THE UNIVERSITY OF CALGARY

CENTERED FORMS AND THEIR ALGORITHMIC APPROACHES

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "**Centered Forms and Their Algorithmic Approaches**" submitted by Paul G. Bao in partial fulfillment of the requirements for the degree of the Doctor of Philosophy.

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ABSTRACT

The centered form is an efficient and effective tool for evaluating the range of a function with error estimation. This thesis is concerned with centered forms for different kinds of functions and algorithmic approaches to these centered forms.

The thesis provides a systematic theory dealing with complex centered forms for polynomial, rational, analytic and meromorphic functions. Using this theory, the three basic properties of inclusion, convergence and inclusion isotonicity are proven systematically for all the above functions. It generalizes centered forms to multi-variate functions and presents low complexity algorithms for evaluating these forms. The thesis shows for the first time that for a polynomial there are a great number of different centered forms evaluated over different arrangements of variables. An asymptotic formula for estimation of this number is given. An algorithm is furthermore presented to generate all the variable arrangements which produce different forms for a polynomial. The thesis also defines a new type of low complexity Taylor forms for multi-variate functions. Evaluation of these forms needs no computation of the derivatives and only requires the repetitive division of the polynomials and the rational functions by scalar linear factors. A low

complexity algorithm for evaluating these forms is given. Finally the thesis defines a type of special Taylor expansions for multi-variate polynomials and rational functions and a low complexity algorithm for evaluating these forms. The number of zero entries and the number of interval variables replaced by scalar variables in these forms are both estimated.

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$\aleph^n$	n-tuples of natural numbers	61
C	the set of all complex numbers	17
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Chapter 1

Introduction

1.1 Background and Motivation

In science and mathematics (analysis) one usually works with continua, that is, physical quantities and variables that can take on an infinite number of values hence requiring infinite precision for their representation. Representation and numerical calculation on a digital computer can, however, only take place in finite spaces. A transition therefore has to be made from the infinite spaces to the finite spaces via approximations. This transition is most elegantly done by introducing intervals having finitely representable upper and lower limits, but containing real quantities not representable in this manner.

This leads in a natural manner to an interval arithmetic and a corresponding interval analysis.

Interval analysis is therefore recognized as a very useful tool for developing algorithms in numerical analysis. These algorithms have the important property that they are able to account for all data, truncation and computational errors.

The groundwork for interval analysis was laid by the publication of the book "Interval Analysis" by Moore[23]. Many of the basic ideas of interval analysis were here made available to a large audience. A number of books[2, 25, 34, 35] then appeared discussing aspects of interval analysis that were developed since the publication of [23]. Interval analysis has the advantage that it generates both an approximation to the exact solutions to a given problem and the magnitude of the error for the approximation. For example, when evaluating a polynomial at a point using double precision or significant digit arithmetic, we only obtain an approximate value. It is unknown how far this approximation is from the value obtained using infinite precision. Using interval arithmetic, we evaluate the polynomial over an interval containing this point. The result interval contains the polynomial value at that point. If we choose the center of the interval to be an approximation to the true

value, then the error is bounded by the radius of the interval.

Interval arithmetic can easily be built into the software for a computer. For a given elementary operation, a low-order bit is added to the right-end point and a low-order bit is subtracted from the left-end point to compute an interval as small as possible containing the real interval result. This process is called rounded interval arithmetic. If the exact value of initial data can be contained in intervals, we will obtain a final interval containing the exact result using this arithmetic for practical computation with round-off error. Taking the center point as an approximate value, the radius of this computed interval will give an error bound. The details of both exact and rounded interval analysis are well covered in for example [2].

One of the most interesting ideas presented in[23] was that of the centered form, a tool for the computation of the range, or an inclusion for the range, of a function over an interval. This form was introduced by Moore in[23]. When he computed the range of specific functions, Moore noticed that he obtained better results on the average if he developed the functions around the center of the domain of the function. Moore's original definition for centered forms was not precise. Hansen (1969) therefore defined a centered form as the natural extension of the Taylor expansion of the polynomial developed at

a point. Explicit forms were given by Hansen[12] for polynomials and by Ratschek[33] for rational functions. Krawczyk's[19] (1983) recursively defined centered forms are also of importance due to their low complexity.

Alefeld-Herzberger[2] (1974) defined centered forms for real functions. A very convenient and useful version may be found in Krawczyk-Nickle[21] (1982). Centered forms for interval valued polynomials are defined in Rokne[37] (1981). Centered forms for functions in k variables were described by Ratschek-Schröder[36].

The so-called quadratic convergence is a measure of how quickly good inclusions for the range of functions can be computed. It was conjectured that centered forms are quadratically convergent with respect to the width of the domain interval. This was verified by Hansen[12]. Because of the property of quadratic convergence, centered forms are widely used for the inclusion estimate of the range of functions, which guarantees an efficient improvement of the inclusion by subdivision methods.

A natural generalization of real intervals to the complex plane is provided by circular intervals. Based on this generalization, a circular interval analysis was developed by Gargantini-Henrici[10]. It has been proven to be a powerful tool for estimating the range of complex functions as well as roots

of complex functions[2, 11, 45, 46]. Centered forms are also defined in this case[11, 46] and they formed the inspiration to this thesis when a conjecture by L. D. Petkovic[29] (1985) concerning the inclusion isotonicity of the circular centered forms was solved as part of this thesis. It was conjectured that the complex centered forms have properties of inclusion, quadratic convergence and inclusion isotonicity. Inclusion and quadratic convergence were partially verified by Rokne and Wu[46]. Inclusion isotonicity was verified for polynomials by Bao and Rokne[3].

This research was then carried further and the following questions were considered based on the fact that different evaluations on different representations (arrangements) of a function over a circular disk will result in different inclusions:

(1) How many distinct forms will be generated using an evaluation on different arrangement of a function?

(2) Are there forms among them which give sharper estimates for the range of the function over others?

(3) Do these forms have the desirable properties of inclusion, convergence and inclusion isotonicity?

(4) Are there low complexity algorithms to compute these forms?

(5) Is there a theory which unifies the centered forms and their properties for polynomials, rational functions, analytic functions and meromorphic functions?

These problems are very important for centered forms both theoretically and practically. They have remained unsolved in varying degrees although many mathematician and computer scientists have worked on these problems for a couple of decades.

The present thesis provides answers to some of the questions by giving detailed theoretical analysis and practical algorithm design for a large class of centered forms and in doing so, it also presents new insights and new interesting problems.

1.2 Scope and Structure of the Thesis

This thesis provides the theoretical basis for centered forms for polynomials, rational functions, analytic functions and meromorphic functions. With this theoretical basis, we present a number of algorithms for evaluating different forms for different functions. In order not to dilute the main theme of the thesis, we stress the art of designing these algorithms and only discuss time

complexity and the practical implementations of the algorithms briefly. The thesis also gives an asymptotic formula estimating the number of the centered forms obtained from different arrangements of the variables for multi-variate functions. Centered form algorithms for complex functions over complex circular disks are discussed in depth. The corresponding algorithms for real centered forms over real intervals can be considered as special cases of the complex form algorithms using real interval arithmetic instead of the complex circular arithmetic and the algorithms for circular forms remain valid for the real forms when the real interval arithmetic is substituted for circular arithmetic. The real form algorithms will therefore not be discussed explicitly but the corresponding results are included implicitly. For simplicity, however, some numerical realizations will be given for real forms, which are enough to illustrate our methods and algorithms for both real and complex forms.

The structure of the thesis is as follows. The introductory chapter gives the historical background of interval circular analysis and centered forms. The chapter introduces the main development of the centered forms during past two decades and discusses briefly the properties of inclusion and convergence of centered forms for the range of functions.

The following chapter gives the basic definitions of interval and circular

analysis, and lists those properties of interval and circular arithmetic which are required in the sequel. All definitions and proofs of propositions in this chapter can be found in[2]. Therefore we only state them for convenience.

The next chapter is dedicated to the establishment of the centered form theory for polynomials and rational functions, the properties of the inclusion, the convergence and the inclusion isotonicity of the centered forms, and the design and description of the algorithms for calculating the centered forms. In this chapter a number of centered forms with different properties are defined and three important properties of the centered forms: inclusion, quadratic convergence and inclusion isotonicity are proven for all forms.

Based on the results in Chapter 3, a number of algorithms with low complexity are designed in detail and their time complexity and implementations are discussed briefly in Chapter 4. A class of low complexity algorithms for multi-variate Taylor and centered forms are given in detail with their time complexity analysis and numerical realizations. A formula is also derived that estimates the number of distinct centered forms for multi-variate functions. This number is surprisingly large for polynomials with the usual sizes of degrees. Some numerical realizations are given to illustrate the algorithms discussed in the chapter.

Chapter 5 introduces a special Taylor form with a special remainder term which gives a very sharp estimate for the range of functions and which can be generated using a low complexity algorithm. The number of zeros in the remainder term and the number of the scalar variables in the gradient and the remainder term of the special Taylor expansion are estimated and asymptotic formulas for both numbers are given. The algorithm to calculate the special Taylor expansion is given in detail. An application of the algorithm to factorable functions is discussed and a numerical example is presented.

Chapter 6 extends the major results in Chapter 3 to meromorphic functions. It proves the three properties of the centered forms for meromorphic functions. Algorithmic approaches with error estimates for meromorphic functions are discussed. Some numerical results are also given for illustration.

Chapter 7 contains some concluding remarks on the results found in Chapters 3 to 6. It is noted that the results developed in the earlier chapters form the basis for a unified theory of centered forms where the three important properties: inclusion, quadratic convergence and inclusion isotonicity play important roles. The chapter is concluded with a summary of the significant results, and suggestions for the future development, improvement and

application of circular analysis and centered forms as well as the designing of algorithms for calculating centered forms. It also raises a question: how to determine an optimal form for a function which gives the best estimation for the range of the function and how to design an algorithm for calculating the form.

Chapter 2

Interval Analysis and Circular Arithmetic

2.1 Real Interval Arithmetic

In this section, we will define real intervals and their arithmetic. Also we will give some basic properties of real interval arithmetic. The proofs of all the propositions can be found in[2], Chapter 1. Complex circular arithmetic will be defined in Section 2.2.

In the sequel, the field of real numbers is denoted by $\Re$ and real numbers are denoted by lowercase letters $a, b, c, \ldots, x, y, z$.

DEFINITION 2.1 *A subset of* $\Re$ *of the form*

$$I = [a, b] = \{t | a \leq t \leq b\}$$

where $a \leq b, a, b \in \Re$, *is called a closed real interval. The set of all closed intervals is called the interval set, denoted by* $\mathbf{I}(\Re)$, *or more simply by* $\mathbf{I}$.

The members of the interval set $\mathbf{I}$ are denoted by uppercase letters $I, A, B, C, \ldots, X, Y, Z$. Real numbers $a \in \Re$ may be considered special members $[a, a]$ of $\mathbf{I}$ and they will be called point intervals.

DEFINITION 2.2 *Two intervals* $I_1 = [a_1, b_1]$ *and* $I_2 = [a_2, b_2]$ *are called equal, if they are equal in the set theoretical sense.*

From this definition, it follows that

$$I_1 = I_2 \text{ iff } a_1 = a_2, b_1 = b_2.$$

Now we define the arithmetic operations on elements in $\mathbf{I}$.

DEFINITION 2.3 *Let* $* \in \{+, -, \cdot, /\}$ *be a binary operation on the set of real numbers* $\Re$. *If* $I_1, I_2 \in \mathbf{I}$, *then a binary operation on* $\mathbf{I}$ *is defined by*

$$I_1 * I_2 = \{x = a * b | a \in I_1, b \in I_2\}.$$

It is assumed that $0 \notin I_2$ in the case $* = /$ and this assumption will be kept implicitly in the sequel.

The operations on intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ may be calculated explicitly as follows

$$I_1 + I_2 = [a_1 + a_2, b_1 + b_2],$$

$$I_1 - I_2 = [a_1 - b_2, b_1 - a_2],$$

$$I_1 \cdot I_2 = [\min\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}, \max\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}],$$

$$I_1/I_2 = [a_1, b_1] \cdot [1/b_2, 1/a_2].$$

Since the function $f(x, y) = x * y, * \in \{+, -, \cdot, /\}$ is a continuous function (for case $/$, $y \neq 0$) on a compact set, it follows that $I_1 * I_2$ is a closed real interval (if $0 \notin I_2$ for case $/$). Hence the set $\mathbf{I}$ is closed under the allowed operations.

In addition to these operations there are further common operations on intervals.

DEFINITION 2.4 *If $s(x)$ is a continuous unary operation on $\Re$, then*

$$s(X) = \left[\min_{x \in X} s(x), \max_{x \in X} s(x)\right]$$

defines a unary operation on $\mathbf{I}$.

We now give some basic and important properties for interval operations.

PROPOSITION 2.1 *Let* $I_1, I_2, I_3 \in \mathbf{I}$. *Then it follows that*

$$I_1 + I_2 = I_2 + I_1, I_1 \cdot I_2 = I_2 \cdot I_1 \ \textit{(commutativity)}, \tag{2.1}$$

$$(I_1 + I_2) + I_3 = I_1 + (I_2 + I_3) \textit{(associativity)}, \tag{2.2}$$

$$(I_1 \cdot I_2) \cdot I_3 = I_1 \cdot (I_2 \cdot I_3) \textit{(associativity)}. \tag{2.3}$$

The interval $X = [0,0]$ *and* $Y = [1,1]$ *are the unique neutral elements with respect to addition and multiplication.* $\mathbf{I}$ *has no zero divisors.*

$$I = I + X = X + I \textit{ for all } I \in \mathbf{I} \textit{ iff } X = [0,0] \tag{2.4}$$

$$I = I \cdot Y = Y \cdot I \textit{ for all } I \in \mathbf{I} \textit{ iff } X = [1,1] \tag{2.5}$$

(2.6)

$$\begin{cases} I_1(I_2 + I_3) \subseteq I_1 I_2 + I_1 I_3 & \textit{(subdistributivity)} \\ a(I_1 + I_2) = aI_1 + aI_2 \qquad a \in \Re & \\ I_1(I_2 + I_3) = I_1 I_2 + I_1 I_3 & \textit{if } bc \geq 0 \textit{ for all } b \in I_2, c \in I_3. \end{cases} \tag{2.7}$$

Proof. The proof can be found in[2] Theorem 1.4. □

A fundamental property of interval operations is inclusion isotonicity.

PROPOSITION 2.2 *Let* $I_i, J_i \in \mathbf{I}, i = 1,2$ *and assuming that*

$$I_i \subseteq J_i, i = 1,2.$$

Then for $* \in \{+, -, \cdot, /\}$ *it follows that*

$$I_1 * I_2 \subseteq J_1 * J_2.$$

Proof. The proof is in[2], Chapter 1. □

Defining the distance between a pair of intervals, we can view **I** as a metric space. The distance is defined as follows.

DEFINITION 2.5 *Let* $I_1 = [a_1, b_1]$ *and* $I_2 = [a_2, b_2]$. *Then the distance between* I_1 *and* I_2 *is defined by*

$$q(I_1, I_2) = \max\{|a_1 - a_2|, |b_1 - b_2|\}.$$

It is easy to see that the map q introduces a metric in **I** since it satisfies the following properties.

$$q(I_1, I_2) \geq 0,$$

$$q(I_1, I_2) = 0 \text{ iff } I_1 = I_2,$$

$$q(I_1, I_2) \leq q(I_1, I_2) + q(I_2, I_3).$$

For degenerate intervals $I_1 = [a_1, a_1]$ and $I_2 = [a_2, a_2]$, the metric q reduces to the ordinary Euclidean distance between two reals

$$q(I_1, I_2) = |a_1 - a_2|.$$

Now since **I** is a metric space, the concepts of convergence and continuity therefore may be introduced as for a metric space.

DEFINITION 2.6 *Let* $I_i = [a_i, b_i], i = 1, 2, \ldots$. *Then we say that sequence* I_i *converges to* $I = [a, b]$, *iff* $\lim_{i \to \infty} q(I_i, I) = 0$.

From the definition, it is easy to show that the following proposition is valid.

PROPOSITION 2.3 *Let* $\{I_i\}_{i=0}^{\infty}$ *satisfy*

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots.$$

Then the sequence $\{I_i\}_{i=0}^{\infty}$ *converges to the interval* $I = \cap_{i=0}^{\infty} I_i$.

Proof. The proof can be found in[2], Chapter 1. □

Now we define the absolute value of an interval.

DEFINITION 2.7 *The absolute value of an interval* $I = [a, b]$ *is defined as*

$$|I| = q(I, [0, 0]) = \max\{|a|, |b|\}.$$

Clearly, we have $|I| = \max_{a \in I} |a|$ and $I_1 \subseteq I_2 \Rightarrow |I_1| \leq |I_2|$.

We have the following properties related to the metric in **I**.

PROPOSITION 2.4 *Let* $I_1 = [a_1, b_1], I_2 = [a_2, b_2], I_3 = [a_3, b_3], I_4 = [a_4, b_4] \in$ **I**. *Then*

$$q(I_1 + I_2, I_1 + I_3) = q(I_2, I_3), \tag{2.8}$$

$$q(I_1 + I_2, I_3 + I_4) \leq q(I_1, I_3) + q(I_2, I_4), \tag{2.9}$$

$$q(aI_1, aI_2) = |a|q(I_1, I_2), a \in \Re, \tag{2.10}$$

$$q(I_1 I_2, I_1 I_3) \leq |I_1| q(I_2, I_3). \tag{2.11}$$

Proof. The proof can be found in[2], Theorem 2.7. □

Finally we define the width of an interval.

DEFINITION 2.8 *The width of an interval* $I = [a, b]$ *is defined as*

$$d(I) = b - a.$$

We give the following propositions for further use, whose proofs can be found in[2], Chapter 2.

PROPOSITION 2.5 *Let* $I_1, I_2 \in \mathbf{I}$ *be real intervals. Then*

$$d(I_1 I_2) \leq d(I_1)|I_2| + |I_1| d(I_2), \tag{2.12}$$

$$d(I_1 I_2) \geq \max\{|I_1| d(I_2), |I_2| d(I_1)\}, \tag{2.13}$$

$$d(aI_1) = |a| d(I_1), \tag{2.14}$$

$$d(I_1^n) \leq n|I_1|^{n-1}d(I_1), n = 1, 2, \ldots, \tag{2.15}$$

$$d((X - x)^n) \leq 2(d(X))^n \text{ for } x \in X, n = 1, 2, \ldots, \tag{2.16}$$

$$|I_1| \leq d(I_1) \leq 2|I_1|, \text{ if } 0 \in I_1. \tag{2.17}$$

PROPOSITION 2.6 *Let $I_1, I_2 \in \mathbf{I}$ be real intervals and I_1 a symmetric interval; i.e., $I_1 = -I_1$. Then it follows that*

$$I_1 I_2 = |I_2| I_1, \tag{2.18}$$

$$d(I_1 I_2) = |I_2| d(I_1). \tag{2.19}$$

PROPOSITION 2.7 *Let $I_1, I_2 \in \mathbf{I}$ be real intervals. Then*

$$d(I_1) = |I_1 - I_1|, \tag{2.20}$$

$$I_1 \subseteq I_2 \Rightarrow (d(I_2) - d(I_1))/2 \leq q(I_1, I_2) \leq d(I_2) - d(I_1). \tag{2.21}$$

2.2 Complex Interval Arithmetic

Circular intervals and circular arithmetic form the natural extension of real intervals to the complex plane. They were first introduced by Gargnatini and Henrici[10] They applied it to develop a method for the simultaneous inclusion of polynomial zeroes. Subsequently it has been used by a number of

authors (see[2, 27, 40]) for a variety of numerical uses such as the computation of eigenvalues[40] and complex numerical integration[27]. Implementation details were discussed in[2] and variations of the arithmetic in[17].

This section will define the circular complex arithmetic operations, including infinite operations, and give all the properties required in the sequel.

DEFINITION 2.9 *Let* $\mathbf{C}$ *be the set of complex numbers and let* $c \in \mathbf{C}$, *and* $\rho > 0$. *The set*

$$Z = \{z \| |z - c| \leq \rho, z \in \mathbf{C}\}$$

is called a circular interval or disk. A circular disk may also be written as

$$Z = \langle c, \rho \rangle$$

and the set of all circular disks will be denoted by $\mathbf{K}(\mathbf{C})$, *or simpler by* $\mathbf{K}$.

The arithmetic operations over $\mathbf{K}$, introduced by Gargantini and Henrici[10], are defined as follows.

DEFINITION 2.10 *Let* $Z_i = \langle c_i, \rho_i \rangle \in \mathbf{K}$, $i = 1, \ldots, n$ *and* $a \in \mathbf{C}$. *Then it follows that*

$$a + Z = \langle a + c, \rho \rangle, \tag{2.22}$$

$$aZ = \langle ac, |a|\rho\rangle, \tag{2.23}$$

$$\sum_{i=1}^{n} Z_i = \langle \sum_{i=1}^{n} c_i, \sum_{i=1}^{n} \rho_i\rangle, \tag{2.24}$$

$$Z_1 Z_2 = \langle c_1 c_2, |c_1|\rho_2 + |c_2|\rho_1 + \rho_1\rho_2\rangle, \tag{2.25}$$

$$\frac{Z_1}{Z_2} = \frac{1}{|c_2|^2 - \rho_2^2}\langle c_1\overline{c_2}, |c_1|\rho_2 + |c_2|\rho_1 + \rho_1\rho_2\rangle, \text{ if } \rho_2 < |c_2|. \tag{2.26}$$

From the definitions it is easily shown that (see[10])

$$\{c_1 * c_2 | c_1 \in Z_1, c_2 \in Z_2\} \subseteq Z_1 * Z_2, \quad * \in \{+, \cdot, /\}. \tag{2.27}$$

Furthermore if $Z_i \in \mathbf{K}, \quad i = 1, \ldots, 4$ and $Z_1 \subseteq Z_2, Z_3 \subseteq Z_4$ then for $* \in \{+, \cdot, /\}$ we have

$$Z_1 * Z_3 \subseteq Z_2 * Z_4. \tag{2.28}$$

The commutative and associative laws hold for the operations $+$ and $\cdot$, but not the distributive law. Only the subdistributivity

$$Z_1(Z_2 + Z_3) \subseteq Z_1 Z_2 + Z_1 Z_3$$

is valid in general, as was the case for real interval arithmetic (eg. Section 2.1.1).

Let $Z_1 = \langle a_0, \sum_{i=1}^{n} |a_i|\rangle$ and $Z_2 = \langle b_0, \sum_{i=1}^{m} |b_i|\rangle$. If $|b_0|^2 - (\sum_{i=1}^{m} |b_i|)^2 > 0$, we then have the following identity

$$\frac{Z_1}{Z_2} = \frac{\langle a_0\overline{b_0}, \sum_{i=0}^{n} |a_i| \sum_{i=0}^{m} |b_i| - |a_0 b_0|\rangle}{|b_0|^2 - (\sum_{i=1}^{m} |b_i|)^2}. \tag{2.29}$$

These operations are now generalized to the infinite case. Let $Z_i = \langle c_i, \rho_i \rangle \in \mathbf{K}$ and let $\rho_i > 0, \quad i = 1, \ldots$. If the limits $c_\infty = \lim_{i\to\infty} c_i$ and $\rho_\infty = \lim_{i\to\infty} \rho_i$ exist, we define

$$\langle c_\infty, \rho_\infty \rangle = \lim_{i\to\infty} \langle c_i, \rho_i \rangle. \tag{2.30}$$

If the series $\sum_{i=1}^{\infty} c_i$ and $\sum_{i=1}^{\infty} \rho_i$ converge then it follows from (2.30) that

$$\sum_{i=1}^{\infty} Z_i = \langle \sum_{i=1}^{\infty} c_i, \sum_{i=1}^{\infty} \rho_i \rangle. \tag{2.31}$$

For the infinite case, the subdistributivity

$$Z \sum_{i=1}^{\infty} Z_i \subseteq \sum_{i=1}^{\infty} Z Z_i \tag{2.32}$$

holds and furthermore

$$Z_\infty \subseteq Y_\infty, \text{ if } Z_i \subseteq Y_i \text{ for } i \geq N \tag{2.33}$$

where N is some positive number.

Let also $Z_1 = \langle a_0, \sum_{i=1}^{\infty} |a_i| \rangle$ and $Z_2 = \langle b_0, \sum_{i=1}^{\infty} |b_i| \rangle$. If $|b_0|^2 - (\sum_{i=1}^{\infty} |b_i|)^2 > 0$, we then have the identity

$$\frac{Z_1}{Z_2} = \frac{\langle a_0 \overline{b_0}, \sum_{i=0}^{\infty} |a_i| \sum_{i=0}^{\infty} |b_i| - |a_0 b_0| \rangle}{|b_0|^2 - (\sum_{i=1}^{\infty} |b_i|)^2}, \tag{2.34}$$

which will be used later.

Chapter 3

Centered Forms for Polynomials and Rational Functions

In this chapter, the idea of a centered form, introduced by Moore[23], is briefly discussed. A formal definition for centered forms is then given for real and complex functions defined by power-series. The three important properties: inclusion, quadratic convergence and inclusion isotonicity are shown to hold for these forms. Some of results presented have already appeared in[3, 33,

41, 46].

Centered forms for uni-variate polynomials and rational functions over real intervals and their properties are treated briefly for completeness. These forms are then generalized to uni-variate complex circular centered forms for functions defined by finite complex power-series. Centered forms for multi-variate polynomials and rational functions will be discussed in Chapter 4 when the corresponding algorithms are presented. Generalized centered forms for meromorphic functions and their properties will be given in Chapter 6.

3.1 Centered Forms

In this section, the centered form for polynomials is defined first. Then the definition is extended to the rational functions. Before defining the complex circular forms, we discuss the real centered forms briefly. The results presented in this section can be found in[33, 41, 46].

Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial, $X \in \mathbf{I}$ an interval and $c = m(X)$. If $p(x)$ is written as

$$p_c(x) = p(c) + g(x)(x - c)$$

then $g(x)$ is a unique polynomial of degree $n-1$. If now each occurrence of the variable x is replaced by X and the arithmtic defined (2.23) to (2.26) is used then the resulting interval $p_c(X)$ is is called *centered form evaluation* of $p(x)$ over X, where the actual interval obtained depends on the order of the evaluation of $g(X)$ (it may itself be a centered form evaluation).

Independently of the evaluation of $g(X)$, we have

$$\overline{p}(X) = \{p(x) | x \in X\} \subseteq p_c(X).$$

Let $f(x) = p(x)/q(x)$, where $p(x) = \sum_{k=0}^{n} a_k x^k$ and $q(x) = \sum_{k=0}^{m} b_k x^k$ are two polynomials, be a rational function, $X \in \mathbf{I}$ an interval and $c = m(X)$.

$$f_{c(0)}(X) = \frac{p_c(X)}{q_c(X)}$$

is called the zeroth order centered form evaluation, or semi-centered form evaluation, of $f(x)$ over X based on the centered form evaluations $p_c(X)$ and $q_c(X)$ over X.

If we now write

$$f(x) = \frac{p(c)}{q(c)} + g(x)(x-c)$$

then

$$f_{c(1)}(X) = \frac{p(c)}{q(c)} + g_{c(0)}(X)(X-c)$$

is a first order centered form evaluation of $f(x)$ over X, where $g_{c(0)}(X)$ is some zeroth order centered form evaluation of the polynomial $g(x)$ defined by

$$g(x) = \left[f(x) - \frac{p(c)}{q(c)} \right] / (x - c).$$

Generally the k-th order centered form evaluation for $f(x)$ can be defined as follows

$$f_{c(k)}(X) = \frac{p(c)}{q(c)} + g_{c(k-1)}(X)(X - c).$$

It is easy to show[33] that

$$\overline{f}(X) \subseteq f_{c(k)}(X).$$

A more specific definition of a set of centered forms is given as follows (see p. 42 of [34]).

Let $f(x) = p(x)/q(x)$ be a rational function, where $p(x)$ and $q(x)$ are polynomials of degree n and m. Let X be a compact interval with $c = m(X)$. For any natural number k, let

$$t_{kh} = p^{(h)}(c) - \sum_{i=0}^{k-1} \begin{pmatrix} h \\ i \end{pmatrix} f^{(i)}(c) q(h-i)(c), h = k, \ldots, k + l$$

where $l = \max(n, m)$.

