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INVERSION OF THE POISSON-HANKEL TRANSFORM

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ABSTRACT. The Poisson-Hankel transform is defined as an integral transform of the initial temperature function, with the kernel as the source solution of the generalized heat equation. In this paper a technique involving integral and differential operators has been used to effect the inversion of the Poisson-Hankel transform.

KEY WORDS AND PHRASES. Heat equation, Poisson-Hankel transform, Gamma function, Bessel functions, Whittaker function.

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1. INTRODUCTION.

The Poisson-Hankel transform has as its kernel the fundamental solution of the generalized heat equation. A special case of the Poisson-Hankel transform, called the reduced Poisson-Hankel transform has been studied in [1], where a differential operator of Laguerre-Polya class [2] has been used to effect its inversion. A general theory of these type of operators has been developed by Widder [2], but can not be applied to the more general Poisson-Hankel transform. Our object in this paper is to establish a procedure for the inversion of this transform in its general form. Our technique consists of applying an integral operator and a differential operator on the transform successively to retrieve the unknown function, cf [3]. The differential operator is of the Laguerre-Polya class.

We shall also deduce the inversions of the Weierstrass Hankel transform and the reduced Poisson-Hankel transform as special cases of our inversion algorithm.

In the end we give an example to illustrate the result of the main theorem.

2. DEFINITIONS AND PRELIMINARIES.

The generalized heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\nu}{x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial t} u(x,t), \qquad \nu \geq 0. \tag{2.1}$$

A C^2 solution of (2.1) is called a generalized temperature. The fundamental solution of (2.1) is the function

$$G(x;t) = (2t)^{-\nu-1/2} e^{-x^2/4t}.$$

We define the function associated with $G(x;t)$ by

$$G(x,y;t) = 2^{\nu-3/2}\,\Gamma(\nu + \tfrac{1}{2})t^{-1}\,(xy)^{1/2-\nu} e^{-\frac{x^2+y^2}{4t}}\, I_{\nu-1/2}\left(\frac{xy}{2t}\right), \quad \nu \geq 0, \tag{2.2}$$

$I_\nu(z)$ being the Bessel function of imaginary argument and order ν. The function $G(x,y;t)$ is the source solution of the generalized heat equation (2.1). Note that $G(x,0;t) = G(x;t)$.

The Poisson-Hankel transform is defined by

$$U(x,t) = \int_0^\infty G(x,y;t)\phi(y)d\mu(y), \quad 0 < t < \infty \tag{2.3}$$

where $\quad d\mu(y) = \dfrac{2^{1/2-\nu}}{\Gamma(\nu + \frac{1}{2})} y^{2\nu} dy$.

The convergent Poisson-Hankel transform defines a generalized temperature $U(x,t)$ with initial temperature

$$U(x,0+) = \phi(x).$$

Next, some operational considerations.

From the Euler product of the gamma function

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots (n-1)}{z(z+1)\cdots(z+n-1)} n^z$$

one can easily show that

$$\frac{1}{\Gamma(\alpha - \beta z)} = \lim_{n\to\infty} \frac{\beta^n}{(n-1)!} n^{-\alpha+\beta z} \prod_{k=1}^{n} \left(\frac{\alpha + k - 1}{\beta} - z\right)$$

$$= \lim_{n\to\infty} \frac{\beta^n}{(n-1)!} n^{-\alpha+\beta z} P_n(z) ,$$

$p_n(z)$ being a polynomial in z of order n. Now we define the operator

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} = \lim_{n \to \infty} \frac{n^{-\nu-1/2}}{(n-1)!2^n} n^{\theta/2} p_n(\theta) ,$$

where $\theta = -x \frac{d}{dx}$. Except for the factor $n^{\theta/2}$, $\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)}$ is the Euler differential operator. To obtain the intended interpretation of the operator $n^{\theta/2}$, we write

$$n^{\theta/2} = e^{\theta \ln n/2} = \lim_{N \to \infty} \sum_{k=0}^{N} \left(\frac{\ln n}{2} \right)^k \frac{1}{k!} \theta^k$$

