$ and $a, b \in \mathbb{C}$, the linearity and associativity of multiplication in $\mathcal{A}$ makes $\pi_\rho$ linear and multiplicative, so it suffices to check that $\pi_\rho(A^*) = \pi_\rho(A)^*$:

$$\langle \pi_\rho(A)(B + \mathcal{L}_\rho), C + \mathcal{L}_\rho\rangle = \langle AB + \mathcal{L}_\rho, C + \mathcal{L}_\rho\rangle = \rho(C^*AB) =$$

$$= \rho((A^*C)^*B) = \langle B + \mathcal{L}_\rho, A^*C + \mathcal{L}_\rho\rangle = \langle B + \mathcal{L}_\rho, \pi_\rho(A^*)(C + \mathcal{L}_\rho)\rangle$$

from which $\pi_\rho(A^*) = \pi_\rho(A)^*$. So $\pi_\rho$ is a *-representation of $\mathcal{A}$ on $\mathcal{H}_\rho$ (since it is on the quotient and extends continuously by density of the quotient).

Now let $x_\rho$ be the vector $I + \mathcal{L}_\rho$,

$$\pi_\rho(A)x_\rho = \pi_\rho(A)(I + \mathcal{L}_\rho) = A + \mathcal{L}_\rho.$$

This means that $\pi_\rho(\mathcal{A})x_\rho$ is $\mathcal{A}/\mathcal{L}_\rho$ which is dense in $\mathcal{H}_\rho$, so $x_\rho$ is a cyclic vector for $\pi_\rho$. Now

$$\langle \pi_\rho(A)x_\rho, x_\rho \rangle = \langle A + \mathcal{L}_\rho, I + \mathcal{L}_\rho \rangle = \rho(A).$$

In particular, $\|x_\rho\|^2 = \rho(I) = 1$. ∎

**Proposition 2.5.5.** *[47], 4.5.3. Suppose that $\rho$ is a state of a $C^*$ algebra $\mathcal{A}$ and $\pi$ is a cyclic representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ such that $\rho = \omega_x \circ \pi$ for some unit cyclic vector $x$ for $\pi$. If $\mathcal{H}_\rho$, $\pi_\rho$, and $x_\rho$ are the Hilbert space, cyclic representation, and unit cyclic vector produced from the GNS construction, there is an isomorphism $U$ from $\mathcal{H}_\rho$ onto $\mathcal{H}$ such that $x = Ux_\rho$ and $\pi(A) = U\pi_\rho(A)U^*$ for each $A \in \mathcal{A}$. That is, $\pi$ and $\pi_\rho$ are* ***unitarily equivalent****.*

*Proof.* For each $A \in \mathcal{A}$,

$$\|\pi(A)x\|^2 = \langle \pi(A)x, \pi(A)x \rangle = \langle \pi(A^*A)x, x \rangle = \rho(A^*A)$$

$$\rho(A^*A) = \langle \pi_\rho(A^*A)x_\rho, x_\rho \rangle = \langle \pi_\rho(A)x_\rho, \pi_\rho(A)x_\rho \rangle = \|\pi_\rho(A)x_\rho\|^2;$$

consequently $\|\pi(A)x\| = \|\pi_\rho(A)x_\rho\|$. Let $A, B \in \mathcal{A}$ such that $\pi_\rho(A)x_\rho = \pi_\rho(B)(x_\rho)$; this means that $\pi_\rho(A - B)x_\rho = 0$ so that $\pi(A - B)x_\rho = 0$; that is $\pi(A)x = \pi(B)x$. This means that

$$U_0\left(\pi_\rho(A)x_\rho\right) = \pi(A)x$$

defines a norm-preserving linear operator from $\pi_\rho(\mathcal{A})x_\rho$ onto $\pi(\mathcal{A})x$. Since these are dense in $\mathcal{H}_\rho$ and $\mathcal{H}$ and $U_0$ is (uniformly) continuous, $U_0$ extends to an isomorphism $U : \mathcal{H}_\rho \to \mathcal{H}$

where

$$Ux_\rho = U_0\pi_\rho(I)x_\rho = \pi(I)x = x.$$

For $A, B \in \mathcal{A}$,

$$U\pi_\rho(A)\pi_\rho(B)x_\rho = U\pi_\rho(AB) = \pi(AB)x = \pi(A)\pi(B)x = \pi(A)U\pi_\rho(B)x_\rho;$$

this means that $U\pi_\rho(A) = \pi(A)U$ on a dense subset. Therefore, $U\pi_\rho(A) = \pi(A)U$ and $\pi(A) = U\pi_\rho(A)U^*$. ■

## 2.6 Hilbert $C^*$-modules

**Definition 2.6.1.** *[54], 1.2.1. A* **pre-Hilbert $A$-module** *is a right $A$-module $M$ equipped with a sesquilinear form $\langle\cdot,\cdot\rangle : M \times M \to A$ with the following properties:*

1. *$\langle x, x\rangle \geq 0$ for any $x \in M$;*
2. *$\langle x, x\rangle = 0$ implies that $x = 0$;*
3. *$\langle y, x\rangle = \langle x, y\rangle^*$ for any $x, y \in M$;*
4. *$\langle x, ya\rangle = \langle x, y\rangle a$ for any $x, y \in M$ and $a \in A$.*

*The map $\langle\cdot,\cdot\rangle$ is called an* **$A$-valued inner product***. Note that unlike an inner product, it is conjugate linear in the first variable, and linear in the second.*

*The associate map $\|x\| = \sqrt{\langle x, x\rangle}$ is called the* **$A$-valued norm***. It satisfies the following properties:*

$$\|x\| = 0 \text{ if and only if } x = 0, \tag{2.2}$$

$$\|xy\| \leq \|x\|\,\|y\| \text{ for } x \in A, y \in M. \tag{2.3}$$

A pre-Hilbert $A$-module which is complete relative to this $A$-valued norm is called a

Hilbert $A$-module.

**Definition 2.6.2.** *[54], 2.1. An operator $T : \mathcal{E} \to \mathcal{F}$ between Hilbert $A$-modules $\mathcal{E}, \mathcal{F}$ is* **adjointable** *if there exists an operator $T^* : \mathcal{F} \to \mathcal{E}$ such that*

$$\langle T^* x, y \rangle_{\mathcal{E}} = \langle x, Ty \rangle_{\mathcal{F}} .$$

*The set of (necessarily bounded) adjointable operators from $\mathcal{E}, \mathcal{F}$ is denoted $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Note that $\mathcal{L}(\mathcal{E}, \mathcal{E})$ is denoted as $\mathcal{L}(\mathcal{E})$.*

*Proof.* Following [54], it needs to shown that $T$ is bounded. By the closed graph theorem, it suffices to show that $T$ has a closed graph. Fix some arbitrary $x \in \mathcal{E}$, and let $y = Tx$. Take a net $x_\alpha$ such that $x_\alpha \to x$ and $Tx_\alpha \to Tx$. Now

$$0 = \langle T^*(y - Tx), x_\alpha \rangle - \langle T^*(y - Tx_\alpha), x_\alpha \rangle = \langle y - Tx, Tx_\alpha \rangle - \langle T^*(y - Tx), Tx \rangle ,$$

taking the limits of both sides as nets gives

$$0 = \langle y - Tx, y \rangle - \langle T^*(y - Tx), x \rangle = \langle y - Tx, y - Tx \rangle ,$$

so that $y = Tx$. So the graph is closed. ∎

**Definition 2.6.3.** *[54], 2.2. The* **compact operators** *from a Hilbert $A$-module $\mathcal{E}$ to another Hilbert $A$-module $\mathcal{F}$ is the closed linear span of the* **elementary operators** *$\theta_{x,y}$ which are defined as follows:*

$$\theta_{x,y}(z) = x \langle y, z \rangle_{\mathcal{E}}$$

*where $x \in \mathcal{F}$ and $y, z \in \mathcal{E}$. They are denoted as $\mathcal{K}(\mathcal{E}, \mathcal{F})$. Those from the module to itself are denoted $\mathcal{K}(\mathcal{E})$.*

A special case of interest concerns a Hilbert $C(X)$-module where $X$ is a finite set. In this case, the module can be interpreted as having an inner product $\langle \cdot, \cdot \rangle_x$ for each point $x \in X$, and the condition for an operator from $\mathcal{E}$ to itself to be adjointable now says that there must exist a bounded $T^*$ such that

$$\langle T^* a, b \rangle_x = \langle a, Tb \rangle_x$$

for each $x \in X$, $a, b \in \mathcal{E}$.

## 2.7 $C^*$ Dynamical Systems and KMS States

**Definition 2.7.1.** *[9], 136. A $C^*$ dynamical system is a triple $(\mathcal{A}, G, \alpha)$ where $\mathcal{A}$ is a $C^*$-algebra, $G$ is a locally compact group, and $\alpha$ is a strongly continuous representation of $G$ in the automorphism group of $\mathcal{A}$. This is, $\alpha : G \to Aut(\mathcal{A})$ is a group homomorphism and*

$$g \mapsto \alpha_g(A)$$

*is norm continuous for each $A \in \mathcal{A}$.*

**Definition 2.7.2.** *An element $A$ in a $C^*$-dynamical system $(\mathcal{A}, \mathbb{R}, \alpha)$ is* **entire analytic** *if there exists a holomorphic function $f$ from the complex plane $\mathbb{C}$ to $\mathcal{A}$ such that*

$$\alpha_t(A) = f(t).$$

*This may be denoted as $f(z) = \alpha_z(A)$ for $z \in I_\lambda$.*

**Proposition 2.7.3.** *Adapted from [9], 100. The entire analytic elements of a $C^*$ system $(\mathcal{A}, \mathbb{R}, \alpha_t)$ form an $\alpha_t$ invariant norm-dense *-subalgebra of $\mathcal{A}$.*

*Proof.* First of all, $\alpha_t(A)$ is an analytic element whenever $A$ is; let $z \mapsto f(z)$ be the holomorphic function corresponding to $A$; then $z \mapsto f(z+t)$ corresponds to $\alpha_t(A)$. (If $f(z)$ was entire then certainly $f(z+t)$ is as well.)

It is clear that entire analytic elements form a subalgebra since if $A$ and $B$ are analytic elements with $f, g : \mathbb{C} \to \mathcal{A}$ satisfying

$$f(t) = \alpha_t(A)$$

and

$$g(t) = \alpha_t(B),$$

then

$$(f + g)(t) = \alpha_t(A) + \alpha_t(B) = \alpha_t(A + B),$$

$$(fg)(t) = \alpha_t(A)\alpha_t(B) = \alpha_t(AB).$$

Since $f + g$ and $fg$ are entire analytic functions, $A + B$ and $AB$ are entire analytic elements. Define an analytic function $h(z) = f(\overline{z})^*$ by reflection, then this corresponds to the element $A^*$ since for $t \in \mathbb{R}$

$$h(t) = f(\overline{t})^* = \alpha_t(A^*),$$

so entire analytic elements form a *-subalgebra.

Now, in close analogy to ( [9], 101), given an arbitrary $A$, construct $A_n$ as

$$A_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \alpha_t(A) e^{-nt^2} \, dt.$$

These then correspond to the entire analytic functions

$$f_n(z) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \alpha_t(A) e^{-n(z-t)^2} \, dt$$

which are defined since $t \mapsto e^{-n(t-z)^2}$ is integrable. It is analytic since given and $t \in \mathbb{R}$, the function $z \mapsto \alpha_t(A) e^{-n(t-z)^2}$ is analytic.

Now

$$\|A_n\| \leq \sup_t \{\|\alpha_t(A)\|\} \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2}\, dt = \|A\|$$

and

$$\|A_n - A\| = \left\| \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \left(\alpha_t(A) - A\right)\, dt \right\|.$$

In order to carry out an approximate identity argument fix $\epsilon > 0$ and select $\delta > 0$ so that $\|\alpha_t(A)) - A\| < \epsilon$ whenever $|t| < \delta$. Choose $N$ so that for $n \geq N$

$$\sqrt{\frac{t}{\pi}} \int_{|t| \geq \delta} e^{-Nt^2}\, dt < \epsilon.$$

This then means that for $n \geq N$,

$$\|A_n - A\| \leq \sqrt{\frac{n}{\pi}} \int_{|t| \leq \delta} e^{-nt^2}\, \|\alpha_t(A) - A\|\, dt + \sqrt{\frac{n}{\pi}} \int_{|t| \geq \delta} e^{-nt^2}\, \|\alpha_t(A) - A\|\, dt$$

This then is

$$\leq \epsilon \sqrt{\frac{n}{\pi}} \int_{|t| \leq \delta} e^{-nt^2}\, dt + 2\, \|A\|\, \epsilon < (1 + 2\, \|A\|)\epsilon$$

showing that $A_n \to A$ in the norm topology. ■

**Proposition 2.7.4.** *Let $(\mathcal{A}, \alpha_t)$ be a $C^*$-dynamical system and $f : \mathbb{R} \to \mathbb{R}$ an integrable function which extends to a holomorphic function on the complex plane $\mathbb{C}$. The element $A \in \mathcal{A}$*

$$\tau_f(A) = \int_{t=-\infty}^{\infty} f(t)\, \alpha_t(A)\, dt$$

*defines an entire analytic element.*

**Definition 2.7.5.** *[21], 3.6, 3.7. [10], 5.3.7.*

*A state $\phi$ of a $C^*$ system $(\mathcal{A}, \mathbb{R}, \alpha_t)$ at inverse temperature $0 < \beta < \infty$ is called a* **KMS state** *if for each $A, B \in \mathcal{A}$ there exists a bounded complex function $f_{AB}$ analytic in the strip*

*$\{z|0 < Im\ z < \beta\}$ and continuous on its closure satisfying*

$$f_{AB}(t + i\beta) = \phi(\alpha_t(A)B) \tag{2.4}$$

$$f_{AB}(t) = \phi(B\alpha_t(A)), \tag{2.5}$$

*for each $t \in \mathbb{R}$. Similarly, a KMS state for $-\infty < \beta < 0$ is analytic and continuous on the closed strip $\{z|\beta \leq Im\ (z) \leq 0\}$.*

*The special cases $\beta = 0, \pm\infty$ have their own definitions:*

- *[60],8.12.2. $\phi$ is a KMS state for $\beta = 0$ if it is an $\alpha$-invariant trace (also known as a* **chaotic state***)*
- *$\phi$ is a* **ground state** *if for each pair $A, B \in \mathcal{A}$ there exists a complex function $g_{AB}$ bounded and analytic on the closed upper half-plane satisfying*

$$g_{AB}(t) = \phi(B\alpha_t(A)). \tag{2.6}$$

- *$\phi$ is, analogously, a* **ceiling state** *if for each pair $A, B \in \mathcal{A}$, there exists a complex function $g_{AB}$ bounded and analytic on the closed lower half-plane satisfying*

$$g_{AB}(t) = \phi(B\alpha_t(A)). \tag{2.7}$$

- *$\phi$ is a KMS state for $\beta = \infty$ if it is a weak* limit of KMS states as $\beta \to \infty$. (Note that each of these states is a ground state, but not every ground state is an $\infty$-KMS state.)*
- *$\phi$ is a KMS state for $\beta = -\infty$ if it is a weak* limit of KMS states as $\beta \to -\infty$. (Like previously, every $-\infty$-KMS state is a ceiling state, but not every ceiling state is a $-\infty$-KMS state.)*

**Remark 2.7.6.** *Note a $-\beta$-KMS state $\phi_{-\beta}$ is a $\beta$-KMS state for the "time-reversed" dy-*

*namical system $(\mathcal{A}, \mathcal{R}, \alpha_{-t})$ by taking $F'_{A,B}(z) = F_{A,B}(-z)$. Similarly, a ceiling state is a ground state for the time-reversed system and vice versa. Consequently, it suffices to show most results for ground states, invariant traces, and $\beta$-KMS states for $0 < \beta < \infty$.*

**Remark 2.7.7.** *The bounded holomorphic functions $g_{AB}$ for a ground state $\phi$ are in fact bounded above by $\|A\| \, \|B\|$. This is because a bounded holomorphic function can be reconstructed from its boundary values by the use of the Poisson kernel (1.9.16). In fact,*

$$|g_{AB}(t)| = |\phi(B\alpha_t(A))| \leq \|B\alpha_t(A)\| \leq \|A\| \, \|B\| \, .$$

*Similarly, the bounded holomorphic functions $g_{AB}$ for a ceiling state $\phi$ are bounded above by $\|A\| \, \|B\|$ since one can simply consider $g_{AB}(-z)$ and use the same result.*

*Similarly (see 1.9.17), the harmonic measure on a horizontal strip $S$ may be used to construct bounded harmonic functions from their boundary values as*

$$u(w) = \oint_{\partial S} u(z) \, d\mu_w(z);$$

*for the horizontal strip $S = \{x + iy : 0 \leq y \leq \pi\}$, this harmonic measure on the lower line $y = 0$ is*

$$d\mu_w = \frac{e^{x+t} \, \sin s \, dt}{e^{2x} - 2e^{x+t} \cos s + e^{2t}}$$

*and the harmonic measure on the upper line $y = \pi$ is*

$$d\mu_w = \frac{e^{x+t} \, \sin s \, dt}{e^{2x} + 2e^{x+t} \cos s + e^{2t}}.$$

*where $w = x + is$. In particular, a bounded harmonic function on the horizontal strip is bounded by its boundary values.*

**Proposition 2.7.8.** *A state $\phi$ is $\beta$-KMS if and only if the following equation holds for all*

*$A, B$ in a dense set $S$ of entire analytic elements:*

$$\phi(AB) = \phi(B\alpha_{i\beta}(A)).$$

*Proof.* Suppose that this holds for $S$. Then as

$$\phi(\alpha_t(A)B) = \phi(B\alpha_t\alpha_{i\beta}(A)) = \phi(B\alpha_{t+i\beta}(A))$$

the holomorphic functions $\phi(B\alpha_z(A))$ and $\phi(\alpha_{z-i\beta}(A)B)$ agree on the horizontal line $\{z = x+iy|y = \beta\}$, so they correspond by the identity principle. Define $f_{AB}(z)$ to be this common entire function. This function satisfies the $\beta$-KMS conditions for $A, B \in \mathcal{A}$

$$f_{AB}(t) = \phi(B\alpha_t(A))$$

$$f_{AB}(t + i\beta) = \phi(\alpha_t(A)B).$$

This shows that $\phi$ satisfies the $\beta$-KMS condition on a norm dense subset. Note that $f_{AB}(t)$ are bounded above by their boundary values (see 2.7.7). so that

$$|f_{AB}(z)| \leq \max\left\{\sup_{t\in\mathbb{R}} |\phi(B\alpha_t(A))|, \sup_{t\in\mathbb{R}} |\phi(\alpha_t(A)B)|\right\} \leq \max\{\|B\alpha_t(A)\|, \|\alpha_t(A)B\|\} \leq \|A\| \, \|B\|.$$

Now choose arbitrary $A, B \in \mathcal{A}$. The construction (2.7.3) which shows that entire analytic elements are dense also gives a recipe for constructing a uniformly bounded sequence of functions $f_{A_n,B_n}(z)$. By Montel's theorem, this sequence has a subsequence $f_{A_{k_n},B_{k_n}}$ which converges uniformly on any compact subset of $\mathbb{C}$ to a holomorphic function which I will denote as $f_{AB}(z)$. Fix any $T > 0$ as then the rectangle

$$R = \{x + iy = z \in \mathbb{C}| - T \leq x \leq T, 0 \leq y \leq \beta\}$$

is a compact subset to which this applies and $F_{AB}(z)$ is uniformly bounded by $\|A\|\,\|B\|$ since the subsequence of functions converging to it are. Now I need to show that $F_{AB}(z)$ satisfies the conditions required. Choose $n$ such that $\max(\|A_{k_n} - A\|, \|B_{k_n} - B\|) < \epsilon$ and such that $\left\|F_{AB}(z) - F_{A_{k_n}B_{k_n}}(z)\right\| < \epsilon$ on the rectangle $R$. Now

$$\|\phi(B\alpha_t(A)) - F_{AB}(t)\| \leq \|\phi((B - B_n)\alpha_t(A))\| + \|\phi(B_n\alpha_t(A - A_n))\| + \cdots$$

$$\cdots \|\phi(B_n\alpha_t(A_n)) - F_{A_nB_n}(t)\| + \|F_{A_nB_n}(t) - F_{AB}(t)\|$$

so that

$$\|\phi(B\alpha_t(A)) - F_{AB}(t)\| < \epsilon\,\|A\| + \epsilon\,\|B\| + 0 + \epsilon = \epsilon(\|A\| + \|B\| + 1)$$

which shows that $\phi(B\alpha_t(A)) = F_{AB}(t)$ on $R$. But the width of the rectangle $R$ is arbitrary as is $\epsilon$ so this expression actually equals zero, so $\phi(B\alpha_t(A)) = F_{AB}(t)$ holds for any $t \in \mathbb{R}$. The relation $F_{AB}(t + i\beta) = \phi(\alpha_t(A)B)$ holds by the same type of computation.

On the other hand, if $A$ and $B$ are entire analytic elements and $\phi$ is a $\beta$-KMS state then I have the holomorphic functions $\phi(B\alpha_z(A))$ and $f_{AB}(z)$ agreeing on the real line. By the identity principle they correspond on all of $\mathbb{C}$. Now for $z = i\beta$, $\phi(B\alpha_{i\beta}(A)) = \phi(AB)$. As $A, B$ are arbitrary, this holds for the set $S$ of entire analytic elements. ■

**Proposition 2.7.9.** *A KMS state $\phi$ is invariant under the automorphisms $\alpha_t$.*

*Proof.* Using the technique of ( [48], 610), assume that $\mathcal{A}$ is unital. Now since $\phi$ is a KMS state, there exists an analytic element $f$ for such $A, I$, such that

$$f(t) = \phi(I\alpha_t(A)) = \phi(\alpha_t(A)),$$

$$f(t + i\beta) = \phi(\alpha_t(A)I) = \phi(\alpha_t(A))$$

continuous on and analytic in the complex strip $\mathbb{R} \times [0, \beta]$. Extend $f$ by periodicity to be a bounded holomorphic function on $\mathbb{C}$. By Liouville's theorem, $f(z) = c$. This means that

$$\phi(\alpha_t(A)) = f(t) = f(0) = \phi(A),$$

showing the required result.