DEFINITION 3.1 *The interval*

$$fc_{(k)}(X) = f(c) + f'(c)(X-c) + \cdots + f^{(k-1)}(c)\frac{(X-c)^{k-1}}{(k-1)!} \quad (3.1)$$

$$+ \frac{t_{kk}(X-c)^k/k! + \cdots + t_{k,k+l1}(X-c)^{k+l-1}/(k+l-1)!}{q(c) + q'(c)(X-c) + \cdots + q^{(l)}(c)(X-c)^l/i!} \quad (3.2)$$

is called the standard centered form of order k for $f(x)$ over X.

For theoretical and practical purposes, the properties of inclusion, convergence and inclusion isotonicity are of importance for these forms. These properties are verified by the following three theorems which are valid for the forms in Definition 3.1.

THEOREM 3.1 *For $k = 2, 3, \ldots$ the inclusions*

$$\overline{f}(X) \subseteq f_{c(k)}(X) \subseteq f_{c(k-1)}(X)$$

hold.

Proof. The proof can be found in[34], p.45. □

THEOREM 3.2 *Let $f(x)$ be a rational function and $X \in \mathbf{I}$ an interval. Then we have*

$$w[f_{c(k)}(X)] - w[\overline{f}(X)] \leq M(w(X))^2, k = 0, 1, 2, \ldots$$

where $M > 0$ is a constant.

Proof. The proof can be found in[34], Chapter 3. □

THEOREM 3.3 *Let $f(x)$ be a polynomial. Let $X, Y \in \mathbf{I}$ be two intervals satisfying $X \subseteq Y$. Then*

$$f_c(X) \subseteq f_c(Y).$$

Proof. See[21]. □

Now we turn our attention to complex forms. Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial over C. The circular form for $p(z)$ is defined as

$$p(Z) = \sum_{k=0}^{n} a_k Z^k,$$

where $Z \in \mathbf{K}$.

DEFINITION 3.2 *Let $p(z)$ be a complex polynomial. The centered form of $p(z)$ is defined as*

$$p_c(z) = \sum_{k=0}^{n} \frac{p^{(k)}(c)}{k!}(z-c)^k.$$

The evaluation of $p_c(z)$ over Z is denoted by

$$p_c(Z) = \sum_{k=0}^{n} \frac{p^{(k)}(c)}{k!}(Z-c)^k.$$

It can be explicitly written[46, 47] as

$$p_c(Z) = \langle p(c), \sum_{k=1}^{n} |p^{(k)}(c)\rho^k|/k! \rangle.$$

In general, the exact range of $p(z)$ over Z, denoted by $\overline{p}(Z)$, is not a circular interval. However we do have the following inclusion

$$\overline{p}(Z) \subseteq p_c(Z)$$

independent of the method used for evaluating $p_c(Z)$ because of (2.27).

The circular complex centered form can be extended to analytic functions. Here we only present the definition. Its properties will be included in Chapter 6 as a special case of meromorphic functions.

DEFINITION 3.3 *Let $f(z)$ be an analytic function over* **C**. *Writing $f(z)$ as*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(z-c)^k,$$

then a centered form evaluation of $f_c(z)$ over $Z = \langle c, \rho \rangle$ using the arithmetic (2.23) to (2.31) is

$$f_c(Z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(Z-c)^k.$$

It will be seen in Chapter 6 that

$$f_c(Z) = \langle f(c), \sum_{k=1}^{\infty} |f^{(k)}(c)|\rho^k/k! \rangle.$$

Now we generalize the centered forms to complex rational functions. Let $p(z) = \sum_{k=0}^{n} a_k z^k$ and $q(z) = \sum_{k=0}^{m} b_k z^k$ be two complex polynomials. Then $f(z) = p(z)/q(z)$ is a complex rational function.

The range of $f(z)$ over a circular disk is defined as

$$\overline{f}(Z) = \{f(z) | z \in Z\}.$$

Generally $\overline{f}(Z)$ is not a circular disk. How to compute a good circular outer estimate of the range is an interesting and important problem. As for the polynomial case, for each rational function there is an infinite number of expressions which can be used to compute outer disks. The centered form of $f(z)$ will give some advantages over others.

Before defining the centered form evaluations for rational functions, we give the power-sum form evaluations of $f(z)$ over a disk $Z = \langle c, \rho \rangle$.

Let the power-sum form evaluations for the polynomials $p(z) = \sum_{i=0}^{n} a_i z^i$ and $q(z) = \sum_{i=0}^{n} b_i z^i$ be given by

$$\sum_{i=0}^{n} a_i Z^i, \quad \sum_{i=0}^{n} b_i Z^i$$

respectively (see[41]). Then applying the disk division we arrive at a disk

$$f_{p(0)}(Z) = \left(\sum_{i=0}^{n} a_i Z^i\right) / \left(\sum_{i=0}^{m} b_i Z^i\right).$$

This disk is called the zero order power-sum form evaluation for the rational function $f(z)$. In the following theorem the explicit formula for this form is given.

THEOREM 3.4 *Let $f(z) = p(z)/q(z)$ be a rational function. Then the zeroth order power-sum form evaluation*

$$f_{p(0)}(Z) = \langle c_{p(0)}, \rho_{p(0)} \rangle$$

satisfies

$$c_{p(0)} = \frac{1}{A} p(c)\overline{q(c)}, \tag{3.3}$$

$$\rho_{(0)} = \frac{1}{A}\left[(|p(c)| + \sum_{i=1}^{n} |a_i|((|c|+\rho)^i - |c|^i))(|q(c)| + \sum_{i=1}^{m} |b_i|((|c|+\rho)^i - |c|^i) - |p(c)q(c)|\right] \tag{3.4}$$

where

$$A = |q(c)|^2 - \left[\sum_{i=1}^{m} |b_i|((|c|+\rho)^i - |c|^i)\right]^2 \tag{3.5}$$

and assuming $A > 0$. Furthermore $\overline{f}(Z) \subseteq f_{p(0)}(Z)$.

Proof. In[46] it was shown that if $Z_n = \langle c_n, \rho_n \rangle$ is the power-sum form evaluation of the polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$ over $Z = \langle c, \rho \rangle$ then it satisfies

$$c_p = p(c)$$

$$\rho_p = \sum_{i=0}^{n} |a_i|((|c| + \rho)^i - |c|^i).$$

Similarly for $q(z)$ we obtain $Z_q = \langle c_q, \rho_q \rangle$ where

$$c_q = q(c)$$

$$\rho_q = \sum_{i=0}^{m} |b_i|((|c| + \rho)^i - |c|^i).$$

Since $f_{p(0)}(Z) = Z_p/Z_q$ we obtain the explicit formulas (3.3) and (3.4) using the identity (2.29).

The inclusion property is an immediate consequence of the inclusion isotonicity of the operations (2.23) to (2.31). □

We note that in the sequel A is always defined by (3.5).

The higher order power-sum form evaluations for rational functions are now defined. Let

$$f(z) = \frac{p(z)}{q(z)}$$

be rearranged as

$$f(z) = \frac{p(c)}{q(c)} + \frac{s(z)}{q(z)}. \tag{3.6}$$

Then

$$s(z) = p(z) - \frac{p(c)}{q(c)} q(z)$$

with $s(c) = 0$. Therefore a polynomial

$$r(z) = r(z)/(z - c)$$

may be defined and $f(z)$ may be written as

$$f(z) = \frac{p(c)}{q(c)} + \frac{r(z)}{q(z)}(z - c). \tag{3.7}$$

Suppose now that the rational function $g_1(z) = r(z)/q(z)$ has the zeroth order power-sum form evaluation $g_{p(0)}(Z)$ over the disk Z. Using this we obtain a disk

$$f_{p(1)}(Z) = \frac{p(c)}{q(c)} + g_{p(0)}(Z)(Z - c)$$

where the disk operation are given by (2.23) to (2.31).

This disk is called the first order power-sum form evaluation for the rational function $f(z)$.

Generally the k-th order power-sum form evaluation for $f(z)$ is defined by

$$f_{p(k)}(Z) = \frac{p(c)}{q(c)} + g_{p(k-1)}(Z)(Z - c) \tag{3.8}$$

where $g_{p(k-1)}(Z)$ is the $(k-1)$-th order power-sum form evaluation for the function $g(z)$. This is a recursive definition.

It is possible to develop explicit formulae for higher order power-sum form evaluations. In the following theorem the explicit formulae for the first and second order power-sum form evaluations are given.

THEOREM 3.5 *Let* $f(z) = p(z)/q(z)$, $p(z) = \sum_{i=0}^{n} a_i z^i, q(z) = \sum_{i=0}^{m} b_i z^i$, *be a rational function. The the explicit formulas for its first and second order power-sum form evaluations*

$$f_{p(1)}(Z) = \langle c_{p(1)}, \rho_{p(1)} \rangle, \quad f_{p(2)}(Z) = \langle c_{p(2)}, \rho_{p(2)} \rangle$$

satisfy

$$c_{p(1)} = \frac{p(c)}{q(c)} \tag{3.9}$$

$$\rho_{p(1)} = \frac{\rho}{A}\left[|r(c)| + \sum_{i=1}^{h-1} |r_i|((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{m} |b_i|((|c|+\rho)^i - |c|^i)\right] \tag{3.10}$$

$$c_{p(2)} = \frac{p(c)}{q(c)} \tag{3.11}$$

$$\rho_{p(2)} = \frac{\rho^2}{A}\left[|t(c)| + \sum_{i=1}^{l-1} |t_i|((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{m} |b_i|((|c|+\rho)^i - |c|^i)\right] + \rho|r(c)/q(c)| \tag{3.12}$$

where $r(c) = \sum_{i=0}^{h} r_i c^i, h = \max(n,m)$ *and* $t(c) = \sum_{i=0}^{l} t_i c^i, l = \max(h,m)$ *satisfy* $f(z) = f(c) + r(z)/q(z)(z-c)$ *and* $r(z)/q(z) = r(c)/q(c) + t(z)/q(z)(z-$

c) respectively and where $r_i, i = 0, 1, \ldots, t_i, i = 0, 1, \ldots,$ *are the Taylor coefficients of* $r(z)$ *and* $t(z)$.

Furthermore $\overline{f}(Z) \subseteq f_{p(k)}(Z)$.

Proof. The first order power-sum form evaluation of $f(z)$ can be written as

$$f_{p(1)}(Z) = f(c) + \frac{r_p(Z)}{q_p(Z)}(Z - c)$$

where $r_p(Z)$ and $q_p(Z)$ are the power-sum form evaluations for $r(z)$ and $q(z)$. From Theorem 3.4, the zeroth order power-sum form evaluation for $g(z) = r(z)/q(z)$ is

$$\frac{r_p(Z)}{q_p(Z)} = \langle c_1, \rho_1 \rangle$$

where

$$c_1 = \frac{1}{A} r(c)\overline{q(c)}$$

and

$$\begin{aligned} \rho_1 \;=\; & \frac{1}{A}\Bigg[(|r(c)| + \sum_{i=1}^{n} |r_i|((|c| + \rho)^i - |c|^i)) \\ & (|q(c)| + \sum_{i=1}^{m} |b_i|((|c| + \rho)^i - |c|^i) - |r(c)q(c)|\Bigg] \end{aligned}$$

where the coefficients r_i are the Taylor coefficients. From the definitions of

the disk operations it follows that

$$f_{p(1)}(Z) = \langle c_{p(1)}, \rho_{p(1)} \rangle,$$

where

$$c_{p(1)} = \frac{p(c)}{q(c)} \tag{3.13}$$

$$\rho_{p(1)} = \frac{\rho}{A}\left[|r(c)| + \sum_{i=1}^{h-1} |r_i|((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{m} |b_i|((|c|+\rho)^i - |c|^i)\right]. \tag{3.14}$$

Continuing this process we develop the first order power-sum form evaluation for $g(z)$ as

$$g_{p(1)}(Z) = g(c) + \frac{t_p(Z)}{q_p(Z)}(Z - c).$$

We obtain from (3.14) and the definitions of disk operations that

$$g_{p(1)}(Z) = \langle c_2, \rho_2 \rangle$$

where

$$c_2 = r(c)/q(c)$$

and

$$\rho_2 = \frac{\rho}{A}\left[|t(c)| + \sum_{i=1}^{l-1} |t_i|((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{m} |b_i|((|c|+\rho)^i - |c|^i)\right].$$

where the t_i are the Taylor coefficients of $t(z)$ at c.

The second order power-sum form evaluation for $f(z)$ now can be written as

$$f_{p(2)}(Z) = f(c) + g_{p(1)}(Z)(Z - c) = \langle c_{p(2)}, \rho_{p(2)} \rangle$$

where

$$c_{p(2)} = p(c)/q(c) \tag{3.15}$$

and

$$\begin{aligned} \rho_{p(2)} &= \frac{\rho^2}{A}\left[|t(c)| + \sum_{i=1}^{l-1} |t_i((|c| + \rho)^i - |c|^i)\right] \\ &\quad \left[|q(c)| + \sum_{i=1}^{m} |b_i|((|c| + \rho)^i - |c|^i)\right] + \rho|r(c)/q(c)|. \end{aligned} \tag{3.16}$$

The inclusion property is again a direct consequence of the inclusion isotonicity. □

Now we are in a position to define the centered forms for complex rational functions.

Let

$$p_c(Z) = \langle p(c), \sum_{k=1}^{n} |p^{(k)}(c)|\rho^k/k! \rangle \text{ and } q_c(Z) = \langle q(c), \sum_{k=1}^{m} |q^{(k)}(c)|\rho^k/k! \rangle$$

be the centered form evaluations of $p(z)$ and $q(z)$ over $Z = \langle c, \rho \rangle$ (see[27, 46,

47]). We define a disk

$$f_{c(0)}(Z) = p_c(Z)/q_c(Z)$$

where the disk division follows formula (2.29). This disk is called the zeroth order centered form evaluation, or semi-centered form evaluation, of the rational function $f(z)$.

Similarly, the first order centered form evaluation of $f(z)$ over Z can be defined as

$$f_{c(1)}(Z) = \frac{p(c)}{q(c)} + g_{c(0)}(Z)(Z - c)$$

where $g(z)$ satisfies

$$f(z) = \frac{p(c)}{q(c)} + g(z)(z - c).$$

Generally the k-th order centered form evaluation of the meromorphic function $f(z) = p(z)/q(z)$ is defined by

$$f_{c(k)}(Z) = \frac{p(c)}{q(c)} + g_{c(k-1)}(Z)(Z - c)$$

where $g_{c(k-1)}(Z)$ is the $(k-1)$-th order centered form evaluation of $g(z)$ over Z.

Explicit formulas for k-th order centered form evaluation can be derived easily. First the explicit formula of the semi-centered form evaluation is given.

In the sequel

$$B = |q(c)|^2 - \left(\sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i\right)^2.$$

and assuming $B > 0$.

THEOREM 3.6 *Let $f_{c(0)}(Z) = \langle c_{c(0)}, \rho_{c(0)}\rangle$ be the zeroth order centered form evaluation for the rational function $f(z) = p(z)/q(z)$ over the disk $Z = \langle c, \rho\rangle$. Then we have*

$$c_{c(0)} = \frac{1}{B}\frac{(p(c)}{\overline{q(c)}},$$

$$\rho_{c(0)} = \frac{1}{B}\left[\sum_{i=1}^{n} \frac{|p^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i - |p(c)q(c)|\right]$$

and

$$\overline{f}(Z) \subseteq f_{c(0)}(Z).$$

Proof. From[46] the centered form evaluations of $p(z)$ and $q(z)$ over Z are

$$\begin{aligned} p_c(Z) &= \langle p(c), \sum_{k=1}^{n} |p^{(k)}(c)|\rho^k/k!\rangle, \\ q_c(Z) &= \langle q(c), \sum_{k=1}^{m} |q^{(k)}(c)|\rho^k/k!\rangle. \end{aligned}$$

Using disk division (2.34) we obtain explicit formulas for $c_{c(0)}$ and $\rho_{c(0)}$. The inclusion property is an immediate consequence of the inclusion isotonicity of the operations (2.23) to (2.31). □

The explicit formulas of the first and second order centered form evaluations for the rational function $f(z)$ are now given.

THEOREM 3.7 *Let $f_{c(k)}(Z) = \langle c_{c(k)}, \rho_{c(k)} \rangle$ be the centered form evaluation of order k for the rational function $f(z) = p(z)/q(z)$ over the disk $Z = \langle c, \rho \rangle$. Then we have*

$$
\begin{aligned}
c_{c(k)} &= f(c), k = 1, 2, \ldots \\
\rho_{c(1)} &= \frac{\rho}{B}\left[\sum_{i=0}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=0}^{h-1} \frac{\rho^i}{|q(c)|(i+1)!}|p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)|\right] \\
\rho_{c(2)} &= \frac{\rho^2}{B}\left[\sum_{i=0}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=0}^{h-1} \frac{\rho^i}{|q(c)|(i+1)!}|r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|\right] + \frac{|r(c)|}{|q(c)|}\rho
\end{aligned}
$$

where

$$r(z) = \sum_{i=0}^{h-1} \frac{r^{(i)}(c)}{i!}(z-c)^i$$

and

$$r^{(i)}(c) = \frac{1}{q(c)(i+1)}[p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)](z-c)^i$$

and where $h = \max(n, m)$. Furthermore

$$\overline{f}(Z) \subseteq f_{c(k)}(Z), k = 1, 2, \ldots,$$

Proof. The first order centered form evaluation of $f(z)$ can be written as

$$f_{c(1)}(Z) = f(c) + \frac{r_c(Z)}{q_c(Z)}(Z - c)$$

where the Taylor coefficients of $r(z)$ are

$$r^{(i)}(c) = \frac{1}{q(c)(i+1)!}[p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)], \quad i = 1, 2, \ldots.$$

From Theorem 3.6, the zeroth order centered form evaluation for $g(z) = r(z)/q(z)$ has the explicit representation

$$\frac{r_c(Z)}{q_c(Z)} = \frac{1}{B}\langle r(c)\overline{q(c)}, \sum_{i=0}^{h-1} \frac{|r^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i - |r(c)q(c)|\rangle.$$

It follows that

$$f_{c(1)}(Z) = \langle f(c), \frac{\rho}{B}\sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=0}^{h-1} \frac{|r^{(i)}(c)|}{i!}\rho^i\rangle$$

$$= \langle f(c), \frac{\rho}{B}\sum_{i=0}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{h-1} \frac{|p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)|}{|q(c)|(i+1)!}\rho^i\rangle. \qquad (3.17)$$

Continuing this process we develop the first centered form evaluation for $g(z)$ as

$$g_{c(1)}(Z) = g(c) + \frac{s_c(Z)}{q_c(Z)}(Z - c)$$

where the Taylor coefficients of $s(z)$ are

$$s^{(i)}(c) = \frac{1}{q(c)(i+1)}[r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)], i = 1, 2, \ldots.$$

It follows from (3.17) that

$$g_{c(1)}(Z) = \langle \frac{r(c)}{q(c)}, \frac{\rho}{B} \sum_{i=0}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{l-1} \frac{|r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|}{|q(c)|(i+1)!}\rho^i \rangle. \quad (3.18)$$

The second order centered form evaluation for $f(z)$ now can be written as

$$\begin{aligned} f_{c(2)}(Z) &= f(c) + g_{c(1)}(Z)(Z-c) \\ &= \langle f(c), \frac{\rho^2}{B} \sum_{i=0}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \\ &\quad \sum_{i=1}^{l-1} \frac{|r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|}{|q(c)|(i+1)!}\rho^i + |r(c)/q(c)|\rho \rangle. \quad (3.19) \end{aligned}$$

The inclusion property is again a direct consequence of the inclusion isotonicity. □

There are different methods for evaluating $p_c(Z)$, which may result in different $p_c(Z)$. There is, however, the simplest evaluation method, power-sum evaluation, to evaluate $p_c(Z)$. We will assume that in the sequel $p_c(Z)$ is evaluated using power-sum evaluation over the circular disk $Z - c$. The power-sum evaluation has an important property as follows.

PROPOSITION 3.1 *Let the complex polynomial* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be writ-*

ten in the centered form

$$p_c(z) = \sum_{k=0}^{n} \frac{p^{(k)}(c)}{k!}(z-c)^k.$$

Then the result of the power sum evaluation of this polynomial over the circular disk $Z = \langle c, \rho \rangle$ *is still a circular disk* $Z' = \langle c', \rho' \rangle$ *where*

$$c' = p(c),$$

$$\rho' = \sum_{k=0}^{n} \left| \frac{p^{(k)}(c)}{k!} \right| \rho^k.$$

Proof. The proof can be found in[46]. □

COROLLARY 1 *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a complex polynomial. Then the evaluation of* $p(z)$ *over* $Z = \langle c, \rho \rangle$ *using the Horner-scheme results in a circular disk* $Z' = \langle c', \rho' \rangle$ *where*

$$c' = p(c),$$

$$\rho' = \sum_{k=0}^{n} \left| \frac{p^{(k)}(c)}{k!} \right| \rho^k.$$

If we evaluate polynomial $p(z)$ directly using power-sum evaluation, we still obtain a circular disk as follows.

PROPOSITION 3.2 *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a complex polynomial. Then the result of the power sum evaluation of this polynomial over the circular disk* $Z = \langle c, \rho \rangle$ *is still a circular disk* $Z' = \langle c', \rho' \rangle$ *where*

$$c' = p(c),$$

$$\rho' = \sum_{k=0}^{n} |a_k|((|c| + \rho)^k - |c|^k).$$

Proof. The proof can be found in[46]. □

Based on these propositions we have the following interesting inclusion relation.

THEOREM 3.8 *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a complex polynomial. Let* $Z' = \langle c', \rho' \rangle$ *be the disk evaluated from* $p(z)$ *over* Z *using the power-sum evaluation and* $Z'' = \langle c'', \rho'' \rangle$ *the disk evaluated from the centered form of* $p(z)$ *using the power-sum evaluation. Then*

$$c' = c'',$$

$$\rho' \geq \rho''.$$

This implies that $Z'' \subseteq Z'$.

Proof. The proof is in[46]. □

3.2 Inclusion Properties

In[41] Rokne showed that the the complex centered forms of higher order for rational functions are included in the lower order forms.

THEOREM 3.9 *Let* $f(z) = p(z)/q(z), p(z) = \sum_{i=0}^{n} a_i z^i, q(z) = \sum_{i=0}^{m} b_i z^i$ *be a rational function and let* $f_{c(1)}(Z)$ *and* $f_{c(2)}(Z)$ *be its first and second centered form evaluations over the disk* $Z = \langle c, \rho \rangle$. *Then*

$$\overline{f}(Z) \subseteq f_{c(2)}(Z) \subseteq f_{c(1)}(Z).$$

Proof. Let $f_{c(1)}(Z) = \langle z_1, \rho_1 \rangle$ and $f_{c(2)}(Z) = \langle z_2, \rho_2 \rangle$. Since $z_1 = z_2$, it is sufficient to prove that $\rho_2 < \rho_1$.

From Theorem 3.6, we have

$$\begin{aligned} D &= \rho_1 - \rho_2 \\ &= \frac{\rho}{B} \sum_{i=0}^{m} \frac{|q^{(i)}(c)|}{i!} \rho^i \sum_{j=0}^{k-1} \frac{|r^{(j)}(c)|}{j!} \rho^j \\ &- \frac{\rho^2}{B} \sum_{i=0}^{m} \frac{|q^{(i)}(c)|}{i!} \rho^i \sum_{j=0}^{k-1} \frac{|r^{(j+1)}(c)q(c) - q^{(j+1)}(c)r(c)|}{|q(c)|(j+1)!} \rho^j - |r(c)/q(c)|\rho \end{aligned}$$

where again $k = \max(m, n)$.

Let

$$\begin{aligned} B^+ &= |q(c)| + \sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i, \\ B^- &= |q(c)| - \sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i, \end{aligned}$$

then $B = B^+B^-$. Since $B > 0$ and $B^+ > 0$, it follows that $B^- > 0$.

Let

$$\sigma = \sum_{i=0}^{k-1} \frac{|q(c)r^{(i)}(c)| - |r^{(i)}(c)q(c) - q^{(i)}(c)r(c)|}{i!}\rho^i.$$

Multiplying D by $B^-|q(c)|/\rho$ we obtain after some rearrangements

$$\begin{aligned} DB^-|q(c)|/\rho &= \sigma + |q(c)r(c)| - |r(c)|\left[|q(c)| - |\sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i|\right] \\ &= \sigma + |q(c)r(c)| - |r(c)q(c)| + |r(c)|\sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \\ &= \sigma + |r(c)|\sum_{i=1}^{m} \frac{|q^{(i)}(c)|}{i!}\rho^i \\ &= \sum_{i=0}^{k-1} \frac{|r(c)q^{(i)}(c)| + |q(c)r^{(i)}(c)| - |r^{(i)}(c)q(c) - q^{(i)}(c)r(c)|}{i!}\rho^i \geq 0. \end{aligned}$$

This proves the theorem. □

Let $f_{c(k)}(Z)$ be the k-th order centered form evaluation of $f(z)$ over Z.

We then have

COROLLARY 2

$$\overline{f}(Z) \subseteq f_{c(k)}(Z) \subseteq f_{c(k-1)}(Z) \subseteq \cdots \subseteq f_{c(1)}(Z).$$

Proof. According to the recursive definition of the centered forms, the $(k-1)$-th order centered form evaluation $f_{c(k-1)}(Z)$ can be written as

$$f_{c(k-1)}(Z) = f(c)+f'(c)(Z-c)+\cdots+\frac{f^{(k-2)}(c)}{(k-2)!}(Z-c)^{k-2}+g_{c(0)}(Z)(Z-c)^{k-1}$$

where $g(z)$ is a rational function. Therefore the k-th and $(k+1)$-th order form evaluations of $f(z)$ can be written as

$$f_{c(k)}(Z) = f(c)+f'(c)(Z-c)+\cdots+\frac{f^{(k-2)}(c)}{(k-2)!}(Z-c)^{k-2}+g_{c(1)}(Z)(Z-c)^{k-1},$$

and

$$f_{c(k+1)}(Z) = f(c)+f'(c)(Z-c)+\cdots+\frac{f^{(k-2)}(c)}{(k-2)!}(Z-c)^{k-2}+g_{c(2)}(Z)(Z-c)^{k-1}.$$

Applying Theorem 3.9 to $g(z)$ we obtain $g_{c(2)}(Z) \subseteq g_{c(1)}(Z)$ which along with the inclusion isotonicity of the operations gives

$$f_{c(k+1)}(Z) \subseteq f_{c(k)}(Z).$$

The corollary follows. □

The power-sum forms and centered forms both have the property that they give increasingly better inclusion for the range of the function. Numerical examples show, however, that there are functions $f(z)$ and disks Z such that both $\rho_{c(k)} < \rho_{p(k)}$ and $\rho_{c(k)} > \rho_{p(k)}$ are possible, where $\rho_{c(k)}$ and $\rho_{p(k)}$ are radii of $f_{c(k)}(Z)$ and $f_{p(k)}(Z)$ respectively.

3.3 Quadratic Convergence

It was shown in[41] that the centered forms for complex rational functions were quadratically convergent to the range as the size of the domain tended to zeroth in the sense of diameter inclusion.

In this section we will show centered forms of all orders are quadratically convergent in the norm of DI-pseudo-distance. This distance is defined as follows (see[46, 47]).

DEFINITION 3.4 *Let* $Z_1, Z_2 \in \mathbf{K}$. *If there exist* $d_1, d_2 \in Z_1$ *such that* $|d_1 - d_2| = 2rad(Z_1)$ *and* $d_1, d_2 \in Z_2$ *then* Z_1 *is diameter included in* Z_2 *written as* $Z_1 \prec Z_2$

DEFINITION 3.5 *Let* $Z_1 = \langle c_1, \rho_1 \rangle, Z_2 = \langle c_2, \rho_2 \rangle \in \mathbf{K}$. *Then the DI-pseudo-distance of two disks is defined as*

$$\Phi(Z_1, Z_2) = \begin{cases} |\rho_1 - \rho_2| & \text{if } Z_1 \prec Z_2 \text{ or } Z_2 \prec Z_1 \\ |\rho_1 - \rho_2| + |c_1 - c_2| & \text{otherwise .} \end{cases}$$

THEOREM 3.10 *Let* $f_{c(1)}(Z)$ *be the first order centered form evaluation of the rational function* $f(z) = p(z)/q(z)$ *over* $Z = \langle c, \rho \rangle$. *If* $W \in \mathbf{K}$ *is the smallest circle containing* $\overline{f}(Z)$ *then*

$$\Phi(W, f_{c(1)}(Z)) = O(\rho^2).$$

Proof. Let $u, v \in Z$ be chosen such that

$$|f(u) - f(v)| = \max_{z_1, z_2 \in Z} |f(z_1) - f(z_2)|$$

and $Y = \langle \frac{1}{2}(f(u) + f(v)), \frac{1}{2}|f(u) - f(v)| \rangle$.