$$= \lim_{N \to \infty} q_N(\theta) ,$$

q_N, a polynomial in θ of degree N. To see the effect of $n^{\theta/2}$ on a function x^α, where α is a constant, first note that

$$\theta^n [x^\alpha] = (-\alpha)^n x^\alpha ,$$

and hence

$$p_n(\theta)[x^\alpha] = p_n(-\alpha) x^\alpha ,$$ where p_n is

a polynomial of degree n. Now,

$$n^{\theta/2} [x^\alpha] = \lim_{N \to \infty} q_N(\theta)[x^\alpha]$$

$$= \lim_{N \to \infty} q_N(-\alpha) x^\alpha$$

$$= n^{-\alpha/2} x^\alpha .$$

With this understanding, one can readily see that

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} [x^\alpha] = \frac{1}{\Gamma(\nu + \frac{1}{2} + \frac{1}{2}\alpha)} x^\alpha \tag{2.4}$$

Thus, $\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)}$ will be called a linear differential operator of infinite order and the effect of this operator on a function x^α is to reproduce it with a constant factor. This operator is of Laguerre-Pólya class and further properties of the operator of this class are well known, cf [5].

Next we give two applications of this operator for future reference. First,

$$\frac{1}{\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\theta)} e^{-a^2x^2} = \frac{1}{\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\theta)} \sum_{k=0}^{\infty} \frac{(-\alpha^2x^2)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-\alpha^2)^k}{k!} \frac{1}{\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\theta)} [x^{2k}].$$

$$= \sum_{k=0}^{\infty} \frac{(-\alpha^2)^k}{k!} \cdot \frac{x^{2k}}{\Gamma(\nu+\frac{1}{2}+k)}$$

$$= (\alpha x)^{1/2-\nu} \cdot J_{\nu-1/2}(2\alpha x) \qquad (2.5)$$

Also, for $\nu + \frac{1}{2} > 0$,

$$\frac{1}{\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\theta)} (1+\frac{x^2}{a})^{-(\nu+1/2)} = \frac{1}{\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\theta)} \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(\nu+\frac{1}{2}+k)}{k!\; \Gamma(\nu+\frac{1}{2})} (\frac{x^2}{a})^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(\nu+\frac{1}{2}+k)}{k!\; \Gamma(\nu+\frac{1}{2})} (\frac{1}{a})^k \frac{1}{\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\theta)} [x^{2k}]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(\nu+\frac{1}{2}+k)}{k!\; \Gamma(\nu+\frac{1}{2})} (\frac{1}{a})^k \frac{x^{2k}}{\Gamma(\nu+\frac{1}{2}+k)}$$

$$= \frac{1}{\Gamma(\nu+\frac{1}{2})} e^{\frac{-x^2}{a}} \qquad (2.6)$$

We shall now consider some properties of the function $G(s,y;t)$, $s = \sigma + i\tau$, defined as

$$G(s,y:t) = 2^{\nu-3/2}\, \Gamma(\nu+\frac{1}{2})t^{-1} e^{-(\frac{s^2+y^2}{4t})} (sy)^{1/2-\nu} I_{\nu-1/2}(\frac{sy}{2t}), \qquad (2.7)$$

based on the equation (2.2) above, where $\nu \geq 0$, $t > 0$, $y > 0$.

LEMMA 2.1. If $G_\nu(s,y;t) = G(s,y,t)$ is the function defined in the equation (2.7), and A_ν and B_ν are some constants, then

(i) $$\left|G(s,y;t)\right| \leq \left|A_\nu t^{-1/2} y^{-\nu} (\sigma^2+\tau^2)^{-\nu/2}\right| e^{-\frac{(\sigma-y)^2-\tau^2}{4t}}, \qquad (2.8)$$

and

$$\text{(ii)}\quad \left|\frac{\partial}{\partial s} G(s,y;t)\right| = \left|\frac{s}{2t}\left[\frac{y^2}{2\nu+1} G_{\nu+1}(s,y;t) - G(s,y;t)\right]\right|$$

$$\leq t^{-3/2}\, |y|^{-\nu}\, (\sigma^2+\tau^2)^{-\nu/2}\, e^{-\frac{(\sigma-y)^2-\tau^2}{4t}}$$

$$\left[\left|A_\nu y\right|(\sigma^2+\tau^2)^{1/2} + \left|B_\nu\right|\right]. \tag{2.9}$$

PROOF. By using the asymptotic expansion of the Bessel function $I(z) \sim \frac{e^z}{(2\pi z)^{1/2}}$, $|z| \to \infty$ and definition (2.7) conclusion (i) follows immediately.

