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Viscosity Solution Theory for Mean Field Stochastic Controls and Feynman-Kac Representation for Nonlinear Time-dependent Schrödinger Equations

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Viscosity Solution Theory for Mean Field Stochastic Controls and Feynman-Kac Representation for Nonlinear Time-dependent Schrödinger Equations

by

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Abstract

Mean field (McKean–Vlasov) control problems model strategic decision-making in large populations of (approximately) symmetric agents interacting through an aggregate, or the so called mean field term. The dependence on the population measure in both the state dynamics and the objective naturally leads to infinite-dimensional formulations. As in finite-dimensional cases, the associated Hamilton–Jacobi–Bellman equations typically do not admit classical solutions, making viscosity solutions a natural and powerful framework. In the first part of this thesis, we explore the theory of viscosity solutions for mean field control problems.

In the second part, we study a novel Feynman–Kac representation. Traditionally, Feynman–Kac formulas connect parabolic partial differential equations (PDEs) with stochastic differential equations (SDEs). We extend this connection to Schrödinger-type equations, which are prototypical non-parabolic PDEs, and demonstrate how this approach facilitates efficient simulation of high-dimensional PDEs.

Preface

This thesis consists of two submitted works, one published work, and one unfinished work that will be submitted after further polishing.

Chapter 2 of this thesis has been published as Cheung *et al.* (2025), under the title *"Viscosity Solutions of a Class of Second-Order Hamilton–Jacobi–Bellman Equations in the Wasserstein Space"* in the journal *Applied Mathematics & Optimization*.

Chapter 3 of this thesis has been submitted to the *SIAM Journal on Control and Optimization*, entitled *"Viscosity Solutions of Fully Second-Order HJB Equations in the Wasserstein Space"*, and is currently under minor revision.

Chapter 4 of this thesis has been submitted to *Stochastic Processes and their Applications*, entitled *"Feynman–Kac Formula for Time-Dependent Nonlinear Schrödinger Equations with Applications in Numerical Approximations"*, and is currently under minor revision.

Chapter 5 of this thesis is titled *"A Viscosity Solution Theory of Stochastic Hamilton-Jacobi-Bellman equations in the Wasserstein Space"* and is under polishing, as our future work.

Dedication

This thesis is dedicated to my uncle, 張珍誼 (Chun Yi Cheung, Sam), who taught me about life—and, sadly, about death.

Acknowledgements

I am deeply grateful to my supervisors, Professor Jinniao Qiu and Professor Alexandru Badescu, for their steadfast guidance and invaluable support since the outset of my Ph.D. journey in January 2022. At that time, I was on the verge of abandoning my intellectual pursuits, and they gave me the opportunity to continue; for this, I owe them my greatest gratitude. I am someone who often asks what may seem like naïve questions, yet they never gave up on me. I am also prone to anxiety, and they always seemed to recognize this, frequently reminding me with patience and kindness that there was "no rushing". Their dedication has greatly enriched my academic understanding and profoundly shaped my development as a researcher. I am especially thankful for their continual encouragement in advancing my research.

I am also deeply grateful to the members of my supervisory committee, Professor Antony Ware and Professor Anatoliy Swishchuck, for their valuable insights and constructive feedback on both my research proposal and my thesis.

I wish to thank my friend and collaborator, Dr. Ho Man Tai, for sharing in the challenges of research and for working with me to complete our papers. Through our collaboration, I have learned not only how to write academic articles in a rigorous style, but also how much determination, persistence, and mutual support matter in research.

I also extend my sincere thanks to my friends, Andrew Cha, Henry Chen, Siu-Hang (Hugo) Cheng, Shaghayegh Ebrahimzade, Yi-Lin Gu, Kenji Ieong, Han-Qing Li, Cai-Feng Li, Keivan Mirzaei, Hasib Molla, Ye Wang, Yang Yang, Natalie Zayvoda, Ge-Ming Zhang,

Dr. Xin Zhang, Ze-Kun Zhang, and Zhong-Hao Zhang. I truly cherished the time spent with you and am deeply grateful for the joy you have brought into my life. The memories we shared are among my greatest treasures.

I would like to express my deepest gratitude to my family for their unwavering love, patience, and support throughout my PhD journey. In the first year of my studies (2022), my uncle Sam passed away in an accident. He lived in Beijing, and I was unable to visit him because of the pandemic. I hope that when the day comes that I see him again, he will not resent me, but will recognize me and call my nickname.

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Chapter 1

Introduction

The purpose of this chapter is to offer heuristic insights and motivation; a rigorous development will follow in later chapters.

1.1 Part I: Viscosity Solutions of Mean Field Control Problems

1.1.1 Mean Field Control Problems and its related Hamilton-Jacobi-Bellman PDEs

We introduce over $\mathbb{R}^d$ the space of probability measures $\mathcal{P}(\mathbb{R}^d)$ and its subset $\mathcal{P}_p(\mathbb{R}^d)$ consisting of those with finite p-th moment for $p \geq 1$. The space $\mathcal{P}_p(\mathbb{R}^d)$ is equipped with the p-Wasserstein distance

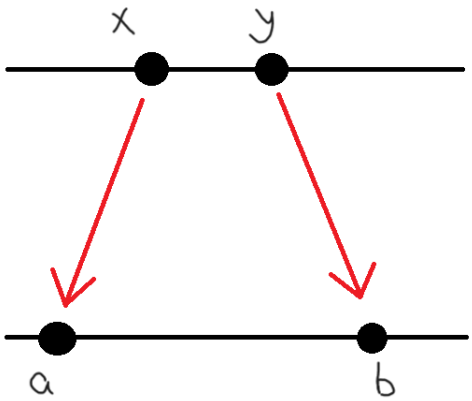
$$\mathcal{W}_p(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \pi(dx,dy) \right)^{\frac{1}{p}}, \quad \text{for } \mu,\nu \in \mathcal{P}_p(\mathbb{R}^d),$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ such that for any $\pi \in \Pi(\mu, v)$, $\pi(\mathbb{R}^d \times \cdot) = \mu(\cdot)$ and $\pi(\cdot \times \mathbb{R}^d) = \nu(\cdot)$. We call $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p)$ the p-th Wasserstein space over $\mathbb{R}^d$, and it is a Polish space.

The above discussion is somewhat abstract, so let us illustrate the idea with a simple example. Suppose we want to transport two point masses located at (x, y) to two point masses located at (a, b).

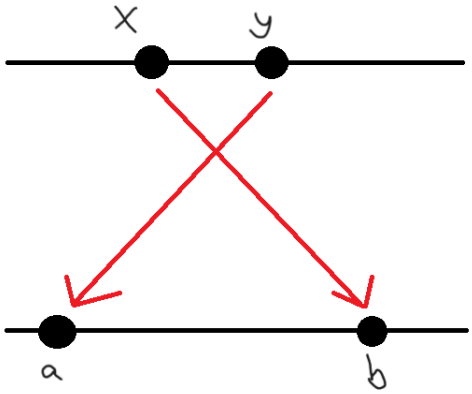
First way:

$$\pi = \tfrac{1}{2}\delta_{(x,a)} + \tfrac{1}{2}\delta_{(y,b)},$$



Second way:

$$\pi = \tfrac{1}{2}\delta_{(x,b)} + \tfrac{1}{2}\delta_{(y,a)},$$



Naturally, we prefer the first transport plan over the second one, as it requires less 'effort' to transport x and y to a and b. This leads us to the notion of a *distance*: the minimal cost required to transfer a distribution μ into another distribution ν. Although the Wasserstein distance is only a metric, we can equip it with a notion of derivative as follows.

Definition 1.1.1. *The function* $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *is said to be first order Lions-differentiable (L-differentiable) if its lifting* $F : [0,T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$*;* $F(t,\xi) :=$

$f(t, \mathcal{L}(\xi))$ admits a continuous Fréchet derivative, where $\mathcal{L}(\xi)$ is the law of ξ. We denote $\partial_\mu f$ for first order derivative, $\partial_\mu^2 f$ for second order derivative.

Now, we consider the solution $X^{t,\xi,\alpha}$ of the following state dynamics:

$$X_s = \xi + \int_t^s b\big(r, X_r, \mathbb{P}_{X_r}^{W^0}, \alpha_r\big)\, dr + \int_t^s \sigma\big(r, X_r, \alpha_r\big)\, dW_r + \int_t^s \sigma^0(r)\, dW_r^0, \tag{1.1.1}$$

for $s \in [t, T]$, $t \geq 0$, where $\mathbb{P}_{X_r}^{W^0}$ denotes the conditional distribution of X_r given W^0. In (1.1.1), the function b is the drift, W_r is the idiosyncratic noise, and W_r^0 is the common noise, which is independent of W_r, and $\alpha \in \mathcal{A}_t$, α being the control, while $\mathcal{A}_t$ denotes the set of admissible controls initiated at time t. The coefficients b, σ, and σ^0 are assumed to be deterministic. The dynamics in (1.1.1) can be interpreted as the mean-field limit of the following n-particle system: for each $i = 1, 2, \ldots, n$, consider

$$X_s^i = \xi^i + \int_t^s b\big(r, X_r^i, \widehat{\mu}_r^n, \alpha_r^i\big)\, dr + \int_t^s \sigma\big(r, X_r^i, \alpha_r^i\big)\, dW_r^i + \int_t^s \sigma^0(r)\, dW_r^0,$$

where

$$\widehat{\mu}_r^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_r^i},$$

and each ξ^i has the same law as ξ. In this framework, W^0 is the noise shared by all players, and is therefore called the *common noise*, while the collection $\{W^i\}_{i=1}^n$ represents the *idiosyncratic noises*, as each W^i is distinct for each player i.

We are subject to the payoff function:

$$J(t, \xi, \alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{t,\xi,\alpha}, \mathbb{P}_{X_s^{t,\xi,\alpha}}^{W^0}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}_{X_T^{t,\xi,\alpha}}^{W^0})\right].$$

Finally, we define the value function V to be

$$V(t, \xi) := \sup_{\alpha \in \mathcal{A}_t} J(t, \xi, \alpha), \quad \text{for any } (t, \xi) \in [0, T] \times L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d). \tag{1.1.2}$$

It could be shown that $V(t, \xi) = V(t, \mathcal{L}(\xi))$, where $\mathcal{L}(\xi)$ is the law of ξ, thus V is a function on the Wasserstein space. We have the following standard toy model to motivate the mean field control problem.

Example A set $\{1,\dots,N\}$ of firms compete in an economy where greenhouse gas emissions are regulated over a fixed period $[0,T]$, for some $T>0$. For each firm i, let X_t^i denote at time t its perceived cumulative emissions up to maturity T. We assume that its dynamics satisfy

$$dX_t^i = \big(b_t^i - \alpha_t^i\big)\,dt + \sigma_t^i\,dW_t^i + \sigma_t^{0;i}\,dW_t^0, \quad t\in[0,T], \quad X_0^i = x^{(0)}.$$

Here b_t^i represents the individual emission rate, σ_t^i the idiosyncratic volatility, $\sigma_t^{0;i}$ the systematic (common) volatility, and α_t^i the abatement rate of firm i, which can be viewed as the control variable governing its emissions output.

Each firm i is allocated Λ_i allowances. The regulator imposes a cap $\Lambda^{(N)} = \sum_{i=1}^N \Lambda_i$ as the aggregate emissions target for the economy over $[0,T]$. If at the end of the period the total emissions exceed the cap, i.e.,

$$\sum_{i=1}^N X_T^i > \Lambda^{(N)},$$

then each firm i must offset its own emissions X_T^i by paying a penalty λ for each unit not covered by a certificate. In this case, firm i incurs the payment

$$\lambda_i = \lambda\big(X_T^i - \Lambda_i\big)^+ \mathbf{1}_{\left(\Lambda^{(N)},\infty\right)}\left(\sum_{j=1}^N X_T^j\right).$$

Suppose there is a (large) upper bound $U>0$ on the penalty. Then firm i at time T pays

$$\lambda_i = \min\left\{\lambda\big(X_T^i - \Lambda_i\big)^+ \mathbf{1}_{\left(\Lambda^{(N)},\infty\right)}\left(\sum_{j=1}^N X_T^j\right), U\right\}.$$

Each firm seeks to maximize its terminal wealth

$$W_T^i = \mathbb{E}\left[w^i - \int_0^T c^i(\alpha_t^i)\,dt - \min\left\{\lambda\big(X_T^i - \Lambda_i\big)^+ \mathbf{1}_{\left(\Lambda^{(N)},\infty\right)}\left(\sum_{j=1}^N X_T^j\right), U\right\}\right],$$

where w^i denotes the initial wealth and c^i the cost associated with controlling emissions via the abatement policy α^i. We further assume that there is also a large cap on c^i.

Assume now that all firms are identical and that N is very large. Conditional on the common noise, the variables $\{X_T^i\}_{i=1}^N$ are i.i.d. In this regime,

$$w^i - \min\Big\{ \lambda\big(X_T^i - \Lambda_i\big)^+ \mathbf{1}_{\frac{1}{N}\sum_{j=1}^N X_T^j > \frac{\Lambda^{(N)}}{N}}, U \Big\} \longrightarrow w - \min\Big\{ \lambda\big(X_T - \Lambda\big)^+ \mathbf{1}_{\mathbb{E}[X_T \,|\, W_T^0] > \Lambda}, U \Big\},$$

which is of the form $g\big(X_T, \mathbb{P}_{X_T}^{W^0}\big)$, w being the common initial wealth. To conclude, we arrive at the following mean-field type control problem:

$$dX_t = \big(b_t - \alpha_t\big)\, dt + \sigma_t\, dW_t + \sigma_t^0\, dW_t^0, \quad t \in [0,T], \quad X_0 = x^{(0)},$$

subject to the cost functional

$$W_T = \mathbb{E}\bigg[- \int_0^T c(\alpha_t)\, dt + w - \min\Big\{ \lambda\big(X_T - \Lambda\big)^+ \mathbf{1}_{\mathbb{E}[X_T \,|\, W_T^0] > \Lambda}, U \Big\}\bigg],$$

with $-c$ being the running cost and $w - \min\Big\{ \lambda\big(X_T - \Lambda\big)^+ \mathbf{1}_{\mathbb{E}[X_T \,|\, W_T^0] > \Lambda}, U \Big\}$ being the terminal cost. □

These control problems often enjoy the dynamic programming principle, i.e., the value function V satisfies the dynamic programming principle: for every $t \in [0,T]$ and $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$, it holds that

$$\begin{aligned} V(t,\xi) &= \sup_{\alpha \in \mathcal{A}_t} \left\{ \inf_{s \in [t,T]} \mathbb{E}\left[\int_t^s f(r, X_r^{t,\xi,\alpha}, \mathbb{P}_{X_r^{t,\xi,\alpha}}^{W^0}, \alpha_r) dr + V(s, X_s^{t,\xi,\alpha}) \right] \right\} \\ &= \sup_{\alpha \in \mathcal{A}_t} \left\{ \sup_{s \in [t,T]} \mathbb{E}\left[\int_t^s f(r, X_r^{t,\xi,\alpha}, \mathbb{P}_{X_r^{t,\xi,\alpha}}^{W^0}, \alpha_r) dr + V(s, X_s^{t,\xi,\alpha}) \right] \right\}. \end{aligned}$$

This formally leads to the Hamilton-Jacobi-Bellman (HJB) equations:

$$\begin{cases} \partial_t V(t,\mu) + \displaystyle\int_{\mathbb{R}^d} \sup_{a \in A} \Big\{ f(t,x,\mu,a) + b(t,x,\mu,a) \cdot \partial_\mu V(t,\mu)(x) \\ \quad + \dfrac{1}{2} \mathrm{tr}\Big(\sigma(t,x,a) \big[\sigma(t,x,a)\big]^\top \partial_x \partial_\mu V(t,\mu)(x) \Big) \Big\} \mu(dx) + \dfrac{1}{2} \mathrm{tr}\Big[\sigma^0(t) [\sigma^0(t)]^\top \mathcal{H} V(t,\mu) \Big] \\ \quad = 0; \\ V(T,\mu) = \displaystyle\int_{\mathbb{R}^d} g(x,\mu) \mu(dx), \end{cases}$$

where $\mathcal{H}$ is defined as Bayraktar *et al.* (2023b),

$$\mathcal{H}u(\mu) := \frac{d^2}{dw^2} u((w + I_d)_\sharp \mu)\Big|_{w=0} \in \mathbb{R}^{d\times d},$$

and $(I_d + w)_\sharp \mu$ is the pushforward measure associated with the map $x \mapsto x + w$; and $\frac{d}{dw}$ is the usual gradient operator on $\mathbb{R}^d$. For sufficiently smooth functions $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, it holds that:

$$\mathcal{H}u(\mu) = \int_{\mathbb{R}^d} \partial_x \partial_\mu u(\mu)(x)\mu(dx) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial^2_\mu u(\mu)(x, \tilde{x})\mu(dx)\mu(d\tilde{x}).$$

It is well-known that HJB equations in Wasserstein space generally do not have classical solutions. Here we provide a simple example. Define $k(x) := \max\{-1, \min\{1, x\}\}$. Consider the following simple dynamics:

$$\begin{cases} dX_s = \alpha_s k(X_s)ds, \\ X_t = \xi, \end{cases} \tag{1.1.3}$$

where $\alpha_s \in [-1, 1]$ and is adapted to $\sigma(\xi, W_s)$ (although W_s is not in the dynamics). We consider the cost functional to be $J(t, \xi, \alpha) := \mathbb{E}[k(X_T)]$. The ODE has no other sorts of randomness besides ξ (as we can choose α_s as a function of ξ only), we can simply think of it as a collection of ODEs starting at $\xi(\omega)$. We consider cases, if $1 < \xi(\omega)$, in order to maximize it, we choose $\alpha = 1$, then X_s will be $\xi + s - t$. If $0 < \xi(\omega) < 1$, we still choose $\alpha = 1$, then $X_s = \xi e^{\tau^+ - t} + s - \tau^+$, where τ^+ is when X_s hit 1. On the other hand, if $\xi(\omega) < -1$, in order to maximize it, we choose $\alpha = -1$, then X_s will be $[\xi + (\tau^- - t)]1_{\tau^- = s} + -1e^{-(s-\tau^-)}1_{\tau^- < s}$, where τ^- is the time when X_s hits -1. If $-1 < \xi(\omega) < 0$, we still choose $\alpha = -1$, then X_s will be $\xi e^{-(s-t)}$. Thus

$$\begin{aligned} X_T = &\Big[\xi + T - t\Big]1_{\xi>1} + \Big[\xi e^{\tau^+ - t} + T - \tau^+\Big]1_{0<\xi<1} + \xi e^{-(T-t)}1_{-1<\xi<0} \\ &+ \Big[[\xi + (\tau^- - t)]1_{\tau^- = T} + -1e^{-(T-\tau^-)}1_{\tau^- < T}\Big]1_{\xi\le -1}, \end{aligned}$$

with $\tau^+ := \min\{T, t - \log \xi\}$, $\tau^- := \min\{T, -1 - \xi + t\}$. Thus the value function V of this

control problem reads

$$\begin{aligned}V(t,\mu) =&\mathbb{E}k\Bigg(\Big[\xi+T-t\Big]1_{\xi\geq 1}+\Big[\xi e^{\tau^+-t}+T-\tau^+\Big]1_{0<\xi<1}\\&+\xi e^{-(T-t)}1_{-1<\xi<0}+\Big[[\xi+(\tau^- -t)]1_{\tau^-=T}+-1e^{-(T-\tau^-)}1_{\tau^-<T}\Big]1_{\xi\leq -1}\Bigg)\\=&\int_{[e^{t-T},\infty)}\mu(dx)+\int_{(0,e^{t-T})}xe^{T-t}\,\mu(dx)-\int_{(-\infty,-T-1+t]}1\,\mu(dx)\\&-\int_{(-T-1+t,-1]}e^{-(T-(-1-x+t))}\,\mu(dx)+e^{-(T-t)}\int_{(-1,0)}x\,\mu(dx).\end{aligned}$$

It can be easily seen that V is not Lions differentiable at measure with $\mu(\{0\})>0$, hence it is not possible for V to be a classical solution to any PDE. However, V can still be shown to be the solution of the following PDE in the sense of *viscosity solution*, which is our main focus of this thesis.

$$\begin{cases}\partial_t u(t,\mu)+\displaystyle\int_{\mathbb{R}}\sup_{a\in[-1,1]}\Big[ak(x)\cdot\partial_\mu u(t,\mu)(x)\Big]\mu(dx)=0,\\ u(T,\mu)=\displaystyle\int k(x)\mu(dx).\end{cases}\tag{1.1.4}$$

1.1.2 Literature Review

Mean field type (McKean–Vlasov) control problems and the closely related mean field games (MFGs), introduced independently by Huang *et al.* (2006) and Lasry & Lions (2007), model strategic decision making in large populations of (approximately) symmetric agents interacting through an aggregate, or *mean field*, term. The measure dependence in both the state dynamics and the objective naturally leads to infinite-dimensional formulations. We refer the reader to the monographs of Bensoussan *et al.* (2013), Carmona & Delarue (2018a,b), and Gomes *et al.* (2016) for comprehensive overviews.

Two classical approaches underpin stochastic control: the Pontryagin maximum principle and the dynamic programming principle leading to Hamilton–Jacobi–Bellman (HJB) equations. The maximum principle provides necessary (and sometimes sufficient) optimality conditions, typically via mean-field forward–backward SDEs; see, among many others, An-

dersson & Djehiche (2011); Li (2012); Buckdahn *et al.* (2011); Carmona & Delarue (2015); Bensoussan *et al.* (2023a,b); Gangbo *et al.* (2022); Mou & Zhang (n.d.); Chassagneux *et al.* (2022); Bayraktar *et al.* (2018, 2023c). The PDE route connects the value function to an HJB equation. Beyond special cases (e.g., linear–quadratic settings), classical solutions to HJB equations are generally out of reach; the viscosity-solution framework of Crandall *et al.* (1992); Lions (1983) therefore plays a central role. Foundational results in Euclidean and Hilbert spaces include Crandall & Lions (1983, 1985); Cannarsa & Soner (1987).

In the Wasserstein space of probability measures, viscosity theory faces two structural obstacles: the lack of local compactness and the non-smoothness of the Wasserstein distance. Several lines of work address these. Pham & Wei (2017) developed dynamic programming with controls adapted to the common-noise filtration and proved well-posedness by *lifting* the HJB to a Hilbert space of random variables. To avoid lifting, Wu & Zhang (2020) proposed a notion of viscosity solutions with maxima/minima enforced over compact subsets of the Wasserstein space, enabling a comparison principle in the first-order case. In another direction, Cosso *et al.* (2024) tried to adapt the Crandall–Lions definition using the Borwein–Preiss variant of Ekeland's variational principle to overcome local compactness issues via smooth gauge perturbations.

Common noise introduces second-order (L-) derivatives in the HJB. Exploiting that the second-order L-derivative enters through a finite-dimensional integral operator (the "partial Hessian"), Bayraktar *et al.* (2023b)—building on the Fourier–Wasserstein metric from Soner & Yan (2024a)—established an Ishii lemma directly in the measure space and obtained comparison for a class of second-order equations, albeit under strong regularity assumptions tailored to that metric. Using a finite-dimensional approximation and delicate smoothing estimates, Daudin *et al.* (2023) proved well-posedness for semi-linear Hamilton–Jacobi equations on the torus; see also convergence results in Bayraktar *et al.* (2024). More recently, Zhou *et al.* (2024) introduced a new viscosity notion on the lifted *process* space, adding a

singular component to test functions to handle second-order terms without Ishii's lemma, and proved existence and comparison under Lipschitz assumptions. Related intrinsic approaches include Bertucci (2024), who established well-posedness for equations of the form $H(\mu, \partial_\mu u) + \frac{\sigma(t)^2}{2} \mathcal{H} u$ under suitable structure, and Bertucci & Lions (2024), who obtained comparison for equations involving $\partial_\mu u$ and $\nabla_x \partial_\mu u$ via sup-convolution regularity.

1.1.3 Summary of our main results

We prove that the value function V in (1.1.2) is the unique viscosity solution of some HJB equations under different circumstances.

In Chapter 2, we are concerned with the functions:

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \quad \sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times d}, \quad \sigma^0 : [0,T] \to \mathbb{R}^{d\times d},$$

$$f : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}, \qquad g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}.$$

We use the following assumptions:

Assumption. (A). *The functions b, σ, σ^0, f and g satisfy the following:*

(1). the functions b, σ, σ^0, f and g are continuous;

(2). there exist constants $K \geq 0$ and $\beta \in (0,1]$ such that for any $a \in A$, (t,x,μ), $(t',x',\mu') \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, it holds that

$$\begin{aligned}
&\big|b(t,x,\mu,a) - b(t',x',\mu',a)\big| + \big|\sigma(t,x,a) - \sigma(t',x',a)\big| + \big|\sigma^0(t) - \sigma^0(t')\big| \\
&+ \big|f(t,x,\mu,a) - f(t',x',\mu',a)\big| + \big|g(x,\mu) - g(x',\mu')\big| \leq K\big[|x-x'| + |t-t'|^\beta + \mathcal{W}_2(\mu,\mu')\big],
\end{aligned}$$

$$\big|b(t,x,\mu,a)\big| + \big|\sigma(t,x,a)\big| + \big|\sigma^0(t)\big| + \big|f(t,x,\mu,a)\big| + \big|g(x,\mu)\big| \leq K.$$

Assumption. (B). *For any $a \in A$, the function $\sigma(\cdot,\cdot,a)$ belongs to $C^{1,2}\big([0,T] \times \mathbb{R}^d\big)$. Moreover, there exists a constant $K \geq 0$ such that*

$$|\partial_t \sigma(t,x,a)| + |\partial_x \sigma(t,x,a)| + \big|\partial^2_{xx} \sigma(t,x,a)\big| + \big|\partial_t \sigma^0(t)\big| \leq K,$$

for all $(t,x,a) \in [0,T] \times \mathbb{R}^d \times A$.

Assumption. (A*). *Suppose that Assumption (A) holds and we assume further that there exist constants $K \geq 0$ and $\beta \in (0,1]$ such that for any $a \in A$, (t,x,μ), $(t',x',\mu') \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, it holds that*

$$\begin{aligned}&\big|b(t,x,\mu,a) - b(t',x',\mu',a)\big| + \big|f(t,x,\mu,a) - f(t',x',\mu',a)\big| + \big|g(x,\mu) - g(x',\mu')\big| \\ &\leq K\big[|x-x'| + |t-t'|^\beta + \mathcal{W}_1(\mu,\mu')\big].\end{aligned}$$

Remark 1.1.1. *Assumption (A*) is a strengthening assumption of Assumption (A). We write Assumption (A) and Assumption (A*) separately to emphasize that some of the lemmas and theorems still hold without assuming the stronger condition in Assumption (A*).*

Also, we set $\mathbb{F}^t = (\mathcal{F}_s^t)_{s\geq 0} := (\sigma(W_r^0)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$. Let A be a compact subset of the Euclidean space $\mathbb{R}^d$ equipped with the distance d_A. Let $t > 0$ and denote by $\mathcal{A}_t$ the set of $\mathbb{F}^t$-progressively measurable processes on Ω valued in A.

For every $t \in [0,T]$, $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ and $\alpha \in \mathcal{A}_t$, we consider the solution $X^{t,\xi,\alpha}$ of the following state dynamics:

$$X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}_{X_r}^{W^0}, \alpha_r)dr + \int_t^s \sigma(r, X_r, \alpha_r)dW_r + \int_t^s \sigma^0(r)dW_r^0, \quad \text{for } s \in [t,T], \tag{1.1.5}$$

where $\mathbb{P}_{X_r}^{W^0}$ denotes the conditional distribution of X_r given W^0. We are subject to the payoff functional:

$$J(t,\xi,\alpha) := \mathbb{E}\bigg[\int_t^T f\Big(s, X_s^{t,\xi,\alpha}, \mathbb{P}_{X_s^{t,\xi,\alpha}}^{W^0}, \alpha_s\Big)ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}_{X_T^{t,\xi,\alpha}}^{W^0}\Big)\bigg]. \tag{1.1.6}$$

Finally, we define the value function V to be

$$V(t,\xi) := \sup_{\alpha\in\mathcal{A}_t} J(t,\xi,\alpha), \quad \text{for any } (t,\xi) \in [0,T] \times L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d). \tag{1.1.7}$$

We showed that V is the unique viscosity solution to the following HJB:

$$\begin{cases} -\partial_t u(t,\mu) = \dfrac{1}{2}\mathrm{tr}\Big[\sigma^0(t)[\sigma^0(t)]^\top \mathcal{H} u(t,\mu)\Big] \\ + \displaystyle\int_{\mathbb{R}^d} \sup_{a\in A}\Big\{ f(t,x,\mu,a) + b(t,x,\mu,a)\cdot \partial_\mu u(t,\mu)(x) \\ \qquad\qquad + \dfrac{1}{2}\mathrm{tr}\Big(\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top \partial_x\partial_\mu u(t,\mu)(x)\Big)\Big\}\mu(dx)\,, \quad (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \displaystyle\int_{\mathbb{R}^d} g(x,\mu)\mu(dx)\,, \quad \mu\in\mathcal{P}_2(\mathbb{R}^d). \end{cases} \tag{1.1.8}$$

For details, readers are referred to Chapter 2.

In Chapter 3, we are concerned with the functions:

$$b:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}^d, \quad \sigma:[0,T]\times\mathbb{R}^d\times A\to\mathbb{R}^{d\times d}, \quad \sigma^0:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d},$$

$$f:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}, \quad g:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R},$$

where A is a compact subset of the Euclidean space $\mathbb{R}^d$ equipped with the distance d_A. The coefficient functions satisfy the following assumptions:

Assumption. *The functions b, σ, σ^0, f and g are jointly continuous in $(t,x,\mu,a)\in([0,T],|\cdot|)\times(\mathbb{R}^d,|\cdot|)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)\times(A,|\cdot|)$. Moreover, there exist constants $K\geq 0$, $\rho\in[0,1)$ and $\beta\in(0,1]$ such that for any $a\in A$, (t,x,μ), $(t',x',\mu')\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, it holds that*

(1). $\big|b(t,x,\mu,a)-b(t',x',\mu',a)\big| + \big|\sigma(t,x,a)-\sigma(t',x',a)\big| + \big|\sigma^0(t,x)-\sigma^0(t',x')\big|$
$+\big|f(t,x,\mu,a)-f(t',x',\mu',a)\big| + \big|g(x,\mu)-g(x',\mu')\big| \leq K\big[|x-x'|+|t-t'|^\beta+\mathcal{W}_1(\mu,\mu')\big];$

(2). $\big|b(t,x,\mu,a)\big| + \big|\sigma(t,x,a)\big| + \big|\sigma^0(t,x)\big| \leq K(1+|x|^\rho);$

(3). $\big|f(t,x,\mu,a)\big| + \big|g(x,\mu)\big| \leq K.$

Assumption. *For any $a\in A$, the functions $\sigma(\cdot,\cdot,a)$ and $\sigma^0(\cdot,\cdot)$ belong to $C^{1,2}\big([0,T]\times\mathbb{R}^d\big)$. Moreover, there exists a constant $K\geq 0$ such that*

$$|\partial_t\sigma(t,x,a)| + |\nabla_x\sigma(t,x,a)| + \big|\nabla^2_{xx}\sigma(t,x,a)\big| + \big|\partial_t\sigma^0(t,x)\big| + \big|\nabla_x\sigma^0(t,x)\big| + \big|\nabla^2_{xx}\sigma^0(t,x)\big| \leq K,$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times A$.

Remark 1.1.2. *The growth rate condition* $\rho \in [0, 1)$ *in Assumption 3.1.2 will be used for proving the uniqueness of viscosity solution in Theorem 3.3.3. Specifically, when controlling the error term of the perturbation of solution in* (3.3.21)*, we are unable to achieve a sufficient decay rate if* $\rho = 1$*. Furthermore, we impose the* $\mathcal{W}_1$*-Lipschitz continuity to obtain better convergence results in the finite-dimensional particle approximation in Lemma 3.2.2. The* $\mathcal{W}_2$*-Lipschitz continuity is insufficient, as noted in (Cheung* et al.*, 2025, Remark 2.5). See also the second part of Remark 3.1.6 for a reason on the compactness issue.*

Also, we set $\mathbb{F}^t = (\mathcal{F}_s^t)_{s\geq 0} := (\sigma(W^0_{r\vee t} - W^0_t)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$. Let A be a compact subset of the Euclidean space $\mathbb{R}^d$ equipped with the distance d_A. Let $t > 0$ and denote by $\mathcal{A}_t$ the set of $\mathbb{F}^t$-progressively measurable processes on Ω valued in A.

We now introduce a mean field control problem related to the HJB equation (3.0.1). For every $t \in [0, T]$, $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ and $\alpha \in \mathcal{A}_t$, we consider the solution $X^{t,\xi,\alpha}$ of the following state dynamics:

$$X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}^{W^0}_{X_r}, \alpha_r)dr + \int_t^s \sigma(r, X_r, \alpha_r)dW_r + \int_t^s \sigma^0(r, X_r)dW^0_r, \quad \text{for } s \in [t, T], \tag{1.1.9}$$

where $\mathbb{P}^{W^0}_{X_r}$ denotes the conditional distribution of X_r given W^0. We are subject to the cost functional:

$$J(t, \xi, \alpha) := \mathbb{E}\bigg[\int_t^T f\Big(s, X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s\Big)ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}\Big)\bigg]. \tag{1.1.10}$$

We define the value function V to be

$$V(t, \xi) := \sup_{\alpha\in\mathcal{A}_t} J(t, \xi, \alpha), \quad \text{for any } (t, \xi) \in [0, T] \times L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d). \tag{1.1.11}$$

We showed that V is the unique viscosity solution to the HJB equation

$$\begin{cases} \partial_t u(t,\mu) + \int_{\mathbb{R}^d} \sup_{a\in A} \Big\{ f(t,x,\mu,a) + b(t,x,\mu,a)\cdot \partial_\mu u(t,\mu)(x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\Big((\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top + \sigma^0(t,x)[\sigma^0(t,x)]^\top)\nabla_x \partial_\mu u(t,\mu)(x)\Big)\Big\}\mu(dx) \\ + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{tr}\Big[\sigma^0(t,x)[\sigma^0(t,y)]^\top \partial_\mu^2 u(t,\mu)(x,y)\Big]\,\mu^{\otimes 2}(dx,dy) = 0, \text{ for any } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \int_{\mathbb{R}^d} g(x,\mu)\mu(dx) \quad \text{for } \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases}$$

Note that, compared to Chapter 2, we allow for the common noise volatility to be dependent on the state rather than only explicitly on time. Moreover, we allow the state dynamics to be unbounded. For details, readers are referred to Chapter 3.

In Chapter 5 (which is our future work), we are in concern with the following settings:

$$b : [0,T]\times\Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}^d, \quad \sigma,\,\sigma^0 : [0,T]\times\Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}^{d\times d},$$

$$f : [0,T]\times\Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}, \quad g : \Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}.$$

For any $t\in[0,T]$, $\xi\in L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)$ and $\alpha\in\mathcal{A}_t$, we consider the process $X^{t,\xi,\alpha}$ evolving as follows:

$$\begin{cases} dX_s^{t,\xi,\alpha} &= b_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + \sigma_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s \\ &\quad + \sigma_s^0(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s^0, \quad \text{for } s\in[t,T]; \\ X_t^{t,\xi,\alpha} &= \xi. \end{cases} \tag{1.1.12}$$

The cost functional J on $[0,T]\times L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)\times\mathcal{A}_t\times\Omega^0$ is defined by

$$J(t,\xi,\alpha) := \mathbb{E}\Big[\int_t^T f_s\Big(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s\Big)ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}\Big)\,\Big|\,\mathcal{F}_t^0\Big]. \tag{1.1.13}$$

We study the corresponding value function v on $[0,T]\times L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)\times\Omega^0$:

$$v(t,\xi) := \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_t} J(t,\xi,\alpha). \tag{1.1.14}$$

The following assumptions will be used throughout our work.

Assumption ($\mathcal{A}1$)**.** *The coefficient functions g and $h = f$, b, σ or σ^0 satisfy the following:*

(1). For all $a \in A$, $\big(h_t(\cdot,\cdot,a)\big)_{0\le t\le T}$ is $(\mathcal{F}^0_t)_{t\ge 0}$-adapted. (or progressively measurable)

(2). For $\mathcal{L}^1 \otimes \mathbb{P}^0$-a.e. $(t,\omega^0) \in [0,T]\times\Omega^0$, $h_t(\cdot,\cdot,\cdot)$ is continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$.

(3). There exists $K > 0$ such that for all x, $x' \in \mathbb{R}^d$, $t \in [0,T]$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ such that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\omega^0\in\Omega^0} |g(x,\mu)| + \operatorname*{ess\,sup}_{\omega^0\in\Omega^0} \Big(\sup_{a\in A} |h_t(x,\mu,a)| \Big) \le K \quad \text{and} \\
&\operatorname*{ess\,sup}_{\omega^0\in\Omega^0} |g(x,\mu) - g(x',\mu')| + \operatorname*{ess\,sup}_{\omega^0\in\Omega^0} \Big(\sup_{a\in A} |h_t(x,\mu,a) - h_t(x',\mu',a)| \Big) \\
&\le K(|x-x'| + \mathcal{W}_2(\mu,\mu')).
\end{aligned}$$

Assumption ($\mathcal{A}1^*$)**.** *This is a strengthening assumption of ($\mathcal{A}1$). In addition to ($\mathcal{A}1$), we further assume that there exists $K > 0$ such that for all x, $x' \in \mathbb{R}^d$, $t \in [0,T]$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, it holds that*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\omega^0\in\Omega^0} |g(x,\mu) - g(x',\mu')| + \operatorname*{ess\,sup}_{\omega^0\in\Omega^0} \Big(\sup_{a\in A} |h_t(x,\mu,a) - h_t(x',\mu',a)| \Big) \\
&\le K(|x-x'| + \mathcal{W}_1(\mu,\mu'))
\end{aligned}$$

Assumption ($\mathcal{A}3$)**.** *We need the following assumption in order to ensure the approximation results like Lemma 5.1.6.*

(1). The map $g : (\Omega^0, \mathcal{F}^0_T, \mathbb{P}^0) \to C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is Bochner measurable.

(2). For all $t \in [0,T]$, the maps $\big(\Omega^0, \mathcal{F}^0_t, \mathbb{P}^0\big) \ni \omega^0 \mapsto b_t(\omega^0,\cdot,\cdot,\cdot) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$, $\big(\Omega^0, \mathcal{F}^0_t, \mathbb{P}^0\big) \ni \omega^0 \mapsto f_t(\omega^0,\cdot,\cdot,\cdot) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$ and $\big(\Omega^0, \mathcal{F}^0_t, \mathbb{P}^0\big) \ni \omega^0 \mapsto \sigma_t(\omega^0,\cdot,\cdot) \in C(\mathbb{R}^d \times A)$ are Bochner measurable.

(3). For $h = f$, b, σ, $(\Omega^0 \times [0,T], \mathcal{F}^0 \otimes \mathcal{B}[0,T], \mathbb{P}^0 \otimes ds) \ni (\omega^0, t) \mapsto h_t(\omega^0,\cdot,\cdot,\cdot) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$ is progressively measurable.

Our goal is to showed that v is the unique viscosity solution to the SHJB equation:

$$\begin{cases} -\eth_t v(t,\mu) = \left(\int_{\mathbb{R}^d} \operatorname*{ess\,inf}_{a\in A} \left\{ f_t(x,\mu,a) + \Big\langle b_t(x,\mu,a), \partial_\mu v(t,\mu)(x) \Big\rangle + \frac{1}{2}\mathrm{tr}\Big\{ (\sigma_t\sigma_t^\intercal(x,\mu,a) \right. \right. \\ \left. + \sigma_t^0\sigma_t^{0;\intercal}(x))\partial_x\partial_\mu v(t,\mu)(x) \Big\} + \int_{\mathbb{R}^d} \frac{1}{2}\mathrm{tr}\Big\{ \sigma_t^0(x)\sigma_t^{0;\intercal}(x')\partial_\mu^2 v(t,\mu)(x,x') \Big\}\mu(dx') \\ \left. \left. + \mathrm{tr}\Big\{ \sigma_t^{0;\intercal}(x)\partial_\mu \eth_w v(t,\mu)(x) \Big\} \right\}\mu(dx) \right) dt, \\ v(T,\mu) = \int_{\mathbb{R}^d} g(x,\mu)\mu(dx). \end{cases} \tag{1.1.15}$$

We have finished the existence part, while leaving the uniqueness part. Details can be found in Chapter 5.

1.2 Part II: A Feynman-Kac Formula for Nonlinear Schrödinger Equations

1.2.1 The Feynman–Kac Formula for Linear Parabolic Equations

Let $(W_s)_{s\geq 0}$ be a one-dimensional Brownian motion on a filtered probability space satisfying the usual conditions. Consider the Itô diffusion

$$dX_s^{t,x} = \mu(s, X_s^{t,x})\, ds + \sigma(s, X_s^{t,x})\, dW_s, \qquad X_t^{t,x} = x,$$

where $\mu, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable and satisfy the standard Lipschitz and linear-growth conditions ensuring a unique strong solution. Define the (time-inhomogeneous) generator

$$(\mathcal{L}_s\phi)(x) := \mu(s,x)\, \partial_x\phi(x) + \tfrac{1}{2}\sigma^2(s,x)\, \partial_{xx}\phi(x).$$

Given $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ of at most polynomial growth, consider the terminal-value problem

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}_t u(t,x) + f(t,x) = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{1.2.1}$$

Assume $u \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ with polynomial growth.

Theorem 1.2.1 (Feynman–Kac)**.** *For all* $(t,x) \in [0,T] \times \mathbb{R}$*,*

$$u(t,x) = \mathbb{E}\left[\psi\big(X_T^{t,x}\big) + \int_t^T f\big(s, X_s^{t,x}\big)\, ds\right]. \tag{1.2.2}$$

Proof sketch. By Itô's formula applied to $s \mapsto u(s, X_s^{t,x})$,

$$du(s, X_s^{t,x}) = \Big(\partial_t u + \mathcal{L}_s u\Big)(s, X_s^{t,x})\, ds + \partial_x u(s, X_s^{t,x})\, \sigma(s, X_s^{t,x})\, dW_s.$$

Using (1.2.1) gives

$$du(s, X_s^{t,x}) = -f(s, X_s^{t,x})\, ds + \partial_x u(s, X_s^{t,x})\, \sigma(s, X_s^{t,x})\, dW_s.$$

Integrate from t to T and take conditional expectation given $X_t^{t,x} = x$; the stochastic integral is a martingale of mean zero, yielding (1.2.2). □

It is often convenient to write the representation in forward–backward SDE form. Define

$$Y_s := u(s, X_s^{t,x}), \qquad Z_s := \partial_x u(s, X_s^{t,x})\, \sigma(s, X_s^{t,x}).$$

Then (X, Y, Z) satisfies

$$\begin{cases} dX_s = \mu(s, X_s)\, ds + \sigma(s, X_s)\, dW_s, & X_t = x, \\ dY_s = -f(s, X_s)\, ds + Z_s\, dW_s, & Y_T = \psi(X_T), \end{cases}$$

and in particular $Y_t = u(t,x)$.

Remarks. (1) The multidimensional extension has $\mathcal{L}_t u = \mu \cdot \nabla u + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^\top)\nabla^2 u\big)$. (2) Discounting, killing, and boundary conditions are incorporated by adding a term $-r(t,x)u$ to the PDE or by stopping at boundary hitting times.

1.2.2 Schrödinger Equations and its Feynman-Kac Representation

The Schrödinger equation is a fundamental equation in quantum mechanics that describes the temporal evolution of the quantum state of a physical system. It is central to the theory of quantum mechanics and essential for the development of quantum theory, with significant

applications in fields such as chemistry, materials science, and quantum computing. In this work, we consider the following Cauchy problem for deterministic backward nonlinear time-dependent Schrödinger equation for the wave function of a quantum mechanical system:

$$i\partial_t u(t,x) = \frac{\nu}{2}\Delta u(t,x) + f(t,x,u), \quad t \le T; \quad u(T,x) = G(x), \tag{1.2.3}$$

which is equivalent to the classical Schrödinger equation via the time-reversing transformation: $u(t,x) \longmapsto u(T-t,x)$, for $t \le T$, $x \in \mathbb{R}^d$. Here and throughout this work, $\nu > 0$ is a real constant and $f(t,x,u)$ is a continuous function with polynomial growth in u, for instance, as in Kato (1987). The unknown function $u : [0,T] \times \mathbb{R}^d \to \mathbb{C}$ is complex-valued and is written as $u(t,x) = u^R(t,x) + iu^I(t,x)$, where both u^R and u^I are real-valued and defined on $[0,T] \times \mathbb{R}^d$. Analogously, we write

$$f(t,x,y) = f^R(t,x,y) + if^I(t,x,y), \quad G(x) = G^R(x) + iG^I(x), \quad (t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{C},$$

with f^R, f^I, G^R, and G^I being real-valued continuous functions. Then the nonlinear Schrödinger equation above may be written as a coupled system of partial differential equations:

$$\begin{cases} -\partial_t u^R(t,x) = -\dfrac{\nu}{2}\Delta u^I(t,x) - f^I(t,x,u), \quad t \le T; \quad u^R(T,x) = G^R(x); \\ -\partial_t u^I(t,x) = \dfrac{\nu}{2}\Delta u^R(t,x) + f^R(t,x,u), \quad t \le T; \qquad u^I(T,x) = G^I(x). \end{cases} \tag{1.2.4}$$

Associated to (1.2.4), we propose the following backward stochastic differential equation (BSDE):

$$\begin{cases} -dY_s^R(t,x) = -\sqrt{\nu}(Z_s^R(t,x) + Z_s^I(t,x)) * dW_s \\ \qquad\qquad - f^I(s, \sqrt{\nu}(W_s - W_t) + x, Y_s(t,x))\, ds - \sqrt{\nu} Z_s^R(t,x)\, dW_s; \\ -dY_s^I(t,x) = \sqrt{\nu}(Z_s^R(t,x) - Z_s^I(t,x)) * dW_s \\ \qquad\qquad + f^R(s, \sqrt{\nu}(W_s - W_t) + x, Y_s(t,x))\, ds - \sqrt{\nu} Z_s^I(t,x)\, dW_s; \\ Y_T(t,x) = G(\sqrt{\nu}(W_T - W_t) + x), \end{cases} \tag{1.2.5}$$

where $Y_s(t,x) = Y_s^R(t,x) + iY_s^I(t,x)$, $Z_s(t,x) = Z_s^R(t,x) + iZ_s^I(t,x)$, and $*dW_s := \circ dW_s - dW_s$ with $\circ dW_s$ and dW_s being Fisk-Stratonovich integral and Itô integral, respectively. They are

related to each other by the following:

$$Y_s(t,x) = u(s, \sqrt{\nu}(W_s - W_t) + x), \quad Z_s(t,x) = \nabla u(s, \sqrt{\nu}(W_s - W_t) + x), \tag{1.2.6}$$

for all $(s,x) \in [t,T] \times \mathbb{R}^d$. The system (1.2.5) can be viewed as an uncoupled forward-backward stochastic differential equation (FBSDE) but is distinguished by the inclusion of nonstandard integrals $*dW_s$. The term "uncoupled" indicates that the backward process does not affect the forward dynamics that is nothing but the Wiener process $(W_t)_{t\geq 0}$, in contrast to "coupled" systems where such an interaction is present.

1.2.3 Literature Review

It is now well-established that FBSDEs are intrinsically linked to a large class of parabolic partial differential equations (PDEs) with real-valued coefficients, which may be linear or nonlinear (see, for example, Cheridito *et al.* (2007); Delarue (2002); Delbaen *et al.* (2015); Hu & Peng (1995); Ma *et al.* (1994); Pardoux & Peng (1992); Pardoux & Tang (1999)). These links yield generalized Feynman-Kac formulas for the corresponding parabolic PDEs, enabling their resolutions to be reformulated as solving FBSDEs (see Cheridito *et al.* (2007); Lejay & González (2020) for instance). Recently, a class of deep learning-based (also called deep-BSDE) methods has been introduced to address the curse of dimensionality often encountered in numerical approximations of such parabolic PDEs; see Beck *et al.* (2019); E *et al.* (2017); Han *et al.* (2018); Huré *et al.* (2020); Zhang (2004) among others. However, this framework does not extend to time-dependent Schrödinger equations such as (1.2.3), due to the presence of the imaginary coefficient before the time derivative. As a result, existing deep-BSDE methods are not directly applicable in this context. This limitation along with the broader challenge in numerical approximations of high-dimensional time-dependent nonlinear Schrödinger equations, exists as an open problem in the seminal work Han *et al.* (2018) and further discussed in Han *et al.* (2024).

This chapter addresses the gap in the availability of a Feynman-Kac representation for non-

linear time-dependent Schrödinger equations. Based on this representation, we propose and investigate learning-based approaches which effectively tackle the curse of dimensionality in numerical simulations. To the best of our knowledge, we are the first to give a Feynman-Kac formula for time-dependent nonlinear Schrödinger equations using BSDEs. This is novel even for the linear case as both complex time transformation and Feynman path integrals (cf. (Albeverio *et al.* (1976); Feynman (1948); Kac (1949); Yan (1994))) are avoided in the representation. Inspired by the deep-BSDE methods introduced in seminal works Beck *et al.* (2019); E *et al.* (2017); Han *et al.* (2018); Huré *et al.* (2020), we propose a novel learning-based approach for numerical approximations for the Schrödinger equation (1.2.3). Our numerical experiments on such BSDEs demonstrate strong agreement with the explicit solutions of the linear and nonlinear Schrödinger equations in both low- and high-dimensional settings, as demonstrated in Section 4.2.2. Furthermore, a convergence analysis is provided to support these findings. In the numerical experiments and convergence analysis, we utilize neural network approximations for the unknown functions, enabling the algorithm and convergence analysis to be extended to high-dimensional time-dependent Schrödinger equations; in this context, deep neural networks are instrumental in addressing the curse of dimensionality (cf. Han *et al.* (2018); Isozaki (2004)).

The integrals with $*dW_t$ in BSDE (1.2.5), which combine Fisk-Stratonovich and Itô integrals, capture the leading differential operators, thereby reflecting imaginary coefficient and wave-like behavior of Schrödinger equations. This feature poses significant challenges in both theoretical study and numerical analysis. On the theoretical side, the Feynman-Kac formula is proved for both classical and weak solutions, while several open questions are raised regarding the pure probabilistic resolution of BSDE (1.2.5) in Section 4.1.3. From a numerical perspective, this feature distinguishes the proposed learning-based approach (see Algorithm 1 and Remark 4.2.1) from those tailored to non-Schrödinger-type parabolic PDEs (cf. Beck *et al.* (2019); E *et al.* (2017); Han *et al.* (2018); Huré *et al.* (2020)). For the

challenges in numerical analysis, some non-standard calculations are introduced to derive estimates in the convergence analysis (see Remark 4.3.2). For other numerical methods for nonlinear Schrödinger equations, we refer to Bao & Wang (2024); Ji *et al.* (2024); Li & Wu (2021); Ostermann & Yao (2022) and references therein.