Then clearly

$$Y \prec W, \ \overline{f}(Z) \subseteq W \text{ and } \overline{f}(Z) \subseteq f_{c(1)}(Z).$$

We now write $f(z)$ as

$$f(z) = f(c) + \frac{r(z)}{q(z)}(z - c)$$

where

$$r(z) = \sum_{i=0}^{\infty} \frac{r^{(i)}(c)}{i!}(z - c)^i$$

and

$$r^{(i)}(c) = \frac{p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)}{q(c)(i+1)}, \ \ i = 0, 1, \ldots.$$

Let $z_1, z_2 \in Z$ be any two points. We have

$$\begin{aligned} |f(z_1) - f(z_2)| &= |\frac{r(z_1)}{q(z_1)}(z_1 - c) - \frac{r(z_2)}{q(z_2)}(z_2 - c)| \\ &= |\frac{r(c)}{q(c)}(z_1 - c) - \frac{r(c)}{q(c)}(z_2 - c) + \frac{t(z_1)}{q(z_1)}(z_1 - c)^2 - \frac{t(z_2)}{q(z_2)}(z_2 - c)^2| \\ &\geq |\frac{r(c)}{q(c)}||z_1 - z_2| - |\frac{t(z_1)}{q(z_1)}(z_1 - c)^2 - \frac{t(z_2)}{q(z_2)}(z_2 - c)^2| \end{aligned}$$

where $t(z)$ satisfies

$$\frac{r(z)}{q(z)} = \frac{r(c)}{q(c)} + \frac{t(z)}{q(z)}(z-c).$$

Let z_1, z_2 to be chosen such that $|z_1 - z_2| = 2\rho$. Then we have

$$\begin{aligned} |f(u) - f(v)| &\geq |f(z_1) - f(z_2)| \\ &\geq 2|\frac{r(c)}{q(c)}|\rho - |\frac{t(z_1)}{q(z_1)}\rho^2| - |\frac{t(z_2)}{q(z_2)}\rho^2|. \end{aligned}$$

The radius $\rho_{c(1)}$ of $f_{c(1)}(Z)$ may be expanded as

$$\begin{aligned} \rho_{c(1)} &= (\frac{\rho}{B}\sum_{i=0}^{m}\frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{k-1}\frac{|p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)|}{q(c)(i+1)!}\rho^i)/(|q(c)|^2 - |\sum_{i=1}^{m}\frac{|q^{(i)}(c)|}{i!}\rho^i|^2) \\ &= \frac{\rho}{B}\sum_{i=0}^{m}\frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{k-1}\frac{|p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)|}{q(c)(i+1)!}\rho^i \frac{\rho}{|q(c)|^2}(1+O(\rho)) \\ &= (|q(c)| + O(\rho))\left[\frac{|p'(c)q(c) - q'(c)p(c)|}{|q(c)|} + O(\rho)\right]\frac{\rho}{|q(c)|^2}(1+O(\rho)) \\ &= \left[\frac{|p'(c)q(c) - q'(c)p(c)|}{|q(c)|} + O(\rho)\right]\rho + O(\rho^2). \end{aligned}$$

Now the DI-pseudo-distance between Y and $f_{c(1)}(Z)$ can be estimated as

$$\begin{aligned} \Phi(Y, f_{c(1)}(Z)) &\leq \left[\frac{|p'(c)q(c) - q'(c)p(c)|}{|q(c)|} + O(\rho)\right]\rho \\ &+ O(\rho^2) - \rho|\frac{r(c)}{q(c)}| - 1/2\rho^2(|\frac{t(z_1)}{q(z_1)}| + |\frac{t(z_2)}{q(z_2)}|) \\ &= -1/2\rho^2\left(|\frac{t(z_1)}{q(z_1)}| + |\frac{t(z_2)}{q(z_2)}|\right) + O(\rho^2). \end{aligned}$$

Since $t(z)/q(z) \to t(c)/q(c)$ when $\rho \to 0$ for any $z \in Z$. We have

$$\Phi(Y, f_{c(1)}(Z)) = O(\rho^2).$$

It follows from $\Phi(W, f_{c(1)}(Z)) \leq \Phi(Y, f_{c(1)}(Z))$ that $\Phi(W, f_{c(1)}(Z)) = O(\rho^2)$. □

COROLLARY 3 $\Phi(W, f_{c(k)}(Z)) = O(\rho^2)$ *for all* $k \geq 1$.

Proof. From Corollary 2 $f_{c(k)}(Z) \subseteq f_{c(1)}(Z)$. The corollary follows. □

In[41] Rokne showed the zeroth order centered form is also quadratically convergent for complex rational functions given certain conditions. Here we present this theorem.

THEOREM 3.11 *Let* $f_{c(0)}(Z)$ *be the zeroth order centered form of the rational function* $f(z) = p(z)/q(z)$ *over* $Z = \langle c, \rho \rangle$. *If* $W \in \mathbf{K}$ *is the smallest circle containing* $\overline{f}(Z)$ *then*

$$\Phi(W, f_{c(0)}(Z)) = O(\rho^2).$$

whenever $\arg(p'(c)q(c)) = -\arg(q'(c)p(c))$.

Proof. Let $W = \langle c_W, \rho_W \rangle$. From the fact $\mathrm{rad}(Y) \leq \mathrm{rad}(W)$, where Y is defined in Theorem 3.10, and a result of Theorem 3.10, we have

$$\rho_W \geq \left[|\frac{r(c)}{q(c)}|\rho + O(\rho^2) \right] \geq |\frac{r(c)}{q(c)}|\rho + O(\rho^2).$$

From Theorem 3.6, we have

$$\rho_{c(0)} = \frac{|p'(c)q(c)| + |q'(c)p(c)|}{|q(c)|^2}\rho + O(\rho^2).$$

It then follows from $\arg(p'(c)q(c)) = -\arg(q'(c)p(c))$ that

$$\begin{aligned}
\Phi(W, f_{c(0)}(Z)) &= \rho_{c(0)} - \rho_W \\
&\leq \frac{|p'(c)q(c)| + |q'(c)p(c)|}{|q(c)|^2}\rho - \frac{|p'(c)q(c) - q'(c)p(c)|}{|q(c)|^2}\rho + O(\rho^2) \\
&\leq \frac{|p'(c)q(c)| + |q'(c)p(c)| - |p'(c)q(c) - q'(c)p(c)|}{|q(c)|^2}\rho + O(\rho^2) \\
&= O(\rho^2).
\end{aligned}$$

This proves the theorem. □

3.4 Inclusion Isotonicity

It was stated in [8] that the circular complex centered form for polynomials was inclusion isotone. In this section, we will prove that the circular complex centered form for polynomials and rational functions has this property. In fact, we will prove a more general theorem, which states that the circular centered form for arbitrary analytic functions is inclusion isotone. Then we will extend this result to rational functions.

Inclusion isotonicity[23] is of fundamental importance for the applications. It was stated without proof in[29] which these centered forms were inclusion isotone. In this section this claim is verified both for polynomials and analytic functions in full generality.

DEFINITION 3.6 *Let $Z_1, Z_2 \in \mathbf{K}$. Let $f(z)$ be any function. Then the centered form evaluation $f_c(Z)$ of $f(z)$ is said to be inclusion isotone over* $\mathbf{K}$ *if*

$$Z_1 \subseteq Z_2 \rightarrow f_c(Z_1) \subseteq f_c(Z_2). \tag{3.20}$$

In[29] some special cases were verified, however, the general theorem has remained unproven up to now. In this section, we present a rigorous proof for the theorem.

If $f(z)$ is an analytic function then the generalization of the centered form above is given by

$$f(Z) = \langle f(c), \sum_{k=1}^{\infty} \frac{|f^{(k)}(c)|\rho^k}{k!} \rangle$$

for $Z \in \mathbf{K}$. This form was first discussed in[23]. It is shown here that this form is also inclusion isotone.

This more general statement is equivalent to the following theorem:

THEOREM 3.12 *Let z_1 and z_2 be any complex numbers and ρ_1 and ρ_2 be any real numbers. Let $f(z)$ be analytic in $|z - z_2| \leq \rho_2$. If $|z_1 - z_2| \leq \rho_2 - \rho_1$ then*

$$|f(z_1) - f(z_2)| \leq \sum_{k=1}^{\infty} \frac{|f^{(k)}(z_2)|}{k!} \rho_2^k - \sum_{k=1}^{\infty} \frac{|f^{(k)}(z_1)|}{k!} \rho_1^k.$$

We first prove the inclusion isotonicity for the circular complex centered form for polynomials, then for the more general form for analytic functions. The theorem for polynomials is of course a corollary of the more general theorem for analytic functions. Both proofs are given, however, since the polynomial theorem introduces the ideas in a simpler framework than what is required for the analytic case.

It is easy to see that the conjecture (3.20) is equivalent to the following theorem.

THEOREM 3.13 *Let z_1 and z_2 be any complex numbers and ρ_1 and ρ_2 be any real numbers. Let $p(z)$ be any complex polynomial of degree n. If*

$$|z_1 - z_2| \leq \rho_2 - \rho_1,$$

then

$$|p(z_1) - p(z_2)| \leq \sum_{k=1}^{n} \frac{|p^{(k)}(z_2)|}{k!}\rho_2^k - \sum_{k=1}^{n} \frac{|p^{(k)}(z_1)|}{k!}\rho_1^k.$$

First we prove the following lemma.

LEMMA 3.1 *For a polynomial $p(z)$ of degree n, we have*

$$|p(z_1) - p(z_2)| \leq \sum_{k=1}^{n} \frac{|p^{(k)}(z_2)|}{k!}|z_1 - z_2|^k.$$

Proof. Expanding $p(z)$ in Taylor series at z_2, we have

$$p(z) = p(z_2) + \sum_{k=1}^{n} \frac{p^{(k)}(z_2)}{k!}(z - z_2)^k.$$

This means that

$$|p(z) - p(z_2)| \leq \sum_{k=1}^{n} \frac{|p^{(k)}(z_2)|}{k!}|z - z_2|^k.$$

With $z = z_1$ the lemma follows. □

The next theorem proves an inequality that is somewhat more general than the required inequality.

THEOREM 3.14 *Let $\lambda = |z_1 - z_2|$. Then the following inequality is valid:*

$$|p(z_1) - p(z_2)| \leq \sum_{k=1}^{n} \frac{|p^{(k)}(z_2)|}{k!}(\rho_1 + \lambda)^k - \sum_{k=1}^{n} \frac{|p^{(k)}(z_1)|}{k!}\rho_1^k.$$

Proof. Let

$$A(\rho_1) = \sum_{k=1}^{n} \frac{|p^{(k)}(z_2)|}{k!}(\rho_1 + \lambda)^k - \sum_{k=1}^{n} \frac{|p^{(k)}(z_1)|}{k!}\rho_1^k - |p(z_1) - p(z_2)|.$$

When $\rho_1 = 0$, from Lemma 3.1 we have $A(0) \geq 0$. For $\rho_1 > 0$, taking the derivative with respect to ρ_1, we arrive at

$$A'(\rho_1) = \sum_{k=0}^{n-1} \frac{|p^{(k+1)}(z_2)|}{k!}(\rho_1 + \lambda)^k - \sum_{k=0}^{n-1} \frac{|p^{(k+1)}(z_1)|}{k!}\rho_1^k.$$

It is now shown that $A'(\rho_1) \geq 0$ for $\rho_1 > 0$. Expanding $p^{(k+1)}(z)$ at z_2 for k in the range $1 \leq k \leq n-1$ yields

$$p^{(k+1)}(z) = \sum_{j=0}^{n-k-1} \frac{p^{(k+j+1)}(z_2)(z - z_2)^j}{j!}.$$

Since $p(z)$ is a polynomial of degree n, it follows that $p^{(k+j+1)}(z) = 0$ for $j > n - k - 1$. It is therefore only necessary to sum over j's running from 0 to $n - k - 1$.

Setting $z = z_1$ and taking the absolute values on both sides, it follows that

$$|p^{(k+1)}(z_1)| \leq \sum_{j=0}^{n-k-1} \frac{|p^{(k+j+1)}(z_2)||z_1 - z_2|^j}{j!}$$

$$= \sum_{j=0}^{n-k-1} \frac{|p^{(k+j+1)}(z_2)|\lambda^j}{j!}.$$

Summing over k yields

$$\sum_{k=0}^{n-1} \frac{|p^{(k+1)}(z_1)|\rho_1^k}{k!} \leq \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \frac{|p^{(k+j+1)}(z_2)|\rho_1^k\lambda^j}{j!k!}$$

and replacing $k+j$ by k_1, the formula

$$\sum_{k=0}^{n-1} \frac{|p^{(k+1)}(z_1)|\rho_1^k}{k!} \leq \sum_{k_1=0}^{n-1} |p^{(k_1+1)}(z_2)| \sum_{\substack{0 \leq k \leq n-1 \\ 0 \leq j \leq n-1 \\ j+k=k_1}} \frac{\rho_1^{k_1-j}\lambda^j}{j!k!}$$

is obtained. The inner summation is a sum over j and k with $j+k=k_1$ (i.e. constant). This means that it can be rewritten as

$$\sum_{\substack{0 \leq k \leq n-1 \\ 0 \leq j \leq n-1 \\ j+k=k_1}} \frac{\rho_1^{k_1-j}\lambda^j}{j!k!} = \sum_{k=0}^{k_1-1} \frac{\rho_1^k\lambda^{k_1-k}}{(k_1-k)!k!}$$

and it follows that

$$\sum_{k=0}^{n-1} |p^{(k+1)}(z_1)|\frac{\rho_1^k}{k!} \leq \sum_{k_1=0}^{n-1} |p^{(k_1+1)}(z_2)| \sum_{j+k=k_1} \frac{\rho_1^{k_1-j}\lambda^j}{j!k!} =$$

$$= \sum_{k_1=0}^{n-1} |p^{(k_1+1)}(z_2)| \frac{1}{k_1!} \sum_{k=0}^{k_1} \frac{\rho_1^k \lambda^{k_1-k} k_1!}{(k_1-k)!k!} =$$

$$= \sum_{k_1=0}^{n-1} |p^{(k_1+1)}(z_2)| \frac{(\rho_1+\lambda)^{k_1}}{k_1!}. \tag{3.21}$$

Equation (3.21) implies that $A'(\rho_1) \geq 0$, for $\rho_1 > 0$ which in turn implies that $A(\rho_1) \geq 0$ □

Proof of theorem 3.13. This theorem is a direct corollary of Theorem 3.14. □

Now we show that the circular centered forms for analytic functions are inclusion isotone.

LEMMA 3.2 *Let $f(z)$ be an analytic function in $|z - z_2| \leq \rho_2$. Then for $|z - z_2| \leq \rho_2$ the series*

$$\sum_{k=1}^{\infty} \frac{|f^{(k)}(z)|}{k!} \rho_2^k$$

has a finite sum. Hence the series

$$\sum_{k=1}^{\infty} \frac{f^{(k)}(z_2)}{k!} (z - z_2)^k$$

is absolutely convergent in the domain $|z - z_2| \leq \rho_2$.

Proof. Since the function $f(z)$ is analytic in the domain $|z - z_2| \leq \rho_2$, so is the function $f^{(2)}(z)$. Thus we have $M_2 = \max |z - z_2| \leq \rho_2 |f^{(2)}(z)| < +\infty$

. From the Cauchy theorem, it follows that $|f^{(n+2)}(z)| \leq M_2 \frac{n!}{\rho_2^n}$. Therefore

$$\sum_{n=0}^{\infty} |f^{(n)}(z)| \frac{\rho_2^n}{n!} = \sum_{n=0}^{1} |f^{(n)}(z)| \frac{\rho_2^n}{n!} + \sum_{n=2}^{\infty} |f^{(n+2)}(z)| \frac{\rho_2^{n+2}}{(n+2)!}$$

$$\leq \sum_{n=0}^{1} |f^{(n)}(z)| \frac{\rho_2^n}{n!} + M_2 \rho_2^2 \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} < +\infty.$$

Thus the series

$$\sum_{k=1}^{\infty} \frac{|f^{(k)}(z)|}{k!} \rho_2^k$$

is convergent in the domain $|z - z_2| < \rho_2$. □

LEMMA 3.3 *Let $f(z)$ be an analytic function in $|z - z_2| \leq \rho_2$ and assume that $|z_1 - z_2| \leq \rho_2$. Then*

$$|f(z_1) - f(z_2)| \leq \sum_{k=1}^{\infty} \frac{|f^{(k)}(z_2)|}{k!} |z_1 - z_2|^k.$$

Proof. Since $f(z)$ is analytic in $|z - z_2| \leq \rho_2$, it can be expanded in a Taylor series around z_2 and we get

$$f(z) = f(z_2) + \sum_{k=1}^{\infty} \frac{f^{(k)}(z_2)}{k!} (z - z_2)^k$$

or with a small rearrangement

$$f(z) - f(z_2) = \sum_{k=1}^{\infty} \frac{f^{(k)}(z_2)}{k!} (z - z_2)^k.$$

From the previous lemma it follows

$$|f(z-f(z_2)| = |\sum_{k=1}^{\infty} \frac{f^{(k)}(z_2)}{k!}(z-z_2)^k| \leq \sum_{k=1}^{\infty} \frac{|f^{(k)}(z_2)|}{k!}|z-z_2|^k < \infty.$$

Replacing z by z_1 we get

$$|f(z_1)-f(z_2)| \leq \sum_{k=1}^{\infty} \frac{|f^{(k)}(z_2)|}{k!}|z_1-z_2|^k.$$

□

We now introduce a few notations for convenience, namely:

$$\sigma_{m,n} = \sum_{k=1}^{n} \frac{|f^{(m+k)}(z_2)|}{k!}|z_1-z_2|^k - |f^{(m)}(z_1)-f^{(m)}(z_2)|,$$

$$A_n = \sum_{k=1}^{n} \frac{|f^{(k)}(z_2)|}{k!}\rho^k - \sum_{k=1}^{n} \frac{|f^{(k)}(z_1)|}{k!}\rho_1^k - |f(z_1)-f(z_2)|.$$

with $\rho = \rho_1 + \lambda$ and the obvious interpretations for $\sigma_{m,\infty}$ and A_∞.

We have from Lemmas 3.2 and 3.3

$$A_\infty = \lim_{n\to\infty} A_n,$$

$$\sigma_{m,\infty} = \lim_{n\to\infty} \sigma_{m,n} \geq 0.$$

LEMMA 3.4 *For any real number ϵ there exists a number $N_0(\epsilon)$ such that*

$$\sum_{k=1}^{n} \frac{\rho_1^k}{k!}\sigma_{k,n-k} \geq -\epsilon$$

for any $n \geq N_0(\epsilon)$.

Proof. There exists obviously a constant $N_1 > 0$ such that

$$\sum_{k>N_1}^{\infty} (\frac{\rho_1}{\rho_2})^k < \frac{\epsilon}{2M}.$$

The following estimate can then be performed

$$\sum_{k>N_1}^{\infty} \frac{\rho_1^k}{k!} |f^{(k)}(z_1) - f^{(k)}(z_2)|$$

$$\leq \sum_{k>N_1}^{\infty} (|f^{(k)}(z_1)| + |f^{(k)}(z_2)|) \frac{\rho_1^k}{k!} \leq \sum_{k>N_1}^{\infty} (\frac{\rho_1}{\rho_2})^k < 2M\epsilon/(4M) < \epsilon/2.$$

Since the series

$$\sum_{j=1}^{\infty} \frac{|f^{(k+j)}(z_2)|}{j!} |z_1 - z_2|^j$$

is convergent, there exists a N_2 such that if $n > N_2$ then

$$\sum_{j=n+1}^{\infty} \frac{|f^{(k+j)}(z_2)|}{j!} |z_1 - z_2|^j < \frac{\epsilon}{2e^{\rho_1}}$$

holds for all k. Therefore if $n > N_2$ it follows that

$$\sum_{j=1}^{n} \frac{|f^{(k+j)}(z_2)|}{j!} |z_1 - z_2|^j - |f^{(k)}(z_1) - f^{(k)}(z_2)|$$

$$= - \sum_{j=n+1}^{\infty} \frac{|f^{(k+j)}(z_2)|}{j!} |z_1 - z_2|^j$$

$$+ \left(\sum_{j=1}^{\infty} \frac{|f^{(k+j)}(z_2)|}{j!} |z_1 - z_2|^j - |^{(k)}(z_1) f^{(k)}(z_2)| \right)$$

$$\geq - \sum_{j=n+1}^{\infty} \frac{|f^{(k+j)}(z_2)|}{j!} |z_1 - z_2|^j > - \frac{\epsilon}{2e^{\rho_1}}$$

using Lemma 3.3. Taking $N_0(\epsilon) = \max(N_1, N_2)$ and applying both inequalities above, we obtain for any $n > 2N_0(\epsilon)$ that

$$\sum_{k=1}^{n} \frac{\rho_1^k}{k!}\sigma_{k,n-k} = \sum_{k=1}^{N_0(\epsilon)} \frac{\rho_1^k}{k!}\sigma_{k,n-k} + \sum_{k=N_0(\epsilon)+1}^{n} \frac{\rho_1^k}{k!}\sigma_{k,n-k}$$

$$\geq \sum_{k=1}^{N_0(\epsilon)} \frac{\rho_1^k}{k!}\sigma_{k,n-k} - \sum_{k>N_0(\epsilon)}^{\infty} \frac{\rho_1^k}{k!}|f^{(k)}(z_1) - f^{(k)}(z_2)|$$

$$\geq \sum_{k=1}^{N_0(\epsilon)} \frac{\rho_1^k}{k!}\left(\sum_{j=1}^{n-k} \frac{|f^{(k+j)}(z_2)|}{j!}|z_1 - z_2|^j - |f^{(k)}(z_1) - f^{(k)}(z_2)|\right) - \frac{\epsilon}{2}$$

$$\geq \left(-\frac{\epsilon}{2e^{\rho_1}}\right)\sum_{k=1}^{N_0(\epsilon)} \frac{\rho_1^k}{k!} - \frac{\epsilon}{2} > \left(-\frac{\epsilon}{2e^{\rho_1}}\right)\sum_{k=1}^{\infty} \frac{\rho_1^k}{k!} - \frac{\epsilon}{2} = -\epsilon.$$

□

Proof of Theorem 3.12. Let $A_\infty = \lim_{n\to\infty} A_n = \delta < 0$. Take $\epsilon < -\frac{\delta}{2}$. From Lemma 3.3, there exists a number N_3 such that

$$\sum_{k=1}^{n} \frac{|f^{(k)}(z_2)|}{k!}|z_1 - z_2|^k - |f(z_1) - f(z_2)| > -\epsilon \tag{3.22}$$

whenever $n > N_3$. Similar to the polynomial case, we have

$$A_n = \sum_{k=1}^{n} \frac{|f^{(k)}(z_2)|}{k!}((\rho_1 + |z_1 - z_2|)^k - \rho_1^k)$$

$$+ \sum_{k=1}^{n} \frac{|f^{(k)}(z_2)| - |f^{(k)}(z_1)|}{k!}\rho_1^k - |f(z_1) - f(z_2)|$$

$$= \sum_{k=1}^{n} \frac{|f^{(k)}(z_2)|}{k!}\sum_{i=1}^{k-1} C_k^i \rho_1^i |z_1 - z_2|^{k-i} + \sum_{k=1}^{n} \frac{|f^{(k)}(z_2)| - |f^{(k)}(z_1)|}{k!}\rho_1^k$$

$$+\left(\sum_{k=1}^{n}\frac{|f^{(k)}(z_2)|}{k!}|z_1-z_2|^k-|f(z_1)-f(z_2)|\right).$$

From formula (3.22), the last term in above formula is greater then $+\frac{\delta}{2}$. Furthermore

$$|f^{(k)}(z_2)|-|f^{(k)}(z_1)|\geq -|f^{(k)}(z_2)-f^{(k)}(z_1)|.$$

Therefore

$$A_n\geq\sum_{k=1}^{n}\frac{|f^{(k)}(z_2)|}{k!}\sum_{i=1}^{k-1}C_k^i\rho_1^i|z_1-z_2|^{k-i}-\sum_{k=1}^{n}\frac{|f^{(k)}(z_2)-f^{(k)}(z_1)|}{k!}\rho_1^k+\frac{\delta}{2}. \tag{3.23}$$

Changing the order of summation in the first term of (3.23) we find after straightforward reductions that it can be written as $(j=i-k)$

$$\sum_{i=1}^{n}\frac{\rho_1^i}{i!}\sum_{j=1}^{n-i}\frac{|f^{(i+j)}(z_2)|}{j!}|z_1-z_2|^j.$$

Therefore whenever $n>N=\max(N_0(\epsilon),N_3)$ we have, using Lemma 3.4, that

$$A_n\geq\sum_{i=1}^{n}\frac{\rho_1^i}{i!}\sum_{j=1}^{n-i}\frac{|f^{(i+j)}(z_2)|}{j!}|z_1-z_2|^j-\sum_{k=1}^{n}\frac{|f^{(k)}(z_2)-f^{(k)}(z_1)|}{k!}\rho_1^k+\frac{\delta}{2}$$

$$\sum_{k=1}^{n}\frac{\rho_1^k}{k!}\left(\sum_{j=1}^{n-k}\frac{|f^{(k+j)}(z_2)|}{j!}|z_1-z_2|^j-|f^{(k)}(z_1)-f^{(k)}(z_2)|\right)+\frac{\delta}{2}$$

$$>-\epsilon+\frac{\delta}{2}>\delta.$$

Since this is true for any $n > N$ it follows that in the limit that $A_\infty \geq -\epsilon + \frac{\delta}{2} > \delta$. This is a contradiction. Therefore $A_\infty \geq 0$. □

3.5 Summary

This chapter gives the formal definitions of centered forms for polynomials and rational functions over circular disks. Power-sum forms for polynomials and rational functions are also defined and compared with centered forms. Then the explicit formulas of the zero, first and second power-sum forms and centered forms are presented for rational functions. In the last three sections, the properties of inclusion, quadratic convergence and inclusion isotonicity are proven for centered forms discussed in this chapter.

Chapter 4

Algorithmic Approaches for Centered Forms

In this chapter, a number of algorithms of low complexity are developed. These algorithms compute centered form evaluations for uni-variate and multi-variate functions. Time complexity and implementations of these algorithms are discussed briefly. A formula is also derived that estimates the number of distinct centered form evaluations for multi-variate functions. This number is surprisingly large even for polynomials of moderate degrees. Some numerical realizations are given to illustrate the algorithms discussed in the chapter.

4.1 A Generalization of the Centered Form and Its Algorithms

4.1.1 Generalization of the Centered Form

In this section, a general centered form for multi-variate polynomials is defined. This centered form is based on the concept of an arrangement of the variables of the polynomial. A formula for the number of arrangements of a term and of a polynomial is given. Some examples are computed showing that even polynomials with moderately many variables may have a very large number of possible centered forms.

This section focuses on a discussion of the algorithm given in Rokne[39]. A generalization of this algorithm is first presented showing that many different centered forms can be derived based on the original algorithm in[39]. From those forms it is possible to choose the most appropriate form for a particular application. The different centered forms are generated from the different arrangements of the variables of a polynomial. A formula is given for the number of possible arrangements of the variables from which different forms can be derived. An algorithm and a corresponding program in Prolog

based on the formula is given in[44]. The program generates all the different centered forms for any multi-variate polynomial. The rational forms are obtained as a simple extension of the polynomial forms in the same manner as in[39].

The notation $\aleph$ is used for the set of integers, $\mathbf{C}$ for the set of complex numbers and $\mathbf{K}$ for the set of compact intervals over $\mathbf{C}$. The sets $\aleph^k$, $\mathbf{C}^k$ and $\mathbf{K}^k$ are the k-dimensional versions of the above. In general we write

$$\mathbf{z} = (z_1, \ldots, z_k) \in \mathbf{C}^k$$

and

$$f = (f_1, \ldots, f_k) : \mathbf{C}^k \to \mathbf{C}.$$

Members of $\mathbf{K}$ and $\mathbf{K}^k$ are Z and $\mathbf{Z}$ respectively. The discussion in this section will mainly focus on the notion of arrangements of polynomials over $\mathbf{C}^k$. These arrangements provide algorithmic information that may be used to compute inclusions in $\mathbf{K}^k$.