This result also holds for the non-unital case. We can unitize the algebra $\mathcal{A}$ and then extend automorphisms and states using the rule

$$\beta(a + \alpha 1) = \beta(a) + \alpha 1$$

$$\phi(a + \alpha 1) = \phi(a) + \alpha.$$

Now the KMS state $\phi$ extends to a KMS state $\phi_0$ on the unitization of $\mathcal{A}$. The result applies, so that $\phi_0$ is stationary. In particular, $\phi$ is also stationary. ■

**Proposition 2.7.10.** *A ground or ceiling state $\phi$ is invariant under the automorphisms $\alpha_t$. [10], 5.3.19.*

*Proof.* Without loss of generality, take $\mathcal{A}$ to be unital, let $\phi$ be a ground state and $A$ be an arbitrary self-adjoint element of $\mathcal{A}$. Consider the function $h(z) = \phi(G_{AI}(z))$ bounded for $\text{Im}(z) \geq 0$. Because bounded holomorphic functions can be reconstructed from their boundary values (see 2.7.7),

$$|h(z)| \leq \sup_{t \in \mathbb{R}} |\phi(\alpha_t(A))| \leq \|A\|.$$

Now $\phi$ is hermitian and $A$ is self-adjoint, so $h(t)$ is real-valued on the real axis. By Schwarz reflection, $h(z)$ extends to a holomorphic function on the lower half plane by $h(\overline{z}) = \overline{h(z)}$ so $h$ extends to a bounded holomorphic function on $\mathbb{C}$. By Liouville's theorem, $h$ is constant. Consequently, for $t \in \mathbb{R}$

$$\phi(\alpha_t(A)) = h(t) = h(0) = \phi(A).$$

By linearity, this extends to arbitrary elements of $\mathcal{A}$ and $\phi$ is $\alpha_t$-invariant. ■

**Proposition 2.7.11.** *Let $(\mathcal{A}, \mathbb{R}, \alpha_t)$ be a $C^*$-dynamical system. The set of KMS states for $\beta \in [-\infty, \infty]$ is convex and weak$^*$ compact. Similarly, ground states and ceiling states are also convex and weak$^*$ compact.*

*Proof.* Alaoglu's theorem says that the unit ball of linear functionals

$$\{f : X \to \mathbb{C} : \|f\| \leq 1\}$$

is weak$^*$ compact. By [47], Thm. 4.3.2, the states are precisely the linear functionals $\phi$ such that $\phi(I) = \|\phi\|$ and $\phi(A) \geq 0$ for all $A$ in the positive cone $\mathcal{A}^+$. Now the set of states is weak$^*$ closed since, given a net $\phi_i$ converging weak$^*$ to $\phi$,

$$\phi(I) = \lim_i \phi_i(I) = \lim_i 1 = 1,$$

$$\phi(A) = \lim_i \phi_i(A) \geq 0$$

for a positive element $A \in \mathcal{A}$; consequently, states are weak$^*$ compact. It is clear that states are convex. Next, $\beta$-KMS states are convex because given a pair $\phi, \psi$, there exist for any $A, B \in \mathcal{A}$ a pair of holomorphic functions $F_{AB}(z), G_{AB}(z)$ such that

$$F_{AB}(t) = \phi(B\alpha_t(A))$$

$$G_{AB}(t) = \psi(B\alpha_t(A))$$

$$F_{AB}(t + i\beta) = \phi(\alpha_t(A)B)$$

$$G_{AB}(t + i\beta) = \psi(\alpha_t(A)B).$$

Then fixing $0 < t < 1$, $\phi' = t\phi + (1-t)\psi$ set $F'_{AB}(z) = tF_{AB}(z) + (1-t)G_{AB}(z)$ as then

$$F'_{AB}(t) = tF_{AB}(t) + (1-t)G_{AB}(t) = t\phi(B\alpha_t(A)) + (1-t)\psi(B\alpha_t(A)) = \phi'(B\alpha_t(A)),$$

$$F'_{AB}(t+i\beta) = tF_{AB}(t+i\beta)+(1-t)G_{AB}(t+i\beta) = t\phi(\alpha_t(A)B)+(1-t)\psi(\alpha_t(A)B) = \phi'(\alpha_t(A)B).$$

Now I need to show that the KMS property persists under these. Let $\phi_k$ be a net of states where $\phi_k$ is a $\beta_k$ state where the net $\beta_k \to \beta \in \mathbb{R}$. Without loss of generality, each $\beta_k \in \mathbb{R}$ also. So for each $k$ in the indexing set $I$ for the net and an arbitrary pair of $\alpha$-analytic elements $A, B$,

$$\phi_k(AB) = \phi_k(B\alpha_{i\beta_k}(A)),$$

so

$$\phi(AB) = \lim_k \phi_k(AB) = \lim_k \phi_k(B\alpha_{i\beta_k}(A)) = \lim_k \phi_k(B\alpha_{i\beta}(A)) = \phi(B\alpha_{i\beta}(A))$$

so that $\phi$ is a $\beta$-KMS state.

Next, I need to show that this is true when $\beta = \infty$. WLOG, by passing to subnets if required, let $\beta_l \geq \beta_k$ whenever $l \geq k$ in $I$. In this case, take $F_{A,B,k}(z)$ be the holomorphic function on the strip

$$\mathcal{D}_{\beta_k} = \{z : 0 \leq \text{Im}(z) \leq \beta_k\}$$

corresponding to the $\beta_k$-state $\phi_k$. Then define a bounded holomorphic function $F_{A,B}$ on each $\mathcal{D}_{\beta_k}$ by normal families (since $F_{A,B,l}$ is uniformly bounded for each $l \geq k$). Now this defines a bounded holomorphic functions $F_{A,B}$ such that

$$F_{A,B}(t) = \lim_k F_{A,B,k}(t) = \lim_k \phi_k(B\alpha_t(A)) = \phi(B\alpha_t(A)).$$

Finally, ground states are closed since I may take a net $\phi_k$ of ground states converging to $\phi$ which have corresponding bounded holomorphic functions $F_{A,B}(z)$ on $\overline{\mathcal{H}}$, the closed upper halfplane. These again are uniformly bounded, so I may take a normal limit to obtain a bounded holomorphic function $F_{A,B}$ and

$$F_{A,B}(t) = \lim_k F_{A,B,k}(t) = \lim_k \phi_k(B\alpha_t(A)) = \phi(B\alpha_t(A)),$$

as required.

These cases of $\beta = -\infty$ and of ceiling states follow from reversing time as usual. ∎

**Proposition 2.7.12.** *Let $(\mathcal{A}, \alpha_t)$ be $C^*$-dynamical system. The set of $\beta$-KMS states for $\beta \in [-\infty, \infty]$ form a simplex in the set of states. Similarly, the set of ground states (or ceiling states) form a face of it.*

*Proof.* To show that the set of $\beta$-KMS states for $\beta \in \mathbb{R}$ form a simplex, first note that the cone over these consists of positive linear functionals satisfying the $\beta$-KMS condition. Take an arbitrary pair of $\beta$-KMS states $\omega$ and $\phi$. Let $\rho$ be $\rho = \omega + \phi$. Then, as $\omega \leq \rho$ and $\phi \leq \rho$, both $\phi$ and $\omega$ are $\pi_\rho$-normal. This means that by (2.4.6) applied to the GNS representation of the state $\rho/2$ there exist positive operators $T_\phi, T_\omega$ in the centre of the von Neumann algebra $\pi_\rho(\mathcal{A})''$ such that $\phi(A) = \hat{\rho}(AT_\phi)$ and $\omega(A) = \hat{\rho}(AT_\omega)$, where $\hat{\rho}$ is the normal extension of $\rho$ to $\pi_\rho(\mathcal{A})''$. Now define a new $\beta$-KMS positive linear functional

$$(\phi \wedge \omega)(A) = \hat{\rho}(AT_\phi T_\omega)$$

for $A \in \mathcal{A}$. This shows that the set of $\beta$-KMS states for $\beta \in \mathbb{R}$ forms a simplex.

To show that ground states form a face, it suffices to show that if a state $\phi \leq \lambda\omega$ where $\lambda \in \mathbb{R}^+$ and $\omega$ a ground state, then $\phi$ is also a ground state. Let $f$ be an arbitrary rapidly decaying test function such that $\operatorname{supp}(\hat{f}) \subseteq (-\infty, 0)$, and $A$ an arbitrary element in $\mathcal{A}$. Then

$$0 \leq \phi(\tau_f(A)\tau_f(A)^*) \leq \lambda\omega(\tau_f(A)\tau_f(A)^*) = 0,$$

so

$$\phi(\tau_f(A)\tau_f(A)^*) = 0;$$

consequently, $\phi$ is a ground state. As $\phi$ was arbitrary, the ground states form of face of the set of all states. ■

**Remark 2.7.13.** *An application of KMS states in von Neumann algebras is to show the existence of a noncommutative Radon-Nikodỳm theorem for states of von Neumann algebras. This theorem made possible the classification of type $III_\lambda$ von Neumann algebras. (See [20], chapter 5 section 5 for details.)*

## 2.8 KMS states in quantum mechanics

[9], 6. [1], 77. [64], 268. The basic structure for quantum mechanics consists of a Hilbert space $\mathcal{H}$, a $C^*$-algebra of operators acting on it, and the Hamiltonian $H$ describes how the system changes. In this case,

- the **state** is given by a unit length vector in $\mathcal{H}$,
- an **observable** $A$ is self-adjoint operators which evolves in time as

$$\alpha_t(A) = e^{itH}Ae^{-itH},$$

- the **expectation value** $\langle A\rangle$ of an operator is given by the inner product $\langle x|Ax\rangle$,
- a **transition probability** from state $\psi$ to state $\phi$ is given by $|\langle\psi|\phi\rangle|^2$,
- an operator $U$ which represents a **symmetry** of the system must satisfy $\langle Ux|Uy\rangle = \langle U^*Ux|y\rangle = \langle x|UU^*y\rangle = \langle x|y\rangle$ and so is represented by a unitary $U$. (Not all symmetries are represented this way however.)

[1], Def. 5.9, 90. [1], 146. [1], 5.3, 90. [1], 77. [1], Def. 9.15, 96.

To describe quantum statistical systems, this is generalized:

- a **state** of the system is given by a state of the $C^*$-algebra of observables. For example, a state vector $|\psi\rangle$ in $C^n$ generates the pure state on $M_n(\mathbb{C})$

$$E \mapsto \mathrm{Tr}\left(|\psi\rangle \langle\psi| E\right);$$

  so this generalizes the density matrix description.

- an **observable** again corresponds to a self-adjoint element of $\mathcal{A}$,

- the **expectation value** of an observable $A$ in a state $\phi$ is $\phi(A)$,

- a **symmetry** of the system is implemented (on the state $\phi$) by an automorphism $\gamma$ so that

$$\phi'(A) = \phi(\gamma(A)).$$

  Often it will take the form of an inner automorphism

$$\phi'(A) = \phi(u^* A u)$$

  for $u$ a unitary.

- There is a representation $\alpha_t$ from $\mathbb{R}$ into the automorphism group $\mathrm{Aut}(\mathcal{A})$ which describes the **time evolution** of the system.

## 2.9 KMS states of a finite dimensional matrix algebra

Here consider a $C^*$ system consisting of $\mathcal{M}_n(\mathbb{C})$ acted on by the automorphisms

$$\alpha_t(A) = e^{itH} A e^{-itH}$$

where $H$ is a self-adjoint Hamiltonian. Note that every matrix is an analytic element of this system since I can define

$$\alpha_z(A) = e^{izH} A e^{-izH}$$

for any $A$.

The following concepts arise from quantum statistical mechanics.

**Definition 2.9.1.** *[1], 171. [1], 9.2.1, 181. Consider the $C^*$-dynamical system $M_n(\mathbb{C})$ equipped with the action generated by self-adjoint $H$. The* **entropy** *$S(\phi)$ associated to a state $\phi$ (corresponding to the density matrix $Q$ via $\phi(E) = Tr(QE)$) is*

$$S(\phi) = -Tr(Q \log Q).$$

*The* **relative entropy** *$S(\phi|\psi)$ is*

$$S(\phi|\psi) = Tr(Q_\phi \log Q_\phi - Q_\phi \log Q_\psi)$$

*provided that $ker(\psi) \subseteq ker(\phi)$, with $S(\phi|\psi) = \infty$ otherwise. It is clear that the relative entropy of a state with respect to itself is 0. The* **free energy** *$F(\phi)$ associated to a state $\phi$ is*

$$F(\phi) = Tr(Q_\phi H) - \frac{1}{\beta} S(Q_\phi).$$

**Proposition 2.9.2.** *[1], Ex. 9.19, 182. The* **Gibbs state** *$\omega_\beta$ at inverse temperature $\beta$ of a $C^*$-algebra $\mathcal{M}_n(\mathbb{C})$ whose time evolution $\alpha_t$ is generated by a self-adjoint Hamiltonian $H$ is defined by*

$$\omega_\beta(E) = \frac{Tr(Ee^{-\beta H})}{Tr(e^{-\beta H})}.$$

*It is the unique state $\omega$ having minimal free energy $F(\omega) = -\frac{1}{\beta} \log Tr(e^{-\beta H})$.*

*Proof.* Let $\phi$ be any state having minimal free energy. By (2.2.6), we get the inequality

$$\text{Tr}(A \log A - A \log B) \geq \text{Tr}(A - B)$$

for positive matrices $A, B$, which for density matrices $Q_\omega, Q_\phi$ becomes

$$\text{Tr}(Q_\phi \log Q_\phi - Q_\phi \log Q_\omega) \geq \text{Tr}(Q_\phi) - \text{Tr}(Q_\omega) = 0$$

which shows that the relative entropy $S(\phi|\omega)$ is always non-negative (noting that density matrices have a trace of 1). Now when $Q_\omega$ is the Gibbs density matrix

$$S(\phi|\omega) = \text{Tr}(Q_\phi \log Q_\phi - Q_\phi \log Q_\omega) \geq 0$$

$$S(\phi|\omega) = \text{Tr}\left(Q_\phi \log Q_\phi - Q_\phi \left(-\beta H - I \log(\text{Tr}(e^{-\beta H}))\right)\right) \geq 0$$

$$S(\phi|\omega) = \text{Tr}(Q_\phi \log Q_\phi + \beta H Q_\phi) \geq -\log(\text{Tr}(e^{-\beta H}) \geq 0$$

But the first part is a positive multiple of the free energy of $\phi$, so

$$S(\phi|\omega) = \beta F(\phi) + \log(\text{Tr}(e^{-\beta H})) \geq 0,$$

and

$$F(\phi) \geq -\frac{1}{\beta} \log(\text{Tr}(e^{-\beta H})).$$

Then if $\phi$ is a state having minimal free energy then $S(\phi|\omega) = 0$ from which I may conclude that $Q_\phi = Q_\omega$ (by (2.2.6) again). That is, $\phi$ is the Gibbs state $\omega_\beta$. ∎

**Proposition 2.9.3.** *The Gibbs state $\omega_\beta$ is the unique $\beta$-KMS state on $\mathcal{M}_n(\mathbb{C})$ (for fixed $\beta$).*

*Proof.* Using 2.7.8, let $\phi$ be any state satisfying

$$\phi(AB) = \phi(B\alpha_{i\beta}(A)) = \phi(Be^{-\beta H}Ae^{\beta H}).$$

As $Q_\phi$ is the density matrix corresponding to $\phi$,

$$\mathrm{Tr}(ABQ_\phi) = \mathrm{Tr}(Be^{-\beta H}Ae^{\beta H}Q_\phi)$$

By the cyclic property

$$\mathrm{Tr}(ABQ_\phi) = \mathrm{Tr}(Ae^{\beta H}Q_\phi Be^{-\beta H})$$

Since $A$ is arbitrary

$$BQ_\phi = e^{\beta H}Q_\phi Be^{-\beta H}$$

$$BQ_\phi e^{\beta H} = e^{\beta H}Q_\phi B$$

From the case where $B = I$ I learn that $Q$ and $e^{\beta H}$ commute:

$$Q_\phi e^{\beta H} = e^{\beta H}Q_\phi,$$

which allows me to rewrite the previous general equation as

$$B\left(Q_\phi e^{\beta H}\right) = \left(Q_\phi e^{\beta H}\right)B$$

where $B$ is, as usual, arbitrary. Then $Q_\phi e^{\beta H}$ is in the centre of $\mathcal{M}_n(\mathbb{C})$, so that

$$Q_\phi e^{\beta H} = \lambda I,$$

$$Q_\phi = \lambda e^{-\beta H}.$$

For $\phi$ to be a state,

$$\text{Tr}(Q_\phi) = \lambda \text{Tr}(e^{-\beta H}) = 1,$$

so that any state satisfying the condition is exactly the Gibbs state:

$$Q_\phi = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})} = Q_\omega.$$

■

**Remark 2.9.4.** *Far more generally, the existence of $\beta$-KMS states and ground states for any $C^*$-dynamical system $(\mathcal{A}, \alpha)$ with dynamics which are strongly continuous and approximately inner is demonstrated in [63]. Approximately inner means that there exist Hermitian $H_n \in \mathcal{A}$ such that*

$$\left\| e^{itH_n} A e^{-itH_n} - \alpha_t(A) \right\| \to 0$$

*as $n \to \infty$ for each $A \in \mathcal{A}$.*

## 2.10 A Number-Theoretic Application

The number-theoretic Bost-Connes system [6] is defined from Hecke correspondences (essentially double cosets) associated to commensurability of lattices in $\mathbb{C}$; in particular, of the Hecke pair $(\mathbb{Z}^+, \mathbb{Q}^+)$. From [51], [11], it has the structure of a semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z})\mathbb{N}^*$, from which it acquires the following presentation by partial isometries $\{\mu_n | n \in \mathbb{N}^\times\}$ and unitaries $\{e(\gamma)\}$ for $\gamma \in \mathbb{Q}/\mathbb{Z}$:

$$\mu_n^* \mu_n = 1 = e(0 + \mathbb{Z}),$$

$$\mu_{mn} = \mu_m \mu_n,$$

$$\mu_n \mu_m^* = \mu_m^* \mu_n \text{ whenever } \gcd(m, n) = 1,$$

$$e(\gamma)^* = e(-\gamma),$$

$$e(\gamma_1 + \gamma_2) = e(\gamma_1)e(\gamma_2),$$

$$e(\gamma)\mu_n = \mu_n e(n\gamma),$$

$$\mu_n e(\gamma)\mu_n^* = \frac{1}{n}\sum_{n\delta=\gamma} e(\delta),$$

whenever $m, n \in \mathbb{N}^\times$ and $\gamma, \gamma_1, \gamma_2 \in \mathbb{Q}/\mathbb{Z}$. Its dynamics are given by the Hamiltonian $H = \log N$ where $N$ is the number operator of this quantum system. For this presentation, the resulting action $\alpha$ acts on the generators as $\alpha_t(e_\delta) = e^{2\pi i \frac{p}{q} t} e_\delta$ where $\delta = \frac{p}{q} + \mathbb{Z}$ and $\alpha_t(\mu_a) = \mu_a$. Its partition function is equal to the Riemann $\zeta$ function and its symmetry group is the idele class group. It has phase transition at $\beta = 1$ where the dimension of its simplex of KMS states changes.

More generally [39] this construction can be done for a general Shimura variety such as number field. In this case, the partition function is the $\zeta$ function related to the variety and its symmetry group is the idele class group for the variety.

## 2.11 Graph Algebras

**Definition 2.11.1.** *[4], 4. [65]. A* **graph** *is a quadruple $(V, E, s, r)$ where $V, E$ are countable sets of vertices and edges, respectively, $r$ is the range function which carries an edge to its ending vertex, and $s$ is the source function which carries an edge onto its starting vertex. A* **source** *is a vertex that only has edges leaving it. A* **sink** *is a vertex that only has edges entering it. An* **isolated vertex** *has no edges leaving or entering it.*

**Definition 2.11.2.** *[4], 696. An* **adjacency matrix** *$A$ of a graph, whose vertices are* *enumerated* $1, \ldots, N$ *or by* $\mathbb{N}$, *as the case may be, takes the form*

$$a_{ij} = \begin{cases} n & \text{there are } n \text{ directed edges leading out of vertex } i \text{ into vertex } j \\ 0 & \text{there is no edge from } i \text{ to } j. \end{cases}.$$

**Remark 2.11.3.** *Note that for a simple digraph* $G$, *that* $a_{ij} \in \{0, 1\}$ *for any vertex pair* $(i, j)$ *and that* $a_{jj} = 0$ *for any vertex* $j$.

**Definition 2.11.4.** *[4], 1.4. A* **walk** $\lambda$ *in a graph* $G$ *is an alternating sequence of vertices and edges*

$$\lambda = v_1 e_1 v_2 e_2 \ldots v_n$$

*such that* $v_k = s(e_k)$ *and* $r(e_k) = v_{k+1}$ *for each integer* $1 \leq k < n$*; that is,* $v_k$ *enters* $e_k$ *while* $v_{k+1}$ *leaves it. Extend the definition of range and source by letting* $s(\lambda) = v_1$ *and* $r(\lambda) = v_n$*. The length* $|\lambda|$ *of* $\lambda$ *is* $n - 1$.

*Concatenation of two walks* $\lambda = v_1 e_1 \ldots v_n$, $\mu = w_1 f_1 \ldots w_m$ *when* $v_n = w_1$ *is* $\lambda\mu$ *where*

$$\lambda\mu = v_1 e_1 \ldots v_n f_1 \ldots w_m.$$

**Definition 2.11.5.** *Walks will be partially ordered as follows: given a pair of walks,* $\lambda$ *and* $\mu$, $\lambda \leq \mu$ *if there exists a walk* $\mu'$ *such that* $\mu = \lambda\mu'$.

**Definition 2.11.6.** *[50], Def. 1.3. A* $C^*$**-correspondence** $\mathcal{E}$ *over* $\mathcal{A}$ *is a Hilbert* $\mathcal{A}$*-module equipped with a* $^*$*-homomorphism* $\phi_{\mathcal{E}} : A \to \mathcal{L}(X)$*. The function* $\phi_{\mathcal{E}}$ *is called the* **left action**.