1.2.4 Main Goal

we present a novel Feynman-Kac formula and investigate learning-based methods for approximating general nonlinear time-dependent Schrödinger equations which may be high-dimensional. Details can be found in Chapter 4.

Chapter 2

Viscosity Solutions of a class of Second Order Hamilton-Jacobi-Bellman Equations in the Wasserstein Space

The primary objective of this chapter is to establish the well-posedness of viscosity solutions for second order Hamilton-Jacobi-Bellman (HJB) equations, which arise from control problems in the space of probability measures over $\mathbb{R}^d$, under a newly proposed notion (see Definition 2.5.1 and Remark 2.5.2 for the rationale behind this definition). This is achieved under mild assumptions on the coefficients, as detailed in Section 2.3. The main HJB equation of interest is given by:

$$\begin{cases} -\partial_t u(t,\mu) = \frac{1}{2}\mathrm{tr}\Big[\sigma^0(t)[\sigma^0(t)]^\top \mathcal{H}u(t,\mu)\Big] \\ + \int_{\mathbb{R}^d} \sup_{a\in A}\Big\{ f(t,x,\mu,a) + b(t,x,\mu,a)\cdot \partial_\mu u(t,\mu)(x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\Big(\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top \partial_x\partial_\mu u(t,\mu)(x)\Big)\Big\}\mu(dx)\,, \quad (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \int_{\mathbb{R}^d} g(x,\mu)\mu(dx)\,, \quad \mu\in\mathcal{P}_2(\mathbb{R}^d). \end{cases} \tag{2.0.1}$$

In the above, the operator $\mathcal{H}$ is defined as in Bayraktar *et al.* (2023b), that is,

$$\mathcal{H}u(\mu) := \frac{d^2}{dw^2} u\Big((I_d + w)_\sharp \mu\Big)\Big|_{w=0} \in \mathbb{R}^{d\times d}, \tag{2.0.2}$$

where $(I_d + w)_\sharp \mu$ is the pushforward measure associated with map $x \mapsto x + w$; and $\frac{d}{dw}$ is the usual gradient operator on $\mathbb{R}^d$. Indeed, for sufficiently smooth functions $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, it holds that (see Lemma 2.1.5):

$$\mathcal{H}u(\mu) = \int_{\mathbb{R}^d} \partial_x \partial_\mu u(\mu)(x)\mu(dx) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_\mu^2 u(\mu)(x,\tilde{x})\mu(dx)\mu(d\tilde{x}),$$

and thus $\mathcal{H}$ can be viewed as the second order differential operator with respect to the measure variable in a "weaker" sense. We refer readers to Section 2 for notations used in the above. This HJB equation characterizes the value function for the following mean field type control problem with common noise:

$$\sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[\int_t^T f\Big(s, X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s\Big) ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}\Big)\right]$$

subject to the dynamics

$$X_s^{t,\xi,\alpha} = \xi + \int_t^s b\Big(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r\Big) dr + \int_t^s \sigma(r, X_r^{t,\xi,\alpha}, \alpha_r) dW_r + \int_t^s \sigma^0(r) dW_r^0\,,$$

where W and W^0 are independent Wiener processes representing idiosyncratic and common noises respectively; and $\mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}$ denotes the conditional distribution of $X_r^{t,\xi,\alpha}$ given W^0.

To establish the uniqueness of the viscosity solution to (2.0.1), a pivotal idea is to seek a suitable comparison function with adequate regularity and favorable estimates on its derivatives, for the purpose of the comparison theorem. Drawing inspiration from Cosso *et al.* (2024), we choose a gauge-type function to construct the comparison function via the Borwein-Preiss variational principle. However, the gauge function used in Cosso *et al.* (2024) encounters difficulties when dealing with the second order L-derivative, particularly in addressing mean field-type control problems involving common noise. To resolve this, we adopt the Gaussian regularized sliced Wasserstein distance proposed in Bayraktar *et al.* (2023a) to compose our gauge function. This provides a metric over the space of high-dimensional probability distributions using their one-dimensional projections, offering the advantage of explicitly deriving the optimal transport map, which facilitates estimates of the derivatives. We observe that

when the state and the law of the first order L-derivative of this metric move together in a deterministic direction (which is captured by the operator $\mathcal{H}$, see Lemma 2.3.7), the gauge function behaves linearly with respect to the perturbed direction, enabling the estimation of the second order L-derivative of this gauge function in a "weaker" sense. As this gauge function will be used as a test function, we have to define a customized function space which is $PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ (see Definition 2.1.4) as our set of test functions. Then, we perturb the HJB equation and mollify the coefficients as in Cosso *et al.* (2024), together with the favorable estimates derived for the gauge function to complete the comparison theorem, as presented in Theorem 2.5.3. By the specific form of the optimal transport map in $\mathbb{R}$, we provide further regularity results of this metric for the purpose of the relaxed Itô's formula in Theorem 2.1.6, which is crucial in establishing the existence of viscosity solutions. Moreover, this gauge function lacks standard linear or quadratic growth, which requires a more subtle analysis in the proof of the existence of viscosity solution compared to standard test functions. We note that our proposed definition of the viscosity supersolution in Definition 2.5.1 is stronger than that of Crandall-Lions. It aims to overcome technical difficulties arising from the supremum taken in equation (2.0.1), which leads to scarce choice of test functions in the proof of the comparison for the supersolution. The cost functional with a fixed control is used to construct the test function for supersolutions as in Cosso *et al.* (2024), but with the domain of $[0,T]\times\mathcal{P}_2(\mathbb{R}^d\times A)$ rather than $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. The strong dependency of the initial random variable and the control makes it hard to use only the Crandall-Lions' definition to draw the conclusion. See Remark 2.5.2 for more details.

This chapter is organized as follows: Section 2 covers the foundational elements: the control problem set-up, some definitions of our tools, Itô's formula and the standing assumptions. In Section 3, we explore fundamental properties of the value function and introduce the dynamic programming principle. Moving to Section 4, we provide some estimates of the Gaussian regularized sliced Wasserstein distance and state the Borwein-Preiss variational

principle. In Section 5, we outline the finite-dimensional approximation scheme of the value function, drawing inspiration from Cosso *et al.* (2024). Section 6 is dedicated to proving the existence and uniqueness of the viscosity solution, which stands as the value function of our control problem.

2.1 Preliminary

2.1.1 Probabilistic Setting and Notations

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form $(\Omega^0 \times \Omega^1, \mathcal{F}^0 \otimes \mathcal{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$. The space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ supports a d-dimensional Brownian motion W^0 which we regard as the common noise. For $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, it is of the form $(\tilde{\Omega}^1 \times \hat{\Omega}^1, \mathcal{G} \otimes \hat{\mathcal{F}}^1, \tilde{\mathbb{P}}^1 \otimes \hat{\mathbb{P}}^1)$. On $(\hat{\Omega}^1, \hat{\mathcal{F}}^1, \hat{\mathbb{P}}^1)$, there lives a d-dimensional Brownian motion W which we regard as the idiosyncratic noise, whereas $(\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1)$ is where the initial random variables live. We assume that the probability space $(\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1)$ is rich enough to support all probability laws in $\mathbb{R}^d$, i.e., for any probability law μ in $\mathbb{R}^d$, there exists $X(\omega) : \tilde{\Omega}^1 \to \mathbb{R}^d$ such that the law of X, denoted by $\mathcal{L}(X)$, is μ.

We write $\omega \in \Omega$ as $\omega = (\omega^0, \omega^1)$, and regard the Brownian motions $W(\omega) = W(\omega^1)$, $W^0(\omega) = W^0(\omega^0)$. We denote by $\mathbb{E}$ the expectation under $\mathbb{P}$ and by $\mathbb{E}^0$ (resp. $\mathbb{E}^1$) the expectation under $\mathbb{P}^0$ (resp. $\mathbb{P}^1$). Also, we set $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0} := (\sigma(W^0_r)_{0\leq r\leq s} \vee \sigma(W_r)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$, $\mathbb{F}^t = (\mathcal{F}^t_s)_{s\geq 0} := (\sigma(W^0_r)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$, $\mathbb{F}^{W^0} = (\mathcal{F}^{W^0}_s)_{s\geq 0} := (\sigma(W^0_r))_{0\leq r\leq s}$, and $\mathbb{F}^1 = (\mathcal{F}^1_s)_{s\geq 0} := (\sigma(W_s) \vee \mathcal{G})_{0\leq r\leq s}$. Without loss of generality, we assume they are $\mathbb{P}$-complete.

Let A be a compact subset of the Euclidean space $\mathbb{R}^d$ equipped with the distance d_A. Let $t > 0$ and denote by $\mathcal{A}$ (resp. $\mathcal{A}_t$) the set of $\mathbb{F}$-progressively measurable processes (resp. $\mathbb{F}^t$-progressively measurable processes) on Ω valued in A. Note that both $\mathcal{A}$ and $\mathcal{A}_t$ are separable metric spaces endowed with the Krylov distance $\Delta(\alpha, \beta) := \mathbb{E}\big[\int_0^T d_A(\alpha_r, \beta_r)dr\big]$. Denote by $\mathcal{B}_{\mathcal{A}}$ (resp. $\mathcal{B}_{\mathcal{A}_t}$) the Borel σ-algebra of $\mathcal{A}$ (resp. $\mathcal{A}_t$). We assume that $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$

is the canonical space, i.e., $\Omega^0 = C(\mathbb{R}_+, \mathbb{R}^d)$, the set of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$. For any $\omega^0, \hat{\omega}^0 \in \Omega^0$, $r \in [0,T]$, we set

$$\hat{\omega}^0 \otimes_r \omega^0(s) := \hat{w}^0(s)\mathbb{1}_{[0,r)}(s) + \left[\hat{w}^0(r) + \omega^0(s) - \omega^0(r)\right]\mathbb{1}_{[r,T]}(s).$$

Let $r \in [0,T]$, for $\mathbb{P}^0$-a.s. $\tilde{\omega}^0$, $\alpha \in \mathcal{A}_t$ for some $t \in [0,T]$, we define:

$$\alpha^{r,\tilde{\omega}^0}(\omega^0, \omega^1) := \alpha(\tilde{\omega}^0 \otimes_r \omega^0, \omega^1).$$

For any $x \in \mathbb{R}^d$, we use $|x|$ to denote the Euclidean norm of x in $\mathbb{R}^d$, x_i (sometimes we also use $(x)_i$ or $[x]_i$) to denote the i-th component of x, $\langle \cdot, \cdot \rangle$ (or simply the $x \cdot y$ for $x, y \in \mathbb{R}^d$) to denote the standard scalar product on $\mathbb{R}^d$. Let $n \in \mathbb{N}$ and $x^1, x^2, \ldots, x^n \in \mathbb{R}^d$, we use $\overline{x} \in \mathbb{R}^{dn}$ to denote $\overline{x} := (x^1, x^2, \ldots, x^n)$. For any matrix $M \in \mathbb{R}^{d\times d}$, we use $\mathrm{tr}M := \sum_{i=1}^d M_{ii}$ to denote its trace, $M^\top$ to denote its transpose, and $|M| := \left[\mathrm{tr}\left(MM^\top\right)\right]^{1/2} = \left(\sum_{i,j}^d |M_{ij}|^2\right)^{1/2}$ to denote the Frobenius norm of M. If $M^0 \in \mathbb{R}^{d\times d}$ is another matrix, we use $M^{0;\top}$ to denote the transpose of M^0. We denote the identity matrix over $\mathbb{R}^d$ by $I_d \in \mathbb{R}^{d\times d}$. If $x \in \mathbb{R}$ is a scalar variable, the notation $\partial_x h \in \mathbb{R}$ means the usual partial derivative of the scalar function h with respect to x; if $x \in \mathbb{R}^d$ is a vector variable, then $\partial_x h \in \mathbb{R}^d$ means the gradient vector of h.

Remark 2.1.1. *Here we explain a little bit about the definition of $\mathbb{F}^t$. It makes no technical difference whether we consider the control sets which is progressively measurable with respect to $\mathbb{F}^t = (\mathcal{F}_s^t)_{s\geq 0} := (\sigma(W_r^0)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$ or $\mathring{\mathbb{F}}^t = (\mathring{\mathcal{F}}_s^t)_{s\geq 0} := (\sigma(W_{r\vee t}^0 - W_t^0)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$. All the arguments in this article remain valid with slight or no modification. In fact, denoting $\mathring{\mathcal{A}}_t$ to be the set of all $\mathring{\mathbb{F}}^t$-progressively measurable processes taking value in A, and the value function $\mathring{V}(t,\xi) := \sup_{\alpha\in\mathring{\mathcal{A}}_t} J(t,\xi,\alpha)$ (see (2.1.4) for the definition), we actually have*

$$\mathring{V}(t,\xi) = V(t,\xi),$$

where V is defined in (2.1.5). This can be seen from the following: First of all, as $\mathring{\mathcal{A}}_t \subset \mathcal{A}_t$, it follows that $\mathring{V}(t,\xi) \leq V(t,\xi)$. To show the other direction of the inequality, let $\alpha \in \mathcal{A}_t$,

then

$$\begin{aligned}
J(t,\xi,\alpha) &= \mathbb{E}\bigg[\int_t^T f(s, X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}\Big)\bigg] \\
&= \mathbb{E}^0\mathbb{E}\bigg[\int_t^T f(s, X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}\Big)\Big|\mathcal{F}_t^0\bigg] \\
&= \mathbb{E}^0[J(t,\xi,\alpha^{t,\omega^0})] \\
&\leq \mathbb{E}^0[\mathring{V}(t,\xi)] \\
&= \mathring{V}(t,\xi).
\end{aligned}$$

Taking supremum with respect to $\alpha \in \mathcal{A}_t$ we conclude the desired equality.

The requirement that the controls are adapted to $\mathcal{G}$ for every $s \geq 0$ is a bit technical. The condition that the controls have to be adapted to $\mathcal{G}$ is used in the existence theorem. Because of the nonlocal nature of the HJB equations in the Wasserstein space, assuming that φ is our test function, in the proof of existence of viscosity solution we can only conclude that

$$\begin{aligned}
&\partial_t\varphi(t_0,\mu_0) + \sup_{\alpha'\in\mathcal{M}_t}\mathbb{E}\Big\{f(t_0,\xi,\mu_0,\alpha') + \partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,\alpha') \\
&+ \frac{1}{2}\operatorname{tr}\left[\partial_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,\alpha')\sigma(t_0,\xi,\alpha')^\top\right]\Big\} + \frac{1}{2}\operatorname{tr}\left[\mathcal{H}\varphi(t_0,\mu_0)\sigma^0(t_0)\sigma^0(t_0)^\top\right] \leq (or\ \geq)\ 0,
\end{aligned} \tag{2.1.1}$$

where $\mathcal{L}(\xi) = \mu_0$, and $\mathcal{M}_t$ is the set of $\mathcal{F}_t^t$-measurable random variables. The condition of requiring the control set to be adapted to $\mathcal{G}$ makes the controls rich enough such that we can choose controls to make the above equivalent to

$$\begin{aligned}
&\partial_t\varphi(t_0,\mu_0) + \int_{\mathbb{R}^d}\sup_{a\in A}\bigg\{f(t_0,x,\mu_0,a) + b(t_0,x,\mu_0,a)\cdot\partial_\mu\varphi(t_0,\mu_0)(x) \\
&+ \frac{1}{2}\operatorname{tr}\Big[\partial_x\partial_\mu\varphi(t_0,\mu_0)(x)\sigma(t_0,x,a)[\sigma(t_0,x,a)]^\top\Big]\bigg\}\mu_0(dx) + \frac{1}{2}\operatorname{tr}\Big\{\mathcal{H}\varphi(t_0,\mu_0)\sigma^0(t_0)[\sigma^0(t_0)]^\top\Big\} \leq (or\ \geq)\ 0,
\end{aligned} \tag{2.1.2}$$

see Theorem 2.5.2.

2.1.2 Sense of Differentiability in the Wasserstein Space

We introduce over $\mathbb{R}^d$ the space of probability measures $\mathcal{P}(\mathbb{R}^d)$ and its subset $\mathcal{P}_p(\mathbb{R}^d)$ consisting of those with finite p-th moment for $p \geq 1$. The space $\mathcal{P}_p(\mathbb{R}^d)$ is equipped with the p-Wasserstein distance

$$\mathcal{W}_p(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \pi(dx,dy) \right)^{\frac{1}{p}}, \quad \text{for } \mu,\nu \in \mathcal{P}_p(\mathbb{R}^d),$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ such that for any $\pi \in \Pi(\mu, v)$, $\pi(\mathbb{R}^d \times \cdot) = \mu$ and $\pi(\cdot \times \mathbb{R}^d) = \nu$. We call $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p)$ the p-Wasserstein space over $\mathbb{R}^d$, and it is a Polish space. Finally, we denote by $\mathrm{Supp}(\mu)$ the support of $\mu \in \mathcal{P}(\mathbb{R}^d)$. For a map $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we adopt the notion of L-derivative (see Cardaliaguet *et al.* (2019) for instance) which is recalled as follows:

Definition 2.1.1. *The function $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be first order L-differentiable if its lifting $F : [0,T] \times L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \to \mathbb{R}$; $F(t,\xi) := f(t,\mathcal{L}(\xi))$ admits a continuous Fréchet derivative $D_\xi F : [0,T] \times L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \to L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$.*

Remark 2.1.2. *By (Carmona & Delarue, 2018a, Proposition 5.25), if f is first order L-differentiable, then it can be shown that there is a measurable function, denoted by $\partial_\mu f(t,\mu)(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, such that $D_\xi F(t,\xi) = \partial_\mu f(t,\mu)(\xi)$ for any $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\xi \in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ with $\mathcal{L}(\xi) = \mu$. We say that $\partial_\mu f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is the first order L-derivative of f.*

Definition 2.1.2. *The function $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be second order L-differentiable if f is first order L-differentiable and for any $x \in \mathbb{R}^d$, the function $\mu \mapsto \partial_\mu f(t,\mu)(x)$ is L-differentiable, i.e., the lifting $F' : L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \to \mathbb{R}^d$ of $\mu \mapsto \partial_\mu f(t,\mu)(x)$ admits a continuous Fréchet derivative $D_\xi F' : [0,T] \times L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \to L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^{d\times d})$.*

Remark 2.1.3. *Similar to Remark 2.1.2, if f is second order L-differentiable, then there is a measurable function, denoted by $\partial^2_\mu f(t,\mu)(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, such that $D_\xi F'(t,\xi') =$*

$\partial_\mu^2 f(t,\mu)(x,\xi')$ *for any* $(t,\mu,x) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d$ *and* $\xi' \in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ *with* $\mathcal{L}(\xi') = \mu$. *We say that* $\partial_\mu^2 f : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ *is the second order* L*-derivative of* f.

As in Bayraktar *et al.* (2023b), we define the set of fully second order L-differentiable functions and the set of partially second order L-differentiable functions.

Definition 2.1.3. *The set* $C_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ *consists of all continuous functions* $f : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ *satisfying the following:*

(1). the derivatives $\partial_t f(t,\mu)$, $\partial_\mu f(t,\mu)(x)$, $\partial_x\partial_\mu f(t,\mu)(x)$, $\partial_\mu^2 f(t,\mu)(x,x')$ *exist and are jointly continuous in the respective arguments;*

(2). there is a constant $C_f \geq 0$ *such that for any* $(t,\mu,x,x') \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\times\mathbb{R}^d$, *we have*

$$|\partial_\mu f(t,\mu)(x)| \leq C_f\Big(1+|x|\Big);$$
$$|\partial_t f(t,\mu)| + |\partial_x\partial_\mu f(t,\mu)(x)| + \Big|\partial_\mu^2 f(t,\mu)(x,x')\Big| \leq C_f.$$

We define the operator $\mathcal{H}$:

$$\mathcal{H}u(\mu) := \frac{d^2}{dw^2}u((I_d+w)_\sharp\mu)\Big|_{w=0} \in \mathbb{R}^{d\times d},$$

where $(I_d+w)_\sharp\mu$ is the pushforward measure associated with map $I_d+w : \mathbb{R}^d\to\mathbb{R}^d$ defined by $x\mapsto x+w$ for $w\in\mathbb{R}^d$; and we are differentiating with respect to finite-dimensional $w\in\mathbb{R}^d$.

Definition 2.1.4. *The set* $PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ *consists of all continuous functions* $f : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ *satisfying the following:*

(1). the derivatives $\partial_t f(t,\mu)$, $\partial_\mu f(t,\mu)(x)$, $\partial_x\partial_\mu f(t,\mu)(x)$, $\mathcal{H}f(\mu)$ *exist and are jointly continuous in the respective arguments;*

(2). there exists $C_f \geq 0$ *such that for any* $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, *it holds that*

$$\int_{\mathbb{R}^d}|\partial_\mu f(t,\mu)(x)|^2\,\mu(dx) \leq C_f\left(1+\int_{\mathbb{R}^d}|x|^2\mu(dx)\right);$$

$$|\partial_t f(t,\mu)| + |\mathcal{H} f(t,\mu)| + \int_{\mathbb{R}^d} |\partial_x \partial_\mu f(t,\mu)(x)|^2 \mu(dx) \leq C_f.$$

We have the following lemma relating $PC_1^{1,2}([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$ and $C_1^{1,2}([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$.

Lemma 2.1.5. *If $u \in C_1^{1,2}([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$, then we have*

$$\mathcal{H} u(t,\mu) = \int_{\mathbb{R}^d} \partial_x \partial_\mu u(t,\mu)(x) \mu(dx) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_\mu^2 u(t,\mu)(x,\tilde{x}) \mu(dx) \mu(d\tilde{x}),$$

and hence $u \in PC_1^{1,2}([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$.

Proof. Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that $\mathcal{L}(X) = \mu$. We denote the lifting of $u \in C_1^{1,2}([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$ to Hilbert space by u^*, we have

$$\frac{d}{dw} u^*(t, X+w) = \mathbb{E}\Big[\partial_\mu u(t, \mathcal{L}(X+w))(X+w)\Big].$$

Besides, we consider

$$\begin{aligned}
&\frac{d}{dw} \partial_\mu u(t, \mathcal{L}(X+w))(X+w) \\
&= \partial_x \partial_\mu u(t, \mathcal{L}(X+w))(X+w) + \bar{\mathbb{E}}\Big[\partial_\mu^2 u(t, \mathcal{L}(X+w))(X+w, \bar{X}+w)\Big],
\end{aligned}$$

where $\bar{X}$ is an independent copy of X and $\bar{\mathbb{E}}$ is the expectation with respect to $\bar{X}$. We assume that $|w| \leq 1$ and use the fact that $u \in C_1^{1,2}([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$ to yield that

$$\left| \frac{d}{dw} \partial_\mu u(t, \mathcal{L}(X+w))(X+w) \right| \leq C_u,$$

for some constant $C_u > 0$ independent of w and X. By the dominated convergence theorem we have

$$\begin{aligned}
\frac{d}{dw}\frac{d}{dw} u^*(t, X+w)\Big|_{w=0} &= \frac{d}{dw} \mathbb{E}\Big[\partial_\mu u(t, \mathcal{L}(X+w))(X+w)\Big]\Big|_{w=0} \\
&= \mathbb{E}\Big\{ \partial_x \partial_\mu u(t, \mathcal{L}(X+w))(X+w) + \bar{\mathbb{E}}\Big[\partial_\mu^2 u(t, \mathcal{L}(X+w))(X+w, \bar{X}+w)\Big] \Big\}\Big|_{w=0} \\
&= \int_{\mathbb{R}^d} \partial_x \partial_\mu u(t,\mu)(x) \mu(dx) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_\mu^2 u(t,\mu)(x,\tilde{x}) \mu(dx) \mu(d\tilde{x}).
\end{aligned}$$

□

We have the following extension of Itô's formula for functions in $PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$. This is inspired by Bayraktar *et al.* (2023b).

Theorem 2.1.6. *Let $b_t \in \mathbb{R}^d$ and $\sigma_t \in \mathbb{R}^{d\times d}$ be two $\mathbb{F}$-adapted processes such that there exists $L > 0$ satisfying $|b_t| \vee |\sigma_t| \leq L$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. Let $\sigma_t^0 : [0,T] \to \mathbb{R}^{d\times d}$ be deterministic and $\xi \in L^2(\Omega,\mathcal{F},\mathbb{P})$. Consider the following $\mathbb{R}^d$-valued Itô process:*

$$X_t = \xi + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \sigma_s^0 dW_s^0, \quad \text{for } t \in [0,T].$$

We write $\mu_t^{W^0}$ for $\mathbb{P}_{X_t}^{W^0}$, then it holds $\mathbb{P}^0$-a.s. that for $\varphi \in PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$,

$$\begin{aligned}\varphi(t,\mu_t^{W^0}) =& \varphi(0,\mu_0^{W^0}) + \int_0^t \partial_t\varphi(s,\mu_s^{W^0})ds + \int_0^t \mathbb{E}^1\big[\partial_\mu\varphi(s,\mu_s^{W^0})(X_s)\cdot b_s\big]ds \\ &+ \int_0^t \mathbb{E}^1\big[\sigma_s^{0;\top}\partial_\mu\varphi(s,\mu_s^{W^0})(X_s)\big]\cdot dW_s^0 + \frac{1}{2}\int_0^t \mathbb{E}^1\Big\{\operatorname{tr}\big[\partial_x\partial_\mu\varphi(s,\mu_s^{W^0})(X_s)\sigma_s\sigma_s^\top\big]\Big\}ds \\ &+ \frac{1}{2}\int_0^t \operatorname{tr}\Big[\mathcal{H}\varphi(s,\mu_s^{W^0})\sigma_s^0\sigma_s^{0;\top}\Big]ds.\end{aligned}$$

Proof. For $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$, we consider

$$X_t(\omega^0,\cdot) = \xi(\omega^0,\cdot) + \left(\int_0^t b_r dr\right)(\omega^0,\cdot) + \left(\int_0^t \sigma_r dW_r\right)(\omega^0,\cdot) + \left(\int_0^t \sigma_r^0 dW_r^0\right)(\omega^0).$$

We define $Y_t^{\omega^0}(\cdot) := X_t(\omega^0,\cdot) - \big(\int_0^t \sigma_r^0 dW_r^0\big)(\omega^0)$. Applying the standard mean field Itô's formula (see (Carmona & Delarue, 2018a, Proposition 5.102)) to $\varphi(t,\mathcal{L}(Y_t^{\omega^0}+y))$ for any $y\in\mathbb{R}^d$ and $\varphi \in PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$, we conclude that for $\mathbb{P}^0$-a.s. $\omega^0\in\Omega^0$,

$$\begin{aligned}\varphi(t,\mathcal{L}(Y_t^{\omega^0}+y)) =& \varphi(0,\mathcal{L}(Y_0^{\omega^0}+y)) + \int_0^t \partial_t\varphi(s,\mathcal{L}(Y_s^{\omega^0}+y))ds \\ &+ \int_0^t \mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathcal{L}(Y_s^{\omega^0}+y))(Y_s^{\omega^0}+y)\cdot b_s\Big]ds \\ &+ \frac{1}{2}\int_0^t \mathbb{E}^1\Big\{\operatorname{tr}\big[\partial_x\partial_\mu\varphi(s,\mathcal{L}(Y_s^{\omega^0}+y))(Y_s^{\omega^0}+y)\sigma_s\sigma_s^\top\big]\Big\}ds.\end{aligned}$$

We define $\Psi(s,y) := \varphi(s,\mathcal{L}(Y_s^{\omega^0}+y))$. By the finite-dimensional Itô-Kunita-Wentzell formula (Dos Reis & Platonov, 2022, Theorem 2.3), we have

$$\Psi\left(t,\int_0^t \sigma_s^0 dW_s^0\right)$$

$$
\begin{aligned}
&= \Psi(0,0) + \int_0^t \partial_t \varphi\Big(s, \mathcal{L}\Big(Y_s^{\omega^0} + \int_0^s \sigma_r^0 dW_r^0\Big)\Big) ds \\
&\quad + \int_0^t \mathbb{E}^1\Big[\partial_\mu \varphi\Big(s, \mathcal{L}\Big(Y_s^{\omega^0} + \int_0^s \sigma_r^0 dW_r^0\Big)\Big)\Big(Y_s^{\omega^0} + \int_0^s \sigma_r^0 dW_r^0\Big) \cdot b_s\Big] ds \\
&\quad + \frac{1}{2}\int_0^t \mathbb{E}^1\Big[\operatorname{tr}\Big\{\partial_x \partial_\mu \varphi\Big(s, \mathcal{L}\Big(Y_s^{\omega^0} + \int_0^s \sigma_r^0 dW_r^0\Big)\Big)\Big(Y_s^{\omega^0} + \int_0^s \sigma_r^0 dW_r^0\Big)\sigma_s \sigma_s^\top\Big\}\Big] ds \\
&\quad + \int_0^t \mathbb{E}^1\Big[\sigma_s^{0;\top} \partial_\mu \varphi\Big(s, \mathcal{L}\Big(Y_s^{\omega^0} + \int_0^s \sigma_r^0 dW_r^0\Big)\Big)\Big(Y_s^{\omega^0} + \int_0^s \sigma_r^0 dW_r^0\Big)\Big] \cdot dW_s^0 \\
&\quad + \frac{1}{2}\int_0^t \Big[\operatorname{tr}\Big\{\partial_y^2 \varphi\Big(s, \mathcal{L}\Big(Y_s^{\omega^0} + \int_0^t \sigma_s^0 dW_s^0\Big)\Big)\sigma_s^0 \sigma_s^{0;\top}\Big\}\Big] ds \\
&= \varphi(0, \mu_0^{W^0}) + \int_0^t \partial_t \varphi(s, \mu_s^{W^0}) ds + \int_0^t \mathbb{E}^1\big[\partial_\mu \varphi(s, \mu_s^{W^0})(X_s) \cdot b_s\big] ds \\
&\quad + \frac{1}{2}\int_0^t \mathbb{E}^1\big[\operatorname{tr}\{\partial_x \partial_\mu \varphi(s, \mu_s^{W^0})(X_s)\sigma_s \sigma_s^\top\}\big] ds + \int_0^t \mathbb{E}^1\big[\sigma_s^{0;\top} \partial_\mu \varphi(s, \mu_s^{W^0})(X_s)\big] \cdot dW_s^0 \\
&\quad + \frac{1}{2}\int_0^t \Big[\operatorname{tr}\Big\{\mathcal{H}\varphi(s, \mu_s^{W^0})\sigma_s^0 \sigma_s^{0;\top}\Big\}\Big] ds.
\end{aligned}
$$

□

2.1.3 Assumptions and the Control Problem

Define the functions:

$$
b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \quad \sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times d}, \quad \sigma^0 : [0,T] \to \mathbb{R}^{d\times d},
$$

$$
f : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}, \qquad g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}.
$$

Throughout this work, we use the following assumptions:

Assumption. (A). *The functions b, σ, σ^0, f and g satisfy the following:*

(1). the functions b, σ, σ^0, f and g are continuous;

(2). there exist constants $K \geq 0$ and $\beta \in (0,1]$ such that for any $a \in A$, (t,x,μ), $(t',x',\mu') \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, it holds that

$$
\begin{aligned}
&\big|b(t,x,\mu,a) - b(t',x',\mu',a)\big| + \big|\sigma(t,x,a) - \sigma(t',x',a)\big| + \big|\sigma^0(t) - \sigma^0(t')\big| \\
&\quad + \big|f(t,x,\mu,a) - f(t',x',\mu',a)\big| + \big|g(x,\mu) - g(x',\mu')\big| \leq K\big[|x-x'| + |t-t'|^\beta + \mathcal{W}_2(\mu,\mu')\big],
\end{aligned}
$$

$$\big|b(t,x,\mu,a)\big| + \big|\sigma(t,x,a)\big| + \big|\sigma^0(t)\big| + \big|f(t,x,\mu,a)\big| + \big|g(x,\mu)\big| \leq K.$$

Assumption. (B). *For any $a \in A$, the function $\sigma(\cdot,\cdot,a)$ belongs to $C^{1,2}\big([0,T]\times\mathbb{R}^d\big)$. Moreover, there exists a constant $K \geq 0$ such that*

$$|\partial_t\sigma(t,x,a)| + |\partial_x\sigma(t,x,a)| + \big|\partial^2_{xx}\sigma(t,x,a)\big| + \big|\partial_t\sigma^0(t)\big| \leq K,$$

for all $(t,x,a) \in [0,T]\times\mathbb{R}^d\times A$.

Assumption. (A*). *Suppose that Assumption (A) holds and we assume further that there exist constants $K \geq 0$ and $\beta \in (0,1]$ such that for any $a \in A$, (t,x,μ), $(t',x',\mu') \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, it holds that*

$$\begin{aligned}&\big|b(t,x,\mu,a) - b(t',x',\mu',a)\big| + \big|f(t,x,\mu,a) - f(t',x',\mu',a)\big| + \big|g(x,\mu) - g(x',\mu')\big|\\ &\leq K\big[|x-x'| + |t-t'|^\beta + \mathcal{W}_1(\mu,\mu')\big].\end{aligned}$$

Remark 2.1.4. *Assumption (A*) is a strengthening assumption of Assumption (A). We write Assumption (A) and Assumption (A*) separately to emphasize that some of the lemmas and theorems still hold without assuming the stronger condition in Assumption (A*).*

Remark 2.1.5. *Here, the coefficients b, σ, f and g are assumed to be $\mathcal{W}_1$-Lipschitz continuous in Assumption (A*), which is different from the $\mathcal{W}_2$-Lipschitz continuity assumption in (Cosso* et al.*, 2024, Assumption (A)). The main reason of adopting $\mathcal{W}_1$-Lipschitz continuity instead of $\mathcal{W}_2$ is to provide a better estimate when we establish the smooth approximation of the coefficients in Lemma 2.4.2. We note that the inequalities (Cosso* et al.*, 2024, (A.6) and (A.7)) are invalid if only $\mathcal{W}_2$-Lipschitz continuity is assumed as in (Cosso* et al.*, 2024, Assumption (A)). This leads to flaws in the proofs of (Cosso* et al.*, 2024, Theorem 5.1, Theorem A.7). See more discussions in Remark 2.4.1.*

For every $t \in [0,T]$, $\xi \in L^2(\Omega^1,\mathcal{F}^1_t,\mathbb{P}^1;\mathbb{R}^d)$ and $\alpha \in \mathcal{A}_t$, we consider the solution $X^{t,\xi,\alpha}$ of the

following state dynamics:

$$X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}_{X_r}^{W^0}, \alpha_r)dr + \int_t^s \sigma(r, X_r, \alpha_r)dW_r + \int_t^s \sigma^0(r)dW_r^0, \quad \text{for } s \in [t, T], \tag{2.1.3}$$

where $\mathbb{P}_{X_r}^{W^0}$ denotes the conditional distribution of X_r given W^0. We are subject to the cost functional:

$$J(t, \xi, \alpha) := \mathbb{E}\bigg[\int_t^T f\Big(s, X_s^{t,\xi,\alpha}, \mathbb{P}_{X_s^{t,\xi,\alpha}}^{W^0}, \alpha_s\Big)ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}_{X_T^{t,\xi,\alpha}}^{W^0}\Big)\bigg]. \tag{2.1.4}$$

Finally, we define the value function V to be

$$V(t, \xi) := \sup_{\alpha \in \mathcal{A}_t} J(t, \xi, \alpha), \quad \text{for any } (t, \xi) \in [0, T] \times L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d). \tag{2.1.5}$$

2.2 Basic Properties

This section collects some basic properties. First we give some standard results concerning the regularity of the solution of the SDE in (2.1.3) and the value function V. Proofs are standard and therefore omitted, and readers are referred to Yong & Zhou (1999).

Proposition 2.2.1. *Suppose that Assumption (A) holds. For every $t \in [0, T]$, $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$, $\alpha \in \mathcal{A}_t$, there exists a unique (up to $\mathbb{P}$-indistinguishability) continuous $\mathbb{F}$-progressively measurable solution $X^{t,\xi,\alpha} = (X_s^{t,\xi,\alpha})_{s\in[t,T]}$ of equation (2.1.3). Moreover, there is a constant C depending only on p, K, T, d such that*

$$\mathbb{E}\Big[\sup_{s\in[t,T]} |X_s^{t,\xi,\alpha}|^2\Big]^{1/p} \leq C\Big(1 + \mathbb{E}|\xi|^2\Big)^{1/2}; \tag{2.2.1}$$

$$\mathbb{E}\Big[\sup_{s\in[t,T]} |X_s^{t,\xi,\alpha} - X_s^{t,\xi',\alpha}|^2\Big]^{1/2} \leq C\mathbb{E}\Big(|\xi - \xi'|^2\Big)^{1/2}; \tag{2.2.2}$$

$$\mathbb{E}\Big[\sup_{s\in[t,t+h]} |X_s^{t,\xi,\alpha} - \xi|^2\Big] \leq Ch, \tag{2.2.3}$$

for any $t \in [0, T]$, $h \in [0, T - t]$, $\xi, \xi' \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ and $\alpha \in \mathcal{A}_t$.

Proposition 2.2.2. *Suppose that Assumption (A) holds. The function V satisfies the following properties:*

(1). V is bounded and jointly continuous;

(2). there exists a constant $C \geq 0$ such that for any $t, t' \in [0,T]$, ξ, $\xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$,

$$|V(t,\xi) - V(t',\xi')| \leq C\Big[\mathbb{E}\big(|\xi - \xi'|^2\big)^{1/2} + |t - t'|^{1/2}\Big].$$

The constant C depends on d, K, T and independent of t, t',ξ, ξ', α.

Proposition 2.2.3 (Law Invariance)**.** *Suppose that Assumption (A) holds. For every $t \in [0,T]$ and ξ, $\eta \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$, with $\mathcal{L}(\xi) = \mathcal{L}(\eta)$, it holds that $V(t,\xi) = V(t,\eta)$.*

Proof. This proof is modified from (Cosso *et al.*, 2023, Theorem 3.6) to suit the case with common noise. We only highlight the differences, and interested readers are referred to (Cosso *et al.*, 2023, Theorem 3.6). Fix $t \in [0,T]$. Let $\xi, \eta \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ with $\mathcal{L}(\xi) = \mathcal{L}(\eta)$. By Proposition 2.2.2, the function $\xi \mapsto V(t,\xi)$ is continuous. Thus, without loss of generality (see (Cosso *et al.*, 2023, Theorem 3.6, Substep 2.2)), we consider the case where ξ, η are discrete, i.e.,

$$\mathcal{L}(\xi) = \mathcal{L}(\eta) = \sum_{i=1}^{m} p_i \delta_{x^i},$$

for some $\{x^1, x^2, \ldots, x^m\} \subset \mathbb{R}^d$ with x^i being distinct. Here δ_{x^i} is the Dirac measure at x^i. For $i = 1, 2, \ldots, m$, the numbers p_i satisfy $p_i > 0$ and $\sum_{i=1}^m p_i = 1$. By (Cosso *et al.*, 2023, Lemma B.3), there exist two $\mathcal{F}_t^1$-measurable random variables U_ξ and U_η with uniform distribution on $[0,1]$, such that ξ and U_ξ are independent, as are η and U_η. Let $\mathbb{F}^{W,W^0,t}$ be the $\mathbb{P}$-completion of the filtration generated by $(W_{s\vee t} - W_t)_{s\geq 0}$ and $(W_s^0)_{s\geq 0}$ and let $Prog(\mathbb{F}^{W,W^0,t})$ be the σ-algebra of $[0,T]\times\Omega$ of all $\mathbb{F}^{W,W^0,t}$-progressively measurable sets. Fixing $\alpha \in \mathcal{A}_t$, we claim that there exists a measurable function:

$$a : \Big([0,T]\times\Omega\times\mathbb{R}^d\times[0,1], Prog(\mathbb{F}^{W,W^0,t}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}([0,1])\Big) \to (A, \mathcal{B}(A))$$

such that

$$\begin{aligned}&\mathcal{L}\Big(\xi,(a_s(\xi,U_\xi))_{s\in[t,T]},(W_s^0)_{s\geq 0},(W_s-W_t)_{s\in[t,T]}\Big)\\=&\mathcal{L}\Big(\xi,(\alpha_s)_{s\in[t,T]},(W_s^0)_{s\geq 0},(W_s-W_t)_{s\in[t,T]}\Big).\end{aligned}\tag{2.2.4}$$

This claim can be seen from the following: denote

$$\hat{\Omega}=[0,T]\times\Omega,\qquad \hat{\mathcal{F}}=\mathcal{B}([0,T])\otimes\mathcal{F},\qquad \hat{\mathbb{P}}=\lambda_T\otimes\mathbb{P},\qquad \bar{E}=[0,T]\times\Omega,\qquad \bar{\mathscr{E}}=Prog(\mathbb{F}^{W,W^0,t}),$$
$$E=\bar{E}\times\mathbb{R}^d,\qquad \mathscr{E}=\bar{\mathscr{E}}\otimes\mathcal{B}(\mathbb{R}^d),$$

with λ_T being the uniform distribution on $([0,T],\mathcal{B}([0,T]))$. Consider the canonical extension of U_ξ to $\hat{\Omega}$, denoted by $\hat{U}_\xi$. Define $\mathcal{I}^{W,W^0,t}:(\hat{\Omega},\hat{\mathcal{F}})\to(\bar{E},\bar{\mathscr{E}})$ the identity map and $\Gamma:=(\mathcal{I}^{W,W^0,t},\xi)$. Then $\Gamma:(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}})\to(E,\mathscr{E})$ is independent of U_ξ as U_ξ is $\mathcal{F}_t^1$ measurable. Applying (Kallenberg, 2006, Theorem 6.10), we get a map $\bar{a}:[0,T]\times\Omega\times\mathbb{R}^d\times[0,1]\to(A,\mathcal{B}(A))$, being measurable with respect to the σ-algebra $Prog(\mathbb{F}^{W,W^0,t})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}([0,1])$ such that

$$\mathcal{L}(\Gamma,\bar{a}(\Gamma,\hat{U}_\xi))=\mathcal{L}(\Gamma,\alpha).$$

From the definition of Γ we conclude that

$$\begin{aligned}&\mathcal{L}\Big(\xi,(\bar{a}_s(\xi,U_\xi))_{s\in[0,T]},(W_s-W_t)_{s\in[t,T]},(W_s^0)_{s\geq 0}\Big)\\=&\mathcal{L}\Big(\xi,(\alpha_s)_{s\in[0,T]},(W_s-W_t)_{s\in[t,T]},(W_s^0)_{s\geq 0}\Big).\end{aligned}$$

Given an arbitrary $u_0\in A$, we can choose the required measurable function $a:[0,T]\times\Omega\times\mathbb{R}^d\times[0,1]\to A$ to be

$$a_s(\omega,x,u):=u_0\mathbb{1}_{[0,t)}(s)+\bar{a}_s(\omega,x,u)\mathbb{1}_{[t,T]}(s),\quad \text{for any } (s,\omega,x,u)\in[0,T]\times\Omega\times\mathbb{R}^d\times A.$$

Then $(a_s(\xi,U_\xi))_{s\in[0,T]}$ is $\mathbb{F}$-progressively measurable and (2.2.4) is satisfied. The measurable function a satisfies

$$\mathcal{L}\Big(\xi,(\alpha_s)_{s\in[t,T]},(W_s-W_t)_{s\in[t,T]},(W_s^0)_{s\geq 0}\Big)$$

$$
\begin{aligned}
&= \mathcal{L}\Big(\xi, (a_s(\xi, U_\xi))_{s\in[t,T]}, (W_s - W_t)_{s\in[t,T]}, (W^0_s)_{s\geq 0}\Big)\\
&= \mathcal{L}\Big(\eta, (a_s(\eta, U_\eta))_{s\in[t,T]}, (W_s - W_t)_{s\in[t,T]}, (W^0_s)_{s\geq 0}\Big)\\
&= \mathcal{L}\Big(\eta, (\beta_s)_{s\in[t,T]}, (W_s - W_t)_{s\in[t,T]}, (W^0_s)_{s\geq 0}\Big),
\end{aligned}
$$

where $\beta_s := a_s(\eta, U_\eta)$. It can be shown therefore that

$$
\mathcal{L}\Big((X^{t,\xi,\alpha}_s)_{s\in[t,T]}, (\alpha_s)_{s\in[t,T]}, (W^0_s)_{s\geq 0}\Big) = \mathcal{L}\Big((X^{t,\eta,\beta}_s)_{s\in[t,T]}, (\beta_s)_{s\in[t,T]}, (W^0_s)_{s\geq 0}\Big),
$$

and therefore $J(t,\xi,\alpha) = J(t,\eta,\beta)$, implying $V(t,\xi) = V(t,\eta)$. □

Theorem 2.2.4 (Dynamic Programming Principle)**.** *Suppose that Assumption (A) holds. The value function V satisfies the dynamic programming principle: for every $t \in [0,T]$ and $\xi \in L^2(\Omega^1, \mathcal{F}^1_t, \mathbb{P}^1; \mathbb{R}^d)$, it holds that*

$$
\begin{aligned}
V(t,\xi) &= \sup_{\alpha\in\mathcal{A}_t}\left\{\inf_{s\in[t,T]} \mathbb{E}\left[\int_t^s f\Big(r, X^{t,\xi,\alpha}_r, \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_r}, \alpha_r\Big)dr + V(s, X^{t,\xi,\alpha}_s)\right]\right\}\\
&= \sup_{\alpha\in\mathcal{A}_t}\left\{\sup_{s\in[t,T]} \mathbb{E}\left[\int_t^s f\Big(r, X^{t,\xi,\alpha}_r, \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_r}, \alpha_r\Big)dr + V(s, X^{t,\xi,\alpha}_s)\right]\right\}.
\end{aligned}
$$

Proof. Put

$$
\begin{aligned}
\underline{\Lambda}(t,\xi) &:= \sup_{\alpha\in\mathcal{A}_t}\left\{\inf_{s\in[t,T]} \mathbb{E}\left[\int_t^s f\Big(r, X^{t,\xi,\alpha}_r, \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_r}, \alpha_r\Big)dr + V(s, X^{t,\xi,\alpha}_s)\right]\right\},\\
\overline{\Lambda}(t,\xi) &:= \sup_{\alpha\in\mathcal{A}_t}\left\{\sup_{s\in[t,T]} \mathbb{E}\left[\int_t^s f\Big(r, X^{t,\xi,\alpha}_r, \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_r}, \alpha_r\Big)dr + V(s, X^{t,\xi,\alpha}_s)\right]\right\}.
\end{aligned}
$$

We shall show $\overline{\Lambda}(t,\xi) \leq V(t,\xi) \leq \underline{\Lambda}(t,\xi)$. We start with $V(t,\xi) \leq \underline{\Lambda}(t,\xi)$. Let $\alpha \in \mathcal{A}_t$ be arbitrary. Note that our value function $V(t,\xi)$ is in fact defined only for $\xi \in L^2(\Omega^1, \mathcal{F}^1_t, \mathbb{P}^1; \mathbb{R}^d)$, thus $V(s, X^{t,\xi,\alpha}_s)$ is in fact random in the sense that $V(s, X^{t,\xi,\alpha}_s)(\omega^0) = V(s, X^{t,\xi,\alpha}_s(\omega^0, \cdot))$. Note that by the uniqueness of the SDE, for $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$, the process $\big(X^{t,\xi,\alpha}_r(\omega^0,\cdot)\big)_{r\in[s,T]}$ is the unique solution of

$$
X_r(\omega^0,\cdot) = \Big(X^{t,\xi,\alpha}_s + \int_s^r b(k, X_k, \mathbb{P}^{W^0}_{X_k}, \alpha_k)dk + \int_s^r \sigma(k, X_k, \alpha_k)dW_k + \int_s^r \sigma^0(k)dW^0_k\Big)(\omega^0,\cdot),
$$

thus for $\mathbb{P}^0 \otimes \mathbb{P}^0$-a.s. $(\omega^0, \omega^{00}) \in \Omega^0 \times \Omega^0$, $\big(X_r^{t,\xi,\alpha}(\omega^0 \otimes_s \omega^{00}, \cdot)\big)_{r\in[s,T]}$ solves

$$X_r(\omega^0 \otimes_s \omega^{00}, \cdot) = \Big(X_s^{t,\xi,\alpha} + \int_s^r b(k, X_k, \mathbb{P}^{W^0}_{X_k}, \alpha_k)dk + \int_s^r \sigma(k, X_k, \alpha_k)dW_k + \int_s^r \sigma^0(k)dW_k^0\Big)(\omega^0 \otimes_s \omega^{00}, \cdot). \tag{2.2.5}$$

Since $X_s^{t,\xi,\alpha}(\omega^0, \cdot)$ is $\mathcal{F}_s^t$-measurable, we have $X_s^{t,\xi,\alpha}(\omega^0 \otimes_s \omega^{00}, \cdot) = X_s^{t,\xi,\alpha}(\omega^0, \cdot)$ and for $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$, (2.2.5) is equivalent to

$$\begin{aligned}
&X_r(\omega^0 \otimes_s \omega^{00}, \cdot)\\
&= X_s^{t,\xi,\alpha}(\omega^0, \cdot) + \Big(\int_s^r b(k, X_k, \mathbb{P}^{W^0}_{X_k}, \alpha_k)dk + \int_s^r \sigma(k, X_k, \alpha_k)dW_k + \int_s^r \sigma^0(k)dW_k^0\Big)(\omega^0 \otimes_s \omega^{00}, \cdot).
\end{aligned}$$

For $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$ and $\tilde{\omega} = (\tilde{\omega}^0, \tilde{\omega}^1) \in \Omega^0 \times \Omega^1$, we define $\hat{\alpha}_r^{\omega^0}(\tilde{\omega}) := \alpha_r(\omega^0 \otimes_s \tilde{\omega}^0, \tilde{\omega}^1)$. Then for $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$, $X_r^{t,\xi,\alpha}(\omega^0 \otimes_s \omega^{00}, \cdot)$ solves

$$\begin{aligned}
Y_r(\omega^{00}, \omega^1) =& X_s^{t,\xi,\alpha}(\omega^0, \omega^1)\\
&+ \Big(\int_s^r b(k, Y_k, \mathbb{P}^{W^0}_{Y_k}, \hat{\alpha}_k^{\omega^0})dk + \int_s^r \sigma(k, Y_k, \hat{\alpha}_k^{\omega^0})dW_k + \int_s^r \sigma^0(k)dW_k^0\Big)(\omega^{00}, \omega^1).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}V(s, X_s^{t,\xi,\alpha}) =& \mathbb{E}^0\Big[V(s, X_s^{t,\xi,\alpha}(\omega^0, \cdot))\Big]\\
\geq& \mathbb{E}^0\Bigg\{\mathbb{E}^{00}\mathbb{E}^1\Bigg[\int_s^T f\Big(r, X_r^{t,\xi,\alpha}(\omega^0 \otimes_s \omega^{00}, \omega^1), \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}(\omega^0 \otimes_s \omega^{00})}, \hat{\alpha}_r^{\omega^0}(\omega^{00}, \omega^1)\Big)\, dr\\
&\qquad + g\Big(X_T^{t,\xi,\alpha}(\omega^0 \otimes_s \omega^{00}, \omega^1), \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}(\omega^0 \otimes_s \omega^{00})}\Big)\Bigg]\Bigg\}\\
=& \mathbb{E}\Bigg[\int_s^T f(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r)dr + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}\Big)\Bigg].
\end{aligned}$$

Therefore

$$\mathbb{E}\Bigg[\int_t^s f\Big(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r\Big)dr + V(s, X_s^{t,\xi,\alpha})\Bigg] \geq J(t, \xi, \alpha),$$

since $s \in [t, T]$ is arbitrary, we have

$$\inf_{s\in[t,T]} \mathbb{E}\Bigg[\int_t^s f\Big(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r\Big)dr + V(s, X_s^{t,\xi,\alpha})\Bigg] \geq J(t, \xi, \alpha).$$

As α is arbitrary, so

$$\underline{\Lambda}(t,\xi) \geq V(t,\xi).$$

On the other hand, for any $\beta \in \mathcal{A}_t$, $s \in [t,T]$, by the continuity of $J(\cdot,\cdot,\cdot)$ and $V(\cdot,\cdot)$, we use the measurable selection theorem (see Wagner (1977) for instance) to find a $\gamma : (\Omega^0, \mathcal{F}_s^{W^0}, \mathbb{P}^0) \to (\mathcal{A}_s, \mathcal{B}_{\mathcal{A}_s})$ such that

$$\begin{aligned} V(s, X_s^{t,\xi,\beta}(\omega^0)) \leq \mathbb{E}\Bigg[\int_s^T f\left(r, X_r^{s,X_s^{t,\xi,\beta}(\omega^0),\gamma(\omega^0)}, \mathbb{P}^{W^0}_{X_r^{s,X_s^{t,\xi,\beta}(\omega^0),\gamma(\omega^0)}}, \gamma_r(\omega^0)\right) dr \\ + g\left(X_T^{s,X_s^{t,\xi,\beta}(\omega^0),\gamma(\omega^0)}, \mathbb{P}^{W^0}_{X_T^{s,X_s^{t,\xi,\beta}(\omega^0),\gamma(\omega^0)}}\right)\Bigg] + \varepsilon. \end{aligned}$$

Put $\Gamma_r(\omega^0,\omega^1) := \beta_r(\omega^0,\omega^1)\mathbb{1}_{[0,s)}(r) + \gamma_r(\omega^0)(\omega^0,\omega^1)\mathbb{1}_{[s,T]}(r)$. Then by (Soner & Touzi, 2002, Lemma 2.1), we have $\Gamma \in \mathcal{A}_t$ and

$$\begin{aligned} &\mathbb{E}\left[\int_t^s f(r, X_r^{t,\xi,\beta}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\beta}}, \beta_r)dr + V(s, X_s^{t,\xi,\beta})\right] \\ &\leq \mathbb{E}\left[\int_t^T f(r, X_r^{t,\xi,\Gamma}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\Gamma}}, \Gamma_r)dr + g(X_T^{t,\xi,\Gamma}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\Gamma}})\right] + \varepsilon \\ &\leq V(t,\xi) + \varepsilon. \end{aligned}$$

Taking supremum in $s \in [t,T]$ on both sides gives

$$\sup_{s\in[t,T]} \mathbb{E}\left[\int_t^s f(r, X_r^{t,\xi,\beta}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\beta}}, \beta_r)dr + V(s, X_s^{t,\xi,\beta})\right] \leq V(t,\xi) + \varepsilon$$

and by the arbitrariness of $\varepsilon > 0$ and $\beta \in \mathcal{A}_t$, we have $\overline{\Lambda}(t,\xi) \leq V(t,\xi)$. □

From Proposition 2.2.3, we can define a function $v(t,\mu) : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that for any $t \in [0,T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, it is given by

$$v(t,\mu) := V(t,\xi) \tag{2.2.6}$$

for any $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ such that $\mathcal{L}(\xi) = \mu$, where $V(t,\xi)$ is defined in (2.1.5). From Theorem 2.2.4 and (2.2.6), the dynamic programming principle can therefore be recast as

Theorem 2.2.5 (Dynamic Programming Principle)**.** *Suppose that Assumption (A) hold. The value function v satisfies the dynamic programming principle: for every $t \in [0,T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, it holds that*

$$\begin{aligned} v(t,\mu) &= \sup_{\alpha\in\mathcal{A}_t} \sup_{s\in[t,T]} \left\{ \mathbb{E}\left[\int_t^s f\Big(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r\Big) dr + v\Big(s, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}\Big) \right] \right\} \\ &= \sup_{\alpha\in\mathcal{A}_t} \inf_{s\in[t,T]} \left\{ \mathbb{E}\left[\int_t^s f\Big(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r\Big) dr + v\Big(s, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}\Big) \right] \right\}, \end{aligned}$$

for any $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ such that $\mathcal{L}(\xi) = \mu$.