The centered forms first discussed in[23] are now introduced. If $f : \mathbf{C}^k \to \mathbf{C}$ is given then a possible centered form function for $f(\mathbf{z})$ over $\mathbf{Z} \in \mathbf{K}^k$ is given by an expression of the form

$$f(\mathbf{z}) = f(\mathbf{c}) + g(\mathbf{z}, \mathbf{c}) \cdot (\mathbf{z} - \mathbf{c})$$

where g:$\mathbf{C}^{2k} \rightarrow \mathbf{C}^k$ (using an inner product notation) is a vector-valued function. Evaluating this centered form function at $\mathbf{Z}$ in the interval arithmetic sense we obtain the centered form evaluation

$$f(\mathbf{Z}) = f(\mathbf{c}) + g(\mathbf{Z}, \mathbf{c}) \cdot (\mathbf{Z} - \mathbf{c}) \supseteq \{f(\mathbf{z}) | \mathbf{z} \in \mathbf{Z}\}.$$

We note that for a given function f and point c there might be many different centered form functions depending on the choice of $g(\mathbf{z}, \mathbf{c})$. In fact, let $f(\mathbf{z}) = p(\mathbf{z}) = z_1 z_2^2$. Then two possible centered form functions for $p(\mathbf{z})$ at $\mathbf{c}$ are

$$p(\mathbf{z}) = p(\mathbf{c}) + g_1(\mathbf{z}, \mathbf{c}) \cdot (\mathbf{z} - \mathbf{c})$$

with

$$g_1(\mathbf{z}, \mathbf{c}) = (z_2^2, c_1(z_2 + c_2))$$

and

$$p(\mathbf{z}) = p(\mathbf{c}) + g_2(\mathbf{z}, \mathbf{c}) \cdot (\mathbf{z} - \mathbf{c})$$

with

$$g_2(\mathbf{z}, \mathbf{c}) = (c_2 z_2, (z_1 z_2 + c_1 c_2)).$$

Here, although $p(\mathbf{z})$ and $\mathbf{c}$ are unchanged the functions $g_1(\mathbf{z}, \mathbf{c})$ and $g_2(\mathbf{z}, \mathbf{c})$ are different. The functions g_1 and g_2 are called the kernels of the centered

form functions and we note that the centered form functions are different if their kernels are different.

It should be noted that the centered form evaluations of one centered form function may yield different results depending on the order of the operations in the evaluation even though the kernel is unchanged as a function. In the sequel, however, we will only be concerned with the kernels that are different as functions so this difference will not be taken into account. For brevity we also shorten the term "centered form function" to "centered form" in the remainder of the thesis since there will be no confusion.

4.1.2 Low Complexity Algorithm

In this section, a low complexity algorithm for calculating multi-variate centered form evaluations is given. The centered form evaluation for multivariate polynomials and rational functions were given in general in the previous section (see also [8]). Here we focus on the developments that resulted in a computationally efficient form in[39]. This form is now summarized by the following algorithm where we employ the notation of multi-indices (see for example [8]).

Algorithm 1. (Description)

Let $p(\mathbf{z}) = \sum_{|\lambda|=0}^{n} a_\lambda \mathbf{z}^\lambda$ be a polynomial in k variables of degree n, where $\mathbf{z} \in \mathbf{C}^k$, $a_\lambda \in \mathbf{C}$, $\lambda \in \aleph^k$. We write the polynomial as $p(\mathbf{z}) = p_1(z_1) = \sum_{i=0}^{n} d_i(1) z_1^i$, where $d_i(1) = d_i(z_2, \ldots, z_k)$. Then $p_1(z_1)$ is a polynomial in one variable z_1 with coefficients that are functions of $z_2, \ldots, z_k$. Using the small Horner-scheme applied to $p_1(z_1)$ at the point c_1 we obtain

$$p_1(z_1) = p_1(c_1) + s_1(z_1, c_1)(z_1 - c_1)$$

where

$$s_1(z_1, c_1) = \sum_{i=0}^{n-1} \Big(\sum_{j=i+1}^{n} d_j(1) c_1^{j-i-1} \Big) z_1^i.$$

Since $p_1(c_1)$ is a function of $z_2, \ldots, z_k$, we may write

$$p_1(c_1) = p_2(z_2) = \sum_{i=0}^{n} d_i(2) z_2^i,$$

where $d_i(2) = d_i(c_1, z_3, \ldots, z_k)$, and using same the method as above we obtain

$$p_2(z_2) = p_2(c_2) + s_2(z_2, c_2)(z_2 - c_2).$$

Continuing in this manner we finally arrive at

$$p_k(z_k) = p_k(c_k) + s_k(z_k, c_k)(z_k - c_k)$$

where $p_k(c_k) = p(\mathbf{c})$, $c = (c_1, \ldots, c_k) \in \mathbf{C}^k$. Combining all the above steps we obtain

$$p(\mathbf{z}) = p_1(c_1)+s_1(z_1, c_1)(z_1-c_1) = p_2(c_2)+s_2(z_2, c_2)(z_2-c_2)+s_1(z_1, c_1)(z_1-c_1) =$$

$$p(\mathbf{c}) + s_k(z_k, c_k)(z_k - c_k) + \ldots + s_1(z_1, c_1)(z_1 - c_1)$$

where $s_i(z_i, c_i) = s_i(c_1, \ldots, c_i, z_i, \ldots, z_k)$. This may be written as

$$p(\mathbf{z}) = p(\mathbf{c}) + (s_1, ..., s_k) \cdot ((z_1 - c_1), ..., (z_k - c_k)), \tag{4.1}$$

using the inner product notation where the kernel $g(z, c) = (s_1, \ldots, s_k)$. This is therefore a centered form as defined in[21].

If this algorithm is applied to the polynomial $p(z) = z_1 z_2^2$ discussed above then the resulting centered form is given by the form having the kernel g_1 above.

4.1.3 Generalization of the Algorithm

In the previous section it was shown that a function may have more than one possible centered form at a point. Furthermore, an algorithm was given that generated one centered form for a multi-variate polynomial. In this section this algorithm is generalized to an algorithm that generates a centered form

at a point c for a given rearrangement of the variables of a multi-variate polynomial, a concept that will be made precise below.

In order to simplify the development of the general algorithm it is also assumed without loss of generality that the polynomials $p(\mathbf{z}) = \sum_{|\lambda|=0}^{n} a_\lambda \mathbf{z}^\lambda$ have the constant term $a_0 = 0$.

In Rokne's algorithm of the previous section examples show that the kernel of a centered form depends on the order and to which power the variables of the polynomial are considered. Different arrangements of variables may therefore generate different kernels and hence centered forms for a polynomial. A precise definition of an arrangement of variables is therefore first introduced. Based on this definition a new algorithm is developed that generates a centered form given an arrangement of the variables. In the next section it will then be shown that the forms generated from different arrangements are different in that they have kernels that are different as functions.

DEFINITION 4.1 *An ordered set* $(z_{j_1}^{m_1}, z_{j_2}^{m_2}, \ldots, z_{j_\tau}^{m_\tau})$ *is called an arrangement of a set of variables* $(z_1, \ldots, z_k)$ *of degree* $(n_1, \ldots, n_k)$, *if it satisfies the conditions :*

$$(1) \qquad 1 \leq j_i \leq k, i = 1, \ldots \tau.$$

$$(2) \quad z_{j_i} \neq z_{j_i+1}, i = 1, \ldots, \tau - 1.$$

$$(3) \quad \sum_{m_\mu = \text{exponent of } z_i} m_\mu = n_i.$$

$$(4) \quad m_i \geq 1, i = 1, \ldots, \tau.$$

Two arrangements $(z_{j_1}^{m_1}, \ldots, z_{j_\tau}^{m_\tau})$ *and* $(z_{l_1}^{d_1}, \ldots, z_{l_\rho}^{d_\rho})$ *are different if one of the following three conditions are satisfied:*

$$(i) \quad \tau \neq \rho.$$

$$(ii) \quad \tau = \rho \text{ and } j_i \neq l_i \text{ for some } i, 1 \leq i \leq \tau.$$

$$(iii) \quad \tau = \rho, j_i = l_i \text{ for all } i, 1 \leq i \leq \tau \text{ and } m_i \neq d_i \text{for some } i.$$

For example, the ordered set $(z_1^2, z_2^3, z_1, z_3^4, z_2^3)$ is an arrangement of 3 variables (z_1, z_2, z_3) of degree (3,6,4) whereas the ordered set $(z_1^2, z_2^3, z_1, z_3^4, z_2^2)$ is not an arrangement of this set of variables, because the sum of exponents of z_2 in this ordered set is 5. Furthermore $(z_1^2, z_2^3, z_1, z_3^4, z_2^3)$ is different from $(z_1, z_2^3, z_1^2, z_3^4, z_2^3)$

Having made the notion of an arrangement of a set of variables precise, the new algorithm can now be presented.

Algorithm 2. (Description)

Let $p(\mathbf{z})$ be a polynomial with variables $(z_1, \ldots, z_k)$ of maximum degrees

$(n_1, \ldots, n_k)$ and let $(z_{j_1}^{m_1}, z_{j_2}^{m_2}, \ldots, z_{j_\tau}^{m_\tau})$ be an arrangement of this set of variables. We write $p(\mathbf{z})$ as

$$p(\mathbf{z}) = p_1(z_{j_1}) = \sum_{i=0}^{n_{j_1}} d_i(j_1) z_{j_1}^i, \tag{4.2}$$

where $d_i(j_1)$ does not contain z_{j_1}. The polynomial $p_1(z_{j_1})$ is now evaluated at $z_{j_1} = c_{j_1}$ as follows:

$$p_1(c_{j_1}) = \sum_{i=0}^{m_1} d_i(j_1) c_{j_1}^i + \sum_{i=m_1+1}^{n_{j_1}} d_i(j_1) c_{j_1}^{m_1} z_{j_1}^{i-m_1}.$$

Dividing $p_1(z_{j_1}) - p_1(c_{j_1})$ by $(z_{j_1} - c_{j_1})$ the expansion

$$p_1(z_{j_1}) = p_1(c_{j_1}) + s_1(z_{j_1}, c_{j_1})(z_{j_1} - c_{j_1}) \tag{4.3}$$

is obtained. Since $p_1(c_{j_1})$ is a function of z_{j_2} , we can write $p_1(c_{j_1})$ as

$$p_1(c_{j_1}) = p_2(z_{j_2}) = \sum_{i=0}^{n_{j_2}} d_i(j_2) z_{j_2}^i,$$

where $d_i(j_2)$ does not contain z_{j_2}. We obtain

$$p_2(z_{j_2}) = p_2(c_{j_2}) + s_2(z_{j_2}, c_{j_2})(z_{j_2} - c_{j_2}).$$

We assume now that at stage l $p_l(c_{j_l})$ and $s_l(z_{j_l}, c_{j_l})$ have been obtained. At the $(l+1)$th stage $p_l(c_{j_l})$ is written as

$$p_l(c_{j_l}) = p_{l+1}(z_{j_l+1}) = \sum_{i=0}^{\hat{n}_{j_l}} d_i(j_{l+1}) z_{j_l+1}^i,$$

where $\hat{n}_{j_l}$ is n_{j_l} minus the sum of all exponents of z_{j_l} appearing before $z_{j_l}^{m_l}$ in the arrangement. Using the same method as above we obtain

$$p_{l+1}(z_{j_{l+1}}) = p_{l+1}(c_{j_{l+1}}) + s_{l+1}(z_{j_{l+1}}, c_{j_{l+1}})(z_{j_{l+1}} - c_{j_{l+1}}).$$

Continuing in this manner we finally obtain

$$p_\tau(z_{j_\tau}) = p_\tau(c_{j_\tau}) + s_\tau(z_{j_\tau}, c_{j_\tau})(z_{j_\tau} - c_{j_\tau}),$$

where $p_\tau(c_{j_\tau}) = p(\mathbf{c})$. Combining all the above steps we obtain

$$p(\mathbf{z}) = p(\mathbf{c}) + s_\tau(z_{j_\tau}, c_{j_\tau})(z_{j_\tau} - c_{j_\tau}) + \ldots + s_1(z_{j_1}, c_{j_1})(z_{j_1} - c_{j_1}).$$

Finally combining terms with same factor $(z_{j_i} - c_{j_i})$ we have

$$p(\mathbf{z}) = p(\mathbf{c}) + s_{\rho_1}(z_1, c_1)(z_1 - c_1) + \cdots + s_{\rho_k}(z_k, c_k)(z_k - c_k).$$

Using the inner product notation this can be written as

$$p(\mathbf{z}) = p(\mathbf{c}) + (s_{\rho_1}, \ldots, s_{\rho_k}) \cdot ((z_1 - c_1), \ldots, (z_k - c_k)).$$

Letting $g(\mathbf{z}, c) = (s_{\rho_1}, \ldots, s_{\rho_k})$ be the kernel, a centered form is obtained.

As an example, let $p(\mathbf{z}) = z_1 z_2^3 + z_2$ and let an arrangement be $A = (z_2, z_1, z_2^2)$. This means that $\tau = 3$. In the first step we consider $z_{j_1} = z_2$ and we get

$$p_1(z_2) = (z_2) + z_2^2(z_1), \;\; p_1(c_2) = c_2 + c_2(z_1 z_2^2)$$

where extra parentheses are used to separate out the quantities used in (4.2). In (4.3) we get $s_1 = (z_1 z_2^2 + 1)$. We continue with $z_{j_2} = z_1$ and get

$$p_2(z_1) = c_2 + c_2 z_2^2 z_1, \;\; p_2(c_1) = c_2 + c_2 z_2^2 c_1$$

and

$$s_2 = c_2 z_2^2.$$

Finally with $z_{j_3} = z_2$ we get

$$p_3(z_2^2) = c_2 + c_1 c_2 z_2^2, \;\; p_3(c_2^2) = c_2 + c_1 c_2^3$$

and

$$s_3 = c_1 c_2 (z_2 + c_2),$$

which results in

$$p(\mathbf{z}) = p(\mathbf{c}) + (z_1 z_2^2 + 1)(z_2 - c_2) + c_2 z_2^2 (z_1 - c_1) + (c_1 c_2 z_2 + c_1 c_2^2)(z_2 - c_2),$$

and finally

$$p(\mathbf{z}) = p(\mathbf{c}) + ((c_2 z_2^2), (z_1 z_2^2 + c_1 c_2 z_2 + c_1 c_2^2 + 1)) \cdot (\mathbf{z} - \mathbf{c}),$$

where $g(\mathbf{z}, c) = ((c_2 z_2^2), (z_1 z_2^2 + c_1 c_2 z_2 + c_1 c_2^2 + 1))$ is the kernel of the centered form. If $\mathbf{Z} = (Z_1, Z_2) \in \mathbf{K}^2$ such that $\mathrm{mid}(Z_i) = c_i$, $i = 1, 2$ (the midpoints of the intervals) then $\mathrm{p}(\mathbf{Z})$ would provide an outer estimate for the range of $p(\mathbf{z})$ over $\mathbf{Z}$ (see also[39]).

4.2 The Number of the Centered Forms

The algorithm in Section 4.1.3 computes a centered form given an arrangement of the variables of a polynomial. It is then natural to ask whether different arrangements generate different centered forms. Since this question is answered affirmatively below, the number of centered forms (that is the number of distinct kernels for a given $p(\mathbf{z})$ and $\mathbf{c}$) can then be counted by counting the number of arrangements. A lemma that gives a formula for the number of arrangements for a set of variables is first proven.

The number of different arrangements $F_k(n_1, \ldots, n_k)$ of a set of variables $(z_1, \ldots, z_k)$ with degrees $(n_1, \ldots, n_k)$ is given by the following lemma.

LEMMA 4.1

$$F_k(n_1, \ldots, n_k) = \frac{(n_1 + \ldots + n_k)!}{n_1! \ldots n_k!}.$$

Proof. The total number of arrangements of variable set $(z_1, \ldots, z_k)$ with degrees $(n_1, \ldots, n_k)$ is $(n_1 + \ldots + n_k)!$. Among them, the permutations within same variable name $z_i^{n_i}$ does not yield different arrangements. Therefore the total number should be divided by the number of all permutations for each variable name, i.e., $n_i!, 1 \leq i \leq k$. It follows that

$$F_k(n_1, \ldots, n_k) = \frac{(n_1 + \ldots + n_k)!}{n_1! \ldots n_k!}.$$

□

DEFINITION 4.2 *A centered form for a polynomial is called the centered form corresponding to an arrangement, if it is generated by Algorithm 2 with this arrangement.*

THEOREM 4.1 *The number of centered forms for a uni-term polynomial*

$$p(z) = a_\lambda z_1^{n_1} \dots z_k^{n_k}$$

is

$$N(p) = F_k(n_1, \dots, n_k).$$

Proof. We only need to prove that the centered forms corresponding to different arrangements are different.

Let A and B therefore be two different arrangements of k variables $(z_1, \dots, z_k)$ of degrees $(n_1, \dots, n_k)$. Let $z_a^{m_a}$ and $z_b^{m_b}$ be the first powers in A and B respectively that are different and let $i, 0 \leq i \leq \tau_1, \tau_2$ denote their position. Then the following two cases have to be considered:

Case 1 : $z_a = z_b = z, m_a \neq m_b$;

Case 2 : $z_a \neq z_b = z$.

Case 1. Assuming that $m_a < m_b$. Then, term $c_{a,1}^{m_{a,1}} \cdots c_{a,i-1}^{m_{a,i-1}} c^{m_a} z_{a,i+1}^{m_{a,i+1}} \cdots z_{a,r_1}^{m_{a,r_1}}$ occurs in form A does not occur in form B, since all terms from z_b in form B contains c with its exponent $> m_a$.

Case 2. Similarly term $c_{a,1}^{m_{a,1}} \cdots c_{a,i-1}^{m_{a,i-1}} c^{m_a} z_{a,i+1}^{m_{a,i+1}} \cdots z_{a,r_1}^{m_{a,r_1}}$ in form A never occurs in form B, since any term in form B from z_b must contain factor $c_b^{m_b}$, which is not in above term.

Therefore, centered forms derived from different variable arrangements are different. This completes the proof.

□

As an example, polynomial $p(\mathbf{z}) = z_1 z_2^3$ is considered. We have

$$N(p) = F_2(1,3) = 4.$$

which means that the polynomial has four different centered forms. Their kernels are

(1) $g_1(\mathbf{z},\mathbf{c}) = (s_{c_1}, s_{c_2}) = (z_2^3, c_1(z_2^2 + z_2 c_2 + c_2^2))$

(2) $g_2(\mathbf{z},\mathbf{c}) = (s_{c_1}, s_{c_2}) = (c_2 z_2^2, z_1 z_2^2 + c_1 c_2 z_2 + c_1 c_2^2)$

(3) $g_3(\mathbf{z},\mathbf{c}) = (s_{c_1}, s_{c_2}) = (z_2 c_2^2, c_1 c_2^2 + z_1 z_2 c_2 + z_1 z_2^2)$

(4) $g_4(\mathbf{z},\mathbf{c}) = (s_{c_1}, s_{c_2}) = (c_2^3, z_1(c_2^2 + c_2 z_2 + z_2^2))$.

Some definitions are now necessary prior to proving the main theorem of

the section.

DEFINITION 4.3 *A product* $t = z_{j_1}^{n_1} \dots z_{j_z}^{n_z}$ *of powers of variables* z_{j_i} *is called a variable product.*

Thus a term $a_\lambda z^\lambda$ of a polynomial is the product of a variable product and a real constant.

DEFINITION 4.4 *A variable product* $t = z_{j_1}^{n_1} \dots z_{j_z}^{n_z}$ *is called a common variable product of two variable products* t_1 *and* t_2, *if* t *divides* t_1 *and* t_2 .

DEFINITION 4.5 *A variable product* $t = z_{j_1}^{n_1} \dots z_{j_z}^{n_z}$ *is called the greatest common variable product of two variable products* t_1 *and* t_2 , *denoted by* $t = \{t_1, t_2\}$, *if* t *satisfies the conditions*

(1) t *is a common variable product of* t_1 *and* t_2 .

(2) If t_0 *is any common variable product of* t_1 *and* t_2 *then* t_0 *divides* t.

For example,

$$z_1^3 z_2^2 z_1 = \{z_1^4 z_2^2 z_1^4, z_1^3 z_2^5 z_1\}.$$

Consider now a polynomial

$$p(\mathbf{z}) = \sum_{|\lambda|=0}^{n} a_\lambda \mathbf{z}^\lambda.$$

The polynomial may be written as $p(\mathbf{z}) = \sum_{i=0}^{\eta} t_i$ where $t_i = a_\lambda \mathbf{z}^\lambda$ and for any two terms $t_i = a_\lambda \mathbf{z}^\lambda$ and $t_j = a_\mu \mathbf{z}^\mu$, if $i \neq j$ then $\lambda \neq \mu$. Only essentially different centered forms are therefore counted below. As an illustration let $p(\mathbf{z}) = 2z_1^2 z_2^2 z_3$. This polynomial may be written as

$$p(\mathbf{z}) = p_1(\mathbf{z}) + p_2(\mathbf{z})$$

where $p_1(\mathbf{z}) = z_1^2 z_2^2 z_3$ and $p_2(\mathbf{z}) = z_1^2 z_2^2 z_3$. There are 15 centered forms for each of $p(\mathbf{z})$, $p_1(\mathbf{z})$ and $p_2(\mathbf{z})$ corresponding to the different arrangements of $z_1^2 z_2^2 z_3$. Further centered forms for $p(\mathbf{z})$ could therefore be generated by considering combinations of forms for $p_1(\mathbf{z})$ and $p_2(\mathbf{z})$. Such hybrid forms are not counted in the sequel.

The number of centered forms for a term $t = c z_1^{n_1} \dots z_i^{n_i}$ is $N(t) = F_i(n_1, \dots, n_i)$ from Theorem 4.1. Also if $c \in \mathbf{C}$ is a constant then $F(c) = 0$. Using this the main theorem of the section is now proven.

THEOREM 4.2 *Let* $p(z) = \sum_{|\lambda|=0}^{n} a_\lambda z^\lambda = \sum_{i=0}^{\eta} t_i = p_{\eta-1}(z) + t_\eta$. *Then the number* $N(p)$ *of centered forms of the polynomial* p *satisfies the recurrence equation*

$$N(p) = N(p_{\eta-1}) N(z_1 \dots z_\eta u) - \sum_{0 \leq i < j < \eta} N(z\{\frac{t_i}{r_i}, \frac{t_j}{r_j}\}) \left(N(z\{\frac{t_i}{r_i}, \frac{t_j}{r_j}\}) - 1 \right), \tag{4.4}$$

where $r_0 = \{t_\eta, t_0\}$, $r_i = \left\{\frac{t_\eta}{r_1 \ldots r_{i-1}}, \frac{t_i}{\{t_i, r_{i-1}\}}\right\}, i = 1, \ldots, \eta - 1$; $u = \frac{t_\eta}{r_1 \ldots r_{\eta-1}}$ *and where* z *is any single variable and* z_i *is any single variable when* $r_i \neq 1$ *or* $z_i = 1$ *when* $r_i = 1, i = 1, \ldots, \eta$.

Proof. Let A be an arrangement of the variables of $p_{\eta-1}$. Since $r_i = \left\{\frac{t_\eta}{r_1 \ldots r_i - 1}, \frac{t_i}{\{t_i, r_i - 1\}}\right\}$ it follows that the variable product r_i in t_η appears in term t_i. When deriving centered forms for term t_η the order among the variables in r_i can therefore not be rearranged and they must obey the order among these variables in A. Therefore when A is fixed r_i can only be considered as one variable. The discussion above is valid for all i's , $0 \leq i \leq \eta - 1$. It follows that term t_η only has

$$N(z_1 \ldots z_\eta u)$$

centered forms corresponding to the arrangement A. The total number of centered forms is therefore

$$N(p) = N(p_{\eta-1}) N(z_1 \ldots z_\eta u). \tag{4.5}$$

In this number, however, some variable arrangements for term t_η are inconsistent with the arrangement A for p_η , because the orders between some pairs of r_i and r_j are fixed in the arrangement A and cannot be changed

in the term t_η. Assuming that $\frac{t_i}{r_i}$ and $\frac{t_j}{r_j}$ include a common variable product $v = \left\{\frac{t_i}{r_i}, \frac{t_j}{r_j}\right\} \neq 1$. Only orders $(\ldots v_1 r_i v_2 \ldots)$ and $(\ldots v_1 r_j v_2 \ldots)$ in A, where $v_1 v_2 = v, v_1 \neq 1, v_2 \neq 1$, do not restrict an order between r_i and r_j. If some v_1 is in the orders $(\ldots v_1 r_i \ldots)$ and $(\ldots r_j v_1 \ldots)$, r_i and r_j is in the order $(r_j, \ldots, r_i)$ then the corresponding variable arrangement for term t_η is inconsistent with the arrangement A and must be excluded. The number of arrangements containing orders $(\ldots v_1 r_i v_2 \ldots)$ and $(\ldots v_1 r_j v_2 \ldots)$ is $N(vz)$, where z is any variable and the total number of different pairs of ordered sets (r_i, v) and (r_j, v) is $N(vz)^2$. So the number of inconsistent variable arrangements for t_η is $N(vz)^2 - N(vz)$. This number has to be subtracted from the total number (4.5). Subtracting this number for each pair $(i,j), 0 \leq i < j \leq \eta - 1$, the final result is

$$N(p) = N(p_{\eta-1})N(z_1 \ldots z_\eta u) - \sum_{1 \leq i < j \leq \eta} N(z\{\frac{t_i}{r_i}, \frac{t_j}{r_j}\}) \left(N(z\{\frac{t_i}{r_i}, \frac{t_j}{r_j}\}) - 1 \right).$$

□

For example, for the polynomial $p(\mathbf{z}) = z_1 z_2 + z_2 z_3 + z_1 z_3$ we have

$$N(p_1) = N(z_1 z_2) = 2,$$

$$N(p_2) = N(p_1)N(z_1 z_3) = 2 \cdot 2 = 4$$

and finally

$$N(p) = N(p_2)N(z_1z_2) - N(z_1z_2)(N(z_1z_2) - 1) = 8 - 2 = 6.$$

This polynomial therefore has 6 different centered forms given by the following kernels and arrangements of variables:

	kernels	arrangements
1 :	$g_1(x,c) = (x_2 + x_3, c_1 + x_3, c_1 + c_2)$	(x_1, x_2, x_3)
2 :	$g_2(x,c) = (x_2 + x_3, c_1 + c_3, c_1 + x_2)$	(x_1, x_3, x_2)
3 :	$g_3(x,c) = (c_2 + x_3, x_1 + x_3, c_1 + c_2)$	(x_2, x_1, x_3)
4 :	$g_4(x,c) = (c_2 + x_3, x_1 + x_3, x_1 + c_2)$	(x_2, x_3, x_1)
5 :	$g_5(x,c) = (c_2 + c_3, x_1 + c_3, x_1 + x_2)$	(x_3, x_2, x_1)
6 :	$g_6(x,c) = (x_2 + c_3, c_1 + c_3, x_1 + x_2)$	(x_3, x_1, x_2)

These are all centered forms for the polynomial in the sense described above.

A theorem is now proven that forms the basis for an algorithm for obtaining all the arrangements that derive different centered forms. Let $S(p)$ denote the set of all the arrangements of the variables in p which derive different centered forms; this set $S(p)$ is called *the arrangement set* of p(z). Obviously, when p is a uni-term polynomial, $S(p)$ is the set of all arrangements

of variables in p.

DEFINITION 4.6 *An ordered set U is called the ordered union of two ordered sets S_1 and S_2 , denoted by $U = S_1 \biguplus S_2$, if U is the (usual) union of S_1 and S_2 and the orders between any pair of elements of U obey the orders both in S_2 and set S_1 . If the order between two elements in one set contradicts the order between same pair elements in another set then U is empty.*

For example,

$$(z_1^2, z_2^3, z_3) \biguplus (z_1^2, z_5^2, z_3) = (z_1^2, z_2^3, z_5^2, z_3).$$

and

$$(z_1, z_2^2, z_3^2) \biguplus (z_3^2, z_1, z_4^2) = \phi.$$

THEOREM 4.3 *Let $p = p_n + t$, where p_n is a n-term polynomial and t is a uni-term polynomial. We have*

$$S(p) = \{s_1 \biguplus s_2 | s_1 \in S(p_n), s_2 \in S(t)\}.$$

Proof. This theorem is a corollary of Theorem 4.2. □

Based on Theorem 4.3, an algorithm to obtain all those arrangements which derive the distinct centered forms for a polynomial is designed as follows.