**Definition 2.11.7.** *[50], Def. 2.1, Def. 2.3. A* **representation** *of a* $C^*$*-correspondence* $\mathcal{E}$ *over* $\mathcal{A}$ *(on a* $C^*$*-algebra* $\mathfrak{B}$*) is a pair* $(\pi, t)$ *where* $\pi$ *is a* $^*$*-homomorphism* $\mathcal{A} \to \mathfrak{B}$ *and a* *linear map $t : \mathcal{E} \to \mathfrak{B}$ such that*

$$t(\xi)^* t(\eta) = \pi\left(\langle \xi, \eta\rangle_{\mathcal{E}}\right) \tag{2.8}$$

$$\pi(a)t(\xi) = t(\phi_{\mathcal{E}}(a)\xi) \tag{2.9}$$

*for any $\xi, \eta \in \mathcal{E}$, $a \in \mathcal{A}$.*

*Define a $^*$-homomorphism $\phi_t : \mathcal{K}(\mathcal{E}) \to \mathfrak{B}$ as $\phi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$.*

*Now let $C^*(\pi, t)$ denote the $C^*$-algebra generated by the images of $\pi$ of $t$ in $\mathfrak{B}$.*

**Definition 2.11.8.** *[50], Def. 3.2, Def. 3.4. Let $(\pi, t)$ be a representation of a $C^*$-correspondence $\mathcal{E}$. Then $(\pi, t)$ is* **covariant** *(with respect to $J_{\mathcal{E}}$) if $\pi(a) = \psi_t(\phi_{\mathcal{E}}(a))$ for all $a \in J_{\mathcal{E}}$ where*

$$J_{\mathcal{E}} = \phi_{\mathcal{E}}^{-1}(\mathcal{K}(\mathcal{E})) \cap \left(ker(\phi_{\mathcal{E}})\right)^{\perp},$$

$$J_{\mathcal{E}} = \left\{a \in \mathcal{A}\colon \phi_{\mathcal{E}}(a) \in \mathcal{K}(\mathcal{E}), ab = 0 \text{ for all } b \in ker(\phi_{\mathcal{E}})\right\}.$$

**Definition 2.11.9.** *To a given finite directed graph $G$ having vertices $v_1, \ldots, v_n$ and directed edges $e_1, \ldots, e_m$ associate the bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}}$ which is defined as follows. Let $r$ and $s$ denote the range and source functions as usual. Take $\mathcal{A} = C(\{v_1, v_2, \ldots, v_n\})$ and $\mathcal{E} = C(\{e_1, e_2, \ldots, e_m\})$ The algebra $\mathcal{A}$ and module $\mathcal{E}$ satisfy the following relations*

$$v_i \cdot v_k = \delta_{ik} v_k \tag{2.10}$$

$$v_i \cdot e_j = \delta_{is(e_j)} e_j \tag{2.11}$$

$$e_j \cdot v_k = \delta_{jr(e_k)} e_j \tag{2.12}$$

$$\langle e_j, e_l\rangle = \delta_{r(e_j)r(e_l)} v_{r(j)} \tag{2.13}$$

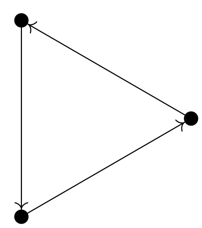
*where $\delta_{mn}$ is the Kronecker delta and $1 \leq i, k \leq n$, $1 \leq j, l \leq m$.*

**Remark 2.11.10.** *This takes the form of a $C^* -$correspondence where the left action $\phi_{\mathcal{E}}$ is*

$$\phi_{\mathcal{E}}(v_j) = \sum_{\{e|s(e)=v_j\}} \theta_{e,e}.$$

*Note that in the case of finite graph that $J_{\mathcal{E}}$ is just the ideal of nonsinks of $G$.*

**Example 2.11.11.** *A simple cycle looks like*



*The corresponding Hilbert $C^*$-module can be realized as $\mathbb{C}^3$ with the actions of $\mathbb{C}^3$: the left and right actions and inner product are*

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \end{bmatrix} \quad , \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} bx \\ cy \\ az \end{bmatrix} \quad , \quad \left\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle = \begin{bmatrix} x_2^* y_2 \\ x_3^* y_3 \\ x_1^* y_1 \end{bmatrix}.$$

**Definition 2.11.12.** *Specialized from [50], Def. 1.4. Let $\mathcal{E}$ and $\mathcal{C}$ be graph bimodules over $\mathcal{A}$. Then their tensor product $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{C}$ is the quotient of the algebraic tensor product by the subspace generated by terms of the form $(\xi \cdot a) \otimes \zeta - \xi \otimes (a \cdot \zeta)$ for $\xi \in \mathcal{E}$, $\zeta \in \mathcal{C}$ and $a \in \mathcal{A}$. Now it is equipped with a left action, a right action, and $\mathcal{A}$-valued inner product as follows:*

$$a\,(\xi \otimes \zeta) = (a \cdot \xi) \otimes \zeta,$$

$$(\xi \otimes \zeta)\, a = \xi \otimes (\zeta \cdot a)\,,$$

*and*

$$\langle \xi \otimes \zeta, \xi' \otimes \zeta' \rangle_{\mathcal{E} \otimes \mathcal{C}} = \langle \zeta, \langle \xi, \xi' \rangle_{\mathcal{C}} \cdot \zeta' \rangle_{\mathcal{E}}\,.$$

**Proposition 2.11.13.** *Let $\mathcal{E}$ and $\mathcal{C}$ be graph bimodules over $\mathcal{A}$ having edges $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$, respectively. Then a term $e_j \otimes f_k \in \mathcal{E} \otimes_{\mathcal{A}} \mathcal{C}$ is nonzero if and only if $r(e_j) = s(f_k)$.*

*Proof.* As then

$$e_j \otimes f_k = (e_j \cdot v_{r(j)}) \otimes f_k = e_j \otimes (v_{r(j)} \cdot f_k) = \delta_{r(j),s(k)} e_j \otimes f_k.$$

∎

**Corollary 2.11.14.** *Let $\mathcal{E}$ be a graph bimodule over $\mathcal{A}$. Then the element*

$$v_{k_1} \otimes v_{k_2} \otimes \ldots v_{k_n}$$

*in the repeated tensor product $E^{\otimes n}$ (which are defined as*

$$E^{\otimes 0} = \mathcal{A}$$

$$E^{\otimes n} = E \otimes E^{\otimes n-1})$$

*is non-zero if and only it corresponds to a walk of the associated graph $G$.*

*Proof.* As $v_{k_1} \otimes v_{k_2} \otimes \cdots v_{k_n}$ is non-zero, it follows that $r(k_j) = s(k_{j+1})$ for $1 \leq j < n$. On the other hand, if $r(k_j) = s(k_{j+1})$ for all $1 \leq j < n$ then $v_{k_1} \otimes v_{k_2} \cdots v_{k_n}$ is non-zero by the properties used in the previous proof. ∎

**Definition 2.11.15.** *Let $\mathcal{E}$ be a graph bimodule over $\mathcal{A}$. Then the Fock bimodule over it is*

$$\mathcal{F}(\mathcal{E}) = \bigoplus_{n \geq 0} E^{\otimes n}.$$

**Proposition 2.11.16.** *The Fock bimodule of a graph bimodule is the vector space of all possible walks of $G$. So its edges take the form $e_\lambda$ where $\lambda$ is a walk of $G$, and its left and* *right actions are*

$$v_a \cdot e_\lambda = \delta_{a,s(\lambda)} e_\lambda,$$

$$e_\lambda \cdot v_b = \delta_{r(\lambda),b} e_\lambda.$$

**Definition 2.11.17.** *[50], Def. 4.2. The Fock representation $(\phi_\infty, \tau_\infty)$ of the graph bimodule $\mathcal{E}$ onto $\mathcal{L}(\mathcal{F}(\mathcal{E}))$ is given by*

$$\phi_\infty(v_a)e_\lambda = \begin{cases} e_\lambda & \text{when } s(\lambda) = v_a \\ 0 & \text{else,} \end{cases}$$

$$\tau_\infty(e_k)e_\lambda = \begin{cases} e_{k\lambda} & \text{when } s(\lambda) = r(e_k) \\ 0 & \text{else.} \end{cases}$$

**Definition 2.11.18.** *The* **covariant representation** *is given by $(\pi, \tau) = (\sigma \circ \pi_\infty, \sigma \circ \tau_\infty)$ where $\sigma$ is the quotient map* $\mathcal{L}(\mathcal{F}(\mathcal{E})) \xrightarrow{\sigma} \frac{\mathcal{L}(\mathcal{F}(\mathcal{E}))}{\mathcal{K}(\mathcal{F}(\mathcal{E})J_X)}$.

**Definition 2.11.19.** *[65]. A* **Cuntz-Krieger $\mathcal{E}$-family** *for a graph $G = (E, V, s, r)$ having edge set $E$ and vertex set $V$ consists of a set $P_v$ of projections and a set of partial isometries $T_e$ such that*

- $T_e^* T_e = P_{r(e)}$ *for each edge* $e \in E$,
- $T_e T_e^* \leq P_{s(e)}$ *for each edge* $e \in E$,
- $\sum_{\{e|s(e)=u\}} T_e T_e^* = P_u$ *for each vertex* $v \in V$ *such that*

$$0 < |\{e \in E : s(e) = v\}| < \infty.$$

**Definition 2.11.20.** *[50], Def. 3.5. The* **graph algebra** $\mathcal{O}_\mathcal{E}$ *is the universal $C^*$ algebra generated by $P_v$ and $T_e$ satisfying these conditions. This means that for any $C^*$-algebra $\mathcal{A}$*

*generated by $P'_v, T'_e$ satisfying these conditions, there exists a $^*$-morphism $\phi : \mathcal{O}_{\mathcal{E}} \to \mathcal{A}$ such that $\phi(P_v) = P'_v$ for each vertex $v$, and $\phi(T_e) = T'_e$ for each edge $e$.*

**Proposition 2.11.21.** *[50], Prop. 6.5. The graph algebra is isomorphic to $C^*(\pi_X, t_X)$ where $(\pi_X, t_X)$ is the universal covariant representation of the associated $C^*$-correspondence $\mathcal{E}$ (a Hilbert bimodule in this case). For any covariant representation $(\pi, t)$ there exists a surjection $\rho : \mathcal{O}_{\mathcal{E}} \to C^*(\pi, t)$, by the universal property concerned. Also the covariant representation generates $C^*(\phi, \tau)$ which is isomorphic to $\mathcal{O}_{\mathcal{E}}$.*

**Proposition 2.11.22.** *The* **gauge action** *of the Cuntz-Krieger algebra is defined on the generators by*

$$\alpha_t(P_v) = P_v,$$

$$\alpha_t(T_e) = e^{it} T_e,$$

*for any $v \in V, e \in E$. For walks $\alpha, \beta$ of length $m, n$, respectively, that*

$$\alpha_t(S_\alpha S_\beta^*) = e^{it(m-n)} S_\alpha S_\beta^*.$$

*These have a dense linear span in the generated $C^*$-algebra $\mathcal{A}$, consequently this action extends to all of $\mathcal{A}$.*

*Also, it is clear that $S_\alpha S_\beta^*$ constitute analytic elements since $\alpha_z(S_\alpha S_\beta^*) = e^{i(m-n)z} S_\alpha S_\beta^*$ is defined everywhere in $\mathbb{C}$. Consequently, the $^*$-subalgebra $\mathcal{A}_0$ is a dense subset of entire elements.*

**Remark 2.11.23.** *[61], §3. There are two conventions for the partial isometries associated with edges. We use the ($\mathcal{E}$-family) convention that the initial projection $v^*v$ of $T_e$ is the projection $P_{r(e)}$ for the vertex $r(e)$. The other ($\Lambda$-family) convention has $v^*v$ corresponding to the projection $P_{s(e)}$ for the vertex $s(e)$.*

**Remark 2.11.24.** *For computational convenience, given a Cuntz-Kreiger $\mathcal{E}$-family, define* *for each walk* $\lambda = v_1 e_1 \dots v_n$ *the operators*

$$S_\lambda = T_{e_1} T_{e_2} \cdots T_{e_n}$$

*for* $|\lambda| > 0$ *and*

$$S_\lambda = P_{v_1} = P_{v_n}$$

*otherwise. Consequently, when* $|\lambda| > 0$

$$S^*_\lambda = T^*_{e_n} \dots T^*_{e_2} T^*_{e_1}.$$

*(When* $|\lambda| = 0$*,* $S_\lambda = S^*_\lambda$*.) These satisfy the relations* $S_{\lambda\mu} = S_\lambda S_\mu$ *and* $S^*_{\lambda\mu} = S^*_\lambda S^*_\mu$*.*

**Proposition 2.11.25.** *The linear span of* $\left\{ S_\lambda S^*_\mu | \lambda, \mu \text{ walks in } G \right\}$ *is dense in* $C^*(G)$*.*

*Proof.* Note that

$$S^*_\lambda S_\mu = \begin{cases} P_v = S_\lambda = S^*_\lambda & \lambda = \mu \text{ where } v = r(\lambda), \\ S^*_{\lambda'} & \lambda = \mu\lambda', \\ S_{\mu'} & \mu = \lambda\mu', \\ 0 & \text{otherwise.} \end{cases}$$

A dense $^*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ is given by

$$\mathcal{A}_0 = \left\{ \sum_{k=0}^{N} c_j S_{\lambda_j} S^*_{\mu_j} \right\}$$

since each $S^*_\lambda S_\mu$ takes one of the forms $S_\nu$ or $S^*_\nu$ for some walk $\nu$. ∎

**Proposition 2.11.26.** *The* $\alpha$*-invariant* $C^*$*-algebra* $\mathcal{A}_\alpha$ *of* $\mathcal{A}$ *is given by*

$$\mathcal{A}_\alpha = \left\{ a \in \mathcal{A} : \alpha_t(a) = a \text{ for all } t \in \mathbb{R} \right\}.$$

*There is a conditional expectation $E : \mathcal{A} \to \mathcal{A}_\alpha$ given by*

$$a \mapsto E(a) = \int_0^{2\pi} \alpha_t(a) \frac{dt}{2\pi}.$$

*Furthermore, the $^*$-subalgebra $\mathcal{A}_{0,\alpha}$ of $\alpha$-invariant elements of $\mathcal{A}_0$ is dense in $\mathcal{A}_\alpha$.*

*Proof.* It is clear that $E$ is linear, so as it is bounded,

$$\|E(a)\| = \left\| \int_0^{2\pi} \alpha_t(a) \frac{dt}{2\pi} \right\| \leq \int_0^{2\pi} \|\alpha_t(a)\| \frac{dt}{2\pi} = \|a\|,$$

$E$ is continuous. Now for $a \in \mathcal{A}_\alpha$,

$$E(a) = \int_{t=0}^{2\pi} \alpha_t(a) \frac{dt}{2\pi} = \int_{t=0}^{2\pi} a \frac{dt}{2\pi} = a;$$

similarly given an arbitrary $b \in \mathcal{A}$,

$$E(ab) = \int_{t=0}^{2\pi} \alpha_t(ab) \frac{dt}{2\pi} = \int_{t=0}^{2\pi} a\alpha_t(b) \frac{dt}{2\pi} = a \int_{t=0}^{2\pi} \alpha_t(b) \frac{dt}{2\pi} = aE(b).$$

Next note that $E(P_v) = P_v$ and

$$E(S_\gamma S_{\gamma'}^*) = \int_0^{2\pi} e^{it(m-n)} S_\gamma S_{\gamma'}^* \frac{dt}{2\pi} = \begin{cases} S_\gamma S_{\gamma'}^* & m = n \\ 0 & m \neq n \end{cases}.$$

Finally, let $a_i$ be a net in $\mathcal{A}_0$ converging to the element $a \in \mathcal{A}_\alpha$. By continuity of $E$, $a$ is the limit of the net $E(a_i)$ in $\mathcal{A}_{0,\alpha}$. ■

**Proposition 2.11.27.** *Let $\phi$ be a linear functional on the graph algebra $\mathcal{A}$. Then an $\alpha$-invariant linear functional $\phi$ satisfying the condition $\phi(AB) = \phi(B\alpha_{i\beta}(A))$ is determined by its values $\phi(P_v)$ on vertex projections and is positive if and only if $\phi(P_v) \geq 0$ for each vertex*

*$v$ and for each infinite emitter $w$ whose edges enter vertices $v_j$ it satisfies*

$$\phi(p_w) \geq e^{-\beta} \sum_{k=0}^{\infty} \phi(p_{v_j}).$$

*In particular, a KMS state $\phi_\beta$ is determined entirely by its values on projections.*

*Proof.* To determine the functional $\phi$ on the graph algebra $\mathcal{A}$ it suffices to consider elements of the form $S_\gamma S_{\gamma'}^*$, by density of $\mathcal{A}_{0,\alpha}$.

Now since $|\gamma| = |\gamma'|$,

$$\phi(S_\gamma S_{\gamma'}^*) = \phi(S_{\gamma'}^* \alpha_{i\beta}(S_\gamma)) = e^{-\beta|\gamma|}\phi(S_{\gamma'}^* S_\gamma) = \begin{cases} e^{-n\beta}\phi(P_{r(\gamma)}) & \gamma = \gamma' \\ 0 & \gamma \neq \gamma', \end{cases}$$

showing that $\phi$ is entirely determined by its values on the vertex projections.

Now I need to show that if $\phi$ is positive on the projections that it is positive. Let $a \geq 0$ and let $a_k$ be a net of elements of $\mathcal{A}_{0,\alpha}$, so that

$$\lim_k a_k = a,$$

now as $\|\mathrm{Re}(x)\| \leq \|x\|$ and the real part of an element of $\mathcal{A}_{0,\alpha}$ is also in this algebra,

$$\lim_k \mathrm{Re}(a_k) = a,$$

and because $\|x^+\| \leq \|x\|$,

$$\lim_k \mathrm{Re}(a_k)^+ = a.$$

Now each of these terms takes the form

$$\sum_{k=0}^{N} \left( c_k S_{\mu_k} S_{\mu_k'}^* + \overline{c_k} S_{\mu_k'} S_{\mu_k}^* \right) + \sum_{l=0}^{M} \alpha_k S_{\nu_k} S_{\nu_k}^*$$

where $(\mu_k, \mu'_k)$ are pairs of walks, $\nu_k$ are walks and $\alpha_k \geq 0$. This means that

$$\phi(a_k) = \sum_{l=0}^{M} \alpha_k \phi(S_{\nu_k} S_{\nu_k}^*) \geq 0$$

so

$$\phi(a) = \lim_k \phi(a_k) \geq 0.$$

The other direction is clear. ■

**Proposition 2.11.28.** *Ground states are exactly those $\alpha$-invariant states which vanish except on projections corresponding to vertices which are sinks or infinite emitters. Ceiling states are exactly those $\alpha$-invariant states which vanish except on projections corresponding to vertices which are sources.*

*Proof.* Let $\lambda, \mu$ be a pair of walks in $G$. By $\alpha$-invariance,

$$\phi(\alpha_t(S_\lambda S_\mu^*)) = e^{it(|\lambda|-|\mu|)}\phi(S_\lambda S_\mu^*) = \phi(S_\lambda S_\mu^*),$$

so unless $|\lambda| = |\mu|$, $\phi(S_\lambda S_\mu^*) = 0$.

Now for $A, B$ analytic and $\phi$ a ground state, the function $\phi(B\alpha_z(A))$ is bounded on the upper halfplane. Noting that $\alpha_z(T_e) = e^{iz}T_e$ and $\alpha_z(T_e^*) = e^{-iz}$ and that these have moduli $e^{-y}$ and $e^y$, respectively, taking $A = T_e, B = T_e^*$, I may conclude that

$$\phi(T_e \alpha_z(T_e^*)) = e^{-iz}\phi(T_e T_e^*) = 0$$

since this function otherwise is unbounded. So $\phi(T_e T_e^*) = 0$ for any edge and for each finite-emitting vertex $v$,

$$P_v = \sum_{s(e)=v} T_e T_e^*$$

and

$$\phi(P_v) = \sum_{s(e)=v} \phi(T_e T_e^*) = 0.$$

So every finite-emitting vertex whose corresponding projection is not in the kernel of $\phi$ is a sink.

On the other hand, if $\phi$ is nonvanishing only on projections corresponding to sinks or infinite emitters. I need to show that $F_{A,B}(z) = \phi(B\alpha_z(A))$ is bounded on the upper halfplane for every entire pair $A, B$. Now let $A = S_\mu S_\nu^*$, $B = S_{\mu'} S_{\nu'}^*$, so that

$$\phi\left(S_{\mu'} S_{\nu'}^* e^{iz(|\mu|-|\nu|)} S_\mu S_\nu^*\right)$$

where $|\mu| = 0 = |\nu|$ if $\phi$ is to be nonzero on this expression. So we are looking at

$$e^{i|\mu|z}\phi(S_{\nu'}^* S_\mu)$$

which is bounded since $\left|e^{i|\mu|z}\right| = e^{-|\mu|y}$. ■

In [26] Enomoto, Fujii, and Watatani showed that for an irreducible adjacency matrix $A$ the Cuntz-Krieger algebra (equipped with its gauge action) has a unique $\beta$-KMS state occurring for $\beta = \ln r$ where $r$ is the spectral radius of $A$. This result is extended in a number of papers including [49] and [44]. An easy case of the general form follows.

**Proposition 2.11.29.** *The Cuntz-Krieger algebra for a finite directed graph G (without infinite emitters) having vertex set $\{1, \dots, n\}$ and edge set $\{1, \dots, m\}$ can be characterized by its adjacency matrix A as follows. Use an ordering of the vertices in which the sinks come last. Then the adjacency matrix and the vector c of $\phi_\beta$ on the projections may be partitioned as follows*

$$A = \begin{bmatrix} A_n & A_s \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} c_n \\ c_s \end{bmatrix}.$$

*This produces the equation*

$$\left(e^{\beta}I - A_n\right)c_n = A_s c_s$$

*requiring non-negative solution vectors $c_n, c_s$. In particular, for a graph without sinks, this takes the form*

$$\left(e^{\beta}I - A\right)c = 0.$$

*Proof.* The graph algebra satisfies the condition

$$T_i^* T_i = P_{r(i)} = \sum_j a_{s(i)j} T_j T_j^*$$

for each vertex $i$ which is not a sink. Without loss of generality, the sinks are the vertices such that $s < j \leq n$ where $s$ is some integer.