2.3 Smooth Variational Principle

This section is devoted to the study of the regularity of our chosen gauge function for the comparison theorem of viscosity solutions. As outlined in Cosso *et al.* (2024), the key to the comparison theorem of viscosity subsolutions/supersolutions is the existence of a smooth gauge function, so that when the value function is slightly perturbed by the gauge function, it can attain its maximum/minimum, as indicated by the Borwein-Preiss variational principle. To this purpose, we shall make use of the Gaussian regularized sliced Wasserstein distance defined in (Bayraktar *et al.*, 2023a, Section 2) as our gauge function. By a gauge function, we mean the following:

Definition 2.3.1. *Let d_2 be a metric on $\mathcal{P}_2(\mathbb{R}^d)$ such that $(\mathcal{P}_2(\mathbb{R}^d), d_2)$ is complete. Consider the set $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ endowed with the metric $((t,\mu),(s,\nu)) \mapsto |t-s| + d_2(\mu,\nu)$. A map $\rho : ([0,T] \times \mathcal{P}_2(\mathbb{R}^d))^2 \to [0,+\infty)$ is said to be a gauge-type function if the following hold:*

(1). $\rho((t,\mu),(t,\mu)) = 0$, for every $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$;

(2). ρ is continuous on $([0,T] \times \mathcal{P}_2(\mathbb{R}^d))^2$;

(3). for any $\varepsilon > 0$, there exists $\eta > 0$ such that, for any $(t,\mu),(s,\nu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, the inequality $\rho((t,\mu),(s,\nu)) \le \eta$ implies $|t-s| + d_2(\mu,\nu) \le \varepsilon$.

2.3.1 Properties of the Smooth Gauge Function

We recall the definition of the Gaussian regularized sliced Wasserstein distance in (Bayraktar *et al.*, 2023a, Section 2).

Definition 2.3.2 (Gaussian regularized sliced Wasserstein distance)**.** *For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\theta \in \mathcal{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$, we define the mapping $P_\theta : \mathbb{R}^d \to \mathbb{R}$ by $P_\theta(x) := \theta^\top x$ and the pushforward measure $\mu_\theta := P_\theta \sharp \mu \in \mathcal{P}_2(\mathbb{R})$. For any $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the sliced Wasserstein distance is defined by*

$$SW_2(\mu, \nu) := \left\{ \int_{\mathcal{S}^{d-1}} \left[\mathcal{W}_2(\mu_\theta, \nu_\theta) \right]^2 d\theta \right\}^{1/2},$$

where the integration is taken with respect to the standard spherical measure on $\mathcal{S}^{d-1}$. Moreover, the Gaussian regularized sliced Wasserstein distance is defined by

$$SW_2^\sigma(\mu, \nu) := SW_2(\mu^\sigma, \nu^\sigma),$$

where $\mu^\sigma := \mu * \mathcal{N}_\sigma$ and $\mathcal{N}_\sigma \in \mathcal{P}_2(\mathbb{R}^d)$ is the normal distribution with variance $\sigma^2 I_d$ for $\sigma \in (0, \infty)$. By abuse of notation, $\mathcal{N}_\sigma(y)dy$ also denotes the 1-dimensional normal distribution with mean 0 and variance σ^2. We have that $\mu_\theta^\sigma := (\mu^\sigma)_\theta = (\mu * \mathcal{N}_\sigma)_\theta = \mu_\theta * \mathcal{N}_\sigma = (\mu_\theta)^\sigma$, where the first Gaussian is d-dimensional and the second one is 1-dimensional.

Denoting the cumulative distribution function of μ by F_μ, it is well known that (for example, see (Villani, 2003, Theorem 2.18)) in the one-dimensional case, the optimal transport map from μ_θ^σ to ν_θ^σ is given by

$$T_{\theta,\mu}^\sigma(x) := F_{\nu_\theta^\sigma}^{-1}\Big(F_{\mu_\theta^\sigma}(x)\Big),$$

which satisfies $\left[\mathcal{W}_2(\mu_\theta^\sigma, \nu_\theta^\sigma)\right]^2 = \int_{\mathbb{R}} \frac{1}{2} \left| x - T_{\theta,\mu}^\sigma(x) \right|^2 \mu_\theta^\sigma(dx)$. Note that because it is the optimal transport map from μ_θ^σ to ν_θ^σ, for any $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that $\mathcal{L}(X) = \mu$,

$$\mathcal{L}\Big(T_{\theta,\mu}^\sigma(\theta^\top(X + N_\sigma))\Big) = \nu_\theta^\sigma, \tag{2.3.1}$$

where $N_\sigma \in \mathbb{R}^d$ is a normal random variable independent of X, following the distribution $\mathcal{N}_\sigma$.

The following lemma is taken from (Bayraktar *et al.*, 2023a, Lemmas 2.1).

Lemma 2.3.3. *For any $\sigma > 0$, $(\mathcal{P}_2(\mathbb{R}^d), SW_2^\sigma)$ is a complete metric space, and it is equal to $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ as a topological space.*

In order to establish the regularity of the derivatives of this metric, we require an estimate on the optimal transport map as outlined below.

Lemma 2.3.4. *Fix $\sigma > 0$. Let $\mathcal{V}, \mathcal{U} \subset \mathcal{P}_2(\mathbb{R}^d)$ be any compact sets such that all measures in $\mathcal{V}, \mathcal{U}$ are of the form $\mu * \mathcal{N}_\sigma$ for some $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Let $\mathcal{X} \subset \mathbb{R}^d$ be any compact set. There is a positive constant $C_{\mathcal{U},\mathcal{V},\mathcal{X},\sigma}$ depending only on $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and σ such that*

$$|T^\sigma_{\theta,\mu}(z)| \exp\left(\frac{-(z-\theta^\top x)^2}{2\sigma^2}\right) \leq C_{\mathcal{U},\mathcal{V},\mathcal{X},\sigma} \left\{\exp\left(\frac{-z^2}{4\sigma^2}\right) \vee \left[\left(|z|^{1/2}+1\right)\exp\left(\frac{-z^2}{8\sigma^2}\right)\right]\right\},$$

for all $\nu^\sigma \in \mathcal{V}$, $\mu^\sigma \in \mathcal{U}$, $x \in \mathcal{X}$, $z \in \mathbb{R}$ and $\theta \in \mathcal{S}^{d-1}$.

Proof. The values of the constants in this proof may change from line to line in this proof, but they still depend on the same set of parameters as mentioned. First of all, we show that the map $\mathcal{U} \times \mathcal{S}^{d-1} \ni (\mu^\sigma, \theta) \mapsto \mu^\sigma_\theta \in \mathcal{P}_2(\mathbb{R})$ is continuous. Let $(\mu^\sigma_n, \theta_n)_{n\in\mathbb{N}}$ be a sequence converging to (μ^σ, θ). For a L^2 random variable X such that $\mathcal{L}(X) = \mu^\sigma$, it yields that

$$\begin{aligned}\mathcal{W}_2(\mu^\sigma_{n,\theta_n}, \mu^\sigma_\theta) &\leq \mathcal{W}_2(\mu^\sigma_{n,\theta_n}, \mu^\sigma_{\theta_n}) + \mathcal{W}_2(\mu^\sigma_{\theta_n}, \mu^\sigma_\theta)\\ &\leq \mathcal{W}_2(\mu^\sigma_n, \mu^\sigma) + \sqrt{\mathbb{E}|\theta_n^\top X - \theta^\top X|^2},\end{aligned}$$

where $\mu^\sigma_{n,\theta_n} := (\mu_n)^\sigma_{\theta_n}$ and $\mu^\sigma_n := (\mu_n)^\sigma$. Taking $n \to \infty$ and using the dominated convergence theorem, we conclude our desired continuity. It also implies the continuity of the map $\mathcal{V} \times \mathcal{S}^{d-1} \ni (\nu^\sigma, \theta) \mapsto \int_{\mathbb{R}} |y|^2 \nu^\sigma_\theta(dy)$ and hence $C_{\mathcal{V},\sigma} := \sup_{\nu^\sigma \in \mathcal{V}, \theta \in \mathcal{S}^{d-1}} \int_{\mathbb{R}} |y|^2 \nu^\sigma_\theta(dy) < \infty$ by the compactness of $\mathcal{V}$. By Markov's inequality, we have

$$F_{\nu^\sigma_\theta}(z) \leq C_{\mathcal{V},\sigma}/z^2, \text{ for any } z < 0,$$

and hence for $z = F^{-1}_{\nu^\sigma_\theta}(y) < 0$ with $0 < y < F_{\nu^\sigma_\theta}(0)$, we have

$$|F^{-1}_{\nu^\sigma_\theta}(y)| \le \sqrt{C_{\mathcal{V},\sigma}/y}, \text{ for } 0 < y < F_{\nu^\sigma_\theta}(0). \tag{2.3.2}$$

Let $m_{\sigma,\theta,\mu}$ be the median of μ^σ_θ, i.e., $m_{\sigma,\theta,\mu} := F^{-1}_{\mu^\sigma_\theta}(1/2)$. We have that $C_{\mathcal{U},\sigma} := \sup_{\mu^\sigma \in \mathcal{U}, \theta \in \mathcal{S}^{d-1}} \left|F^{-1}_{\mu^\sigma_\theta}(1/2)\right| < \infty$ due to the continuity of the map $\mathcal{U} \times \mathcal{S}^{d-1} \ni (\mu^\sigma, \theta) \mapsto F^{-1}_{\mu^\sigma_\theta}(1/2)$ (see (Van der Vaart, 2000, Lemma 21.2)). Therefore, there is a large enough positive constant $L_{\mathcal{U},\sigma} > 0$ depending only on $\sigma, \mathcal{U}$ such that for any $z < -L_{\mathcal{U},\sigma}$, $\mu^\sigma \in \mathcal{U}$ and $\theta \in \mathcal{S}^{d-1}$, we have

$$\begin{aligned}
F_{\mu^\sigma_\theta}(z) &= \int_{z-m_{\sigma,\theta,\mu}}^{\infty} F_{\mu_\theta}(z-y)\mathcal{N}_\sigma(y)dy + \int_{-\infty}^{z-m_{\sigma,\theta,\mu}} F_{\mu_\theta}(z-y)\mathcal{N}_\sigma(y)dy \\
&\ge \int_{-\infty}^{z-m_{\sigma,\theta,\mu}} F_{\mu_\theta}(m_{\sigma,\theta,\mu})\mathcal{N}_\sigma(y)dy \\
&= \frac{1}{2}\int_{-\infty}^{z-m_{\sigma,\theta,\mu}} \mathcal{N}_\sigma(y)dy \\
&\ge \frac{\sigma^2}{2|z-m_{\sigma,\theta,\mu}|}\left(1 - \frac{\sigma^2}{|z-m_{\sigma,\theta,\mu}|^2}\right)\exp\left(-\frac{|z-m_{\sigma,\theta,\mu}|^2}{2\sigma^2}\right) \\
&\ge \frac{\sigma^2}{4|z-m_{\sigma,\theta,\mu}|}\exp\left(-\frac{|z-m_{\sigma,\theta,\mu}|^2}{2\sigma^2}\right),
\end{aligned} \tag{2.3.3}$$

where the second last inequality comes from, for instance, (Durrett, 2019, Theorem 1.2.6). Note that $\inf_{\nu^\sigma \in \mathcal{V}, \theta \in \mathcal{S}^{d-1}} F_{\nu^\sigma_\theta}(0) > 0$ since every measures ν^σ is Gaussian regularized, and the infimum can actually be attained due to compactness. There is a $L_{\mathcal{U},\mathcal{V},\sigma} \ge L_{\mathcal{U},\sigma}$ such that $0 < F_{\mu^\sigma_\theta}(z) < F_{\nu^\sigma_\theta}(0)$ for any $z < -L_{\mathcal{U},\mathcal{V},\sigma}$. Thus, by (2.3.2),

$$\left|T^\sigma_{\theta,\mu}(z)\right| = \left|F^{-1}_{\nu^\sigma_\theta}(F_{\mu^\sigma_\theta}(z))\right| \le \frac{2C^{1/2}_{\mathcal{V},\sigma}|z-m_{\sigma,\theta,\mu}|^{1/2}}{\sigma}\exp\left(\frac{|z-m_{\sigma,\theta,\mu}|^2}{4\sigma^2}\right)$$

for any $z < -L_{\mathcal{U},\mathcal{V},\sigma}$. The case for the right tail is similar. By Markov's inequality, we have

$$|F^{-1}_{\nu^\sigma_\theta}(y)| \le \sqrt{C_{\mathcal{V},\sigma}/(1-y)}, \text{ for } F_{\nu^\sigma_\theta}(0) < y < 1. \tag{2.3.4}$$

Taking $L_{\mathcal{U},\mathcal{V},\sigma} > 0$ large enough, we see that for any $z > L_{\mathcal{U},\mathcal{V},\sigma}$,

$$\begin{aligned}
F_{\mu^\sigma_\theta}(z) &= \int_{z-m_{\sigma,\theta,\mu}}^{\infty} F_{\mu_\theta}(z-y)\mathcal{N}_\sigma(y)dy + \int_{-\infty}^{z-m_{\sigma,\theta,\mu}} F_{\mu_\theta}(z-y)\mathcal{N}_\sigma(y)dy \\
&\le \int_{z-m_{\sigma,\theta,\mu}}^{\infty} \frac{1}{2}\mathcal{N}_\sigma(y)dy + \int_{-\infty}^{z-m_{\sigma,\theta,\mu}} \mathcal{N}_\sigma(y)dy
\end{aligned}$$

$$= 1 - \int_{z-m_{\sigma,\theta,\mu}}^{\infty} \frac{1}{2}\mathcal{N}_\sigma(y)dy,$$

and therefore

$$1 - F_{\mu_\theta^\sigma}(z) \geq \frac{1}{2}\int_{z-m_{\sigma,\theta,\mu}}^{\infty} \mathcal{N}_\sigma(y)dy.$$

By the symmetry of the distribution of $\mathcal{N}_\sigma$, a similar estimate as (2.3.3) can be derived. Therefore, together with (2.3.2) and (2.3.4), we have that for any $|z| > L_{\mathcal{U},\mathcal{V},\sigma}$,

$$\left|T_{\theta,\mu}^\sigma(z)\right| = \left|F_{\nu_\theta^\sigma}^{-1}(F_{\mu_\theta^\sigma}(z))\right| \leq \frac{2C_{\mathcal{V},\sigma}^{1/2}|z - m_{\sigma,\theta,\mu}|^{1/2}}{\sigma}\exp\left(\frac{|z - m_{\sigma,\theta,\mu}|^2}{4\sigma^2}\right).$$

Since both the distribution and quantile functions of probability measures in $\mathcal{U}$, $\mathcal{V}$ are continuous and strictly increasing on the spatial variable, $T_{\theta,\mu}^\sigma(z)$ is strictly increasing. For the region $|z| \leq L_{\mathcal{U},\mathcal{V},\sigma}$, as the map $T_{\theta,\mu}^\sigma(z)$ is continuous on μ and θ, $K := \sup_{\mu^\sigma\in\mathcal{U},\theta\in\mathcal{S}^{d-1}} |T_{\theta,\mu}^\sigma(L_{\mathcal{U},\mathcal{V},\sigma})| \vee |T_{\theta,\mu}^\sigma(-L_{\mathcal{U},\mathcal{V},\sigma})| < \infty$. Since $T_{\theta,\mu}^\sigma(z)$ is strictly increasing in z, we have $C_{\mathcal{U},\mathcal{V},\sigma} := \sup_{\mu^\sigma\in\mathcal{U},\theta\in\mathcal{S}^{d-1}} \sup_{|z|\leq L_{\mathcal{U},\mathcal{V},\sigma}} |T_{\theta,\mu}^\sigma(z)| < \infty$. Therefore, for any $x \in \mathbb{R}^d$ and $z \in \mathbb{R}$, we have

$$\begin{aligned}
&|T_{\theta,\mu}^\sigma(z)|\exp\left(-\frac{|z - \theta^\top x|^2}{2\sigma^2}\right)\\
&\leq \left[C_{\mathcal{U},\mathcal{V},\sigma}\exp\left(-\frac{|z - \theta^\top x|^2}{2\sigma^2}\right)\right] \vee \left[\frac{2C_{\mathcal{V},\sigma}^{1/2}|z - m_{\sigma,\theta,\mu}|^{1/2}}{\sigma}\exp\left(\frac{|z - m_{\sigma,\theta,\mu}|^2 - 2|z - \theta^\top x|^2}{4\sigma^2}\right)\right].
\end{aligned}$$

Recall that $|m_{\sigma,\theta,\mu}| = |F_{\mu_\theta^\sigma}^{-1}(1/2)| \leq C_{\mathcal{U},\sigma}$. Thus, we can use Young's inequality to obtain that

$$\begin{aligned}
&|T_{\theta,\mu}^\sigma(z)|\exp\left(-\frac{|z - \theta^\top x|^2}{2\sigma^2}\right)\\
&\leq \left[C_{\mathcal{U},\mathcal{V},\sigma}\exp\left(\frac{-z^2/2 + |\theta^\top x|^2}{2\sigma^2}\right)\right] \vee \left[\frac{2C_{\mathcal{V},\sigma}^{1/2}\left(|z|^{1/2} + C_{\mathcal{U},\sigma}^{1/2}\right)}{\sigma}\exp\left(\frac{-z^2/2 + 14|\theta^\top x|^2 + 5C_{\mathcal{U},\sigma}^2}{4\sigma^2}\right)\right].
\end{aligned}$$

As x is in the compact set $\mathcal{X}$, we conclude the required result. □

We have the following lemmas for the derivatives of $\left[SW_2^\sigma(\mu,\nu)\right]^2$, when fixing $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma > 0$:

Lemma 2.3.5. *Let $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma > 0$ be fixed. It holds that*

(1). the mapping $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \big[SW_2^\sigma(\mu,\nu)\big]^2$ *is L-differentiable with*

$$\partial_\mu\big[SW_2^\sigma(\mu,\nu)\big]^2(x) = \int_{\mathcal{S}^{d-1}} \theta\Big\{\theta^\top x - \mathbb{E}\big[T^\sigma_{\theta,\mu}(\theta^\top(x+N_\sigma))\big]\Big\}d\theta;$$

(2). there is a constant $C_d > 0$ *depending only on* d *such that*

$$\int_{\mathbb{R}^d}\Big|\partial_\mu\big[SW_2^\sigma(\mu,\nu)\big]^2(x)\Big|^2\mu(dx) \le C_d\Big(\int_{\mathbb{R}^d}|x|^2\mu(dx) + \int_{\mathbb{R}^d}|y|^2\nu^\sigma(dy)\Big);$$

(3). the mapping $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,\mu) \mapsto \partial_\mu\big[SW_2^\sigma(\mu,\nu)\big]^2(x)$ *is jointly continuous.*

Proof. We follow (Bayraktar *et al.*, 2023a, Lemma 2.2) to obtain items (1) and (2). To prove (3), we let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ and $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ such that $|x_n - x| \to 0$ and $\mathcal{W}_2(\mu_n,\mu) \to 0$ as $n \to \infty$. It is sufficient to consider

$$\begin{aligned}
&\int_{\mathcal{S}^{d-1}} \theta\Big\{\mathbb{E}\big[T^\sigma_{\theta,\mu_n}\big(\theta^\top(x_n+N_\sigma)\big) - T^\sigma_{\theta,\mu}\big(\theta^\top(x+N_\sigma)\big)\big]\Big\}d\theta\\
&= \int_{\mathcal{S}^{d-1}} \theta\Big\{\mathbb{E}\big[T^\sigma_{\theta,\mu_n}\big(\theta^\top x_n + N_\sigma\big) - T^\sigma_{\theta,\mu}\big(\theta^\top x + N_\sigma\big)\big]\Big\}d\theta\\
&= \int_{\mathcal{S}^{d-1}} \theta\int_{\mathbb{R}}\Big[T^\sigma_{\theta,\mu_n}\big(\theta^\top x_n + y\big) - T^\sigma_{\theta,\mu}\big(\theta^\top x + y\big)\Big]\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\Big(-\frac{y^2}{2\sigma^2}\Big)\,dyd\theta\\
&= \frac{1}{(2\pi\sigma^2)^{1/2}}\int_{\mathcal{S}^{d-1}}\int_{\mathbb{R}}\theta T^\sigma_{\theta,\mu_n}(y)\exp\Big(-\frac{|y-\theta^\top x_n|^2}{2\sigma^2}\Big) - \theta T^\sigma_{\theta,\mu}(y)\exp\Big(-\frac{|y-\theta^\top x|^2}{2\sigma^2}\Big)\,dyd\theta,
\end{aligned}$$

where the normal random variable in the first line is d-dimensional and that in the second line is 1-dimensional. By Lemma 2.3.4, the pointwise convergence of $T^\sigma_{\theta,\mu_n}(z)$ and the Lebesgue dominated convergence theorem, we conclude that the above term converges to zero as $n \to \infty$, which implies the desired continuity.

□

Lemma 2.3.6. *Let* $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ *and* $\sigma > 0$ *be fixed. It holds that*

(1). for each $\mu \in \mathcal{P}_2(\mathbb{R}^d)$*, the mapping* $\mathbb{R}^d \ni x \mapsto \partial_\mu\big[SW_2^\sigma(\mu,\nu)\big]^2(x)$ *is differentiable with respect to* x *with*

$$\begin{aligned}
\partial_x\partial_\mu\big[SW_2^\sigma(\mu,\nu)\big]^2(x) &= \mathbb{E}\Big[\int_{\mathcal{S}^{d-1}}\theta\theta^\top\Big(1 - \frac{d}{dz}T^\sigma_{\theta,\mu}(z)\Big|_{z=\theta^\top(x+N_\sigma)}\Big)\,d\theta\Big]\\
&= \int_{\mathcal{S}^{d-1}}\theta\theta^\top d\theta + \int_{\mathbb{R}}\int_{\mathcal{S}^{d-1}}\Big(\theta\theta^\top\frac{(\theta^\top x - y)}{\sigma^2}T^\sigma_{\theta,\mu}(y)\mathcal{N}_\sigma(\theta^\top x - y)\Big)\,d\theta dy;
\end{aligned}$$

(2). there is a constant $C_d > 0$ depending only on d such that

$$\int_{\mathbb{R}^d} \left|\partial_x \partial_\mu \left[SW_2^\sigma(\mu,\nu)\right]^2(x)\right|^2 \mu(dx) \leq C_d \left(1 + \frac{1}{\sigma^2}\int_{\mathbb{R}^d} |x|^2 \nu^\sigma(dx)\right);$$

(3). the mapping $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,\mu) \mapsto \partial_x \partial_\mu \left[SW_2^\sigma(\mu,\nu)\right]^2(x)$ is jointly continuous.

Proof. Item (1) is obtained by directly differentiating (1) in Lemma 2.3.5 and the integration by part. The interchange of the differentiation and the integration is justified in the proof of (Bayraktar *et al.*, 2023a, Lemma 2.4). To prove (2), we use (2.3.1) to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\partial_x \partial_\mu \left[SW_2^\sigma(\mu,\nu)\right]^2(x)\right|^2 \mu(dx) \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathcal{S}^{d-1}} \theta\theta^\top d\theta + \int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \left(\theta\theta^\top \frac{(\theta^\top x - y)}{\sigma^2} T^\sigma_{\theta,\mu}(y) \mathcal{N}_\sigma(\theta^\top x - y)\right) d\theta dy \right|^2 \mu(dx) \\
&\leq 2 \left| \int_{\mathcal{S}^{d-1}} \theta\theta^\top d\theta \right|^2 + 2 \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \left(\theta\theta^\top \frac{(\theta^\top x - y)}{\sigma^2} T^\sigma_{\theta,\mu}(y) \mathcal{N}_\sigma(\theta^\top x - y)\right) d\theta dy \right|^2 \mu(dx) \\
&\leq C_d + \frac{2}{\sigma^4} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}\times\mathcal{S}^{d-1}} (\theta^\top x - y)^2 \mathcal{N}_\sigma(\theta^\top x - y) d\theta dy \right) \left(\int_{\mathbb{R}\times\mathcal{S}^{d-1}} \left[T^\sigma_{\theta,\mu}(y)\right]^2 \mathcal{N}_\sigma(\theta^\top x - y) d\theta dy \right) \mu(dx) \\
&\leq C_d \left(1 + \frac{1}{\sigma^2} \int_{\mathbb{R}^d} |x|^2 \nu^\sigma(dx)\right).
\end{aligned}$$

To prove (3), we let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ and $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ such that $|x_n - x| \to 0$ and $\mathcal{W}_2(\mu_n,\mu) \to 0$ as $n \to \infty$. It is sufficient to consider

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \theta\theta^\top \left(\frac{(\theta^\top x_n - y)}{\sigma^2} T^\sigma_{\theta,\mu_n}(y) \mathcal{N}_\sigma(\theta^\top x_n - y) - \frac{(\theta^\top x - y)}{\sigma^2} T^\sigma_{\theta,\mu}(y) \mathcal{N}_\sigma(\theta^\top x - y) \right) d\theta dy \\
&= \int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \theta\theta^\top \frac{(\theta^\top x_n - \theta^\top x)}{\sigma^2} T^\sigma_{\theta,\mu_n}(y) \mathcal{N}_\sigma(\theta^\top x_n - y) d\theta dy \\
&\quad + \int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \theta\theta^\top \frac{(\theta^\top x - y)}{\sigma^2} \left[T^\sigma_{\theta,\mu_n}(y) - T^\sigma_{\theta,\mu}(y)\right] \mathcal{N}_\sigma(\theta^\top x_n - y) d\theta dy \\
&\quad + \int_{\mathbb{R}} \int_{\mathcal{S}^{d-1}} \theta\theta^\top \frac{(\theta^\top x - y)}{\sigma^2} T^\sigma_{\theta,\mu}(y) \left[\mathcal{N}_\sigma(\theta^\top x_n - y) - \mathcal{N}_\sigma(\theta^\top x - y)\right] d\theta dy.
\end{aligned}$$

By Lemma 2.3.4, the pointwise convergence of $T^\sigma_{\theta,\mu_n}(y)$ and the Lebesgue dominated convergence theorem, we conclude that the above term converges to zero as $n \to \infty$, which implies the desired continuity. □

We have the following estimate when acting $\mathcal{H}$ on the metric:

Lemma 2.3.7. *Let $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma > 0$ be fixed. We have*

$$\Big|\mathcal{H}\Big[SW_2^\sigma(\mu,\nu)\Big]^2\Big| \leq C_d.$$

Proof. For ease of notation, let h denote the map:

$$h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\,;\, \mu \longmapsto h(\mu) := \Big[SW_2^\sigma(\mu,\nu)\Big]^2.$$

For each $X \in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ such that $\mu = \mathcal{L}(X)$, we denote $T^\sigma_{\theta,X}$ to be the map $T^\sigma_{\theta,X}(x) := F^{-1}_{\nu^\sigma_\theta}\Big(F_{\mu^\sigma_\theta}(x)\Big)$. For $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mu = \mathcal{L}(X)$, we also denote h^* to be the lifting of h to Hilbert space, then $h^*(X+w) = h((I_d+w)_\sharp\mu)$, where $I_d + w : \mathbb{R}^d \to \mathbb{R}^d$ is the map $x \mapsto x + w$ for $w \in \mathbb{R}^d$, and

$$\frac{d}{dw}h^*(X+w) = \mathbb{E}[\partial_\mu h(\mathcal{L}(X+w))(X+w)],$$

where $\dfrac{d}{dw}$ is the usual gradient operator on $\mathbb{R}^d$. Therefore, as the laws of $T^\sigma_{\theta,X+w}\Big(\theta^\top(X+w+\bar{N}_\sigma)\Big)$ and $T^\sigma_{\theta,X}\Big(\theta^\top(X+\bar{N}_\sigma)\Big)$ are the same by (2.3.1), we have

$$\begin{aligned}
&\mathbb{E}[\partial_\mu h(\mathcal{L}(X+w))(X+w)] - \mathbb{E}[\partial_\mu h(\mathcal{L}(X))(X)]\\
=&\mathbb{E}\int_{\mathcal{S}^{d-1}} \theta\left\{\theta^\top(X+w) - \bar{\mathbb{E}}\Big[T^\sigma_{\theta,X+w}\Big(\theta^\top(X+w+\bar{N}_\sigma)\Big)\Big]\right\}d\theta\\
&-\mathbb{E}\int_{\mathcal{S}^{d-1}} \theta\left\{\theta^\top X - \bar{\mathbb{E}}\Big[T^\sigma_{\theta,X}\Big(\theta^\top(X+\bar{N}_\sigma)\Big)\Big]\right\}d\theta\\
=&\int_{\mathcal{S}^{d-1}} \theta\theta^\top w d\theta,
\end{aligned}$$

and thus

$$\mathcal{H}h = \int_{\mathcal{S}^{d-1}} \theta\theta^\top d\theta,$$

from which $|\mathcal{H}h| \leq C_d$ is obvious. □

For each $\sigma > 0$, we define $\rho_\sigma : ([0,T]\times\mathcal{P}_2(\mathbb{R}^d))^2 \to \mathbb{R}$ by

$$\rho_\sigma((s,\mu),(t,\nu)) := |t-s|^2 + \Big[SW_2^\sigma(\mu,\nu)\Big]^2. \tag{2.3.5}$$

It is a gauge type function on $([0,T]\times\mathcal{P}_2(\mathbb{R}^d),(\rho_\sigma)^{1/2})$ by Definition 2.3.1. We conclude the following smooth variational principle:

Theorem 2.3.8. *Fix $\delta > 0$ and let $G : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be upper semicontinuous and bounded from above. If there exists $\lambda > 0$ and $(t_0, \mu_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that*

$$\sup_{(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)} G(t,\mu) - \lambda \leq G(t_0,\mu_0),$$

then there exists $(\tilde{t},\tilde{\mu}) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ and a sequence $\{(t_n,\mu_n)\}_{n\in\mathbb{N}} \subset [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that:

(1). $\rho_{1/\delta}\big((\tilde{t},\tilde{\mu}),(t_n,\mu_n)\big) \leq \dfrac{\lambda}{2^n\delta^2}$ *for any $n = 0,1,2,\ldots$;*

(2). $G(t_0,\mu_0) \leq G(\tilde{t},\tilde{\mu}) - \delta^2\varphi_\delta(\tilde{t},\tilde{\mu})$ *with $\varphi_\delta : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to [0,+\infty)$ given by*

$$\varphi_\delta(t,\mu) := \sum_{n=0}^{+\infty} \frac{1}{2^n}\rho_{1/\delta}\left((t,\mu),(t_n,\mu_n)\right), \quad \text{for any } (t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d); \tag{2.3.6}$$

(3). $G(t,\mu) - \delta^2\varphi_\delta(t,\mu) < G(\tilde{t},\tilde{\mu}) - \delta^2\varphi_\delta(\tilde{t},\tilde{\mu})$, *for every $(t,\mu) \in \big([0,T] \times \mathcal{P}_2(\mathbb{R}^d)\big) \setminus \{(\tilde{t},\tilde{\mu})\}$;*

(4). $\varphi_\delta \in PC_1^{1,2}([0,T] \times \mathcal{P}_2(\mathbb{R}^d))$ *and there exists a constant $C_d > 0$ depending only on d such that for any $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, we have*

$$\begin{aligned}
|\partial_t\varphi_\delta(t,\mu)| &\leq 4T;\\
\int_{\mathbb{R}^d} |\partial_\mu\varphi_\delta(t,\mu)(x)|^2\,\mu(dx) &\leq C_d\left(\frac{1+\lambda}{\delta^2} + \int_{\mathbb{R}^d}|x|^2\mu(dx) + \int_{\mathbb{R}^d}|x|^2\mu_0(dx)\right);\\
\int_{\mathbb{R}^d} |\partial_x\partial_\mu\varphi_\delta(t,\mu)(x)|^2\,\mu(dx) &\leq C_d\delta^2\Big(\frac{1+\lambda}{\delta^2} + \int_{\mathbb{R}^d}|x|^2\mu_0(dx)\Big);\\
|\mathcal{H}\varphi_\delta(t,\mu)| &\leq C_d.
\end{aligned}$$

Proof. For items (1) to (3), we refer readers to (Jonathan M. Borwein, 2005, Theorem 2.5.2). As there is a constant $\kappa_d > 0$ depending only on d such that $\int_{\mathcal{S}^{d-1}} \theta\theta^\top d\theta = \kappa_d I_d$, by a simple application of the triangle inequality we conclude that

$$\kappa_d\int_{\mathbb{R}^d}|x|^2\nu(dx) = \int_{\mathcal{S}^{d-1}}\int_{\mathbb{R}}|y|^2\nu_\theta(dy)d\theta \leq C_d\left(\int_{\mathbb{R}^d}|x|^2\mu(dx) + \Big[SW_2(\mu,\nu)\Big]^2\right). \tag{2.3.7}$$

The constant C_d changes from line to line in this proof, but it still depends only on d. Replacing ν and μ with $\mu_n^{1/\delta}$ and $\mu_0^{1/\delta}$ respectively in the above, we have

$$\int_{\mathbb{R}^d}|x|^2\mu_n^{1/\delta}(dx) \leq C_d\left(\int_{\mathbb{R}^d}|x|^2\mu_0^{1/\delta}(dx) + \Big[SW_2(\mu_n^{1/\delta},\tilde{\mu}^{1/\delta})\Big]^2 + \Big[SW_2(\tilde{\mu}^{1/\delta},\mu_0^{1/\delta})\Big]^2\right)$$

$$\leq C_d \left(\int_{\mathbb{R}^d} |x|^2 \mu_0(dx) + \int_{\mathbb{R}^d} |y|^2 \mathcal{N}_{1/\delta}(dy) + \frac{\lambda}{2^n \delta^2} + \frac{\lambda}{\delta^2} \right)$$
$$\leq C_d \left(\int_{\mathbb{R}^d} |x|^2 \mu_0(dx) + \frac{1+\lambda}{\delta^2} + \frac{\lambda}{2^n \delta^2} \right). \tag{2.3.8}$$

From Lemmas 2.3.5-2.3.7, we see that the derivatives $\partial_t \rho_{1/\delta}\left((t,\mu),(t_n,\mu_n)\right)$, $\partial_\mu \rho_{1/\delta}\left((t,\mu),(t_n,\mu_n)\right)(x)$, $\partial_x \partial_\mu \rho_{1/\delta}\left((t,\mu),(t_n,\mu_n)\right)(x)$ and $\mathcal{H}\rho_{1/\delta}\left((t,\mu),(t_n,\mu_n)\right)$ exist for each $n = 0, 1, 2, \ldots$ and also

$$|\partial_t \rho_{1/\delta}((t,\mu),(t_n,\mu_n))| = 2t \text{ and } \left| \mathcal{H}\Big(\rho_{1/\delta}((t,\mu),(t_n,\mu_n))\Big) \right| \leq C_d. \tag{2.3.9}$$

The interchange of the differentiation and the infinite sum for $\partial_t \varphi_{1/\delta}(t,\mu)$ and $\mathcal{H}\varphi_{1/\delta}(t,\mu)$ are obvious by the above bounds and Lemma 2.3.4. Therefore, $\partial_t \varphi_{1/\delta}(t,\mu)$ and $\mathcal{H}\varphi_{1/\delta}(t,\mu)$ exist and have the desired upper bounds by using the definition of $\varphi_{1/\delta}(t,\mu)$ and (2.3.9). Moreover, we are safe to interchange the measure differentiation and the infinite sum for $\partial_\mu \varphi_{1/\delta}(t,\mu)(x)$ by the proof of (Bayraktar *et al.*, 2023a, Proposition 2.5), thus the derivative $\partial_\mu \varphi_{1/\delta}(t,\mu)(x)$ also exists. According to Lemmas 2.3.4 and 2.3.6, we see that the derivative $\partial_x \partial_\mu \varphi_{1/\delta}(t,\mu)(x)$ also exists by interchanging the differentiation and the infinite sum. The joint continuity of these derivatives follows from the dominating function constructed in Lemma 2.3.4, (3) in Lemma 2.3.5, (3) in Lemma 2.3.6, Lemma 2.3.7, the definition of $\varphi_{1/\delta}(t,\mu)$ and the dominated convergence theorem. Finally, from Lemmas 2.3.5-2.3.7 and (2.3.8), we have

$$\int_{\mathbb{R}^d} |\partial_\mu \rho_{1/\delta}((t,\mu),(t_n,\mu_n))(x)|^2 \mu(dx) \leq C_d \Big(\frac{1+\lambda}{\delta^2} + \frac{\lambda}{2^n \delta^2} + \int_{\mathbb{R}^d} |x|^2 \mu(dx) + \int_{\mathbb{R}^d} |x|^2 \mu_0(dx) \Big),$$
$$\int_{\mathbb{R}^d} |\partial_x \partial_\mu \rho_{1/\delta}((t,\mu),(t_n,\mu_n))(x)|^2 \mu(dx) \leq C_d \Big(1 + \lambda + \frac{\lambda}{2^n} + \delta^2 \int_{\mathbb{R}^d} |x|^2 \mu_0(dx) \Big).$$

By the above bounds, (2.3.9) and the definition of $\varphi_{1/\delta}(t,\mu)$, we conclude the desired bounds.

□

2.4 Smooth Finite-dimensional Approximations of the Value Function

This section is to carry out the smooth finite-dimensional approximations of the value function through the n-particles approximation of the measure argument and the mollification of the coefficient functions b, f and g.

2.4.1 Infinite-dimensional Approximation

Fix a complete probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$, also of the form $(\check{\Omega}^0 \times \check{\Omega}^1, \check{\mathcal{F}}^0 \otimes \check{\mathcal{F}}^1, \check{\mathbb{P}}^0 \otimes \check{\mathbb{P}}^1)$. The space $(\check{\Omega}^0, \check{\mathcal{F}}^0, \check{\mathbb{P}}^0)$ supports a d-dimensional Brownian motion $\check{W}^0$. For $(\check{\Omega}^1, \check{\mathcal{F}}^1, \check{\mathbb{P}}^1)$, it is of the form $(\check{\tilde{\Omega}}^1 \times \check{\tilde{\Omega}}^1, \check{\mathcal{G}} \otimes \check{\tilde{\mathcal{F}}}^1, \check{\tilde{\mathbb{P}}}^1 \otimes \check{\tilde{\mathbb{P}}}^1)$. On $(\check{\tilde{\Omega}}^1, \check{\tilde{\mathcal{F}}}^1, \check{\tilde{\mathbb{P}}}^1)$, there lives d-dimensional Brownian motions $\check{W}$ and $\check{B}$. The space $(\check{\tilde{\Omega}}^1, \check{\mathcal{G}}, \check{\tilde{\mathbb{P}}}^1)$ is where the initial random variables live. We assume that $(\check{\tilde{\Omega}}^1, \check{\mathcal{G}}, \check{\tilde{\mathbb{P}}}^1)$ is rich enough to support all probability laws in $\mathbb{R}^d$, i.e., for any probability law μ in $\mathbb{R}^d$, there exists $X \in \check{\tilde{\Omega}}^1$ such that $\mathcal{L}(X) = \mu$. The expectation $\mathbb{E}$ is taken with respect to $\check{\mathbb{P}}^0 \otimes \check{\mathbb{P}}^1$.

Set $\check{\mathbb{F}} = (\check{\mathcal{F}}_s)_{s\geq 0} := \big(\sigma(\check{W}^0_r)_{0\leq r\leq s} \vee \sigma(\check{W}_r)_{0\leq r\leq s} \vee \sigma(\check{B}_r)_{0\leq r\leq s} \vee \check{\mathcal{G}}\big)_{s\geq 0}$, $\check{\mathbb{F}}^t = (\mathcal{F}^t_s)_{s\geq 0} := \big(\sigma(\check{W}^0_r)_{0\leq r\leq s} \vee \sigma(\check{W}_{r\vee t} - \check{W}_t)_{0\leq r\leq s} \vee \sigma(\check{B}_{r\vee t} - \check{B}_t)_{0\leq r\leq s} \vee \check{\mathcal{G}}\big)_{s\geq 0}$. Without loss of generality, we assume that they are $\check{\mathbb{P}}$-complete. Let $t > 0$, denote $\check{\mathcal{A}}$ (resp. $\check{\mathcal{A}}_t$) the set of $\check{\mathbb{F}}$-progressively measurable process (resp. $\check{\mathbb{F}}^t$-progressively measurable process) $\check{\alpha}$ valued in A.

Letting $\varepsilon > 0$, $t \in [0, T)$, $\check{\xi} \in L^2(\check{\Omega}^1, \check{\mathcal{F}}^1, \check{\mathbb{P}}^1; \mathbb{R}^d)$ and $\check{\alpha} \in \check{\mathcal{A}}$, we consider the unique solution $\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}} = (\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_s)_{s\in[t,T]}$ of the system of the perturbed equation:

$$\begin{aligned} X_s =& \check{\xi} + \int_t^s b(r, X_r, \mathbb{P}^{\check{W}^0}_{X_r}, \check{\alpha}_r)dr + \int_t^s \sigma(r, X_r, \check{\alpha}_r)d\check{W}_r + \int_t^s \sigma^0(r)d\check{W}^0_r \\ &+ \varepsilon(\check{B}_s - \check{B}_t). \end{aligned} \tag{2.4.1}$$

For any $t \in [0,T]$ and $\check{\xi} \in L^2(\check{\Omega}^1, \check{\mathcal{F}}^1, \check{\mathbb{P}}^1; \mathbb{R}^d)$, we consider the value function:

$$V_\varepsilon(t,\check{\xi}) = \sup_{\check{\alpha}\in\check{\mathcal{A}}_t} J_\varepsilon(t,\check{\xi},\check{\alpha}) := \sup_{\check{\alpha}\in\check{\mathcal{A}}_t} \mathbb{E}\left[\int_t^T f\Big(s, \check{X}_s^{\varepsilon,t,\check{\xi},\check{\alpha}}, \mathbb{P}^{\check{W}^0}_{\check{X}_s^{\varepsilon,t,\check{\xi},\check{\alpha}}}, \check{\alpha}_s\Big)ds + g\Big(\check{X}_T^{\varepsilon,t,\check{\xi},\check{\alpha}}, \mathbb{P}^{\check{W}^0}_{\check{X}_T^{\varepsilon,t,\check{\xi},\check{\alpha}}}\Big)\right]. \tag{2.4.2}$$

By the law invariance property, we can define a function $v_\varepsilon(t,\mu) : [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that

$$v_\varepsilon(t,\mu) := V_\varepsilon(t,\check{\xi}), \tag{2.4.3}$$

for any $\check{\xi} \in L^2(\check{\Omega}^1, \check{\mathcal{F}}^1, \check{\mathbb{P}}^1; \mathbb{R}^d)$ such that $\mathcal{L}(\check{\xi}) = \mu$.

Lemma 2.4.1. *Suppose that Assumption (A) holds. There exists a constant $C_5 = C_5(d,K,T)$ such that for any $\varepsilon \geq 0$ and $(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, it holds that $|v_\varepsilon(t,\mu) - v_0(t,\mu)| \leq C_5\varepsilon$.*

The proof is standard by using Assumption (A), equation (2.4.1) and the definition in (2.4.2). We omit it here.

2.4.2 Finite-dimensional Approximation

In this section, we will illustrate the finite-dimensional approximation. Consider a complete probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, it is also of the form $(\overline{\Omega}^0 \times \overline{\Omega}^1, \overline{\mathcal{F}}^0 \otimes \overline{\mathcal{F}}^1, \overline{\mathbb{P}}^0 \otimes \overline{\mathbb{P}}^1)$. The space $(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0)$ supports a d-dimensional Brownian motion $\overline{W}^0$. For $(\overline{\Omega}^1, \overline{\mathcal{F}}^1, \overline{\mathbb{P}}^1)$, it is of the form $(\overline{\tilde{\Omega}}^1 \times \overline{\hat{\Omega}}^1, \overline{\mathcal{G}} \otimes \overline{\hat{\mathcal{F}}}^1, \overline{\tilde{\mathbb{P}}}^1 \otimes \overline{\hat{\mathbb{P}}}^1)$. Let $n \in \mathbb{N}$, there lives d-dimensional Brownian motions $\overline{W}^1, \ldots, \overline{W}^n$ and $\overline{B}^1, \ldots, \overline{B}^n$ on $(\overline{\hat{\Omega}}^1, \overline{\hat{\mathcal{F}}}^1, \overline{\hat{\mathbb{P}}}^1)$. We require $\overline{W}^1, \ldots, \overline{W}^n, \overline{B}^1, \ldots, \overline{B}^n$ to be mutually independent. The space $(\overline{\tilde{\Omega}}^1, \overline{\mathcal{G}}, \overline{\tilde{\mathbb{P}}}^1)$ is where the initial random variables live. We assume that $(\overline{\tilde{\Omega}}^1, \overline{\mathcal{G}}, \overline{\tilde{\mathbb{P}}}^1)$ is rich enough to support all probability laws in $\mathbb{R}^d$, i.e., for any probability law μ in $\mathbb{R}^d$, there exists $X \in \overline{\tilde{\Omega}}^1$ such that $\mathcal{L}(X) = \mu$.

Put $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_s)_{s\geq 0} := \Big(\sigma(\overline{W}^0_r)_{0\leq r\leq s} \vee \sigma(\overline{W}^i_r)_{0\leq r\leq s, i=1,\ldots,n} \vee \sigma(\overline{B}^i_r)_{0\leq r\leq s, i=1,\ldots,n} \vee \overline{\mathcal{G}}\Big)_{s\geq 0}$, $\overline{\mathbb{F}}^t = (\mathcal{F}^t_s)_{s\geq 0} := \Big(\sigma(\overline{W}^0_r)_{0\leq r\leq s} \vee \sigma(\overline{W}^i_{r\vee t} - \overline{W}^i_t)_{0\leq r\leq s, i=1,\ldots,n} \vee \sigma(\overline{B}^i_{r\vee t} - \overline{B}^i_t)_{0\leq r\leq s, i=1,\ldots,n} \vee \overline{\mathcal{G}}\Big)_{s\geq 0}$. Without loss of generality, we assume that they are $\overline{\mathbb{P}}$-complete. Let $t > 0$ and denote $\overline{\mathcal{A}}^n$ (resp.