Algorithm 3.

Let $p(\mathbf{z}) = \sum_{|\lambda|=0}^{n} a_\lambda \mathbf{z}^\lambda$ be a m-term polynomial in k variables of degree n, where $\mathbf{z} \in \mathbf{C}^k$, $a_\lambda \in \mathbf{C}$, $\lambda \in \aleph^k$.

Step 1. Set $i = 1$ and $S = \Phi$. Go to Step 2.

Step 2. Let t_i be the i-th term of $p(\mathbf{z})$. Generate all permutations of the variable product of t_i and remove all duplicate permutations to form the arrangement set $S(t_i)$. Go to Step 3.

Step 3. Union S and $S(t_i)$ to form S. If $i = m$, output S, which is the arrangement set $S(p)$ of $p(\mathbf{z})$; otherwise go back step 2.

We have written this algorithm in Pascal codes that gives all these arrangements for any polynomial. Then for a given variable arrangement, Algorithm 2 can be used to derive the corresponding centered form.

The numbers of centered forms obtained from Algorithm 3 was now calculated for three simple polynomials using the results of this section. These numbers and their corresponding polynomials are given in Table 4.1. These

$p(\mathbf{z})$	number of forms
$z_1 z_2^2 z_3^3$	60
$x_1 z_2^2 z_3^3 z_4^4 z_5^5$	37837800
$z_1 z_2^2 z_3^3 z_4^4 z_5^5 z_6^6$	$2.0532304e^{12}$

Table 4.1: Number of different centered forms

numbers are surprisingly large.

Remark

In this section an algorithm is given that produces a very large number of genuinely different centered forms for a polynomial, where the kernels are different from each other as functions. From this set of centered forms it may be possible to chose a form that is most suitable for a given application.

From formula (4.4) and the numerical examples it is clear that the number of different centered forms of this type increases very fast with the number of terms and with the degree of the polynomial. The choice of the most effective centered form for a particular application might therefore be quite time-consuming.

It is now interesting to reflect briefly on the history of interval methods for range inclusions. The natural interval extension was first discussed by

Moore[23]. From these extensions he conjectured that the subset of centered forms gave good results. In this thesis we have shown that a particular class of these forms contain a rather large number of centered forms that are obtained simply by manipulating the expressions algebraically (that is, no "coarsening" has taken place prior to the actual interval evaluation). It is now an open question as to which of these good forms can be selected to be the best one prior to an actual evaluation of all the possible forms.

4.3 Low Complexity Algorithms for Multivariate Taylor Forms

Taylor expansions are special cases of centered forms, called *Taylor forms.* This section gives algorithms for the computation of Taylor forms in k variables for polynomials and for rational functions at a point. These algorithms do not require the explicit computation of the derivatives, they are simply based on the repetitive division of the polynomials and the rational functions by scalar linear factors. Some sample calculations are also performed using these algorithms.

The work in[39] and[38] forms the background for these methods. Related ideas are found in[14] and[1].

Taylor series are useful tools in numerical computations. For example, in many iterative methods for solving nonlinear equations the first term, or the Jacobean is required. In global optimization problems solved iteratively the second term, or the Hessian occurs frequently(see for example [31] and[16]).

An interesting discussion on the difficulties encountered when analytically calculating the gradient is given in[50]. Such difficulties are avoided when the techniques presented in this section are implemented as computer codes.

4.3.1 Centered Form Algorithm

Let $p(\mathbf{z}) = \sum_{|\lambda|=0}^{n} a_\lambda \mathbf{z}^\lambda$ be a polynomial in k variables and let maximum degree of z_i be n_i , $0 \leq i \leq k$. Rokne[39] developed a low complexity algorithm to obtain centered forms(see[34] for a definition) for $p(\mathbf{z})$ by using the small Horner-scheme. This algorithm was given is Section 4.1.2 as Algorithm 1.

The expansion (4.1) in Algorithm 1 resembles a Taylor expansion of one term where the inner product takes the place of the error term. In the following sections the resemblance is taken advantage of in the development

of an actual Taylor expansion for both polynomials and rational functions.

Before discussing the algorithms for Taylor expansion, we generalize the above algorithm to rational functions.

Let $p(\mathbf{z}) = \sum_{\lambda=0}^{n} a_\lambda \mathbf{z}^\lambda$ and $q(\mathbf{z}) = \sum_{\lambda=0}^{m} b_\lambda \mathbf{z}^\lambda$ be two polynomials. Then

$$f(\mathbf{z}) = \frac{p(\mathbf{z})}{q(\mathbf{z})}$$

is a rational function of degree $l = \max(n, m)$. The polynomials $p(\mathbf{z})$ and $q(\mathbf{z})$ are now developed as centered form expressions

$$p(\mathbf{z}) = p(\mathbf{c}) + s(\mathbf{z})(\mathbf{z} - \mathbf{c})$$

$$q(\mathbf{z}) = q(\mathbf{c}) + t(\mathbf{z})(\mathbf{z} - \mathbf{c}).$$

If we write

$$f(\mathbf{z}) = f(\mathbf{c}) + \frac{r(\mathbf{z})}{q(\mathbf{z})}(\mathbf{z} - \mathbf{c})$$

then it follows immediately that $f(\mathbf{c}) = p(\mathbf{c})/q(\mathbf{c})$. As in[44] $r(\mathbf{z})$ is computed as follows

$$(f(\mathbf{z}) - f(\mathbf{c}))q(\mathbf{z}) = r(\mathbf{z})(\mathbf{z} - \mathbf{c})$$

or using the centered form expressions for $f(\mathbf{z})$ and $q(\mathbf{z})$ we obtain

$$(\mathbf{z} - \mathbf{c})r(\mathbf{z}) = \left(\frac{p(\mathbf{c}) + s(\mathbf{z})(\mathbf{z} - \mathbf{c})}{q(\mathbf{c}) + t(\mathbf{z})(\mathbf{z} - \mathbf{c})} - f(\mathbf{c}) \right) q(\mathbf{z})$$

$$= \frac{1}{q(\mathbf{c})}((p(\mathbf{c}) + s(\mathbf{z})(\mathbf{z}-\mathbf{c}))q(\mathbf{c}) - p(\mathbf{c})(q(\mathbf{c}) + t(\mathbf{z})(\mathbf{z}-\mathbf{c})))$$

$$= \frac{s(\mathbf{z})q(\mathbf{c}) - t(\mathbf{z})p(\mathbf{c})}{q(\mathbf{c})}(\mathbf{z}-\mathbf{c})$$

$$\left(s(\mathbf{z}) - \frac{t(\mathbf{z})p(\mathbf{c})}{q(\mathbf{c})}\right)(\mathbf{z}-\mathbf{c})$$

where the vector $(\mathbf{z}-\mathbf{c})$ is explicitly factored out since $p(\mathbf{z})$ and $q(\mathbf{z})$ were written as centered form expressions. Therefore the computations for a rational centered form expression only involves the computation of two polynomial centered forms as well as one scalar multiplication of a vector and a vector addition. The processing may be continued using

$$\frac{r(\mathbf{z})}{q(\mathbf{z})}$$

hence successively computing higher order forms for rational functions. These are not discussed here explicitly. In Chapter 6, we will discuss the higher order forms in the sense of meromorphic functions. The rational forms may be considered as special cases of meromorphic forms.

The following is an example to illustrate the centered form algorithm for rational functions. For simplicity, the example is given for real function.

Let

$$f(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

where

$$p(\mathbf{x}) = 3 + x_1 + x_2 + x_1^2 x_2$$

and

$$q(\mathbf{x}) = x_1^2 x_2^2 + x_2^2.$$

Let also

$$\mathbf{X} = ([1-\epsilon, 1+\epsilon], [2-\epsilon, 2+\epsilon],), \epsilon > 0$$

with $c = (1, 2)$. The centered form expressions for $p(\mathbf{x})$ and $q(\mathbf{x})$ are

$$p(\mathbf{x}) = 8 + (x_1 x_2 + 1 + x_2, 2) \cdot ((x_1 - 1)(x_2 - 2))$$

and

$$q(\mathbf{x}) = 8 + (x_1 x_2^2 + x_2^2, 2x_2 + 4) \cdot ((x_1 - 1)(x_2 - 2))$$

respectively. Hence

$$r(\mathbf{x}) = (x_1 x_2 + 1 + x_2 - x_1 x_2^2 - x_2^2, -2 - 2x_2)$$

and the centered form expression for $f(\mathbf{x})$ is

$$f(\mathbf{x}) = 1 + \frac{r(\mathbf{x})}{q(\mathbf{x})} \cdot ((x_1 - 1)(x_2 - 2)).$$

4.3.2 Algorithm for the Polynomial Taylor Expansion

It is known that a multi-variate function can be expanded into its Taylor expansion if this function is *holomorphic* (see [9] p. 39). Since both polynomials and rational functions are holomorphic, their Taylor expansions exist. Based on this, the centered form algorithm outlined in the previous section is generalized to an algorithm that computes the Taylor expansion for a multi-variate polynomial in this section.

Algorithm 4. (Description)

In Section 4.1.2 the inner product expansion (4.1) was obtained for a polynomial $p(z_1, \ldots, z_k)$. In this expansion the quantity $s_1(z_1, \ldots, z_k)$ is a polynomial in the variables $z_1, \ldots, z_k$, the quantity $s_2(c_1, z_2, \ldots, z_k)$ is a polynomial in the variables $z_2, \ldots, z_k$; and so on. Clearly, each of these polynomials also have centered forms which can be computed using the centered form algorithm from the first section. The centered forms for the first two polynomials are

$$
\begin{aligned}
& s_1(c_1, z_1, \ldots, z_k) = \\
& \quad (s_1(c_1, c_1, \ldots, c_k), s_{1,1}(c_1, z_1, \ldots, z_k), \ldots, s_{1,k}(c_1, c_1, \ldots, c_k, z_k)) \\
& \quad \cdot (1, (z_1 - c_1), \ldots, (z_k - c_k)) \qquad (4.6)
\end{aligned}
$$

and

$$s_2(c_1, c_2, z_2, \ldots, z_k) =$$

$$(s_2(c_1, c_2, c_2, c_3, \ldots, c_k), s_{2,1}(c_1, c_2, z_2, \ldots, z_k), \ldots, s_{2,k-1}(c_1, c_2, c_2, \ldots, , c_k, z_k))$$

$$\cdot (1, (z_2 - c_2), \ldots, (z_k - c_k)) \tag{4.7}$$

where a second subscript has been introduced to indicate that this is the second set of centered forms.

The above scheme for labelling the successive polynomials does not suffice for the algorithm to be developed below, however. The following notation is therefore introduced. The polynomial $p(z_1, \ldots, z_k)$ is written as

$$s_{0,\ldots,0}(z_1, \ldots, z_k).$$

Furthermore the polynomial

$$s_i(c_1, \ldots, c_i, z_i, \ldots, z_k)$$

is written as

$$s_{0,\ldots,0,1,0,\ldots,0}(c_1, \ldots, c_{i-1}, z_i, \ldots, z_k)$$

where the 1 in the subscript is in the ith position in the subscript array. In this manner the successive polynomials will have k explicit parameters

$c_1, \ldots, c_{i-1}, z_i, \ldots, z_k$. The parameter c_i is implicit. Consider now the polynomial

$$s_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,0}(c_1, \ldots, c_{i-1}, z_i, \ldots, z_k).$$

If the centered form algorithm is applied to this polynomial then the

$$s_1(c_1, \ldots, c_{j-1}, z_j, \ldots, z_k), i \leq j \leq k$$

generated in equation (4.1) corresponding to this polynomial is denoted by

$$s_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1, \ldots, c_{i-1}, z_i, \ldots, z_k), j = i$$

$$s_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,1,\ldots,0}(c_1, \ldots, c_{j-1}, z_j, \ldots, z_k), i+1 \leq j \leq k.$$

Using this notation formulas (4.6) and (4.7) can be written as

$$\begin{aligned} & s_{1,0,\ldots,0}(z_1, \ldots, z_k) \\ &= (s_{1,0,\ldots,0}(c_1, \ldots, c_k), s_{2,0,\ldots,0}(z_1, \ldots, z_k), \ldots s_{1,0,\ldots,0,1}(c_1, \ldots, c_{k-1}, z_k)) \\ &\cdot (1, (z_1 - c_1), \ldots, (z_k - c_k)) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} & s_{0,1,\ldots,0}(c_1, z_2, \ldots, z_k) \\ &= (s_{0,1,\ldots,0}(c_1, \ldots, c_k), \end{aligned}$$

$$
\begin{aligned}
& s_{0,2,0,\ldots,0}(c_1, z_2, \ldots, z_k), \\
& \vdots \\
& , s_{0,1,0,\ldots,0,1}(c_1, \ldots, c_{k-1}, z_k)) \\
& \cdot (1, (z_2 - c_2), \ldots, (z_k - c_k)) \,. \qquad (4.9)
\end{aligned}
$$

In general the following, somewhat complicated formula is obtained

$$
\begin{aligned}
& s_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1, \ldots, c_{i-1}, z_i, \ldots, z_k) \\
& = (s_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1, \ldots c_k), \\
& \quad s_{\lambda_1,\ldots,\lambda_{i-1},\lambda_i+1,0,\ldots,0}(c_1, \ldots, c_{i-1}, z_i, \ldots, z_k), \\
& \quad s_{\lambda_1,\ldots,\lambda_i,1,0,\ldots,0}(c_1, \ldots, c_i, z_{i+1}, \ldots, z_k), \\
& \quad \vdots \\
& \quad , s_{\lambda_1,\ldots,\lambda_i,0,\ldots,0,1}(c_1, \ldots, c_{k-1}, z_k)) \\
& \quad \cdot (1, (z_i - c_i), \ldots, (z_k - c_k)) \,. \qquad (4.10)
\end{aligned}
$$

Each application of the centered form procedure reduces the degree of at least one variable in the resulting polynomials by one. This implies that the algorithm will terminate in a finite number of steps. Eventually the terms

$$
s_{\lambda_1,\ldots,\lambda_{k-1},\lambda_{n_k}}(c_1, \ldots, c_{k-1}, z_k), 0 \leq \lambda_1 \leq n_1, \ldots, 0 \leq \lambda_{k-1} \leq n_{k-1}
$$

are arrived at. These terms are constants, that is,

$$s_{\lambda_1,\ldots,\lambda_{k-1},\lambda_{n_k}}(c_1,\ldots,c_{k-1},z_k) = s_{\lambda_1,\ldots,\lambda_{k-1},\lambda_{n_k}}(c_1,\ldots,c_k),$$

$$0 \leq \lambda_1 \leq n_1,\ldots,0 \leq \lambda_{k-1} \leq n_{k-1}.$$

Plugging formula (4.10) into the first expression step by step and expanding the expression

$$p(z_1,\ldots,z_k) =$$

$$\sum_{0\leq\lambda_1\leq n_1,\ldots,0\leq\lambda_k\leq n_k} s_{\lambda_1,\ldots,\lambda_k}(c_1,\ldots,c_k)(z_1-c_1)^{\lambda_1}\cdots(z_k-c_k)^{\lambda_k} \qquad (4.11)$$

is obtained.

In a slightly unconventional manner the notation

$$\frac{\partial^{\lambda_1+\cdots+\lambda_k} p(c_1,\ldots,c_k)}{\partial z_1^{\lambda_1}\partial z_2^{\lambda_2}\cdots\partial z_k^{\lambda_k}}$$

is now used for the partial derivative of $p(z_1,\ldots,z_k)$ with respect to z_1 λ_1 times and so on up to the partial derivative with respect to z_k λ_k times, then evaluated at $c_1,\ldots,c_k$.

It is now shown, using the compact partial derivative notation, that expression (4.11) is the Taylor expansion of the polynomial $p(z_1,\ldots,z_k)$ at $(c_1,\ldots,c_k)$.

THEOREM 4.4

$$s_{\lambda_1,\ldots,\lambda_k}(c_1,\ldots,c_k) = \frac{1}{\lambda_1!\lambda_2!\cdots\lambda_k!}\frac{\partial^{\lambda_1+\cdots+\lambda_k}p(c_1,\ldots,c_k)}{\partial z_1^{\lambda_1}\partial z_2^{\lambda_2}\cdots\partial z_k^{\lambda_k}}. \tag{4.12}$$

Proof. Taking the partial differentials in (4.11) and evaluating at $\mathbf{z} = \mathbf{c}$ the expression

$$\frac{\partial^{\lambda_1+\cdots+\lambda_k}p(c_1,\ldots,c_k)}{\partial z_1^{\lambda_1}\partial z_2^{\lambda_2}\cdots\partial z_k^{\lambda_k}} = \lambda_1!\lambda_2!\cdots\lambda_k!s_{\lambda_1,\ldots,\lambda_k}(c_1,\ldots,c_k).$$

is obtained. A slight rearrangement gives

$$s_{\lambda_1,\ldots,\lambda_k}(c_1,\ldots,c_k) = \frac{1}{\lambda_1!\lambda_2!\cdots\lambda_k!}\frac{\partial^{\lambda_1+\cdots+\lambda_k}p(c_1,\ldots,c_k)}{\partial z_1^{\lambda_1}\partial z_2^{\lambda_2}\cdots\partial z_k^{\lambda_k}}$$

which is the desired expression. □

A recursive definition of the Taylor coefficients for a multi-variate polynomial is now given based on the last theorem.

THEOREM 4.5

$$s_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots c_{i-1},z_i,\ldots,z_k) =$$

$$\frac{1}{(z_i-c_i)}\left[s_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,0}(c_1,\ldots,c_{i-1},z_i,\ldots,z_k) - s_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)\right].$$

Proof. This expression follows directly from algorithm in Section 4.3.2.

□

4.3.3 Algorithm for Rational Functions

The usual method for obtaining the Taylor series for a multi-variate polynomial is to evaluate all the required partial derivatives. The polynomial algorithm in the previous section does not improve significantly on the time-complexity of this process. The technique using the partial derivatives is normally also used to form the Taylor series for a rational function. Here, however, evaluating the partial derivatives for a function $f(z) = (p(z)/q(z))$ is a task of high time-complexity. The algorithm given in this section provides a low complexity alternative.

A low complexity algorithm for the Taylor series for a rational function in one variable was developed in Rokne[38]. This algorithm is given as follows.

Algorithm 5. (Description)

Let $p(z)$ and $q(z)$ be polynomials of degree n and degree m respectively. Then

$$f(z) = \frac{p(z)}{q(z)}$$

is a rational function. It may be written as

$$f(z) = \frac{p(c)}{q(c)} + \frac{s_1(z)}{q(z)}$$

where

$$s_1(z) = p(z) - q(z)\frac{p(c)}{q(c)}.$$

Clearly $s_1(z)$ is divisible by $(z-c)$ since $s_1(c) = 0$ and therefore

$$s_1(z) = p_1(z)(z-c).$$

It follows that

$$f(z) = \frac{p(c)}{q(c)} + \frac{p_1(z)}{q(z)}(z-c)$$

with

$$f'(c) = \frac{p_1(c)}{q(c)}.$$

Suppose now that the Taylor expansions

$$p(z) = \sum_{i=0}^{n} \frac{p^{(i)}(c)}{i!}(z-c)^i$$

and

$$q(z) = \sum_{i=0}^{n} \frac{q^{(i)}(c)}{i!}(z-c)^i$$

for $p(z)$ and $q(z)$ are already known. Then $p_1(z)$ can be calculated as follows

$$p_1(z) = \sum_{i=0}^{k} \frac{1}{i!}\left[p^{(i+1)}(c) - q^{(i+1)}(c)\frac{p(c)}{q(c)}\right](z-c)^i,$$

where $k = \max(n, m)$. Repeating this process we obtain

$$f(z) = f(c) + \frac{p_1(c)}{q(c)}(z-c) + \cdots + \frac{p_l(c)}{q(c)}\frac{(z-c)^l}{l!} + \frac{p_{l+1}(z)}{q(z)}\frac{(z-c)^{l+1}}{(l+1)!}.$$

where

$$f^{(i)}(c) = \frac{p_i(c)}{q(c)}.$$

The quantity $p_i(c)$ is furthermore defined via

$$p_i(z) = \sum_{j=0}^{k-i+1} \frac{p_i^{(j)}(c)}{j!}(z-c)^j$$

and

$$p_i^{(j)}(c) = p_{i-1}^{(j+1)}(c) - q^{(j+1)}(c)\frac{p(c)}{q(c)}.$$

This simple method for computing the Taylor expansion of a rational function in one variable is now generalized to multi-variate rational functions.

Algorithm 6. (Description)

Let

$$f(z_1, \ldots, z_k) = \frac{p(z_1, \ldots, z_n)}{q(z_1, \ldots, z_m)}$$

be a multivariate rational function and let $k = \max(n, m)$.

Considering this function as a rational function of one variable z_1 and applying the above algorithm the expression

$$\begin{aligned} & f(z_1, \ldots, z_k) \\ &= f_{0,\ldots,0}(c_1, z_2, \ldots, z_k) \\ &+ f_{1,\ldots,0}(c_1, z_2, \ldots, z_k)(z_1 - c_1) \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&+ \quad f_{n_1,\ldots,0}(c_1, z_2, \ldots, z_k)\frac{(z_1 - c_1)^{n_1}}{n_1!} \\
&+ \quad f_{n_1+1,\ldots,0}(z_1, z_2, \ldots, z_k)\frac{(z_1 - c_1)^{n_1+1}}{(n_1+1)!},
\end{aligned}
$$

is obtained where

$$f_i(c_1, z_2, \ldots, z_k) = \frac{p_i(c_1, z_2, \ldots, z_k)}{q(c_1, z_2, \ldots, z_k)}.$$

Since $p_i(c_1, z_2, \ldots, z_k)$ is a rational function in the variables $z_2, \ldots, z_k$ it follows that $f_{i,0,\ldots,0}(c_1, z_2, \ldots, z_k)$ is as well.

Considering now $p_{i,0,\ldots,0}(c_1, z_2, \ldots, z_k)$ as a function of the variable z_2 and applying the algorithm the expression

$$
\begin{aligned}
&p_{i,0,\ldots,0}(c_1, z_2, \ldots, z_k) \\
&= \quad \frac{p_{i,0,\ldots,0}(c_1, c_2, \ldots, z_k)}{q(c_1, c_2, \ldots, z_m)} \\
&+ \quad \frac{p_{i,1,\ldots,0}(c_1, c_2, \ldots, z_k)}{q(c_1, c_2, \ldots, z_m)}(z_2 - c_2) \\
&\vdots \\
&+ \quad \frac{p_{i,n_2,\ldots,0}(c_1, c_2, \ldots, z_k)}{q(c_1, c_2, \ldots, z_m)}\frac{(z_2 - c_2)^{n_2}}{n_2!} \\
&+ \quad \frac{p_{i,n_2+1,\ldots,0}(c_1, z_2, \ldots, z_k)}{q(c_1, z_2, \ldots, z_m)}\frac{(z_2 - c_2)^{n_2+1}}{(n_2+1)!}.
\end{aligned}
$$

is obtained.

The general step is given by

$$\begin{aligned}
&\frac{p_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}{q(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)} \\
&= \frac{p_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_{i+1},z_{i+2},\ldots,z_k)}{q(c_1,\ldots,c_{i+1},z_{i+2},\ldots,z_k)} \\
&+ \frac{p_{\lambda_1,\ldots,\lambda_i,1,\ldots,0}(c_1,\ldots,c_{i+1},z_{i+2},\ldots,z_k)}{q(c_1,\ldots,c_{i+1},z_{i+2},\ldots,z_k)}(z_{i+1}-c_{i+1}) \\
&\vdots \\
&+ \frac{p_{\lambda_1,\ldots,\lambda_i,n_{i+1},0,\ldots,0}(c_1,\ldots,c_{i+1},z_{i+2},\ldots,z_k)}{q(c_1,\ldots,c_{i+1},z_{i+2},\ldots,z_k)}\frac{(z_{i+1}-c_{i+1})^{n_{i+1}}}{n_{i+1}!} \\
&+ \frac{p_{\lambda_1,\ldots,\lambda_i,n_{i+1}+1,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}{q(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}\frac{(z_{i+1}-c_{i+1})^{n_{i+1}+1}}{(n_{i+1}+1)!}.
\end{aligned}$$

Expanding all the expressions

$$\frac{p_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}{q(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}, 1 \le i \le k$$

in this manner until $\lambda_1+\cdots+\lambda_i = N$ and combining all the expansions, the expression

$$\begin{aligned}
f(\mathbf{z}) = \\
&\textstyle\sum_{0\le\lambda_1+\cdots+\lambda_k<N} \frac{p_{\lambda_1,\ldots,\lambda_k}(c_1,\ldots,c_k)}{q(c_1,\ldots,c_k)} \prod_{j=0}^{j} \frac{(z_j-c_j)^{\lambda_j}}{\lambda_j!} + \\
&\textstyle\sum_{\lambda_1+\cdots+\lambda_i=N,1\le i\le k} \frac{p_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_{i-1},z_i,\ldots,z_k)}{q(c_1,\ldots,c_{i-1},z_i,\ldots,z_k)} \prod_{j=1}^{i} \frac{(z_i-c_i)^{\lambda_i}}{\lambda_i!} \qquad (4.13)
\end{aligned}$$

is obtained. From formula (4.13) we get for $0 \le \lambda_1+\cdots+\lambda_k \le N$ that

$$\frac{p_{\lambda_1,\ldots,\lambda_k}(c_1,\ldots,c_k)}{q(c_1,\ldots,c_m)} = \frac{\partial^{\lambda_1+\cdots+\lambda_k} f(c_1,\ldots,c_k)}{\partial^{\lambda_1} z_1 \cdots \partial^{\lambda_k} z_k}.$$

A set of operators $\mathbf{\Phi}_i^{\lambda_i}, 1 \leq i \leq k$ is now introduced. These operators are defined as follows on constants

$$\begin{aligned}&\left[(z_1 - c_1)^{\lambda_1}\mathbf{\Phi}_1^{\lambda_1} \cdots (z_k - c_k)^{\lambda_k}\mathbf{\Phi}_k^{\lambda_k}\right](c_1, \ldots, c_k) \\ &= \frac{p_{\lambda_1,\ldots,\lambda_k}(c_1, \ldots, c_k)}{q(c_1, \ldots, c_k)} \prod_{j=0}^{k}(z_j - c_j)^{\lambda_j}\end{aligned}$$

and on variables

$$\begin{aligned}&\left[(z_1 - c_1)^{\lambda_1}\mathbf{\Phi}_1^{\lambda_1} \cdots (z_i - c_i)^{\lambda_i}\mathbf{\Phi}_i^{\lambda_i}\right](z_1, \ldots, z_k) \\ &= \frac{p_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1, \ldots, c_{i-1}, z_i, \ldots, z_k)}{q(c_1, \ldots, c_{i-1}, z_k, \ldots, z_k)} \prod_{j=0}^{i}(z_j - c_j)^{\lambda_j}, 1 \leq i \leq k.\end{aligned}$$

The operators also satisfy the commutativity conditions

$$(z_i - c_i)\mathbf{\Phi}_i(z_j - c_j)\mathbf{\Phi}_j = (z_i - c_i)(z_j - c_j)\mathbf{\Phi}_i\mathbf{\Phi}_j = (z_i - c_i)(z_j - c_j)\mathbf{\Phi}_j\mathbf{\Phi}_i.$$

For example,

$$\left[(z_1 - c_1)^{\lambda_1}\mathbf{\Phi}_1^{\lambda_1}\right](c_1, \ldots, c_k) = \frac{p_{\lambda_1,0,\ldots,0}(c_1, \ldots, c_k)(z_1 - c_1)^{\lambda_1}}{q(c_1, \ldots, z_k)}.$$

Then the expression (4.13) may be written as

$$\begin{aligned}f(\mathbf{z}) &= \sum_{l=0}^{N-1}\frac{1}{l!}\left[((z_1 - c_1)\mathbf{\Phi}_1 + \cdots + (z_k - c_k)\mathbf{\Phi}_k)^l\right](c_1, \ldots, c_k) \\ &+ \frac{1}{N!}\left[((z_1 - c_1)\mathbf{\Phi}_1 + \cdots + (z_k - c_k)\mathbf{\Phi}_k)^N\right](z_1, \ldots, z_k). \quad (4.14)\end{aligned}$$

This formula is the Taylor expansion of function $f(\mathbf{z})$ with a remainder.