Conclusion (ii) follows by direct differentiation and making use of conclusion (i).

As direct consequences of the above lemma, we have that

$$\text{(i)}\quad \left|G(x,y;t)\right| \leq \left|A_\nu\right| t^{-1/2}\, (xy)^{-\nu}\, e^{-\frac{(x-y)^2}{4t}}, \tag{2.10}$$

(ii) $\frac{\partial}{\partial s} G(s,y;t)$ is a continuous function of the variables s and y.

LEMMA 2.2. Let $\int_0^\infty y^\nu e^{-\alpha y^2} |\Phi(y)|\, dy < \infty$, for positive α and $\nu \geq 0$.

Then
$$U(x,t) = \int_0^\infty G(x,y;t)\Phi(y)\,d\mu(y)$$

exists for $0 \leq x < \infty$ and can be analytically extended into the complex plane so that U(s,t) is analytic for $\sigma = \mathrm{Re}(s) \geq 0$.

PROOF. Using the estimate (2.10) and the value

$$d\mu(y) = \frac{2^{1/2-\nu}}{\Gamma(\nu+\frac{1}{2})}\, y^{2\nu}\, dy$$

we have

$$\left|U(x,t)\right| \leq \int_0^\infty \left|G(x,y;t)\ \phi(y)\ d\mu(y)\right|$$

$$\leq \left|A_\nu(x,t)\right| \int_0^\infty y^\nu\, e^{-\frac{(x-y)^2}{4t}} \left|\phi(y)\right| dy$$

Since
$$(x-y)^2 \geq \frac{1}{2} y^2 - x^2, \qquad 0 \leq y < \infty, \tag{2.11}$$

therefore, $$e^{-\frac{(x-y)^2}{4t}} \le e^{\frac{x^2}{4t} - \frac{y^2}{8t}},$$

and $$\left|U(x,t)\right| \le \left|B_\nu(x,t)\right| \int_0^\infty y^\nu e^{-\frac{y^2}{8t}} \left|\phi(y)\right| dy < \infty$$

due to the hypothesis with $\alpha = \frac{1}{8t}$, $t > 0$. Hence, the integral defining the function $U(x,t)$ exists and is, in fact, absolutely convergent. Now we consider

$$U(s,t) = \int_0^\infty G(s,y;t)\, \phi(y) d\mu(y), \quad s = \sigma + i\tau.$$

Now using the estimate (2.8) of $G(s,y:t)$, we have

$$\left|U(s,t)\right| \le \int_0^\infty \left|G(s,y;t)\, \phi(y)\, d\mu(y)\right|$$

$$\le A_\nu(\sigma,\tau,t) \int_0^\infty y^\nu e^{-\frac{(\sigma-y)^2}{4t}} \left|\phi(y)\right| dy$$

$$\le A_\nu(\sigma,\tau,t) \int_0^\infty y^\nu e^{-\frac{y^2}{8t}} \left|\phi(y)\right| dy < \infty.$$

using the inequality (2.11) and the hypothesis. Hence, the function $U(s,t)$ exists and is defined by an absolutely convergent integral. Now to prove that $U(s,t)$ is analytic in the half-plane $\sigma \ge 0$, we need to show that

$$\int_0^\infty \frac{\partial}{\partial s} G(s,y;t)\, \phi(y)\, d\mu(y)$$

converges uniformly in the region $\sigma \ge 0$.

By making use of the estimate (2.9), we obtain

$$\left| \int_0^\infty \frac{\partial}{\partial s} G(s,y:t) \phi(y) d\mu(y) \right| \le \left|A_\nu(\sigma,\tau,t)\right| \int_0^\infty y^{\nu+1} e^{-\frac{(\sigma-y)^2}{4t}} \left|\phi(y)\right| dy$$

$$+ \left|B_\nu(\sigma,\tau,t)\right| \int_0^\infty y^\nu e^{-\frac{(\sigma-y)^2}{4t}} \left|\phi(y)\right| dy$$

Now due to the hypothesis and using the inequality (2.11), both the integrals on the right hand side above, converge for all s and for $t > 0$, giving us the desired result and hence the lemma.