$\overline{\mathcal{A}}_t^n$) the set of $\overline{\mathbb{F}}$-progressively measurable process (resp. $\overline{\mathbb{F}}^t$-progressively measurable process) $\overline{\alpha} = (\overline{\alpha}^1, \dots, \overline{\alpha}^n)$ valued in A^n.

Let $\overline{\xi} = (\overline{\xi}^1, \dots, \overline{\xi}^n) \in \mathbb{R}^{dn}$, we consider the solution $\overline{X}_s^{\varepsilon,t,\overline{\xi},\overline{\alpha}} = (\overline{X}_s^{1,\varepsilon,t,\overline{\xi},\overline{\alpha}}, \dots, \overline{X}_s^{n,\varepsilon,t,\overline{\xi},\overline{\alpha}}) \in \mathbb{R}^{dn}$ of the system of the perturbed equation:

$$\begin{aligned} X_s^i =& \overline{\xi}^i + \int_t^s b(r, X_r^i, \widehat{\mu}_r^n, \overline{\alpha}_r^i)dr + \int_t^s \sigma(r, X_r^i, \overline{\alpha}_r^i)d\overline{W}_r^i + \int_t^s \sigma^0(r)d\overline{W}_r^0 \\ &+ \varepsilon(\overline{B}_s^i - \overline{B}_t^i), \end{aligned} \tag{2.4.4}$$

for each $i = 1, 2, \dots, n$ and $X_s = (X_s^1, \dots, X_s^n)$ with

$$\widehat{\mu}_r^n = \frac{1}{n}\sum_{i=1}^n \delta_{\overline{X}_r^i}.$$

Denoting $\widehat{\mu}_s^{n,\varepsilon,t,\overline{\xi},\overline{\alpha}} := \dfrac{1}{n}\displaystyle\sum_{i=1}^n \delta_{\overline{X}_s^{i,\varepsilon,t,\overline{\xi},\overline{\alpha}}}$, we consider the value function:

$$\begin{aligned} \widetilde{v}_{\varepsilon,n}(t,\overline{\mu}) :=& \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} J_{\varepsilon,n}(t,\overline{\xi},\overline{\alpha}) \\ :=& \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\int_t^T f(s, \overline{X}_s^{i,\varepsilon,t,\overline{\xi},\overline{\alpha}}, \widehat{\mu}_s^{n,\varepsilon,t,\overline{\xi},\overline{\alpha}}, \overline{\alpha}_s^i)ds + g(\overline{X}_T^{i,\varepsilon,t,\overline{\xi},\overline{\alpha}}, \widehat{\mu}_T^{n,\varepsilon,t,\overline{\xi},\overline{\alpha}})\right], \end{aligned}$$

for any $t \in [0,T]$ and $\overline{\mu} \in \mathcal{P}_2(\mathbb{R}^{dn})$ such that $\mathcal{L}(\overline{\xi}) = \overline{\mu}$.

Now we introduce the smooth approximation of the coefficients. For $n, m \in \mathbb{N}$ and $i = 1, \dots, n$, we define $b_{n,m}^i : [0,T]\times\mathbb{R}^{dn}\times A \to \mathbb{R}^d$, $f_{n,m}^i : [0,T]\times\mathbb{R}^{dn}\times A \to \mathbb{R}^d$, $g_{n,m}^i : \mathbb{R}^{dn} \to \mathbb{R}^d$ be the smooth approximation of b,f and g respectively such that

$$b_{n,m}^i(t,\overline{x},a) := m^{dn+1}\int_{\mathbb{R}^{dn+1}} b\left((t-s)^+\wedge T, x^i - y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j-y^j}, a\right)\phi(ms)\prod_{k=1}^n \Phi(my^k)dy^k ds; \tag{2.4.5}$$

$$f_{n,m}^i(t,\overline{x},a) := m^{dn+1}\int_{\mathbb{R}^{dn+1}} f\left((t-s)^+\wedge T, x^i - y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j-y^j}, a\right)\phi(ms)\prod_{k=1}^n \Phi(my^k)dy^k ds; \tag{2.4.6}$$

$$g^i_{n,m}(\overline{x}) := m^{dn} \int_{\mathbb{R}^{dn}} g\left(x^i - y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}\right) \prod_{k=1}^n \Phi(my^k) dy^k, \tag{2.4.7}$$

where $t \in [0,T]$, $a \in A$ and $\overline{x} = (x^1, \ldots, x^n)$. The functions $\phi : \mathbb{R} \to \mathbb{R}^+$, $\Phi : \mathbb{R}^d \to \mathbb{R}^+$ are two compactly supported smooth functions satisfying $\int_{\mathbb{R}} \phi(s)ds = 1$ and $\int_{\mathbb{R}^d} \Phi(y)dy = 1$. First, we establish some basic properties of these approximation functions.

Lemma 2.4.2. *Suppose that Assumption (A*) holds and define*

$$\widehat{\mu}^{n,\overline{x}} := \frac{1}{n}\sum_{j=1}^n \delta_{x^j}.$$

For any $i = 1, 2, \ldots, n$, we have the following

(1). **Upper Bound:** *for any $n, m \in \mathbb{N}$ and $(t, \overline{x}, a) \in [0,T] \times \mathbb{R}^{dn} \times A$, we have*

$$|b^i_{n,m}(t,\overline{x},a)| \vee |f^i_{n,m}(t,\overline{x},a)| \vee |g^i_{n,m}(\overline{x})| \le K;$$

(2). **Convergence:** *it holds that*

$$\lim_{m\to\infty} b^i_{n,m}(t,\overline{x},a) = b(t,x^i,\widehat{\mu}^{n,\overline{x}},a), \quad \lim_{m\to\infty} f^i_{n,m}(t,\overline{x},a) = f(t,x^i,\widehat{\mu}^{n,\overline{x}},a)$$
$$\lim_{m\to\infty} g^i_{n,m}(\overline{x}) = g(x^i,\widehat{\mu}^{n,\overline{x}}),$$

uniformly in $(t,\overline{x},a) \in [0,T] \times \mathbb{R}^{dn} \times A$;

(3). **Convergence Rate:** *for any $n, m \in \mathbb{N}$, $(t,\overline{x},a) \in [0,T] \times \mathbb{R}^{dn} \times A$, we have the estimates*

(i). for b and f:

$$\begin{aligned}
&|b^i_{n,m}(t,\overline{x},a) - b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)| \vee |f^i_{n,m}(t,\overline{x},a) - f(t,x^i,\widehat{\mu}^{n,\overline{x}},a)| \\
&\le Km\int_{\mathbb{R}} \left|t - \left[T \wedge (t-s)^+\right]\right|^\beta \phi(ms)ds + Km^{dn}\int_{\mathbb{R}^{dn}} \left(|y^i| + \frac{1}{n}\sum_{j=1}^n |y^j|\right)\prod_{k=1}^n \Phi(my^k)dy^k;
\end{aligned}$$

(ii). for g:

$$|g^i_{n,m}(\overline{x}) - g(x^i,\widehat{\mu}^{n,\overline{x}})| \le Km^{dn}\int_{\mathbb{R}^{dn}} \left(|y^i| + \frac{1}{n}\sum_{j=1}^n |y^j|\right)\prod_{k=1}^n \Phi(my^k)dy^k;$$

(4). **Continuity:** *for any* $n, m \in \mathbb{N}$, $(t, \overline{x}, \overline{x}', a) \in [0,T] \times \mathbb{R}^{dn} \times \mathbb{R}^{dn} \times A$, *we have the estimate*

$$
\begin{aligned}
&|b^i_{n,m}(t,\overline{x},a) - b^i_{n,m}(t,\overline{x}',a)| \vee |f^i_{n,m}(t,\overline{x},a) - f^i_{n,m}(t,\overline{x}',a)| \vee |g^i_{n,m}(\overline{x}) - g^i_{n,m}(\overline{x}')| \\
&\leq K\left[|x^i - x'^i| + \frac{1}{n}\sum_{j=1}^n \left|x^j - x'^j\right|\right]. \qquad (2.4.8)
\end{aligned}
$$

Proof. Assertion (1) is obvious by the definitions in (2.4.5)-(2.4.7) and Assumption (A*). We prove assertion (3) for g by considering

$$
\left|g(x^i, \widehat{\mu}^{n,\overline{x}}) - g^i_{n,m}(\overline{x})\right| \leq m^{dn}\int_{\mathbb{R}^{dn}} \left|g(x^i, \widehat{\mu}^{n,\overline{x}}) - g\Big(x^i - y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}\Big)\right| \prod_{k=1}^n \Phi(my^k)dy^k.
$$

Using the Lipschitz continuity of g in Assumption (A*) and the fact that

$$
\begin{aligned}
\mathcal{W}_1\Big(\widehat{\mu}^{n,\overline{x}}, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}\Big) &= \mathcal{W}_1\Big(\frac{1}{n}\sum_{j=1}^n \delta_{x^j}, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}\Big) \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \left[\frac{1}{n}\sum_{j=1}^n \delta_{(x^j, x^j - y^j)}(dx, dy)\right] \\
&= \frac{1}{n}\sum_{j=1}^n |y^j|, \qquad (2.4.9)
\end{aligned}
$$

we obtain that

$$
\left|g(x^i, \widehat{\mu}^{n,\overline{x}}) - g^i_{n,m}(\overline{x})\right| \leq Km^{dn}\int_{\mathbb{R}^{dn}} \Big(|y^i| + \frac{1}{n}\sum_{j=1}^n |y^j|\Big) \prod_{k=1}^n \Phi(my^k)dy^k.
$$

We prove assertion (3) for b by considering (the proof for f is exactly the same)

$$
\begin{aligned}
&|b(t, x^i, \widehat{\mu}^{n,\overline{x}}, a) - b^i_{n,m}(t,\overline{x},a)| \\
&\leq m^{dn+1}\int_{\mathbb{R}^{dn+1}} \left|b(t, x^i, \widehat{\mu}^{n,\overline{x}}, a) - b\Big(T \wedge (t-s)^+, x^i - y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}, a\Big)\right| \phi(ms) \prod_{k=1}^n \Phi(my^k)dy^k ds \\
&\leq m\int_{\mathbb{R}} \left|b(t, x^i, \widehat{\mu}^{n,\overline{x}}, a) - b\left(T \wedge (t-s)^+, x^i, \widehat{\mu}^{n,\overline{x}}, a\right)\right| \phi(ms)ds \\
&\quad + m^{dn+1}\int_{\mathbb{R}^{dn+1}} \left|b\left(T \wedge (t-s)^+, x^i, \widehat{\mu}^{n,\overline{x}}, a\right) - b\Big(T \wedge (t-s)^+, x^i - y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}, a\Big)\right|.
\end{aligned}
$$

$$\phi(ms)\prod_{j=1}^{n}\Phi(my^j)dy^j ds.$$

The inequality in (2.4.9) and Assumption (A*) implies that

$$\begin{aligned}
&|b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)-b^i_{n,m}(t,\overline{x},a)|\\
&\le Km\int_{\mathbb{R}}\left|t-(T\wedge(t-s)^+)\right|^{\beta}\phi(ms)ds+Km^{dn}\int_{\mathbb{R}^{dn}}\Big(|y^i|+\frac{1}{n}\sum_{j=1}^{n}|y^j|\Big)\prod_{k=1}^{n}\Phi(my^k)dy^k.
\end{aligned}$$

Assertion (2) follows immediately from assertion (3). For the proof of assertion (4) for g (the proofs for f and b are exactly the same), we let $\overline{x}$, $\overline{z}\in\mathbb{R}^{dn}$ and estimate

$$\begin{aligned}
&|g^i_{n,m}(\overline{x})-g^i_{n,m}(\overline{z})|\\
&\le m^{dn}\int_{\mathbb{R}^{dn}}\left|g\Big(x^i-y^i,\frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}\Big)-g\Big(z^i-y^i,\frac{1}{n}\sum_{j=1}^{n}\delta_{z^j-y^j}\Big)\right|\prod_{k=1}^{n}\Phi(my^k)dy^k.
\end{aligned}$$

Then the inequality in (2.4.9) yields that

$$\begin{aligned}
|g^i_{n,m}(\overline{x})-g^i_{n,m}(\overline{z})|&\le Km^{dn}\int_{\mathbb{R}^{dn}}\Big[|x^i-z^i|+\frac{1}{n}\sum_{j=1}^{n}|x^j-z^j|\Big]\prod_{k=1}^{n}\Phi(my^k)dy^k\\
&=K\Big[|x^i-z^i|+\frac{1}{n}\sum_{j=1}^{n}|x^j-z^j|\Big].
\end{aligned}$$

□

Remark 2.4.1. *It is worth noting that the coefficients b, f and g are assumed to be $\mathcal{W}_1$-Lipschitz continuous in Assumption (A*), which is stronger than the $\mathcal{W}_2$-Lipschitz continuity assumed in (Cosso* et al.*, 2024, Assumption (A)). The main reason is to provide a better Lipschitz estimate when we establish the smooth approximation of the coefficients in Lemma 2.4.2, which is critical in the proof of Lemma 2.4.5 and Theorem 2.5.3. We note that there are some gaps in the proofs of (Cosso* et al.*, 2024, Theorem 5.1, Theorem A.7) if only $\mathcal{W}_2$-Lipschitz continuity is assumed in (Cosso* et al.*, 2024, Assumption (A)). More precisely, if we assume the coefficients are only $\mathcal{W}_2$-Lipschitz continuous, then we only have*

$$\mathcal{W}_2\Big(\widehat{\mu}^{n,\overline{x}},\frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}\Big)=\mathcal{W}_2\Big(\frac{1}{n}\sum_{j=1}^{n}\delta_{x^j},\frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}\Big)\le\frac{1}{n^{1/2}}\sum_{j=1}^{n}|y^j|, \tag{2.4.10}$$

instead of the estimate in (2.4.9)*. It seems that the corresponding estimate of* $\mathcal{W}_2\Big(\widehat{\mu}^{n,\overline{x}}, \frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}\Big)$ *in the proof of (Cosso* et al.*, 2024, Lemma A.3) is not true, see the counterexample at the end of this remark and the estimate in* (2.4.12)*. Hence, adopting the* $\mathcal{W}_2$*-Lipschitz continuity and the estimate in* (2.4.10)*, assertion (3) of Lemma 2.4.2 becomes*

$$\begin{aligned}&|b^i_{n,m}(t,\overline{x},a)-b^i_{n,m}(t,\overline{x}',a)|\vee|f^i_{n,m}(t,\overline{x},a)-f^i_{n,m}(t,\overline{x}',a)|\vee|g^i_{n,m}(\overline{x})-g^i_{n,m}(\overline{x}')|\\&\leq K\left[|x^i-x'^i|+\frac{1}{n^{1/2}}\sum_{j=1}^{n}\left|x^j-x'^j\right|\right],\end{aligned}\tag{2.4.11}$$

see the corresponding estimate in (Cosso et al.*, 2024, (A.7)). This causes a problem when we apply Lemma 2.4.2 to prove Lemma 2.4.5 and the bound of the first order derivative in* (2.4.18) *(see also the corresponding estimate in (Cosso* et al.*, 2024, (A.12))), which is critical in the proof of Theorem 2.5.3. While (Cosso* et al.*, 2024, Assumption (A)) adopts the* $\mathcal{W}_2$*-Lipschitz continuity assumption, the estimate in (Cosso* et al.*, 2024, (A.7)) is invalid and we should have* (2.4.11) *instead. However, by using the arguments in Lemma 2.4.2, 2.4.5, we showed that the estimates (Cosso* et al.*, 2024, (A.6), (A.7), (A.12)) and the proofs of (Cosso* et al.*, 2024, Theorem 5.1, Theorem A.7) should be valid if (Cosso* et al.*, 2024, Assumption (A)) adopts to the* $\mathcal{W}_1$*-Lipschitz continuity.*

Here is a counterexample to the estimate of $\mathcal{W}_2\Big(\widehat{\mu}^{n,\overline{x}}, \frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}\Big)$ *in the proof of (Cosso* et al.*, 2024, Lemma A.3)) which claims that*

$$\mathcal{W}_2\Big(\widehat{\mu}^{n,\overline{x}}, \frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}\Big)\leq\frac{1}{n}\sum_{j=1}^{n}|y^j|.\tag{2.4.12}$$

Suppose $d=1$, $\overline{x}=(x^1,x^2,\ldots,x^n)^\top=(0,\ldots,0)^\top$ *and* $\overline{y}=(y^1,y^2,\ldots,y^n)^\top=(y^1,0,\ldots,0)^\top$. *Consider the probability measures* $\widehat{\mu}^{n,\overline{x}}=\delta_0$ *and* $\frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}=\frac{1}{n}\sum_{j=1}^{n}\delta_{-y^j}$. *As* $\widehat{\mu}^{n,\overline{x}}=\delta_0$*, all mass should be transported from/into this point only (see (Villani, 2003, (11) in Chapter 1)), therefore, there is only one joint distribution* π *such that* $\pi(A\times\mathbb{R})=\widehat{\mu}^{n,\overline{x}}(A)=\delta_0(A)$ *and* $\pi(\mathbb{R}\times B)=\frac{1}{n}\sum_{j=1}^{n}\delta_{-y^j}(B)$ *for any measurable sets* A *and* B *in* $\mathbb{R}$*. It is given by*

$\pi(A\times B)=\frac{1}{n}\sum_{j=1}^n\delta_0(A)\delta_{-y^j}(B)$. *Thus, for* $n>1$, *we have*

$$\mathcal{W}_2\Big(\widehat{\mu}^{n,\overline{x}},\frac{1}{n}\sum_{j=1}^n\delta_{x^j-y^j}\Big)=\left(\int_{\mathbb{R}}|y|^2\frac{1}{n}\sum_{j=1}^n\delta_{-y^j}(dy)\right)^{1/2}=\left(\frac{1}{n}\sum_{j=1}^n|y^j|^2\right)^{1/2}=\frac{1}{n^{1/2}}\sum_{j=1}^n|y^j|,$$

where we have used $(y^1,y^2,\dots,y^n)^\top=(y^1,0,\dots,0)^\top\in\mathbb{R}^n$ *in the last equality. It contradicts* (2.4.12).

Let $\overline{X}_s^{m,\varepsilon,t,\overline{\xi},\overline{\alpha}}=(\overline{X}_s^{1,m,\varepsilon,t,\overline{\xi},\overline{\alpha}},\dots,\overline{X}_s^{n,m,\varepsilon,t,\overline{\xi},\overline{\alpha}})\in\mathbb{R}^{dn}$ be the solution of

$$\begin{aligned}X_s^i=&\overline{\xi}^i+\int_t^s b_{n,m}^i(r,X_r,\overline{\alpha}_r^i)dr+\int_t^s\sigma(r,X_r^i,\overline{\alpha}_r^i)d\overline{W}_r^i+\int_t^s\sigma^0(r)d\overline{W}_r^0\\&+\varepsilon(\overline{B}_s^i-\overline{B}_t^i),\end{aligned}\tag{2.4.13}$$

for $i=1,\dots,n$, where X_s^i is the ith-component of X_s. We define

$$\begin{aligned}\widetilde{v}_{\varepsilon,n,m}(t,\overline{\mu}):=&\sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n}J_{\varepsilon,n}^*(t,\overline{\xi},\overline{\alpha})\\:=&\sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n}\frac{1}{n}\sum_{i=1}^n\mathbb{E}\bigg[\int_t^T f_{n,m}^i\Big(s,\overline{X}_s^{1,m,\varepsilon,t,\overline{\xi},\overline{\alpha}},\dots,\overline{X}_s^{n,m,\varepsilon,t,\overline{\xi},\overline{\alpha}},\overline{\alpha}_s^i\Big)ds\\&+g_{n,m}^i\Big(\overline{X}_T^{1,m,\varepsilon,t,\overline{\xi},\overline{\alpha}},\dots,\overline{X}_T^{n,m,\varepsilon,t,\overline{\xi},\overline{\alpha}}\Big)\bigg],\end{aligned}\tag{2.4.14}$$

for any $t\in[0,T]$ and $\overline{\mu}\in\mathcal{P}_2(\mathbb{R}^{dn})$ such that $\mathcal{L}(\overline{\xi})=\overline{\mu}$. We define $v_{\varepsilon,n,m}(t,\mu)$, $v_{\varepsilon,n}(t,\mu):[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ by

$$v_{\varepsilon,n,m}(t,\mu):=\widetilde{v}_{\varepsilon,n,m}(t,\mu\otimes\ldots\otimes\mu)\quad\text{and}\quad v_{\varepsilon,n}(t,\mu):=\widetilde{v}_{\varepsilon,n}(t,\mu\otimes\ldots\otimes\mu).\tag{2.4.15}$$

Lemma 2.4.3. *Suppose that Assumption (A*) holds. There exists a constant* $C_6=C_6(K,T,d)$ *such that for any* $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, *we have* $|v_{\varepsilon,n,m}(t,\mu)-v_{0,n,m}(t,\mu)|\le C_6\varepsilon$.

Proof. For every $i=1,\dots,n$, $m\in\mathbb{N}$, $\varepsilon\ge0$, $t\in[0,T]$, $\overline{\xi}\in L^2(\overline{\Omega},\overline{\mathcal{F}}_t,\overline{\mathbb{P}};\mathbb{R}^{nd})$, $\overline{\alpha}\in\overline{\mathcal{A}}_t^n$, as ε is the only parameter that is varying, we write $\overline{X}_s^\varepsilon$ to stand for $(\overline{X}_s^{1;\varepsilon},\dots,\overline{X}_s^{n;\varepsilon})=(\overline{X}_s^{1,m,\varepsilon,t,\overline{\xi},\overline{\alpha}},\dots,\overline{X}_s^{n,m,\varepsilon,t,\overline{\xi},\overline{\alpha}})$ which solves (2.4.13). Then with (2.4.8), it is easily seen that there is some constant $C_{d,K,T}>0$ such that

$$\mathbb{E}\left[|X_s^{i;\varepsilon}-X_s^{i;0}|^2\right]\le C_{d,K,T}\mathbb{E}\bigg[\int_t^s|X_r^{i;\varepsilon}-X_r^{i;0}|^2dr+\frac{1}{n}\sum_{j=1}^n\int_t^s|X_r^{j;\varepsilon}-X_r^{j;0}|^2dr\bigg]+\varepsilon^2d(s-t).$$

Summing on $i = 1, \dots, n$, we have

$$\mathbb{E}\left[\sum_{i=1}^{n} |X_s^{i;\varepsilon} - X_s^{i;0}|^2\right] \leq C_{d,K,T}\left[\mathbb{E}\left(\int_t^s \sum_{i=1}^{n} |X_r^{i;\varepsilon} - X_r^{i;0}|^2 dr\right) + n\varepsilon^2(s-t)\right],$$

and hence Grönwall's inequality gives

$$\mathbb{E}\left[\sum_{i=1}^{n} |X_s^{i;\varepsilon} - X_s^{i;0}|^2\right] \leq C_{d,K,T} n\varepsilon^2 (s-t) e^{C_{d,K,T}(s-t)}.$$

Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and take $\overline{\xi} = (\xi, \dots, \xi) \in \mathbb{R}^{dn}$ such that $\mathcal{L}(\xi) = \mu$, we have

$$\begin{aligned}
&|v_{\varepsilon,n,m}(t,\mu) - v_{0,n,m}(t,\mu)| \\
&\leq \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\int_t^T \left|f_{n,m}^i\left(s, \overline{X}_s^{\varepsilon}, \overline{\alpha}_s^i\right) - f_{n,m}^i\left(s, \overline{X}_s^{0}, \overline{\alpha}_s^i\right)\right| ds + \left|g_{n,m}^i\left(\overline{X}_T^{\varepsilon}\right) - g_{n,m}^i\left(\overline{X}_T^{\varepsilon}\right)\right|\right] \\
&\leq \frac{2K}{n} \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} \sum_{i=1}^{n} \left\{\int_t^T \mathbb{E}\left[\left|\overline{X}_s^{i;\varepsilon} - \overline{X}_s^{i;0}\right|\right] ds + \mathbb{E}\left[\left|\overline{X}_T^{i;\varepsilon} - \overline{X}_T^{i;0}\right|\right]\right\} \\
&\leq \frac{2\sqrt{2}K}{n^{1/2}} \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} \left\{\sum_{i=1}^{n} \int_t^T T\mathbb{E}\left[\left|\overline{X}_s^{i;\varepsilon} - \overline{X}_s^{i;0}\right|^2\right] ds + \mathbb{E}\left[\left|\overline{X}_T^{i;\varepsilon} - \overline{X}_T^{i;0}\right|^2\right]\right\}^{1/2} \\
&\leq 2\sqrt{2}K\varepsilon\Big(C_{d,K,T}(T^3+T)e^{C_{d,K,T}T}\Big)^{1/2}.
\end{aligned}$$

□

Lemma 2.4.4. *Suppose that Assumption (A) holds and let $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$. If there exists $q > 2$ such that $\mu \in \mathcal{P}_q(\mathbb{R}^d)$, then $\lim_{n\to\infty}\lim_{m\to\infty} v_{\varepsilon,n,m}(t,\mu) = v_\varepsilon(t,\mu)$.*

Proof. The proof goes as the same line as (Cosso *et al.*, 2024, Theorem A.6) with (Djete *et al.*, 2022, Theorem 3.1, Theorem 3.6) replacing the limit theory in Cosso *et al.* (2024), because of the appearance of the common noise. □

Lemma 2.4.5. *Suppose that Assumptions (A*) and (B) hold. The function $\overline{v}_{\varepsilon,n,m} : [0,T] \times \mathbb{R}^{dn} \to \mathbb{R}$ defined by*

$$\overline{v}_{\varepsilon,n,m}(t,\overline{x}) = \overline{v}_{\varepsilon,n,m}(t,x^1,\dots,x^n) := \widetilde{v}_{\varepsilon,n,m}(t,\delta_{x^1}\otimes\ldots\otimes\delta_{x^n}) \tag{2.4.16}$$

is a classical solution of the Bellman equation

$$\begin{cases} \partial_t u(t,\overline{x}) + \sup\limits_{\overline{a}\in A^n} \left\{ \dfrac{1}{n}\sum\limits_{i=1}^n f^i_{n,m}(t,\overline{x},a^i) + \dfrac{1}{2}\sum\limits_{i=1}^n \mathrm{tr}\left[\Big((\sigma\sigma^\top)(t,x^i,a^i) + (\sigma^0\sigma^{0;\top})(t) + \varepsilon^2 I_d\Big)\partial^2_{x^ix^i}u(t,\overline{x})\right] \right. \\ \qquad\qquad \left. + \sum\limits_{i=1}^n \left\langle b^i_{n,m}(t,\overline{x},a^i), \partial_{x^i}u(t,\overline{x})\right\rangle + \dfrac{1}{2}\sum\limits_{i,j=1,i\neq j}^n \mathrm{tr}\Big[(\sigma^0\sigma^{0;\top})(t)\partial^2_{x^ix^j}u(t,\overline{x})\Big] \right\} \\ = 0 \ \text{ in } [0,T)\times\mathbb{R}^{dn}; \\ u(T,\overline{x}) = \dfrac{1}{n}\sum\limits_{i=1}^n g^i_{n,m}(\overline{x}) \quad \text{in } \mathbb{R}^{dn}, \end{cases} \tag{2.4.17}$$

where $\overline{a} = (a^1,\dots,a^n)$ *with* $a^i \in A$ *for* $i = 1,2,\dots,n$. *Moreover, there are constants* $C_4 = C_4(d,K,T)$ *and* $C_{n,m}$ *such that for any* $\varepsilon > 0$, $n,m \in \mathbb{N}$, $t,t' \in [0,T]$ *and* $\overline{x},\overline{x}' \in \mathbb{R}^{dn}$, *we have*

$$\begin{aligned} &|\partial_{x^i}\overline{v}_{\varepsilon,n,m}(t,\overline{x})| \leq \frac{C_4}{n} \text{ for every } i = 1,2,\dots,n \text{ and} \\ &-C_{n,m} \leq \partial^2_{\overline{x}_j\overline{x}_k}\overline{v}_{\varepsilon,n,m}(t,\overline{x}) \leq \frac{C_{n,m}}{\varepsilon^2} \text{ for every } j,k = 1,2,\dots,dn, \end{aligned} \tag{2.4.18}$$

where we denote by $\partial^2_{\overline{x}_j\overline{x}_k}v(t,\overline{x}) \in \mathbb{R}$ *the second order derivative with respect to* $\overline{x}_j$ *and* $\overline{x}_k$ *for* $j,k = 1,2,\dots,dn$.

Proof. We refer readers to (Cosso *et al.*, 2024, Theorem A.7) for proving that $\overline{v}_{\varepsilon,n,m}$ is a classical solution of the Bellman equation to (2.4.17). From (Krylov, 2008, Chapter 4.7, Theorem 4), we deduce the estimate of the second order derivative in (2.4.18). It remains to prove the estimate of the first order derivative in (2.4.18). We would like to point out that the proof presented below differs from that in (Cosso *et al.*, 2024, Theorem A.7). Specifically, we were unable to reproduce identity (Cosso *et al.*, 2024, (A.21)) therein, which may be attributed to a possible typographical error. As $\overline{v}_{\varepsilon,n,m}(t,\overline{x}) \in C^{1,2}([0,T]\times\mathbb{R}^{dn})$, we notice that it is enough to establish that

$$|\overline{v}_{\varepsilon,n,m}(t,\overline{x}) - \overline{v}_{\varepsilon,n,m}(t,\overline{z})| \leq \frac{C_4}{n}|\overline{x}-\overline{z}|,$$

when the components of $\overline{x} = (x^1,\dots,x^n)$ and $\overline{z} = (z^1,\dots,z^n)$ are all equal, apart from one component $x^k \neq z^k$ for some $k = 1,2,\dots,n$. Recalling the relations in (2.4.14) and (2.4.16),

we use the continuity in (4) of Lemma 2.4.2 to yield that

$$
\begin{aligned}
&|\overline{v}_{\varepsilon,n,m}(t,\overline{x})-\overline{v}_{\varepsilon,n,m}(t,\overline{z})|\\
&\leq 2K\sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\int_t^T\left|\overline{X}_s^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}}-\overline{X}_s^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}\right|ds+\left|\overline{X}_T^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}}-\overline{X}_T^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}\right|\right]. \quad (2.4.19)
\end{aligned}
$$

Suppose that $\overline{x}$ and $\overline{z}$ differ only for the first component $x^1\neq z^1$. For each $i=1,2,\ldots,n$, the $\mathbb{R}^d$-valued process $\overline{X}^{\overline{x},i}:=(\overline{X}_s^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}})_{s\in[t,T]}$ solves the following equation on $[t,T]$:

$$
\overline{X}_s^{\overline{x},i}=x^i+\int_t^s b_{n,m}^i(r,\overline{X}_r^{\overline{x},1},\ldots,\overline{X}_r^{\overline{x},n},\overline{\alpha}_r^i)dr+\int_t^s\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)d\overline{W}_r^i+\int_t^s\sigma^0(r)d\overline{W}_r^0+\varepsilon(\overline{B}_s^i-\overline{B}_t^i).
$$

Denote $\Delta_{x^1}\overline{X}_s^i:=\overline{X}_s^{\overline{z},i}-\overline{X}_s^{\overline{x},i}$. Then we have

$$
\begin{aligned}
\Delta_{x^1}\overline{X}_s^i=&(z^1-x^1)\delta_{1i}+\int_t^s b_{n,m}^i(r,\overline{X}_r^{\overline{z},1},\ldots,\overline{X}_r^{\overline{z},n},\overline{\alpha}_r^i)-b_{n,m}^i(r,\overline{X}_r^{\overline{x},1},\ldots,\overline{X}_r^{\overline{x},n},\overline{\alpha}_r^i)dr\\
&+\int_t^s\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)d\overline{W}_r^i.
\end{aligned}
$$

Letting $\vartheta>0$, we consider the function $u_\vartheta:\mathbb{R}^d\to\mathbb{R}$ defined by

$$
u_\vartheta(y):=\sqrt{|y|^2+\vartheta^2}.
$$

Direct calculation gives

$$
\partial_y u_\vartheta(y)=\frac{y}{u_\vartheta(y)},\quad \partial_{yy}^2u_\vartheta(y)=\frac{1}{u_\vartheta(y)}I_d-\frac{1}{\left[u_\vartheta(y)\right]^3}yy^\top.
$$

Applying Itô's formula to $u_\vartheta\left(\Delta_{x^1}\overline{X}_s^i\right)$ gives

$$
\begin{aligned}
&du_\vartheta\left(\Delta_{x^1}\overline{X}_s^i\right)\\
=&\left\langle\partial_y u_\vartheta\left(\Delta_{x^1}\overline{X}_s^i\right),d\Delta_{x^1}\overline{X}_s^i\right\rangle\\
&+\frac{1}{2}\mathrm{tr}\left\{\left[\sigma(r,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(r,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\right]^\top\partial_{yy}^2u_\vartheta\left(\Delta_{x^1}\overline{X}_s^i\right)\left[\sigma(r,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(r,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\right]\right\}ds\\
=&\left\langle\frac{\Delta_{x^1}\overline{X}_s^i}{u_\vartheta(\Delta_{x^1}\overline{X}_s^i)},\left(b_{n,m}^i(s,\overline{X}_s^{\overline{z},1},\ldots,\overline{X}_s^{\overline{z},n},\overline{\alpha}_s^i)-b_{n,m}^i(s,\overline{X}_s^{\overline{x},1},\ldots,\overline{X}_s^{\overline{x},n},\overline{\alpha}_s^i)\right)ds\right.\\
&\left.+\left[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\right]d\overline{W}_s^i\right\rangle
\end{aligned}
$$

$$+\frac{1}{2}\mathrm{tr}\Bigg\{\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]^\top \left(\frac{1}{u_\vartheta(\Delta_{x^1}\overline{X}_s^i)}I_d-\frac{\big(\Delta_{x^1}\overline{X}_s^i\big)\big(\Delta_{x^1}\overline{X}_s^i\big)^\top}{\Big[u_\vartheta(\Delta_{x^1}\overline{X}_s^i)\Big]^3}\right)\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]\Bigg\}ds.$$

It is clear that

$$\frac{\Delta_{x^1}\overline{X}_s^i}{u_\vartheta(\Delta_{x^1}\overline{X}_s^i)}\longrightarrow\frac{\Delta_{x^1}\overline{X}_s^i}{\big|\Delta_{x^1}\overline{X}_s^i\big|}\mathbb{1}_{\{\Delta_{x^1}\overline{X}_s^i\neq0\}},\quad\mathbb{P}\text{-a.s. as }\vartheta\to0.$$

Moreover, in view of the Lipschitz-continuity of σ, we have

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}\left\{\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]^\top\left(\frac{1}{u_\vartheta(\Delta_{x^1}\overline{X}_s^i)}I_d\right)\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]\right\}\\ \longrightarrow&\frac{1}{2}\left(\frac{1}{\big|\Delta_{x^1}\overline{X}_s^i\big|}\mathbb{1}_{\{\Delta_{x^1}\overline{X}_s^i\neq0\}}\right)\mathrm{tr}\left\{\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]^\top\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]\right\}\end{aligned}$$

$\mathbb{P}$-a.s. as $\vartheta\to0$. Similarly, it holds that

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}\left\{\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]^\top\left(\frac{\Delta_{x^1}\overline{X}_s^i\big(\Delta_{x^1}\overline{X}_s^i\big)^\top}{\Big[u_\vartheta(\Delta_{x^1}\overline{X}_s^i)\Big]^3}\right)\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]\right\}\\ \longrightarrow&\frac{1}{2}\left(\frac{1}{\big|\Delta_{x^1}\overline{X}_s^i\big|^3}\mathbb{1}_{\{\Delta_{x^1}\overline{X}_s^i\neq0\}}\right)\\ &\quad\mathrm{tr}\left\{\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]^\top\Big(\Delta_{x^1}\overline{X}_s^i\big(\Delta_{x^1}\overline{X}_s^i\big)^\top\Big)\Big[\sigma(s,\overline{X}_s^{\overline{z},i},\overline{\alpha}_s^i)-\sigma(s,\overline{X}_s^{\overline{x},i},\overline{\alpha}_s^i)\Big]\right\}\end{aligned}$$

$\mathbb{P}$-a.s. as $\vartheta\to0$. Therefore, by taking expectation and then passing $\vartheta\to0$, together with the dominated convergence theorem, we conclude that

$$\begin{aligned}&\mathbb{E}\big[|\Delta_{x^1}\overline{X}_s^i|\big]-|z^1-x^1|\delta_{1i}\\ =&\mathbb{E}\left[\int_t^s\left\langle\frac{\Delta_{x^1}\overline{X}_r^i}{\big|\Delta_{x^1}\overline{X}_r^i\big|},b_{n,m}^i(r,\overline{X}_r^{\overline{z},1},\dots,\overline{X}_r^{\overline{z},n},\overline{\alpha}_r^i)-b_{n,m}^i(r,\overline{X}_r^{\overline{x},1},\dots,\overline{X}_r^{\overline{x},n},\overline{\alpha}_r^i)\right\rangle\mathbb{1}_{\{\Delta_{x^1}\overline{X}_r^i\neq0\}}dr\right]\\ &+\frac{1}{2}\mathbb{E}\left\{\int_t^s\frac{1}{\big|\Delta_{x^1}\overline{X}_r^i\big|}\mathrm{tr}\left\{\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]^\top\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]\right\}\mathbb{1}_{\{\Delta_{x^1}\overline{X}_r^i\neq0\}}dr\right\}\\ &-\frac{1}{2}\mathbb{E}\Bigg\{\int_t^s\frac{1}{\big|\Delta_{x^1}\overline{X}_r^i\big|^3}\mathrm{tr}\Bigg\{\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]^\top\Big(\Delta_{x^1}\overline{X}_r^i\big(\Delta_{x^1}\overline{X}_r^i\big)^\top\Big)\end{aligned}$$

$$\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]\Bigg\}\mathbb{1}_{\left\{\Delta_{x^1}\overline{X}_r^i\neq 0\right\}}dr\Bigg\}. \tag{2.4.20}$$

The term in the third line of (2.4.20) can be estimated by Assumption (A):

$$\begin{aligned}&\frac{1}{\left|\Delta_{x^1}\overline{X}_r^i\right|}\mathrm{tr}\left\{\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]^\top\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]\right\}\mathbb{1}_{\left\{\Delta_{x^1}\overline{X}_r^i\neq 0\right\}}\\&\leq K^2\left|\Delta_{x^1}\overline{X}_r^i\right|.\end{aligned}$$

Similarly, the term in the forth and fifth lines of (2.4.20) can be estimated by

$$\begin{aligned}&\frac{1}{\left|\Delta_{x^1}\overline{X}_r^i\right|^3}\mathrm{tr}\Bigg\{\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]^\top\Big(\Delta_{x^1}\overline{X}_r^i\big(\Delta_{x^1}\overline{X}_r^i\big)^\top\Big)\\&\qquad\qquad\Big[\sigma(r,\overline{X}_r^{\overline{z},i},\overline{\alpha}_r^i)-\sigma(r,\overline{X}_r^{\overline{x},i},\overline{\alpha}_r^i)\Big]\Bigg\}\mathbb{1}_{\left\{\Delta_{x^1}\overline{X}_r^i\neq 0\right\}}\\&\leq K^2\left|\Delta_{x^1}\overline{X}_r^i\right|.\end{aligned}$$

Therefore, with (4) in Lemma 2.4.2, (2.4.20) reduces to

$$\begin{aligned}&\mathbb{E}\Big[|\Delta_{x^1}\overline{X}_s^i|\Big]\\&\leq|z^1-x^1|\delta_{1i}+\mathbb{E}\left[\int_t^s\left|b^i_{n,m}(r,\overline{X}_r^{\overline{z},1},\dots,\overline{X}_r^{\overline{z},n},\overline{\alpha}_r^i)-b^i_{n,m}(r,\overline{X}_r^{\overline{x},1},\dots,\overline{X}_r^{\overline{x},n},\overline{\alpha}_r^i)\right|+K^2\left|\Delta_{x^1}\overline{X}_r^i\right|dr\right]\\&\leq|z^1-x^1|\delta_{1i}+\mathbb{E}\left[\int_t^s K\left(\left|\Delta_{x^1}\overline{X}_r^i\right|+\frac{1}{n}\sum_{j=1}^n\left|\Delta_{x^1}\overline{X}_r^i\right|\right)+K^2\left|\Delta_{x^1}\overline{X}_r^i\right|dr\right].\end{aligned}$$

Summing over $i=1,2,\dots,n$, we have

$$\mathbb{E}\left[\sum_{i=1}^n\left|\Delta_{x^1}\overline{X}_s^i\right|\right]\leq|z^1-x^1|+\mathbb{E}\left[\int_t^s(2K+K^2)\sum_{i=1}^n\left|\Delta_{x^1}\overline{X}_r^i\right|dr\right].$$

Grönwall's inequality yields

$$\mathbb{E}\left[\sum_{i=1}^n\left|\Delta_{x^1}\overline{X}_s^i\right|\right]\leq e^{(2K+K^2)T}|z^1-x^1|,\quad\text{for every } s\in[t,T]. \tag{2.4.21}$$

From (2.4.19), (2.4.21), we obtain that

$$|\overline{v}_{\varepsilon,n,m}(t,\overline{x})-\overline{v}_{\varepsilon,n,m}(t,\overline{z})|$$

$$\leq 2K \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\int_t^T \left|\overline{X}_s^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}} - \overline{X}_s^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}\right| ds + \left|\overline{X}_T^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}} - \overline{X}_T^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}\right|\right]$$
$$\leq \frac{C_{d,K,T}}{n}|z^1 - x^1|.$$

□

Theorem 2.4.6. *Suppose that Assumptions (A)-(B) hold. For every $\varepsilon > 0$, $n, m \in \mathbb{N}$, the function $\overline{v}_{\varepsilon,n,m}$ defined in* (2.4.16) *and the function $v_{\varepsilon,n,m}$ defined in* (2.4.15) *satisfy the following:*

(1). for any $(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, we have

$$v_{\varepsilon,n,m}(t,\mu) = \int_{\mathbb{R}^{dn}} \overline{v}_{\varepsilon,n,m}(t,x^1,\ldots,x^n)\mu(dx^1)\otimes\ldots\otimes\mu(dx^n)\,; \tag{2.4.22}$$

(2). $v_{\varepsilon,n,m}(t,\mu) \in C_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$;

(3). there is a constant $\ell_2 = \ell_2(K,T)$ such that for any $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, we have $|v_{\varepsilon,n,m}(t,\mu)| \leq \ell_2$;

(4). $v_{\varepsilon,n,m}(t,\mu)$ solves the following equation in the classical sense

$$\begin{cases}
\partial_t u(t,\mu) \\
+\displaystyle\int_{\mathbb{R}^{dn}} \sup_{\overline{a}\in A^n} \Bigg\{ \frac{1}{n}\sum_{i=1}^n f_{n,m}^i(t,\overline{x},a^i) + \frac{1}{2}\sum_{i=1}^n \mathrm{tr}\left[\Big((\sigma\sigma^\top)(t,x^i,a^i) + (\sigma^0\sigma^{0;\top})(t) + \varepsilon^2 I_d\Big)\partial^2_{x^ix^i}\overline{v}_{\varepsilon,n,m}(t,\overline{x})\right] \\
\qquad\qquad + \frac{1}{2}\displaystyle\sum_{i,j=1,i\neq j}^n \mathrm{tr}\Big[(\sigma^0\sigma^{0;\top})(t)\partial^2_{x^ix^j}\overline{v}_{\varepsilon,n,m}(t,\overline{x})\Big] \\
\qquad\qquad + \displaystyle\sum_{i=1}^n \left\langle b_{n,m}^i(t,\overline{x},a^i), \partial_{x^i}\overline{v}_{\varepsilon,n,m}(t,\overline{x})\right\rangle \Bigg\}\mu(dx^1)\otimes\ldots\otimes\mu(dx^n) \\
= 0 \text{ for any } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\
u(T,\mu) = \displaystyle\frac{1}{n}\sum_{i=1}^n \int_{\mathbb{R}^{dn}} g_{n,m}^i(\overline{x})\mu(dx^1)\otimes\ldots\otimes\mu(dx^n) \text{ for any } \mu\in\mathcal{P}_2(\mathbb{R}^d).
\end{cases}$$

Proof. The proof follows from Lemma 2.4.5 and (Cosso *et al.*, 2024, Theorem A.7). □

2.5 Viscosity Solution Theory

Our main goal in this chapter is to show that the value function defined by (2.2.6) is the unique viscosity solution to the following HJB equation:

$$\begin{cases} \partial_t u(t,\mu) + \int_{\mathbb{R}^d} \sup_{a\in A} \Big\{ f(t,x,\mu,a) + b(t,x,\mu,a)\cdot \partial_\mu u(t,\mu)(x) \\ \quad + \frac{1}{2}\mathrm{tr}\Big(\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top \partial_x \partial_\mu u(t,\mu)(x)\Big)\Big\}\mu(dx) + \frac{1}{2}\mathrm{tr}\Big[\sigma^0(t)[\sigma^0(t)]^\top \mathcal{H} u(t,\mu)\Big] \\ \quad = 0 \text{ for } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \int_{\mathbb{R}^d} g(x,\mu)\mu(dx) \quad \text{for } \mu\in\mathcal{P}_2(\mathbb{R}^d). \end{cases} \tag{2.5.1}$$

We choose the set of test function to be $PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$. Now, we are ready to define our notion of viscosity subsolution (resp. supersolution).

Definition 2.5.1. *A continuous function $u : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is called a viscosity subsolution of equation* (2.5.1) *if*

(1a). $u(T,\mu)\le\int_{\mathbb{R}^d} g(x,\mu)\mu(dx)$, for every $\mu\in\mathcal{P}_2(\mathbb{R}^d)$;

(1b). for any $(t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$ and $\varphi\in PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ such that $u-\varphi$ attains a maximum with a value of 0 at (t,μ), then the first equation of (2.5.1) *holds with the inequality sign $\ge$ replacing the equality sign and with φ replacing u.*

For any function $h:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$, it could be naturally interpreted as a function with domain $\mathcal{P}_2(\mathbb{R}^d\times A)$ by $h(\nu) := h(\mu)$, with μ being the marginal distribution of ν on $\mathbb{R}^d$ such that $\nu(\cdot\times A)=\mu(\cdot)$. We define the respective projections $\pi_d:\mathbb{R}^{2d}\to\mathbb{R}^d$ and $\pi_{d\times d}:\mathbb{R}^{2d\times 2d}\to\mathbb{R}^{d\times d}$ such that for any $y=(y_1,y_2,\dots,y_{2d})^\top\in\mathbb{R}^{2d}$ and $M\in\mathbb{R}^{2d\times 2d}$, it holds that $\pi_d(y)=(y_1,y_2,\dots,y_d)^\top$ and $(\pi_{d\times d}(M))_{ij}=M_{ij}$. A continuous function $u:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is called a viscosity supersolution of equation (2.5.1) *if*

(2a). $u(T,\mu) \geq \int_{\mathbb{R}^d} g(x,\mu)\mu(dx)$, *for every* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;

(2b). for any $s_0 \in [0,T]$, $(t,\nu) \in [0,s_0) \times \mathcal{P}_2(\mathbb{R}^d \times A)$ *and* $\varphi \in PC_1^{1,2}([0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A))$ *such that* $u-\varphi$ *attains a minimum with a value of* 0 *at* (t,ν) *over* $[0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$, *the following inequality holds:*

$$\begin{aligned}
&\partial_t\varphi(t,\nu) + \int_{\mathbb{R}^d \times A} \Big\{ f(t,x,\mu,a) + b(t,x,\mu,a) \cdot \partial_\mu \varphi(t,\nu)(x,a) \\
&+ \frac{1}{2}\mathrm{tr}\Big((\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top) \partial_x \partial_\mu \varphi(t,\nu)(x,a) \Big) \Big\} \nu(dx,da) + \frac{1}{2}\mathrm{tr}\Big[\sigma^0(t)[\sigma^0(t)]^\top \mathcal{H}_{d\times d}\varphi(t,\nu)\Big] \\
&\leq 0, \qquad (2.5.2)
\end{aligned}$$

where μ *is the marginal distribution of* ν *on* $\mathbb{R}^d$, $\partial_\mu\varphi(t,\nu)(\cdot,\cdot) := \pi_d(\partial_\nu\varphi(t,\nu)(\cdot,\cdot))$, $\mathcal{H}_{d\times d}\varphi := \pi_{d\times d}(\mathcal{H}\varphi)$, *recalling that* $\partial_\nu\varphi(t,\nu)(\cdot,\cdot) : \mathbb{R}^d \times A \to \mathbb{R}^d \times A$ *and* $\mathcal{H}$ *maps functions on* $\mathcal{P}_2(\mathbb{R}^d \times A)$ *to* $\mathbb{R}^{2d\times 2d}$.