The Taylor coefficients may also be calculated using the following recursive formula

$$p_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_{i-1},z_i,\ldots,z_k) =$$

$$[p_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,0}(c_1,\ldots,c_{i-1},z_i,\ldots,z_k) -$$

$$q(c_1,\ldots,c_{i-1},z_i,\ldots,z_k)\frac{p_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}{q(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}] \times \frac{1}{(z_i-c_i)}. \quad (4.15)$$

If the Taylor expansions of $p(\mathbf{z})$ and $q(\mathbf{z})$,

$$p(\mathbf{z}) = \sum_{0\le\lambda_1+\cdots+\lambda_n\le N} \frac{\partial^{\lambda_1+\cdots+\lambda_n}p(c)}{\partial^{\lambda_1}z_1\cdots\partial^{\lambda_n}z_n}\frac{(z_1-c_1)^{\lambda_1}\cdots(z_n-c_n)^{\lambda_n}}{\lambda_1!\cdots\lambda_n!},$$

and

$$q(\mathbf{z}) = \sum_{0\le\lambda_1+\cdots+\lambda_m\le M} \frac{\partial^{\lambda_1+\cdots+\lambda_m}q(c)}{\partial^{\lambda_1}z_1\cdots\partial^{\lambda_m}z_m}\frac{(z_1-c_1)^{\lambda_1}\cdots(z_m-c_m)^{\lambda_m}}{\lambda_1!\cdots\lambda_m!},$$

are known, where N and M are the degrees of $p(z)$ and $q(z)$, the above recursive formula may be written as

$$p_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_{i-1},z_i,\ldots,z_k) =$$

$$\sum_{j=1}^{l-1}\frac{1}{j!}p^{(j)}_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)(z_i-c_i)^j, \quad (4.16)$$

where

$$p^{(j)}_{\lambda_1,\ldots,\lambda_i,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k) =$$

$$p^{(j+1)}_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)$$

$$- \quad q^{(j+1)}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)\frac{p_{\lambda_1,\ldots,\lambda_i-1,0,\ldots,0}(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}{q(c_1,\ldots,c_i,z_{i+1},\ldots,z_k)}. \qquad (4.17)$$

In next section two examples are given showing the application of the above algorithms.

4.3.4 Time Complexity Analysis and Numerical Examples

From formula (4.15) we see that in this algorithm we only need a polynomial division by a factor $(\mathbf{z}-\mathbf{c})$ and an evaluation at $(z_i = c_i, \ldots, z_k = c_k)$ for each coefficient of the Taylor form. The coefficients are obtained recursively by formula (4.15). If we know the Taylor expansions of $p(\mathbf{z})$ and $q(\mathbf{z})$, we only need a polynomial multiplication by a constant and an addition between two polynomials for each Taylor coefficient. Therefore this algorithm is very low complexity and much better than the algorithm obtained by evaluating the analytic derivatives..

Now an example is given for illustration of the algorithms.

Let

$$f(z,w) = \frac{p(z,w)}{q(z,w)} = \frac{z^2w + zw + 1}{zw + 1}, \mathbf{c} = (1,1).$$

We want to obtain the Taylor expansion of $f(z,w)$ up to degree 2. If the Taylor's forms of $p(z,w)$ and $q(z,w)$ are unknown, we can use formula (4.15) to calculate the Taylor coefficients of $f(z,w)$ recursively.

We have all $p_{i,j}$ with $i+j \leq 3$ as the following

$$p_{1,0}(z,w) = \frac{z^2w + zw + 1 - (zw+1)\frac{2w+1}{w+1}}{(z-1)} = zw + \frac{w}{w+1}.$$

$$p_{2,0}(z,w) = \frac{zw + \frac{w}{w+1} - (zw+1)w + \frac{w}{w+1}w + 1}{z-1} = \frac{w}{(w+1)^2}.$$

$$p_{3,0}(z,w) = \frac{\frac{w}{(w+1)^2} - (zw+1)\frac{w}{(w+1)^3}}{z-1} = -\frac{w^2}{(w+1)^3}.$$

$$p_{0,1}(1,w) = \frac{2w+1-(w+1)\frac{3}{2}}{w-1} = \frac{1}{2}.$$

$$p_{0,2}(1,w) = \frac{\frac{1}{2} - (w+1)\frac{1}{4}}{w-1} = \frac{1}{4}.$$

$$p_{0,3}(1,w) = \frac{\frac{1}{4} - (w+1)\frac{1}{8}}{w-1} = \frac{1}{8}.$$

$$p_{1,1}(1,w) = \frac{w^2 + 2w - (w+1)^2\frac{3}{4}}{w-1} = \frac{w+3}{4}.$$

$$p_{1,2}(1,w) = \frac{\frac{w+3}{4} - (w+1)^2\frac{1}{4}}{w-1} = -\frac{w+2}{4}.$$

$$p_{2,1}(1,w) = \frac{w - (w+1)^3\frac{1}{8}}{w-1} = w^2 + 4w - 1.$$

Therefore we obtain by formula (4.13)

$$f(z,w) = \frac{3}{2} + \frac{(z-1)}{1!} + \frac{1}{4}\frac{(z-1)^2}{2!} + \frac{1}{2}\frac{(w-1)}{1!} + \frac{1}{4}\frac{(w-1)^2}{2!} + \frac{(z-1)(w-1)}{1!1!}$$

$$-\frac{w^2}{(w+1)^3}\frac{(z-1)^3}{3!}+\frac{1}{8}\frac{(w-1)^3}{3!}$$
$$-\frac{w+2}{4}\frac{(z-1)(w-1)^2}{1!2!}+(w^2+4w-1)\frac{(z-1)^2(w-1)}{2!1!}.$$

This is the Taylor expansion of $f(z,w)$ up to degree 3.

4.4 Summary

In Section 4.2, the centered forms were generalized to multi-variate polynomials. A low complexity algorithm is presented for evaluating these generalized forms. It was shown in Section 4.3 that centered forms evaluated over different arrangement of the variables of a polynomial may be different. The number of the different centered forms was estimated asymptotically. Some numerical examples were given showing that this number is surprisingly large. An efficient algorithm was designed for generating the all the arrangements of the variables which result in different centered forms. Based on the idea of multi-variate centered forms, a type of computationally efficient multi-variate Taylor forms were given in Section 4.4. Low complexity algorithms for evaluating these forms for both polynomials and rational functions were presented in detail. Also a detailed time complexity analysis for these algorithms was

given and some numerical examples were shown for the illustration.

Chapter 5

A Special Multi-variate Taylor Form

In this chapter, a special Taylor form for real multi-variate functions is discussed. It employs a particular remainder term in a Taylor expansion. Compared with the standard Taylor remainder term, this remainder term has very few nonzero terms and some of these nonzero are scalar terms. The algorithm generating this special Taylor form is given, which has a very low time complexity because of this special remainder term. The number of zeros and the number of scalar variables in the remainder term are estimated asymptotically. An application of the special Taylor form to the factorable

functions is presented in detail and some numerical examples are given.

5.1 Special Taylor Form

In this section we introduce a special Taylor expansion for *real functions*. The corresponding algorithm generating this expansion will be given in Section 5.2.

In interval analytic methods it is important to be able to replace as many interval variables as possible with scalar variables in order to reduce the width of the interval computed in a given computation. This was recognized early both by Moore[24] and Hansen[14]. In Hansen[14] an expansion for the gradient and the hessian of a multi-variate function was therefore developed that contains a number of interval variables replaced by scalars as well as some components replaced by zeros. This expansion is here generalized to an arbitrary order and estimates for the number of zero components and interval variables replaced by scalars are given.

It is also of importance to compute non-linear functions in a convenient and effective manner. This is possible for the so-called factorable functions. These functions, introduced by McCormick[22] are important in that their

value and derivatives may be computed in a natural manner.

A method for computing the range of a factorable function using the above expansion is then developed.

Thus the advantages of using an expansion of the Hansen type can be retained for factorable functions resulting in an expansion both having low implementation complexity and small width.

In this section we therefore give a class of *special Taylor expansions.* Let $f : \Re^n \rightarrow \Re$ be a multi-variate function, differentiable as required below. Hansen proposed an expansion for the gradient and the hessian of f of a special form in[14]. The suggested form has zero elements in the hessian and some interval variables replaced by scalars both in the gradient and the hessian. Thus it was a form suitable for use in interval methods of various kinds. This form is here generalized to an expansion up to the m-th term. Furthermore an asymptotic estimate is found for the number of zero elements of the interval remainder term. The number of variables replaced by scalars in the remainder term is also estimated.

The following theorem gives the generalization of the expansion found in Hansen[14]. It differs from the usual Taylor expansions in the remainder terms.

THEOREM 5.1 *Let $f(x_1, x_2, \ldots, x_n)$ be a multi-variate function and let $\mathbf{X} \in \mathbf{I}^n$ be an interval. Let furthermore $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Then for $m \geq 1$ the following expansion is valid.*

$$f(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n} (y_i - x_i) \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)$$

$$+ \ldots +$$

$$+ \sum_{\lambda_1 + \ldots + \lambda_n = m-1} \left(\prod_{k=1}^{n} \frac{(y_k - x_k)^{\lambda_k}}{\lambda_k!} \right) \frac{\partial^{m-1}}{\partial x} f(x_1, x_2, \ldots, x_n)$$

$$\sum_{\substack{1 \leq j_1 \leq \ldots \leq j_m \\ 1 \leq j_i \leq n}} \frac{\prod_{k=1}^{m} (y_{j_k} - x_{j_k})}{k_1! \ldots k_n!} \frac{\partial^m f}{\partial x_{j_1} \cdots \partial x_{j_m}} (y_1, \ldots, y_{j_m - 1}, \xi_{j_1, \ldots, j_m}, x_{j_m + 1}, \ldots, x_n) \tag{5.1}$$

where $\xi_{j_1, \ldots, j_m} \in X_{j_m}$ and k_i is the number of indices $j_r, 1 \leq r \leq m$, equal to i in the corresponding term.

Proof. Consider $f(\mathbf{y})$ to be a function of the last variable y_n only and expand it in a Taylor series up to the m-th order obtaining

$$\begin{aligned} f(y_1, \ldots, y_n) &= f(y_1, \ldots, y_{n-1}, x_n) + (y_n - x_n) \frac{\partial f}{\partial x_n}(y_1, \ldots, y_{n-1}, x_n) \\ &+ \cdots + \frac{(y_n - x_n)^m}{m!} \frac{\partial^m f}{\partial x_n^m}(y_1, \ldots, y_{n-1}, \xi_{n, \ldots, n}). \end{aligned} \tag{5.2}$$

In the next step the first $(n-1)$ terms on the right-side of (5.2) are expanded with respect to the variable y_{n-1} in m-th degree Taylor expansions. The first of these terms expanded results in

$$\begin{aligned} f(y_1,\ldots,y_{n-1},x_n) &= f(y_1,\ldots,y_{n-2},x_{n-1},x_n) \\ &+ (y_{n-1}-x_{n-1})\frac{\partial f}{\partial x_{n-1}}(y_1,\ldots,y_{n-2},x_{n-1},x_n) \\ &+ \cdots + \frac{(y_{n-1}-x_{n-1})^m}{m!}\frac{\partial^m f}{\partial x_{n-1}^m}(y_1,\ldots,y_{n-2},\xi_{n-1,\ldots,n-1},x_n). \end{aligned}$$

The remaining terms are expanded in the same manner until the $(m-1)$-th term which results in the expansion

$$\begin{aligned} \frac{\partial^{m-1} f}{\partial x_n^{m-1}}(y_1,\ldots,y_{n-1},x_n) &= \frac{\partial^{m-1} f}{\partial x_n^{m-1}}(y_1,\ldots,y_{n-2},x_{n-1},x_n) \\ &+ (y_{n-1}-x_{n-1})\frac{\partial^m f}{\partial x_n^{m-1}x_{n-1}}(y_1,\ldots,y_{n-2},\xi_{n,\ldots,n,n-1},x_n). \end{aligned}$$

This sequential expansion is continued until every term with degree less than m is evaluated at the point x. When these results are combined equation(5.1) is obtained. □

In equation(5.1) the term,

$$\frac{\partial^m f}{\partial x_{j_1}\ldots\partial x_{j_m}}(y_1,\ldots,y_{j_m-1},\xi_{j_1,\ldots,j_m},x_{j_m+1},\ldots,x_n)$$

can be considered as an element of an m-dimensional remainder tensor

$$f_{j_1,\ldots,j_m}.$$

It should be noted that the coefficient of the terms are zero in the remainder tensor if $j_1 < j_2 < \cdots < j_m$.

If the expansion is started with y_1 then the terms of the remainder tensor are zero if $j_1 > j_2 > \cdots > j_m$.

Below the example of this expansion from[48] is given for $n = 3$ and $m = 2$.

$$\begin{aligned} f(\mathrm{y}) &= f(\mathrm{x}) + (\mathrm{y}-\mathrm{x})^T \mathrm{g}(\mathrm{x}) \\ &+ \frac{1}{2}(\mathrm{y}-\mathrm{x})^T \begin{bmatrix} h_{11}(\xi_{11}, x_2, x_3) & 0 & 0 \\ h_{21}(\xi_{21}, x_2, x_3) & h_{22}(y_1, \xi_{22}, x_3) & 0 \\ h_{31}(\xi_{31}, x_2, x_3) & h_{32}(y_1, \xi_{32}, x_3) & h_{33}(y_1, y_2, \xi_{33}) \end{bmatrix} (\mathrm{y}-\mathrm{x}). \end{aligned}$$

Here $g(\mathrm{x})$ is the gradient of $f(\mathrm{x})$ evaluated at x with components $g_i(\mathrm{x}) = \partial f(\mathrm{x})/\partial x_i, i = 1, 2, \ldots, n$ and the components of the hessian are the appropriate second partial derivatives evaluated as required.

The matrix in the above formula is the special hessian matrix used for example in the global optimization algorithms discussed in[48],[14].

The tensor defined above has non-zero and zero elements. Below the number of zero elements are estimated by an asymptotic formula.

THEOREM 5.2 *Let $N(n,m)$ be the number of the zero elements in the tensor $f_{j_1,\ldots,j_m}$ then*

$$N(n,m) = (1 - 1/m!)n^m + O(n^m). \tag{5.3}$$

In particular, if $m = 2$, then

$$N(n,2) = \frac{n(n-1)}{2}. \tag{5.4}$$

This theorem is proven below. A supporting lemma is first developed.

LEMMA 5.1

$$\sum_{\lambda=1}^{n} \lambda^m = \frac{n^m + 1}{m+1} + O(n^m).$$

Proof.

We have

$$\int_{\lambda-1}^{\lambda} x^m dx \leq \sum_{\lambda}^{\lambda+1} \lambda^m \leq \int_{\lambda}^{\lambda+1} x^m dx.$$

Summing up from $\lambda = 1$ to n, we get

$$\int_0^n x^m dx \leq \sum_{\lambda=1}^{n} \lambda^m \leq \int_1^{n+1} x^m dx$$

or

$$\frac{n^{m+1}}{m+1} \leq \sum_{\lambda=1}^{n} \lambda^m \leq \frac{(n+1)^m + 1 - 1}{m+1}. \tag{5.5}$$

From (5.2) we get

$$\sum_{\lambda=1}^{n} \lambda^m = \frac{n^{m+1}}{m+1} + O(n^m).$$

□

Proof of Theorem 2.24. We have

$$N = \sum_{1 \leq n_1 < \cdots < n_m \leq n} 1 = n^m - \sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} 1. \tag{5.6}$$

We use induction to prove

$$\sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} 1 = \frac{n^m}{m!} + O(n^{m-1}). \tag{5.7}$$

For $m = 1$, $\sum_{n \geq n_1 \geq 1} 1 = n$, so (5.7) is obvious. Assuming

$$\sum_{n \geq n_1 \geq \cdots \geq n_{m-1} \geq 1} 1 = \frac{n^{m-1}}{(m-1)!} + O(n^{m-2})$$

is true. From lemma, we have

$$\begin{aligned} \sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} 1 &= \sum_{n_1=1}^{n} \left(\sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 1} 1 \right) \\ &= \sum_{n_1=1}^{n} \left(\frac{n_1^{m-1}}{(m-1)!} + O(n_1^{m-2}) \right) \\ &= \frac{n^m}{m!} + O(n^{m-1}). \end{aligned}$$

This is the formula (5.7). So from (5.6) and (5.7), we finally get

$$N = (1 - 1/m!)n^m + O(n^m).$$

Formula (5.2) is obvious. □

Compared to the regular remainder term in the Taylor expansion up to the second order the new remainder has almost half the terms zero. As $m \to \infty$ the remainder tends to a zero tensor.

THEOREM 5.3 *Let M(n,m) be the number of variables y and ξ in the remainder tensor, then*

$$M(n,m) = \frac{n^{m+1}}{(m+1)!} + O(n^m). \tag{5.8}$$

In particular, when $m = 2$, then

$$M(n,2) = \frac{n(n+1)(n+2)}{6}. \tag{5.9}$$

Proof. We know that in an element $f_{j_1,\ldots,j_m}$, the number of variables y and ξ is j_m . This means that the number of all these variables in the tensor is

$$M(n,m) = \sum_{n \geq j_1 \cdots \geq j_m \geq 1} j_m = \sum_{n \geq j_1 \cdots \geq j_m \geq j \geq 1} 1 = \frac{n^{m+1}}{(m+1)!} + O(n^m).$$

For $m = 2$, we have

$$M(n,2) = \sum_{n \geq j_1 \geq j_2 \geq 1} j_2$$
$$= \sum_{j_1=1}^{n} \frac{j_1(j_1+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

□

The standard Taylor form as given in for example[34] has n^m elements in the remainder, each element being a function of n variables. If the new form is used as a Taylor form up to the degree m for an interval $\mathbf{X} \in \mathbf{I}^n$ then clearly the width of the form is going to be less than the width of the corresponding form from[34].

Now we present an application of the Taylor form to real *factorable functions.* In McCormick[22] it is rightly pointed out that the effort expended on developing methods for solving non-linear problems in several variables has not been paralelled by a similar effort in finding suitable means of representing non-linear functions. He points out that the process of implementing a non-linear multivariate algorithm is in general seriously hampered by the effort required to actually code the algorithm and that there exists no general scheme for this coding. As a means of overcomming this problem he introduced the class of factorable functions (see[22] Chapter 3, and references

found there). This class includes all functions explicitly representable by a computer program. Exceptions are functions given implicitly or not included in a list of primitive functions such as sin, cos etc. Table 3.2.3 of[22] gives a typical example of such a list. The class of factorable functions is now defined below.

DEFINITION 5.1 *A function $f(x_1, \ldots, x_n)$ is a factorable function of several variables if it can be represented as the last in a finite sequence of functions $\{f_j(x)\}$. This sequence consists of the basis functions*

$$f_j(x) = x_j,\ j = 1, \ldots, n.$$

Using the basis functions the functions $f_j(x), j > n$ are formed by

$$f_j(x) = \begin{cases} either & f_k(x) + f_l(x),\ k, l < j \\ or & f_k(x) \times f_l(x),\ k, l < j \\ or & T[f(x)],\ k < j, \end{cases}$$

where $T(f)$ is a function of a single variable included in a table of primitive functions.

The function

$$f(x) = [b - x_3(2\pi)^{-1/2} \int_{-\infty}^{\frac{(a-x_1)}{2} x_2^{-2}} e^{-t^2/2} dt]^2, \tag{5.10}$$

Function Sequence	Form of Composition	$T(f)$	$\dot{T}(f)$	$\ddot{T}(f)$
$f_j(x) = x_j$ $j = 1,2,3$				
$f_4(x) =$	$a - f_1(x)$	$a - f$	-1	0
$f_5(x) =$	$[f_2(x)]^{-2}/2$	$1/2f^{-2}$	$-f^{-3}$	$3f^{-4}$
$f_6(x) =$	$f_4(x)f_5(x)$			
$f_7(x) =$	$\frac{\int_{-\infty}^{f_6(x)} e^{-t^2/2}dt}{(2\pi)^{1/2}}$	$\frac{\int_{-\infty}^{f} e^{-t^2/2}dt}{(2\pi)^{1/2}}$	$\frac{fe^{-f^2/2}}{(2\pi)^{1/2}}$	$-\frac{fe^{-f^2/2}}{(2\pi)^{1/2}}$
$f_8(x) =$	$f_3(x)f_7(x)$			
$f_9(x) =$	$(b - f_8(x))^2$	$(b-f)^2$	$-2(b-f)$	2

Table 5.1: Factorable function example

from McCormick [22] is an example of a factorable function. The sequence of steps for the evaluation of the function is given in Table 5.1 where $\dot{T}(f) = \partial T(f)/\partial f$ and $\ddot{T}(f) = \partial^2 T(f)/\partial f^2$ which are required later are also included.

Again following[22] it is clear that a factorable function has a first and second derivative. The first derivative is formed for each step in the function sequence using the rules

$$f_j'(x) = \begin{cases} f_k'(x) + f_l'(x) & \text{if } f_j(x) = f_k(x) + f_l(x) \\ f_k(x)f_l'(x) + f_k'(x)f_l(x) & \text{if } f_j(x) = f_k(x)f_l(x) \\ \dot{T}[f_k(x)]f_k'(x) & \text{if } f_j(x) = T[f_k(x)] \end{cases} \tag{5.11}$$

and the second derivative using the rules

$$f_j''(x) = \begin{cases} f_k''(x) + f_l''(x) & \text{if } f_j(x) = f_k(x) + f_l(x) \\ f_k(x)f_l''(x) + f_k''(x)f_l(x) & \\ \quad + f_k'(x)^T f_l'(x) + f_l'(x)^T f_k(x) & \text{if } f_j(x) = f_k(x)f_l(x) \\ f_k''(x)\dot{T}[f_k(x)] + f_k'(x)^T \ddot{T}[f_k(x)]f_k'(x) & \text{if } f_j(x) = T[f_k(x)]. \end{cases} \tag{5.12}$$

The first derivative of the function (5.10) is developed in Table 5.2 as an example where P^* denotes the previous entry in the same column. Second and higher derivatives are developed in the same manner.

j	Function $f_j(x)$	$\partial f_j(x)/\partial x_1$	$\partial f_j(x)/\partial x_2$	$\partial f_j(x)/\partial x_1$
1	x_1	1	0	0
2	x_2	0	1	0
3	x_3	0	0	1
4	$a - f_1(x)$	-1	0	0
5	$[f_2(x)]^2/2$	0	f_2	0
6	$f_4(x)f_5(x)$	$-f_5$	$f_4 f_2$	0
7	$\frac{\int_{-\infty}^{f_6(x)} e^{-t^2/2} dt}{(2\pi)^{1/2}}$	$\frac{e^{-f_6^2/2} dt}{(2\pi)^{1/2}} P^*$	$\frac{e^{-f_6^2/2} dt}{(2\pi)^{1/2}} P^*$	0
8	$f_3(x)f_7(x)$	$f_3 P^*$	$f_3 P^*$	f_7
9	$(b - f_8(x))^2$	$-2(b - f_8(x))^2 P^*$	$-2(b - f_8(x))P^*$	$-2(b - f_8(x))P^*$

Table 5.2: Derivative Example

5.2 Taylor Form Algorithm

5.2.1 Algorithm

In this section a twice differentiable function $f : \Re^n \rightarrow \Re$ is considered. The range of f over $X \in \mathbf{I}^n$, written by $\overline{f}(\mathrm{X})$, is

$$\overline{f}(X) = \{f(\mathrm{x}) | \mathrm{x} \in \mathrm{X}\}.$$

Since f is twice differentiable it can be expanded in a Taylor series

$$f(\mathrm{y}) = f(\mathrm{x}) + f'(\mathrm{x})(\mathrm{y} - \mathrm{x}) + 1/2(\mathrm{y} - \mathrm{x})^T f''(\xi)(\mathrm{y} - \mathrm{x})$$

up to the first order with a remainder of second order where $\xi \in \mathrm{x} \vee \mathrm{y}$ (that is the interval spanned by x and y).

In[34] it was shown that this series could be used to compute an including estimate

$$f_T(\mathbf{X}) = f(\mathbf{c}) + f'(\mathbf{c})(\mathbf{X} - \mathbf{c}) + 1/2(\mathbf{X} - \mathbf{c})^T f''(\mathbf{X})(\mathbf{X} - \mathbf{c})$$

for the range of f over an interval $\mathbf{X}$ where $f''(\mathrm{X})$ is an including evaluation of the Hessian over the domain interval X and $\mathbf{c} = mid\mathbf{X} \in X$ is the evaluation vector. This estimate is known as the second order Taylor form. In[34] it was shown that the second order Taylor forms are quadratically convergent

if the remainder tensor is bounded. The first order forms (see[34]) require difficult to establish Lipschitz conditions in order to achieve the same order of convergence. Higher order forms may also be defined as in[34], but this is not considered here.

With this in mind the Taylor form for a factorable function using the expansion (5.1) is developed. The resulting Taylor form is

$$f_S(X) = f(\mathbf{c}) + \mathbf{f}'(\mathbf{c})(\mathbf{X} - \mathbf{c}) + 1/2(\mathbf{X} - \mathbf{c})^T \mathbf{H}''(\mathbf{c}, \mathbf{X}, \mathbf{X})(\mathbf{X} - \mathbf{c}) \quad (5.13)$$

where H(x, X, X) is defined to be the special Hessian

$$H(\mathbf{x}, \mathbf{X}, \mathbf{X}) = \begin{bmatrix} h_{11}(X_1, c_2, \ldots, c_n) & 0 & \ldots & 0 \\ h_{21}(X_1, c_2, \ldots, c_n) & h_{22}(X_1, \ldots, X_{n-1}, c_n) & \ldots & 0 \\ \vdots & & & \vdots \\ h_{n1}(X_1, c_2, \ldots, c_n) & h_{n2}(X_1, \ldots, X_{n-1}, c_n) & \ldots & h_{nn}(X_1, \ldots, X_n) \end{bmatrix} \quad (5.14)$$

From the definitions of $f_T(X)$ and $f_S(X)$ it clearly follows that

$$\overline{f}(X) \subseteq f_S(X) \subseteq f_T(X)$$

(see also[34]).

It is therefore of interest to compute the second order Taylor expansion for a factorable function applied to computing the range of a function f

via the form (5.13) as efficiently as possible. An algorithm in a Pascal-like notation is given for the computation of the value and the first derivative at c and the special second derivative inclusion at c and X. Translating this to an actual implementation in Pascal-SC (see Kaucher et al[18]) is fairly simple. The above notation is, however, used both for clarity and brevity.

Essentially the algorithm consists of an implementation of the derivative formulas (5.11) and (5.12).

The input to the algorithm is an array containing the function in a factored form, an evaluation vector c and an interval **X** over which the range is sought. The output is the first derivative in an one-dimensional array and the special interval Hessian in a two dimensional interval array. The final evaluation of the special Taylor form is omitted for brevity. It consists of applying the formula (5.13) given above.

The main procedure **Hessian-expansion** consists of a call to an input procedure reading the factorable function sequence followed by calls to the derivative and second derivative procedures. The factorable function sequence is assumed to be K elements long. A further assumption is that there exists a repertoire of primitive functions T for which the derivative $D(T)$ and the second derivative $D(D(T))$ are known.

Algorithm 7.

procedure Taylor;

var

$\mathcal{F}$: Array to store the factorable function sequence;

$\mathcal{D}$: Array to store the derivative sequence;

$\mathcal{H}$: Array to store the hessian sequence;

X : Developing interval; Developing vector is $\mathbf{c}$=mid($\mathbf{X}$);

begin

Input the function in factorable form into array $\mathcal{F}$;

Number of functions is K, final function is $\mathcal{F}(K)$;

Input interval vector $\mathbf{X}$ and developing vector $\mathbf{c}$;

value($\mathcal{F}$, $\mathcal{V}$, $\mathbf{c}$);

derivative($\mathcal{F}$, $\mathcal{V}$, $\mathcal{W}$, $\mathbf{c}$);

special-hessian($\mathcal{F}$, $\mathcal{H}$, $\mathbf{X}$, $\mathbf{c}$);

evaluate (5.13) using $\mathcal{V}$, $\mathcal{W}$, $\mathcal{H}$, $\mathbf{X}$, $\mathbf{c}$.

output resulting interval $\mathbf{X}$;

end.