As corollaries of Lemma 2.2, we have

$$\left|U(s,t)\right| \le A_\nu(t)\ (\sigma^2 + \tau^2)^{-\nu/2}\ e^{\frac{\sigma^2 + \tau^2}{4t}}, \tag{2.12}$$

where $s = \sigma + i\tau$ and $t > 0$; and

$$U(ix,t) = O(e^{\frac{x^2}{4t}}), \quad x \to \infty$$
$$= O(1), \quad x \to 0\ . \tag{2.13}$$

3. THE INVERSION.

We give below a lemma which is a direct consequence of a general result, [2; Theorem 2.1].

LEMMA 3.1. If $f(x) = 2\int_0^\infty \phi(t)\ \frac{1}{t}\ (\frac{t}{x})^{2\nu+1}\ e^{-t^2/x^2}\ dt$, $x > 0$, $\nu > 0$, then

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)}\ [f(x)] = \phi(x), \qquad 0 < x < \infty.$$

PROOF. We write the above integral as

$$f(x) = \int_0^\infty \phi(t)\ \frac{1}{t}\ k(\frac{x}{t})\ dt,$$

where

$$k(x) = 2e^{-1/x^2}\ x^{-(2\nu + 1)}.$$

Now the Mellin transform of $k(x)$ is $k^*(s) = \Gamma(\nu + \frac{1}{2} - \frac{1}{2}s)$, $\sigma < 2\nu + 1$ and $\frac{1}{k^*(s)}$ is of Laguerre-Pólya class. Thus

$$\frac{1}{k^*(\theta)}\ f(x) = \phi(x) \quad \text{or} \quad \frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\ )}\ f(x) = \phi(x), \qquad x > 0\ .$$

The Main Theorem: Let $\int_0^\infty \left|y^\nu\ e^{-\alpha y^2}\ \phi(y)\right| dy < \infty$, $\nu > 0$, $\alpha > 0$ and

$$U(x,t) = \int_0^\infty G(x,y:t)\ \phi(y)\ d\mu(y)$$

be the Poisson-Hankel transform. If

$$R(x;t) = \Gamma(\nu + \tfrac{1}{2}) \int_0^\infty e^{-\frac{v^2x^2}{16t^2}} G(v;t)\ U(iv,t)\ d\mu(v) \tag{3.1}$$

then

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\phi)} R(x;t) = e^{-x^2/4t}\ \phi(x), \qquad x > 0,\ t > 0,$$

where the functions $G(v;t)$, $G(x,y;t)$ and $d\mu(v)$ are defined above.

PROOF. From the result (2.8) and the definitions of the functions $G(v:t)$ and $d\mu(v)$, it is clear that the integral defining $R(x;t)$ exists. Also note that

$$U(iv,t) = \int_0^\infty G(iv,y;t)\ \phi(y)\ d\mu(y)$$

exists due to Lemma 2.2.

Then we can write

$$R(x,t) = \Gamma(\nu + \tfrac{1}{2}) \int_0^\infty e^{-\frac{v^2x^2}{16t^2}} G(v;t)\ d\mu(v) \int_0^\infty G(iv,y;t)\ \phi(y)\ d\mu(y)$$

$$= \Gamma(\nu + \tfrac{1}{2}) \int_0^\infty \phi(y)\ d\mu(y) \int_0^\infty e^{-\frac{v^2x^2}{16t^2}} G(v;t)\ G(iv,y;t)\ d\mu(v), \tag{3.2}$$

the change of order of integration can be justified by absolute convergence; we need only to observe that

$$\int_0^\infty \left| e^{-\frac{v^2x^2}{16t^2}} G(v;t)\ d\mu(v)\right| \int_0^\infty \left|G(iv,y;t)\ \phi(y)\ d\mu(y)\right|$$

$$\leq K\ t^{-(\nu + 1)} \int_0^\infty \left|e^{-\frac{v^2x^2}{16t^2}} v^\nu\ dv\right| \int_0^\infty \left| y^\nu\ e^{-y^2/4t}\ \phi(y)\ dy\right| < \infty,$$

by hypothesis.