So any functional on the graph algebra satisfies

$$\phi(T_i^* T_i) = \sum_j a_{s(i)j} \phi(T_j T_j^*)$$

for each nonsink $i \in V$. In particular, for a KMS state $\phi_\beta$

$$\phi_\beta(P_{r(i)}) = e^{-\beta} \sum_j a_{s(i)j} \phi_\beta(P_j)$$

for each nonsink $i \in V$. So

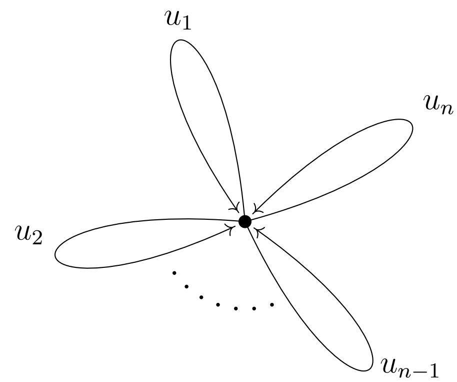
$$\phi_\beta(P_{r(i)}) = e^{-\beta} \sum_{j<s} a_{s(i)j} \phi_\beta(P_j) + e^{-\beta} \sum_{j \geq s} a_{s(i)j} \phi_\beta(P_j),$$

and

$$e^{\beta} \left( \phi_\beta(P_{r(i)}) - \sum_{j<s} a_{s(i)j} \phi_\beta(P_j) \right) = \sum_{j \geq s} a_{s(i)j} \phi_\beta(P_j).$$

If there are no sinks, the right hand side vanishes. ∎

**Example 2.11.30.** *[23] The Cuntz algebra $\mathcal{O}_n$ is the graph algebra of the following graph:*




There are no sinks here so the solutions here take the form

$$\left(e^{\beta}1 - \begin{bmatrix} n \end{bmatrix}\right)\begin{bmatrix} c_1 \end{bmatrix} = 0$$

subject to positivity ($c_1 \geq 0$) and normalization ($c_1 = 1$). Consequently, there is a unique KMS state at $\beta = \ln n$ satisfying $\phi_\beta(P_1) = 1$.

This can also be modelled with the graph on n-vertices having edges between every pair of vertices and edges from each vertex to itself.

In this case, it takes the form

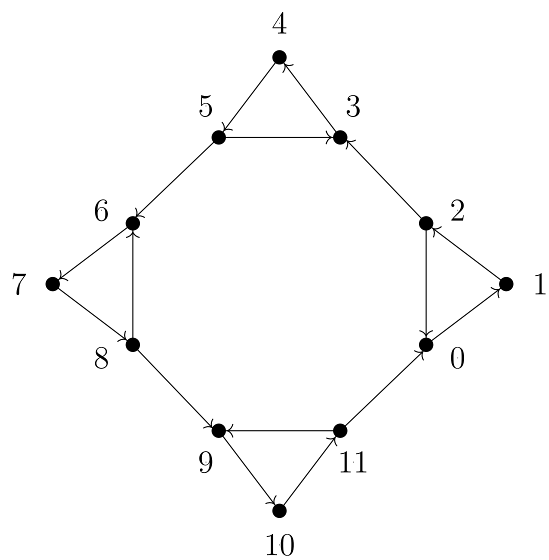
$$\left(e^{\beta}I_n - \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right)\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The eigenvalues $\lambda = n, 0$ are associated to the eigenvectors

$$n : \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \; 0 : \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix}.$$

I conclude that I have a unique $\beta$-KMS state at $\beta = \ln n$ where $\phi_\beta(P_k) = n^{-1}$.

**Example 2.11.31.**




The adjacency matrix of this graph takes the form

$$A = \begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & A_1 & A_2 \\ A_2 & 0 & 0 & A_1 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $\lambda$ be an eigenvalue of $A$ and $u$ a (suitably partitioned) eigenvector of $A$. That is,

$$\begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & A_1 & A_2 \\ A_2 & 0 & 0 & A_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

Note now that

$$\begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & A_1 & A_2 \\ A_2 & 0 & 0 & A_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \lambda \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

and consequently

$$\begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & A_1 & A_2 \\ A_2 & 0 & 0 & A_1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_1 \end{bmatrix} = \lambda \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_1 \end{bmatrix}.$$

This means that if I define $u_0 = u_1 + u_2 + u_3 + u_4$, then

$$\begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & A_1 & A_2 \\ A_2 & 0 & 0 & A_1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_0 \\ u_0 \\ u_0 \end{bmatrix} = \lambda \begin{bmatrix} u_0 \\ u_0 \\ u_0 \\ u_0 \end{bmatrix}.$$

To find eigenvalues which correspond to nonnegative eigenvectors, then it suffices to check the eigenvalues of the matrix $A_1 + A_2$:

$$A_1 + A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Now

$$|\lambda I - (A_1 + A_2)| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & 0 & \lambda \end{vmatrix} = \lambda^3 - 2.$$

The only real eigenvalue $\lambda = e^\beta = 2^{1/3}$; so the only possible inverse temperature $\beta$ for a KMS state is $\frac{1}{3}\ln 2$. From looking at its null space I find that $u_0$ is an eigenvector of $A_1 + A_2$ and $u$

$$u = \begin{bmatrix} u_0 \\ u_0 \\ u_0 \\ u_0 \end{bmatrix} \quad u_0 = \begin{bmatrix} 1 \\ 2^{1/3} \\ 2^{2/3} \end{bmatrix}$$

when normalized by the sum of its elements will generate the unique KMS state $\phi_\beta$. That is,

$$\phi_\beta(P_j) = \begin{cases} \frac{1}{4(1+2^{1/3}+2^{2/3})} & \text{for } j \equiv 0 \pmod 3, \\ \frac{2^{1/3}}{4(1+2^{1/3}+2^{2/3})} & \text{for } j \equiv 1 \pmod 3, \\ \frac{2^{2/3}}{4(1+2^{1/3}+2^{2/3})} & \text{for } j \equiv 2 \pmod 3. \end{cases}$$

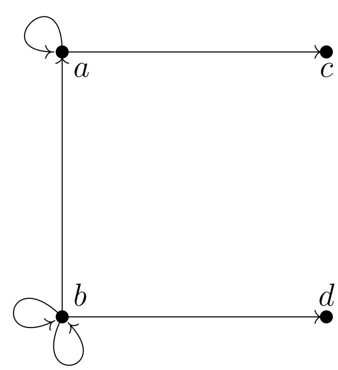
(Uniqueness can be shown from a theorem or demonstrated directly. First of all each $Au_i$ has to be in the image of $A$ which means that $u_i$ is an eigenvector of $A_1 + A_2$ in this case.

Consequently, as

$$\begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & A_1 & A_2 \\ A_2 & 0 & 0 & A_1 \end{bmatrix} \begin{bmatrix} \nu_1 u_0 \\ \nu_2 u_0 \\ \nu_3 u_0 \\ \nu_4 u_0 \end{bmatrix} = \begin{bmatrix} 2^{1/3}\nu_1 u_0 + (\nu_2 - \nu_1)e_3 \\ 2^{1/3}\nu_2 u_0 + (\nu_3 - \nu_2)e_3 \\ 2^{1/3}\nu_3 u_0 + (\nu_4 - \nu_3)e_3 \\ 2^{1/3}\nu_4 u_0 + (\nu_1 - \nu_4)e_3 \end{bmatrix},$$

this shows that $\nu_1 = \nu_2 = \nu_3 = \nu_4$ so there are no eigenvectors independent of the one already constructed.)

**Example 2.11.32.** *Next I will give an example which has two phase transitions (that is, points where the dimension of the space of KMS states changes): the graph algebra of the following digraph.*




This multigraph has an adjacency matrix which I will partition according to nonsinks/sinks so that

$$\begin{bmatrix} A & A' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the values of a $\beta$-KMS state on the projections must satisfy

$$\left(\begin{bmatrix} e^{\beta} & 0 \\ 0 & e^{\beta} \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) \begin{bmatrix} c_a \\ c_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_c \\ c_d \end{bmatrix} = \begin{bmatrix} c_c \\ c_d \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} c_a \\ c_b \end{bmatrix} = \frac{1}{\left(e^{\beta}-1\right)\left(e^{\beta}-2\right)} \begin{bmatrix} e^{\beta}-2 & 1 \\ 0 & e^{\beta}-1 \end{bmatrix} \begin{bmatrix} c_c \\ c_d \end{bmatrix}.$$

So given $t, u \geq 0$, the values on the projection $p_a, p_b, p_c, p_d$ are

$$\begin{bmatrix} c_a \\ c_b \\ c_c \\ c_d \end{bmatrix} = \begin{bmatrix} t + (e^{\beta}-1)^{-1} \\ u \\ (e^{\beta}-1)t \\ (e^{\beta}-2)u \end{bmatrix}.$$

- When $\beta < 0$, then $t = u = 0$, which means that no KMS states exist in this case.
- When $0 < \beta < \ln 2$, then $u = 0$, and there is a unique KMS state at $\beta$: the one taking the following values on the projections. Namely,

$$\begin{bmatrix} e^{-\beta} & 0 & 1 - e^{\beta} & 0 \end{bmatrix}^{T}.$$

- When $\beta > \ln 2$, each term is positive, so each of these when normalized so that the sum is 1 works. Consequently, there is a 1-dimensional simplex of KMS states for each $\beta$.

This takes cares of $\beta \neq 0, \ln 2$, leaving:

- For $\beta = 0$, the equation takes the form

$$\begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_a \\ c_b \end{bmatrix} = \begin{bmatrix} c_c \\ c_d \end{bmatrix}.$$

  This means that $-c_b = c_c = c_d$ so that $c_b = c_c = c_d = 0$. Setting $c_a = 1$, I obtain the tracial state

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T.$$

- For $\beta = \ln 2$, it takes the form

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_a \\ c_b \end{bmatrix} = \begin{bmatrix} c_c \\ c_d \end{bmatrix}.$$

  The states take values on the projections

$$\begin{bmatrix} s+t, & s, & t, & 1-2(s+t) \end{bmatrix}^T$$

  where $1 - 2(s+t) \geq 0$. This produces a 1-dimensional simplex spanned by

$$\frac{1}{4}\begin{bmatrix} 1 & 0 & 1 & 2 \end{bmatrix}^T, \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 & 2 \end{bmatrix}^T.$$

- The existence of ceiling states is precluded by the non-existence of sources in the graph.

- There is a 1-dimensional simplex of ground states because of the sinks the graph has, however. These take the form

$$\begin{bmatrix} 0 & 0 & t & 1-t \end{bmatrix}^T$$

for $0 \leq t \leq 1$.

# Chapter 3

# Part II: Corona Problem for Planar Domains

## 3.1 Maximal Ideals and the Spectrum

**Proposition 3.1.1.** *[32], 1.1-1.2. Let $\mathcal{A}$ be a unital Banach algebra. Then spectrum $\sigma(f)$ of an element $f \in \mathcal{A}$ is a non-empty closed subset of $\mathbb{C}$ enclosed in the closed disk $\overline{B}_{\|f\|}(0)$.*

*Proof.* Let the resolvent $\rho$ be the complement of $\sigma(f)$, $\mathbb{C} - \sigma(f)$, and define the function

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{f^n}{\lambda^{n+1}}$$

for $|\lambda| \geq \|f\|$; then for each $N$,

$$\sum_{n=0}^{N} \frac{f^n}{\lambda^{n+1}} (\lambda - f) = \sum_{n=0}^{N} \frac{f^n}{\lambda^n} - \sum_{n=1}^{N+1} \frac{f^n}{\lambda^n} = 1 - \frac{f^{N+1}}{\lambda^{N+1}} \to 1$$

as $N \to \infty$ so that $g(\lambda)(\lambda - f) = 1$ for $|\lambda| \geq \|f\|$. This means that $(\lambda - f)^{-1}$ is analytic at $\infty$ (taking the value 0 there) showing that $\sigma_{\mathcal{A}}(f) \subset \overline{B}_{\|f\|}(0)$.

Next take an arbitrary $\lambda_0 \in \rho(f)$ and define

$$h(\lambda) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 - f)^{-(n+1)}$$

where this is defined. Now

$$\left\|(\lambda_0 - \lambda)(\lambda_0 - f)^{-1}\right\| \leq |\lambda_0 - \lambda| \left\|(\lambda_0 - f)^{-1}\right\|$$

which means that $h(\lambda)$ is defined whenever

$$|\lambda_0 - \lambda| \left\|(\lambda_0 - f)^{-1}\right\| < 1$$

$$|\lambda_0 - \lambda| < \frac{1}{\|(\lambda_0 - f)^{-1}\|};$$

that is, whenever, $\lambda_0 \in B_{\|(\lambda_0 - f)^{-1}\|^{-1}}$. This shows that $(\lambda - f)^{-1}$ is holomorphic at each point and that the resolvent is open. Consequently, the spectrum is closed.

Suppose that the spectrum is empty. Then for an arbitrary bounded linear functional $L$, $L((\lambda - f)^{-1})$ is a bounded holomorphic function defined on the (extended) complex plane. By Liouville's theorem, it is constant and as it equals 0 at infinity, $L((\lambda - f)^{-1}) = 0$ for each $L$. By the Hahn-Banach theorem, $(\lambda - f)^{-1} = 0$, but $(\lambda - f)^{-1}(\lambda - f) = 1$, which is a contradiction to the spectrum being empty. ■

**Lemma 3.1.2.** *Gelfand-Mazur Theorem. [32], 1.4. A unital Banach algebra $\mathcal{A}$ which is a division algebra is isometrically isomorphic to $\mathbb{C}$.*

*Proof.* Suppose that $\mathcal{A}$ is a division algebra, and let $a$ be an arbitrary element in it. Then there is some $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is singular. Because $\mathcal{A}$ is a division algebra, $a - \lambda 1 = 0$, so $a = \lambda 1$. Therefore, $\mathcal{A} \cong \mathbb{C}$. ■

**Remark 3.1.3.** *[32], 2.1. Recall that any proper ideal of a unital ring is contained in a maximal ideal (by Zorn's lemma applied to the proper ideals) and that an ideal $J$ (in a commutative unital ring $\mathcal{A}$) is maximal if and only if the ring $\mathcal{A}/J$ is a field.*

**Lemma 3.1.4.** *[32], 2.2. Every maximal ideal of a unital Banach algebra $\mathcal{A}$ is closed.*

*Proof.* Note that an element $f \in A$ such that $\|1 - f\| < 1$ is invertible since then

$$(1 - f)(1 + f + f^2 + \cdots) = 1,$$

so a singular element $f$ must have $\|1 - f\| \geq 1$.

Let $I$ be a proper ideal in $\mathcal{A}$; for each element $f \in I$, $\|1 - f\| \geq 1$. Then any $f$ in the closure $\overline{I}$ must also satisfy $\|1 - f\| \geq 1$ by continuity of the norm, so $\overline{I}$ is also a proper ideal. Consequently, for a maximal ideal $\mathcal{M}$ its closure $\overline{\mathcal{M}}$ is proper, so by maximality $\overline{\mathcal{M}} = \mathcal{M}$ and $\mathcal{M}$ is closed. ∎

**Corollary 3.1.5.** *[32], 2.2. If $J$ is a maximal ideal in a commutative unital Banach algebra $\mathcal{A}$, then $\mathcal{A}/J \cong \mathbb{C}$.*

*Proof.* As $J$ is maximal, $J$ is closed by (3.1.4). Consequently $\mathcal{A}/J$ is a Banach space when equipped with the norm

$$\|f + J\| = \inf \{\|f + g\| : g \in J\}.$$

Now as

$$\{\|(f + k_1)(g + k_2)\| : k_1, k_2 \in J\} = \{\|fg + k)\| : k + J\}$$

since $(f + k_1)(g + k_2) = fg + (k_1 g + k_2 f + k_1 k_2)$

$$\|fg + J\| = \inf \{\|fg + k\| : k \in J\} = \inf \{\|(f + k_1)(g + k_2)\| : k_1, k_2 \in J\}$$

$$\|fg + J\| \leq \inf \{\|f + k_1\| \|g + k_2\| : k_1, k_2 \in J\} = \inf \{\|f + k_1\| : k_1 \in J\} \inf \{\|g + k_2\| : k_2 \in J\}$$

$$\|fg + J\| \leq \|f + J\| \|g + J\|,$$

it follows that $\mathcal{A}/J$ is a Banach algebra. Now $\|1 + k\| \geq 1$ for all $k \in J$, so $\|1 + J\| = 1$. Then $1 + J$ is an identity for $\mathcal{A}/J$. Now because $\mathcal{A}$ is commutative and $J$ is maximal, $J$ is a prime ideal and $\mathcal{A}/J$ is a division algebra. So $\mathcal{A}/J$ is a unital Banach algebra which is a division algebra, so it is isometrically isomorphic to $\mathbb{C}$ by Gelfand-Mazur (3.1.2). ∎

**Proposition 3.1.6.** *[32], 2.3. Let $\mathcal{A}$ be a unital commutative Banach algebra $\mathcal{A}$. The correspondence $\phi \mapsto ker(\phi)$ is a bijective correspondence of non-zero (algebra) homomorphisms $\mathcal{A} \to \mathbb{C}$ and maximal ideals of $\mathcal{A}$.*

*Proof.* Let $J$ be a maximal ideal of $\mathcal{A}$. This ideal is prime by commutativity, so $\mathcal{A}/J$ is isomorphic to $\mathbb{C}$ by (3.1.5) and we get a homomorphism $\phi_J : \mathcal{A} \to \mathbb{C}$.

$$\begin{array}{ccc} \mathcal{A} & & \\ \downarrow q & \searrow^{\phi_J} & \\ \frac{\mathcal{A}}{J} & \xrightarrow{\cong} & \mathbb{C} \end{array}$$

On the other hand, let $\phi$ be a non-zero homomorphism from $\mathcal{A}$ to $\mathbb{C}$. Then $\mathcal{A}/\ker(\phi)$ is isomorphic to $\mathbb{C}$, so $\ker(\phi)$ is a maximal ideal since $\mathbb{C}$ only has trivial (complex) subalgebras.

■

**Proposition 3.1.7.** *[32], 2.4. Each (non-zero) homomorphism $\phi$ of a unital Banach algebra is continuous, having a norm $\|\phi\| = 1$.*

*Proof.* First of all, $\phi(1) = \phi(1^2) = \phi(1)^2$, so $\phi(1) = 0$ or $\phi(1) = 1$. As $\phi$ is non-zero, $\phi(1) = 1$.

Choose some $f \in A$ and assume that $\lambda \in \mathbb{C}$ has modulus $|\lambda| > \|f\|$. Then as $\lambda - f$ is invertible,

$$\phi((\lambda - f)(\lambda - f)^{-1}) = \phi(\lambda - f)\phi((\lambda - f)^{-1}) = 1$$

which means that $\phi(\lambda - f) \neq 0$; consequently, $\phi(f) \neq \lambda$. As $\lambda$ was an arbitrary complex number having a modulus bigger than $\|f\|$, $\phi(f) \leq \|f\|$, and $\|\phi\| = 1$. Therefore, $\phi$ is continuous. ■

**Definition 3.1.8.** *[32], 2.5. Let $\mathcal{A}$ be a commutative unital Banach algebra. Since its* **maximal ideal space** *$M_{\mathcal{A}}$ precisely corresponds to its non-zero multiplicative functionals, $M_{\mathcal{A}}$ inherits a weak* topology from the unit ball of algebra's dual space $\mathcal{A}^*$. This topology is known as the* **Gelfand topology***, and $M_{\mathcal{A}}$ is a compact space under it, since the weak* limit of homomorphisms satisfying $\phi(1) = 1$ is also a homomorphism and the unit ball of $\mathcal{A}^*$ is weak* compact by Alaoglu's theorem.*

**Definition 3.1.9.** *[32], 2.6. The* **Gelfand transform** *is the linear operation which carries* *each element $a \in \mathcal{A}$ to the function on the maximal ideal space*

$$\phi \mapsto \hat{a}(\phi) = \phi(a).$$

*This homomorphism embeds $\mathcal{A}$ into $C(M_{\mathcal{A}})$ in the unital case, $C_0(M_{\mathcal{A}})$ in the non-unital case. Also $\left\|\hat{f}\right\| \leq \|f\|$.*

**Definition 3.1.10.** *A* **uniform algebra** *is a commutative Banach algebra for which the Gelfand transform is isometric. That is, $\|f\| = \left\|\hat{f}\right\|$.*

**Remark 3.1.11.** *[32], §5.3. This is actually equivalent to the condition that $\|f^2\| = \|f\|^2$ for each $f \in \mathcal{A}$.*

*Proof.* For the one direction, use that

$$\|f^2\| = \left\|\hat{f^2}\right\| = \left\|\hat{f}^2\right\| = \left\|\hat{f}\right\|^2 = \|f\|^2 ;$$

for the other note that $2^n$ is a cofinal subset of $\mathbb{N}$ and that $\left\|f^{2^n}\right\| = \|f\|^{2^n}$. Now (see [32], §5.2)

$$\left\|\hat{f}\right\| = \lim_{n\to\infty} \left\|\hat{f}^n\right\|^{1/n} = \|f\| .$$

∎

**Example 3.1.12.** *For a subalgebra of $C(X)$ equipped with its supremum norm $\|\cdot\|_\infty$ this requires it to be closed under uniform convergence. Both the algebra $C_b(X)$ of bounded continuous function on a Hausdorff space $X$ and the algebra $H^\infty(\mathcal{R})$ of bounded holomorphic functions on a Riemann surface are uniform algebras.*

**Definition 3.1.13.** *[33] For the uniform algebra defined by $H^\infty(D)$ for some domain $D \subseteq \mathbb{C}$, the* **fibre** *$\mathcal{M}_x$ of the maximal ideal space consists of those multiplicative functionals which carry the function $z$ to the point $x \in \overline{D}$. They partition $\mathcal{M}_x$ since $z - \phi(z)$, being in the kernel of $\phi$, must be singular. This only occurs when $x$ has a distance from $D$ of 0.*

## 3.2 The Hyperbolic Plane

We alternate a good deal between the Poincaré disk model of hyperbolic space and the upper halfplane model. This is because questions in one will be equivalent to questions in the other and often a question will be more easily analyzed in one model than in the other. So in each of them there is

- the domain which models hyperbolic 2-space,
- a distance metric $ds$,
- a pseudohyperbolic metric $\rho$ which is often easier to carry out computations in,
- and maps between them.

| Symbol | Space | Pseudodistance $\rho(a,b)$ | Metric $ds$ | Transformation |
|---|---|---|---|---|
| $\mathbb{D}$ | $\{z \in \mathbb{C} \| \|z\| < 1\}$ | $\left\|\frac{a-b}{1-\bar{a}b}\right\|$ | $\frac{dz}{1-\|z\|^2}$ | $z \mapsto i\frac{1-z}{1+z}$ |
| $\mathcal{H}$ | $\{z \in \mathbb{C} \| \mathrm{Im}\ z > 0\}$ | $\left\|\frac{a-b}{a-\bar{b}}\right\|$ | $\frac{dw}{\mathrm{Im}(w)^2}$ | $w \mapsto \frac{i-w}{i+w}$ |

**Definition 3.2.1.** *( [37],21) The Hardy-Littlewood maximal function $Mf : \mathbb{R} \to \mathbb{R}$ of a function $f : \mathbb{R} \to \mathbb{R}$ which is locally integrable on $\mathbb{R}$ is*

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)|\, dt,$$

*where $I$ ranges over open intervals containing $x$.*

**Theorem 3.2.2.** *The Hardy-Littlewood maximal theorem. If $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, then $Mf(t)$ is finite almost everywhere.*

1. *If $f \in L^1(\mathbb{R})$, then $Mf$ is weak $L^1$.*

$$|\{t \in \mathbb{R}|Mf(t) > \lambda\}| \leq \frac{2}{\lambda} \|f\|_1$$

*for $\lambda > 0$.*

2. *If $f \in L^p(\mathbb{R})$, for $1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R})$ and*

$$\|Mf\|_p \leq A_p \|f\|_p$$

*where $A_p$ depending only on $p$.*

*Proof.* Following Garnett ( [37],24), to prove (1) assume that $f \in L^1$, and let $\lambda > 0$. Then the set $E_\lambda = \{t|Mf(t) > \lambda\}$ is open, and therefore measurable. For each $t \in E_\lambda$, take an open interval $I \ni t$ such that

$$\frac{1}{|I|} \int_I |f|\, ds > \lambda;$$

equivalently,

$$|I| < \frac{1}{\lambda} \int_I |f|\, ds. \tag{3.1}$$

Take $K$ to be a compact subset of $E_\lambda$ which may be then covered by the intervals $I_1, \ldots, I_n$. By Vitali's theorem, we can obtain pairwise disjoint intervals $J_1, \ldots, J_m$ satisying the property (3.1) where

$$\left|\bigcup_{j=1}^{n} I_j\right| \leq 2 \sum_{j=1}^{m} |J_j|.$$

This means that

$$|K| \leq \left|\bigcup_{j=1}^{n} I_j\right| \leq 2 \sum_{j=1}^{m} \frac{1}{\lambda} \int_{J_j} |f|\, ds \leq \frac{2}{\lambda} \int |f|\, ds.$$

By the inner regularity of Lebesgue measure with respect to the compact sets,

$$|E_\lambda| \leq \frac{2}{\lambda} \int |f|\, ds$$

which proves that $Mf$ is weak $L^1$.