The procedure **value** evaluates the factorable function at a point **c**. The result is found in $\mathcal{V}(K)$.

procedure value($\mathcal{F}$, **c**, $\mathcal{V}$);

begin

for $j = 1$ step 1 to n do $\mathcal{V}(j) = c_j$;

for $j = n + 1$ step 1 to K do begin

case $\mathcal{F}(j)$ of

$\mathcal{F}(k) + \mathcal{F}(l)$: $\mathcal{V}(j) = \mathcal{V}(k) + \mathcal{V}(l)$;

$\mathcal{F}(k) * \mathcal{F}(l)$: $\mathcal{V}(j) = \mathcal{V}(k) * \mathcal{V}(l)$;

$T(\mathcal{F}(k))$: $\mathcal{V}(j) = T(\mathcal{V}(k))$;

end;

end;

In the procedure **derivative** the derivative values of a function sequence is calculated using the cases of (5.11). The result is in the $N \times K$ array $\mathcal{W}$.

procedure derivative($\mathcal{F}$, $\mathcal{V}$, $\mathcal{W}$, **c**);

begin

for $i = 1$ step 1 to K do $\mathcal{W}(i) = \mathbf{e}_i$;

for $i = n + 1$ step 1 to K do begin

case $\mathcal{F}(j)$ of

$\mathcal{F}(k) + \mathcal{F}(l)$: $\mathcal{W}(j) = \mathcal{W}(k) + \mathcal{W}(l)$;

$\mathcal{F}(k) * \mathcal{F}(l)$: $\mathcal{W}(j) = \mathcal{V}(k) * \mathcal{W}(l) + \mathcal{W}(k) * \mathcal{V}(l)$;

$T(\mathcal{F}(k))$: $\mathcal{W}(j) = \mathcal{V}(D(T(\mathcal{V}(k)))) * \mathcal{W}(k)$

{ $\mathcal{V}(D(T(\mathcal{V}(j))))$; is the value of the derivative of the built-in function T at $\mathcal{V}(j)$ };

$\mathbf{xe}_J^T$: $\mathcal{W}(j) = \mathbf{e}_J$;

end;

end;

In the procedure **hessian** the values of the hessian of a function sequence is calculated using the cases of (5.12).

procedure hessian($\mathcal{F}$, $\mathcal{V}$, $\mathcal{W}$, $\mathcal{H}$, $\mathbf{X}$, $\mathbf{c}$);

for $i = 1$ step 1 to K do $\mathcal{W}(i) = 0$;

for $i = n + 1$ step 1 to K do begin

case $\mathcal{F}(j)$ of

$\mathcal{F}(k) + \mathcal{F}(l)$: $\mathcal{H}(j) = \mathcal{H}(k) + \mathcal{H}(l)$;

$\mathcal{F}(k) * \mathcal{F}(l)$: $\mathcal{H}(j) = \mathcal{W}(k) * \mathcal{H}(l) + \mathcal{H}(k) * \mathcal{W}(l)$;

$T(\mathcal{F}(k))$: $\mathcal{H}(j) = \mathcal{W}(D(T(\mathcal{W}(k)))) * \mathcal{H}(k)$;

{ $\mathcal{W}(D(T(\mathcal{W}(j))))$ is the value of the derivative of the built-in function T at $\mathcal{W}(j)$ };

$\mathbf{xe}_J^T : \mathcal{H}(j) = \mathbf{e}_J$;

end;

end;

The procedure **special-hessian** now calculates the special hessian given in (5.14).

procedure special-hessian($\mathcal{F}$, $\mathcal{H}$, X, **c**);

var : Auxiliary arrays $\mathcal{VV}$, $\mathcal{WW}$, $\mathcal{HH}$, Xx;

begin

for $i = 1$ step 1 to N do Xx(i)=c(i);

for $i = 1$ step 1 to N do

begin

Xx(i)=X(i);

value($\mathcal{F}$, Xx, $\mathcal{VV}$);

derivative($\mathcal{F}$, $\mathcal{VV}$, $\mathcal{WW}$);

hessian($\mathcal{F}$, $\mathcal{VV}$, $\mathcal{WW}$, $\mathcal{HH}$);

for $j = i$ step 1 to N do $\mathcal{H}(i,j) = \mathcal{HH}(i,j)$;

end;

end;

The above procedures form the skeleton of the algorithm for the special Taylor-form range computation. It is of course assumed that the appropriate menu of fundamental functions and their first two derivatives are available.

5.2.2 Numerical Realizations

In this section, a numerical example is given illustrating the use of the algorithm discussed in the previous section for a *real* factorable function.

The inclusions $f_T(\mathbf{X})$ and $f_S(\mathbf{X})$ were calculated for the factorable function

$$f(\mathbf{x}) = [b - x_3(2\pi)^{-1/2} \int_{-\infty}^{\frac{x_2(a-x_1)}{2}} e^{-t^2/2} dt]^2 \tag{5.15}$$

using an implementation of the pseudo-Pascal code given in the last section in FORTRAN. As an example of the calculation procedure some symbolic calculations are given in Table 5.3.

With the input values $a = 0.1$, $b = 1.0$ and

$$\mathbf{X} = \{[0.2, 1.5], [1.0, 2.0], [1.0, 2.0]\}$$

Function Sequence	$f_i(x)$	$df_i(x)$	$hf_i(x)$
i = 1	$f_1(x)$	1	0
i = 2	$f_2(x)$	0	0
i = 3	$f_3(x)$	0	0
i = 4	$a - f_1(x)$	$(a - f_1(x))df_1(x)$	$(a - f_1)hd_1 - [df_1]^2$
i = 5	$[f_2(x)]^2/2$	$df_2(x)/2[f_2(x)]^2$	$hf_2/2[f_2]^2 - [df_2]^2/[f_2]^3$
i = 6	$f_4(x)f_5(x)$	$df_4(x)f_5(x) + f_4(x)df_5(x)$	$f_4hf_5 + hf_4f_5 + df_4df_5$
i = 7	$\frac{\int_{-\infty}^{f_6(x)} e^{-t^2/2}dt}{(2\pi)^{-1/2}}$	$\frac{\int_{-\infty}^{f_6} e^{-t^2/2}dt}{(2\pi)^{-1/2}}df_6(x)$	$\frac{\int_{-\infty}^{f_6} e^{-t^2/2}dt}{(2\pi)^{-1/2}}hf_6 + \frac{f_6 e^{-f_6^2/2}}{(2\pi)^{-1/2}}df_6^2$
i = 8	$f_3(x)f_7(x)$	$df_3(x)f_7(x) + f_3(x)df_7(x)$	$f_3hf_7 + hf_3f_7 + df_3df_7$
i = 9	$(b - f_8(x))^2$	$-2(b - f_8)df_8(x)$	$-2(b - f_8)hf_8 + 2[df_8]^2$

Table 5.3: $f(\mathrm{x})$, $f'(\mathrm{x})$(1st. component), $f''(\mathrm{x})$(component 1,1) (when i=9)

the results were

$$f_T(\mathbf{X}) = [-1.9973993, 2.2413865], \quad f_S(\mathbf{X}) = [-1.0345017, 1.2784888].$$

Similarly with the inputs $a = 0.1$, $b = 2.5$ and

$$\mathbf{X} = \{[0.4, 2.1], [0.4, 2.1], [0.4, 2.1]\}$$

the results were

$$f_T(\mathbf{X}) = [-1545.8111, 1554.2526], \quad f_S(\mathbf{X}) = [-451.01445, 459.45599].$$

These and further calculations show that the width of the inclusions $f_S(\mathbf{X})$ is 0.5 to 0.25 of the width of the inclusions $f_T(\mathbf{X})$ for reasonably large input intervals.

5.3 Summary

In this chapter, a special Taylor form for multi-variate polynomials is discussed in detail as a special case of centered forms. This special Taylor form can be evaluated with very low complexity using interval arithmetic. An asymptotic estimate is given for the number of zero entries in the interval remainder terms of this form. Also the number of variables replaced by

scalars in the remainder term is estimated. The implementation of the new form for factorable functions is given and some numerical examples are presented showing that significant improvements are obtained for one particular factorable function.

In this chapter a low complexity algorithm has been developed for the special Taylor expansion of real multi-variate polynomials and rational functions. In the polynomial case, the algorithm is comparable to the direct derivative algorithm. The new algorithm for the rational case provides, however, a significant improvement in time-complexity.

The resulting Taylor expansion for rational functions consists of main terms and a remainder. These remainders, which differ from the usual remainders, have the forms

$$\frac{\partial^{\lambda_1+\cdots+\lambda_i} f(c_1,\ldots,c_{i-1},x_i,\ldots,x_k)}{\partial x_1^{\lambda_1}\cdots\partial x_i^{\lambda_i}} \frac{(x_1-c_1)^{\lambda_1}\cdots(x_k-c_k)^{\lambda_k}}{\lambda_1!\cdots\lambda_k!}, 1 \le i \le k$$

that is they are exact. Practically, this algorithm provides a computational method to calculate Taylor coefficients for rational functions. Based on recursive formulae (5.13) and (5.14), the algorithm is easily implemented as computer programs.

Chapter 6

Centered Forms for Meromorphic Functions

In this chapter, power-sum forms and centered forms are generaliized to meromorphic functions. The three properties of the centered forms: inclusion, quadratic convergence and inclusion isotonicity are shown to hold for meromorphic functions. Algorithmic approaches with error estimates for meromorphic functions are discussed briefly. Some numerical results are given for illustration.

6.1 Meromorphic Functions

6.1.1 Forms for Meromorphic Functions

In this chapter, the centered forms will be extended to meromorphic functions and the corresponding properties for these forms will be proven. Also some algorithms for calculating these forms will be given briefly.

We first define the power-sum forms for meromorphic functions. Then based on them we define centered forms and give their properties of inclusion, convergence and inclusion isotonicity. Some results of this section appeared in[6, 7]. Let $p(z)$ and $q(z)$ be entire functions on the complex plane. Then $f(z) = p(z)/q(z)$ is a meromorphic function on the complex plane. Expanding $p(z)$ and $q(z)$ in Taylor series at $z = 0$ we get $p(z) = \sum_{i=0}^{\infty}(p^{(i)}(0)/(i!))z^i$ and $q(z) = \sum_{i=1}^{\infty}(q^{(i)}(0)/(i!))z^i$ and we may write $f(z)$ as

$$f(z) = \frac{\sum_{i=0}^{\infty}(p^{(i)}(0)/(i!))z^i}{\sum_{i=0}^{\infty}(q^{(i)}(0)/(i!))z^i}.$$

In the sequel the range of $f(z)$ over the disk $Z = \langle c, \rho \rangle$ is denoted by

$$\overline{f}(Z) = \{f(z) | z \in Z\}.$$

For notational convenience we also define

$$a_i = \frac{p^{(i)}(0)}{i!}, \quad b_i = \frac{q^{(i)}(0)}{i!}.$$

If $f(z)$ is now evaluated by replacing the variable z by a circular interval Z using the operations (2.23) to (2.26) obtaining a circular interval $f(Z)$ then

$$\overline{f}(Z) \subseteq f(Z)$$

because of (2.27) and (2.28). Both c and ρ may vary, however, depending on the organization of the operations. This was first realized by Moore[23] for real functions. In the following some forms are defined for meromorphic functions and some interesting and useful properties are verified. Results in this section may be compared to those results for rational functions in Section 4.1.

The first forms we consider are the power-sum forms starting with the zeroth order power-sum form. For this we first develop the power-sum forms for the polynomials $p(z) = \sum_{i=1}^{n} a_i z^i$ and $q(z) = \sum_{i=1}^{n} b_i z^i$ given by

$$\sum_{i=1}^{n} a_i Z^i, \quad \sum_{i=1}^{n} b_i Z^i$$

respectively (see[41]). Then letting n tend to infinity we obtain

$$\sum_{i=1}^{\infty} a_i Z^i, \quad \sum_{i=1}^{\infty} b_i Z^i.$$

Applying the disk division we arrive at a disk

$$f_{p(0)}(Z) = \left(\sum_{i=1}^{\infty} a_i Z^i\right) / \left(\sum_{i=1}^{\infty} b_i Z^i\right).$$

This disk is called the zeroth order power-sum form for the meromorphic function $f(z)$. In the following theorem the explicit formula for this form is given.

THEOREM 6.1 *Let $f(z) = p(z)/q(z)$ be a meromorphic function. Then the zeroth order power-sum form*

$$f_{p(0)}(Z) = \langle c_{p(0)}, \rho_{p(0)} \rangle$$

satisfies

$$c_{p(0)} = \frac{1}{A} p(c)\overline{q(c)},$$

$$\rho_{(0)} = \frac{1}{A}\left[(|p(c)| + \sum_{i=1}^{\infty} |a_i|((|c|+\rho)^i - |c|^i))(|q(c)| + \sum_{i=1}^{\infty} |b_i|((|c|+\rho)^i - |c|^i) - |p(c)q(c)|\right]$$

where

$$A = |q(c)|^2 - \left[\sum_{i=1}^{\infty} |b_i|((|c|+\rho)^i - |c|^i)\right]^2. \tag{6.1}$$

Furthermore $\overline{f}(Z) \subseteq f_{p(0)}(Z)$.

Proof. In[46] it was shown that if $Z_n = \langle c_n, \rho_n \rangle$ is the power-sum form of the polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$ over $Z = \langle c, \rho \rangle$ then it satisfies $c_n = p(c)$

and $\rho_n = \sum_{i=0}^{n} |a_i|((|c| + \rho)^i - |c|^i)$. Let $Z_p = \langle c_p, \rho_p \rangle$ and $Z_q = \langle c_q, \rho_q \rangle$ be the power-sum forms for $p(z)$ and $q(z)$ and $Z_{p,n} = \langle c_{p,n}, \rho_{p,n} \rangle$ and $Z_{q,n} = \langle c_{q,n}, \rho_{q,n} \rangle$ for $\sum_{i=0}^{n} a_i z^i$ and $\sum_{i=0}^{n} b_i z^i$. Using the above results we have

$$c_p = \lim_{n\to\infty} c_{p,n} = p(c), \quad \rho_p = \lim_{n\to\infty} \rho_{p,n} = \sum_{i=0}^{\infty} |a_i|((|c| + \rho)^i - |c|^i)$$

and

$$c_q = \lim_{n\to\infty} c_{q,n} = q(c), \quad \rho_q = \lim_{n\to\infty} \rho_{q,n} = \sum_{i=0}^{\infty} |b_i|((|c| + \rho)^i - |c|^i).$$

Since $f_{p(0)}(Z) = Z_p/Z_q$ we obtain the explicit formulas above using the identity (2.34).

The inclusion property is an immediate consequence of the inclusion isotonicity of the operations (2.23) to (2.31). □

We note that in the sequel A is always defined by (6.1).

The higher order power-sum forms for meromorphic functions are now defined. Let

$$f(z) = \frac{p(z)}{q(z)}$$

be rearranged as

$$f(z) = \frac{p(c)}{q(c)} + \frac{s(z)}{q(z)}. \tag{6.2}$$

Then

$$s(z) = p(z) - \frac{p(c)}{q(c)}q(z)$$

with $s(c) = 0$. Therefore

$$s(z) = r(z)(z - c)$$

and $f(z)$ may be written as

$$f(z) = \frac{p(c)}{q(c)} + \frac{r(z)}{q(z)}(z - c). \tag{6.3}$$

Suppose now that the meromorphic function $g_1(z) = r(z)/q(z)$ has the zeroth order power-sum form $g_{p(0)}(Z)$ over the disk Z. Using this form we obtain a disk

$$f_{p(1)}(Z) = \frac{p(c)}{q(c)} + g_{p(0)}(Z)(Z - c)$$

where the disk operation are given by (2.23) to (2.31).

This disk is called the first order power-sum form for the meromorphic function $f(z)$.

Generally the k-th order power-sum form for $f(z)$ is defined by

$$f_{p(k)}(Z) = \frac{p(c)}{q(c)} + g_{p(k-1)}(Z)(Z - c) \tag{6.4}$$

where $g_{p(k-1)}(Z)$ is the $(k-1)$-th order power-sum form for the function $g(z)$. This is a recursive definition.

It is possible to develop explicit formulae for higher order power-sum forms. In the following theorem the explicit formulae for the first and second order power-sum forms are given.

THEOREM 6.2 *Let $f(z) = p(z)/q(z)$ be a meromorphic function. The the explicit formulas for its first and second order power-sum forms*

$$f_{p(1)}(Z) = \langle c_{p(1)}, \rho_{p(1)} \rangle, \quad f_{p(2)}(Z) = \langle c_{p(2)}, \rho_{p(2)} \rangle$$

satisfy

$$c_{p(1)} = \frac{p(c)}{q(c)} \tag{6.5}$$

$$\rho_{p(1)} = \frac{\rho}{A}\left[|r(c)| + \sum_{i=1}^{\infty} r_i|((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{\infty} |b_i|((|c|+\rho)^i - |c|^i)\right] \tag{6.6}$$

$$c_{p(2)} = \frac{p(c)}{q(c)} \tag{6.7}$$

$$\rho_{p(2)} = \frac{\rho^2}{A}\left[|t(c)| + \sum_{i=1}^{\infty} t_i((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{\infty} |b_i|((|c|+\rho)^i - |c|^i)\right] + \rho|r(c)/q(c)| \tag{6.8}$$

where $r(c) = \sum_{i=0}^{\infty} r_i c^i$ and $t(c) = \sum_{i=0}^{\infty} t_i c^i$ satisfy $f(z) = f(c) + r(z)/q(z)(z - c)$ and $r(z)/q(z) = r(c)/q(c) + t(z)/q(z)(z-c)$ respectively and where $t_i, i = 0, 1, \ldots, t_i, i = 0, 1, \ldots$, are the Taylor coefficients of $r(z)$ and $t(z)$.

Furthermore $\overline{f}(Z) \subseteq f_{p(k)}(Z)$.

Proof. The first order power-sum form of $f(z)$ can be written as

$$f_{p(1)}(Z) = f(c) + \frac{r_p(Z)}{q_p(Z)}(z - c)$$

where $r_p(Z)$ and $q_p(Z)$ are the power-sum forms for $r(z)$ and $q(z)$. From Theorem 6.1, the zeroth order power-sum form for $g(z) = r(z)/q(z)$ is

$$\frac{r_p(Z)}{q_p(Z)} = \langle c_1, \rho_1 \rangle$$

where

$$c_1 = \frac{1}{A} r(c)\overline{q(c)}$$

and

$$\begin{aligned} \rho_1 &= \frac{1}{A}\Bigg[(|r(c)| + \sum_{i=1}^{\infty} |r_i|((|c| + \rho)^i - |c|^i)) \\ &\quad (|q(c)| + \sum_{i=1}^{\infty} + |b_i|((|c| + \rho)^i - |c|^i) - |r(c)q(c)|\Bigg]. \end{aligned}$$

and where the coefficients r_i are the Taylor coefficients of $r(z)$ expanded around $z_0 = c$. From the definitions of the disk operations it follows that

$$f_{p(1)}(Z) = \langle c_{p(1)}, \rho_{p(1)} \rangle,$$

where

$$c_{p(1)} = \frac{p(c)}{q(c)} \tag{6.9}$$

$$\rho_{p(1)} = \frac{\rho}{A}\left[|r(c)| + \sum_{i=1}^{\infty} r_i|((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{\infty} |b_i|((|c|+\rho)^i - |c|^i)\right]. \quad (6.10)$$

Continuing this process we develop the first order power-sum form for $g(z)$ as

$$g_{p(1)}(Z) = g(c) + \frac{t_p(Z)}{q_p(Z)}(Z-c).$$

We obtain from (6.10) and the definitions of disk operations that

$$g_{p(1)}(Z) = \langle c_2, \rho_2 \rangle$$

where

$$c_2 = r(c)/q(c)$$

and

$$\rho_2 = \frac{\rho}{A}\left[|t(c)| + \sum_{i=1}^{\infty} t_i|((|c|+\rho)^i - |c|^i)\right] \left[|q(c)| + \sum_{i=1}^{\infty} |b_i|((|c|+\rho)^i - |c|^i)\right].$$

where the t_i are the Taylor coefficients of $t(z)$ at c.

The second order power-sum form for $f(z)$ now can be written as

$$f_{p(2)}(Z) = f(c) + g_{p(1)}(Z)(Z-c) = \langle c_{p(2)}, \rho_{p(2)} \rangle$$

where

$$c_{p(2)} = p(c)/q(c) \tag{6.11}$$

and

$$\begin{aligned} \rho_{p(2)} &= \frac{\rho^2}{A}\left[|t(c)| + \sum_{i=1}^{\infty} t_i((|c|+\rho)^i - |c|^i)\right] \\ &\quad \left[|q(c)| + \sum_{i=1}^{\infty} |b_i|((|c|+\rho)^i - |c|^i)\right] + \rho|r(c)/q(c)|. \end{aligned} \tag{6.12}$$

The inclusion property is again a direct consequence of the inclusion isotonicity. □

Now we extend the notion of a centered form to meromorphic functions. Let

$$p_c(Z) = \langle p(c), \sum_{k=1}^{\infty} |p^{(k)}(c)|\rho^k/k!\rangle \text{ and } q_c(Z) = \langle q(c), \sum_{k=1}^{\infty} |q^{(k)}(c)|\rho^k/k!\rangle$$

be the centered forms of $p(z)$ and $q(z)$ over $Z = \langle c, \rho\rangle$ (see[27]). We define a disk

$$f_{c(0)}(Z) = p_c(Z)/q_c(Z)$$

where the disk division follows formula (2.34). This disk is called the zeroth order centered form, or semi-centered form, of the meromorphic function $f(z)$.

Similarly, the first order centered form of $f(z)$ can be defined as

$$f_{c(1)}(Z) = \frac{p(c)}{q(c)} + g_{c(0)}(Z)(Z - c)$$

where $g(z)$ satisfies

$$f(z) = \frac{p(c)}{q(c)} + g(z)(z - c).$$

Generally the k-th order centered form of the meromorphic function $f(z) = p(z)/q(z)$ is defined by

$$f_{c(k)}(Z) = \frac{p(c)}{q(c)} + g_{c(k-1)}(Z)(Z - c)$$

where $g_{c(k-1)}(Z)$ is the $(k-1)$-th order centered form of $g(z)$.

Explicit formulas for k-th order centered form can be derived easily. First we will give the explicit formula of the semi-centered form.

In the sequel

$$B = |q(c)|^2 - \left(\sum_{i=1}^{\infty} \frac{q^{(i)}(c)|}{i!}\rho^i\right)^2$$

and assuming $B > 0$.

THEOREM 6.3 *Let $f_{c(0)}(Z) = \langle c_{c(0)}, \rho_{c(0)} \rangle$ be the zeroth order centered form for the meromorphic function $f(z) = p(z)/q(z)$ over the disk $Z = \langle c, \rho \rangle$. Then we have*

$$c_{c(0)} = \frac{1}{B}(p(c)\overline{q(c)},$$

$$\rho_{c(0)} = \frac{1}{B}\left[\sum_{i=1}^{\infty}\frac{|p^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i - |p(c)q(c)|\right]$$

and $\overline{f}(Z) \subseteq f_{c(0)}(Z)$.

Proof. From[46] the centered forms of $p(z)$ and $q(z)$ over Z are

$$\begin{aligned} p_c(Z) &= \langle p(c), \sum_{k=1}^{\infty}|p^{(k)}(c)|\rho^k/k!\rangle, \\ q_c(Z) &= \langle q(c), \sum_{k=1}^{\infty}|q^{(k)}(c)|\rho^k/k!\rangle. \end{aligned}$$

Using (2.34) we obtain explicit formulas for $c_{c(0)}$ and $\rho_{c(0)}$. The inclusion property is an immediate consequence of the inclusion isotonicity of the operations (2.23) to (2.31). □

The explicit formulas of the first and second order centered forms for the meromorphic function $f(z)$ are now given.

THEOREM 6.4 *Let* $f_{c(k)}(Z) = \langle c_{c(k)}, \rho_{c(k)}\rangle$ *be the centered form of order* k *for the meromorphic function* $f(z) = p(z)/q(z)$ *over the disk* $Z = \langle c, \rho\rangle$. *Then we have*

$$\begin{aligned} c_{c(k)} &= f(c), k = 1, 2, \ldots \\ \rho_{c(1)} &= \frac{\rho}{B}\left[\sum_{i=0}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=0}^{\infty}\frac{\rho^i}{|q(c)|(i+1)!}|p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)|\right] \\ \rho_{c(2)} &= \frac{\rho^2}{B}\left[\sum_{i=0}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=0}^{\infty}\frac{\rho^i}{|q(c)|(i+1)!}|r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|\right] + \frac{|r(c)|}{|q(c)|}\rho \end{aligned}$$

where

$$r(z) = \sum_{i=0}^{\infty} \frac{r^{(i)}(c)}{i!}(z-c)^i$$

and

$$r^{(i)}(c) = \frac{1}{q(c)(i+1)}[p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)](z-c)^i.$$

Furthermore

$$\overline{f}(Z) \subseteq f_{c(k)}(Z), k = 1, 2, \ldots.$$

Proof. The first order centered form of $f(z)$ can be written as

$$f_{c(1)}(Z) = f(c) + \frac{r_c(Z)}{q_c(Z)}(Z-c)$$

where the Taylor coefficients of $r(z)$ are

$$r^{(i)}(c) = \frac{1}{q(c)(i+1)}[p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)], \quad i = 1, 2, \ldots.$$

From Theorem 6.3, the zeroth order centered form for $g(z) = r(z)/q(z)$ has the explicit representation

$$\frac{r_c(Z)}{q_c(Z)} = \frac{1}{B}\langle r(c)\overline{q(c)}, \sum_{i=0}^{\infty} \frac{|r^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{\infty} \frac{|q^{(i)}(c)|}{i!}\rho^i - |r(c)q(c)|\rangle.$$

It follows that

$$f_{c(1)}(Z) = \langle f(c), \frac{\rho}{B}\sum_{i=0}^{\infty} \frac{|q^{(i)}(c)|}{i!}\rho^i \sum_{i=1}^{\infty} \frac{|r^{(i)}(c)|}{i!}\rho^i \rangle$$

$$= \langle f(c), \frac{\rho}{B} \sum_{i=0}^{\infty} \frac{|q^{(i)}(c)|}{i!} \rho^i \sum_{i=1}^{\infty} \frac{|p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)|}{q(c)(i+1)!} \rho^i \rangle. \qquad (6.13)$$

Continuing this process we develop the first centered form for $g(z)$ as

$$g_{c(1)}(Z) = g(c) + \frac{s_c(Z)}{q_c(Z)}(Z - c)$$

where the Taylor coefficients of $s(z)$ are

$$s^{(i)}(c) = \frac{1}{q(c)(i+1)}[r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)], i = 1, 2, \ldots.$$

It follows from (6.13) that

$$g_{c(1)}(Z) = \langle \frac{r(c)}{q(c)}, \frac{\rho}{B} \sum_{i=0}^{\infty} \frac{|q^{(i)}(c)|}{i!} \rho^i \sum_{i=1}^{\infty} \frac{|q^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|}{|q(c)|(i+1)!} \rho^i \rangle. \quad (6.14)$$

The second order centered form for $f(z)$ now can be written as

$$\begin{aligned} f_{c(2)}(Z) &= f(c) + g_{c(1)}(Z)(z - c) \\ &= \langle f(c), \frac{\rho^2}{B} \sum_{i=0}^{\infty} \frac{|q^{(i)}(c)|}{i!} \rho^i \\ &\quad \sum_{i=1}^{\infty} \frac{|r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|}{|q(c)|(i+1)!} \rho^i + |r(c)/q(c)|\rho \rangle. \quad (6.15) \end{aligned}$$

The inclusion property is again a direct consequence of the inclusion isotonicity. □

6.1.2 Inclusion

In Section 3.2 it was shown that the the complex centered forms of higher order for rational functions are included in the lower order forms.

In this section, we show that the centered forms for meromorphic functions have the same property.

THEOREM 6.5 *Let $f(z) = p(z)/q(z)$ be a meromorphic function and let $f_{c(1)}(Z)$ and $f_{c(2)}(Z)$ be its first and second centered forms over the disk $Z = \langle c, \rho \rangle$. Then*

$$\overline{f}(Z) \subseteq f_{c(2)}(Z) \subseteq f_{c(1)}(Z).$$

Proof. Let $f_{c(1)}(Z) = \langle z_1, \rho_1 \rangle$ and $f_{c(2)}(Z) = \langle z_2, \rho_2 \rangle$. Since $z_1 = z_2$, it is sufficient to prove that $\rho_2 < \rho_1$.