A continuous function $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *is called a viscosity solution of* (2.5.1) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 2.5.1. *We note that this definition of viscosity supersolution is stronger than Crandall-Lions' definition which is adopted by Cosso* et al. *(2024). In fact, items (2a)-(2b) in Definition 2.5.1 are sufficient conditions for the usual Crandall-Lions' definition of supersolution. Assume* u *is a viscosity supersolution satisfying items (2a)-(2b) in Definition 2.5.1. Let* $\varphi \in PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ *such that* $u-\varphi$ *attains a minimum with a value of* 0 *at* (t_0,μ_0). *Now, we interpret both* u, φ *as functions on* $[0,T] \times \mathcal{P}_2(\mathbb{R}^d \times A)$. *Let* $\xi \in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$, $\mathcal{L}(\xi) = \mu_0$, *and* $\alpha' \in \mathcal{M}_{t_0}$ *be arbitrary, where* $\mathcal{M}_t$ *is the set of* $\mathcal{F}_t^t$*-measurable random variables. Then for* $\nu_0 := \mathcal{L}(\xi,\alpha') \in \mathcal{P}_2(\mathbb{R}^d \times A)$, $u-\varphi$ *attains a minimum at* (t_0,ν_0), *therefore from* (2.5.2)

$$\begin{aligned}
&\partial_t\varphi(t_0,\mu_0) + \mathbb{E}\Big\{ f(t_0,\xi,\mu_0,\alpha') + \big[\partial_\mu\varphi(t_0,\mu_0)(\xi) \cdot b(t_0,\xi,\mu_0,\alpha')\big] \\
&+ \frac{1}{2}\,\mathrm{tr}\Big[\partial_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,\alpha')[\sigma(t_0,\xi,\alpha')]^\top\Big] + \frac{1}{2}\mathrm{tr}\Big[\sigma^0(t_0)[\sigma^0(t_0)]^\top \mathcal{H}\varphi(t_0,\mu_0)\Big]\Big\} \leq 0.
\end{aligned}$$

Since $\alpha' \in \mathcal{M}_{t_0}$ is arbitrary, we have

$$\partial_t\varphi(t_0,\mu_0) + \sup_{\alpha'\in\mathcal{M}_{t_0}} \mathbb{E}\Big\{ f(t_0,\xi,\mu_0,\alpha') + \partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,\alpha') \\ + \frac{1}{2}\operatorname{tr}\Big[\partial_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,\alpha')[\sigma(t_0,\xi,\alpha')]^\top\Big] + \frac{1}{2}\mathrm{tr}\Big[\sigma^0(t_0)[\sigma^0(t_0)]^\top\mathcal{H}\varphi(t_0,\mu_0)\Big]\Big\} \le 0.$$

We claim that the above is equivalent to the following:

$$\partial_t\varphi(t_0,\mu_0) + \int_{\mathbb{R}^d}\sup_{a\in A}\Big\{ f(t_0,x,\mu_0,a) + b(t_0,x,\mu_0,a)\cdot\partial_\mu\varphi(t_0,\mu_0)(x) \\ + \frac{1}{2}\mathrm{tr}\Big[\partial_x\partial_\mu\varphi(t_0,\mu_0)(x)\sigma(t_0,x,a)[\sigma(t_0,x,a)]^\top\Big]\Big\}\mu_0(dx) + \frac{1}{2}\mathrm{tr}\Big[\sigma^0(t_0)[\sigma^0(t_0)]^\top\mathcal{H}\varphi(t_0,\mu_0)\Big] \le 0. \tag{2.5.3}$$

Define $H(t,x,\mu,a) := f(t,x,\mu,a)+\partial_\mu\varphi(t,\mu)(x)\cdot b(t,x,\mu,a)+\frac{1}{2}\operatorname{tr}\big[\partial_x\partial_\mu\varphi(t,\mu)(x)\sigma(t,x,a)\sigma(t,x,a)^\top\big]$, we aim to show that

$$\sup_{\alpha'\in\mathcal{M}_{t_0}}\mathbb{E}H(t_0,\xi,\mu_0,\alpha') = \int_{\mathbb{R}^d}\sup_{a\in A}H(t_0,x,\mu_0,a)\mu_0(dx), \qquad \text{for } \xi\in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \text{ such that } \mathcal{L}(\xi)=\mu_0.$$

To see this, we observe that for any $\alpha'\in\mathcal{M}_{t_0}$,

$$\mathbb{E}H(t_0,\xi,\mu_0,\alpha') \le \int_{\mathbb{R}^d}\sup_{a\in A}H(t_0,x,\mu_0,a)\mu_0(dx),$$

and thus $\sup_{\alpha'\in\mathcal{M}_{t_0}}\mathbb{E}H(t_0,\xi,\mu_0,\alpha') \le \int_{\mathbb{R}^d}\sup_{a\in A}H(t_0,x,\mu_0,a)\mu_0(dx)$. Invoking the regularity of f, φ, b and σ, for any $\varepsilon>0$, we apply the measurable selection theorem (see for instance Wagner (1977)) to find a $\alpha^\varepsilon\in\mathcal{M}_{t_0}$ such that

$$\sup_{\alpha'\in\mathcal{M}_{t_0}}\mathbb{E}H(t_0,\xi,\mu_0,\alpha') \ge \mathbb{E}H(t_0,\xi,\mu_0,\alpha^\varepsilon) \ge \int_{\mathbb{R}^d}\sup_{a\in A}H(t_0,x,\mu_0,a)\mu_0(dx) - \varepsilon.$$

Since ε is arbitrary, we conclude the claim in (2.5.3)*, which is the classical Crandall-Lions' definition.*

Remark 2.5.2. *We impose this stronger definition because when proving the comparison theorem for the viscosity supersolution, we encounter technical difficulties that the Crandall-Lions' definition alone cannot resolve. We believe that a similar issue arises in Step II of*

the proof of (Cosso et al.*, 2024, Theorem 5.1), resulting in certain gaps in the arguments presented therein. Specifically, when we imitate the method used for the subsolution in the proof of Theorem 2.5.3, it requires the inequality* (2.5.19) *but with a reversed sign. Due to the supremum taken and the integration in* (2.5.19)*, only one direction of the inequality is allowed. To bypass this issue, we prove the comparison theorem by showing that for every step control, i.e., controls of the form* $\alpha = \sum_{i=0}^{n-1} \mathfrak{a}_i 1_{[t_i,t_{i+1})}$*, with* $0 =: t_0 < t_1 < \ldots < t_n := T$ *and* $\mathfrak{a}_i \in \mathcal{M}_{t_i}$ *(where* $\mathcal{M}_t$ *is the set of* $\mathcal{F}_t^t$*-measurable random variables), the supersolution* u_2 *satisfies*

$$u_2(t,\mu) \geq J(t,\xi,\alpha), \tag{2.5.4}$$

for every $(t,\xi) \in [0,T] \times L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ *such that* $\mu = \mathcal{L}(\xi)$*. Clearly this leads to* $u_2 \geq v$ *where* v *is the value function defined in* (2.2.6)*. By using inductive arguments, it is sufficient to prove* (2.5.4) *by proving that for every* $\underline{t}, \bar{t} \in [0,T]$ *with* $\underline{t} < \bar{t}$*, it holds that*

$$u_2(\underline{t},\mu) \geq \mathbb{E}\left[\int_{\underline{t}}^{\bar{t}} f\left(r, X_r^{\underline{t},\xi,\mathfrak{a}}, \mathbb{P}_{X_r^{\underline{t},\xi,\mathfrak{a}}}, \mathfrak{a}\right) dr + u_2(\bar{t}, \mathbb{P}^{W^0}_{X_{\bar{t}}^{\underline{t},\xi,\mathfrak{a}}})\right],$$

for every $\xi \in L^2((\Omega,\mathcal{F},\mathbb{P});\mathbb{R}^d)$ *with* $\mathcal{L}(\xi) = \mu$*, and* $\mathfrak{a} \in \mathcal{M}_{\underline{t}}$*. It is not hard to see that the right-hand side is law invariant with respect to the joint law of the initial random variable and the control. Thus,* $v^*(t,\nu) := \mathbb{E}\left[\int_t^{\bar{t}} f\left(r, X_r^{t,\xi,\mathfrak{a}}, \mathbb{P}_{X_r^{t,\xi,\mathfrak{a}}}, \mathfrak{a}\right) dr + u_2(\bar{t}, \mathbb{P}^{W^0}_{X_{\bar{t}}^{t,\xi,\mathfrak{a}}})\right]$ *is a real-valued function on* $[0,\bar{t}] \times \mathcal{P}_2(\mathbb{R}^d \times A)$ *with* ν *being the joint law of the initial random variable and the control. It seems that we must involve the function* $v^*_{n,m}(t,\nu)$*, the smooth and finite-dimensional approximation of* $v^*(t,\nu)$*, as a part of the test function used for comparison. The Crandall-Lions' definition cannot handle cases where the minimum point lies in* $[0,T] \times \mathcal{P}_2(\mathbb{R}^d \times A)$*. A natural approach is to modify the test function on* $[0,T] \times \mathcal{P}_2(\mathbb{R}^d \times A)$ *to be a function on* $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$*. However, finding such a modification that remains within our set of test functions and retains the necessary information (such as the L-derivatives) is challenging. For instance, we are unable to treat* $v^*_{n,m}(t,\nu)$ *as a function on* $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ *for a fixed control* $\mathfrak{a}$ *directly, as* $v^*_{n,m}(t,\nu)$ *is not law invariant with respect to the marginal*

of ν on $\mathbb{R}^d$ (law of ξ). Furthermore, we define $\underline{v}^* : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by

$$\underline{v}^*(t,\mu) = \sup_{\substack{\nu \in \mathcal{P}_2(\mathbb{R}^d \times A) \text{ s.t.} \\ \text{marginal of } \nu \text{ on } \mathbb{R}^d \text{ is } \mu}} v^*_{n,m}(t,\nu),$$

However, there is no guarantee that $\underline{v}^*(t,\mu)$ is L-differentiable with respect to μ. Moreover, even if it is L-differentiable, it is not certain that it could be related to $\partial_\nu v^*_{n,m}$. This issue seems to also arise in Step II of the proof of (Cosso et al.*, 2024, Theorem 5.1). Finally, we remark that assuming non-degenerate volatility does not remedy this issue.*

2.5.1 Existence

Theorem 2.5.2. *Suppose that Assumption (A) holds. The value function v defined in* (2.2.6) *is a viscosity solution of* (2.5.1)*.*

Proof. **Part 1. v is a viscosity supersolution:** For any $s_0' \in [0,T]$ and $\varphi \in PC_1^{1,2}([0,s_0'] \times \mathcal{P}_2(\mathbb{R}^d \times A))$, we assume that $v - \varphi$ attains a minimum at $(t_0,\nu_0) \in [0,s_0'] \times \mathcal{P}_2(\mathbb{R}^d \times A))$ with a value of 0. Recalling that $\mathcal{M}_t$ is the set of $\mathcal{F}_t^t$-measurable random variables, let $\alpha' \in \mathcal{M}_{t_0}$ and $\xi \in L^2((\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1); \mathbb{R}^d)$ such that $\mathcal{L}(\xi,\alpha') = \nu_0$. Letting $a \in A$ be arbitrary, we define $\alpha_s := a\mathbb{1}_{[0,t_0)}(s) + \alpha'\mathbb{1}_{[t_0,T]}(s)$, which belongs to $\mathcal{A}_{t_0}$. Let $(X_s^{t_0,\xi,\alpha})_{s\in[t_0,T]}$ be the solution of the dynamic (2.1.3) with initial time t_0, initial data ξ and control α defined in the above. Then for $h > 0$ small enough, we use the dynamic programming in Theorem 2.2.5 and Itô's formula in Theorem 2.1.6 to obtain that

$$\begin{aligned}
0 &\geq \frac{1}{h}\mathbb{E}\Big[(v-\varphi)(t_0,\nu_0) - (v-\varphi)(t_0+h, \mathbb{P}^{W^0}_{(X^{t_0,\xi,\alpha}_{t_0+h}, \alpha_{t_0+h})})\Big] \\
&\geq \frac{1}{h}\mathbb{E}\Bigg[\int_{t_0}^{t_0+h} f(s, X_s^{t_0,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}}, \alpha_s) + \partial_t\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)}) \\
&\qquad + \mathbb{E}^1\big[\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\cdot b(s, X_s^{t_0,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}}, \alpha_s)\big] \\
&\qquad + \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\big[\partial_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\sigma(s, X_s^{t_0,\xi,\alpha},\alpha_s)\sigma^\top(s, X_s^{t_0,\xi,\alpha},\alpha_s)\big]\Big\} \\
&\qquad + \frac{1}{2}\operatorname{tr}\big[\mathcal{H}_{d\times d}\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})\sigma^0(s)[\sigma^0(s)]^\top\big] ds\Bigg].
\end{aligned}$$

We proceed as follows: for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}^0\left(\sup_{t\in[s,s+h]}\mathcal{W}_2(\mathbb{P}^{W^0}_{X_t^{t_0,\xi,\alpha}},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}})>\varepsilon\right)&\leq\frac{\mathbb{E}^0\sup_{t\in[s,s+h]}\left[\mathcal{W}_2(\mathbb{P}^{W^0}_{X_t^{t_0,\xi,\alpha}},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}})\right]^2}{\varepsilon^2}\\&\leq\frac{\mathbb{E}\sup_{t\in[s,s+h]}|X_t^{t_0,\xi,\alpha}-X_s^{t_0,\xi,\alpha}|^2}{\varepsilon^2}\\&\leq\frac{Ch}{\varepsilon^2},\end{aligned}$$

where in the last inequality we have used Proposition 2.2.1. The above term goes to 0 as $h\to 0$, thus the flow of measure $s\mapsto\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}}$ is also continuous for $\mathbb{P}$-a.s. $\omega^0\in\Omega^0$ and for any $s\in[t_0,T]$, so is $s\mapsto\mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)}$ for $\mathbb{P}$-a.s. $\omega^0\in\Omega^0$ and any $s\in[t_0,T]$. From the regularity of the coefficients in Assumption (A), the regularity of the test function, the continuity of the process $s\mapsto X_s^{t_0,\xi,\alpha}$ for $\mathbb{P}$-a.s. $\omega\in\Omega$ and the dominated convergence theorem, we have

$$\begin{aligned}&\mathbb{E}\left\{\frac{1}{h}\int_{t_0}^{t_0+h}\left[f(s,X_s^{t_0,\xi,\alpha},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}},\alpha_s)+\partial_t\varphi(s,\mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})+\frac{1}{2}\operatorname{tr}\left[\mathcal{H}_{d\times d}\varphi(s,\mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})\sigma^0(s)[\sigma^0(s)]^\top\right]\right]ds\right\}\\&\longrightarrow\mathbb{E}\Big\{f(t_0,\xi,\mu_0,\alpha')+\partial_t\varphi(t_0,\nu_0)+\frac{1}{2}\operatorname{tr}\left[\mathcal{H}_{d\times d}\varphi(t_0,\nu_0)\sigma^0(s)[\sigma^0(s)]^\top\right]\Big\}\quad\text{as } h\to 0.\end{aligned}\tag{2.5.5}$$

We claim that the map $[t_0,t_0+h]\ni s\mapsto\mathbb{E}\mathbb{E}^1\left[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\cdot b(s,X_s^{t_0,\xi,\alpha},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}},\alpha_s)\right]$ is continuous. Writing $\partial_\mu\varphi_s\cdot b_s:=\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\cdot b(s,X_s^{t_0,\xi,\alpha},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}},\alpha_s)$ for ease of notation, the continuity of $\partial_\mu\varphi$, b, $X_s^{t_0,\xi,\alpha}$ and $\mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)}$ imply the continuity of $\partial_\mu\varphi_s\cdot b_s$ with respect to s for $\mathbb{P}$-a.s. $\omega\in\Omega$. Furthermore, due to Markov's inequality, Definition 2.1.4 and Proposition 2.2.1, we have

$$\begin{aligned}\sup_{s\in[t_0,t_0+h]}\mathbb{E}\mathbb{E}^1\left[|\partial_\mu\varphi_s\cdot b_s|\mathbb{1}_{\{|\partial_\mu\varphi_s\cdot b_s|\geq M\}}\right]&\leq K\sup_{s\in[t_0,t_0+h]}\mathbb{E}\mathbb{E}^1\left[|\partial_\mu\varphi_s|\mathbb{1}_{\{|\partial_\mu\varphi_s\cdot b_s|\geq M\}}\right]\\&\leq K\sup_{s\in[t_0,t_0+h]}\mathbb{E}\mathbb{E}^1\left[|\partial_\mu\varphi_s|\mathbb{1}_{\{K|\partial_\mu\varphi_s|\geq M\}}\right]\\&\leq K\sup_{s\in[t_0,t_0+h]}\sqrt{\mathbb{E}\mathbb{E}^1|\partial_\mu\varphi_s|^2}\sqrt{\mathbb{E}\mathbb{E}^1\mathbb{1}_{\{K|\partial_\mu\varphi_s|\geq M\}}}\\&\leq\frac{K^2}{M}\sup_{s\in[t_0,t_0+h]}\mathbb{E}\mathbb{E}^1|\partial_\mu\varphi_s|^2\\&\leq\frac{K^2}{M}C_\varphi(1+\mathbb{E}|\xi|^2+\mathbb{E}|\alpha'|^2)\to 0\text{ as }M\to\infty.\end{aligned}\tag{2.5.6}$$

The continuity of the concerned map thus follows from Vitali convergence theorem. The continuity of the map $s \mapsto \mathbb{E}\mathbb{E}^1 \operatorname{tr}\left[\partial_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\sigma(s,X_s^{t_0,\xi,\alpha},\alpha_s)\sigma^\top(s,X_s^{t_0,\xi,\alpha},\alpha_s)\right]$ follows similarly. Hence, when $h \to 0$, together with (2.5.5) we conclude that

$$\partial_t\varphi(t_0,\nu_0) + \mathbb{E}\Big\{f(t_0,\xi,\mu_0,\alpha') + \left[\partial_\mu\varphi(t_0,\nu_0)(\xi,\alpha')\cdot b(t_0,\xi,\mu_0,\alpha')\right]$$
$$+ \frac{1}{2}\operatorname{tr}\left[\partial_x\partial_\mu\varphi(t_0,\nu_0)(\xi,\alpha')\sigma(t_0,\xi,\alpha')[\sigma(t_0,\xi,\alpha')]^\top\right] + \frac{1}{2}\operatorname{tr}\left[\mathcal{H}_{d\times d}\varphi(t_0,\nu_0)\sigma^0(t_0)[\sigma^0(t_0)]^\top\right]\Big\} \le 0.$$

Part 2. v **is a viscosity subsolution:** For any $\varphi \in PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$, we assume that $v-\varphi$ attains a maximum at (t_0,μ_0) with value 0. Let $\varepsilon > 0$ and $\xi \in L^2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ satisfy $\mathcal{L}(\xi) = \mu_0$. There is an $\alpha^\varepsilon \in \mathcal{A}_t$ such that

$$\begin{aligned} v(t_0,\mu_0) - \varepsilon &< \inf_{h\in[0,T-t_0]} \mathbb{E}\left[\int_{t_0}^{t_0+h} f(r, X_r^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_r^{t_0,\xi,\alpha^\varepsilon}}, \alpha_r^\varepsilon)dr + v(t_0+h, \mathbb{P}^{W^0}_{X_{t_0+h}^{t_0,\xi,\alpha^\varepsilon}})\right] \\ &\le \mathbb{E}\left[\int_{t_0}^{t_0+h} f(r, X_r^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_r^{t_0,\xi,\alpha^\varepsilon}}, \alpha_r^\varepsilon)dr + v(t_0+h, \mathbb{P}^{W^0}_{X_{t_0+h}^{t_0,\xi,\alpha^\varepsilon}})\right], \ \forall\, h\in[0,T-t_0], \end{aligned}$$

where $(X_s^{t_0,\xi,\alpha^\varepsilon})$ solves the dynamic (2.1.3) with initial time t_0, initial data ξ and control α^ε. Then Theorem 2.1.6 implies

$$\begin{aligned} 0 \le& \frac{1}{h}\mathbb{E}\Big[(v-\varphi)(t_0,\mu_0) - (v-\varphi)(t_0+h, \mathbb{P}^{W^0}_{X_{t_0+h}^{t_0,\xi,\alpha^\varepsilon}})\Big] \\ <& \frac{1}{h}\mathbb{E}\Bigg\{\int_{t_0}^{t_0+h} f(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, \alpha_s^\varepsilon) + \partial_t\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}) \\ &+ \mathbb{E}^1\Big[\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, \alpha_s^\varepsilon)\Big] \\ &+ \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\left[\partial_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \alpha_s^\varepsilon)\sigma^\top(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \alpha_s^\varepsilon)\right]\Big\} \\ &+ \frac{1}{2}\operatorname{tr}\left[\mathcal{H}\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})\sigma^0(s)\sigma^{0;\top}(s)\right]ds\Bigg\} + \varepsilon \\ \le& \frac{1}{h}\mathbb{E}\Bigg\{\int_{t_0}^{t_0+h} \sup_{a\in A}\Big(f(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a) + \partial_t\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}) \\ &+ \mathbb{E}^1\left[\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a)\right] \\ &+ \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\left[\partial_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s, X_s^{t_0,\xi,\alpha^\varepsilon}, a)\sigma^\top(s, X_s^{t_0,\xi,\alpha^\varepsilon}, a)\right]\Big\} \\ &+ \frac{1}{2}\operatorname{tr}\left[\mathcal{H}\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})\sigma^0(s)\sigma^{0;\top}(s)\right]\Big)ds\Bigg\} + \varepsilon. \end{aligned}$$

Arguing as in (2.5.5), we have

$$\frac{1}{h}\mathbb{E}\left\{\int_{t_0}^{t_0+h}\partial_t\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})+\frac{1}{2}\operatorname{tr}\Big[\mathcal{H}\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})\sigma^0(s)\sigma^{0;\top}(s)\Big]ds\right\}$$
$$\longrightarrow \partial_t\varphi(t_0,\mu_0)+\frac{1}{2}\operatorname{tr}\Big[\mathcal{H}\varphi(t_0,\mu_0)\sigma^0(t_0)\sigma^{0;\top}(t_0)\Big] \text{ as } h\to 0.$$

We estimate the following terms:

$$\begin{aligned}
\mathrm{I} :=& \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{f(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)-f(t_0,\xi,\mu_0,a)\Big\}ds;\\
\mathrm{II} :=& \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)\\
&\qquad -\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\Big]\Big\}ds;\\
\mathrm{III} :=& \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\partial_x\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\sigma^\top(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\Big]\Big\}\\
&\qquad -\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\partial_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,a)\sigma^\top(t_0,\xi,a)\Big]\Big\}\Big\}ds.
\end{aligned}$$

Then the above gives

$$\begin{aligned}
&\frac{1}{h}\mathbb{E}\Big\{\int_{t_0}^{t_0+h}\sup_{a\in A}\Big(f(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)+\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)\Big]\\
&\qquad +\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\partial_x\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\sigma^\top(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\Big]\Big\}\Big)ds\Big\}\\
&\le \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathbb{E}\sup_{a\in A}\Big\{f(t_0,\xi,\mu_0,a)+\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\\
&\qquad +\frac{1}{2}\operatorname{tr}\Big[\partial_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,a)\sigma^\top(t_0,\xi,a)\Big]\Big\}.
\end{aligned}$$

We now show that all these three terms I, II, III $\to 0$ as $h\to 0$. We first investigate I:

$$\begin{aligned}
\mathrm{I} &= \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{f(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)-f(t_0,\xi,\mu_0,a)\Big\}ds\\
&\le \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}K\Big\{|X_s^{t_0,\xi,\alpha^\varepsilon}-\xi|+\mathcal{W}_2(\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},\mu_0)+|s-t_0|^\beta\Big\}ds\\
&\longrightarrow 0,
\end{aligned}$$

as $h\to 0$ by Proposition 2.2.1 and Assumption (A). For II, we have

$$\mathrm{II}=\frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)$$

$$
\begin{aligned}
&\qquad - \partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\big]\Big\}ds\\
&\le \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)\Big]\\
&\qquad - \mathbb{E}^1\Big[\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)\Big]\Big\}ds\\
&\quad + \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\mathbb{E}^1\Big[\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)-\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\Big]ds\\
&\le \frac{K}{h}\mathbb{E}\int_{t_0}^{t_0+h}\mathbb{E}^1\Big[\Big|\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})-\partial_\mu\varphi(t_0,\mu_0)(\xi)\Big|\Big]ds\\
&\quad + \frac{K}{h}\mathbb{E}\int_{t_0}^{t_0+h}\mathbb{E}^1\Big[\Big|\partial_\mu\varphi(t_0,\mu_0)(\xi)\Big|\Big(|s-t_0|^\beta+|X_s^{t_0,\xi,\alpha^\varepsilon}-\xi|+\mathcal{W}_2(\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},\mu_0)\Big)\Big]ds\\
&\le \frac{K}{h}\mathbb{E}\int_{t_0}^{t_0+h}\mathbb{E}^1\Big[\Big|\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})-\partial_\mu\varphi(t_0,\mu_0)(\xi)\Big|\Big]ds \qquad (2.5.7)\\
&\quad + \frac{\sqrt{3}K}{h}\sqrt{\mathbb{E}\mathbb{E}^1|\partial_\mu\varphi(t_0,\mu_0)(\xi)|^2}\int_{t_0}^{t_0+h}\sqrt{\mathbb{E}\mathbb{E}^1\Big(|s-t_0|^{2\beta}+|X_s^{t_0,\xi,\alpha^\varepsilon}-\xi|^2+\Big[\mathcal{W}_2(\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},\mu_0)\Big]^2\Big)}ds.
\end{aligned}
$$

The convergence of the term in (2.5.7) is due to the continuity as proved in (2.5.6). Thus, the term II converges to zero as $h\to 0$ with the aid of Proposition 2.2.1. Similar estimate holds for the term III, thus as $h\to 0$,

$$
\begin{aligned}
0\le{}&\partial_t\varphi(t_0,\mu_0)+\mathbb{E}\sup_{a\in A}\Big\{f(t_0,\xi,\mu_0,a)+\big[\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,a,\mu_0)\big]\\
&+\frac{1}{2}\operatorname{tr}\big[\partial_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,a)\sigma^\top(t_0,\xi,a)\big]\Big\}+\frac{1}{2}\operatorname{tr}\big[\mathcal{H}\varphi(t_0,\mu_0)\sigma^0(t_0)\sigma^{0;\top}(t_0)\big]+\varepsilon.
\end{aligned}
$$

□

2.5.2 Comparison Theorem and Uniqueness

Theorem 2.5.3. *Suppose that Assumptions (A*) and (B) hold. Let u_1, $u_2:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ be bounded functions such that they are the viscosity subsolution and supersolution (in the sense of Definition 2.5.1) of equation* (2.5.1) *respectively. Then it holds that $u_1\le u_2$ on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. Hence, the viscosity solution of equation* (2.5.1) *is unique.*

Proof. The following proof is inspired by (Cosso *et al.*, 2024, Theorem 5.1). Recalling the function defined in (2.4.3) with $\varepsilon=0$, we shall prove that $u_1\le v_0$ and $v_0\le u_2$ on

$[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$.

Part 1. Proof of $u_1 \leq v_0$:

We prove by contradiction and suppose that there exists $(t_0, \widetilde{\mu}_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that

$$(u_1 - v_0)(t_0, \widetilde{\mu}_0) > 0.$$

Let $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that $\mathcal{L}(\xi) = \widetilde{\mu}_0$. For any $k \in \mathbb{N}$, we let $\mu_0^k \in \mathcal{P}_2(\mathbb{R}^d)$ be the law of $\xi \mathbb{1}_{\{|\xi| \leq k\}}$. We see that $\mu_0^k \in \mathcal{P}_q(\mathbb{R}^d)$ for any $q \geq 1$ and

$$\Big[\mathcal{W}_2(\mu_0^k, \widetilde{\mu}_0)\Big]^2 \leq \mathbb{E}\Big[|\xi \mathbb{1}_{\{|\xi| \leq k\}} - \xi|^2\Big] = \int_{|x|>k} |x|^2 \widetilde{\mu}_0(dx) \longrightarrow 0$$

as $k \to \infty$. Therefore, as both u_1 and v_0 are continuous on $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, we can find a $k \in \mathbb{N}$ large enough such that $\mu_0 := \mu_0^k \in \mathcal{P}_q(\mathbb{R}^d)$ for any $q \geq 1$ and

$$(u_1 - v_0)(t_0, \mu_0) > 0. \tag{2.5.8}$$

The purpose of adopting this $\mu_0 := \mu_0^k$ instead of just working with $\widetilde{\mu}_0$ is to apply the approximation theorem in Lemma 2.4.4 which requires the point (t, μ) with μ having a higher moment $q > 2$.

Step 1A: Choice of the comparison function:

For any $\varepsilon > 0$, $n, m \in \mathbb{N}$, we define $\breve{u}_1(t,\mu) := e^{t-t_0} u_1(t,\mu)$ for any $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$. Recalling the definition in (2.4.15), (2.4.6), (2.4.7), we define similarly for $\breve{v}_{\varepsilon,n,m}$, $\breve{f}^i_{n,m}$, $\breve{f}$ from $v_{\varepsilon,n,m}$, $f^i_{n,m}$, f respectively; we also define $\breve{g} := e^{T-t_0} g$ and $\breve{g}^i_{n,m} := e^{T-t_0} g^i_{n,m}$. By direct

computation, we see that $\breve{u}_1$ is a viscosity subsolution of the equation

$$\begin{cases} \partial_t u(t,\mu) + \displaystyle\int_{\mathbb{R}^d} \sup_{a\in A} \Big\{ \breve{f}(t,x,\mu,a) + \frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(t,x,a)\big]\partial_x\partial_\mu u(t,\mu)(x)\Big\} \\ \qquad\qquad + \Big\langle b(t,x,\mu,a), \partial_\mu u(t,\mu)(x)\Big\rangle \Big\}\mu(dx) + \frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma^0\sigma^{0;\top})(t)\big]\mathcal{H}u(t,\mu)\Big\} - u(t,\mu) \\ = 0 \text{ for any } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \displaystyle\int_{\mathbb{R}^d} \breve{g}(x,\mu)\mu(dx) \text{ for any } \mu\in\mathcal{P}_2(\mathbb{R}^d). \end{cases} \tag{2.5.9}$$

Besides, recalling the definition in (2.4.16), we define $\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x}) := e^{t-t_0}\overline{v}_{\varepsilon,n,m}(t,\overline{x})$ where $\overline{x} = (x^1,\dots,x^n)$ for each $x^i\in\mathbb{R}^d$ with $i = 1,2,\dots,n$. By Theorem 2.4.6, we obtain that $\breve{v}_{\varepsilon,n,m}$ solves the following equation in the classical sense:

$$\begin{cases} \partial_t u(t,\mu) \\ + \displaystyle\int_{\mathbb{R}^{dn}} \sup_{\overline{a}\in A^n} \Big\{ \frac{1}{n}\sum_{i=1}^n \breve{f}^i_{n,m}(t,\overline{x},a^i) + \frac{1}{2}\sum_{i=1}^n \mathrm{tr}\Big[\Big((\sigma\sigma^\top)(t,x^i,a^i) + (\sigma^0\sigma^{0;\top})(t) + \varepsilon^2 I_d\Big)\partial^2_{x^ix^i}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big] \\ \qquad + \frac{1}{2}\displaystyle\sum_{i,j=1,i\neq j}^n \mathrm{tr}\Big[(\sigma^0\sigma^{0;\top})(t)\partial^2_{x^ix^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big] \\ \qquad + \displaystyle\sum_{i=1}^n \Big\langle b^i_{n,m}(t,\overline{x},a^i), \partial_{x^i}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big\rangle \Big\} \bigotimes_{k=1}^n \mu(dx^k) - u(t,\mu) \\ = 0 \text{ for any } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n \int_{\mathbb{R}^{dn}} \breve{g}^i_{n,m}(\overline{x}) \bigotimes_{k=1}^n \mu(dx^k) \text{ for any } \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{2.5.10}$$

where we write $\displaystyle\bigotimes_{k=1}^n \mu(dx^k) := \mu(dx^1)\otimes\ldots\otimes\mu(dx^n)$ and $\overline{a} = (a^1,\dots,a^n)$ for each $a^i\in A$ with $i = 1,2,\dots,n$. As $v_{\varepsilon,n,m}$, u_1 and $G := \breve{u}_1 - \breve{v}_{\varepsilon,n,m}$ are bounded by a constant independent of ε, m, n and continuous due to Theorem 2.4.6 and the assumption of this theorem, there is a $\lambda_0 > 0$ (depending only on K, u_1, T) such that

$$\sup_{(t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)} G(t,\mu) = \sup_{(t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)} \Big[\breve{u}_1(t,\mu) - \breve{v}_{\varepsilon,n,m}(t,\mu)\Big] \le \breve{u}_1(t_0,\mu_0) - \breve{v}_{\varepsilon,n,m}(t_0,\mu_0) + \lambda_0. \tag{2.5.11}$$

Recalling the definitions of $\rho_{1/\delta}$ and φ_δ in (2.3.5) and (2.3.6) for $\delta > 0$ respectively, we apply Theorem 2.3.8 to show that there exists $(\widetilde{t}, \widetilde{\mu}) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ and a sequence $\{(t_n, \mu_n)\}_{n\in\mathbb{N}} \subset [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ converging to $\left(\widetilde{t}, \widetilde{\mu}\right)$ such that

$$\left[SW_2^{1/\delta}(\widetilde{\mu}, \mu_0)\right]^2 \leq \rho_{1/\delta}\left((\widetilde{t}, \widetilde{u}), (t_0, \mu_0)\right) \leq \lambda_0/\delta^2.$$

Therefore, by the stability result stated in (Nietert *et al.*, 2021, Lemma 1), there is a constant $C_d > 0$ depending only on d such that

$$SW_2(\widetilde{\mu}, \mu_0) \leq \sqrt{2} SW_2^{1/\delta}(\widetilde{\mu}, \mu_0) + \frac{C_d}{\delta} \leq \frac{\sqrt{2\lambda_0} + C_d}{\delta}.$$

The values of C_d may change from line to line in this proof, but it still depends only on d. Furthermore, the first equality in (2.3.7) yields that $\kappa_d \mathcal{W}_2(\mu, \delta_0) = SW_2(\mu, \delta_0)$ for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, which implies

$$\begin{aligned}
\mathcal{W}_2(\widetilde{\mu}, \mu_0) &\leq \mathcal{W}_2(\widetilde{\mu}, \delta_0) + \mathcal{W}_2(\delta_0, \mu_0) \\
&\leq C_d SW_2(\widetilde{\mu}, \mu_0) + C_d SW_2(\mu_0, \delta_0) + \mathcal{W}_2(\delta_0, \mu_0) \\
&\leq C_d \frac{\sqrt{\lambda_0} + 1}{\delta} + C_d SW_2(\mu_0, \delta_0) + \mathcal{W}_2(\delta_0, \mu_0) \\
&\leq C_d \left(\frac{\sqrt{\lambda_0} + 1}{\delta} + SW_2(\mu_0, \delta_0) \right). \qquad (2.5.12)
\end{aligned}$$

Besides, item (2) in Theorem 2.3.8 implies that

$$\begin{aligned}
G(t_0, \mu_0) = (\breve{u}_1 - \breve{v}_{\varepsilon,n,m})(t_0, \mu_0) = (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) &\leq \breve{u}_1(\widetilde{t}, \widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(\widetilde{t}, \widetilde{\mu}) - \delta^2 \varphi_\delta(\widetilde{t}, \widetilde{\mu}) \\
&\leq \breve{u}_1(\widetilde{t}, \widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(\widetilde{t}, \widetilde{\mu}). \qquad (2.5.13)
\end{aligned}$$

Step 1B. Proof of $\widetilde{t} < T$:

In this step, we shall prove that $\widetilde{t} < T$. Suppose not, we have $\widetilde{t} = T$ and hence (2.5.13) implies that $u_1(t_0, \mu_0) - v_{\varepsilon,n,m}(t_0, \mu_0) \leq \breve{u}_1(T, \widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(T, \widetilde{\mu})$. Thus, for $\widehat{\mu}^{n,\overline{x}} := \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \delta_{x^j}$, we can use equations (2.5.9), (2.5.10) to obtain that

$$u_1(t_0, \mu_0) - v_{\varepsilon,n,m}(t_0, \mu_0) \leq \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \left[\int_{\mathbb{R}^{dn}} \left(g(x^i, \widetilde{\mu}) - g^i_{n,m}(x^1, \ldots, x^n) \right) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \right]$$

$$= \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \left[\int_{\mathbb{R}^{dn}} \left(g(x^i, \widetilde{\mu}) - g(x^i, \widehat{\mu}^{n,\overline{x}}) \right) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \right]$$
$$+ \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \left[\int_{\mathbb{R}^{dn}} \left(g(x^i, \widehat{\mu}^{n,\overline{x}}) - g^i_{n,m}(x^1, \ldots, x^n) \right) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \right].$$

Using the Lipschitz property of g and (3) of Lemma 2.4.2, we further have

$$u_1(t_0, \mu_0) - v_{\varepsilon,n,m}(t_0, \mu_0) \leq K e^{T-t_0} \left[\int_{\mathbb{R}^{dn}} \mathcal{W}_1(\widetilde{\mu}, \widehat{\mu}^{n,\overline{x}}) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \right]$$
$$+ \frac{2K e^{T-t_0}}{n} \left[m^{dn} \int_{\mathbb{R}^{dn}} \left(\sum_{i=1}^{n} |y^i| \right) \prod_{j=1}^{n} \Phi(m y^j) dy^j \right]. \tag{2.5.14}$$

From (Fournier & Guillin, 2015, Theorem 1), there is a constant $C_d > 0$ depending on d only and a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}^{dn}} \mathcal{W}_1(\widetilde{\mu}, \widehat{\mu}^{n,\overline{x}}) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \leq C_d \left[\int_{\mathbb{R}^d} |x|^{q_0} \widetilde{\mu}(dx) \right]^{1/q_0} h_n. \tag{2.5.15}$$

The sequence h_n and the number q_0 are given by

$$h_n = \begin{cases} n^{-1/2} + n^{-(q_0-1)/q_0} & \text{if } d = 1; \\ n^{-1/2} \log(1+n) + n^{-(q_0-1)/q_0} & \text{if } d = 2; \\ n^{-1/d} + n^{-(q_0-1)/q_0} & \text{if } d > 2, \end{cases} \quad \text{with} \quad q_0 = \begin{cases} 3/2 & \text{if } d = 1, 2; \\ 5/3 & \text{if } d > 2, \end{cases}$$

where h_n satisfies $\lim_{n\to\infty} h_n = 0$. Together with (2.5.12), we further have

$$\frac{1}{C_d h_n} \int_{\mathbb{R}^{dn}} \mathcal{W}_1(\widetilde{\mu}, \widehat{\mu}^{n,\overline{x}}) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \leq \left[\int_{\mathbb{R}^d} |x|^2 \widetilde{\mu}(dx) \right]^{1/2} = \mathcal{W}_2(\widetilde{\mu}, \delta_0) \leq C_d \left[\frac{\sqrt{\lambda_0} + 1}{\delta} + SW_2(\mu_0, \delta_0) \right]. \tag{2.5.16}$$

Hence, from (2.5.14), we have,

$$u_1(t_0, \mu_0) - v_{\varepsilon,n,m}(t_0, \mu_0) \leq K e^{T-t_0} C_d h_n \left[\frac{\sqrt{\lambda_0} + 1}{\delta} + SW_2(\mu_0, \delta_0) \right]$$
$$+ \frac{2K m^{dn} e^{T-t_0}}{n} \left[\int_{\mathbb{R}^{dn}} \left(\sum_{i=1}^{n} |y^i| \right) \prod_{j=1}^{n} \Phi(m y^j) dy^j \right].$$

Passing $m \to \infty$ and then $n \to \infty$ subsequently, we use the fact that $h_n \to 0$ to yield that

$$u_1(t_0, \mu_0) - \lim_{n\to\infty} \lim_{m\to\infty} v_{\varepsilon,n,m}(t_0, \mu_0) \leq 0.$$

Finally, using Lemmas 2.4.1 and 2.4.4, we pass $\varepsilon \to 0$ to conclude that $u_1(t_0,\mu_0) - v_0(t_0,\mu_0) \leq 0$ which contradicts (2.5.8), and thus $\widetilde{t} < T$.

Step 1C. Estimate of $u_1 - v_{\varepsilon,n,m}$:

Recalling from Theorems 2.3.8, 2.4.6, it implies that $\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta$ is in $PC_1^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ (see Definition 2.1.4). From item (3) of Theorem 2.3.8 with $G := \breve{u}_1 - \breve{v}_{\varepsilon,n,m}$ and (2.5.11), we observe that $\breve{u}_1 - \breve{v}_{\varepsilon,n,m} - \delta^2\varphi_\delta$ attains the maximum at $(\widetilde{t},\widetilde{\mu})$. Suppose the maximum value is $M^* \in \mathbb{R}$, then $\breve{u}_1 - \breve{v}_{\varepsilon,n,m} - \delta^2\varphi_\delta - M^*$ attains the maximum with value zero at $(\widetilde{t},\widetilde{\mu})$. We use the fact that $\breve{u}_1$ is the viscosity subsolution of (2.5.9) to see that

$$
\begin{aligned}
0 \leq\ & \partial_t(\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu}) - (\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu}) - M^* \\
&+ \int_{\mathbb{R}^d} \sup_{a\in A} \Big\{ \breve{f}(\widetilde{t},x,\widetilde{\mu},a) + \frac{1}{2}\mathrm{tr}\Big\{ \big[(\sigma\sigma^\top)(\widetilde{t},x,a)\big]\partial_x\partial_\mu(\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu})(x)\Big\} \\
&\qquad + \frac{1}{2}\mathrm{tr}\big[(\sigma^0\sigma^{0;\top})(\widetilde{t})\mathcal{H}(\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu})\big] \\
&\qquad + \Big\langle b(\widetilde{t},x,\widetilde{\mu},a), \partial_\mu(\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu})(x)\Big\rangle \Big\}\widetilde{\mu}(dx).
\end{aligned}
$$

Therefore, as $\breve{u}_1(\widetilde{t},\widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu}) - \delta^2\varphi_\delta(\widetilde{t},\widetilde{\mu}) - M^* = 0$ and $\breve{v}_{\varepsilon,n,m}$ solves (2.5.10), we further have

$$
\begin{aligned}
&(\breve{u}_1 - \breve{v}_{\varepsilon,n,m})(\widetilde{t},\widetilde{\mu}) \\
&\leq \delta^2\partial_t\varphi_\delta(\widetilde{t},\widetilde{\mu}) + \int_{\mathbb{R}^d}\sup_{a\in A}\Big\{ \breve{f}(\widetilde{t},x,\widetilde{\mu},a) + \frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},x,a)\big]\partial_x\partial_\mu(\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu})(x)\Big\} \\
&\qquad + \frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma^0\sigma^{0;\top})(\widetilde{t})\big]\mathcal{H}(\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu})\Big\} \\
&\qquad + \Big\langle b(\widetilde{t},x,\widetilde{\mu},a), \partial_\mu(\breve{v}_{\varepsilon,n,m} + \delta^2\varphi_\delta)(\widetilde{t},\widetilde{\mu})(x)\Big\rangle\Big\}\widetilde{\mu}(dx) \\
&- \int_{\mathbb{R}^{dn}}\sup_{\overline{a}\in A^n}\Big\{ \frac{1}{n}\sum_{i=1}^n \breve{f}^i_{n,m}(\widetilde{t},\overline{x},a^i) + \frac{1}{2}\sum_{i=1}^n \mathrm{tr}\Big[\Big((\sigma\sigma^\top)(\widetilde{t},x^i,a^i) + (\sigma^0\sigma^{0;\top})(\widetilde{t}) + \varepsilon^2 I_d\Big)\partial^2_{x^ix^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\Big] \\
&\qquad + \frac{1}{2}\sum_{i,j=1,i\neq j}^n \mathrm{tr}\big[(\sigma^0\sigma^{0;\top})(\widetilde{t})\partial^2_{x^ix^j}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\big]
\end{aligned}
$$

$$+\sum_{i=1}^{n}\Big\langle b^i_{n,m}(\widetilde{t},\overline{x},a^i),\partial_{x^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\Big\rangle\Big\}\bigotimes_{k=1}^{n}\widetilde{\mu}(dx^k). \tag{2.5.17}$$

We estimate term by term. First, Theorem 2.3.8 and Assumption (A*) tell us that

$$\begin{aligned}
&\partial_t\varphi_\delta(\widetilde{t},\widetilde{\mu})+\int_{\mathbb{R}^d}\sup_{a\in A}\Big\{\Big\langle b(\widetilde{t},x,\widetilde{\mu},a),\partial_\mu\varphi_\delta(\widetilde{t},\widetilde{\mu})(x)\Big\rangle+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},x,a)\big]\partial_x\partial_\mu\varphi_\delta(\widetilde{t},\widetilde{\mu})(x)\Big\}\Big\}\widetilde{\mu}(dx)\\
&+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma^0\sigma^{0;\top})(\widetilde{t})\big]\mathcal{H}\varphi_\delta(\widetilde{t},\widetilde{\mu})\Big\}\\
&\le 4T+\int_{\mathbb{R}^d}K\Big|\partial_\mu\varphi_\delta(\widetilde{t},\widetilde{\mu})(x)\Big|+\frac{K^2}{2}\Big|\partial_x\partial_\mu\varphi_\delta(\widetilde{t},\widetilde{\mu})(x)\Big|\widetilde{\mu}(dx)+\frac{K^2}{2}\Big|\mathcal{H}\varphi_\delta(\widetilde{t},\widetilde{\mu})\Big|\\
&\le 4T+K\sqrt{C_d}\bigg(\int_{\mathbb{R}^d}|x|^2\widetilde{\mu}(dx)+\int_{\mathbb{R}^d}|x|^2\mu_0(dx)+\frac{1+\lambda_0}{\delta^2}\bigg)^{1/2}\\
&\quad+\frac{K^2}{2}\sqrt{C_d}\delta\bigg(\int_{\mathbb{R}^d}|x|^2\mu_0(dx)+\frac{1+\lambda_0}{\delta^2}\bigg)^{1/2}+\frac{K^2}{2}C_d.
\end{aligned}\tag{2.5.18}$$

Second, we recall the representation of $\breve{v}_{\varepsilon,n,m}(t,\mu)=e^{t-t_0}v_{\varepsilon,n,m}(t,\mu)$ in (2.4.22) and also (1), (2) of Theorem 2.4.6. Inequality in (2.4.18) shows that we can directly compute that

$$\partial_\mu\breve{v}_{\varepsilon,n,m}(t,\mu)(x)=\sum_{i=1}^{n}\int_{\mathbb{R}^{d(n-1)}}\partial_{x^i}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^i=x}\bigotimes_{k=1,k\neq i}^{n}\mu(dx^k);$$

and (2) in Theorem 2.4.6, Lemma 2.1.5 yield that

$$\mathcal{H}\breve{v}_{\varepsilon,n,m}(t,\mu)=\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}^{dn}}\partial^2_{x^ix^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\bigotimes_{k=1}^{n}\mu(dx^k).$$

Hence, we estimate the term

$$\begin{aligned}
&\int_{\mathbb{R}^d}\sup_{a\in A}\Big\{\breve{f}(\widetilde{t},y,\widetilde{\mu},a)+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},y,a)\big]\partial_x\partial_\mu\breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu})(y)\Big\}\\
&\qquad+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma^0\sigma^{0;\top})(\widetilde{t})\big]\mathcal{H}\breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu})\Big\}+\Big\langle b(\widetilde{t},y,\widetilde{\mu},a),\partial_\mu\breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu})(y)\Big\rangle\Big\}\widetilde{\mu}(dy)\\
&=\int_{\mathbb{R}^d}\sup_{a\in A}\bigg\{\sum_{j=1}^{n}\int_{\mathbb{R}^{d(n-1)}}\frac{1}{n}\breve{f}(\widetilde{t},y,\widetilde{\mu},a)\bigotimes_{k=1,k\neq j}^{n}\widetilde{\mu}(dx^k)\\
&\qquad+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},y,a)\big]\sum_{j=1}^{n}\int_{\mathbb{R}^{d(n-1)}}\partial^2_{x^jx^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^j=y}\bigotimes_{k=1,k\neq j}^{n}\widetilde{\mu}(dx^k)\Big\}\\
&\qquad+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma^0\sigma^{0;\top})(\widetilde{t})\big]\sum_{j=1}^{n}\sum_{l=1}^{n}\int_{\mathbb{R}^{dn}}\partial^2_{x^jx^l}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\bigotimes_{k=1}^{n}\widetilde{\mu}(dx^k)\Big\}
\end{aligned}$$

$$+\Big\langle b(\widetilde{t},y,\widetilde{\mu},a),\sum_{j=1}^n\int_{\mathbb{R}^{d(n-1)}}\partial_{x^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^j=y}\bigotimes_{k=1,k\neq j}^n\widetilde{\mu}(dx^k)\Big\rangle\Big\}\widetilde{\mu}(dy)$$

$$\leq\sum_{j=1}^n\int_{\mathbb{R}^{dn}}\sup_{a^j\in A}\Big\{\frac{1}{n}\breve{f}(\widetilde{t},x^j,\widetilde{\mu},a^j)+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},x^j,a^j)\big]\partial^2_{x^jx^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big\}$$

$$+\Big\langle b(\widetilde{t},x^j,\widetilde{\mu},a^j),\partial_{x^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big\rangle\Big\}\bigotimes_{k=1}^n\widetilde{\mu}(dx^k)$$

$$+\frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma^0\sigma^{0;\top})(\widetilde{t})\big]\sum_{j=1}^n\sum_{l=1}^n\int_{\mathbb{R}^{dn}}\partial^2_{x^jx^l}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\bigotimes_{k=1}^n\widetilde{\mu}(dx^k)\Big\}. \tag{2.5.19}$$

Putting (2.5.18) and (2.5.19) into (2.5.17), we use (2.4.18) and Theorem 2.3.8 to deduce that

$$\begin{aligned}
&(\breve{u}_1-\breve{v}_{\varepsilon,n,m})(\widetilde{t},\widetilde{\mu})\\
&\leq 4T\delta^2+K\delta^2\sqrt{C_d}\bigg(\int_{\mathbb{R}^d}|x|^2\widetilde{\mu}(dx)+\int_{\mathbb{R}^d}|x|^2\mu_0(dx)+\frac{1+\lambda_0}{\delta^2}\bigg)^{1/2}\\
&\quad+\frac{K^2\delta^2}{2}\sqrt{C_d}\bigg(1+\lambda_0+\delta^2\int_{\mathbb{R}^d}|x|^2\mu_0(dx)\bigg)^{1/2}+\frac{K^2\delta^2C_d}{2}\\
&\quad+\int_{\mathbb{R}^{dn}}\sum_{i=1}^n\sup_{a^i\in A}\bigg\{\frac{1}{n}\breve{f}(\widetilde{t},x^i,\widetilde{\mu},a^i)-\frac{1}{n}\breve{f}^i_{n,m}(\widetilde{t},\overline{x},a^i)-\frac{\varepsilon^2}{2}\mathrm{tr}\partial^2_{x^ix^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\\
&\qquad\qquad+\Big\langle b(\widetilde{t},x^i,\widetilde{\mu},a^i)-b^i_{n,m}(\widetilde{t},\overline{x},a^i),\partial_{x^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\Big\rangle\bigg\}\bigotimes_{k=1}^n\widetilde{\mu}(dx^k)\\
&\leq 4T\delta^2+K\delta^2\sqrt{C_d}\bigg(\int_{\mathbb{R}^d}|x|^2\widetilde{\mu}(dx)+\int_{\mathbb{R}^d}|x|^2\mu_0(dx)+\frac{1+\lambda_0}{\delta^2}\bigg)^{1/2}\\
&\quad+\frac{K^2\delta^2}{2}\sqrt{C_d}\bigg(1+\lambda_0+\delta^2\int_{\mathbb{R}^d}|x|^2\mu_0(dx)\bigg)^{1/2}+\frac{K^2\delta^2C_d}{2}\\
&\quad+\int_{\mathbb{R}^{dn}}\sum_{i=1}^n\sup_{a^i\in A}\bigg\{\frac{e^{\widetilde{t}-t_0}}{n}\Big|f(\widetilde{t},x^i,\widetilde{\mu},a^i)-f^i_{n,m}(\widetilde{t},\overline{x},a^i)\Big|-\frac{\varepsilon^2}{2}\mathrm{tr}\partial^2_{x^ix^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\\
&\qquad\qquad+\frac{C_4e^{\widetilde{t}-t_0}}{n}\Big|b(\widetilde{t},x^i,\widetilde{\mu},a^i)-b^i_{n,m}(\widetilde{t},\overline{x},a^i)\Big|\bigg\}\bigotimes_{k=1}^n\widetilde{\mu}(dx^k).
\end{aligned}\tag{2.5.20}$$

We use Assumption (A*) and Lemma 2.4.2 to estimate the following term:

$$\begin{aligned}
&\Big|f(\widetilde{t},x^i,\widetilde{\mu},a^i)-f^i_{n,m}(\widetilde{t},\overline{x},a^i)\Big|+C_4\Big|b(\widetilde{t},x^i,\widetilde{\mu},a^i)-b^i_{n,m}(\widetilde{t},\overline{x},a^i)\Big|\\
&\leq\Big|f(\widetilde{t},x^i,\widetilde{\mu},a^i)-f(\widetilde{t},x^i,\widehat{\mu}^{n,\overline{x}},a^i)\Big|+\Big|f(\widetilde{t},x^i,\widehat{\mu}^{n,\overline{x}},a^i)-f^i_{n,m}(\widetilde{t},\overline{x},a^i)\Big|\\
&\quad+C_4\Big|b(\widetilde{t},x^i,\widetilde{\mu},a^i)-b(\widetilde{t},x^i,\widehat{\mu}^{n,\overline{x}},a^i)\Big|+C_4\Big|b(\widetilde{t},x^i,\widehat{\mu}^{n,\overline{x}},a^i)-b^i_{n,m}(\widetilde{t},\overline{x},a^i)\Big|\\
&\leq K(1+C_4)\mathcal{W}_1(\widetilde{\mu},\widehat{\mu}^{n,\overline{x}})+K(1+C_4)m\int_{\mathbb{R}}\Big|\widetilde{t}-\big[T\wedge(\widetilde{t}-s)^+\big]\Big|^\beta\phi(ms)ds
\end{aligned}$$

$$+ K(1+C_4)m^{dn}\int_{\mathbb{R}^{dn}}\left(|y^i|+\frac{1}{n}\sum_{j=1}^{n}|y^j|\right)\prod_{k=1}^{n}\Phi(my^k)dy^k. \tag{2.5.21}$$

Putting (2.5.21), (2.5.13) and (2.5.16) into (2.5.20), we see that

$$\begin{aligned}
&(u_1 - v_{\varepsilon,n,m})(t_0,\mu_0)\\
&\le 4T\delta^2 + K\delta^2\sqrt{C_d}\left(\left[\frac{\sqrt{\lambda_0}+1}{\delta}+SW_2(\mu_0,\delta_0)\right]^2+\int_{\mathbb{R}^d}|x|^2\mu_0(dx)+\frac{1+\lambda_0}{\delta^2}\right)^{1/2}\\
&\quad+\frac{K^2\delta^2}{2}\sqrt{C_d}\left[1+\lambda_0+\delta^2\int_{\mathbb{R}^d}|x|^2\mu_0(dx)\right]^{1/2}+\frac{K^2\delta^2C_d}{2}\\
&\quad-\int_{\mathbb{R}^{dn}}\frac{\varepsilon^2}{2}\sum_{i=1}^{n}\mathrm{tr}\partial^2_{x^ix^i}\breve{\breve{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\bigotimes_{k=1}^{n}\widetilde{\mu}(dx^k)+e^{T-t_0}C_dh_nK(1+C_4)\left[\frac{\sqrt{\lambda_0}+1}{\delta}+SW_2(\mu_0,\delta_0)\right]\\
&\quad+e^{T-t_0}K(1+C_4)m\int_{\mathbb{R}}\left|\widetilde{t}-\left[T\wedge(\widetilde{t}-s)^+\right]\right|^{\beta}\phi(ms)ds\\
&\quad+2e^{T-t_0}K(1+C_4)\frac{m^{dn}}{n}\int_{\mathbb{R}^{dn}}\left(\sum_{j=1}^{n}|y^j|\right)\prod_{k=1}^{n}\Phi(my^k)dy^k.
\end{aligned}$$

Using Lemma 2.4.3, 2.4.5 and (2.4.18), we first take $\varepsilon\to 0^+$ and then $m\to\infty$ to obtain that

$$\begin{aligned}
&(u_1-\lim_{m\to\infty}v_{0,n,m})(t_0,\mu_0)\\
&\le 4T\delta^2+K\delta^2\sqrt{C_d}\left(\left[\frac{\sqrt{\lambda_0}+1}{\delta}+SW_2(\mu_0,\delta_0)\right]^2+\int_{\mathbb{R}^d}|x|^2\mu_0(dx)+\frac{1+\lambda_0}{\delta^2}\right)^{1/2}\\
&\quad+\frac{K^2\delta^2}{2}\sqrt{C_d}\left[1+\lambda_0+\delta^2\int_{\mathbb{R}^d}|x|^2\mu_0(dx)\right]^{1/2}+\frac{K^2\delta^2C_d}{2}\\
&\quad+e^{T-t_0}C_dh_nK(1+C_4)\left[\frac{\sqrt{\lambda_0}+1}{\delta}+SW_2(\mu_0,\delta_0)\right].
\end{aligned}$$

By (2.5.15), (2.5.16), Lemmas 2.4.4 and 2.4.1, we take $n\to\infty$ and then $\delta\to 0^+$ to obtain that

$$(u_1-v_0)(t_0,\mu_0)=\left(u_1-\lim_{n\to\infty}\lim_{m\to\infty}v_{0,n,m}\right)(t_0,\mu_0)\le 0,$$

which contradicts (2.5.8).