From the definitions of the functions $G(v;t)$, $G(iv,y;t)$ and $d\mu(v)$, the v-integral in (3.2) can be written as

$$(2t)^{-(\nu + 3/2)} e^{-y^2/4t} y^{1/2-\nu} \int_0^\infty e^{-\frac{v^2x^2}{16t^2}} v^{\nu+1/2} J_{\nu-1/2}\left(\frac{vy}{2t}\right) dv$$

$$= 2^{\nu+1/2} e^{-y^2/4t-y^2/x^2} x^{-(2\nu + 1)}, \qquad [2, p.29).$$

and we then obtain,

$$R(x,t) = 2x^{-(2\nu + 1)} \int_0^\infty e^{-y^2/4t-y^2/x^2} y^{2\nu} \phi(y)\, dy$$

$$= 2 \int_0^\infty e^{-y^2/4t} \phi(y) e^{-y^2/x^2} \frac{1}{y} \left(\frac{y}{x}\right)^{2\nu + 1} dy .$$

Now the Lemma (3.1) is applicable and hence

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} R(x,t) = e^{-x^2/4t} \phi(x) , \tag{3.3}$$

establishing the inversion of the Poisson-Hankel transform.

It is to be noted that the transforming function $R(x,t)$ defined by (3.1) is in fact the modified Laplace transform of $U(ix,t)$. This can be recognized if we simplify and write

$$R(x,t) = (4t)^{-(\nu+1/2)} \int_0^\infty e^{-p\gamma(x,t)} p^{\nu-1/2} U(2p^{1/2},t)\, dp$$

where $\gamma(x,t) = \frac{x^2}{16t^2} + \frac{1}{4t}$, $t > 0$. Also note that the above inversion algorithm is valid for the entire function ϕ having a series expansion. The condition on ϕ simply restricts its growth.

Next we shall discuss some special cases. Let $\lim_{t \to 1} U(x,t) = f(x)$. Then the Poisson-Hankel transform (2.3) becomes the Weierstrass-Hankel transform and is given by

$$f(x) = \int_0^\infty G(x,y;1)\, \phi(y)\, d\mu(y).$$

Now write $R(x,1) = R(x)$, so from (3.1)

$$R(x) = \Gamma(\nu + \tfrac{1}{2}) \int_0^\infty e^{-\frac{v^2x^2}{16}} G(v;1)\, f(iv)\, d\mu(v).$$

which on simplifying gives

$$R(x) = \frac{1}{4^{\nu+1/2}} \int_0^\infty v^{2\nu} e^{-\frac{v^2x^2}{16} - \frac{v^2}{4}} f(iv)\, dv.$$

According to the inversion algorithm (3.3), we have

$$e^{-x^2/4}\,\phi(x) = \frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} R(x)$$

$$= \frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} \cdot \frac{1}{4^{\nu+1/2}} \int_0^\infty v^{2\nu} e^{-\frac{v^2x^2}{16} - \frac{v^2}{4}} f(iv)\,dv$$

$$= \frac{1}{4^{\nu+1/2}} \int_0^\infty v^{2\nu} e^{-v^2/4} f(iv)\,dv \cdot \frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} e^{-\frac{v^2x^2}{16}},$$

formally. Now using the result (2.5), we obtain,

$$e^{-x^2/4}\,\phi(x) = \frac{1}{4^\nu} \int_0^\infty e^{-\nu^2/4}\, v^{\nu+1/2}\, x^{1/2-\nu}\, J_{\nu-1/2}\left(\frac{vx}{2}\right) f(iv)\,dv.$$

Thus,

$$\Phi(x) = \int_0^\infty G(ix,v;1)\, f(iv)\, d\mu(v),$$

giving the inversion of the Weierstrass-Hankel transform and agreeing with the inversion given in [4].