To show (2), start by showing that each of the conditions needed for the Marcinkiewicz interpolation theorem holds.

- $Mf$ is subadditive since

$$M(f+g)(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f+g|\, dt \le \sup_{I \ni x} \frac{1}{|I|} \int_I |f|\, dt + \sup_{I \ni x} \frac{1}{|I|} \int_I |g|\, dt$$

- The earlier part (1), shows that

$$|\{x||(Mf)(x)| > \lambda\}| \le \frac{2}{\lambda} \|f\|_1$$

  for $f \in L^1$.

- Since

$$\|Mf\|_\infty \le \|f\|_\infty ,$$

  the final condition holds as well.

Now by the Marcinkiewicz interpolation theorem,

$$\|Mf\|_p \le A_p \|f\|_p$$

for $1 < p \le \infty$, which is what was needed. Also, it then follows that $Mf < \infty$ almost everywhere. ■

**Definition 3.2.3.** *( [37],30) A measure $\sigma$ on $\mathcal{H}$ which has a constant $N(\sigma)$ such that*

$$\sigma(Q) \le N(\sigma)h$$

*on squares $Q$ taking the form*

$$Q_{x_0,h} = \{(x,y)|x_0 < x < x_0 + h, 0 < y < h\}$$

*is called a* **Carleson measure**. *On the Poincaré disk $\mathbb{D}$ this is a measure having a constant $N(\sigma)$ such that*

$$\sigma(U) \leq N(\sigma)\delta$$

*on domains of the form*

$$U_{\theta_0,\delta} = \left\{re^{i\theta}|1-\delta < r < 1, \theta_0 - \delta < \theta < \theta_0 + \delta\right\}.$$

## 3.3 Möbius Transformations and Fuchsian Groups

**Definition 3.3.1.** *The transformations $S^2 \to S^2$ which take the form*

$$z \mapsto \frac{az+b}{cz+d}$$

*are the Möbius transformations. In projective coordinates they take the form*

$$\begin{bmatrix} z \\ w \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}.$$

*They describe all of the conformal self-maps of the sphere $S^2$.*

**Remark 3.3.2.** *Let $U$ be a domain in $S^2$; the Möbius transformations which carry $U$ into itself and its boundary $\partial U$ into itself are the conformal self-maps of $U$. These are a subgroup of the Möbius transformations.*

**Definition 3.3.3.** *A* **Fuchsian group** *is a discrete subgroup of the projective special linear group $PSL_n(\mathbb{R})$.*

**Remark 3.3.4.** *The group $PSL_2(\mathbb{R})$ describes the conformal self-maps of the upper half plane $\mathcal{H}$. Given an element $A \in PSL_2(\mathbb{R})$, its action on a complex number $z$ is given by*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}.$$

## 3.4 Characters

**Definition 3.4.1.** *[52], VI, §4. Let $G$ be a monoid, and $K$ be a field. A* **character** *of $G$ in $K$ is a homomorphism $\chi : G \to K^*$ from $G$ to the multiplicative group of $K$.*

**Example 3.4.2.** *[52], VI, §4. The* **trivial character** *$\chi_1$ takes the constant value 1.*

**Example 3.4.3.** *[52], VI, §4. Continuous characters of $\mathbb{R}/\mathbb{Z}$ in $\mathbb{C}$ are given by*

$$f_m(x) = e^{2\pi i m x}$$

*for $m \in \mathbb{Z}$.*

**Theorem 3.4.4.** *[52], VI, §4, Thm. 4.1. Let $G$ be a monoid, $K$ a field. Let $\chi_1, \ldots, \chi_n$ be distinct characters of $G$ in $K$. Then they are linearly independent over $K$.*

**Definition 3.4.5.** *[62] Let $\Gamma$ be a Fuchsian group and $f \in H^\infty(\mathbb{D})$ be a bounded holomorphic function satisfying*

$$f(\gamma(z)) = f(z)\chi(z)$$

*for some character $\chi : \Gamma \to S^1$. Then $f$ is* **character-automorphic**. *Furthermore, the algebra of holomorphic functions which are character-automorphic for the character $\chi$ of the Fuchsian group $\Gamma$ is denoted*

$$H^\infty(\Gamma, \chi) = \{f \in H^\infty(\mathbb{D}) : f(\gamma(z)) = f(z)\chi(\gamma) \text{ for all } \gamma \in \Gamma\}.$$

**Example 3.4.6.** *If* $\Gamma$ *is a Fuchsian group of convergent type, then the Blaschke product*

$$B(z) = \prod_{\gamma \in \Gamma} \frac{|\gamma(0)|}{\gamma(0)} \frac{z + \gamma(0)}{1 + \overline{\gamma(0)} z}$$

*is character-automorphic.*

*Proof.* Define the character $\chi : \Gamma \to S^1$ by

$$\chi(\zeta) = \prod_{\gamma} \frac{|\gamma(\zeta^{-1}(0))| \, \gamma(0)}{\gamma(\zeta^{-1}(0)) \, |\gamma(0)|}.$$

This is a character of $\Gamma$ in $S^1$ since

$$\chi(\zeta\xi) = \prod_{\gamma \in \Gamma} \frac{|\gamma(\xi^{-1}(\zeta^{-1}(0)))| \, \gamma(0)}{\gamma(\xi^{-1}(\zeta^{-1}(0))) \, |\gamma(0)|} = \prod_{\gamma \in \Gamma} \frac{|\gamma(\xi^{-1}(\zeta^{-1}(0)))| \, \gamma(\xi^{-1}(0))}{\gamma(\xi^{-1}(\zeta^{-1}(0))) \, |\gamma(\xi^{-1}(0))|}$$

$$\chi(\zeta\xi) = \prod_{\gamma \in \Gamma} \frac{|\gamma(\xi^{-1}(0))| \, \gamma(0)}{\gamma(\xi^{-1}(0)) \, |\gamma(0)|} \prod_{\gamma \in \Gamma} \frac{|\gamma(\zeta^{-1}(0))| \, \gamma(0)}{\gamma(\zeta^{-1}(0)) \, |\gamma(0)|} = \chi(\zeta)\chi(\xi).$$

Next note that for $\zeta \in \Gamma$,

$$B(\zeta(z)) = \prod_{\gamma \in \Gamma} \frac{|\gamma(0)|}{\gamma(0)} \gamma(\zeta(z)) = \prod_{\gamma \in \Gamma} \frac{|\gamma(\zeta^{-1}(0))|}{\gamma(\zeta^{-1}(0))} \gamma(z),$$

$$B(\zeta(z)) = \prod_{\gamma \in \Gamma} \frac{|\gamma(\zeta^{-1}(0))| \, \gamma(0)}{\gamma(\zeta^{-1}(0)) \, |\gamma(0)|} \prod_{\gamma \in \Gamma} \frac{|\gamma(0)|}{\gamma(0)} \gamma(z),$$

$$B(\zeta(z)) = \chi(\zeta) B(z).$$

■

## 3.5 Riemann Surfaces and Green's Functions

**Definition 3.5.1.** *[28], 1.1. Let* $X$ *be a two-dimensional manifold. A* **complex chart** *is a homeomorphism* $\phi : U \to V$ *where* $U$ *is open in* $X$ *and* $V$ *is open in* $\mathbb{C}$*. Two complex charts*

*$\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are* **holomorphically compatible** *if*

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

*is biholomorphic.*

*A* **complex atlas** *on $X$ is a system $\mathfrak{A} = \{\phi_i : U_i \to V_i, i \in I\}$ of charts which are holomorphically compatible and cover X; that is, $\cup_{i \in I} U_i = X$.*

*Two atlases $\mathfrak{A}, \mathfrak{A}'$ are analytically equivalent if each chart of $\mathfrak{A}$ is holomorphically compatible with every chart of $\mathfrak{A}'$.*

**Definition 3.5.2.** *[28], 1.3. A* **complex structure** *on a two-dimensional manifold $X$ is an equivalence class of analytically equivalent atlases in $X$. The equivalence class of an atlas corresponds to the unique maximal atlas containing it. (The maximal atlas $\mathfrak{A}^*$ containing an atlas $\mathfrak{A}$ consists of all complex charts which are holomorphically compatible with all of the charts in $\mathfrak{A}$.)*

**Definition 3.5.3.** *[28], 1.4. A* **Riemann surface** *is a pair $(X, \Sigma)$ where $X$ is a connected two-dimensional manifold and $\Sigma$ is a complex structure on it. A chart on a Riemann surface means a chart in its maximal atlas.*

**Definition 3.5.4.** *[28], 1.9. Suppose $X, Y$ are Riemann surfaces. Then a continuous mapping $f : X \to Y$ is* **holomorphic** *if for each pair of charts where $\psi_1 : U_1 \to V_1$ is on $X$ and $\psi_2 : U_2 \to V_2$ is on $Y$,*

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2$$

*is holomorphic (on $\mathbb{C}$).*

**Definition 3.5.5.** *[28], 1.12. Let $Y$ be an open subset of a Riemann surface $X$. A* **meromorphic function** *$f$ on $Y$ is a holomorphic function $f : Y' \to \mathbb{C}$ such that*

- *$Y \setminus Y'$ contains only isolated points,*

- *for each point $p \in Y \setminus Y'$,*

$$\lim_{x \to p} |f(x)| = \infty.$$

*The points of $Y \setminus Y'$ are the* **poles** *of $f$.*

**Definition 3.5.6.** *[36], 422. A subsurface $R$ of a Riemann surface $S$ having boundary $\partial R$ is a* **finite bordered subsurface** *if $R \cup \partial R$ is a compact subset of $S$ and $\partial R$ consists of a finite number of disjoint simple closed analytic curves in $S$.*

**Definition 3.5.7.** *[36], 441. An analytic map $\pi : S \to R$ where $R, S$ are Riemann surfaces is called a* **covering map** *if each point $p \in R$ has an open neighbourhood $U$ whose inverse image $\pi^{-1}(U)$ is a union of disjoint open subsets $V_\alpha$ of $S$ each of which is mapped injectively by $\pi$ into $U$.*

*The triple $(S, \pi, R)$ is now called a* **covering space***.*

*The inverse image $\pi^{-1}(\{x\})$ of a point $x$ is called the* **fibre over** *$x$. A map $f$ from $S \to S$ such that $p \circ f = p$ is* **fibre-preserving***.*

**Theorem 3.5.8.** *[28], 4.6. Let $Y$ be a Hausdorff topological space, $X$ a Riemann surface and $p : Y \to X$ a local homeomorphism. There exists a unique complex structure $\Sigma$ for $Y$ such that $p$ is holomorphic.*

**Theorem 3.5.9.** *[28], 4.9. Let $X, Y, Z$ be Riemann surfaces, $p : Y \to X$ a holomorphic map, $f : Z \to X$ a holomorphic map, and $g : Z \to Y$ a continuous map such that the following diagram commutes:*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & \searrow^{f} & \downarrow{p} \\ & & X \end{array}$$

*Then $g$ is holomorphic.*

*Proof.* Choose an arbitrary $c \in Z$, let $b = g(c)$ and $a = p(b)$. Also choose open neighbourhoods $U, V$ of $a, b$, respectively such that $p|_V$ a homeomorphism. By the continuity of $g$, there is a neighbourhood $W$ of $c$ such that $g(W) \subseteq V$. Now as $p|_V \circ g|_W = f|_W$ and $p|_V$ is a homeomorphism from $V$ to $U$, it now follows that $g|_W = (p|_V)^{-1} \circ f|_W$ and $g$ is holomorphic since it is a composition of holomorphic functions. ■

**Corollary 3.5.10.** *Let $(S, \pi, R)$ be a covering space of Riemann surfaces. Then any fibre-preserving map $f : S \to S$ is holomorphic. Also, any fibre-preserving homeomorphism $f : S \to S$ is biholomorphic (conformal).*

*Proof.* Form the diagram $\begin{array}{ccc} S & \xrightarrow{f} & S \\ & \searrow^{\pi} & \downarrow^{\pi} \\ & & R \end{array}$ where $\pi$ is unbranched and holomorphic and $f$ is continuous. Consequently, $f$ is holomorphic.

If $f$ is also a homeomorphism, then $f^{-1}$ satisfies the conditions as well, which means that $f$ is conformal. ■

**Example 3.5.11.** *Let $S$ be the vertical strip in the plane $\mathbb{C}$*

$$S = \{z \in \mathbb{C} | a < Re(z) < b\}$$

*and $R$ be the annulus*

$$R = \left\{z \in \mathbb{C} \middle| e^a < |z| < e^b\right\}.$$

*Then $\pi : S \to R$ defined as $z \mapsto e^z$ is a covering map, and $(S, \pi, R)$ is a covering space.*

**Definition 3.5.12.** *The* **universal cover** *of a Riemann surface is a triple $(S, \pi, R)$ which is a covering space in which $S$ is simply connected. It is one of the Riemann sphere $S^2$, the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$, unique up to conformal equivalence.*

*Proof.* As a topological space $R$ has a universal cover $S$ where $\pi$ is a local homeomorphism since complex manifolds are clearly locally simply connected. So the universal covering constructed ( [41], 63) is applicable here. By (3.5.8) $S$ is a Riemann surface. ■

**Definition 3.5.13.** *A* **covering transformation** *$\phi$ of a covering space $(S, \pi, R)$ (also known as a* **deck transformation***) is an injective analytic map $\phi : S \to S$ such that $\pi \circ \phi = \pi$.*

*That is, the following diagram commutes:*

$$\begin{array}{ccc} S & \xrightarrow{\phi} & S \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi} & \\ R & & \end{array}$$

*The deck transformations form a discrete subgroup of the conformal self-maps of $S$.*

*Proof.* The universal cover is a Riemann surface, so by (3.5.10), its deck transformations are conformal self-maps of $S$. Now the composition of deck transformations is a deck transformation and deck transformations are invertible, so they form a group. They act discontinuously, so they comprise a discrete subgroup of the conformal self-maps. ∎

**Example 3.5.14.** *In example 3.5.11 where $R$ is an annulus, $S$ is a vertical strip, and the projection map $\pi$ is $z \mapsto e^z$, the deck transformations are the vertical translations $z \mapsto z + 2k\pi i$ where $k$ is an integer.*

**Theorem 3.5.15.** *Uniformization Theorem. [36], 439. Each simply connected Riemann surface is conformally equivalent to one of the open unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$ or the Riemann sphere $S^2$.*

**Definition 3.5.16.** *A Riemann surface $R$ is Carathéodory hyperbolic if $H^\infty(R)$ separates the points of $R$.*

**Proposition 3.5.17.** *Any Carathéodory hyperbolic Riemann surface $R$ has the open unit disk as its universal cover $(S, r, R)$.*

*Proof.* Let $f$ be a non-constant bounded analytic function on $R$. Then $f \circ r$ is a non-constant bounded analytic function on the universal cover $S$. But neither $\mathbb{C}$ nor $S^2$ admit any, so the universal cover must be $\mathbb{D}$ (up to conformal equivalence). ∎

A consequence of this is that the bounded analytic maps on $\mathcal{R}$ correspond to the bounded analytic functions on the disk which are invariant under $\Gamma$: that is,

$$r^*(H^\infty(\mathcal{R}) \cong H^\infty(\Gamma, \chi_1) = H^\infty_\Gamma(\mathbb{D}).$$

**Definition 3.5.18.** *[36], 408. A* **Green's function** *for a bounded domain $D \subset \mathbb{C}$ having a pole at $\zeta$ ($G(z,\zeta)$) is a real-valued function on $D$ satisfying the following properties:*

- *$G(z,\zeta)$ must be harmonic (in $z$) on $D \setminus \{\zeta\}$,*
- *$G(z,\zeta) - \log\left(\frac{1}{|z-\zeta|}\right)$ must be harmonic (in $z$) at $z = \zeta$, and*
- *$G(z,\zeta)$ must tend to zero as $z$ approaches the boundary of the domain $\partial D$.*

**Example 3.5.19.** *[36], 408. The Green's function for the unit disk with a pole at $z = 0$ is*

$$G(z,0) = \log \frac{1}{|z|}.$$

*Then*

$$G(z,\zeta) = \log \frac{1}{|\rho(z,\zeta)|} = -\log|\rho(z,\zeta)|\,,$$

$$G(z,\zeta) = -\log \frac{|z-\zeta|}{|1-\bar{\zeta}z|}.$$

**Definition 3.5.20.** *[36], 410. A point $z_0$ is a critical point of the Green's function $G(z,\zeta)$ if the gradient (with respect to $z$) $\nabla_z G(z,\zeta) = 0$.*

**Definition 3.5.21.** *The Green's function $g(z,\zeta)$ of an arbitrary Riemann surface $\mathcal{R}$ exhausted by a sequence of finite bordered Riemann surfaces $\mathcal{R}_n$ equipped with Green's functions $g_n(z,\zeta)$ is*

$$g(z,\zeta) = \lim_n g_n(z,\zeta).$$

**Proposition 3.5.22.** *[62] If $\mathcal{R}$ is a Riemann surface having $\mathbb{D}$ as its universal cover, and $\Gamma \cong \pi_1(\mathcal{R})$ as its associated Fuchsian group, then the Green's function of $\mathcal{R}$ having a pole at*

*$\pi(0)$ is given by $g(\pi(z)) = -\log|B(z)|$ where $B(z)$ is the Blaschke product*

$$B(z) = \prod_{\gamma\in\Gamma} \frac{|\gamma(0)|}{\gamma(0)}\gamma(z).$$

*Proof.* Since $B(z)$ is holomorphic, $u(z) = -\log|B(z)|$ is a harmonic function on $\mathbb{D}$ except where $B(z)$ has a root. A consequence of the relation $B(\gamma(z)) = B(z)\chi(\gamma)$ (see 3.4.6), where $\chi$ is a character of $\Gamma$ in $\mathbb{C}$, is that $|B(\gamma(z))| = |B(z)|$. This means that $u(z)$ is $\Gamma$-invariant, so it induces a function $g(z)$ on the Riemann surface $\mathcal{R}$ which is harmonic except at $\pi(0)$.