Part 2. Proof of $u_2 \geq v_0$:

This step mainly follows the same pattern as in Step II of (Cosso *et al.*, 2024, Theorem 5.1) as the presence of the common noise does not alter the arguments of the approximated control problems there. However, due to the technical gap presented in Remark 2.5.2, we give a sketch of the new proof here. Following the arguments of the first part of Step II of the proof of (Cosso *et al.*, 2024, Theorem 5.1), showing $u_2 \geq v_0$ is equivalent to showing

$$u_2(t,\mu) \geq v^s(t,\nu) := \mathbb{E}\left[\int_t^s f\left(r, X_r^{t,\xi,\mathfrak{a}}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\mathfrak{a}}}, \mathfrak{a}\right) dr\right] + \mathbb{E}u_2\left(s, \mathbb{P}^{W^0}_{X_s^{t,\xi,\mathfrak{a}}}\right),$$

for every $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d), s \in (t,T], \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, with $\mathcal{L}(\xi) = \mu$, and $\mathfrak{a} \in \mathcal{M}_t$, where $\mathcal{M}_t$ denotes the set of $\mathcal{F}_t^t$-measurable random variables $\alpha : \Omega \to A$ and $\nu := \mathcal{L}(\xi, \mathfrak{a})$. Moreover, it is without loss of generality that $u_2(s,\cdot)$ is Lipschitz continuous for every $s \in [0,T]$ by the arguments of the first part of Step II of the proof of (Cosso *et al.*, 2024, Theorem 5.1). Suppose that there exist $t_0 \in [0,T)$, $s_0 \in (t_0, T]$ and $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $\nu_0 \in \mathcal{P}_2(\mathbb{R}^d \times A)$, with μ_0 being the marginal of ν_0 on $\mathbb{R}^d$, such that

$$v^{s_0}(t_0, \nu_0) > u_2(t_0, \mu_0).$$

As in the beginning of Part 1 of this proof, we can suppose that there exists some $q > 2$ such that $\nu_0 \in \mathcal{P}_q(\mathbb{R}^d \times A)$.

The function $\breve{u}_2(t,\mu) := e^{t-t_0} u_2(t,\mu)$ is a viscosity supersolution of the following equation

$$\begin{cases} \partial_t u(t,\mu) + \displaystyle\int_{\mathbb{R}^d} \sup_{a \in A} \Big\{ \breve{f}(t,x,\mu,a) + b(t,x,\mu,a) \cdot \partial_\mu u(t,\mu)(x) \\ \quad + \dfrac{1}{2}\mathrm{tr}\Big(\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top \partial_x \partial_\mu u(t,\mu)(x)\Big)\Big\} \mu(dx) + \dfrac{1}{2}\mathrm{tr}\Big[\sigma^0(t)[\sigma^0(t)]^\top \mathcal{H} u(t,\mu)\Big] - u(t,\mu) \\ \quad = 0 \text{ for } (t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \displaystyle\int_{\mathbb{R}^d} \breve{g}(x,\mu)\mu(dx) \quad \text{for } \mu \in \mathcal{P}_2(\mathbb{R}^d). \end{cases}$$

where $\breve{f}(t,x,\mu,a) := e^{t-t_0} f(t,x,\mu,a)$ and $\breve{g}(x,\mu) := e^{t-t_0} g(x,\mu)$. That is, for any $s_0 \in [0,T]$, for any $\varphi \in PC_1^{1,2}([0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A))$ such that $\breve{u}_2 - \varphi$ attains a minimum with a value of

0 at $(t^*, \nu^*) \in [0, s_0) \times \mathcal{P}_2(\mathbb{R}^d \times A)$, then the following inequality holds:

$$\begin{aligned} 0 \geq \partial_t \varphi(t^*, \nu^*) + \int_{\mathbb{R}^d \times A} \Big\{ & \breve{f}(t^*, x, \mu^*, a) + b(t^*, x, \mu^*, a) \cdot \partial_\mu \varphi(t^*, \nu^*)(x, a) \\ & + \frac{1}{2} \mathrm{tr}\Big[\sigma(t^*, x, a)\big[\sigma(t^*, x, a)\big]^\top \partial_x \partial_\mu \varphi(t^*, \nu^*)(x, a)\Big] \Big\} \nu^*(dx, da) \\ & + \frac{1}{2} \mathrm{tr}\Big[\sigma^0(t^*)[\sigma^0(t^*)]^\top \mathcal{H}_{d\times d} \varphi(t^*, \nu^*)\Big] - \varphi(t^*, \nu^*), \end{aligned} \tag{2.5.22}$$

where μ^* is the marginal of ν^* on $\mathbb{R}^d$. We regularize the coefficients for the control variable. Let $\Psi : \mathbb{R}^d \to \mathbb{R}^+$ be a compactly supported smooth function satisfying $\int_{\mathbb{R}^d} \Psi(y) dy = 1$. We extend b and f trivially (for example, $f(t, x, \mu, a) = 0$ when $a \in \mathbb{R}^d$ is not in A) on the space $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$, these extensions are still denoted by b and f. We further define the functions $\widetilde{b}^i_{n,m}$ and $\widetilde{f}^i_{n,m}$ by

$$\widetilde{b}^i_{n,m}(t, \overline{x}, a) = m^d \int_{\mathbb{R}^d} b^i_{n,m}(t, \overline{x}, a - a') \Psi(ma')\, da',$$

$$\widetilde{f}^i_{n,m}(t, \overline{x}, a) = m^d \int_{\mathbb{R}^d} f^i_{n,m}(t, \overline{x}, a - a') \Psi(ma')\, da',$$

for any $n, m \in \mathbb{N}$, $i = 1, 2, \ldots, n$, $\overline{x} = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^{dn}$ and $(t, a) \in [0, T] \times A$. Recalling the compactly supported smooth function Φ defined in Section 2.4.2, we also define

$$u_{n,m}(t, \overline{x}) = m^{nd} \int_{\mathbb{R}^{dn}} u_2\left(t, \frac{1}{n} \sum_{j=1}^n \delta_{x^j - y^j}\right) \prod_{j=1}^n \Phi(my^j)\, dy^j,$$

and

$$\begin{aligned} v^{s_0}_{n,m}(t, \nu) := \frac{1}{n} \sum_{i=1}^n \mathbb{E}\Big[& \int_t^{s_0} \widetilde{f}^i_{n,m}\left(r, \overline{\widetilde{X}}_r^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}, \ldots, \overline{\widetilde{X}}_r^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}, \mathfrak{a}^i_0\right) dr \\ & + u_{n,m}\left(s_0, \overline{\widetilde{X}}_{s_0}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}, \ldots, \overline{\widetilde{X}}_{s_0}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}\right)\Big], \end{aligned}$$

for any $t \in [0, s_0]$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d \times A)$, where $\overline{\xi} = (\overline{\xi}^1, \overline{\xi}^2, \ldots, \overline{\xi}^n) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{dn})$, $\overline{\mathfrak{a}}_0 = (\mathfrak{a}^1_0, \mathfrak{a}^2_0, \ldots, \mathfrak{a}^n_0) \in (\mathcal{M}_t)^n$ such that $\mathcal{L}(\overline{\xi}, \overline{\mathfrak{a}}_0) = \nu \otimes \ldots \otimes \nu$ and $\overline{\widetilde{X}}_s^{m,t,\overline{\xi},\overline{\mathfrak{a}}_0} = \left(\overline{\widetilde{X}}_s^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}, \ldots, \overline{\widetilde{X}}_s^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}\right)$ is the solution to (2.4.4) with $\overline{\alpha} = \overline{\mathfrak{a}}_0$, $\varepsilon = 0$ and b replacing by $\widetilde{b}^i_{n,m}$ for $s \in [t, s_0]$.

For every $n, m \in \mathbb{N}$, $(t, \mu) \in [0, s_0] \times \mathcal{P}_2(\mathbb{R}^d)$, we define $\breve{v}^{s_0}_{n,m} := e^{t-t_0} v^{s_0}_{n,m}$ and similarly for $\breve{\widetilde{f}}^{i}_{n,m}$ and $\breve{u}_{n,m}$ from $\widetilde{f}^{i}_{n,m}$ and $u_{n,m}$. Let

$$\overline{v}^{s_0}_{n,m}(t, \overline{x}, \overline{a}) := \widetilde{v}^{s_0}_{n,m}(t, \delta_{(x^1,a^1)} \otimes \ldots \otimes \delta_{(x^n,a^n)})$$

with $\overline{x} = (x^1, \ldots, x^n) \in \mathbb{R}^{dn}$ and $\overline{a} = (a^1, \ldots, a^n) \in A^n$, where

$$\begin{aligned}\widetilde{v}^{s_0}_{n,m}(t, \overline{\nu}) :=& \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\Bigg[\int_t^{s_0} \widetilde{f}^{i}_{n,m} \left(s, \overline{\widetilde{X}}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_s, \ldots, \overline{\widetilde{X}}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_s, \overline{\mathfrak{a}}^i_0 \right) ds \\ &+ u_{n,m} \left(\overline{\widetilde{X}}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_{s_0}, \ldots, \overline{\widetilde{X}}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_{s_0} \right) \Bigg], \end{aligned} \tag{2.5.23}$$

with $\overline{\nu} = \mathcal{L}(\overline{\xi}, \overline{\mathfrak{a}}_0)$. Moreover, by (Cosso *et al.*, 2024, Theorem A.8), we deduce that $\breve{v}^{s_0}_{n,m}$ can be represented by

$$\breve{v}^{s_0}_{n,m}(t, \nu) = e^{t-t_0} \int_{\mathbb{R}^{dn} \times A^n} \overline{v}^{s_0}_{n,m}(t, \overline{x}, \overline{a}) \nu(dx^1, da^1) \ldots \nu(dx^n, da^n),$$

and it could be shown by following the proofs of Lemma 2.4.5, Theorem 2.4.6 and (Cosso *et al.*, 2024, Theorem A.8) that

(1). $\overline{v}^{s_0}_{n,m} \in C^{1,2}\left([0, s_0] \times \mathbb{R}^{dn} \times A^n\right)$ and $v^{s_0}_{n,m} \in C^{1,2}\left([0, s_0] \times \mathcal{P}_2\left(\mathbb{R}^d \times A\right)\right)$;

(2). for any $i = 1, \ldots, n$ and $(t, \overline{x}, \overline{a}) \in [0, s_0] \times \mathbb{R}^{dn} \times A^n$, it holds that

$$\left| \partial_{x^i} \overline{v}^{s_0}_{n,m}(t, \overline{x}, \overline{a}) \right| \leq \frac{C_K}{n}$$

where the constant $C_K \geq 0$ depends on K, but independent of n, m;

(3). if $t \in [0, s_0]$ and $\nu \in \mathcal{P}_q\left(\mathbb{R}^d \times A\right)$ for some $q > 2$, then

$$\lim_{n \to +\infty} \lim_{m \to +\infty} v^{s_0}_{n,m}(t, \nu) = v^{s_0}(t, \nu);$$

(4). the function $\breve{v}^{s_0}_{n,m}(t,\nu)$ solves the following equation classically:

$$\begin{cases} \partial_t u(t,\nu) + \overline{\mathbb{E}}\Bigg[\sum_{i=1}^{n} \Big\{ \frac{1}{n} \breve{\breve{f}}^i{}_{n,m}(t,\overline{\xi},\mathfrak{a}_0^i) + \Big\langle \widetilde{b}^i_{n,m}(t,\overline{\xi},\mathfrak{a}_0^i), \partial_{x^i} \overline{v}^{s_0}_{n,m}(t,\overline{\xi},\overline{\mathfrak{a}}_0) \Big\rangle \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\Big(\Big[(\sigma\sigma^\top)(t,\overline{\xi}^i,\mathfrak{a}_0^i) + (\sigma^0\sigma^{0;\top})(t) \Big] \partial^2_{x^ix^i} \overline{v}^{s_0}_{n,m}(t,\overline{\xi},\overline{\mathfrak{a}}_0) \Big) \Big\} \\ \qquad + \frac{1}{2} \sum_{i,j=1, i\neq j}^{n} \mathrm{tr}\Big[(\sigma^0\sigma^{0;\top})(t) \partial^2_{x^ix^j} \overline{v}^{s_0}_{n,m}(t,\overline{\xi},\overline{\mathfrak{a}}_0) \Big] \Bigg] \\ - u(t,\nu) = 0; \\ u(s_0,\nu) = \overline{\mathbb{E}}[\breve{u}_{n,m}(s_0,\overline{\xi})], \end{cases}$$

for any $t \in [0,s_0)$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d \times A)$, where $\overline{\xi} = (\overline{\xi}^1, \overline{\xi}^2, \ldots, \overline{\xi}^n) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{dn})$, $\overline{\mathfrak{a}}_0 = (\mathfrak{a}_0^1, \mathfrak{a}_0^2, \ldots, \mathfrak{a}_0^n) \in (\mathcal{M}_t)^n$ such that $\mathcal{L}(\overline{\xi}, \overline{\mathfrak{a}}_0) = \nu \otimes \ldots \otimes \nu$.

Now, notice that $v^{s_0}_{n,m}$ is bounded by a constant independent of n, m. As a consequence, there exists $\lambda \geq 0$, independent of n, m, satisfying

$$\sup_{[0,T]\times\mathcal{P}_2(\mathbb{R}^d\times A)} (\breve{v}^{s_0}_{n,m} - \breve{u}_2) \leq (\breve{v}^{s_0}_{n,m} - \breve{u}_2)(t_0,\nu_0) + \lambda. \tag{2.5.24}$$

Since $\breve{v}^{s_0}_{n,m} - \breve{u}_2$ is bounded and continuous, by (2.5.24) and Theorem 2.3.8 applied on $[0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$ with $G = \breve{v}^{s_0}_{n,m} - \breve{u}_2$, we obtain that for every $\delta > 0$ there exist $\{(t_k,\nu_k)\}_{k\geq 1} \subset [0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$ converging to some $(\widetilde{t},\widetilde{\nu}) \in [0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$ and a function φ_δ such that items (1)-(4) of Theorem 2.3.8 hold. It deduces that there is $M_* \in \mathbb{R}$ such that $\breve{v}^{s_0}_{n,m} - \breve{u}_2 - \delta^2\varphi_\delta - M_*$ attains the maximum with a value of 0 at $(\widetilde{t},\widetilde{\mu})$. If $\widetilde{t} = s_0 = T$, then we proceed as in Step 1B to get a contradiction. If $\widetilde{t} = s_0 < T$, then it is straightforward to draw the contradiction by following (2.5.13) and using the definition of $\widetilde{v}^{s_0}_{n,m}(t,\overline{\nu})$ in (2.5.23). If $\widetilde{t} < s_0$, we apply Definition 2.5.1 of supersolution and put $\varphi = \breve{v}^{s_0}_{n,m} - \delta^2\varphi_\delta - M_*$ in (2.5.22). Then we proceed as in Step 1C and utilize items (1)-(4) in the above to find a contradiction and conclude the proof. □

Chapter 3

Viscosity Solutions of Fully second-order HJB Equations in the Wasserstein Space

This chapter establishes the existence and uniqueness of viscosity solution to the following Hamilton-Jacobi-Bellman (HJB) equation in the Wasserstein space arising from mean field control problems with common noise

$$\begin{cases} \partial_t u(t,\mu) + \displaystyle\int_{\mathbb{R}^d} \sup_{a\in A} \Big\{ f(t,x,\mu,a) + b(t,x,\mu,a)\cdot \partial_\mu u(t,\mu)(x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\Big((\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top + \sigma^0(t,x)[\sigma^0(t,x)]^\top)\nabla_x \partial_\mu u(t,\mu)(x)\Big)\Big\}\mu(dx) \\ + \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{tr}\Big[\sigma^0(t,x)[\sigma^0(t,y)]^\top \partial_\mu^2 u(t,\mu)(x,y)\Big]\,\mu^{\otimes 2}(dx,dy) = 0,\ \ \forall\,(t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d), \\ u(T,\mu) = \displaystyle\int_{\mathbb{R}^d} g(x,\mu)\mu(dx) \quad \text{for } \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{3.0.1}$$

where A is the control space, and $b, \sigma, \sigma^0, f, g$ are coefficients of the associated optimization problem. Here $\partial_\mu u$ and $\partial_\mu^2 u$ denote the respective first and second-order L-derivatives of u Carmona & Delarue (2018a); Cardaliaguet *et al.* (2019), and (3.0.1) is a degenerate fully second-order equation in the Wasserstein space. By "fully" second-order, we mean that the second-order term is infinite-dimensional, rather than a finite-dimensional operator as in Bayraktar *et al.* (2023b); Daudin *et al.* (2023); Cheung *et al.* (2025). Our manuscript is the

first work in the literature to address the viscosity solution directly in the Wasserstein space and to allow for unbounded dynamics and state-dependent common noise volatility.

In this chapter, we address the HJB equations directly in the Wasserstein space, establishing the existence and uniqueness of viscosity solution to equation (3.0.1) under a Crandall-Lions-like Definition 3.1.5. A viscosity solution to equation (3.0.1) is provided by the value function v of the corresponding mean-field control problem; see Theorem 3.3.2. To prove uniqueness, we adopt the idea of smooth approximation $v_n \to v$, as $n \to \infty$ from Cosso *et al.* (2024); Cheung *et al.* (2025). For any subsolution u_1, we aim to show that $u_1 \leq v$. Suppose the contrary, $u_1(t_0, \mu_0) > v(t_0, \mu_0)$ at some (t_0, μ_0) in the interior. Therefore, for sufficiently large n, $\sup_{(t,\mu)} u_1(t, \mu) - v_n(t, \mu) > 0$. To show a contradiction, we want to find a maximum point of $(t, \mu) \mapsto u_1(t, \mu) - v_n(t, \mu)$ and apply the definition of viscosity solution. However, the Wasserstein space is not locally compact and the maximizer may not be obtained. Therefore, Cosso *et al.* (2024); Cheung *et al.* (2025) applied the smooth variational principle so that perturbed functions achieve their global maximum. In Cosso *et al.* (2024); Cheung *et al.* (2025); Bayraktar *et al.* (2023a), the perturbation terms are not second-order L-differentiable, and hence they were not able to prove uniqueness for fully second-order HJB equations in the Wasserstein space. In this chapter, to overcome the compactness issue of the Wasserstein space, instead of applying the smooth variational principle, we adopt the idea of moment penalization from Soner & Yan (2024b). For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we define $M_2(\mu) := \int x^2 \, \mu(dx)$. We can always find a small $\delta > 0$ such that $u_1(t_0, \mu_0) - v_n(t_0, \mu_0) - \delta M_2(\mu_0) > 0$. This implies that the maximum point of $u_1 - v_n - \delta M_2$, if exists, must be attained in the set of measures with bounded second moments. As shown in Lemma 3.1.1, such a set is compact with respect to the finer topology induced by 1-Wasserstein metric. Moreover, the coefficient functions are assumed to be Lipschitz continuous under the 1-Wasserstein metric in our framework, which implies the same property for the value function. Therefore, the maximum point of $u_1 - v_n - \delta M_2$ is attained, and we derive the desired contradiction using the definition of

viscosity solution. Another advantage of this approach is that $M_2(\mu)$ possesses L-derivatives up to second-order in a very clean form, $\partial_\mu M_2(\mu)(x) = 2x$ and $\partial^2_\mu M_2(\mu)(x, y) = 0$, which allows unbounded state dynamics and state dependent common noise in our estimate.

We note that the definition of viscosity subsolution in this chapter follows the standard Crandall-Lions framework, but we need to modify the definition of viscosity supersolution in a stronger sense when dealing with equations with mean field terms. This modification aims to address the technical difficulty posed by the supremum in equation (3.0.1), which leads to scarce choice of test functions in the proof of the comparison for the supersolution. Similarly to Cosso *et al.* (2024), we construct smooth test functions for supersolutions using the cost functional with fixed control, but with the domain of $[0, T] \times \mathcal{P}_2(\mathbb{R}^d \times A)$ rather than $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$. The strong dependence on the initial random variable and the control makes it difficult to rely solely on the Crandall-Lions definition to draw the conclusion. We note that this modification of the definition is only required for the comparison theorem; however, the value function can still be shown to be the viscosity solution under the Crandall-Lions definition. Nonetheless, for consistency, we shall work on the Crandall-Lions-like definition for both the existence and the uniqueness theorem. See Remark 3.1.5 for more details.

The rest of the chapter is organized as follows. In Section 2, we present the problem formulation along with some preliminary results. Section 3 introduces the smooth finite-dimensional approximation of the value function and provides the related estimates. Section 4 contains the proof of the existence and uniqueness of the viscosity solution.

3.1 Preliminaries

In this section, we introduce the framework for setting up the HJB equation in (3.0.1) and recall some basic results. Before this, we shall introduce some notations which are frequently used in the article. For any random variable X, the law of X is denoted by $\mathcal{L}(X)$. For any $x \in \mathbb{R}^d$, we denote its Euclidean norm by $|x|$, and its i-th component by x_i or

$(x)_i$. The standard scalar product of $x, y \in \mathbb{R}^d$ is written as $\langle x, y\rangle$ or $x \cdot y$. Let $n \in \mathbb{N}$ and $x^1, x^2, \dots, x^n \in \mathbb{R}^d$, we denote the vector $\overline{x} \in \mathbb{R}^{dn}$ by $\overline{x} = (x^1, x^2, \dots, x^n)^\top$. For any matrix $M \in \mathbb{R}^{d\times d}$, the trace of M is denoted by $\mathrm{tr}(M) := \sum_{i=1}^d M_{ii}$, its transpose by $M^\top$, and its Frobenius norm by $|M| := \left[\mathrm{tr}(MM^\top)\right]^{1/2}$. If $M^0 \in \mathbb{R}^{d\times d}$ is another matrix, its transpose is denoted by $M^{0;\top}$. The identity matrix in $\mathbb{R}^d$ is denoted by I_d. For a scalar variable $x \in \mathbb{R}$, $\partial_x h \in \mathbb{R}$ refers to the partial derivative of the scalar function h with respect to x. For a vector $x \in \mathbb{R}^d$, $\nabla_x h \in \mathbb{R}^d$ denotes the gradient of h.

3.1.1 Compactness and Differentiation in the Wasserstein space

We introduce over $\mathbb{R}^d$ the space of probability measures $\mathcal{P}(\mathbb{R}^d)$ and define the p-th moment of any $\mu \in \mathcal{P}(\mathbb{R}^d)$ by

$$M_p(\mu) := \int_{\mathbb{R}^d} |x|^p \, \mu(dx), \quad \text{for any } p \geq 1, \tag{3.1.1}$$

if it is finite. We also denote $\mathcal{P}_p(\mathbb{R}^d)$ the subset of $\mathcal{P}(\mathbb{R}^d)$ consisting of those with finite p-th moment for $p \geq 1$. The space $\mathcal{P}_p(\mathbb{R}^d)$ is typically equipped with the q-Wasserstein distance with $q \in [1, p]$ and

$$\mathcal{W}_q(\mu, \nu) := \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^q \pi(dx, dy) \right)^{\frac{1}{q}}, \quad \text{for } \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d),$$

where $\Pi(\mu, \nu)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $\pi(\cdot \times \mathbb{R}^d) = \mu$ and $\pi(\mathbb{R}^d \times \cdot) = \nu$. We present a simple compactness criterion for some subsets of the Wasserstein space, which is crucial for our uniqueness result.

Lemma 3.1.1. *Let $K > 0$ and $p_2 > p_1 \geq 1$. The set $V_K^{p_1,p_2} := \{\mu \in \mathcal{P}_{p_1}(\mathbb{R}^d) : M_{p_2}(\mu) \leq K\} = \{\mu \in \mathcal{P}_{p_2}(\mathbb{R}^d) : M_{p_2}(\mu) \leq K\}$ is compact in $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_{p_1})$ for any $p \in [p_1, p_2]$.*

Proof. Consider the closed ball $\overline{B_R}$ centered at 0 with radius $R > 0$ in $\mathbb{R}^d$. Then for any $\mu \in V_K^{p_1,p_2}$ and any random variable X^μ with law μ, we have

$$\mu(\overline{B_R}) = \mathbb{P}(X^\mu \in \overline{B_R}) = 1 - \mathbb{P}(|X^\mu| > R) \geq 1 - K/R^{p_2},$$

by Markov's inequality. $V_K^{p_1,p_2}$ is thus tight. Therefore, by Prokhorov's theorem, for any sequence $\{\mu_n\}_{n\in\mathbb{N}} \subseteq V_K^{p_1,p_2}$, there exists a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ and $\mu^* \in \mathcal{P}_{p_1}(\mathbb{R}^d)$ such that $\mu_{n_k} \to \mu^*$ weakly in the sense that $\int_{\mathbb{R}^d} f(x)\mu_{n_k}(dx) \to \int_{\mathbb{R}^d} f(x)\mu^*(dx)$ as $k \to \infty$, for any bounded and continuous function $f : \mathbb{R}^d \to \mathbb{R}$. Hence, it holds that $\int_{\mathbb{R}^d} \Big(|x|^{p_2} \wedge M\Big)\mu^*(dx) \leq K$ as $\int_{\mathbb{R}^d} \Big(|x|^{p_2} \wedge M\Big)\mu_{n_k}(dx) \leq K$ for any $k \in \mathbb{N}$. Passing $M \to \infty$, the monotone convergence theorem implies that $\int_{\mathbb{R}^d} |x|^{p_2}\mu^*(dx) \leq K$, therefore $\mu^* \in V_K^{p_1,p_2}$.

We claim that also $\mathcal{W}_{p_1}(\mu_{n_k}, \mu) \to 0$. This can be easily seen from the fact (see (Villani, 2003, Theorem 7.12)) that $\mathcal{W}_{p_1}(\mu_{n_k}, \mu) \to 0$ if and only if

(1). $\mu_{n_k} \to \mu$ in weak convergence;

(2). $\lim\limits_{R\to\infty} \limsup\limits_{k\to\infty} \int_{|x|\geq R} |x|^{p_1}\mu_{n_k}(dx) = 0.$

Take a L^{p_1}-random variable X^{n_k} with its law $\mathcal{L}(X^{n_k}) = \mu_{n_k}$, the second condition in the above follows from the fact that

$$\begin{aligned}\int_{|x|\geq R} |x|^{p_1}\mu_{n_k}(dx) = \mathbb{E}\Big[|X^{n_k}|^{p_1}\mathbb{1}_{\{|X^{n_k}|\geq R\}}\Big] \leq &\Big\{\mathbb{E}\Big[|X^{n_k}|^{p_2}\Big]\Big\}^{p_1/p_2} \Big\{\mathbb{E}\Big[\mathbb{1}_{\{|X^{n_k}|\geq R\}}\Big]\Big\}^{1-p_1/p_2} \\ \leq &\frac{K}{R^{p_2-p_1}},\end{aligned}$$

hence $V_K^{p_1,p_2}$ is compact in $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_{p_1})$ for any $p \in [p_1, p_2]$. □

Remark 3.1.1. *Note that if $\mu_n, \mu \in \mathcal{P}_2(\mathbb{R}^d)$ for any $n \in \mathbb{N}$ such that $\mu_n \to \mu$ in the $\mathcal{W}_1$-topology as $n \to \infty$, then μ_n weakly converges to μ, and hence with $\mathcal{L}(X_n) = \mu_n$, $\mathcal{L}(X) = \mu$, it holds that $\liminf_n \mathbb{E}f(X_n) \geq \mathbb{E}f(X)$, for all nonnegative and continuous functions f. In particular, for any $p \geq 1$ we choose $f(x) := |x|^p$ and see that $\liminf_n M_p(\mu_n) \geq M_p(\mu)$, i.e., M_p is lower semicontinuous with respect to $\mathcal{W}_1$.*

In the rest of this subsection, we fix a rich enough probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that it supports all probability laws on $\mathbb{R}^d$, i.e., for any probability law μ on $\mathbb{R}^d$, there exists $X : \Omega \to \mathbb{R}^d$ such that the law of X, denoted by $\mathcal{L}(X)$, is μ. We recall the calculus in the Wasserstein space. For a function $f : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we adopt the notion of L-derivatives (see Cardaliaguet *et al.* (2019) for instance) which are recalled as follows:

Definition 3.1.2. *The function $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be first-order L-differentiable if its lifting $F : [0,T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$; $F(t,\xi) := f(t, \mathcal{L}(\xi))$ admits a continuous Fréchet derivative $D_\xi F : [0,T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$.*

Remark 3.1.2. *By (Carmona & Delarue, 2018a, Proposition 5.25), if f is first-order L-differentiable, then it can be shown that there is a measurable function, denoted by $\partial_\mu f(t,\mu)(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, such that $D_\xi F(t,\xi) = \partial_\mu f(t,\mu)(\xi)$ for any $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ with $\mathcal{L}(\xi) = \mu$. We say that $\partial_\mu f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is the first-order L-derivative of f.*

Definition 3.1.3. *The function $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be second-order L-differentiable if f is first-order L-differentiable and for any $x \in \mathbb{R}^d$, the function $\mu \mapsto \partial_\mu f(t,\mu)(x)$ is L-differentiable, i.e., the lifting $F' : L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}^d$ of $\mu \mapsto \partial_\mu f(t,\mu)(x)$ admits a continuous Fréchet derivative $D_\xi F' : [0,T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{d\times d})$.*

Remark 3.1.3. *Similarly, if f is second-order L-differentiable, then there is a measurable function, denoted by $\partial^2_\mu f(t,\mu)(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, such that $D_\xi F'(t,\xi') = \partial^2_\mu f(t,\mu)(x,\xi')$ for any $(t,\mu,x) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ and $\xi' \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ with $\mathcal{L}(\xi') = \mu$. We say that $\partial^2_\mu f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is the second-order L-derivative of f.*

3.1.2 Problem Formulation

This article aims to provide the proof of existence and uniqueness of viscosity solution of the HJB equation

$$
\begin{cases}
\partial_t u(t,\mu) + \displaystyle\int_{\mathbb{R}^d} \sup_{a\in A} \Big\{ f(t,x,\mu,a) + b(t,x,\mu,a) \cdot \partial_\mu u(t,\mu)(x) \\
\qquad\qquad + \frac{1}{2}\mathrm{tr}\Big((\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top + \sigma^0(t,x)[\sigma^0(t,x)]^\top) \nabla_x \partial_\mu u(t,\mu)(x) \Big) \Big\} \mu(dx) \\
+ \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{tr}\Big[\sigma^0(t,x)[\sigma^0(t,y)]^\top \partial^2_\mu u(t,\mu)(x,y) \Big] \mu^{\otimes 2}(dx,dy) = 0, \ \text{for any } (t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d); \\
u(T,\mu) = \displaystyle\int_{\mathbb{R}^d} g(x,\mu)\mu(dx) \quad \text{for } \mu \in \mathcal{P}_2(\mathbb{R}^d),
\end{cases}
$$

with the coefficient functions

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \quad \sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times d}, \quad \sigma^0 : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d},$$
$$f : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}, \quad g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R},$$

where A is a compact subset of the Euclidean space $\mathbb{R}^d$ equipped with the distance d_A. The coefficient functions satisfy the following assumptions:

Assumption. *The functions b, σ, σ^0, f and g are jointly continuous in $(t,x,\mu,a) \in ([0,T],|\cdot|) \times (\mathbb{R}^d,|\cdot|) \times (\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1) \times (A,|\cdot|)$. Moreover, there exist constants $K \geq 0$, $\rho \in [0,1)$ and $\beta \in (0,1]$ such that for any $a \in A$, (t,x,μ), $(t',x',\mu') \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, it holds that*

(1). $\big|b(t,x,\mu,a) - b(t',x',\mu',a)\big| + \big|\sigma(t,x,a) - \sigma(t',x',a)\big| + \big|\sigma^0(t,x) - \sigma^0(t',x')\big|$
$+ \big|f(t,x,\mu,a) - f(t',x',\mu',a)\big| + \big|g(x,\mu) - g(x',\mu')\big| \leq K\big[|x-x'| + |t-t'|^\beta + \mathcal{W}_1(\mu,\mu')\big]$;

(2). $\big|b(t,x,\mu,a)\big| + \big|\sigma(t,x,a)\big| + \big|\sigma^0(t,x)\big| \leq K(1+|x|^\rho)$;

(3). $\big|f(t,x,\mu,a)\big| + \big|g(x,\mu)\big| \leq K$.

Assumption. *For any $a \in A$, the functions $\sigma(\cdot,\cdot,a)$ and $\sigma^0(\cdot,\cdot)$ belong to $C^{1,2}\big([0,T] \times \mathbb{R}^d\big)$. Moreover, there exists a constant $K \geq 0$ such that*

$$|\partial_t \sigma(t,x,a)| + |\nabla_x \sigma(t,x,a)| + \big|\nabla^2_{xx}\sigma(t,x,a)\big| + \big|\partial_t \sigma^0(t,x)\big| + \big|\nabla_x \sigma^0(t,x)\big| + \big|\nabla^2_{xx}\sigma^0(t,x)\big| \leq K,$$

for all $(t,x,a) \in [0,T] \times \mathbb{R}^d \times A$.

Remark 3.1.4. *The growth rate condition $\rho \in [0,1)$ in Assumption 3.1.2 will be used for proving the uniqueness of viscosity solution in Theorem 3.3.3. Specifically, when controlling the error term of the perturbation of solution in* (3.3.21)*, we are unable to achieve a sufficient decay rate if $\rho = 1$. Furthermore, we impose the $\mathcal{W}_1$-Lipschitz continuity to obtain better convergence results in the finite-dimensional particle approximation in Lemma 3.2.2. The $\mathcal{W}_2$-Lipschitz continuity is insufficient, as noted in (Cheung* et al.*, 2025, Remark 2.5). See also the second part of Remark 3.1.6 for a reason on the compactness issue.*

Before giving the definition of viscosity solution, we introduce the set of test functions adopted in the article.

Definition 3.1.4. *The set* $C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ *consists of all* $([0,T],|\cdot|)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_2)$ *continuous functions* $\varphi:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ *satisfying the following:*

(1). the derivatives $\partial_t\varphi(t,\mu)$, $\partial_\mu\varphi(t,\mu)(x)$, $\nabla_x\partial_\mu\varphi(t,\mu)(x)$, $\partial^2_\mu\varphi(t,\mu)(x,x')$ *exist and are jointly continuous in* $(t,\mu,x,x')\in([0,T],|\cdot|)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_2)\times(\mathbb{R}^d,|\cdot|)\times(\mathbb{R}^d,|\cdot|)$;

(2). there is a constant $C_\varphi\geq 0$ *such that for any* $(t,\mu,x,x')\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\times\mathbb{R}^d$, *we have*

$$|\partial_t\varphi(t,\mu)|+\Big|\partial^2_\mu\varphi(t,\mu)(x,x')\Big|+\Big|\nabla_x\partial_\mu\varphi(t,\mu)(x)\Big|\leq C_\varphi\quad\text{and}\quad\Big|\partial_\mu\varphi(t,\mu)(x)\Big|\leq C_\varphi\Big(1+|x|\Big).$$

We now introduce the notion of viscosity solution used in this article. The subsolution part follows the standard Crandall-Lions' definition. However, for equations involving the mean field term, we have to modify the supersolution part.

Definition 3.1.5. *A bounded* $([0,T],|\cdot|)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$*-continuous function* $u:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ *is called a viscosity subsolution of equation* (3.0.1) *if*

(1a). $u(T,\mu)\leq\int_{\mathbb{R}^d}g(x,\mu)\mu(dx)$, *for every* $\mu\in\mathcal{P}_2(\mathbb{R}^d)$;

(1b). for any $(t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$ *and* $\varphi\in C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ *such that* $u-\varphi$ *attains a maximum with a value of* 0 *at* (t,μ) *over* $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, *the first equation of* (3.0.1) *holds with the inequality sign* $\geq$ *replacing the equality sign and with* φ *replacing* u.

For any function $h:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$, *we extend it to a function on* $\mathcal{P}_2(\mathbb{R}^d\times A)$ *by* $h(\nu)=h(\mu)$, *with* μ *being the marginal distribution of* ν *on* $\mathbb{R}^d$ *such that* $\nu(\cdot\times A)=\mu(\cdot)$. *We define the respective projections* $\pi_d:\mathbb{R}^{2d}\to\mathbb{R}^d$ *and* $\pi_{d\times d}:\mathbb{R}^{2d\times 2d}\to\mathbb{R}^{d\times d}$ *such that for any* $y=(y_1,y_2,\ldots,y_{2d})^\top\in\mathbb{R}^{2d}$ *and* $M\in\mathbb{R}^{2d\times 2d}$, *it holds that* $\pi_d(y)=(y_1,y_2,\ldots,y_d)^\top$ *and*

$(\pi_{d\times d}(M))_{ij} = M_{ij}$ *for* $i,j = 1,\dots,d$. *A bounded* $([0,T],|\cdot|)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$*-continuous function* $u:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ *is called a viscosity supersolution of equation* (3.0.1) *if*

(2a). $u(T,\mu)\geq\int_{\mathbb{R}^d} g(x,\mu)\mu(dx)$*, for every* $\mu\in\mathcal{P}_2(\mathbb{R}^d)$;

(2b). for any $s\in(0,T]$, $(t,\nu)\in[0,s)\times\mathcal{P}_2(\mathbb{R}^d\times A)$ *and* $\varphi\in C^{1,2}([0,s]\times\mathcal{P}_2(\mathbb{R}^d\times A))$ *such that* $u-\varphi$ *attains a minimum with a value of* 0 *at* (t,ν) *over* $[0,s]\times\mathcal{P}_2(\mathbb{R}^d\times A)$*, the following inequality holds:*

$$\begin{aligned}
&\partial_t\varphi(t,\nu)+\int_{\mathbb{R}^d\times A}\Big\{f(t,x,\mu,a)+b(t,x,\mu,a)\cdot\partial_\mu\varphi(t,\nu)(x,a)\\
&\qquad+\frac{1}{2}\mathrm{tr}\Big((\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top+\sigma^0(t,x)[\sigma^0(t,x)]^\top)\nabla_x\partial_\mu\varphi(t,\mu)(x,a)\Big)\Big\}\nu(dx,da)\\
&+\frac{1}{2}\int_{\mathbb{R}^d\times A\times\mathbb{R}^d\times A}\mathrm{tr}\Big[\sigma^0(t,x)[\sigma^0(t,y)]^\top\partial_\mu^2\varphi(t,\mu)(x,a,y,\alpha)\Big]\,\nu^{\otimes 2}(dx,da,dy,d\alpha)\\
&\leq 0, \qquad (3.1.2)
\end{aligned}$$

where μ *is the marginal distribution of* ν *on* $\mathbb{R}^d$*,* $u(t,\nu)$ *is defined as* $u(t,\mu)$*,* $\partial_\mu\varphi(t,\nu)(\cdot,\cdot):=\pi_d(\partial_\nu\varphi(t,\nu)(\cdot,\cdot))$*, and* $\partial_\mu^2\varphi(t,\nu)(\cdot,\cdot):=\pi_{d\times d}(\partial_\nu\varphi(t,\nu)(\cdot,\cdot))$*. Here* $\partial_\nu\varphi(t,\nu)(\cdot,\cdot):\mathbb{R}^d\times A\to\mathbb{R}^{2d}$ *and* $\partial_\nu^2\varphi(t,\nu)(\cdot,\cdot,\cdot,\cdot):\mathbb{R}^d\times A\times\mathbb{R}^d\times A\to\mathbb{R}^{2d\times 2d}$ *are similarly defined as in Definitions 3.1.2, 3.1.3, respectively, but over* $\mathbb{R}^d\times A$.

A function $u:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ *is called a viscosity solution of* (3.0.1) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 3.1.5. *As noted in (Cheung* et al.*, 2025, Remark 6.1), conditions (2a)-(2b) in Definition 3.1.5 provide sufficient criteria for the standard Crandall-Lions' definition of a supersolution. This formulation is adopted because, in proving the comparison theorem for viscosity supersolutions, there are technical difficulties that the Crandall-Lions' definition alone cannot resolve. For further details, see (Cheung* et al.*, 2025, Remark 6.2). It is important to note that this modification is required only for the comparison theorem; the*

value function can still be shown to be a viscosity solution to the HJB equation (3.0.1) *under the standard Crandall-Lions' definition, using essentially the same proof as in Theorem 3.3.2.*

Remark 3.1.6. *There are two key reasons for adopting the $\mathcal{W}_1$-Lipschitz continuity in Definition 3.1.5. First, as highlighted in (Cheung* et al.*, 2025, Remark 2.5), to apply the finite-dimensional particle approximation of the value function v in the comparison theorem, it is inevitable to assume the $\mathcal{W}_1$-Lipschitz continuity of the coefficient functions in Assumption 3.1.2. This assumption ensures that the value function is also $\mathcal{W}_1$-Lipschitz continuity (see Lemma 3.3.1). Hence, it is natural to seek candidate solutions among $\mathcal{W}_1$-continuous functions. Second, the $\mathcal{W}_1$-continuity provides a suitable compactness property, replacing the smooth variational principle to guarantee the existence of extrema for functions in the Wasserstein space. To effectively apply the smooth variational principle, a gauge function that matches the order of the HJB equations is required. However, constructing a gauge function that is smooth up to second-order is challenging, which complicates the extension of viscosity solutions to fully nonlinear second-order HJB equations. Fortunately, this compactness property allows us to bypass the need for a second-order L-differentiable gauge function.*

3.1.3 Associated Mean Field Control Problems

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ structured as $(\Omega^0 \times \Omega^1, \mathcal{F}^0 \otimes \mathcal{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$. The space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ supports a d-dimensional Brownian motion W^0, which represents the common noise. The space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ is of the form $(\tilde{\Omega}^1 \times \hat{\Omega}^1, \mathcal{G} \otimes \hat{\mathcal{F}}^1, \tilde{\mathbb{P}}^1 \otimes \hat{\mathbb{P}}^1)$. On $(\hat{\Omega}^1, \hat{\mathcal{F}}^1, \hat{\mathbb{P}}^1)$, there is a d-dimensional Brownian motion W, which represents the idiosyncratic noise. Meanwhile, $(\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1)$ supports the initial random variables. We further assume, without loss of generality, that the probability space $(\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1)$ is rich enough to support any probability law on $\mathbb{R}^d$. Specifically, for any probability measure μ on $\mathbb{R}^d$, there exists a random variable $X(\omega) : \tilde{\Omega}^1 \to \mathbb{R}^d$ such that the distribution of X, denoted by $\mathcal{L}(X)$, is μ.

We represent $\omega \in \Omega$ as $\omega = (\omega^0, \omega^1)$, where the Brownian motions are given by $W(\omega) = W(\omega^1)$ and $W^0(\omega) = W^0(\omega^0)$. Let $\mathbb{E}$ denote the expectation under $\mathbb{P}$, while $\mathbb{E}^0$ and $\mathbb{E}^1$ represent the expectations under $\mathbb{P}^0$ and $\mathbb{P}^1$, respectively. We define the filtrations as follows:

- $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0} := (\sigma(W^0_r)_{0\leq r\leq s} \vee \sigma(W_r)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$;
- $\mathbb{F}^t = (\mathcal{F}^t_s)_{s\geq 0} := (\sigma(W^0_{r\vee t} - W^0_t)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$;
- $\mathbb{F}^{W^0} = (\mathcal{F}^{W^0}_s)_{s\geq 0} := (\sigma(W^0_r))_{0\leq r\leq s}$;
- $\mathbb{F}^1 = (\mathcal{F}^1_s)_{s\geq 0} := (\sigma(W_s) \vee \mathcal{G})_{0\leq r\leq s}$.

For simplicity, we assume they are $\mathbb{P}$-complete.

Recall that $A \subset \mathbb{R}^d$ is a compact subset equipped with the distance d_A. For $t > 0$, let $\mathcal{A}$ (resp. $\mathcal{A}_t$) denote the set of $\mathbb{F}$-progressively measurable (resp. $\mathbb{F}^t$-progressively measurable) processes on Ω that take values in A. The set $\mathcal{A}$ (resp. $\mathcal{A}_t$) is a separable metric space with the Krylov distance defined as $\Delta(\alpha, \beta) := \mathbb{E}\left[\int_0^T d_A(\alpha_r, \beta_r)\, dr\right]$ (resp. $\Delta_t(\alpha, \beta) := \mathbb{E}\left[\int_t^T d_A(\alpha_r, \beta_r)\, dr\right]$). We denote by $\mathcal{B}_{\mathcal{A}}$ (resp. $\mathcal{B}_{\mathcal{A}_t}$) the Borel σ-algebra on $\mathcal{A}$ (resp. $\mathcal{A}_t$). Without loss of generality, let $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ be the canonical space, i.e., $\Omega^0 = C(\mathbb{R}_+, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^d$.