Now if we write $G(0,y;t) = G(y;t)$ and $U(0,t) = f(t)$, then the Poisson-Hankel transform given by (2.3) becomes

$$f(t) = \int_0^\infty G(y;t)\, \phi(y)\, d\mu(y),$$

and is called the reduced Poisson-Hankel transform. We can write it, using the definitions of G and $d\mu$ and making a suitable change of variable, as

$$f(t^2/4) = \frac{2}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-y^2/t^2} \frac{1}{y} \left(\frac{t}{y}\right)^{-(2\nu+1)} \phi(y)\,dy.$$

Hence by Lemma (3.1), we have

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} f\left(\frac{x^2}{4}\right) = \frac{1}{\Gamma(\nu + \frac{1}{2})} \phi(x)$$

or,

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} U\left(0,\frac{x^2}{4}\right) = \frac{1}{\Gamma(\nu + \frac{1}{2})} U(x,0),$$

This establishes the inversion of the reduced Poisson-Hankel transform as in [1].

Next we shall illustrate the inversion procedure for the Poisson-Hankel transform by an example.

Let

$$\phi(x) = x^{\alpha},\ 2\nu + \alpha > -1,\ \nu > -\frac{1}{2}\ .$$

The function satisfies the condition of the main theorem, and

$$U(ix,t) = \int_0^{\infty} G(ix,y;t)y^{\alpha}\, d\mu(y)$$

$$= (2t)^{-1}\, e^{x^2/4t}\, x^{1/2-\nu} \int_0^{\infty} e^{-y^2/4t}\, y^{\alpha+\nu+1/2}\, J_{\nu-1/2}\left(\frac{xy}{2t}\right) dy,$$

$$= \frac{\Gamma(\nu + \frac{1}{2}\alpha + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} (4t)^{1/2(\alpha+\nu+1/2)} x^{-1/2-\nu} e^{x^2/8t} M_{1/2(\alpha+\nu+1/2),1/2(\nu-1/2)}(x^2/4t),$$

[5, p. 185], M being the Whittaker function.

Now

$$R(x,t) = \Gamma(\nu + \tfrac{1}{2}) \int_0^{\infty} e^{-v^2x^2/16t^2}\, G(v;t)\ U(iv:t)\ d\mu(v)$$

$$= \frac{\Gamma(\nu + \frac{1}{2}\alpha + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} (4t)^{1/2(\alpha-\nu-1/2)} \int_0^{\infty} e^{-v^2\left(\frac{x^2}{16t^2} + \frac{1}{8t}\right)}$$

$$v^{\nu-1/2} M_{1/2(\alpha+\nu+1/2),\ 1/2(\nu-1/2)}(v^2/4t)dv$$

$$= \Gamma(\nu + \tfrac{1}{2}\alpha + \tfrac{1}{2})\ x^{\alpha}\ \left(1 + \frac{x^2}{4t}\right)^{-(\nu+\alpha/2\ +\ 1/2)}$$

[5, p. 215].

Hence,

$$\frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} [R(x,t)] = \Gamma(\nu + \tfrac{1}{2}\alpha + \tfrac{1}{2}) \frac{1}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\theta)} \left| x^{\alpha}\left(1 + \frac{x^2}{4t}\right)^{-(\nu+\alpha/2\ +\ 1/2)} \right|$$

$$= e^{-x^2/4t}\ x^{\alpha} \quad \text{by (2.3)}$$

$$= e^{-x^2/4t}\ \phi(x),$$

according to the main theorem, whence, as predicted,

$$\phi(x) = x^{\alpha}.$$

REFERENCES

1. HAIMO, D.T. and CHOLEWINSKI, F.M. Inversion of the reduced Poisson-Hankel Transform, J. d'Analyse Math. 25 (1972), 323-343.

2. WIDDER, D.V. Inversion of a Convolution Transform by use of Series, J. d'Analyse Math. 21 (1968), 293-312.

3. NASIM, C. An Inversion Formula for Hankel Transform, Pacific J. of Math., Vol. 57, No. 1, 1975.

4. CHOLEWINSKI, F.M. and HAIMO, D.T. The Weierstrass Hankel Convolution Transform, J. d'Analyse Math. 17 (1966), 1-58.

5. ERDELYI, A. et al. Tables of Integral Transform, Vol. 2, New York, 1954.