Next, $g(z)$ has a logarithmic pole at $\pi(0)$, which can be ascertained from the fact that $B$ has a simple pole at $z = 0$. The Blaschke product $B(z)$ has a modulus $|B(z)| = 1$ on the boundary of the disc, so $u(z) \to 0$ as $z \to \partial D$. ∎

**Definition 3.5.23.** *[46], 295.*

*A connected non-parabolic Riemann surface $\mathcal{R}$ having a Green's function $G$ is of* **Widom type** *if*

$$\int_0^\infty b(t)\,dt < \infty, \tag{3.2}$$

*where $b(t)$ is the dimension of the first homology group $H_1(\{x \in X : G(x) > t\})$. Since for a Riemann surface $\mathcal{S}$, the positive integer dim$(H_1(\mathcal{S}))$ is exactly twice the genus, this means that the genus of $\{z : G(z) > t\}$ grows slowly as $t \to 0$.*

**Remark 3.5.24.** *Since dim$(H_1(\mathcal{R}))$ is a positive integer, for the integral (3.2) to converge it is necessary that there exist some $t_0$ for which dim$(H_1(\{z : G(z) > t\})) = 0$ whenever $t \geq t_0$. Also note that if $U_j$ are the open connected components of $\{z : G(z) > t\}$ (for a fixed $t$), that*

$$H_1(\{z : G(z) > t\}) = \bigoplus_j H_1(U_j)$$

*which means that for each component $U_j$, dim$(U_j) = 0$ so $U_j$ is simply connected. So for any $t > t_0$, $\{z : G(z) > t\})$ is a disjoint union of simply connected open sets. Note further that* *any point $z \in U_j$ has each element of its fibre in a different $U_j$. This occurs because each element $\gamma$ of the Fuchsian group is continuous and bijective. Therefore, each $U_j$ is carried into some $U_k$ (where $j \neq k$ whenever $\gamma$ is not the identity element). For the same reason, each $U_j$ belongs to a different sheet.*

**Remark 3.5.25.** *[46], Lemma 2.11. This property is closely related to regularity of the Riemann surface's Green's function. Using exhaustion by finite bordered Riemann surfaces, Jones and Marshall show that any Widom type Riemann surface can be made regular by the addition of a countable number of points.*

## 3.6 Special Ring Properties

**Definition 3.6.1.** *[45], I.2.2. Let $R$ be ring with identity such that for every free $R$-module $F$, any two bases of $F$ have the same cardinality. Then $R$ has the* **invariant dimension property** *and the cardinal number of any basis of $F$ is called the* **dimension** *of $F$ over $R$.*

**Definition 3.6.2.** *[19], 17. A commutative unital ring $R$ is* **projective free** *if it has the invariant dimension property and every finitely-generated projective $R$-module is free.*

**Proposition 3.6.3.** *[19], 18. A commutative unital ring $R$ with the invariant dimension property is projective free if and only if every $n \times n$ idempotent matrix in $R$ is similar to a matrix of the form $I_r \oplus 0_{n-r}$ (for some $0 \leq r \leq n$).*

*Proof.* Suppose that $R$ is projective free and that $e : R^n \to R^n$ corresponds to an idempotent matrix $E$. Then $R^n = e(R^n) \oplus (1-e)(R^n)$ and each of $e(R^n)$ and $(1-e)(R^n)$ are projective. This means that there are isomorphism $e(R^n) \to R^r$ and $e(R^n) \to R^s$ which induce an isomorphism $e(R^n) \oplus (1-e)(R^n) \to R^r \oplus R^s$ so I obtain the diagram

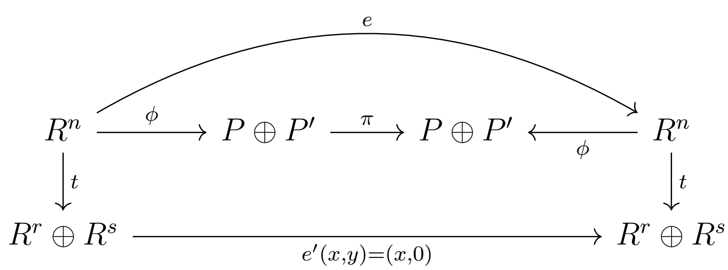
$$\begin{array}{ccccc} R^n & \longrightarrow & e(R^n) \oplus (1-e)(R^n) & \xrightarrow{\phi} & R^r \oplus R^s \\ \downarrow{\scriptstyle e} & & & & \downarrow{\scriptstyle e'} \\ R^n & \longleftarrow & e(R^n) \oplus (1-e)(R^n) & \xrightarrow{\phi} & R^r \oplus R^s \end{array}$$

where $e' = \phi^{-1} \circ e \circ \phi$. But $e'$ just projects on its first factor, so its matrix $E'$ takes the form

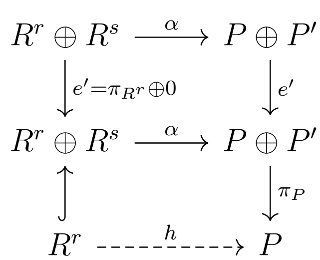
$$E' = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

so that the matrix of $e$ is $\Phi E' \Phi^{-1}$ where $\Phi$ represents $\phi$. So $E$ is diagonalizable as required.

Now suppose that every idempotent matrix ring over $R$ is diagonalizable and that $P$ is a finitely generated projective $R$-module. That is, there exists an $R$-module $P'$ such that $P \oplus P' \cong R^n$ for some $n$ (via the isomorphism $\phi$). Let $\pi$ be the projection $P \oplus P' \to P \oplus P'$ onto the first factor and let $e$ be the idempotent $e = \phi^{-1} \circ \pi\phi$. The isomorphism $t : R^n \to R^r \oplus R^s$ making the diagram commute exists by the diagonalizability of the matrix $E$ corresponding to $e$.




I next obtain the diagram (where $\alpha$ is the isomorphism $\phi \circ t^{-1}$)




so I will take $h : R^r \to P$ to be the morphism $h(x) = \pi_P \circ \alpha(x, 0)$. If I show that $h$ is an isomorphism, then I am done. First suppose that $h(x) = h(y)$; that is, that

$$\pi_P \alpha(x, 0) = \pi_P \alpha(y, 0)$$

$$(\pi_P \oplus 0)(\alpha(x, 0)) = (\pi_P \oplus 0)(\alpha(y, 0))$$

By commutativity of the graph $\alpha \circ e' = (\pi_P \oplus 0) \circ \alpha$

$$\alpha(e'(x,0)) = \alpha(e'(y,0))$$

$$\alpha(x,0) = \alpha(y,0)$$

and as $\alpha$ is monic, $x = y$. As $x$ and $y$ were arbitrary, $h$ is monic.

Now let $p \in P$. Then since $\alpha$ is epic, there exists $(x,y) \in R^r \oplus R^s$ such that

$$\alpha(x,y) = (p,0)$$

$$(\pi_P \oplus 0)\alpha(x,y) = (\pi_P \oplus 0)(p,0) = (p,0)$$

and by commutativity of the diagram

$$\alpha(e'(x,y)) = (p,0)$$

$$\alpha(x,0) = (p,0).$$

That is, $h(x) = p$. As $p$ was arbitrary, $h$ is epic. So

$$h : R^r \to P$$

is an isomorphism and $P \cong R^r$ for some $r \in \mathbb{N}$ as required. ■

**Definition 3.6.4.** *[19], 15. A commutative unital ring $R$ is* **Hermite** *if it has the invariant dimension property and every stably free $R$-module is free.*

**Definition 3.6.5.** *[19], 15. An $m \times n$ matrix $A$ over a ring $R$ is* **completable** *provided one of the following conditions holds:*

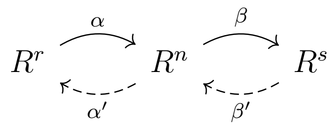
- *$m = n$ and $A$ has an inverse*

- *$m < n$ and there exists an $(n-m) \times n$ matrix $B$ such that* $\begin{bmatrix} A \\ B \end{bmatrix}$ *is invertible,*

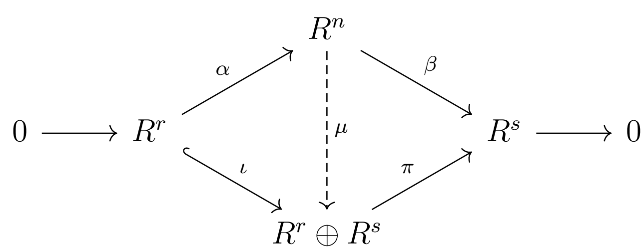
  *or*

- *$m > n$ and there exists an $m \times (m-n)$ matrix $C$ such that* $\begin{bmatrix} A & C \end{bmatrix}$ *is invertible.*

**Lemma 3.6.6.** *[19], §0, 2.4. Let $R$ be a weakly finite ring (for instance, a Hermite ring) and $\alpha, \beta$ be $R$-module homomorphisms*




*where $r + s = n$, $\beta\alpha = 0$, $\alpha$ has a left inverse $\alpha'$ and $\beta$ has a right inverse $\beta'$; that is, $\alpha'\alpha = 1_{R^r}$ and $\beta\beta' = 1_{R^s}$. Then there exists an automorphism $\mu : R^n \to R^n$ such that the follow diagram commutes and has exact rows*




*where $\iota$ is the inclusion $u \mapsto (u, 0)$ and $\pi$ is the projection $(u, v) \mapsto v$.*

*Proof.* Now each $x \in R^n$ may be expressed in the form

$$x = (x - \beta'\beta x) + \beta'\beta x$$

where

$$\ker(\beta) = \{x \in R^n : \beta(x) = 0\} = \{x - \beta'\beta x : x \in R^n\},$$

which means that $R^n = \ker(\beta) \oplus \beta'\beta(R^n)$. Also, as $\mathrm{im}(\alpha) \subseteq \ker(\beta)$

$$\ker(\beta) \cong \mathrm{im}(\alpha) \oplus \frac{\ker(\beta)}{\mathrm{im}(\alpha)}$$

so that

$$\ker(\beta) \cong \mathrm{im}(\alpha) \oplus I;$$

let $I$ be the quotient module noted. Consequently,

$$R^n \cong \ker(\beta) \oplus \beta'\beta(R^n) \cong (\mathrm{im}(\alpha) \oplus I) \oplus \beta'\beta(R^n)$$

$$R^n \cong (R^r \oplus I) \oplus R^s \cong R^n \oplus I.$$

By weak finiteness, $I = 0$. This means that the top sequence is exact at its middle position and exactness at the first and third positions follows from the existence of $\alpha'$ and $\beta'$.

Now I may construct the automorphism $\mu$:

$$\mu(u) = (\alpha'(u - \beta'\beta(u)), \beta(u)).$$

It is clear that it is a homomorphism and I can check that it makes the diagram commute:

$$\mu(\alpha(x)) = (\alpha'(\alpha(x) - \beta'\beta(\alpha(x))), \beta(\alpha(x))) = (x - 0, 0) = \iota(x),$$

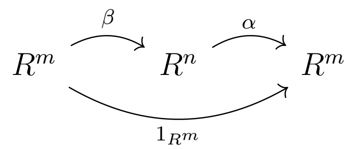
and

$$\pi(\mu(u)) = \pi\left(\alpha'(u - \beta'\beta(u)), \beta(u)\right) = \beta(u).$$

∎

**Theorem 3.6.7.** *[19], §0, 4.1. A ring $R$ is Hermite if and only if every matrix having either a left or right inverse is completable.*

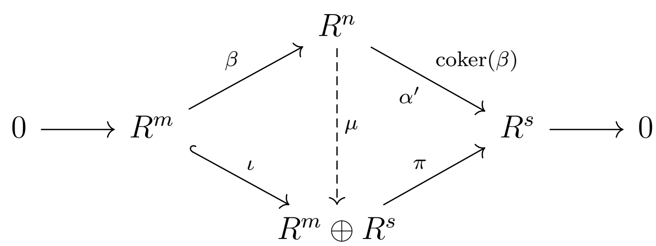
*Proof.* Let $A$, $B$ be a pair of rectangular matrices such that $AB = 1$. Call the corresponding $R$-module homomorphisms $\alpha, \beta$: they may be setup as follows.




Consider the short exact sequence

$$0 \longrightarrow R^m \xrightarrow{\beta} R^n \xrightarrow{\text{coker}(\beta)} P \longrightarrow 0.$$

Now $R^n = \text{im}(\beta) \oplus P \cong R^m \oplus P$, so $P$ is stably free. Consequently, $P \cong R^s$ where $s+m=n$. By (3.6.6), we obtain the diagram




So I will take $A'$ to be matrix for $\alpha'$ and $M$ to be the matrix for $\mu$, so that

$$A' = \begin{bmatrix} e_{m+1}^T \\ \vdots \\ e_n^T \end{bmatrix} M.$$

Now define

$$B' = M^{-1} \begin{bmatrix} e_{m+1} \cdots e_n \end{bmatrix},$$

so that

$$A'B' = \begin{bmatrix} e_{m+1}^T \\ \vdots \\ e_n^T \end{bmatrix} MM^{-1} \begin{bmatrix} e_{m+1} \cdots e_n \end{bmatrix} = I_s.$$

Consequently,

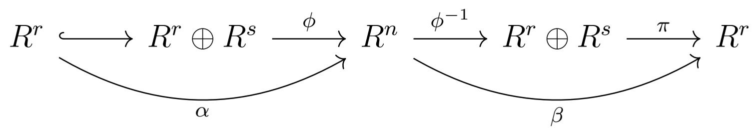
$$\begin{bmatrix} A \\ A' \end{bmatrix} \begin{bmatrix} B & B' \end{bmatrix} = \begin{bmatrix} AB & AB' \\ A'B & A'B' \end{bmatrix} = \begin{bmatrix} I_m & AB' \\ 0 & I_s \end{bmatrix}$$

and postmultiplying by the inverse of the right side

$$\begin{bmatrix} A \\ A' \end{bmatrix} \begin{bmatrix} B & B' - BAB' \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_s \end{bmatrix}.$$

Therefore, $A$ and $B$ are completable.

On the other hand, suppose $P$ is a stably finite $R$-module so that there exist $r, n \in \mathbb{N}$ such that $R^r \oplus P \cong R^n$ via the isomorphism $\phi$. Now considering the diagram



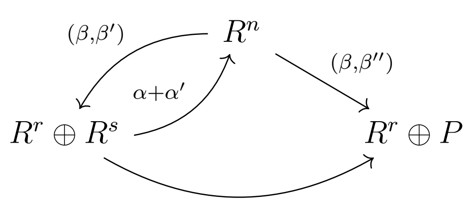

where $\beta\alpha = 1_{R^r}$, I have a pair of linear maps $\alpha$, $\beta$ whose matrices $A$, $B$ satisfy

$$BA = I_r,$$

so by completability there exist $A'$, $B'$ such that

$$\begin{bmatrix} B \\ B' \end{bmatrix} \begin{bmatrix} A & A' \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_s \end{bmatrix}.$$

I now construct the diagram




and obtain an isomorphism $R^r \oplus R^s \to R^r \oplus P$ defined as

$$(u, v) \mapsto (\beta(\alpha(u) + \alpha'(v)), \beta''(\alpha(u) + \alpha'(v))) = (u, \beta''(\alpha(u) + \alpha'(v))).$$

Next it can be shown that the $R$-module morphism $\beta'' \circ \alpha'$ is an isomorphism. It is first monic. Let

$$\beta''(\alpha'(v)) = \beta''(\alpha'(v'))$$

$$(0, \beta''(\alpha(v)) = (0, \beta''(\alpha'(v')))$$

$$\phi(0, v) = \phi(0, v')$$

and since $\phi$ is monic $v = v'$ which shows that $\beta'' \circ \alpha'$ is monic also. Next let $w \in P$. Then

$$(0, w) = \phi(u, v) = (u, \beta''(\alpha'(v) + \alpha(0))) = (0, \beta''(\alpha(v)));$$

that is, $\beta \circ \alpha'(v) = w$ and as $w$ was arbitrary, $\beta'' \circ \alpha'$ is epic also. So $\beta'' \circ \alpha'$ is an isomorphism from $R^s \to P$, and $R$ is Hermite as required. ■

**Proposition 3.6.8.** *A projective free ring $R$ is also Hermite.*

*Proof.* A stably free $R$-module is projective and finitely generated. ■

# Chapter 4

# The Corona and Complement Problems

## 4.1 Background

The foundation for this is Carleson's corona theorem. S. Kakutani originally proposed the problem for the disk in 1941 and Carleson [16] later solved it in 1962 by using an interpolation problem which was equivalent to it. The equivalence of these problems had previously been demonstrated by D. Newman [58] in a paper published in 1959. This problem was subsequently simplified by Hörmander [43] using a Koszul complex to reduce it to $\overline{\partial}$ equation, and by Tom Wolff [35] using duality and Carleson measures.

The next stage is to generalize this problem. It is fairly easy to see that the corona theorem holds for any simply connected domain. In the case of the annulus, the simplest multiply connected domain, it was demonstrated by Scheinberg [69] and Stout [71] that the corona theorem holds.) Gamelin showed in [33] that the corona theorem is local; that is, it suffices to solve it in a neighbourhood of each point of a domain.

It can be extended to Riemann surfaces, of which planar domains are a special case. When a Riemann surface $\mathcal{R}$ admits non-constant bounded holomorphic functions, its universal cover is the unit disc. The deck transformations $\Gamma$ comprise a Fuchsian group in this case.

Alling, Earle and Marden, Forelli, and Stout proved the corona theorem for finite bordered Riemann surfaces in [2], [3], [24], [25], [27], [70], [71], and [72] (corrected in [73]). On the other hand, Riemann surfaces were found that the corona problem does not hold on. (For instance, see [46], [34], [5], or [53].)

One of the most powerful techniques which has been developed is that of pulling corona data $f_k$ back to the disk where they may be solved to obtain corona solution $g_k$ on the disc [46]. These can be projected (using the Forelli projection) into the space $H^\infty_\Gamma$ of bounded analytic functions which are invariant under the Fuchsian group $\Gamma$ (corresponding to the Riemann surface concerned) in order to obtain corona solutions on the finite bordered Riemann surface concerned. Because any Riemann surface may be exhausted by finite bordered ones, if the norms of the corona solutions can be kept bounded on the sequence then the Riemann surface has corona solutions on it also.

Using Green's functions [46], Jones and Marshall showed that it is sufficient to solve the corona problem at the critical points of the Green's function $G$ for the Riemann surface $\mathcal{R}$. They also showed it holds for $\mathcal{R}$ whenever the critical points form an interpolating sequence in $\mathcal{R}$ or if their pre-images form an interpolating sequence in the open disc.

A strong result for the corona problem in the plane is the result of Moore [55], who showed that the corona theorem holds for any domain whose boundary is contained in the graph of a $C^{1+\epsilon}$ function. This is an extension of the result of Garnett and Jones in [38] showing that the corona theorem holds for Denjoy domains (complements of proper closed subsets of $\mathbb{R}$).

## 4.2 The Corona Problem

The most general form of the corona problem asks whether a uniform algebra of continuous functions on a paracompact Hausdorff space $X$ separating points of $X$ has $X$ dense in its maximal ideal space.

For instance, given a noncompact Riemann surface $\mathcal{R}$, the question of whether the point evaluations $\delta_x$ on the uniform algebra consisting of bounded holomorphic functions $H^\infty(\mathcal{R})$ of the Riemann surface are dense in the maximal ideal space (equipped with its weak* topology). In more concrete terms, given bounded holomorphic functions $f_1, f_2, \dots f_n \in H^\infty(\mathcal{R})$ satisfying

$$\sum_{i=1}^{n} |f_i| \geq \delta$$

for some $\delta > 0$, is it possible to find bounded holomorphic corona solutions $g_1, g_2, \dots g_n$ such that

$$f_1g_1 + f_2g_2 + \dots f_ng_n = 1?$$

These conditions are equivalent [37],185. Assume that the point evaluations are dense in the maximal ideal space and that there are functions $f_1, f_2, \dots f_n \in H^\infty(\mathcal{R})$ whose moduli are uniformly bounded away from zero; that is, there is some $\delta > 0$ such that

$$\max_{1\leq i\leq n} |f_i| \geq \delta > 0.$$

Hence, for any point evaluation $\delta_x$, $\max_{1\leq i\leq n} |\delta_x(f_i)| \geq \delta$. By density of point evaluations, $\max_{1\leq i\leq n} |\phi(f_i)| \geq \delta$ for any multiplicative linear functional $\phi$ (since $\phi$ is a weak-star limit of point-evaluations). Therefore, the only ideal containing each of the $f_k$s is $H^\infty$. By definition $1 \in H^\infty$, so there exist $g_k \in H^\infty$ such that

$$f_1g_1 + f_2g_2 + \dots f_ng_n = 1.$$

Suppose on the other hand, that the point evaluations are not dense in the maximal ideal space. Pick some $\phi$ which is outside their closure. Then there exists a subbasic neighbourhood $U(\phi; f_1, \dots, f_n; \epsilon)$ without point evaluations taking the form

$$U = \bigcap_{i=1}^{n} \{\rho | \, |\rho(f_i) - \phi(f_i)| < \epsilon\},$$

which taking $h_i = f_i - \phi(f_i)$ has this form

$$U = \bigcap_{i=1}^{n} \{\rho | \, |\rho(h_i)| < \epsilon\}.$$

As there are no point evaluations in $U$, $\max_{1\leq k\leq n} |h_k(z)| \geq \epsilon$. On the other hand, $\phi$ vanishes on each of $h_1, h_2, \dots, h_n$, so they are contained in a common maximal ideal; consequently, there do not exist $g_k$ such that $\sum_{k=1}^{n} h_k g_k = 1$.

## 4.3 Hermiteness of $H^\infty(\mathbb{D})$

**Proposition 4.3.1.** *Let $\mathcal{A}$ be a complex commutative unital Banach algebra. Consider an $m \times n$ matrix having coefficients in $\mathcal{A}$ which is tall ($m > n$). To extend this matrix to an invertible square matrix it is necessary that its $n$-th order minors lie in no common maximal ideal of $\mathcal{A}$. The algebra $\mathcal{A}$ is* **Hermite** *exactly when this holds.*

**Example 4.3.2.** *The uniform algebra $H^\infty(\mathbb{D})$ is Hermite.*

## 4.4 Wolff's Proof of the Corona Theorem

Here I use the Koszul Complex method (developed by Hörmander and cited in [37], 354-357) to explain Wolff's proof of the Corona Theorem [37], 309-317.

Build a commutative diagram

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Lambda^2(\mathfrak{A}) & \xrightarrow{J} & \Lambda^1(\mathfrak{A}) & \xrightarrow{J} & \mathfrak{A} & \longrightarrow & 0 \\
\downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \\
\Lambda^2(\mathcal{E}) & \xrightarrow{J} & \Lambda^1(\mathcal{E}) & \xrightarrow{J} & \mathcal{E} & \longrightarrow & 0 \\
\downarrow \overline{\partial} & & \downarrow \overline{\partial} & & \downarrow \overline{\partial} & & \\
\Lambda^2(\mathcal{E}_{(0,1)}) & \xrightarrow{J} & \Lambda^1(\mathcal{E}_{(0,1)}) & \xrightarrow{J} & \mathcal{E}_{(0,1)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

where $\Lambda^k(\cdot)$ is the $k$-forms over the algebra $\cdot$, $\mathfrak{A}$ is the algebra of holomorphic functions on $\mathbb{D}$ continuous on the boundary, $\mathcal{E}$ is the algebra of $C^\infty$ functions on $\overline{\mathbb{D}}$, $\mathcal{E}_{(0,1)}$ is the algebra of differentials $f(z, \overline{z})\, d\overline{z}$. The maps $J$ and $\overline{\partial}$ are defined as

$$
J\left(\sum_{i,j} h_{ij} e_i \wedge e_j\right) = \sum_{i,j} g_i h_{ij} e_j - g_j h_{ij} e_i,
$$

$$J\left(\sum_j h_j e_j\right) = \sum_j g_j h_j,$$

and $\overline{\partial}$ acts componentwise

$$\overline{\partial}\left(\sum h_{ij} e_i \wedge e_j\right) = \sum \frac{\partial h_{ij}}{\partial \overline{z}} d\overline{z} e_i \wedge e_j$$

$$\overline{\partial}\left(\sum h_i e_i\right) = \sum \frac{\partial h_i}{\partial \overline{z}} d\overline{z} e_i.$$

Note the following properties of $\overline{\partial}$ and $J$. Note that $\overline{\partial}^2 = 0$ and $\ker(J) = \mathfrak{A}$ by the Cauchy-Riemann equations. Consequently, the columns are exact. Also note that $J^2 = 0$ as

$$J^2\left(\sum_{i,j} h_{ij} e_i \wedge e_j\right) = J\left(\sum_{i,j} h_{ij} g_i e_j - h_{ij} g_j e_i\right) = \sum_{i,j} h_{ij} g_i g_j - h_{ij} g_j g_i = 0;$$

also note that for 1-forms $\alpha, \beta$.

$$J(\alpha \wedge \beta) = J(\alpha) \wedge \beta - \alpha \wedge J(\beta).$$

Because $\overline{\partial}$ just acts componentwise and $J$ behaves as an interior product, these operations commute: $\overline{\partial} J = J \overline{\partial}$.