We now introduce a mean field control problem related to the HJB equation (3.0.1). For every $t \in [0, T]$, $\xi \in L^2(\Omega^1, \mathcal{F}^1_t, \mathbb{P}^1; \mathbb{R}^d)$ and $\alpha \in \mathcal{A}_t$, we consider the solution $X^{t,\xi,\alpha}$ of the following state dynamics:

$$X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}^{W^0}_{X_r}, \alpha_r)dr + \int_t^s \sigma(r, X_r, \alpha_r)dW_r + \int_t^s \sigma^0(r, X_r)dW^0_r, \quad \text{for } s \in [t, T], \tag{3.1.3}$$

where $\mathbb{P}^{W^0}_{X_r}$ denotes the conditional distribution of X_r given W^0. We are subject to the cost functional:

$$J(t, \xi, \alpha) := \mathbb{E}\left[\int_t^T f\Big(s, X^{t,\xi,\alpha}_s, \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_s}, \alpha_s\Big)ds + g\Big(X^{t,\xi,\alpha}_T, \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_T}\Big)\right]. \tag{3.1.4}$$

We define the value function V to be

$$V(t,\xi) := \sup_{\alpha\in\mathcal{A}_t} J(t,\xi,\alpha), \quad \text{for any } (t,\xi)\in[0,T]\times L^2(\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1;\mathbb{R}^d). \tag{3.1.5}$$

As in (Cosso *et al.*, 2023, Appendix B) and (Cheung *et al.*, 2025, Proposition 3.3), it can be shown that the value function is law invariant under Assumption 3.1.2, i.e., for every $t\in[0,T]$ and $\xi,\,\eta\in L^2(\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1;\mathbb{R}^d)$, with $\mathcal{L}(\xi)=\mathcal{L}(\eta)$, it holds that $V(t,\xi)=V(t,\eta)$. Therefore we can define a function $v(t,\mu):[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\mapsto\mathbb{R}$ such that for any $t\in[0,T]$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, it holds that

$$v(t,\mu) := V(t,\xi), \tag{3.1.6}$$

for any $\xi\in L^2(\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1;\mathbb{R}^d)$ such that $\mathcal{L}(\xi)=\mu$, where $V(t,\xi)$ is defined in (3.1.5). The main goal of this article is to prove that the value function $v(t,\mu)$ is the unique viscosity solution to the HJB equation (3.0.1), under Definition 3.1.5. To address this, we need some preliminary results. We first give the regularity of the solution of the SDE (3.1.3) and the value function v. Proofs are standard and therefore omitted, and readers are referred to Yong & Zhou (1999).

Proposition 3.1.6. *Suppose that Assumption 3.1.2 holds. For every $t\in[0,T]$, $\xi\in L^2(\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1;\mathbb{R}^d)$ and $\alpha\in\mathcal{A}_t$, there exists a unique (up to $\mathbb{P}$-indistinguishability) continuous $\mathbb{F}$-progressively measurable solution $X^{t,\xi,\alpha}=(X_s^{t,\xi,\alpha})_{s\in[t,T]}$ of equation (3.1.3). Moreover, there is a constant C depending only on K, T, d such that*

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|X_s^{t,\xi,\alpha}|^2\Big]\le C\big(1+\mathbb{E}|\xi|^2\big),\quad \mathbb{E}\Big[\sup_{s\in[t,T]}|X_s^{t,\xi,\alpha}-X_s^{t,\xi',\alpha}|^2\Big]\le C\mathbb{E}\big(|\xi-\xi'|^2\big)\quad \textit{and}$$
$$\mathbb{E}\Big[\sup_{s\in[t,t+h]}|X_s^{t,\xi,\alpha}-\xi|^2\Big]\le Ch,$$

for any $t\in[0,T]$, $h\in[0,T-t]$, $\xi,\xi'\in L^2(\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1;\mathbb{R}^d)$ and $\alpha\in\mathcal{A}_t$.

Proposition 3.1.7. *Suppose that Assumption 3.1.2 holds. There exists a constant $C\ge0$ such that for any $t,t'\in[0,T]$, μ, $\mu'\in\mathcal{P}_2(\mathbb{R}^d)$,*

$$|v(t,\mu)|\le(1+T)K\quad\textit{and}\quad|v(t,\mu)-v(t',\mu')|\le C\Big[\mathcal{W}_2(\mu,\mu')+|t-t'|^{1/2}\Big]. \tag{3.1.7}$$

The constant C depends on d, K, T and independent of t, t', μ, μ'.

We then state the dynamic programming principle which is essential to verify that the value function v is the viscosity solution to the HJB equation (3.0.1). The proof is standard and can be obtained following arguments similar to those in Pham & Wei (2017).

Theorem 3.1.8 (Dynamic Programming Principle)**.** *Suppose that Assumption 3.1.2 holds. The value function v satisfies the dynamic programming principle: for every $t \in [0,T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, it holds that*

$$\begin{aligned} v(t,\mu) &= \sup_{\alpha\in\mathcal{A}_t} \sup_{s\in[t,T]} \left\{ \mathbb{E}\left[\int_t^s f\Big(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r\Big) dr + v\Big(s, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}\Big) \right] \right\} \\ &= \sup_{\alpha\in\mathcal{A}_t} \inf_{s\in[t,T]} \left\{ \mathbb{E}\left[\int_t^s f\Big(r, X_r^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\alpha}}, \alpha_r\Big) dr + v\Big(s, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}\Big) \right] \right\}, \end{aligned}$$

for any $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ such that $\mathcal{L}(\xi) = \mu$.

We recall the following version of Itô's formula for functions in $C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$, as stated in (Carmona & Delarue, 2018b, Theorem 4.14).

Theorem 3.1.9. *Let $(\widetilde{b}_t)_{t\geq 0}$, $(\widetilde{\sigma}_t)_{t\geq 0}$ and $(\widetilde{\sigma}^0_t)_{t\geq 0}$ be progressively measurable processes with respect to $\mathbb{F}$, with values in $\mathbb{R}^d$, $\mathbb{R}^{d\times d}$ and $\mathbb{R}^{d\times d}$ respectively, such that for any finite horizon $T > 0$,*

$$\mathbb{E}\left[\int_0^T \big(|\widetilde{b}_t|^2 + |\widetilde{\sigma}_t|^4 + |\widetilde{\sigma}^0_t|^4 \big) dt \right] < \infty.$$

Consider the following $\mathbb{R}^d$-valued Itô process:

$$X_t = \xi + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s + \int_0^t \widetilde{\sigma}^0_s dW^0_s, \quad \text{for } t \in [0,T],$$

where $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. It holds $\mathbb{P}^0$-a.s. that for $\varphi \in C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$,

$$\begin{aligned} &\varphi(t, \mathbb{P}^{W^0}_{X_t}) \\ &= \varphi(0, \mathbb{P}^{W^0}_{X_0}) + \int_0^t \partial_t \varphi(s, \mathbb{P}^{W^0}_{X_s}) ds + \int_0^t \mathbb{E}^1\big[\partial_\mu \varphi(s, \mathbb{P}^{W^0}_{X_s})(X_s) \cdot \widetilde{b}_s \big] ds \end{aligned}$$

$$+\int_0^t \mathbb{E}^1\big[\widetilde{\sigma}_s^{0;\top}\partial_\mu\varphi(s,\mathbb{P}_{X_s}^{W^0})(X_s)\big]\cdot dW_s^0+\frac{1}{2}\int_0^t \mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s,\mathbb{P}_{X_s}^{W^0})(X_s)\widetilde{\sigma}_s\widetilde{\sigma}_s^\top\Big]\Big\}ds$$
$$+\frac{1}{2}\int_0^t \mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s,\mathbb{P}_{X_s}^{W^0})(X_s)\widetilde{\sigma}_s^0\widetilde{\sigma}_s^{0;\top}\Big]\Big\}ds+\frac{1}{2}\int_0^t \mathbb{E}^1\breve{\mathbb{E}}^1\Big\{\operatorname{tr}\Big[\partial_\mu^2\varphi(s,\mathbb{P}_{X_s}^{W^0})(X_s,\breve{X}_s)\widetilde{\sigma}_s^0\big(\breve{\widetilde{\sigma}}_s^0\big)^\top\Big]\Big\}ds,$$

where $\breve{\mathbb{E}}^1$ *is the expectation under* $(\breve{\Omega}^1,\breve{\mathcal{F}}^1,\breve{\mathbb{P}}^1)$ *which is a copy of* $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$*; and* $(\breve{X}_t)_{t\geq 0}$*,* $(\breve{\widetilde{b}}_t)_{t\geq 0}$*,* $(\breve{\widetilde{\sigma}}_t)_{t\geq 0}$*,* $(\breve{\widetilde{\sigma}}_t^0)_{t\geq 0}$ *are copies of* $(X_t)_{t\geq 0}$*,* $(\widetilde{b}_t)_{t\geq 0}$*,* $(\widetilde{\sigma}_t)_{t\geq 0}$*,* $(\widetilde{\sigma}_t^0)_{t\geq 0}$ *on* $(\breve{\Omega}^1,\breve{\mathcal{F}}^1,\breve{\mathbb{P}}^1)$*.*

3.2 Smooth Finite-dimensional Approximations of Value Function

In this section, we construct $C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ approximations of the value function v. As similar results have been proved in Cosso *et al.* (2024); Cheung *et al.* (2025) and the notations are quite heavy, we only provide main ideas of the construction in this section and postpone the details to Appendix. The main idea is to first add a small term in the volatility to ensure that the HJB equation (3.0.1) is non-degenerate in the second-order term. Then, we mollify the coefficient functions and use the empirical measure to approximate the measure arguments. It ensures that the approximated coefficient functions are smooth and on a finite-dimensional domain. These procedures allow us to construct a regular enough test function from the approximations of value function v.

3.2.1 Approximation by Non-degenerate Problem

Fix a complete probability space $(\check{\Omega},\check{\mathcal{F}},\check{\mathbb{P}})$, also of the form $(\check{\Omega}^0\times\check{\Omega}^1,\check{\mathcal{F}}^0\otimes\check{\mathcal{F}}^1,\check{\mathbb{P}}^0\otimes\check{\mathbb{P}}^1)$. The space $(\check{\Omega}^0,\check{\mathcal{F}}^0,\check{\mathbb{P}}^0)$ supports a d-dimensional Brownian motion $\check{W}^0$. For $(\check{\Omega}^1,\check{\mathcal{F}}^1,\check{\mathbb{P}}^1)$, it is of the form $(\check{\hat{\Omega}}^1\times\check{\hat{\Omega}}^1,\check{\mathcal{G}}\otimes\check{\hat{\mathcal{F}}}^1,\check{\hat{\mathbb{P}}}^1\otimes\check{\hat{\mathbb{P}}}^1)$. On $(\check{\hat{\Omega}}^1,\check{\hat{\mathcal{F}}}^1,\check{\hat{\mathbb{P}}}^1)$, there lives d-dimensional Brownian motions $\check{W}$ and $\check{B}$. The space $(\check{\hat{\Omega}}^1,\check{\mathcal{G}},\check{\hat{\mathbb{P}}}^1)$ is where the initial random variables live. We assume that $(\check{\hat{\Omega}}^1,\check{\mathcal{G}},\check{\hat{\mathbb{P}}}^1)$ is rich enough to support all probability laws on $\mathbb{R}^d$, i.e., for any probability law μ on $\mathbb{R}^d$, there exists $X:\check{\hat{\Omega}}^1\to\mathbb{R}^d$ such that $\mathcal{L}(X)=\mu$. The expectation $\mathbb{E}$

in this subsection is taken with respect to $\check{\mathbb{P}} = \check{\mathbb{P}}^0 \otimes \check{\mathbb{P}}^1$.

Set $\check{\mathbb{F}} = (\check{\mathcal{F}}_s)_{s\geq 0} := \big(\sigma(\check{W}^0_r)_{0\leq r\leq s} \vee \sigma(\check{W}_r)_{0\leq r\leq s} \vee \sigma(\check{B}_r)_{0\leq r\leq s} \vee \check{\mathcal{G}}\big)_{s\geq 0}$, $\check{\mathbb{F}}^t = (\mathcal{F}^t_s)_{s\geq 0} := \big(\sigma(\check{W}^0_r - \check{W}^0_t)_{0\leq r\leq s} \vee \sigma(\check{W}_{r\vee t} - \check{W}_t)_{0\leq r\leq s} \vee \sigma(\check{B}_{r\vee t} - \check{B}_t)_{0\leq r\leq s} \vee \check{\mathcal{G}}\big)_{s\geq 0}$. Without loss of generality, we assume that they are $\check{\mathbb{P}}$-complete. Let $t > 0$, denote by $\check{\mathcal{A}}$ (resp. $\check{\mathcal{A}}_t$) the set of $\check{\mathbb{F}}$-progressively measurable processes (resp. $\check{\mathbb{F}}^t$-progressively measurable processes) valued in A.

Letting $\varepsilon > 0$, $t \in [0,T)$, $\check{\xi} \in L^2(\check{\Omega}^1, \check{\mathcal{F}}^1, \check{\mathbb{P}}^1; \mathbb{R}^d)$ and $\check{\alpha} \in \check{\mathcal{A}}$, we consider the unique solution $\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}} = (\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_s)_{s\in[t,T]}$ of the perturbed equation:

$$\begin{aligned} X_s =& \check{\xi} + \int_t^s b(r, X_r, \mathbb{P}^{\check{W}^0}_{X_r}, \check{\alpha}_r)dr + \int_t^s \sigma(r, X_r, \check{\alpha}_r)d\check{W}_r + \int_t^s \sigma^0(r, X_r)d\check{W}^0_r \\ &+ \varepsilon(\check{B}_s - \check{B}_t). \end{aligned} \tag{3.2.1}$$

For any $t \in [0,T]$ and $\check{\xi} \in L^2(\check{\Omega}^1, \check{\mathcal{F}}^1, \check{\mathbb{P}}^1; \mathbb{R}^d)$, we consider the value function:

$$V_\varepsilon(t,\check{\xi}) = \sup_{\check{\alpha}\in\check{\mathcal{A}}_t} J_\varepsilon(t,\check{\xi},\check{\alpha}) := \sup_{\check{\alpha}\in\check{\mathcal{A}}_t} \mathbb{E}\bigg[\int_t^T f\Big(s, \check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_s, \mathbb{P}^{\check{W}^0}_{\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_s}, \check{\alpha}_s\Big)ds + g\Big(\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_T, \mathbb{P}^{\check{W}^0}_{\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_T}\Big)\bigg]. \tag{3.2.2}$$

By the law invariance property similar to (Cosso *et al.*, 2023, Appendix B) and (Cheung *et al.*, 2025, Proposition 3.3), we can define a function $v_\varepsilon(t,\mu) : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that

$$v_\varepsilon(t,\mu) := V_\varepsilon(t,\check{\xi}), \tag{3.2.3}$$

for any $\check{\xi} \in L^2(\check{\Omega}^1, \check{\mathcal{F}}^1, \check{\mathbb{P}}^1; \mathbb{R}^d)$ such that $\mathcal{L}(\check{\xi}) = \mu$.

Lemma 3.2.1. *Suppose that Assumption 3.1.2 holds. There exists a constant $C_5 = C_5(d,K,T) > 0$ such that for any $\varepsilon \geq 0$ and $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, it holds that $|v_\varepsilon(t,\mu) - v_0(t,\mu)| \leq C_5\varepsilon$.*

Proof. It is standard to see that $\mathbb{E}\big[|\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_s - \check{X}^{0,t,\check{\xi},\check{\alpha}}_s|^2\big] \leq C\varepsilon^2$ for any $s \in [t,T]$ by using Assumption 3.1.2 and equation (3.2.1). The result follows by

$$|v_\varepsilon(t,\mu) - v_0(t,\mu)|^2 \leq 2K^2\mathbb{E}\left[T\int_t^T \left|\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_s - \check{X}^{0,t,\check{\xi},\check{\alpha}}_s\right|^2 ds + \left|\check{X}^{\varepsilon,t,\check{\xi},\check{\alpha}}_T - \check{X}^{0,t,\check{\xi},\check{\alpha}}_T\right|^2\right] \leq C_5^2\varepsilon^2.$$

□

3.2.2 Smooth Finite-dimensional Approximation of Coefficient Functions

In this section, we illustrate the finite-dimensional approximation. Consider a complete probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, which is also of the form $(\overline{\Omega}^0 \times \overline{\Omega}^1, \overline{\mathcal{F}}^0 \otimes \overline{\mathcal{F}}^1, \overline{\mathbb{P}}^0 \otimes \overline{\mathbb{P}}^1)$. The space $(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0)$ supports a d-dimensional Brownian motion $\overline{W}^0$. For $(\overline{\Omega}^1, \overline{\mathcal{F}}^1, \overline{\mathbb{P}}^1)$, it is of the form $(\overline{\tilde{\Omega}}^1 \times \overline{\hat{\Omega}}^1, \overline{\mathcal{G}} \otimes \overline{\hat{\mathcal{F}}}^1, \overline{\tilde{\mathbb{P}}}^1 \otimes \overline{\hat{\mathbb{P}}}^1)$. Let $n \in \mathbb{N}$. There lives d-dimensional Brownian motions $\overline{W}^1, \dots, \overline{W}^n$ and $\overline{B}^1, \dots, \overline{B}^n$ on $(\overline{\hat{\Omega}}^1, \overline{\hat{\mathcal{F}}}^1, \overline{\hat{\mathbb{P}}}^1)$. We require $\overline{W}^1, \dots, \overline{W}^n, \overline{B}^1, \dots, \overline{B}^n$ to be mutually independent. The space $(\overline{\tilde{\Omega}}^1, \overline{\mathcal{G}}, \overline{\tilde{\mathbb{P}}}^1)$ is where the initial random variables live. We assume that $(\overline{\tilde{\Omega}}^1, \overline{\mathcal{G}}, \overline{\tilde{\mathbb{P}}}^1)$ is rich enough to support all probability laws on $\mathbb{R}^d$, i.e., for any probability law μ on $\mathbb{R}^d$, there exists $X : \overline{\tilde{\Omega}}^1 \to \mathbb{R}^d$ such that $\mathcal{L}(X) = \mu$.

We define $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_s)_{s\geq 0} := \left(\sigma(\overline{W}^0_r)_{0\leq r\leq s} \vee \sigma(\overline{W}^i_r)_{0\leq r\leq s, i=1,\dots,n} \vee \sigma(\overline{B}^i_r)_{0\leq r\leq s, i=1,\dots,n} \vee \overline{\mathcal{G}}\right)_{s\geq 0}$, $\overline{\mathbb{F}}^t = (\mathcal{F}^t_s)_{s\geq 0} := \left(\sigma(\overline{W}^0_r - \overline{W}^0_t)_{0\leq r\leq s} \vee \sigma(\overline{W}^i_{r\vee t} - \overline{W}^i_t)_{0\leq r\leq s, i=1,\dots,n} \vee \sigma(\overline{B}^i_{r\vee t} - \overline{B}^i_t)_{0\leq r\leq s, i=1,\dots,n} \vee \overline{\mathcal{G}}\right)_{s\geq 0}$. Without loss of generality, we assume that they are $\overline{\mathbb{P}}$-complete. Let $t > 0$ and denote by $\overline{\mathcal{A}}^n$ (resp. $\overline{\mathcal{A}}^n_t$) the set of $\overline{\mathbb{F}}$-progressively measurable processes (resp. $\overline{\mathbb{F}}^t$-progressively measurable processes) $\overline{\alpha} = (\overline{\alpha}^1, \dots, \overline{\alpha}^n)$ valued in A^n.

Now we introduce the smooth approximations of the coefficients. Let $m \in \mathbb{N}$, $\phi : \mathbb{R} \to \mathbb{R}^+$ and $\Phi : \mathbb{R}^d \to \mathbb{R}^+$ be two compactly supported smooth functions satisfying $\int_{\mathbb{R}} \phi(s)ds = 1$, $\int_{\mathbb{R}^d} \Phi(y)dy = 1$ and $\int_{\mathbb{R}^d} |y|^\rho \Phi(y)dy \leq C_{\Phi,\rho}$ for some constant $C_{\Phi,\rho} > 0$, where ρ is the parameter given in Assumption 3.1.2. For $n, m \in \mathbb{N}$ and $i = 1, \dots, n$, we define $b^i_{n,m} : [0,T] \times \mathbb{R}^{dn} \times A \to \mathbb{R}^d$, $f^i_{n,m} : [0,T] \times \mathbb{R}^{dn} \times A \to \mathbb{R}^d$, $g^i_{n,m} : \mathbb{R}^{dn} \to \mathbb{R}^d$ by the smooth approximations of b, f and g respectively such that

$$b^i_{n,m}(t, \overline{x}, a) := m^{dn+1} \int_{\mathbb{R}^{dn+1}} b\left((t-s)^+ \wedge T, x^i - y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}, a\right) \phi(ms) \prod_{k=1}^n \Phi(my^k) dy^k ds; \tag{3.2.4}$$

$$f^i_{n,m}(t,\overline{x},a) := m^{dn+1}\int_{\mathbb{R}^{dn+1}} f\left((t-s)^+\wedge T, x^i-y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j-y^j}, a\right)\phi(ms)\prod_{k=1}^n \Phi(my^k)dy^k ds; \tag{3.2.5}$$

$$g^i_{n,m}(\overline{x}) := m^{dn}\int_{\mathbb{R}^{dn}} g\left(x^i-y^i, \frac{1}{n}\sum_{j=1}^n \delta_{x^j-y^j}\right)\prod_{k=1}^n \Phi(my^k)dy^k. \tag{3.2.6}$$

First, we establish some basic properties of these approximation functions.

Lemma 3.2.2. *Suppose that Assumption 3.1.2 holds, and define* $\widehat{\mu}^{n,\overline{x}} := \frac{1}{n}\sum_{j=1}^n \delta_{x^j}$. *For any* $n,m\in\mathbb{N}$, $i=1,2,\ldots,n$ *and* $(t,\overline{x},\overline{z},a)\in[0,T]\times\mathbb{R}^{dn}\times\mathbb{R}^{dn}\times A$, *we have the following:*

(1). $|f^i_{n,m}(t,\overline{x},a)|\vee|g^i_{n,m}(\overline{x})|\le K$ *and* $|b^i_{n,m}(t,\overline{x},a)|\le K\Big(1+C_{\Phi,\rho}m^{-\rho}+|x^i|^\rho\Big)$;

(2). $|b^i_{n,m}(t,\overline{x},a)-b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)|\vee|f^i_{n,m}(t,\overline{x},a)-f(t,x^i,\widehat{\mu}^{n,\overline{x}},a)|$
$\le Km\int_{\mathbb{R}}\Big|t-\big[T\wedge(t-s)^+\big]\Big|^\beta\phi(ms)ds + Km^{dn}\int_{\mathbb{R}^{dn}}\left(|y^i|+\frac{1}{n}\sum_{j=1}^n|y^j|\right)\prod_{k=1}^n\Phi(my^k)dy^k;$
$|g^i_{n,m}(\overline{x})-g(x^i,\widehat{\mu}^{n,\overline{x}})|\le Km^{dn}\int_{\mathbb{R}^{dn}}\left(|y^i|+\frac{1}{n}\sum_{j=1}^n|y^j|\right)\prod_{k=1}^n\Phi(my^k)dy^k;$

(3). $|b^i_{n,m}(t,\overline{x},a)-b^i_{n,m}(t,\overline{z},a)|\vee|f^i_{n,m}(t,\overline{x},a)-f^i_{n,m}(t,\overline{z},a)|\vee|g^i_{n,m}(\overline{x})-g^i_{n,m}(\overline{z})|$
$\le K\left[|x^i-z^i|+\frac{1}{n}\sum_{j=1}^n\big|x^j-z^j\big|\right]$;

(4). $\lim_{m\to\infty} b^i_{n,m}(t,\overline{x},a)=b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)$, $\lim_{m\to\infty} f^i_{n,m}(t,\overline{x},a)=f(t,x^i,\widehat{\mu}^{n,\overline{x}},a)$ *and* $\lim_{m\to\infty} g^i_{n,m}(\overline{x})=g(x^i,\widehat{\mu}^{n,\overline{x}})$. *These limits hold uniformly in* $(t,\overline{x},a)\in[0,T]\times\mathbb{R}^{dn}\times A$.

For the proof of Lemma 3.2.2, please refer to Appendix 3.4.

Let $\overline{X}_s^{m,\varepsilon,t,\overline{x},\overline{\alpha}} = (\overline{X}_s^{1,m,\varepsilon,t,\overline{x},\overline{\alpha}},\ldots,\overline{X}_s^{n,m,\varepsilon,t,\overline{x},\overline{\alpha}})$ be the solution of

$$X^i_s = x^i + \int_t^s b^i_{n,m}(r,X_r,\overline{\alpha}^i_r)dr + \int_t^s \sigma(r,X^i_r,\overline{\alpha}^i_r)d\overline{W}^i_r + \int_t^s \sigma^0(r,X^i_r)d\overline{W}^0_r + \varepsilon(\overline{B}^i_s-\overline{B}^i_t), \tag{3.2.7}$$

where $X_s=(X^1_s,\ldots,X^n_s)$ with $\mathbb{R}^d$-valued processes X^i_s for $i=1,\ldots,n$. We define

$$\overline{v}_{\varepsilon,n,m}(t,\overline{x}) := \sup_{\overline{\alpha}\in\overline{\mathcal{A}}^n_t} J^*_{\varepsilon,n,m}(t,\overline{x},\overline{\alpha})$$

$$:= \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\Bigg[\int_t^T f_{n,m}^i\left(s, \overline{X}_s^{1,m,\varepsilon,t,\overline{x},\overline{\alpha}}, \ldots, \overline{X}_s^{n,m,\varepsilon,t,\overline{x},\overline{\alpha}}, \overline{\alpha}_s^i\right) ds + g_{n,m}^i\left(\overline{X}_T^{1,m,\varepsilon,t,\overline{x},\overline{\alpha}}, \ldots, \overline{X}_T^{n,m,\varepsilon,t,\overline{x},\overline{\alpha}}\right)\Bigg], \tag{3.2.8}$$

for any $t \in [0,T]$ and $\overline{x} \in \mathbb{R}^{dn}$. This approximation of the value function is defined on a finite-dimensional domain. Owing to the non-degeneracy and smoothness of the coefficients, the approximation belongs to the class $C^{1,2}([0,T]\times\mathbb{R}^{dn})$ and possesses the following properties:

Lemma 3.2.3. *Suppose that Assumptions 3.1.2-3.1.2 hold. There are positive constants $C_4 = C_4(d,K,T)$ and $C_{n,m}$ such that the function $\overline{v}_{\varepsilon,n,m} : [0,T]\times\mathbb{R}^{dn} \longmapsto \mathbb{R}$ satisfies the following:*

(1). For any $(t,\overline{x})$ in $[0,T]\times\mathbb{R}^{dn}$ and $i = 1,2,\ldots,n$, we have

$$|\nabla_{x^i}\overline{v}_{\varepsilon,n,m}(t,\overline{x})| \leq \frac{C_4}{n}; \tag{3.2.9}$$

(2). Denoting by $\partial^2_{\overline{x}_i\overline{x}_j} v(t,\overline{x}) \in \mathbb{R}$ the second-order derivative with respect to $\overline{x}_i$ and $\overline{x}_j$, it holds that

$$-C_{n,m} \leq \partial^2_{\overline{x}_i\overline{x}_j}\overline{v}_{\varepsilon,n,m}(t,\overline{x}) \leq \frac{C_{n,m}}{\varepsilon^2}, \tag{3.2.10}$$

for any $(t,\overline{x})$ in $[0,T]\times\mathbb{R}^{dn}$ and every $i,j = 1,2,\ldots,dn$;

(3). The function $\overline{v}_{\varepsilon,n,m} \in C^{1,2}([0,T]\times\mathbb{R}^{dn})$ is the unique classical solution of the Bellman equation

$$\begin{cases} \partial_t u(t,\overline{x}) + \sup\limits_{\overline{a}\in A^n}\Bigg\{ \dfrac{1}{n}\sum\limits_{i=1}^n f_{n,m}^i(t,\overline{x},a^i) + \dfrac{1}{2}\sum\limits_{i=1}^n \mathrm{tr}\Big[\Big((\sigma\sigma^\top)(t,x^i,a^i) + \sigma^0\sigma^{0;\top}(t,x^i) + \varepsilon^2 I_d\Big)\nabla^2_{x^ix^i}u(t,\overline{x})\Big] \\ \qquad + \sum\limits_{i=1}^n \big\langle b_{n,m}^i(t,\overline{x},a^i), \nabla_{x^i}u(t,\overline{x})\big\rangle + \dfrac{1}{2}\sum\limits_{i,j=1,i\neq j}^n \mathrm{tr}\Big[\sigma^0(t,x^i)\sigma^{0;\top}(t,x^j)\nabla^2_{x^ix^j}u(t,\overline{x})\Big]\Bigg\} \\ = 0 \ \text{ in } [0,T)\times\mathbb{R}^{dn}; \\ u(T,\overline{x}) = \dfrac{1}{n}\sum\limits_{i=1}^n g_{n,m}^i(\overline{x}) \quad \text{in } \mathbb{R}^{dn}, \end{cases} \tag{3.2.11}$$

where $\overline{a} = (a^1,\ldots,a^n)$ with $a^i \in A$ for $i = 1,2,\ldots,n$.

The proof of Lemma 3.2.3 is delegated to Appendix 3.4.

3.2.3 Approximation of Value Function

Recalling the definition of $J^*_{\varepsilon,n,m}$ in (3.2.8), we define

$$\widetilde{v}_{\varepsilon,n,m}(t,\overline{\mu}) := \sup_{\overline{\alpha}\in\overline{\mathcal{A}}_t^n} J^*_{\varepsilon,n,m}(t,\overline{\xi},\overline{\alpha})$$

for any $t \in [0,T]$ and $\overline{\mu} \in \mathcal{P}_2(\mathbb{R}^{dn})$ such that $\mathcal{L}(\overline{\xi}) = \overline{\mu}$. It is clear that $\overline{v}_{\varepsilon,n,m}(t,\overline{x}) = \widetilde{v}_{\varepsilon,n,m}(t,\delta_{x^1}\otimes\ldots\otimes\delta_{x^n})$. We define $v_{\varepsilon,n,m}(t,\mu) : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ by

$$v_{\varepsilon,n,m}(t,\mu) := \widetilde{v}_{\varepsilon,n,m}(t,\mu\otimes\ldots\otimes\mu). \tag{3.2.12}$$

Theorem 3.2.4. *Suppose that Assumptions 3.1.2-3.1.2 hold. For every $\varepsilon > 0$, $n,m\in\mathbb{N}$, the function $\overline{v}_{\varepsilon,n,m}$ defined in* (3.2.8) *and the function $v_{\varepsilon,n,m}$ defined in* (3.2.12) *satisfy the following:*

(1). For any $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, we have

$$v_{\varepsilon,n,m}(t,\mu) = \int_{\mathbb{R}^{dn}} \overline{v}_{\varepsilon,n,m}(t,x^1,\ldots,x^n)\mu(dx^1)\otimes\ldots\otimes\mu(dx^n)\,; \tag{3.2.13}$$

(2). $v_{\varepsilon,n,m}(t,\mu)\in C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$;

(3). There is a positive constant $\ell_2 = \ell_2(K,T)$ such that for any $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, we have $|v_{\varepsilon,n,m}(t,\mu)|\le\ell_2$;

(4). The function $v_{\varepsilon,n,m}(t,\mu)$ solves the following equation in the classical sense:

$$\begin{cases} \partial_t u(t,\mu) \\ + \int_{\mathbb{R}^{dn}} \sup_{\overline{a}\in A^n} \Big\{ \frac{1}{n}\sum_{i=1}^n f^i_{n,m}(t,\overline{x},a^i) + \sum_{i=1}^n \big\langle b^i_{n,m}(t,\overline{x},a^i), \nabla_{x^i}\overline{v}_{\varepsilon,n,m}(t,\overline{x})\big\rangle \\ \qquad + \frac{1}{2}\sum_{i=1}^n \mathrm{tr}\Big[\Big((\sigma\sigma^\top)(t,x^i,a^i) + (\sigma^0\sigma^{0;\top})(t,x^i) + \varepsilon^2 I_d\Big)\nabla^2_{x^ix^i}\overline{v}_{\varepsilon,n,m}(t,\overline{x})\Big] \\ \qquad + \frac{1}{2}\sum_{i,j=1,i\neq j}^n \mathrm{tr}\Big[\sigma^0(t,x^i)\sigma^{0;\top}(t,x^j)\nabla^2_{x^ix^j}\overline{v}_{\varepsilon,n,m}(t,\overline{x})\Big]\Big\}\mu(dx^1)\otimes\ldots\otimes\mu(dx^n) \\ = 0 \text{ for any } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \frac{1}{n}\sum_{i=1}^n \int_{\mathbb{R}^{dn}} g^i_{n,m}(\overline{x})\mu(dx^1)\otimes\ldots\otimes\mu(dx^n) \text{ for any } \mu\in\mathcal{P}_2(\mathbb{R}^d). \end{cases}$$

Proof. The proof of item (1) follows from Steps I-II of the proof of (Cosso *et al.*, 2024, Theorem A.7), together with Lemma 3.2.3. The function $\overline{v}_{\varepsilon,n,m}(t,\overline{x})$ is in $C^{1,2}([0,T]\times\mathbb{R}^{dn})$ from Lemma 3.2.3 and its boundedness is ensured by its definition. Then, item (2) follows by differentiating (3.2.13) and item (3) is ensured by the boundedness of $\overline{v}_{\varepsilon,n,m}(t,\overline{x})$. Item (4) is easily obtained by integrating (3.2.11) with $u=\overline{v}_{\varepsilon,n,m}$. □

Lemma 3.2.5. *Suppose that Assumption 3.1.2 holds. There exists a positive constant $C_6 = C_6(K,T,d)$ such that for any $n,m\in\mathbb{N}$ and $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, we have $|v_{\varepsilon,n,m}(t,\mu) - v_{0,n,m}(t,\mu)|\leq C_6\varepsilon$.*

Proof. This proof follows the same approach as the proof of Lemma 3.2.1, with the aid of Lemma 3.2.2. The details are omitted here. □

Lemma 3.2.6. *Suppose that Assumption 3.1.2 holds. Let $\varepsilon>0$ and $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. If there exists $q>2$ such that $\mu\in\mathcal{P}_q(\mathbb{R}^d)$, then $\lim_{n\to\infty}\lim_{m\to\infty} v_{\varepsilon,n,m}(t,\mu) = v_\varepsilon(t,\mu)$ where $v_\varepsilon(t,\mu)$ is defined in* (3.2.3).

Proof. The proof follows the same reasoning as (Cosso *et al.*, 2024, Theorem A.6), but with (Djete *et al.*, 2022, Theorems 3.1, 3.6) replacing the limit theory in Cosso *et al.* (2024) when

passing to the limit $n \to \infty$ in $\lim_{m\to\infty} v_{\varepsilon,n,m}(t,\mu)$, because of the appearance of the common noise.

□

3.3 Viscosity Solution Theory

In this section, we prove the main result of the article, which is the existence and uniqueness of the viscosity solution of the HJB equation in (3.0.1). We first need the following lemma.

Lemma 3.3.1. *The value function $v(t,\mu)$ defined* (3.1.6) *and its finite-dimensional approximation $v_{\varepsilon,n,m}(t,\mu)$ defined* (3.2.12) *satisfy*

$$|v_{\varepsilon,n,m}(t,\mu) - v_{\varepsilon,n,m}(t,\mu')| \le C_4 \mathcal{W}_1(\mu,\mu'), \quad |v(t,\mu) - v(t,\mu')| \le C_4 \mathcal{W}_1(\mu,\mu')$$

for any $(t,\mu,\mu') \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$, where C_4 is given in Lemma 3.2.3.

Proof. Let μ, $\mu' \in \mathcal{P}_2(\mathbb{R}^d)$. Let $(\xi^1,\eta^1),\dots,(\xi^n,\eta^n)$ be a sequence of independent and identically distributed random variables, with $\mathcal{L}(\xi^i) = \mu$, $\mathcal{L}(\eta^i) = \mu'$, for $i = 1,\dots,n$. Theorem 3.2.4 deduces that $v_{\varepsilon,n,m}(t,\mu) = \mathbb{E}\bar{v}_{\varepsilon,n,m}(t,\xi^1,\dots,\xi^n)$ and $v_{\varepsilon,n,m}(t,\mu') = \mathbb{E}\bar{v}_{\varepsilon,n,m}(t,\eta^1,\dots,\eta^n)$. Lemma 3.2.3 implies that

$$\begin{aligned}|v_{\varepsilon,n,m}(t,\mu) - v_{\varepsilon,n,m}(t,\mu')| \le \mathbb{E}|\bar{v}_{\varepsilon,n,m}(t,\xi^1,\dots,\xi^n) - \bar{v}_{\varepsilon,n,m}(t,\eta^1,\dots,\eta^n)| &\le \frac{C_4}{n}\sum_{i=1}^n \mathbb{E}|\xi^i - \eta^i| \\ &= C_4 \mathbb{E}|\xi^1 - \eta^1|.\end{aligned}$$

Since ξ^1 and η^1 are arbitrary random variables such that $\mathcal{L}(\xi^1) = \mu$ and $\mathcal{L}(\eta^1) = \mu'$, we conclude that

$$|v_{\varepsilon,n,m}(t,\mu) - v_{\varepsilon,n,m}(t,\mu')| \le C_4 \mathcal{W}_1(\mu,\mu').$$

Now we consider v. Let $\mathcal{L}(\xi) = \mu$ and $\mathcal{L}(\eta) = \mu'$. By applying Proposition 3.1.7, together with Lemmas 3.2.5 and 3.2.6, we have

$$|v(t,\mu) - v(t,\mu')| = |V(t,\xi) - V(t,\eta)|$$

$$
\begin{aligned}
&= \lim_{k\to\infty} |V(t, \xi \mathbb{1}_{\{|\xi|\le k\}}) - V(t, \eta \mathbb{1}_{\{|\eta|\le k\}})| \\
&= \lim_{k\to\infty} \lim_{\varepsilon\to 0} \lim_{n\to\infty} \lim_{m\to\infty} |v_{\varepsilon,n,m}(t, \mathcal{L}(\xi \mathbb{1}_{\{|\xi|\le k\}})) - v_{\varepsilon,n,m}(t, \mathcal{L}(\eta \mathbb{1}_{\{|\eta|\le k\}}))| \\
&\le C_4 \lim_{k\to\infty} \mathbb{E}|\xi \mathbb{1}_{\{|\xi|\le k\}} - \eta \mathbb{1}_{\{|\eta|\le k\}}| \\
&= C_4 \mathbb{E}|\xi - \eta|.
\end{aligned}
$$

As ξ, η are arbitrary, provided that $\mathcal{L}(\xi) = \mu$ and $\mathcal{L}(\eta) = \mu'$, we conclude that $|v(t,\mu) - v(t,\mu')| \le C_4 \mathcal{W}_1(\mu, \mu')$. □

3.3.1 Existence

We verify that the value function v is a viscosity solution of (3.0.1) by applying the Itô's formula in Theorem 3.1.9 and the dynamic programming in Theorem 3.1.8. Similarly, the value function v can be shown to be a viscosity solution under the standard Crandall-Lions definition by following the approach outlined in the theorem below. However, for consistency with the uniqueness theorem, we provide the proof under Definition 3.1.5.

Theorem 3.3.2. *Suppose that Assumption 3.1.2 holds. The value function v defined in* (3.1.6) *is a viscosity solution of* (3.0.1).

Proof. From Lemma 3.3.1, the value function v is $([0,T], |\cdot|) \times (\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_1)$-continuous, and its boundedness follows from the boundedness of f and g. Now we show separately that it is a viscosity supersolution and a viscosity subsolution of (3.0.1), according to Definition 3.1.5.

Part 1. v **is a viscosity supersolution:** For any $s_0 \in (0,T]$ and $\varphi \in C^{1,2}([0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A))$, we assume that $v - \varphi$ attains a minimum at $(t_0, \nu_0) \in [0, s_0) \times \mathcal{P}_2(\mathbb{R}^d \times A)$ with a value of 0, where $v(t_0, \nu_0)$ is defined as $v(t_0, \mu_0)$ with μ_0 being the marginal of ν_0 on $\mathbb{R}^d$. Recalling that $\mathcal{M}_t$ is the set of $\mathcal{F}_t^t$-measurable random variables, we let $\alpha' \in \mathcal{M}_{t_0}$ and $\xi \in L^2(\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1; \mathbb{R}^d)$ such that $\mathcal{L}(\xi, \alpha') = \nu_0$. Letting $a \in A$ be arbitrary, we define

$\alpha_s := a\mathbb{1}_{[0,t_0)}(s) + \alpha'\mathbb{1}_{[t_0,T]}(s)$, which belongs to $\mathcal{A}_{t_0}$. Let $(X_s^{t_0,\xi,\alpha})_{s\in[t_0,T]}$ be the solution of the dynamic (3.1.3) with initial time t_0, initial data ξ and control α defined in the above. Then for $h > 0$ small enough, we use the dynamic programming in Theorem 3.1.8 and Itô's formula in Theorem 3.1.9 to obtain that

$$
\begin{aligned}
0 &\geq \frac{1}{h}\mathbb{E}\Big[(v-\varphi)(t_0,\nu_0) - (v-\varphi)(t_0+h, \mathbb{P}^{W^0}_{(X^{t_0,\xi,\alpha}_{t_0+h},\alpha_{t_0+h})})\Big] \\
&\geq \frac{1}{h}\mathbb{E}\Bigg[\int_{t_0}^{t_0+h} f(s, X_s^{t_0,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}}, \alpha_s) + \partial_t\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)}) \\
&\qquad + \mathbb{E}^1\big[\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\cdot b(s, X_s^{t_0,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}}, \alpha_s)\big] \\
&\qquad + \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\sigma(s, X_s^{t_0,\xi,\alpha},\alpha_s)\sigma^\top(s, X_s^{t_0,\xi,\alpha},\alpha_s)\Big]\Big\} \\
&\qquad + \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s)\sigma^0(s, X_s^{t_0,\xi,\alpha})\sigma^{0;\top}(s, X_s^{t_0,\xi,\alpha})\Big]\Big\} \\
&\qquad + \frac{1}{2}\mathbb{E}^1\breve{\mathbb{E}}^1\Big\{\operatorname{tr}\Big[\partial_\mu^2\varphi(s, \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)})(X_s^{t_0,\xi,\alpha},\alpha_s,\breve{X}_s^{t_0,\xi,\alpha},\breve{\alpha}_s)\sigma^0(s, X_s^{t_0,\xi,\alpha})\sigma^{0;\top}(s, \breve{X}_s^{t_0,\xi,\alpha})\Big]\Big\}ds\Bigg],
\end{aligned}
$$

where the operators ∂_μ, ∂_μ^2 are defined as in Definition 3.1.5, through the projection. The process $(\breve{X}_s^{t_0,\xi,\alpha}, \breve{\alpha}_s)$ is an independent copy of $(X_s^{t_0,\xi,\alpha}, \alpha_s)$. For any $\varepsilon > 0$, it holds that

$$
\begin{aligned}
\mathbb{P}^0\left(\sup_{t\in[s,s+h]} \mathcal{W}_2(\mathbb{P}^{W^0}_{X_t^{t_0,\xi,\alpha}}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}}) > \varepsilon\right) &\leq \frac{\mathbb{E}^0 \sup_{t\in[s,s+h]}\Big[\mathcal{W}_2(\mathbb{P}^{W^0}_{X_t^{t_0,\xi,\alpha}}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}})\Big]^2}{\varepsilon^2} \\
&\leq \frac{\mathbb{E}\sup_{t\in[s,s+h]}|X_t^{t_0,\xi,\alpha} - X_s^{t_0,\xi,\alpha}|^2}{\varepsilon^2} \\
&\leq \frac{Ch}{\varepsilon^2},
\end{aligned}
\tag{3.3.1}
$$

where in the last inequality we have used Proposition 3.1.6. The above term goes to 0 as $h \to 0^+$, thus the flow of measure $s \mapsto \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha}}$ is continuous on $[t_0, s_0]$ for $\mathbb{P}$-a.s. $\omega^0 \in \Omega^0$, so is $s \mapsto \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)}$ for $\mathbb{P}$-a.s. $\omega^0 \in \Omega^0$. From the regularity of the coefficients in Assumption 3.1.2, the regularity of the test function φ, the continuity of the processes $s \mapsto X_s^{t_0,\xi,\alpha}$, $s \mapsto \mathbb{P}^{W^0}_{(X_s^{t_0,\xi,\alpha},\alpha_s)}$ for $\mathbb{P}$-a.s. $\omega^0 \in \Omega^0$ and the dominated convergence theorem, we have as

$h \to 0^+$,

$$\partial_t\varphi(t_0,\nu_0) + \mathbb{E}\breve{\mathbb{E}}\Big\{ f(t_0,\xi,\mu_0,\alpha') + \big[\partial_\mu\varphi(t_0,\nu_0)(\xi,\alpha')\cdot b(t_0,\xi,\mu_0,\alpha')\big]$$
$$+ \frac{1}{2}\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(t_0,\nu_0)(\xi,\alpha')\sigma(t_0,\xi,\alpha')[\sigma(t_0,\xi,\alpha')]^\top\Big] + \frac{1}{2}\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(t_0,\nu_0)(\xi,\alpha')\sigma^0(t_0,\xi)[\sigma^0(t_0,\xi)]^\top\Big]$$
$$+ \frac{1}{2}\operatorname{tr}\big[\partial_\mu^2\varphi(t_0,\nu_0)(\xi,\alpha',\breve{\xi},\breve{\alpha}')\sigma^0(t_0,\xi)[\sigma^0(t_0,\breve{\xi})]^\top\big]\Big\} \le 0. \tag{3.3.2}$$

Part 2. v **is a viscosity subsolution:** For any $\varphi \in C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$, we assume that $v-\varphi$ attains a maximum at $(t_0,\mu_0)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$ with a value of 0. Let $\varepsilon>0$ and $\xi\in L^2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ satisfying $\mathcal{L}(\xi)=\mu_0$. There is an $\alpha^\varepsilon\in\mathcal{A}_t$ such that for any $h\in(0,T-t_0]$, we have

$$v(t_0,\mu_0) - \varepsilon \le \mathbb{E}\bigg[\int_{t_0}^{t_0+h} f(r, X_r^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_r^{t_0,\xi,\alpha^\varepsilon}}, \alpha_r^\varepsilon)dr + v(t_0+h, \mathbb{P}^{W^0}_{X_{t_0+h}^{t_0,\xi,\alpha^\varepsilon}})\bigg],$$

where $(X_s^{t_0,\xi,\alpha^\varepsilon})_{s\in[t_0,T]}$ solves the dynamic (3.1.3) with initial time t_0, initial data ξ and control α^ε. Then Theorem 3.1.9 implies

$$\begin{aligned}
0 \le{}& \frac{1}{h}\mathbb{E}\Big[(v-\varphi)(t_0,\mu_0) - (v-\varphi)(t_0+h, \mathbb{P}^{W^0}_{X_{t_0+h}^{t_0,\xi,\alpha^\varepsilon}})\Big]\\
<{}& \frac{1}{h}\mathbb{E}\bigg\{\int_{t_0}^{t_0+h} f(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, \alpha_s^\varepsilon) + \partial_t\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})\\
&+ \mathbb{E}^1\Big[\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, \alpha_s^\varepsilon)\Big]\\
&+ \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \alpha_s^\varepsilon)\sigma^\top(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \alpha_s^\varepsilon)\Big]\Big\}\\
&+ \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^0(s, X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^{0;\top}(s, X_s^{t_0,\xi,\alpha^\varepsilon})\Big]\Big\}\\
&+ \frac{1}{2}\mathbb{E}^1\breve{\mathbb{E}}^1\Big\{\operatorname{tr}\Big[\partial_\mu^2\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon}, \breve{X}_s^{t_0,\xi,\alpha^\varepsilon})\sigma^0(s, X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^{0;\top}(s, \breve{X}_s^{t_0,\xi,\alpha^\varepsilon})\Big]\Big\}ds\bigg\} + \varepsilon\\
\le{}& \frac{1}{h}\mathbb{E}\bigg\{\int_{t_0}^{t_0+h}\sup_{a\in A}\Big(f(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a) + \partial_t\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})\\
&+ \mathbb{E}^1\big[\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a)\big]\\
&+ \frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s, X_s^{t_0,\xi,\alpha^\varepsilon}, a)\sigma^\top(s, X_s^{t_0,\xi,\alpha^\varepsilon}, a)\Big]\Big\}\Big)
\end{aligned}$$

$$+\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^0(s,X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^{0,\top}(s,X_s^{t_0,\xi,\alpha^\varepsilon})\Big]\Big\}$$
$$+\frac{1}{2}\mathbb{E}^1\check{\mathbb{E}}^1\Big\{\operatorname{tr}\Big[\partial_\mu^2\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon},\check{X}_s^{t_0,\xi,\alpha^\varepsilon})\sigma^0(s,X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^{0,\top}(s,\check{X}_s^{t_0,\xi,\alpha^\varepsilon})\Big]\Big\}\Big\}ds+\varepsilon. \tag{3.3.3}$$

Arguing as in (3.3.2), we have

$$\begin{aligned}&\frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\Big\{\partial_t\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})+\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^0(s,X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^{0,\top}(s,X_s^{t_0,\xi,\alpha^\varepsilon})\Big]\Big\}\\&\qquad+\frac{1}{2}\mathbb{E}^1\check{\mathbb{E}}^1\Big\{\operatorname{tr}\Big[\partial_\mu^2\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon},\check{X}_s^{t_0,\xi,\alpha^\varepsilon})\sigma^0(s,X_s^{t_0,\xi,\alpha^\varepsilon})\sigma^{0,\top}(s,\check{X}_s^{t_0,\xi,\alpha^\varepsilon})\Big]\Big\}\Big\}ds\\&\longrightarrow\partial_t\varphi(t_0,\mu_0)+\frac{1}{2}\mathbb{E}\check{\mathbb{E}}\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma^0(t_0,\xi)[\sigma^0(t_0,\xi)]^\top\Big]+\operatorname{tr}\Big[\partial_\mu^2\varphi(t_0,\mu_0)(\xi,\check{\xi})\sigma^0(t_0,\xi)[\sigma^0(t_0,\check{\xi})]^\top\Big]\Big\},\end{aligned} \tag{3.3.4}$$

as $h\to0^+$. We define the following terms:

$$\begin{aligned}\mathrm{I}:=&\frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{f(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)-f(t_0,\xi,\mu_0,a)\Big\}ds;\\\mathrm{II}:=&\frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)\\&\qquad-\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\Big]\Big\}ds;\\\mathrm{III}:=&\frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h}\sup_{a\in A}\Big\{\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\sigma^\top(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\Big]\Big\}\\&\qquad-\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,a)\sigma^\top(t_0,\xi,a)\Big]\Big\}\Big\}ds.\end{aligned}$$

Then the above gives

$$\begin{aligned}&\frac{1}{h}\mathbb{E}\Big\{\int_{t_0}^{t_0+h}\sup_{a\in A}\Big(f(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)+\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s,X_s^{t_0,\xi,\alpha^\varepsilon},\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}},a)\Big]\\&\qquad+\frac{1}{2}\mathbb{E}^1\Big\{\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\sigma(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\sigma^\top(s,X_s^{t_0,\xi,\alpha^\varepsilon},a)\Big]\Big\}\Big)ds\Big\}\\&\le\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathbb{E}\sup_{a\in A}\Big\{f(t_0,\xi,\mu_0,a)+\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\\&\qquad+\frac{1}{2}\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,a)\sigma^\top(t_0,\xi,a)\Big]\Big\}.\end{aligned} \tag{3.3.5}$$

We now show that all these three terms I, II, III $\to 0$ as $h \to 0^+$. We first investigate I by

$$\begin{aligned}
\mathrm{I} &= \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h} \sup_{a\in A}\Big\{f(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a) - f(t_0,\xi,\mu_0,a)\Big\}ds \\
&\le \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h} K\Big\{|X_s^{t_0,\xi,\alpha^\varepsilon} - \xi| + \mathcal{W}_1(\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, \mu_0) + |s-t_0|^\beta\Big\}ds \\
&\longrightarrow 0,
\end{aligned}$$

as $h \to 0^+$ by Proposition 3.1.6 and Assumption 3.1.2. For II, we have

$$\begin{aligned}
\mathrm{II} &= \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h} \sup_{a\in A}\Big\{\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a) \\
&\qquad\qquad - \partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\Big]\Big\}ds \\
&\le \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h} \sup_{a\in A}\Big\{\mathbb{E}^1\Big[\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon})\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a)\Big] \\
&\qquad\qquad - \mathbb{E}^1\Big[\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a)\Big]\Big\}ds \\
&\quad + \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h} \sup_{a\in A}\mathbb{E}^1\Big[\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a) - \partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,\mu_0,a)\Big]ds \\
&\le \frac{1}{h}\mathbb{E}\int_{t_0}^{t_0+h} \sup_{a\in A}\mathbb{E}^1\Big[\Big|\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon}) - \partial_\mu\varphi(t_0,\mu_0)(\xi)\Big|\Big|b(s, X_s^{t_0,\xi,\alpha^\varepsilon}, \mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, a)\Big|\Big]ds \\
&\quad + \frac{K}{h}\mathbb{E}\int_{t_0}^{t_0+h} \mathbb{E}^1\Big[\Big|\partial_\mu\varphi(t_0,\mu_0)(\xi)\Big|\Big(|s-t_0|^\beta + |X_s^{t_0,\xi,\alpha^\varepsilon} - \xi| + \mathcal{W}_1(\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, \mu_0)\Big)\Big]ds \\
&\le \frac{K}{h}\mathbb{E}\int_{t_0}^{t_0+h} \mathbb{E}^1\Big[\Big|\partial_\mu\varphi(s,\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}})(X_s^{t_0,\xi,\alpha^\varepsilon}) - \partial_\mu\varphi(t_0,\mu_0)(\xi)\Big|\Big(|1+|X_s^{t_0,\xi,\alpha^\varepsilon}|^\rho\Big)\Big]ds \quad (3.3.6) \\
&\quad + \frac{\sqrt{3}K}{h}\sqrt{\mathbb{E}\mathbb{E}^1|\partial_\mu\varphi(t_0,\mu_0)(\xi)|^2}\int_{t_0}^{t_0+h}\sqrt{\mathbb{E}\mathbb{E}^1\Big(|s-t_0|^{2\beta} + |X_s^{t_0,\xi,\alpha^\varepsilon} - \xi|^2 + \Big[\mathcal{W}_1(\mathbb{P}^{W^0}_{X_s^{t_0,\xi,\alpha^\varepsilon}}, \mu_0)\Big]^2\Big)}ds.
\end{aligned}$$

The convergence of the term in (3.3.6) is due to the continuity as proved in (3.3.1), Proposition 3.1.6, the regularity of the test function φ and the dominated convergence theorem. Thus, the term II converges to zero as $h \to 0^+$ with the aid of Proposition 3.1.6. Similar estimate holds for the term III, thus putting (3.3.4) and (3.3.5) into (3.3.3) and then passing $h \to 0^+$, we obtain that

$$\begin{aligned}
0 \le{}& \partial_t\varphi(t_0,\mu_0) + \mathbb{E}\sup_{a\in A}\Big\{f(t_0,\xi,\mu_0,a) + \big[\partial_\mu\varphi(t_0,\mu_0)(\xi)\cdot b(t_0,\xi,a,\mu_0)\big] \\
&+ \frac{1}{2}\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma(t_0,\xi,a)\sigma^\top(t_0,\xi,a)\Big]\Big\} + \mathbb{E}\breve{\mathbb{E}}\Big\{\frac{1}{2}\operatorname{tr}\Big[\nabla_x\partial_\mu\varphi(t_0,\mu_0)(\xi)\sigma^0(t_0,\xi)[\sigma^0(t_0,\xi)]^\top\Big]
\end{aligned}$$

$$+\frac{1}{2}\operatorname{tr}\left[\partial_\mu^2\varphi(t_0,\mu_0)(\xi,\check{\xi})\sigma^0(t_0,\xi)[\sigma^0(t_0,\check{\xi})]^\top\right]\Bigg\}+\varepsilon.$$

As $\varepsilon>0$ is arbitrary, the claim is proved. □

3.3.2 Uniqueness

This subsection demonstrates that any viscosity subsolution is less than or equal to any viscosity supersolution, leading to the uniqueness of viscosity solution which is the main result of this article.