From Theorem 6.3, we have

$$\begin{aligned} D &= \rho_1 - \rho_2 \\ &= \frac{\rho}{B} \sum_{i=0}^{\infty} \frac{|q^{(i)}(c)|}{i!} \rho^i \sum_{i=1}^{\infty} \frac{|r^{(i)}(c)|}{i!} \rho^i \\ &- \frac{\rho^2}{B} \sum_{i=0}^{\infty} \frac{|q^{(i)}(c)|}{i!} \rho^i \sum_{i=1}^{\infty} \frac{|r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|}{|q(c)|(i+1)!} \rho^i - |r(c)/q(c)|\rho. \end{aligned}$$

Let

$$B^+ = |q(c)| + \sum_{i=1}^{\infty} \frac{|q^{(i)}(c)|}{i!}\rho^i,$$
$$B^- = |q(c)| - \sum_{i=1}^{\infty} \frac{|q^{(i)}(c)|}{i!}\rho^i,$$

then $B = B^+B^-$. Since $B > 0$ and $B^+ > 0$, it follows that $B^- > 0$.

Multiplying D by $B^-|q(c)|/\rho$ we obtain

$$DB^-|q(c)|/\rho = \frac{B^-}{B}|q(c)|\sum_{i=0}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i\sum_{i=0}^{\infty}\frac{r^{(i)}(c)|}{i!}\rho^i - |r(c)|B^- -$$
$$\frac{B^-}{B}|q(c)|\sum_{i=0}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i\sum_{i=0}^{\infty}\frac{|r^{(i+1)}(c)q(c) - q^{(i+1)}(c)r(c)|}{(i+1)!}\rho^{i+1}$$
$$= |q(c)r(c)| - |r(c)q(c)| + \sum_{i=1}^{\infty}\frac{|r^{(i)}(c)q(c)| + |q^{(i)}(c)r(c)| - |r^{(i)}(c)q(c) - q^{(i)}(c)r(c)|}{(i)!}\rho^i \geq 0.$$

This proves the theorem. □

Let $f_{c(k)}(Z)$ be the k-th order centered form of $f(z)$ over Z. We then have

COROLLARY 4

$$\overline{f}Z) \subseteq f_{c(k)}(Z) \subseteq f_{c(k-1)}(Z) \subseteq \cdots \subseteq f_{c(1)}(Z).$$

Proof. According to the recursive definition of the centered forms, the $(k-1)$-th order centered form $f_{c(k-1)}(Z)$ can be written as

$$f_{c(k-1)}(Z) = f(c) + f'(c)(Z-c) + \cdots + \frac{f^{(k-2)}(c)}{(k-2)!}(Z-c)^{k-2} + g_{c(0)}(Z)(Z-c)^{k-1}$$

where $g(z)$ is a meromorphic function. Therefore the (k)-th and $(k+1)$-th order forms of $f(z)$ can be written as

$$f_{c(k)}(Z) = f(c)+f'(c)(Z-c)+\cdots+\frac{f^{(k-2)}(c)}{(k-2)!}(Z-c)^{k-2}+g_{c(1)}(Z)(Z-c)^{k-1},$$

and

$$f_{c(k+1)}(Z) = f(c)+f'(c)(Z-c)+\cdots+\frac{f^{(k-2)}(c)}{(k-2)!}(Z-c)^{k-2}+g_{c(2)}(Z)(Z-c)^{k-1}$$

Applying Theorem 6.5 to $g(z)$ we obtain $g_{c(2)}(Z) \subseteq g_{c(1)}(Z)$ which along with the inclusion isotonicity of the operations gives

$$f_{c(k+1)}(Z) \subseteq f_{c(k)}(Z).$$

The corollary follows. □

The power-sum forms and centered forms both have the property that they give increasingly better inclusion for the range of the function. Numerical examples show, however, that there are functions $f(z)$ and disks Z such that both $\rho_{c(k)} < \rho_{p(k)}$ and $\rho_{c(k)} > \rho_{p(k)}$ are possible, where $\rho_{c(k)}$ and $\rho_{p(k)}$ are radii of $f_{c(k)}(Z)$ and $f_{p(k)}(Z)$ respectively.

6.1.3 Quadratic Convergence

It was shown in Section 3.3 that the centered forms for complex rational functions were quadratically convergent to the range as the size of the domain tended to zeroth in the sense of diameter inclusion.

In this section we will show centered forms of all orders are quadratically convergent in the norm of DI-pseudo-distance. This distance was defined in Section 3.3(see[46]).

THEOREM 6.6 *Let $f_{c(1)}(Z)$ be the first order centered form of the meromorphic function $f(z) = p(z)/q(z)$ over $Z = \langle c, \rho \rangle$. If $W \in \mathbf{K}$ is the smallest circle containing $\overline{f}(Z)$ then*

$$\Phi(W, f_{c(1)}(Z)) = O(\rho^2).$$

Proof. Let $u, v \in Z$ be chosen such that

$$|f(u) - f(v)| = \max_{z_1, z_2 \in Z} |f(z_1) - f(z_2)|$$

and

$$Y = \langle \frac{1}{2}(f(u) + f(v)), \frac{1}{2}|f(u) - f(v)| \rangle.$$

Then clearly

$$Y \prec W, \ \overline{f}(Z) \subseteq W \text{ and } \overline{f}(Z) \subseteq f_{c(1)}(Z).$$

We now write $f(z)$ as

$$f(z) = f(c) + \frac{r(z)}{q(z)}(z-c)$$

where

$$r(z) = \sum_{i=0}^{\infty} \frac{r^{(i)}(c)}{i!}(z-c)^i$$

and

$$r^{(i)}(c) = \frac{p^{(i+1)}(c)q(c) - q^{(i+1)}(c)p(c)}{q(c)(i+1)}, \ \ i = 0, 1, \ldots.$$

Let $z_1, z_2 \in Z$ be any two points. We have

$$\begin{aligned} |f(z_1) - f(z_2)| &= |\frac{r(z_1)}{q(z_1)}(z_1 - c) - \frac{r(z_2)}{q(z_2)}(z_2 - c)| \\ &= |\frac{r(c)}{q(c)}(z_1 - c) - \frac{r(c)}{q(c)}(z_2 - c) + \frac{t(z_1)}{q(z_1)}(z_1 - c)^2 - \frac{t(z_2)}{q(z_2)}(z_2 - c)^2| \\ &\geq |\frac{r(c)}{q(c)}||z_1 - z_2| - |\frac{t(z_1)}{q(z_1)}(z_1 - c)^2 - \frac{t(z_2)}{q(z_2)}(z_2 - c)^2| \end{aligned}$$

where $t(z)$ satisfies

$$\frac{r(z)}{q(z)} = \frac{r(c)}{q(c)} + \frac{t(z)}{q(z)}(z - c).$$

Let z_1, z_2 to be chosen such that $|z_1 - z_2| = 2\rho$. Then we have

$$\begin{aligned} |f(u) - f(v)| &\geq |f(z_1) - f(z_2)| \\ &\geq 2|\frac{r(c)}{q(c)}|\rho - |\frac{t(z_1)}{q(z_1)}\rho^2| - |\frac{t(z_2)}{q(z_2)}\rho^2|. \end{aligned}$$

The radius $\rho_{c(1)}$ of $f_{c(1)}(Z)$ may be expanded as

$$\begin{aligned}
\rho_{c(1)} &= (\frac{\rho}{B}\sum_{i=0}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i\sum_{i=1}^{\infty}\frac{|p^{(i+1)}(c)q(c)-q^(i+1)(c)p(c)|}{q(c)(i+1)!}\rho^i)/(|q(c)|^2-|\sum_{i=1}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i|^2)\\
&= \frac{\rho}{B}\sum_{i=0}^{\infty}\frac{|q^{(i)}(c)|}{i!}\rho^i\sum_{i=1}^{\infty}\frac{|p^{(i+1)}(c)q(c)-q^{(i+1)}(c)p(c)|}{|q(c)|(i+1)!}\rho^i\frac{\rho}{|q(c)|^2}(1+O(\rho))\\
&= (|q(c)|+O(\rho))\left[\frac{|p'(c)q(c)-q'(c)p(c)|}{|q(c)|}+O(\rho)\right]\frac{\rho}{|q(c)|^2}(1+O(\rho))\\
&= \left[\frac{|p'(c)q(c)-q'(c)p(c)|^2}{|q(c)|}+O(\rho)\right]\rho+O(\rho^2).
\end{aligned}$$

The DI-pseudo-distance between Y and $f_{c(1)}(Z)$ can now be estimated as

$$\begin{aligned}
\Phi(Y,f_{c(1)}(Z)) &\leq \left[\frac{|p'(c)q(c)-q'(c)p(c)|}{|q(c)|}+O(\rho)\right]\rho\\
&+ O(\rho^2)-\rho|\frac{r(c)}{q(c)}|-1/2\rho^2(|\frac{t(z_1)}{q(z_1)}|+|\frac{t(z_2)}{q(z_2)}|)\\
&= -1/2\rho^2(\frac{|t(z_1)}{q(z_1)}|+|\frac{t(z_2)}{q(z_2)}|)+O(\rho^2).
\end{aligned}$$

Since $t(z)/q(z)\rightarrow t(c)/q(c)$ when $\rho\rightarrow 0$ for any $z\in Z$. We have

$$\Phi(Y,f_{c(1)}(Z))=O(\rho^2).$$

It follows from $\Phi(W,f_{c(1)}(Z))\leq\Phi(Y,f_{c(1)}(Z))$ that $\Phi(W,f_{c(1)}(Z))=O(\rho^2)$. □

COROLLARY 5 $\Phi(W,f_{c(k)}(Z))=O(\rho^2)$ *for all* $k\geq 1$.

Proof. From Corollary 4 $f_{c(k)}(Z)\subseteq f_{c(1)}(Z)$. The corollary follows. □

In Section 3.3 (see also[41]) it was shown that the zeroth order centered form is also quadratically convergent for complex rational functions given certain conditions. Here we show that it is true for meromorphic functions under the same conditions.

THEOREM 6.7 *Let $f_{c(0)}(Z)$ be the zeroth order centered form of the meromorphic function $f(z) = p(z)/q(z)$ over $Z = \langle c, \rho \rangle$. If $W \in \mathbf{K}$ is the smallest circle containing $\overline{f}(Z)$ then*

$$\Phi(W, f_{c(0)}(Z)) = O(\rho^2).$$

whenever $\arg(p'(c)q(c)) = -\arg(q'(c)p(c))$.

Proof. Let $W = \langle c_W, \rho_W \rangle$. From the fact $\mathrm{rad}(Y) \leq \mathrm{rad}(W)$ (see [41]), where Y is defined in Theorem 6.6, and a result of Theorem 6.6, we have

$$\rho_W \geq \left[|\frac{r(c)}{q(c)}|\rho + O(\rho^2) \right] \geq |\frac{r(c)}{q(c)}|\rho + O(\rho^2).$$

From Theorem 6.3, we have

$$\rho_{c(0)} = \frac{|p'(c)q(c)| + |q'(c)p(c)|}{|q(c)|^2}\rho + O(\rho^2).$$

It then follows from $\arg(p'(c)q(c)) = -\arg(q'(c)p(c))$ that

$$\Phi(W, f_{c(0)}(Z)) \;=\; \rho_{c(0)} - \rho_W$$

$$\begin{aligned} &\leq \frac{|p'(c)q(c)| + |q'(c)p(c)|}{|q(c)|^2}\rho - \frac{|p'(c)q(c) - q'(c)p(c)|}{|q(c)|^2}\rho + O(\rho^2) \\ &\leq \frac{|p'(c)q(c)| + |q'(c)p(c)| - |p'(c)q(c) - q'(c)p(c)|}{|q(c)|^2}\rho + O(\rho^2) \\ &= O(\rho^2). \end{aligned}$$

This proves the Theorem. □

Using a similar method and the explicit formulas (6.1), (6.6) and (6.8) for the power-sum forms we obtain the following two theorems. The proofs are omitted.

THEOREM 6.8 *Let $f_{p(1)}Z$ be the first order power-sum form for the meromorphic function $f(z) = p(z)/q(z)$ over $Z = \langle c, \rho \rangle$. If $W \in \mathbf{K}$ is the smallest circle containing $\overline{f}(Z)$ then*

$$\Phi(W, f_{p(1)}(Z)) = O(\rho^2).$$

THEOREM 6.9 *Let $f_{p(0)}(Z)$ be the zeroth order power-sum form for the meromorphic function $f(z) = p(z)/q(z)$ over $Z = \langle c, \rho \rangle$. If $W \in \mathbf{K}$ is the smallest circle containing $\overline{f}(Z)$ then*

$$\Phi(W, f_{p(0)}(Z) = O(\rho^2)$$

whenever $\arg(p'(c)q(c)) = -\arg(q'(c)p(c))$.

6.1.4 Inclusion Isotonicity

In Section 3.4 (see also[6]) it was shown that the centered form for arbitrary analytic functions is inclusion isotone, that is, if $Z_1, Z_2 \in \mathbf{K}$ and $h(z)$ is an analytic function then

$$Z_1 \subseteq Z_2 \Longrightarrow h_{c(k)}(Z_1) \subseteq h_{c(k)}(Z_2)$$

where the centered forms for analytic functions is defined in a natural manner.

Clearly the property of inclusion isotonicity is of importance in the applications. In fact, if the domain is restricted then it is desirable that the estimate of the range should improve. This property is not always guaranteed as shown for example in Moore[24].

In this note the inclusion isotonicity is verified for all orders of centered forms for rational and meromorphic circular functions.

Obviously the meromorphic case implies the rational case. Therefore we only need to show that it is true for meromorphic functions. In fact we will prove the following theorem which also covers the power-sum forms for meromorphic functions (see[7] for definitions of power sum forms. We use the same notation $f_{p(k)}(Z)$ as in[7] for the k-th power sum form).

THEOREM 6.10 *Let $Z_1, Z_2 \in \mathbf{K}$ be any two disks and $f(z) = p(z)/q(z)$*

be a meromorphic function. If $Z_1 \subseteq Z_2$ *, then*

$$(i) \qquad f_{p(k)}(Z_1) \subseteq f_{p(k)}(Z_2), \quad k = 0, 1, \ldots$$

$$(ii) \qquad f_{c(k)}(Z_1) \subseteq f_{c(k)}(Z_2), \quad k = 0, 1, \ldots.$$

Proof. By induction in both cases.

(i) From the inclusion isotonicity of disk operations and (2.33), it follows that if $Z_1 \subseteq Z_2$ then

$$\begin{aligned} \sum_{i=0}^{\infty} a_i Z_1^i &\subseteq \sum_{i=0}^{\infty} a_i Z_2^i, \\ \sum_{i=0}^{\infty} b_i Z_1^i &\subseteq \sum_{i=0}^{\infty} b_i Z_2^i. \end{aligned}$$

From the isotonicity of disk division, we have

$$f_{p(0)}(Z_1) = \frac{\sum_{i=0}^{\infty} a_i Z_1^i}{\sum_{i=0}^{\infty} b_i Z_1^i} \subseteq f_{p(0)}(Z_2) = \frac{\sum_{i=0}^{\infty} a_i Z_2^i}{\sum_{i=0}^{\infty} b_i Z_2^i}.$$

Assume that $Z_1 \subseteq Z_2$ implies $f_{p(k-1)}(Z_1) \subseteq f_{p(k-1)}(Z_2)$ for any meromorphic function $f(z)$. Thus $g_{p(k-1)}(Z_1) \subseteq g_{p(k-1)}(Z_2)$, where $g_{p(k-1)}(z)$ is the $(k-1)$ th power sum form of the $g(z)$ that satisfies

$$f(z) = f(c) + g(z)(z - c).$$

It follows that

$$f_{p(k)}(Z_1) = f(c)+g_{p(k-1)}(Z_1)(Z_1-c) \subseteq f_{p(k)}(Z_2) = f(c)+g_{p(k-1)}(Z_2)(Z_2-c).$$

This proves (i).

(ii) Let $p_c(Z)$ and $q_c(Z)$ be the centered forms[6] of the analytic functions $p(z)$ and $q(z)$ over Z respectively. From the main theorem in[6], we have

$$Z_1 \subseteq Z_2 \Longrightarrow p_c(Z_1) \subseteq p_c(Z_2), \;\; q_c(Z_1) \subseteq q_c(Z_2).$$

Thus it follows from (2.28) that

$$f_{c(0)}(Z_1) = \frac{p_c(Z_1)}{q_c(Z_1)} \subseteq f_{c(0)}(Z_2) = \frac{p_c(Z_2)}{q_c(Z_2)}$$

Assume that $Z_1 \subseteq Z_2$ implies $f_{c(k-1)}(Z_1) \subseteq f_{c(k-1)}(Z_2)$ for any meromorphic function $f(z)$. We have

$$f_{c(k)}(Z) = f(c) + g_{c(k-1)}(Z)(Z-c)$$

where $g_{c(k-1)}(Z)$ satisfies $g_{c(k-1)}(Z_1) \subseteq g_{c(k-1)}(Z_2)$ from the induction hypothesis.

It follows that

$$f_{c(k)}(Z_1) = f(c)+g_{c(k-1)}(Z_1)(Z_1-c) \subseteq f_{c(k)}(Z_2) = f(c)+g_{c(k-1)}(Z_2)(Z_2-c).$$

This proves (ii) and the theorem. □

6.2 Algorithmic Approaches

6.2.1 Circular Approximation

In this section, the circular approximation to the circular centered forms for analytic functions is defined and the error estimation for the approximation is presented. This definition then is extended to the meromorphic functions. Based on this approximation, all the algorithms discussed in Chapter 4 can be used to compute the approximations to circular centered forms for a meromorphic function. It will be shown that this approximation to the centered form is always better than the centered form obtained from the polynomial approximation to the analytic functions up to same degree as the approximation to the analytic centered form.

Centered forms can be generalized to multi-variate meromorphic functions as to multi-variate rational functions. The corresponding properties hold for multi-variate meromorphic forms. With the above approximation, all the algorithms for multi-variate rational forms discussed in Chapter 4 can be used to evaluate approximately multi-variate meromorphic forms. However, this is not a part of this thesis.

DEFINITION 6.1 *Let $f(z)$ be an analytic function and $Z = \langle c, \rho \rangle$ a circular disk. Then disk*

$$f_n(Z) = \langle f(c), \sum_{k=1}^{n} |f^{(k)}(c)|\rho^k/k! \rangle$$

is called the approximation of degree n to the centered form evaluation $f_c(Z)$ with error

$$\Phi(f_c(Z), f_n(Z)) = \sum_{k>n} |f^{(k)}(c)|\rho^k/k!.$$

Let $p(z, n)$ be the Taylor expansion at $z = c$ of $f(z)$ of degree n and $p_c(Z, n)$ the value of centered form $p_c(z, n)$ over disk $Z = \langle c, \rho \rangle$. Then the following theorem holds.

THEOREM 6.11 *For any analytic function $f(z)$ and circular disk $Z = \langle c, \rho \rangle \in \mathbf{K}$, let $f_c(Z) = \langle c_1, \rho_1 \rangle, f_n(Z) = \langle c_2, \rho_2 \rangle$ and $p_c(Z, n) = \langle c_3, \rho_3 \rangle$. Then*

$$c_1 = c_2 = f(c), \ c_3 = p(c, n), \tag{6.16}$$

$$\rho_2 = \rho_3 = \sum_{k=1}^{n} |f^{(k)}(c)|\rho^k/k!, \tag{6.17}$$

$$\Phi(f_c(Z), f_n(Z)) \leq \Phi(f_c(Z), p_c(Z, n)). \tag{6.18}$$

Proof. Formulae (6.16) and (6.17) are obvious. From these two formulae, it follows $f_n(Z) \prec f_c(Z)$. However, relations $p_c(Z,n) \prec f_c(Z)$ or $f_c(Z) \prec p_c(Z,n)$ may not true. Therefore we have

$$\Phi(f_c(Z), f_n(Z)) = \sum_{k>n} |f^{(k)}(c)|\rho^k/k!$$

and

$$\Phi(f_c(Z), p_c(Z,n)) = \sum_{k>n} |f^{(k)}(c)|\rho^k/k! + \Delta$$

where

$$\Delta = \begin{cases} 0 & \text{if } p_c(Z,n) \prec f_c(Z) \text{ or } f_c(Z) \prec p_c(Z,n) \\ |f(c) - p(c,n)| & \text{otherwise .} \end{cases}$$

It follows that $\Phi(f_c(Z), f_n(Z)) \leq \Phi(f_c(Z), p_c(Z,n))$, which proves the theorem. □

Based on the above definition, the approximations to meromorphic forms are recursively defined as follows.

DEFINITION 6.2 *Let $f(z) = p(z)/q(z)$ be a meromorphic function, where $p(z)$ and $q(z)$ are analytic functions. Let $p_n(z)$ and $q_m(z)$ be the approximations of degree n and m to $p(z)$ and $q(z)$ respectively. Then the approximation of degree (n,m) to the zeroth order centered form evaluation for $f(z)$ is*

defined as

$$f_{c(0)}^{(n,m)}(Z) = p_n(Z)/q_m(Z),$$

and recursively the approximation of degree (n, m) *to the* k*-th order centered form evaluation is defined as*

$$f_{c(k)}^{(n,m)} = \frac{p(c)}{q(c)} + g_{c(k-1)}^{(n,m)}(Z)(Z - c)$$

where $g_{c(k-1)}^{(n,m)}(Z)(Z - c)$ *is the approximation to the* $(k-1)$*-th order centered form evaluation of* $g(z)$ *and* $g(z)$ *satisfies*

$$f(z) = \frac{p(c)}{q(c)} + g(z)(z - c).$$

Similarly, we can define approximations to the power-sum meromorphic form evaluations. Here, however, we will not present the definition.

Based on two definitions above, all the algorithms discussed in Chapter 4 can be used to compute the approximations up to arbitrary degree, i.e., to compute the centered form evaluations for analytic and meromorphic functions up to certain error bound. From theorem 6.11, these approximation are better than those centered form evaluations obtained from Taylor expansions to the functions with same degree.

6.2.2 Numerical Realizations

In[41] examples are given for the numerical realization of rational circular centered forms and in[29] examples are given for the inclusion of standard functions using Taylor series, that is, an application of the circular centered form for functions that are analytic. Here we similarly give examples showing how the the meromorphic forms may be used in the applications.

Let therefore $f(z) = \sin(z)/e^z$ with $Z = \langle c, \rho \rangle$. As usual we wish to find including approximations to $\overline{f}(Z)$. The Taylor series for $\sin(z)$ and e^z are

$$\sin(z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} z^{2i+1}$$

and

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

and we immediately have a first estimate

$$f(Z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} Z^{2i+1} / \sum_{i=0}^{\infty} \frac{z^i}{i!} \supseteq \overline{f}(Z).$$

The centered forms for $\sin(z)$ and e^z over Z are

$$\begin{aligned} \sin_c(Z) &= \langle \sin(c), |\sin(c)| \sum_{i=1}^{\infty} \frac{\rho^{2i}}{(2i)!} + |\cos(c)| \sum_{i=1}^{\infty} \frac{\rho^{2i+1}}{(2i-1)!} \rangle \\ &= \langle \sin(c), |\sin(c)|(e^{\rho} + e^{-\rho})/2 + |\cos(c)|(e^{\rho} - e^{-\rho})/2 \rangle \end{aligned}$$

and

$$e_c(Z) = \langle e^c, \sum_{i=1}^{\infty} \frac{|e^c|}{i!}\rho^i \rangle = \langle e^c, |e^c|(e^\rho - 1)\rangle.$$

Thus we obtain from Theorem 6.3

$$B = |e^c|(|e^c| - (e^\rho - 1)^2)$$

$$c_{c(0)} = (\sin(c)\overline{e^c})/B$$

and

$$\rho_{c(0)} = \frac{|e^c|e^\rho}{2B}((|\sin(c)| + |\cos(c))(e^\rho - e^{-\rho}) - 2|\cos(c)|).$$

From Theorem 6.4 we obtain

$$c_{c(1)} = \sin(c)/e^c$$

and

$$\rho_{c(1)} = \frac{1}{4\rho B}(|\sin(c)|(e^\rho + e^{-\rho}) + |\cos(c)|(e^\rho - e^{-\rho} - 2))(|\cos(c)| - |\sin(c)|)(e^\rho - e^{-\rho}).$$

In the general case we do not expect to be able to obtain closed expressions as above. Instead we would have to use further estimations and coarsenings.

6.3 Summary

This chapter generalizes the power-sum forms and centered forms with all degrees to meromorphic functions. The explicit formulas for the zero, first

and second degree forms are obtained. Similarly to the rational forms, the properties of inclusion, quadratic convergence and inclusion isotonicity for meromorphic forms are proven.

For computational purpose, the disk approximations to a meromorphic form are defined. It is proven that these approximations are always sharper than the forms obtained from approximate polynomials to the meromorphic function with same degree; i.e., these approximations are closer to the original meromorphic form than the polynomial forms in the DI-pseudo-distance.

The disk approximations can be computed using the algorithms discussed in Chapter 4 and the errors are remainders of real serieses which can be estimated easily using usual series estimations. Some numerical examples are presented for illustration of meromorphic forms.

Chapter 7

Conclusion

7.1 Theory of Centered Forms

With the series of theorems in this thesis we have established three important properties for the centered forms of meromorphic functions: inclusion, quadratic convergence and inclusion isotonicity. These properties are of importance in the applications.

It is known that all the rational, analytic and polynomial functions are special cases of the meromorphic functions. Also when a meromorphic function reduces to one of above three kinds of functions, the centered forms reduce to the centered forms of corresponding function (k-th order form re-

duces to k-th order form for rational function; but for analytic and polynomial functions all different order forms reduce to the unique centered form for corresponding functions). All other concepts and relations remain valid after reductions.

Since we have established those three important properties for the centered forms of meromorphic functions: inclusion, quadratic convergence and inclusion isotonicity, simultaneously we have implicitly establish these three properties for centered forms for the three classes of functions; though these properties were obtained for each class of functions by different methods originally. Thus we have an unified theory for the centered forms of all classes of functions.

7.2 Significant Achievements of the Thesis

This thesis provides a systematic theory dealing with centered forms for polynomial, rational, analytic and meromorphic functions. Using this theory, the three basic properties of inclusion, convergence and inclusion isotonicity are proven systematically for all the above functions.

Furthermore, a type of special Taylor expansion forms for multi-variate

polynomials and rational functions has been designed. These expansions can be evaluated efficiently using a low complexity algorithm, since there are a lot of zero entries and scalar variables in the remainder. The number of zero entries and the number of interval variables replaced by scalar variables are both estimated. As an application, these forms are implemented for factorable functions showing the forms can be evaluated very efficiently.

The centered forms have also been generalized to multi-variate functions and the low complexity algorithms for evaluating these forms have been presented.

It has been shown that for a polynomial there are a great number of different centered forms evaluated over different arrangements of variables. An asymptotic formula for estimation of this number is given showing that this number is surprisingly large. An algorithm is presented to generate all the variable arrangements which produce different forms for a polynomial.

A type of low complexity Taylor forms for multi-variate functions has been obtained. Evaluation of this form needs no computation of the derivatives and only requires the repetitive division of the polynomials and the rational functions by scalar linear factors. The low complexity algorithm for evaluating this form is given with comparison of the direct derivative method.

A detailed time complexity is given for the algorithm.

7.3 Suggestions for the Further Research

The centered forms of a function over circular disks have been discussed and researched systematically in the thesis. A number of low complexity algorithms for evaluating these forms are presented. Nevertheless, the centered form theory is far from complete. A lot of problems need to be solved and a lot of new questions are arising. The following are two of them, which would constitute interesting topics of my further research in this area.

(A) For a multi-variate function, there are a great number of different centered forms evaluated with different variable arrangements. Is there an optimal centered form for each function which gives the sharpest estimate for range of the function over a disk? If so, is there an efficient algorithm which generates the variable arrangement producing the sharpest form for a polynomial?

(B) The interval and circular arithmetic used for evaluating forms is just one, maybe the most popular one, of different arithmetic systems. Using the existing forms and algorithms, is there an interval and circular arithmetic

which gives the best inclusion disk for the range of a function?

There is a number of other unsolved problems in this area which are of much importance both theoretically and practically. We will not list them here, which can been found in various references listed in the **Bibliography** below.

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