To show that the second and third rows are exact, consider a 1-form $h = h_1 e_1 + \cdots h_n e_n$ such that the sum $\sum_j h_j g_j = 0$. In a region $U_l$ where the function $g_l(z)$ is bounded away from zero, take the 2-form

$$x = g_l^{-1} \sum_{k=1}^{n} h_k e_l \wedge e_k,$$

so that

$$Jx = g_l^{-1} \sum_{k=1}^{n} h_k J(e_l \wedge e_k) = g_l^{-1} \sum_{k=1}^{n} h_k g_l e_k - g_l^{-1} \sum_{k=1}^{n} h_k g_k e_l = \sum_{k=1}^{n} h_k e_k = h.$$

To complete the argument, construct open sets

$$U_j = \left\{ z \in \mathbb{D} : |f_j(z)| > \frac{\delta}{2} \right\}$$

on which $g_j(z)$ is bounded away from 0, and take a partition of unity $\chi_j(z)$ of the disk $\mathbb{D}$ subordinate to them. Now let

$$x = \sum_{j=1}^{n} \chi_j(z) g_j^{-1}(z) \sum_{k=1}^{n} h_k e_j \wedge e_k.$$

It now follow that the second and third rows are exact.

Solving the corona problem here is equivalent to finding a 1-form $g \in \Lambda^1(\mathfrak{A})$ such that $Jg = 1$. To start out, observe that

$$\sum_{j=1}^{n} f_j \frac{\overline{f_j}}{\sum_{k=1}^{n} |f_k|^2} = 1.$$

This gives us a 1-form $\phi \in \Lambda^1(\mathcal{E})$ such that $J\phi = 1$. Using the fact that $J$ is an antiderivation (i.e, $J(\alpha \wedge \beta) = J(\alpha) \wedge \beta - \alpha \wedge J(\beta)$), it follows that

$$J(\phi \wedge \overline{\partial}\phi) = J(\phi) \wedge \overline{\partial}\phi - \phi \wedge J(\overline{\partial}\phi) = \overline{\partial}\phi.$$

Assume for the moment that there is a solution $a \in \Lambda^2(\mathcal{E})$ to the equation $\overline{\partial} a = \phi \wedge \overline{\partial}\phi$. Then

$$\overline{\partial}(Ja) = J(\overline{\partial}a) = J(\phi \wedge \overline{\partial}\phi) = \overline{\partial}\phi$$

so that

$$\overline{\partial}(\phi - Ja) = \overline{\partial}\phi - \overline{\partial}\phi = 0$$

and

$$J(\phi - Ja) = J\phi - 0 = 1.$$

This means that provided we solved $\overline{\partial} a = \phi \wedge \overline{\partial} \phi$ for $a$, then $\phi - Ja$ is a form whose components are the corresponding corona solutions $g_k$ to our corona data $f_k$. Writing $a = \sum_{j,k} a_{j,k} e_j \wedge e_k$ and $\phi = \sum_k \phi_k e_k$ and $g = \sum_k g_k e_k$ where $g = \phi - Ja$, the differential equations to be solved take the form

$$(a_{j,k} - a_{k,j})\, e_j \wedge e_k = \left( \phi_j \frac{\partial \phi_k}{\partial \overline{z}} - \phi_k \frac{\partial \phi_j}{\partial \overline{z}} \right) e_j \wedge e_k$$

for each $j, k$. Taking $a_{j,k} = -a_{k,j}$ makes the system

$$\frac{\partial a_{j,k}}{\partial \overline{z}} = \frac{1}{2} \left( \phi_j \frac{\partial \phi_k}{\partial \overline{z}} - \phi_k \frac{\partial \phi_j}{\partial \overline{z}} \right)$$

and the solution $g$ takes the form

$$g_l = \phi_l - \sum_k f_k a_{k,l}.$$

**Proposition 4.4.1.** *If the corona theorem holds for the disc algebra $H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ with bounds $C(n, \delta)$ on the solutions depending on $n$ and $\delta$, then it holds for the algebra $H^\infty(\mathbb{D})$ of bounded holomorphic functions as well.*

*Proof.* Let $f_j(z)$ be corona data in $H^\infty$ such that

$$\max_j |f_j(z)| \geq \delta > 0$$

for $z \in \mathbb{D}$. For $0 < r < 1$ define bounded holomorphic functions $f_{r,j}(z) = f_j(rz)$ which are defined in some neighbourhood of $\overline{\mathbb{D}}$ and are therefore continuous on $\mathbb{D}$. By hypothesis, each of these has corresponding corona solutions; denote these as $g_{r,j}(z)$. Then for each $1 \leq j \leq n$, the family of functions $g_{r,j}(z)$ for $0 < r < 1$ is uniformly bounded by $C(n, \delta)$, so by Montel's theorem, the sequence $g_{1-2^{-k},j}(z)$ has a normally convergent subsequence $g_{s_k,j}(z)$ which converges to a holomorphic function on $\mathbb{D}$ which I will denote $g_j(z)$. These functions are bounded since they are normal limits of bounded functions, so I just need to show that they satisfy the required relation. Fix some $0 < R < 1$ and $\epsilon > 0$. Choose $k^*$ such that $\|f_j(z) - f_{s_k,j}(z)\| < \epsilon$ and $\|g_j(z) - g_{s_k,j}(z)\| < \epsilon$ for $|z| \leq R$ whenever $k \geq k^*$. Next consider the expression

$$\left\| \sum_j f_j(z) g_j(z) - 1 \right\|;$$

that is,

$$\left\| \sum_j f_{s_k,j}(z) \left( g_j(z) - g_{s_k,j}(z) \right) + \sum_j f_{s_k,j}(z) g_{s_k,j}(z) - 1 + \sum_j \left( f_j(z) - f_{s_k,j} \right) g_j(z) \right\|.$$

The middle term vanishes since $g_{s_k,j}(z)$ are corona solutions to the corona date $f_{s_k,j}(z)$ for each $k$ so I get the upper bound

$$\left\| \sum_j f_j(z) g_j(z) - 1 \right\| \leq \left\| \sum_j f_{s_k,j}(z) \left( g_j(z) - g_{s_k,j}(z) \right) \right\| + \left\| \sum_j g_j(z) \left( f_j(z) - f_{s_k,j}(z) \right) \right\|$$

$$\left\| \sum_j f_j(z) g_j(z) - 1 \right\| \leq \max_j \|f_{s_k,j}(z)\| \left\| \sum_j \left( g_j(z) - g_{s_k,j}(z) \right) \right\| + \max_j \|g_j(z)\| \left\| \sum_j \left( f_j(z) - f_{s_k,j}(z) \right) \right\|$$

$$\left\| \sum_j f_j(z) g_j(z) - 1 \right\| < Mn\epsilon + MCn\epsilon = nM(C+1)\epsilon$$

where $C = C(n, \delta)$ is the bound on the corona solutions and $M$ is the bound on the corona data. As $\epsilon$ and $R$ were arbitrary,

$$\sum_j f_j(z) g_j(z) = 1$$

on $\mathbb{D}$ and the functions $g_j(z)$ do indeed comprise corona solutions. ■

**Lemma 4.4.2.** *[37], 309-311. Let $G(\zeta)$ be $C^1$ and bounded in $\mathbb{D}$. The set of $C^1$ solutions* *to the inhomogeneous Cauchy-Riemann equation*

$$\frac{\partial F}{\partial \overline{z}} = G(z) \text{ for } |z| < 1 \tag{4.1}$$

*is*

$$\mathcal{F} = \left\{ F_0 + h : h \in H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}}) \right\}$$

*where*

$$F_0(z) = \frac{1}{2\pi i} \iint_{|\zeta|<1} \frac{G(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}.$$

*Proof.* Note first of all that for $\phi \in C^\infty(\mathbb{D})$ having compact support in $\mathbb{D}$ that

$$\frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{\partial \phi}{\partial \overline{z}} \frac{1}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} = -\frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{\partial}{\partial \overline{\zeta}} \left( \frac{\phi(\zeta)}{\zeta - z} \right) d\zeta \wedge d\overline{\zeta} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|\zeta - z| = \epsilon} \frac{\phi(\zeta) d\zeta}{\zeta - z} = \phi(z). \tag{4.2}$$

Next notice that

$$\iint_{\mathbb{D}} F \frac{\partial \phi}{\partial \overline{z}} dz \wedge d\overline{z} + \iint_{\mathbb{D}} \phi \frac{\partial F}{\partial \overline{z}} dz \wedge d\overline{z} = -\iint_{\mathbb{D}} \frac{\partial F \phi}{\partial \overline{z}} d\overline{z} \wedge dz = -\int_{|z|=1} (F\phi) dz = 0$$

which means that

$$\iint_{\mathbb{D}} F \frac{\partial \phi}{\partial \overline{z}} dz \wedge d\overline{z} = -\iint_{\mathbb{D}} \phi \frac{\partial F}{\partial \overline{z}} dz \wedge d\overline{z}. \tag{4.3}$$

On the other hand,

$$\frac{1}{2\pi i} \iint_{\mathbb{D}} F \frac{\partial \phi}{\partial \overline{z}} dz \wedge d\overline{z} = \iint_{|\zeta|<1} G(\zeta) \left( \frac{-1}{2\pi i} \iint_{|z|<1} \frac{\partial \phi}{\partial \overline{z}} \frac{1}{z - \zeta} dz \wedge d\overline{z} \right) d\zeta \wedge d\overline{\zeta}$$

and by Fubini's theorem and (4.2)

$$\iint_{\mathbb{D}} F \frac{\partial \phi}{\partial \overline{z}} dz \wedge d\overline{z} = -\iint_{|\zeta|<1} G(\zeta) \phi(\zeta)\, d\zeta \wedge d\overline{\zeta}. \tag{4.4}$$

Equating (4.3) and (4.4) tells me that

$$\iint_{\mathbb{D}} \phi(z) \frac{\partial F}{\partial \overline{z}} \, dz \wedge d\overline{z} = \iint_{\mathbb{D}} \phi(z) G(z) \, dz \wedge d\overline{z}$$

$$\iint_{\mathbb{D}} \phi(z) \left( \frac{\partial F}{\partial \overline{z}} - G(z) \right) dz \wedge d\overline{z} = 0$$

for any $C^\infty$ $\phi$ having compact support in $\mathbb{D}$.

Suppose that it were possible for some nonzero continuous function $A$ to satisfy

$$\iint_{\mathbb{D}} \phi(z) A(z) \, dz \wedge d\overline{z} = 0 \tag{4.5}$$

for all $\phi \in C_c^\infty(\mathbb{D})$. Then choose some $z_0$ at which $A(z_0) \neq 0$, and without loss of generality assume that $A(z)$ is real-valued and that $A(z_0) = \delta > 0$. By continuity, there exists some $\epsilon > 0$ such that $A(z) > \frac{\delta}{2}$ whenever $|z - z_0| < \epsilon$. Let $\phi \in C^\infty(\mathbb{D})$ be positive, have a finite non-vanishing integral over $\mathbb{D}$, and have compact support in this open ball $B_{\delta/2}(z_0)$. Then

$$\iint_{\mathbb{D}} A(z)\phi(z) \, dz \wedge d\overline{z} = \iint_{B_\epsilon(z_0)} A(z)\phi(z) > \frac{\delta}{2} \iint_{B_\epsilon(z_0)} \phi(z) dz \wedge d\overline{z} > 0,$$

contradicting (4.5).

Let $F_1$ be another $C^1$ solution to (4.1). Then $F_1 - F_0$ is a function in $C^1(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ satisfying the Cauchy-Riemann equation and is therefore a member of the disk algebra $H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$. Consequently, all solutions are elements of $\mathcal{F}$, as required. ∎

**Theorem 4.4.3.** *[37], 312. Let $G(z)$ be bounded and $C^1$ on the disk $\mathbb{D}$ and assume that $|G|^2 \log(1/|z|) \, dx \, dy$ and $\left|\frac{\partial G}{\partial \overline{z}}\right| \log(1/|z|) \, dx \, dy$ are Carleson measures; that is,*

$$\iint_S |G|^2 \log \frac{1}{|z|} \, dx \, dy \leq B_1 \, l(S)$$

*and*

$$\iint_S \left|\frac{\partial G}{\partial \overline{z}}\right| \log\frac{1}{|z|}\,dx\,dy \le B_2\,l(S)$$

*for any sector of the form* $S = \left\{re^{i\theta} : 1 - l(S) < r < 1, |\theta - \theta_0| < l(S)\right\}$*. Then there is a function* $b(z)$ *which is continuous on* $\overline{\mathbb{D}}$ *and* $C^1$ *on* $\mathbb{D}$ *such that*

$$\frac{\partial b}{\partial \overline{z}} = G(z)$$

*for* $|z| < 1$ *and such that*

$$\|b\|_\infty = \sup_{|z|=1} |b(z)| \le C_1\sqrt{B_1} + C_2B_2;$$

*with* $C_1, C_2$ *absolute constants.*

*Proof.* Note that

$$\inf\left\{\|b\|_\infty \left|\frac{\partial b}{\partial \overline{z}} = G\right.\right\} = \sup\left\{\left|\frac{1}{2\pi}\int_0^{2\pi} F k\,d\theta\right| : k \in H_0^1, \|k_1\| \le 1\right\}$$

by the duality of $H_0^1$ with the disk algebra $A_0 = C(S^1) \cap H^\infty(\mathbb{D})$ [37], IV.1.1.

Then

$$\frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})k(e^{i\theta})\,d\theta = \frac{1}{2\pi}\iint_D \Delta(F(z)k(z))\log(1/|z|)dx\,dy$$

by Green's formula. This becomes

$$\frac{2}{\pi}\iint_D \frac{\partial}{\partial z}k(z)\frac{\partial F}{\partial \overline{z}}\log(\frac{1}{|z|})dx\,dy$$

$$\frac{2}{\pi}\iint_D k'(z)G(z)\log\frac{1}{|z|}dx\,dy + \frac{2}{\pi}\iint_D k(z)\frac{\partial G}{\partial z}\log\frac{1}{|z|}dx\,dy;$$

call these terms $I_1$ and $I_2$. Now $|I_1| \le C_2B_2\,\|k\|_1 \le C_2B_2$ and an estimate is needed on the other integral $I_2$. (Note that $C_2$ is the constant of the Carleson measure.) Rewrite $k \in H^1$

as an average of $k_1, k_2 \in H^1$ which are square free on the disk and $\|k_j\| \le 2$ (for $j = 1, 2$). Since these are zero free, there exist $g_1, g_2 \in H^2$ such that $k_j(z) = g_j(z)^2$ and $\|g_j\|_2^2 \le 2$.

This means that

$$\left|\frac{2}{\pi}\iint_D k_j'(z)G(z)\log\frac{1}{|z|}dx\,dy\right| = \left|\frac{4}{\pi}\iint_D g_j(z)g_j'(z)G(z)\log\frac{1}{|z|}dx\,dy\right|$$

since $k_j'(z) = 2g_j(z)g_j'(z)$. This is bounded above by

$$\left(\frac{4}{\pi}\iint_D |g_j'(z)|^2\log\frac{1}{|z|}dx\,dy\right)^{1/2}\left(\frac{4}{\pi}\iint_D |g_j(z)|^2|G(z)|^2\log\frac{1}{|z|}dx\,dy\right)^{1/2},$$

by Cauchy-Schwarz applied to the measure space $L^2(\log\frac{1}{|z|}\,d\mu)$. This has the upper bound

$$\left(\frac{2}{\pi}\iint_{\mathbb{D}} |\nabla g_j(z)|^2\log\frac{1}{|z|}dx\,dy\right)^{1/2}\left(CB_1\|g_j\|_2^2\right)^{1/2};$$

The Littlewood-Paley identity (from [37], VI.3.1) now applies in the first term to give

$$\left(\frac{1}{\pi}\int_{\theta=0}^{2\pi}\left|g_j(e^{i\theta}) - g_j(0)\right|^2\,d\theta\right)^{1/2} \le \sqrt{2}\,\|g_j\|_2\,C\sqrt{B_1}\,\|g_j\|_2 = \sqrt{B_1}C\sqrt{2}\,\|g_j\|_2^2 \le \sqrt{B_1}2C\sqrt{2},$$

so $I_2 \le C_1\sqrt{B_1}$ as required. ∎

**Theorem 4.4.4.** *Wolff's proof of the corona theorem [37], VIII.2.1. If* $f_1, f_2, \ldots, f_n \in H^\infty(\mathbb{D})$ *are bounded holomorphic functions which satisfy* $\max_j |f_j(z)| \ge \delta$ *for any* $z \in \mathbb{D}$ *and* $\|f_j\|_\infty \le 1$ *for each* $1 \le j \le n$*, then there are* $g_1, g_2, \ldots, g_n \in H^\infty(\mathbb{D})$ *solving the corona problem*

$$f_1g_1 + \cdots + f_ng_n = 1$$

*with bounds* $\|g_j\|_\infty \le C(n,\delta)$ *where* $C(n,\delta)$ *is a constant.*

*Proof.* The main technique of Wolff's proof is to show that $|G|^2\log(1/|z|)\,dx\,dy$ and $|\partial G/\partial z|\log(1/|z|)\,dx\,dy$ are Carleson measures in order to apply Wolff's theorem. In both cases, you use the fact that $|f_k'|^2 \log(1/|z|) dx\, dy$ is a Carleson measure since $f_k$ is an element of $L^1$ (see [37], 233).

To show that $|G_{j,k}|^2 \log \frac{1}{|z|} dx\, dy$ is a Carleson measure notice first that since $|\phi_j| \leq \delta^{-1}$,

$$|G_{j,k}|^2 \log \frac{1}{|z|} \leq \delta^{-2} \left| \frac{\partial \phi_k}{\partial \overline{z}} \right|^2 \log \frac{1}{|z|} dx\, dy.$$

Next,

$$\frac{\partial \phi_k}{\partial \overline{z}} = \frac{\sum_l f_l \left( \overline{f_l f_k'} - \overline{f_k f_l'} \right)}{\left( \sum_l |f_l|^2 \right)^2}.$$

Then

$$\left| \frac{\partial \phi_k}{\partial \overline{z}} \right|^2 \leq \frac{2 \left( \sum_l |f_l|^2 \right)^2 \sum_l |f_l'|^2}{\left( \sum_l |f_l|^2 \right)^4} \leq 2\delta^{-4} \sum_l |f_l'|^2$$

As a result,

$$|G_{j,k}|^2 \leq 2\delta^{-6} \sum_l |f_l'|^2;$$

this means that $B_1 \leq Cn\delta^{-6}$. Next, to show that $|\frac{\partial G}{\partial z}| \log \frac{1}{|z|} dx\, dy$ is a Carleson measure, calculate the derivative of $G_{j,k}$:

$$\frac{\partial G_{j,k}}{\partial z} = \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_k}{\partial \overline{z}} + \phi_j \frac{\partial^2 \phi_k}{\partial z \partial \overline{z}}.$$

This becomes

$$\frac{\partial G_{j,k}}{\partial z} = \left( \frac{-\overline{f_j} \sum_l \overline{f_l} f_l'}{\left( \sum_l |f_l|^2 \right)^2} \right) \left( \frac{\sum_l f_l (\overline{f_l f_k'} - \overline{f_k f_l'})}{\left( \sum_l |f_l|^2 \right)^2} \right) \cdots$$

$$\cdots + \frac{\overline{f_j}}{\sum_l |f_l|^2} \left( \frac{\sum_l f_l' \left( \overline{f_l f_k'} - \overline{f_k f_l'} \right)}{\left( \sum_l |f_l|^2 \right)^2} - \frac{2 \sum_l f_l' \overline{f_l} \sum_l f_l \left( \overline{f_l f_k'} - \overline{f_k f_l'} \right)}{\left( \sum_l |f_l|^2 \right)^3} \right),$$

so

$$\left| \frac{\partial G_{j,k}}{\partial z} \right| \leq C \frac{\sum_{p,q} |f_p'| |f_q'|}{\left( \sum_l |f_l|^2 \right)^2}.$$

This shows that

$$\left| \frac{\partial G_{j,k}}{\partial z} \right| \leq Cn\delta^{-4} \sum_l |f_l'|^2$$

and that $B_2 \le Cn\delta^{-4}$. By Wolff's theorem, we have $b_{j,k}$ satisfying

$$|b_{j,k}(z)| \le C(n,\delta) \le Cn^{1/2}\delta^{-3} + Cn\delta^{-4};$$

noting that $a_{j,k}(z) = b_{j,k}(z) - b_{k,j}(z)$ and $g_j(z) = \phi_j(z) - \sum_k a_{j,k}(z)\phi_k(z)$,

$$|g_j(z)| \le |\phi_j(z)| + n\,|a_{j,k}(z)|$$

so that the corona solutions $g_j$ have bounds

$$\|g_j\|_\infty \le C(n,\delta) \le C\left(n^{3/2}\delta^{-3} + n^2\delta^{-4}\right).$$

■

## 4.5 Forelli Projections

**Definition 4.5.1.** *A sequence of points $z_k$ in an open Riemann surface is an* **interpolation sequence** *if given any bounded sequence $\{a_k\} \in l^\infty$, there exists a bounded holomorphic $f \in H^\infty(\mathcal{R})$ such that $f(z_k) = a_k$ for every $k \in \mathbb{N}$.*

**Theorem 4.5.2.** *[16], Thm. 3.1. A sequence $\{z_k\}$ in $\mathbb{D}$ is an interpolation sequence if and only if there exists some $\delta > 0$ such that*

$$\prod_{j,k\neq j} \left|\frac{z_k - z_j}{1 - \overline{z_k} z_j}\right| \ge \delta > 0, \text{ for } j \in \mathbb{N}$$

*The* **constant of interpolation** *$M$ of an interpolation sequence is*

$$M = \sup_{\|a_j\|\le 1} \inf\left\{\|f\|_\infty : \|a_j\|_\infty \le 1, f \in H^\infty, f(z_j) = a_j, j = 1,2,\ldots\right\}.$$

**Theorem 4.5.3.** *Theorem of Varopoulis [37], VII.2.1. Given an interpolation sequence* $\{z_k\}$ *in the upper halfplane* $\mathcal{H}$*, there exist bounded holomorphic functions* $f_j(z) \in H^\infty(\mathcal{H})$ *such that*

$$f_j(z_k) = \delta_{jk},$$

$$\sum_j |f_j(z)| \leq M.$$

Let $(S, r, R)$ be a covering space (of Riemann surfaces). Let $F_z$ (for a point $z \in R$) denote the fibre $r^{-1}(\{z\}$ over $z$, and $c(F_z)$ denote the subspace of $l^\infty(F_z)$ consisting of constant functions.

**Definition 4.5.4.** *[13], Thm. 1.1. A Forelli projection* $P$ *is a bounded linear operator (norm* $\|P\| \leq C < \infty$*)*

$$H^\infty(S) \to r^*(H^\infty(R))$$

*satisfying the following conditions:*

1. *There exists a family of linear projectors* $P_z : l^\infty(F_z) \to c(F_z) \cong \mathbb{C}$ *holomorphically depending on* $z$ *such that* $P[f]|_{F_z} = P_z\left[f|_{F_z}\right]$ *for any* $f \in H^\infty(S)$,
2. $P(fg) = P(f)g$ *whenever* $f \in H^\infty(S), g \in r^*(H^\infty(R))$,
3. *if* $f \in H^\infty(S)$ *such that* $f|_{F_z}$ *is constant then* $P(f)|_{F_z} = f|_{F_z}$,
4. *each* $P_z$ *is continuous in the weak* topology of* $l^\infty(F_z)$,

**Proposition 4.5.5.** *From the conditions, there exists an* $h \in H^\infty(S)$ *such that*

1. $\hat{h}(z) = \sum_{w \in F_z} |h(w)|$ *is continuous on* $R$ *and* $\sup_R \hat{h} \leq C < \infty$,
2. $\sum_{w \in F_z} h(w) = 1$ *for all* $z \in R$,
3. *The projection* $P$ *is defined as*

$$P(f)(z) = \sum_{w \in F_z} f(w)h(w)$$

*so that $P : H^\infty(S) \to r^*(H^\infty(R))$ is weak* continuous; that is, whenever the net $f_k$ in $H^\infty(S)$ satisfies* $\lim_k f_k(z) \to f(z)$ *for every $z \in S$, then it also satisfies* $\lim_k P_k(f)(z) \to P(f)(z)$.

Forelli first discovered that such projections exist for finite bordered Riemann surfaces, and Earle and Marden later made his construction explicit. Carleson established their existence for a class of infinitely connected Riemann surfaces: complement $\mathbb{C}\setminus E$ of homogeneous subsets $E \subseteq \mathbb{R}$. (These satisfy the condition $\mu((x-r, x+r)) \geq \epsilon r$ for some $\epsilon > 0$, all $r > 0, x \in E$.) Jones and Marshall, on the other hand, showed their existence for Riemann surfaces $X$ for which critical points of the Green's function on $X$, or their preimages in $\mathbb{D}$ form interpolating sequences for $H^\infty(X)$ or $H^\infty(\mathbb{D})$, respectively. This class also contains surfaces previously considered in [27] and [17].

Jones and Marshall [46] provide the following construction for Forelli projections. Let $\Gamma$ be the Fuchsian group of deck transformations of the universal cover $\mathbb{D}$ of the Riemann surface $R$. Form the infinite Blaschke product:

$$B(z) = \prod_{\gamma\in\Gamma} \frac{|\gamma(0)|}{\gamma(0)}\gamma(z),$$

which (by logarithmic differentiation) has the derivative

$$B'(z) = B(z)\sum_{\gamma\in\Gamma}\frac{\gamma'(z)}{\gamma(z)}.$$

Then now define the projection on a function $f \in H^\infty(\mathbb{D})$:

$$E(f)(z) = \sum_{\gamma\in\Gamma} f(\gamma(z))\frac{\gamma'(z)}{\gamma(z)}\frac{B(z)}{B'(z)},$$

where

$$h(\gamma(z)) = \frac{\gamma'(z)}{\gamma(z)}\frac{B(z)}{B'(z)}.$$

It should be noted that $E(f)$ is not necessarily holomorphic, let alone bounded; it is however $\Gamma$-invariant. Consider the following expression for an arbitrary $\zeta \in \Gamma$

$$E(f)(\zeta(z)) = \sum_{\gamma\in\Gamma} f(\gamma(\zeta(z)))\frac{\gamma'(\zeta(z))}{\gamma(\zeta(z))}\frac{B(\zeta(z))}{B'(\zeta(z))} = \sum_{\gamma\in\Gamma} f(\gamma(z))\frac{\frac{\gamma'(z)}{\zeta'(\zeta^{-1}(z))}}{\gamma(z)}\frac{B(z)}{\frac{B'(z)}{\zeta'(\zeta^{-1}(z))}},$$

$$E(f)(\zeta(z)) = \sum_{\gamma\in\Gamma} f(\gamma(z))\frac{\gamma'(z)}{\gamma(z)}\frac{B(z)}{B'(z)} = E(f)(z).$$

However, when applied to a function $f$ such that $(f\circ\gamma)(z_k) = f(z_k)$ at each critical point $z_k$ and any $\gamma \in \Gamma$, the result is a bounded holomorphic function since the functions concerned are holomorphic off of the critical points. Consider the projection written in the form

$$E(f) = f + \sum_{\gamma\in\Gamma}(f\circ\gamma - f)\,\frac{\gamma'}{\gamma}\frac{B}{B'};$$

the functions $f\circ\gamma - f$ have zeroes at the critical points which turn the simple poles into removable singularities. Consequently, $E(f)$ is holomorphic. As it is continuous on the boundary $\partial\mathbb{D}$ as well, it is bounded; hence, $E(f) \in r^*\left(H^\infty(\mathcal{R})\right)$. Since the function $E(f) \in H^\infty(\mathbb{D})$, its nontangential limits for $\partial\mathbb{D}$ define a function $g \in L^\infty(\mathbb{D})$ such that $\|g\|_\infty = \|f\|_\infty$. This means that it is sufficient to bound $E(f)$ on $\partial\mathbb{D}$, letting using $g(e^{i\theta})$ for $f$ there. Then

$$E(f)(z) = \sum_{\gamma\in\Gamma} f(\gamma(z))\frac{\gamma'(z)}{\gamma(z)}\frac{B(z)}{B'(z)}.$$

Note that

$$z\frac{\gamma'(z)}{\gamma(z)} = \frac{1-|\gamma(0)|}{|e^{i\theta}-\gamma(0)|^2} > 0$$

so that

$$|E(f)(z)| \le \sum_{\gamma\in\Gamma}|f(\gamma(z))|\left|z\frac{\gamma'(z)}{\gamma(z)}\right|\left|\frac{B(z)}{zB'(z)}\right|$$

$$|E(f)(z)| \le \|f\|_\infty \frac{\sum_{\gamma\in\Gamma}\frac{z\gamma'(z)}{\gamma(z)}}{\sum_{\zeta\in\Gamma}\frac{z\zeta'(z)}{\zeta(z)}} = \|f\|_\infty\,;$$

consequently, $\|E(f)\|_\infty \leq \|f\|_\infty$ since this holds for almost every point $z \in \partial\mathbb{D}$.

**Theorem 4.5.6.** *[46], Thm. 2.10. Suppose that there exists some $N < \infty$ such that the set $\{\zeta \in \mathcal{R} : \sum_{k=1}^{\infty} G(\zeta, \zeta_k) > N\}$ is a disjoint union of simply connected regions where $\zeta_k$ is an enumeration of the critical points of the Green's function. There exists a Forelli projection $P : H^\infty(\mathbb{D}) \to r^*\left(H^\infty(\mathcal{R})\right)$.*

*Proof.* Let $\{\zeta_m\}$ and $\{\gamma_k\}$ be enumerations of the critical points of the Green's function and the Fuchsian group $\Gamma$, respectively. For each $m$, define $\{z_{m,0}\}$ to be a pre-image of $\zeta_m$ under $r$, and for each $k \neq 0$, let $z_{m,k} = \gamma_k(z_{m,0})$. Take the Blaschke product

$$C(z) = \prod_{(m,k)} \frac{|z_{m,k}|}{z_{m,k}} \frac{z - z_{m,k}}{1 - \overline{z_{m,k}} z}$$

and select a set $W = \{w_{n,0}\}$ of $z_{m,j}$ such that $w_{p,0} \neq \gamma(w_{q,0})$ for any $\gamma \in \Gamma$ also satisfying the condition that any $z_{n,j}$ in $\left\{z : |C(z)| < e^{-N}\right\}$ also belongs to $W$. That is, $W$ consists of all critical points on one sheet of $\mathcal{R}$. Next define $w_{m,k} = \gamma_k(w_{m,0})$ and let

$$A = \{f \in H^\infty(\mathbb{D}) : f(w_{m,k}) = f(w_{1,k}), \forall k\}$$

be the algebra of functions which are constant on fibres of critical points. By (4.5.2), there is an interpolating sequence for $A$ with interpolating constant $K(N)$ depending only on $N$. Next by the Varopoulis theorem (4.5.3), there exist $F_k \in A$ such that $F_k(w_{m,j}) = \delta_{jk}$ and $\sum_{k=1}^{\infty} |F_k(z)| \leq K^2$ for all $z \in \mathbb{D}$. Next for $f \in H^\infty(\mathbb{D})$, define

$$T(f)(z) = f + \sum_{\gamma_k \in \Gamma} \left(f \circ \gamma_k^{-1}(z) - f(z)\right) F_k(z).$$

It is clear that this carries 1 into itself, and that it satisfies the property that for any

$f \in r^* (H^\infty(\mathcal{R}))$,

$$T(fg)(z) = fg + \sum_{\gamma_k \in \Gamma} \left( fg \circ \gamma_k^{-1}(z) - f(z)g(z) \right) F_k(z)$$

$$T(fg)(z) = fg + \sum_{\gamma_k \in \Gamma} \left( f(z)(g \circ \gamma_k^{-1})(z) \right)$$

$$T(fg)(z) = f \left( g + \sum_{\gamma_k \in \Gamma} \left( g \circ \gamma_k^{-1}(z) \right) \right) = f(z)T(g)(z).$$

Importantly, $T$ is bounded

$$|T(f)(z)| \leq |f(z)| + \sum_{\gamma_k \in \Gamma} \left| f \circ \gamma_k^{-1}(z) - f(z) \right| |F_k(z)| \leq \|f\|_\infty \left( 1 + \sum_{\gamma_k \in \Gamma} |F_k(z)| \right),$$

$$\|T(f)\|_\infty \leq \|f\|_\infty \left( 2K^2 + 1 \right),$$

and $T(f)$ is constant on fibres of critical points:

$$T(f)(z_{m,k}) = f(z_{m,k}) + \sum_{\gamma_k \in \Gamma} \left( f \circ \gamma_l^{-1}(z_{m,k}) - f(z_{m,k}) \right) F_l(z_{m,k}) = f(z_{m,k}) + f \circ \gamma_k^{-1}(z_{m,k}) - f(z_{m,k})$$

$$T(f)(z_{m,k}) = f(z_{m,k}) + f(z_{m,0}) - f(z_{m,k}) = f(z_{m,0}).$$

But this means that the previous result can be applied. Define $P(f)(z) = E(T(f))(z)$. Then $P$ satisfies the properties of a Forelli projection whose constant $\|P\| \leq (2K^2 + 1)$ since $E$ and $T$ have the other properties. ∎

**Theorem 4.5.7.** *Let $R$ be a connected Carathéodory hyperbolic Riemann surface. If a Forelli projection $P$ exists for the universal covering space $(\mathbb{D}, r, R)$, then the corona theorem holds on $H^\infty(R)$.*

*Proof.* Let $f_1, \ldots, f_n$ be corona data in $H^\infty(R)$ where

$$0 < \delta \leq \max_{1 \leq j \leq n} |f_j(x)| \leq 1$$

for all $x \in R$. Then $r^* f_1, \ldots, r^* f_n$ form corona data in $H^\infty(\mathbb{D})$ in which

$$0 < \delta \leq \max_{1 \leq j \leq n} |r^* f_j(x)| \leq 1$$

for all $x \in \mathbb{D}$.

By Carleson's corona theorem, there are corona solutions $g_1, \ldots, g_n$ in $H^\infty(\mathbb{D})$; that is,

$$\sum_{j=1}^{n} (r^* f_j) g_j = 1$$

with the bound $C(n^{3/2}\delta^{-2} + n^2\delta^{-3})$. Now because $P$ is a Forelli projection

$$P\left(\sum_{j=1}^{n} (r^* f_j) g_j\right) = \sum_{j=1}^{n} P\left((r^* f_j) g_j\right) = P(1) = 1$$

$$\sum_{j=1}^{n} (r^* f_j) P(g_j) = 1;$$

note next that $P(g_j) = r^* h_j$ for some $h_j \in H^\infty(R)$ so that

$$\sum_{j=1}^{n} r^* f_j(x) r^* h_j(x) = 1$$

for all $x \in \mathbb{D}$. But this then means that

$$\sum_{j=1}^{n} f_j(z) h_j(z) = 1$$

for all $z \in R$ (with the bounds $\|P\| \, C(n^{3/2}\delta^{-2} + n^2\delta^{-3})$) and $h_j$ form corona solutions. ■

**Theorem 4.5.8.** *[13], Thm. 1.1. Let $X, \bar{X}$ be unbranched coverings of a relative compact domain $N$ in a connected Stein manifold $M$.*

$$\begin{array}{ccc} \bar{X} & \xrightarrow{r_2} & X \\ & & \big\downarrow{\scriptstyle r_1} \\ & & N & \hookrightarrow & M \end{array}$$

-

*If the fundamental groups $\pi_1(X)$, $\pi_1(\bar{X})$ are isomorphic, then there is a Forelli projection*

$$P : H^\infty(\bar{X}) \to r^* \left(H^\infty(X)\right)$$

*and its norm $\|P\| \leq C(N)$ where $C$ is a constant depending only the domain $N$.*

Brudnyi proposes a general construction for Forelli projections on coverings of Stein manifolds in [13] and, in particular, obtains the following result.

**Corollary 4.5.9.** *[13], Cor. 1.3. Let $X$ be a domain in an unbranched covering $\mathcal{R}$ of a bordered Riemann surface $N$. If the homomorphism $\pi_1(i)$ of fundamental groups induced by the inclusion $i : X \to \mathcal{R}$ is monic, then by the covering homotopy theorem, there is a Forelli projection*

$$P : H^\infty(\mathbb{D}) \to r^* \left(H^\infty(X)\right)$$

*having norm $\|P\|$ bounded above by a constant $C(N)$ depending only on $N$.*

**Example 4.5.10.** *[14], 4. Consider the standard action of $\mathbb{Z} + i\mathbb{Z}$ on $\mathbb{C}$ by translations:*

$$(m + ni) + (x + iy) = (x + m) + i(y + n).$$

*A fundamental compact $Q_{1/2}$ for this action is given by*

$$Q_{1/2} = \left\{ z = x + iy \in \mathbb{C} : \max\left(|x|, |y|\right) \leq \frac{1}{2} \right\}.$$

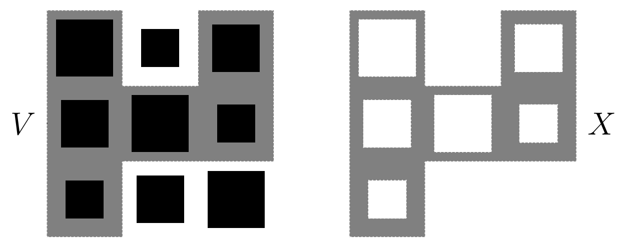
*Define squares of sidelength $t$ homothetic to $Q_{1/2}$,*

$$Q_t = \{z = x + iy \in \mathbb{C} : \max(|x|, |y|) \leq t.\}$$

*Given a function $t : \mathbb{Z} + i\mathbb{Z} \to [1/4, 3/8]$ let $Q : \mathbb{C} + i\mathbb{C} \to \mathcal{P}(\mathbb{C})$ be defined as*

$$Q(m + ni) = (m + ni) + Q_{t(m+ni)}.$$

*This function $Q$ now defines a set of squares centred on elements of the grid $\mathbb{Z} + i\mathbb{Z}$.*

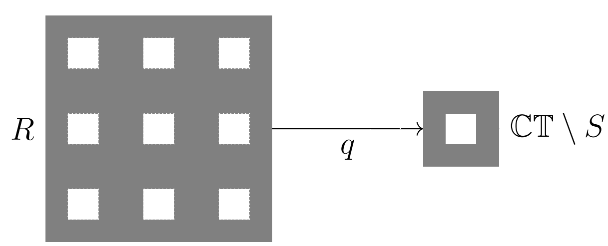



*Now choose some simply connected domain $V \subseteq \mathbb{C}$ satisfying the property that for any $m, n \in \mathbb{Z}$, either $V \cap Q(m + ni) = \emptyset$ or $Q(m + ni) \subseteq V$. That is, $V$ completely contains any square $Q(m + ni)$ which it intersects. Now define the domain $X$*

$$X = V \setminus \bigcup_{(m,n)} Q(m + ni)$$

*and the domain $R$*

$$R = \mathbb{C} \setminus \bigcup_{(m,n)} \left((m + ni) + Q_{1/5}\right).$$




*Recall that the quotient space $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is the torus $\mathbb{CT}$. Let $S$ be the image of $Q_{1/5}$ in the $\mathbb{CT}$ so that $R$ is the regular covering of $CT \setminus S$ with fundamental group $\mathbb{Z} + i\mathbb{Z}$. Now the* *induced map $\pi_1(X \to R)$ is injective since every void of $X$ contains one in $R$. Consequently, there is a weak* continuous Forelli projection $P : H^\infty(\mathbb{D}) \to r^*(H^\infty(X))$.*

**Remark 4.5.11.** *[62] The Widom condition on a Carathéodory hyperbolic Riemann surface $\mathbb{R}$ having an associated Fuchsian group $\Gamma$ is equivalent to the algebra $H^\infty(\Gamma, \chi)$ containing non-constants for each character $\chi : \Gamma \to S^1$. (This is based on reinterpreting $H^\infty$ sections as functions on the universal cover.)*

*[40] Widom type surfaces are the only infinitely connected surfaces for which the theory of Hardy spaces $H^p$ has been developed to any extent, as many concepts such as the boundary, radial lines, Stolz regions used in the study of $H^p(\mathbb{D})$ have analogues in this case. They have many bounded holomorphic functions and these separate points and directions.*

*Jones and Marshall (in [46], 295) observe that results in [74] and [62] imply that if the Forelli projection $P : H^\infty(\mathbb{D}) \to r^*(H^\infty(\mathcal{R}))$ exists for a connected Carathéodory hyperbolic Riemann surface $\mathcal{R}$, then it must be of Widom type.*

**Remark 4.5.12.** *The corona theorem does not hold for all Riemann surfaces. B. Cole (see [34], 46) constructed a non-planar counterexample using a sequence of finite bordered Riemann surfaces $\mathcal{R}^{(k)}$ and functions $f_1^{(k)}, f_2^{(k)}$ comprising corona data in $H^\infty(\mathcal{R}^{(k)})$ for which solutions $g_1^{(k)}, g_2^{(k)} \in \mathcal{R}^{(k)}$ such that $f_1^{(k)} g_1^{(k)} + f_2^{(k)} g_2^{(k)} = 1$ must satisfy $\sup\left(\left\|g_1^{(k)}\right\| + \left\|g_2^{(k)}\right\|\right) = \infty$. B. Oh [59] constructed explicit $\mathcal{R}^{(k)} \subset \mathbb{C}^2$ with such $f_1^{(k)}, f_2^{(k)}$ and Nakai [57] extended Cole's result to obtain an $\mathcal{R}$ of Widom type.*

*In [33], 3.4, Gamelin shows that the existence of such a sequence of planar domains is equivalent to the failure of the corona theorem for a planar theorem and Jones and Marshall (in [46], 295) observe that it can be shown that if the corona theorem fails for a planar domain, it fails for one of Widom type.*

*Other examples were found later by Barrett and Diller [5], who showed that the homology open cover of a compact subset of $S^2$ having positive logarithmic capacity and vanishing length has a corona, and Lárusson [53] who used a construction based on an interpolation result for* *holomorphic functions of exponential growth on unbranched coverings of complex projective manifolds. By using certain vanishing theorems for $L^2$ cohomology groups on the coverings, he showed that the restriction map of holomorphic functions of slow growth from $Y \to X$ is an isometry for $d$ large enough. (See the setup of Brudnyi's related result for $H^\infty$ next.)*

*Brudnyi [12] showed that when $(Y, r, M)$ is an unbranched covering of a complex connected projective manifold $M \subset \mathbb{CP}^n$ of dimension $n \geq 2$ and $C$ is the intersection of $M$ with at most $n-1$ generic hypersurfaces of degree $d$ in $\mathbb{CP}^n$, that the restriction $H^\infty(Y) \to H^\infty(r^{-1}(C))$ is an isometry for $d$ sufficiently large.*

*When $Y$ is either an open Euclidean ball or polydisk in in $\mathbb{C}^n$ for $n \geq 2$, this can be applied to find a compact Riemann surface $S_n$ and its regular covering $S_n \to \mathfrak{S}_n$ such that $\mathfrak{S}_n \hookrightarrow Y$ where this embedding induces an isometry of the $H^\infty$ spaces. Thus the maximal ideal spaces $\mathcal{M}(H^\infty(\mathfrak{S}_n))$ and $\mathcal{M}(H^\infty(Y))$ are homeomorphic. In particular, the covering dimension of $\mathcal{M}(H^\infty(\mathfrak{S}_n))$ is $\geq n$ and the corona theorem fails for $\mathfrak{S}_n$.*

# Chapter 5

# Summary

In this thesis we considered two interesting problems which incorporate bounded holomorphic functions.

Kubo-Martin-Schwinger states were first used to model equilibrium in quantum statistical mechanical systems, but are now an active area of study for number theoretic systems and graph theoretic systems. In the graph theoretic context, they reflect the dynamics of the associated gauge action whether for the basic digraphs I looked at or the more general Cuntz-Pimsner algebras.

The corona problem for a Riemann surface concerns whether the point evaluations are dense in its maximal ideal space. This is solved for the disk using various estimates and from there extended to other Riemann surfaces using Forelli projections to create solutions on their universal cover. The most general cases that can be solved involve a variety of techniques.

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