Theorem 3.3.3. *Suppose that Assumptions 3.1.2-3.1.2 hold. Let u_1, $u_2:[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ be the viscosity subsolution and supersolution (in the sense of Definition 3.1.5) of equation* (3.0.1) *respectively. Then it holds that $u_1\le u_2$ on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. Hence, the viscosity solution of equation* (3.0.1) *is unique.*

Proof. Recalling the function v_0 defined in (3.2.3) with $\varepsilon=0$, we shall prove that $u_1\le v_0$ and $v_0\le u_2$ on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. We prove the cases for subsolution and supersolution separately.

Part 1. Proof of $u_1\le v_0$: Let u_1 be a bounded viscosity subsolution of the equation (3.0.1). To prove $u_1\le v_0$ by contradiction, we assume that there exists $(t_0,\widetilde{\mu}_0)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ such that

$$(u_1-v_0)(t_0,\widetilde{\mu}_0)>0.$$

Let $\xi\in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ such that $\mathcal{L}(\xi)=\widetilde{\mu}_0$. For any $k\in\mathbb{N}$, we let $\mu_0^k\in\mathcal{P}_2(\mathbb{R}^d)$ be the law of $\xi\mathbb{1}_{\{|\xi|\le k\}}$. We see that $\mu_0^k\in\mathcal{P}_q(\mathbb{R}^d)$ for any $q\ge1$ and

$$\left[\mathcal{W}_1(\mu_0^k,\widetilde{\mu}_0)\right]^2\le\left[\mathcal{W}_2(\mu_0^k,\widetilde{\mu}_0)\right]^2\le\mathbb{E}\left[|\xi\mathbb{1}_{\{|\xi|\le k\}}-\xi|^2\right]=\int_{|x|>k}|x|^2\widetilde{\mu}_0(dx)\longrightarrow0$$

as $k\to\infty$. Therefore, as both u_1 and v_0 are continuous on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, we can find a $k\in\mathbb{N}$ large enough such that $\mu_0:=\mu_0^k\in\mathcal{P}_q(\mathbb{R}^d)$ for any $q\ge1$ and

$$(u_1-v_0)(t_0,\mu_0)>0. \tag{3.3.7}$$

Step 1A: Construction of the comparison function: Recalling the approximations defined in (3.2.5), (3.2.6), (3.2.8) and (3.2.13), we define $\breve{u}_1(t,x) := e^{t-t_0}u_1(t,x)$ and similarly for $\breve{\overline{v}}_{\varepsilon,n,m}$, $\breve{v}_{\varepsilon,n,m}$, $\breve{f}^i_{n,m}$, $\breve{f}$ from $\overline{v}_{\varepsilon,n,m}$, $v_{\varepsilon,n,m}$, $f^i_{n,m}$, f respectively. We also define $\breve{g} := e^{T-t_0}g$ and $\breve{g}^i_{n,m} := e^{T-t_0}g^i_{n,m}$. By direct computation, we see that $\breve{u}_1$ is a viscosity subsolution of the equation

$$\begin{cases} \partial_t u(t,\mu) + \displaystyle\int_{\mathbb{R}^d} \sup_{a\in A}\Big\{ \breve{f}(t,x,\mu,a) + b(t,x,\mu,a)\cdot\partial_\mu u(t,\mu)(x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\Big[\big(\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top + \sigma^0(t,x)[\sigma^0(t,x)]^\top\big)\nabla_x\partial_\mu u(t,\mu)(x)\Big]\Big\}\mu(dx) \\ + \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2d}} \mathrm{tr}\Big[\sigma^0(t,x)[\sigma^0(t,y)]^\top\partial^2_\mu u(t,\mu)(x,y)\Big]\,\mu^{\otimes 2}(dx,dy) - u(t,\mu) \\ = 0 \text{ for } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \displaystyle\int_{\mathbb{R}^d}\breve{g}(x,\mu)\mu(dx) \quad \text{for } \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{3.3.8}$$

in the sense of Definition 3.1.5. By Theorem 3.2.4, we obtain that $\breve{v}_{\varepsilon,n,m}$ solves the following equation in the classical sense:

$$\begin{cases} \partial_t u(t,\mu) + \displaystyle\int_{\mathbb{R}^{dn}} \sup_{\overline{a}\in A^n}\Big\{ \frac{1}{n}\sum_{i=1}^n \breve{f}^i_{n,m}(t,\overline{x},a^i) + \sum_{i=1}^n\Big\langle b^i_{n,m}(t,\overline{x},a^i), \nabla_{x^i}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big\rangle \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i=1}^n \mathrm{tr}\Big[\Big((\sigma\sigma^\top)(t,x^i,a^i) + (\sigma^0\sigma^{0;\top})(t,x^i) + \varepsilon^2 I_d\Big)\nabla^2_{x^ix^i}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big] \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i,j=1,i\neq j}^n \mathrm{tr}\Big[\sigma^0(t,x^i)\sigma^{0;\top}(t,x^j)\nabla^2_{x^ix^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big]\Big\}\displaystyle\bigotimes_{k=1}^n\mu(dx^k) - u(t,\mu) \\ = 0 \text{ for } (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n\int_{\mathbb{R}^{dn}}\breve{g}^i_{n,m}(\overline{x})\bigotimes_{k=1}^n\mu(dx^k) \text{ for } \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{3.3.9}$$

where we write $\displaystyle\bigotimes_{k=1}^n\mu(dx^k) := \mu(dx^1)\otimes\ldots\otimes\mu(dx^n)$ and $\overline{a} = (a^1,\ldots,a^n)$ for each $a^i\in A$ with $i = 1,2,\ldots,n$.

Let $l_0 := (u_1 - v_0)(t_0,\mu_0) > 0$. Referring to (3.1.1) for the definition of $M_2(\mu_0)$, we choose a sufficiently small $\delta > 0$, depending on $M_2(\mu_0)$ and l_0 only, such that $u_1(t_0,\mu_0) - v_0(t_0,\mu_0) -$

$\delta M_2(\mu_0) \geq l_0/2$ and thus $u_1(t_0,\mu_0) - v_{\varepsilon,n,m}(t_0,\mu_0) - \delta M_2(\mu_0) \geq l_0/3$, for small enough $\varepsilon > 0$ and large enough n, $m \in \mathbb{N}$ depending on l_0, by Lemmas 3.2.1 and 3.2.6. Thus, it holds that

$$\sup_{(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)} \breve{u}_1(t,\mu) - \breve{v}_{\varepsilon,n,m}(t,\mu) - \delta M_2(\mu) \geq l_0/3, \tag{3.3.10}$$

for small enough $\varepsilon > 0$ and large enough n, $m \in \mathbb{N}$ depending on l_0. Define the set

$$U^2_{\delta,\varepsilon,n,m} := \Big\{(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d) : \breve{u}_1(t,\mu) - \breve{v}_{\varepsilon,n,m}(t,\mu) - \delta M_2(\mu) \geq l_0/3\Big\},$$

which is non-empty as $u_1(t_0,\mu_0) - v_{\varepsilon,n,m}(t_0,\mu_0) - \delta M_2(\mu_0) \geq l_0/3$. It is obvious that

$$\sup_{(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)} \breve{u}_1(t,\mu) - \breve{v}_{\varepsilon,n,m}(t,\mu) - \delta M_2(\mu) = \sup_{(t,\mu)\in U^2_{\delta,\varepsilon,n,m}} \breve{u}_1(t,\mu) - \breve{v}_{\epsilon,n,m}(t,\mu) - \delta M_2(\mu).$$

Since $\breve{u}_1$ and $\breve{v}_{\varepsilon,n,m}$ are bounded independent of ε, n, m by Theorem 3.2.4, it yields that for any $\mu \in U^2_{\delta,\varepsilon,n,m}$,

$$\delta M_2(\mu) \leq l_0/3 + e^T\|u_1\|_\infty + e^T\|v_{\varepsilon,n,m}\|_\infty \leq l_0/3 + e^T(\|u_1\|_\infty + \ell_2) < +\infty, \tag{3.3.11}$$

where ℓ_2 is the constant given in Theorem 3.2.4. Then we see that

$$U^2_{\delta,\varepsilon,n,m} \subset \Big\{(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d) : M_2(\mu) \leq \frac{1}{\delta}\big[l_0/3 + e^T(\|u_1\|_\infty + \ell_2)\big]\Big\} =: U^2_\delta.$$

The set U^2_δ is compact in $\big([0,T],|\cdot|\big)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$ due to Lemma 3.1.1. Note that $\breve{u}_1(t,\mu)$ is $\big([0,T],|\cdot|\big)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$-continuous by our definition of viscosity solution, $\breve{v}_{\varepsilon,n,m}(t,\mu)$ is $\big([0,T],|\cdot|\big)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$-continuous by Lemma 3.3.1, and $-\delta M_2(\mu)$ is $\mathcal{W}_1$-upper semicontinuous from Remark 3.1.1, therefore $\breve{u}_1 - \breve{v}_{\varepsilon,n,m} - \delta M_2$ is upper semicontinuous in $\big([0,T],|\cdot|\big)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$. It implies that the set $U^2_{\delta,\varepsilon,n,m}$ is a closed subset of U^2_δ under $\big([0,T],|\cdot|\big)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$. Therefore, $U^2_{\delta,\varepsilon,n,m}$ is compact in $\big([0,T],|\cdot|\big)\times(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$. The same arguments imply that $U^2_{\delta,\varepsilon,n,m}$ is compact in $\big([0,T],|\cdot|\big)\times(\mathcal{P}_1(\mathbb{R}^d),\mathcal{W}_1)$ as well. Let $\{(t_k,\mu_k)\}_{k\in\mathbb{N}}$ be a sequence in $U^2_{\delta,\varepsilon,n,m}$ such that

$$\breve{u}_1(t_k,\mu_k) - \breve{v}_{\varepsilon,n,m}(t_k,\mu_k) - \delta M_2(\mu_k) > \sup_{(t,\mu)\in U^2_{\delta,\epsilon,n,m}} \breve{u}_1(t,\mu) - \breve{v}_{\varepsilon,n,m}(t,\mu) - \delta M_2(\mu) - \frac{1}{k}. \tag{3.3.12}$$

By the compactness of $U^2_{\delta,\varepsilon,n,m}$ and the upper semicontinuity of $\breve{u}_1 - \breve{v}_{\varepsilon,n,m} - \delta M_2$, there exists $(\widetilde{t},\widetilde{\mu}) \in [0,T] \times U^2_{\delta,\varepsilon,n,m}$ such that $(t_k,\mu_k) \to (\widetilde{t},\widetilde{\mu})$ in $\big([0,T],|\cdot|\big) \times (\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1)$. From (3.3.12), we conclude that

$$\begin{aligned}\breve{u}_1(\widetilde{t},\widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu}) - \delta M_2(\widetilde{\mu}) &\geq \limsup_k \Big[\breve{u}_1(t_k,\mu_k) - \breve{v}_{\varepsilon,n,m}(t_k,\mu_k) - \delta M_2(\mu_k)\Big] \\ &\geq \sup_{(t,\mu)\in U^2_{\delta,\varepsilon,n,m}} \breve{u}_1(t,\mu) - \breve{v}_{\varepsilon,n,m}(t,\mu) - \delta M_2(\mu),\end{aligned}$$

and the maximum is attained at $(\widetilde{t},\widetilde{\mu})$. We note that this maximum point depends on δ,ε,n,m.

Step 1B. Proof of $\widetilde{t} < T$: In this step, we aim to prove that $\widetilde{t} < T$. Suppose, on the contrary, that $\widetilde{t} = T$. The definition of the point $(\widetilde{t},\widetilde{\mu}) = (T,\widetilde{\mu})$ implies that $u_1(t_0,\mu_0) - v_{\varepsilon,n,m}(t_0,\mu_0) - \delta M_2(\mu_0) \leq \breve{u}_1(T,\widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(T,\widetilde{\mu}) - \delta M_2(\widetilde{\mu}) \leq \breve{u}_1(T,\widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(T,\widetilde{\mu})$. Thus, for $\widehat{\mu}^{n,\overline{x}} := \dfrac{1}{n}\sum_{j=1}^n \delta_{x^j}$, we can use the terminal conditions in equations (3.3.8) and (3.3.9) to obtain that

$$\begin{aligned}&u_1(t_0,\mu_0) - v_{\varepsilon,n,m}(t_0,\mu_0) \\ &\leq \frac{e^{T-t_0}}{n}\sum_{i=1}^n \left[\int_{\mathbb{R}^{dn}} \big(g(x^i,\widetilde{\mu}) - g^i_{n,m}(x^1,\dots,x^n)\big) \bigotimes_{k=1}^n \widetilde{\mu}(dx^k)\right] + \delta M_2(\mu_0) \\ &= \frac{e^{T-t_0}}{n}\sum_{i=1}^n \left[\int_{\mathbb{R}^{dn}} \big(g(x^i,\widetilde{\mu}) - g(x^i,\widehat{\mu}^{n,\overline{x}})\big) \bigotimes_{k=1}^n \widetilde{\mu}(dx^k)\right] \\ &\quad + \frac{e^{T-t_0}}{n}\sum_{i=1}^n \left[\int_{\mathbb{R}^{dn}} \big(g(x^i,\widehat{\mu}^{n,\overline{x}}) - g^i_{n,m}(x^1,\dots,x^n)\big) \bigotimes_{k=1}^n \widetilde{\mu}(dx^k)\right] + \delta M_2(\mu_0).\end{aligned}$$

Using the Lipschitz property of g in Assumption 3.1.2 and (2) of Lemma 3.2.2, we further have

$$\begin{aligned}&u_1(t_0,\mu_0) - v_{\varepsilon,n,m}(t_0,\mu_0) \\ &\leq Ke^{T-t_0}\left[\int_{\mathbb{R}^{dn}} \mathcal{W}_1(\widetilde{\mu},\widehat{\mu}^{n,\overline{x}}) \bigotimes_{k=1}^n \widetilde{\mu}(dx^k)\right] + \frac{2Ke^{T-t_0}}{n}\left[m^{dn}\int_{\mathbb{R}^{dn}} \left(\sum_{i=1}^n |y^i|\right)\prod_{j=1}^n \Phi(my^j)dy^j\right] \\ &\quad + \delta M_2(\mu_0). \qquad (3.3.13)\end{aligned}$$

From (Fournier & Guillin, 2015, Theorem 1), there is a constant $C_d > 0$ depending on d only and a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}^{dn}} \mathcal{W}_1(\widetilde{\mu}, \widehat{\mu}^{n,\overline{x}}) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \leq C_d h_n \left[\int_{\mathbb{R}^d} |x|^{q_0} \widetilde{\mu}(dx)\right]^{1/q_0}. \tag{3.3.14}$$

The sequence $\{h_n\}_{n\in\mathbb{N}}$ and the number q_0 are given by

$$h_n = \begin{cases} n^{-1/2} + n^{-(q_0-1)/q_0} & \text{if } d = 1; \\ n^{-1/2}\log(1+n) + n^{-(q_0-1)/q_0} & \text{if } d = 2; \\ n^{-1/d} + n^{-(q_0-1)/q_0} & \text{if } d > 2, \end{cases} \quad \text{with} \quad q_0 = \begin{cases} 3/2 & \text{if } d = 1, 2; \\ 5/3 & \text{if } d > 2, \end{cases}$$

where h_n satisfies $\lim_{n\to\infty} h_n = 0$. Inequalities (3.3.11) and (3.3.14) further imply

$$\frac{1}{C_d h_n} \int_{\mathbb{R}^{dn}} \mathcal{W}_1(\widetilde{\mu}, \widehat{\mu}^{n,\overline{x}}) \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k) \leq \left[\int_{\mathbb{R}^d} |x|^2 \widetilde{\mu}(dx)\right]^{1/2} \leq \frac{1}{\delta^{1/2}} \left[l_0/3 + e^T(\|u_1\|_\infty + \ell_2)\right]^{1/2}. \tag{3.3.15}$$

Hence, from (3.3.13) and (3.3.15), we have,

$$\begin{aligned} &u_1(t_0, \mu_0) - v_{\varepsilon,n,m}(t_0, \mu_0) \\ &\leq \frac{KC_d h_n e^{T-t_0}}{\delta^{1/2}} \left[l_0/3 + e^T(\|u_1\|_\infty + \ell_2)\right]^{1/2} + \frac{2Ke^{T-t_0} m^{dn}}{n} \left[\int_{\mathbb{R}^{dn}} \left(\sum_{i=1}^{n} |y^i|\right) \prod_{j=1}^{n} \Phi(my^j) dy^j\right] \\ &\quad + \delta M_2(\mu_0). \end{aligned}$$

Passing $m \to \infty$ and then $n \to \infty$ subsequently, we use the fact that $h_n \to 0$ to yield that

$$u_1(t_0, \mu_0) - \lim_{n\to\infty} \lim_{m\to\infty} v_{\varepsilon,n,m}(t_0, \mu_0) \leq \delta M_2(\mu_0).$$

Finally, using Lemmas 3.2.1 and 3.2.6, we pass $\varepsilon \to 0^+$ then $\delta \to 0^+$ to conclude that $u_1(t_0, \mu_0) - v_0(t_0, \mu_0) \leq 0$, which contradicts (3.3.7) and thus $\widetilde{t} < T$.

Step 1C. Estimate of $u_1 - v_{\varepsilon,n,m}$: We assume the maximum value of $\breve{u}_1(t, \mu) - \breve{v}_{\varepsilon,n,m}(t, \mu) - \delta M_2(\mu)$ attained at $(\widetilde{t}, \widetilde{\mu})$ over $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ is $M^* \in \mathbb{R}$. As $\breve{v}_{\varepsilon,n,m}(t, \mu) + \delta M_2(\mu) \in$

$C^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$, we use the fact that $\breve{u}_1$ is the viscosity subsolution of (3.3.8) to see that

$$
\begin{aligned}
0 \le\, & \partial_t(\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu}) - (\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu}) - M^* \\
& + \int_{\mathbb{R}^d}\sup_{a\in A}\Bigg\{ \breve{f}(\widetilde{t},x,\widetilde{\mu},a) + \Big\langle b(\widetilde{t},x,\widetilde{\mu},a), \partial_\mu(\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu})(x)\Big\rangle \\
& \qquad + \frac{1}{2}\mathrm{tr}\Big\{\big[\sigma(\widetilde{t},x,a)\big[\sigma(\widetilde{t},x,a)\big]^\top + \sigma^0(\widetilde{t},x)[\sigma^0(\widetilde{t},x)]^\top\big]\nabla_x\partial_\mu(\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu})(x)\Big\}\Bigg\}\widetilde{\mu}(dx) \\
& + \frac{1}{2}\int_{\mathbb{R}^{2d}}\mathrm{tr}\Big[\sigma^0(\widetilde{t},x)[\sigma^0(\widetilde{t},y)]^\top\partial_\mu^2(\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu})(x,y)\Big]\,\widetilde{\mu}^{\otimes 2}(dx,dy).
\end{aligned}
$$

Therefore, as $\breve{u}_1(\widetilde{t},\widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu}) - \delta M_2(\widetilde{\mu}) \le M^*$ and $\breve{v}_{\varepsilon,n,m}$ solves (3.3.9), we further have

$$
\begin{aligned}
& (\breve{u}_1 - \breve{v}_{\varepsilon,n,m})(\widetilde{t},\widetilde{\mu}) \\
\le\, & \int_{\mathbb{R}^d}\sup_{a\in A}\Bigg\{ \breve{f}(\widetilde{t},x,\widetilde{\mu},a) + \Big\langle b(\widetilde{t},x,\widetilde{\mu},a), \partial_\mu(\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu})(x)\Big\rangle \\
& \qquad + \frac{1}{2}\mathrm{tr}\Big\{\big[\sigma(\widetilde{t},x,a)\big[\sigma(\widetilde{t},x,a)\big]^\top + \sigma^0(\widetilde{t},x)[\sigma^0(\widetilde{t},x)]^\top\big]\nabla_x\partial_\mu(\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu})(x)\Big\}\Bigg\}\widetilde{\mu}(dx) \\
& + \frac{1}{2}\int_{\mathbb{R}^{2d}}\mathrm{tr}\Big[\sigma^0(\widetilde{t},x)[\sigma^0(\widetilde{t},y)]^\top\partial_\mu^2(\breve{v}_{\varepsilon,n,m}+\delta M_2)(\widetilde{t},\widetilde{\mu})(x,y)\Big]\,\widetilde{\mu}^{\otimes 2}(dx,dy) \\
& - \int_{\mathbb{R}^{dn}}\sup_{\overline{a}\in A^n}\Bigg\{ \frac{1}{n}\sum_{i=1}^n \breve{f}^i_{n,m}(\widetilde{t},\overline{x},a^i) + \sum_{i=1}^n\Big\langle b^i_{n,m}(\widetilde{t},\overline{x},a^i), \nabla_{x^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\Big\rangle \\
& \qquad + \frac{1}{2}\sum_{i=1}^n \mathrm{tr}\Big[\Big((\sigma\sigma^\top)(\widetilde{t},x^i,a^i) + (\sigma^0\sigma^{0;\top})(\widetilde{t},x^i) + \varepsilon^2 I_d\Big)\nabla^2_{x^ix^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\Big] \\
& \qquad + \frac{1}{2}\sum_{i,j=1,i\neq j}^n \mathrm{tr}\Big[\sigma^0(\widetilde{t},x^i)\sigma^{0;\top}(\widetilde{t},x^j)\nabla^2_{x^ix^j}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\Big]\Bigg\}\bigotimes_{k=1}^n\widetilde{\mu}(dx^k). \qquad (3.3.16)
\end{aligned}
$$

We divide the estimate into three parts: the part involving M_2, the part involving $\breve{\overline{v}}_{\varepsilon,n,m}$ and the term $(\breve{u}_1 - \breve{v}_{\varepsilon,n,m})(\widetilde{t},\widetilde{\mu})$. First, by direct computation, we obtain $\partial_\mu M_2(\mu)(x) = 2x$, $\partial^2_\mu M_2(\mu)(x,y) = 0$ and $\nabla_x\partial_\mu M_2(\mu)(x) = 2I_d$. Assumption 3.1.2 and (3.3.15) tell us that

$$
\begin{aligned}
& \int_{\mathbb{R}^d}\sup_{a\in A}\Big\{\Big\langle b(\widetilde{t},x,\widetilde{\mu},a), \partial_\mu M_2(\widetilde{\mu})(x)\Big\rangle + \frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},x,a) + \sigma^0(\widetilde{t},x)[\sigma^0(\widetilde{t},x)]^\top\big]\nabla_x\partial_\mu M_2(\widetilde{\mu})(x)\Big\}\Big\}\widetilde{\mu}(dx) \\
& + \frac{1}{2}\int_{\mathbb{R}^{2d}}\mathrm{tr}\Big[\sigma^0(\widetilde{t},x)[\sigma^0(\widetilde{t},y)]^\top\partial^2_\mu M_2(\widetilde{\mu})(x,y)\Big]\,\widetilde{\mu}^{\otimes 2}(dx,dy)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \sup_{a\in A} \Big\{ \Big\langle b(\widetilde{t},x,\widetilde{\mu},a), 2x\Big\rangle + \mathrm{tr}\Big[(\sigma\sigma^\top)(\widetilde{t},x,a) + \sigma^0(t,x)[\sigma^0(t,x)]^\top\Big]\Big\}\widetilde{\mu}(dx)\\
&\le \int_{\mathbb{R}^d} \Big[2K(1+|x|^\rho)|x| + 4K^2(1+|x|^{2\rho})\Big]\widetilde{\mu}(dx)\\
&\le C_K \int_{\mathbb{R}^d} (1+|x|^{\rho+1})\widetilde{\mu}(dx)\\
&\le C_K + C_K\big[M_2(\widetilde{\mu})\big]^{(1+\rho)/2}\\
&\le C_K + \frac{C_K}{\delta^{(1+\rho)/2}}\Big[l_0/3 + e^T(\|u_1\|_\infty + \ell_2)\Big]^{(1+\rho)/2},
\end{aligned}
\tag{3.3.17}
$$

for some fixed constant C_K depending only on K. Second, we recall the representation of $\breve{v}_{\varepsilon,n,m}(t,\mu) = e^{t-t_0}v_{\varepsilon,n,m}(t,\mu)$ from (3.2.13), as well as (1) and (2) of Theorem 3.2.4. Using inequalities (3.2.9) and (3.2.10), we can directly compute that

$$
\partial_\mu \breve{v}_{\varepsilon,n,m}(t,\mu)(x) = \sum_{i=1}^n \int_{\mathbb{R}^{d(n-1)}} \nabla_{x^i}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^i=x} \bigotimes_{k=1,k\neq i}^n \mu(dx^k); \tag{3.3.18}
$$

and use (2) in Theorem 3.2.4 to yield that

$$
\partial_\mu^2 \breve{v}_{\varepsilon,n,m}(t,\mu)(x,y) = \sum_{i=1}^n \sum_{j=1,j\neq i}^n \int_{\mathbb{R}^{d(n-2)}} \nabla^2_{x^ix^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^i=x,x^j=y} \bigotimes_{k=1,k\neq i,j}^n \mu(dx^k).
$$

Hence, we estimate the term

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \sup_{a\in A}\Big\{\breve{f}(\widetilde{t},y,\widetilde{\mu},a) + \frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},y,a) + (\sigma^0\sigma^{0;\top})(\widetilde{t},y)\big]\nabla_x\partial_\mu\breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu})(y)\Big\}\\
&\qquad + \Big\langle b(\widetilde{t},y,\widetilde{\mu},a), \partial_\mu\breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu})(y)\Big\rangle\Big\}\widetilde{\mu}(dy)\\
&+ \frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{tr}\Big[\sigma^0(\widetilde{t},z)[\sigma^0(\widetilde{t},y)]^\top \partial_\mu^2\breve{v}_{\varepsilon,n,m}(\widetilde{t},\widetilde{\mu})(z,y)\Big]\,\widetilde{\mu}^{\otimes 2}(dz,dy)\\
&= \int_{\mathbb{R}^d}\sup_{a\in A}\Bigg\{\sum_{j=1}^n \int_{\mathbb{R}^{d(n-1)}} \frac{1}{n}\breve{f}(\widetilde{t},y,\widetilde{\mu},a) \bigotimes_{k=1,k\neq j}^n \widetilde{\mu}(dx^k)\\
&\qquad + \frac{1}{2}\mathrm{tr}\Big\{\big[(\sigma\sigma^\top)(\widetilde{t},y,a) + (\sigma^0\sigma^{0;\top})(\widetilde{t},y)\big]\sum_{j=1}^n \int_{\mathbb{R}^{d(n-1)}} \nabla^2_{x^jx^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^j=y}\bigotimes_{k=1,k\neq j}^n \widetilde{\mu}(dx^k)\Big\}\\
&\qquad + \Big\langle b(\widetilde{t},y,\widetilde{\mu},a), \sum_{j=1}^n \int_{\mathbb{R}^{d(n-1)}} \nabla_{x^j}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^j=y}\bigotimes_{k=1,k\neq j}^n \widetilde{\mu}(dx^k)\Big\rangle\Bigg\}\widetilde{\mu}(dy)\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{tr}\Bigg\{\big[\sigma^0(\widetilde{t},z)\sigma^{0;\top}(\widetilde{t},y)\big]\sum_{j=1}^n\sum_{l=1,l\neq j}^n \int_{\mathbb{R}^{d(n-2)}} \nabla^2_{x^jx^l}\breve{\overline{v}}_{\varepsilon,n,m}(t,\overline{x})\Big|_{x^j=z,x^l=y}\bigotimes_{k=1,k\neq j,l}^n \widetilde{\mu}(dx^k)\Bigg\}\widetilde{\mu}^{\otimes 2}(dz,dy)
\end{aligned}
$$

$$\leq \sum_{j=1}^{n} \int_{\mathbb{R}^{dn}} \sup_{a^j \in A} \left\{ \frac{1}{n} \breve{f}(\widetilde{t}, x^j, \widetilde{\mu}, a^j) + \frac{1}{2} \mathrm{tr} \Big\{ \big[(\sigma\sigma^\top)(\widetilde{t}, x^j, a^j) + (\sigma^0 \sigma^{0;\top})(\widetilde{t}, x^j) \big] \nabla^2_{x^j x^j} \breve{\overline{v}}_{\varepsilon,n,m}(t, \overline{x}) \Big\} \right.$$
$$\left. + \Big\langle b(\widetilde{t}, x^j, \widetilde{\mu}, a^j), \nabla_{x^j} \breve{\overline{v}}_{\varepsilon,n,m}(t, \overline{x}) \Big\rangle \right\} \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k)$$
$$+ \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1, l\neq j}^{n} \int_{\mathbb{R}^{dn}} \mathrm{tr} \left\{ \big[\sigma^0(\widetilde{t}, x^j) \sigma^{0;\top}(\widetilde{t}, x^l) \big] \nabla^2_{x^j x^l} \breve{\overline{v}}_{\varepsilon,n,m}(t, \overline{x}) \right\} \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k). \tag{3.3.19}$$

Third, as $\breve{u}_1 - \breve{v}_{\varepsilon,n,m} - \delta M_2$ attains its maximum over the space $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ at $(\widetilde{t}, \widetilde{\mu})$, it holds that

$$(u_1 - v_{\varepsilon,n,m} - \delta M_2)(t_0, \mu_0) = (\breve{u}_1 - \breve{v}_{\varepsilon,n,m} - \delta M_2)(t_0, \mu_0) \leq \breve{u}_1(\widetilde{t}, \widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(\widetilde{t}, \widetilde{\mu}) - \delta M_2(\widetilde{\mu})$$
$$\leq \breve{u}_1(\widetilde{t}, \widetilde{\mu}) - \breve{v}_{\varepsilon,n,m}(\widetilde{t}, \widetilde{\mu}). \tag{3.3.20}$$

After substituting (3.3.17), (3.3.19) and (3.3.20) into (3.3.16), we make use of (3.2.9) and (3.2.10) to deduce that

$$(u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) - \delta M_2(\mu_0)$$
$$\leq \delta \left[\frac{C_K}{\delta^{(1+\rho)/2}} \big[l_0/3 + e^T (\|u_1\|_\infty + \ell_2) \big]^{(1+\rho)/2} + C_K \right]$$
$$+ \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \sup_{a^i \in A} \left\{ \frac{1}{n} \breve{f}(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - \frac{1}{n} \breve{f}^i_{n,m}(\widetilde{t}, \overline{x}, a^i) - \frac{\varepsilon^2}{2} \mathrm{tr} \nabla^2_{x^i x^i} \breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t}, \overline{x}) \right.$$
$$\left. + \Big\langle b(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - b^i_{n,m}(\widetilde{t}, \overline{x}, a^i), \nabla_{x^i} \breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t}, \overline{x}) \Big\rangle \right\} \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k)$$
$$\leq \delta \left[\frac{C_K}{\delta^{(1+\rho)/2}} \big[l_0/3 + e^T (\|u_1\|_\infty + \ell_2) \big]^{(1+\rho)/2} + C_K \right]$$
$$+ \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \sup_{a^i \in A} \left\{ \frac{e^{\widetilde{t}-t_0}}{n} \big| f(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - f^i_{n,m}(\widetilde{t}, \overline{x}, a^i) \big| - \frac{\varepsilon^2}{2} \mathrm{tr} \nabla^2_{x^i x^i} \breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t}, \overline{x}) \right.$$
$$\left. + \frac{C_4 e^{\widetilde{t}-t_0}}{n} \big| b(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - b^i_{n,m}(\widetilde{t}, \overline{x}, a^i) \big| \right\} \bigotimes_{k=1}^{n} \widetilde{\mu}(dx^k). \tag{3.3.21}$$

We use Assumption 3.1.2 and (2) of Lemma 3.2.2 to estimate the following term:

$$\big| f(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - f^i_{n,m}(\widetilde{t}, \overline{x}, a^i) \big| + C_4 \big| b(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - b^i_{n,m}(\widetilde{t}, \overline{x}, a^i) \big|$$
$$\leq \big| f(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - f(\widetilde{t}, x^i, \widehat{\mu}^{n,\overline{x}}, a^i) \big| + \big| f(\widetilde{t}, x^i, \widehat{\mu}^{n,\overline{x}}, a^i) - f^i_{n,m}(\widetilde{t}, \overline{x}, a^i) \big|$$
$$+ C_4 \big| b(\widetilde{t}, x^i, \widetilde{\mu}, a^i) - b(\widetilde{t}, x^i, \widehat{\mu}^{n,\overline{x}}, a^i) \big| + C_4 \big| b(\widetilde{t}, x^i, \widehat{\mu}^{n,\overline{x}}, a^i) - b^i_{n,m}(\widetilde{t}, \overline{x}, a^i) \big|$$

$$\leq K(1+C_4)\mathcal{W}_1(\widetilde{\mu},\widehat{\mu}^{n,\overline{x}}) + K(1+C_4)m\int_{\mathbb{R}}\left|\widetilde{t}-\left[T\wedge(\widetilde{t}-s)^+\right]\right|^{\beta}\phi(ms)ds$$
$$+ K(1+C_4)m^{dn}\int_{\mathbb{R}^{dn}}\left(|y^i|+\frac{1}{n}\sum_{j=1}^{n}|y^j|\right)\prod_{k=1}^{n}\Phi(my^k)dy^k. \tag{3.3.22}$$

Putting (3.3.22) and (3.3.15) into (3.3.21), we see that

$$\begin{aligned}
&(u_1 - v_{\varepsilon,n,m} - \delta M_2)(t_0,\mu_0)\\
&\leq \delta\left[\frac{C_K}{\delta^{(1+\rho)/2}}\left[l_0/3 + e^T(\|u_1\|_\infty+\ell_2)\right]^{(1+\rho)/2} + C_K\right]\\
&\quad - \int_{\mathbb{R}^{dn}}\frac{\varepsilon^2}{2}\sum_{i=1}^{n}\mathrm{tr}\nabla^2_{x^ix^i}\breve{\overline{v}}_{\varepsilon,n,m}(\widetilde{t},\overline{x})\bigotimes_{k=1}^{n}\widetilde{\mu}(dx^k) + \frac{e^{T-t_0}C_dh_nK(1+C_4)}{\delta^{1/2}}\left[l_0/3+e^T(\|u_1\|_\infty+\ell_2)\right]^{1/2}\\
&\quad + e^{T-t_0}K(1+C_4)m\int_{\mathbb{R}}\left|\widetilde{t}-\left[T\wedge(\widetilde{t}-s)^+\right]\right|^{\beta}\phi(ms)ds\\
&\quad + \frac{2e^{T-t_0}K(1+C_4)m^{dn}}{n}\int_{\mathbb{R}^{dn}}\left(\sum_{j=1}^{n}|y^j|\right)\prod_{k=1}^{n}\Phi(my^k)dy^k.
\end{aligned}$$

Using Lemmas 3.2.3 and 3.2.5, we first take $\varepsilon\to 0^+$ and then $m\to\infty$ to obtain that

$$\begin{aligned}
&(u_1 - \lim_{m\to\infty} v_{0,n,m} - \delta M_2)(t_0,\mu_0)\\
&\leq \delta\left[\frac{C_K}{\delta^{(1+\rho)/2}}\left[l_0/3 + e^T(\|u_1\|_\infty+\ell_2)\right]^{(1+\rho)/2} + C_K\right]\\
&\quad + \frac{e^{T-t_0}C_dh_nK(1+C_4)}{\delta^{1/2}}\left[l_0/3+e^T(\|u_1\|_\infty+\ell_2)\right]^{1/2}.
\end{aligned}$$

By (3.3.14), (3.3.15), Lemmas 3.2.1 and 3.2.6, we take $n\to\infty$ and then $\delta\to 0^+$ to obtain that

$$(u_1 - v_0)(t_0,\mu_0) = \left(u_1 - \lim_{n\to\infty}\lim_{m\to\infty} v_{0,n,m}\right)(t_0,\mu_0) \leq 0,$$

which contradicts (3.3.7).

Part 2. Proof of $u_2 \geq v_0$: Following the arguments of the first part of Step II of the proof of (Cosso *et al.*, 2024, Theorem 5.1), we can assume without loss of generality that $u_2(s,\cdot)$ is $\mathcal{W}_1$-Lipschitz continuous for every $s\in[0,T]$; and showing $u_2\geq v_0$ is equivalent to showing

$$u_2(t,\mu) \geq v^s(t,\nu) := \mathbb{E}\left[\int_t^s f\left(r, X_r^{t,\xi,\mathfrak{a}}, \mathbb{P}^{W^0}_{X_r^{t,\xi,\mathfrak{a}}}, \mathfrak{a}\right)dr\right] + \mathbb{E}u_2\left(s, \mathbb{P}^{W^0}_{X_s^{t,\xi,\mathfrak{a}}}\right), \tag{3.3.23}$$

for every $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, $s \in (t,T]$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ with $\mathcal{L}(\xi) = \mu$, and $\mathfrak{a} \in \mathcal{M}_t$, where $\mathcal{M}_t$ denotes the set of $\mathcal{F}_t^t$-measurable random variables $\alpha : \Omega \to A$ and $\nu := \mathcal{L}(\xi, \mathfrak{a})$. Suppose, for contradiction, that (3.3.23) does not hold. Then, there exist $t_0 \in [0,T)$, $s_0 \in (t_0, T]$, $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $\nu_0 \in \mathcal{P}_2(\mathbb{R}^d \times A)$, with μ_0 being the marginal of ν_0 on $\mathbb{R}^d$, such that

$$v^{s_0}(t_0, \nu_0) > u_2(t_0, \mu_0). \tag{3.3.24}$$

Following the approach outlined at the beginning of Part 1 of this proof, we assume that there exists $q > 2$ such that $\nu_0 \in \mathcal{P}_q(\mathbb{R}^d \times A)$. The function $\breve{u}_2(t,\mu) := e^{t-t_0} u_2(t,\mu)$ is a viscosity supersolution of the following equation:

$$\begin{cases} \partial_t u(t,\mu) + \displaystyle\int_{\mathbb{R}^d} \sup_{a \in A} \Big\{ \breve{f}(t,x,\mu,a) + b(t,x,\mu,a) \cdot \partial_\mu u(t,\mu)(x) \\ \qquad\qquad + \frac{1}{2} \mathrm{tr}\Big((\sigma(t,x,a)\big[\sigma(t,x,a)\big]^\top + \sigma^0(t,x)[\sigma^0(t,x)]^\top) \nabla_x \partial_\mu u(t,\mu)(x) \Big) \Big\} \mu(dx) \\ + \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^{2d}} \mathrm{tr}\Big[\sigma^0(t,x)[\sigma^0(t,y)]^\top \partial_\mu^2 u(t,\mu)(x,y) \Big] \, \mu^{\otimes 2}(dx,dy) - u(t,\mu) = 0 \text{ for } (t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d); \\ u(T,\mu) = \displaystyle\int_{\mathbb{R}^d} \breve{g}(x,\mu)\mu(dx) \quad \text{for } \mu \in \mathcal{P}_2(\mathbb{R}^d), \end{cases}$$

where $\breve{f}(t,x,\mu,a) := e^{t-t_0} f(t,x,\mu,a)$ and $\breve{g}(x,\mu) := e^{T-t_0} g(x,\mu)$. That is, for any $s_1 \in (0,T]$ and $\varphi \in C^{1,2}([0,s_1] \times \mathcal{P}_2(\mathbb{R}^d \times A))$ such that $\breve{u}_2 - \varphi$ attains a minimum with a value of 0 at $(t^*, \nu^*) \in [0,s_1) \times \mathcal{P}_2(\mathbb{R}^d \times A)$, then the following inequality holds:

$$\begin{aligned} 0 \geq \partial_t \varphi(t^*,\nu^*) + \int_{\mathbb{R}^d \times A} \Big\{ & \breve{f}(t^*,x,\mu^*,a) + b(t^*,x,\mu^*,a) \cdot \partial_\mu \varphi(t^*,\nu^*)(x,a) \\ & + \frac{1}{2} \mathrm{tr}\Big[(\sigma(t^*,x,a)\big[\sigma(t^*,x,a)\big]^\top + \sigma^0(t^*,x)[\sigma^0(t^*,x)]^\top) \nabla_x \partial_\mu \varphi(t^*,\nu^*)(x,a) \Big] \Big\} \nu^*(dx,da) \\ & + \frac{1}{2} \int_{\mathbb{R}^d \times A \times \mathbb{R}^d \times A} \mathrm{tr}\Big[\sigma^0(t^*,x)[\sigma^0(t^*,y)]^\top \partial_\mu^2 \varphi(t^*,\nu^*)(x,a,y,\alpha) \Big] \, (\nu^*)^{\otimes 2}(dx,da,dy,d\alpha) - \varphi(t^*,\nu^*), \end{aligned} \tag{3.3.25}$$

where μ^* is the marginal of ν^* on $\mathbb{R}^d$, and the operators ∂_μ, ∂_μ^2 are defined as in Definition 3.1.5, through the projection.

We regularize the coefficients with respect to the control variable. Let $\Psi : \mathbb{R}^d \to \mathbb{R}^+$ be a compactly supported smooth function satisfying $\int_{\mathbb{R}^d} \Psi(y)dy = 1$. We extend b and f to the space $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ by setting $b(t,x,\mu,a) = 0$ and $f(t,x,\mu,a) = 0$ when $a \in \mathbb{R}^d$ is not in A. For simplicity, we continue to denote these extensions by b and f. We further define the functions $\widetilde{b}^i_{n,m}$ and $\widetilde{f}^i_{n,m}$ by

$$\widetilde{b}^i_{n,m}(t,\overline{x},a) := m^d \int_{\mathbb{R}^d} b^i_{n,m}(t,\overline{x},a-a')\Psi(ma')\,da',$$

$$\widetilde{f}^i_{n,m}(t,\overline{x},a) := m^d \int_{\mathbb{R}^d} f^i_{n,m}(t,\overline{x},a-a')\Psi(ma')\,da',$$

for any $n,m \in \mathbb{N}$, $i = 1,2,\ldots,n$, $\overline{x} = (x^1,x^2,\ldots,x^n) \in \mathbb{R}^{dn}$ and $(t,a) \in [0,T] \times A$. Here $b^i_{n,m}$ and $f^i_{n,m}$ are as defined in (3.2.4) and (3.2.5). Recalling the compactly supported smooth function Φ defined in Section 3.2.2, we also define

$$u_{n,m}(t,\overline{x}) := m^{dn} \int_{\mathbb{R}^{dn}} u_2\left(t, \frac{1}{n}\sum_{j=1}^n \delta_{x^j - y^j}\right) \prod_{j=1}^n \Phi(my^j)\,dy^j.$$

We now introduce

$$v^{s_0}_{n,m}(t,\nu) := \frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big[\int_t^{s_0} \widetilde{f}^i_{n,m}\left(r, \overline{\widetilde{X}}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_r, \ldots, \overline{\widetilde{X}}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_r, \mathfrak{a}^i_0\right) dr + u_{n,m}\left(s_0, \overline{\widetilde{X}}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_{s_0}, \ldots, \overline{\widetilde{X}}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_{s_0}\right)\Big], \tag{3.3.26}$$

for any $t \in [0,s_0]$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d \times A)$, where $\overline{\xi} = (\xi^1,\xi^2,\ldots,\xi^n) \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{dn})$, $\overline{\mathfrak{a}}_0 = (\mathfrak{a}^1_0,\mathfrak{a}^2_0,\ldots,\mathfrak{a}^n_0) \in (\mathcal{M}_t)^n$ such that $\mathcal{L}(\overline{\xi},\overline{\mathfrak{a}}_0) = \nu \otimes \cdots \otimes \nu$ and $\overline{\widetilde{X}}^{m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_s = \left(\widetilde{X}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_s, \ldots, \widetilde{X}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_s\right)$ is the solution to (3.2.7) on $[t,s_0]$ with $\overline{\alpha} = \overline{\mathfrak{a}}_0$, $\varepsilon = 0$ and b replaced by $\widetilde{b}^i_{n,m}$. For every $n,m \in \mathbb{N}$, $(t,\nu) \in [0,s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$, we define $\breve{v}^{s_0}_{n,m} := e^{t-t_0}v^{s_0}_{n,m}$ and also similarly define $\breve{\widetilde{f}}^i_{n,m}$ and $\breve{u}_{n,m}$ from $\widetilde{f}^i_{n,m}$ and $u_{n,m}$, respectively. Let

$$\overline{v}^{s_0}_{n,m}(t,\overline{x},\overline{a}) := \widetilde{v}^{s_0}_{n,m}(t, \delta_{(x^1,a^1)} \otimes \cdots \otimes \delta_{(x^n,a^n)})$$

for any $\overline{x} = (x^1,\ldots,x^n) \in \mathbb{R}^{dn}$ and $\overline{a} = (a^1,\ldots,a^n) \in A^n$, where

$$\widetilde{v}^{s_0}_{n,m}(t,\overline{\nu}) := \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\int_t^{s_0} \widetilde{f}^i_{n,m}\left(s, \overline{\widetilde{X}}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_s, \ldots, \overline{\widetilde{X}}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_s, \mathfrak{a}^i_0\right) ds\right.$$

$$+ u_{n,m}\left(\overline{\widetilde{X}}^{1,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_{s_0}, \ldots, \overline{\widetilde{X}}^{n,m,t,\overline{\xi},\overline{\mathfrak{a}}_0}_{s_0}\right)\Bigg], \tag{3.3.27}$$

for any $t \in [0, s_0]$ and $\overline{\nu} \in \mathcal{P}_2\left(\mathbb{R}^{dn} \times A^n\right)$, where $\overline{\xi}$ and $\overline{\mathfrak{a}}_0$ satisfy $\mathcal{L}(\overline{\xi}, \overline{\mathfrak{a}}_0) = \overline{\nu}$. Moreover, by (Cosso *et al.*, 2024, Theorem A.8), we deduce that $\breve{v}^{s_0}_{n,m}$ can be represented by

$$\breve{v}^{s_0}_{n,m}(t, \nu) = e^{t-t_0} \int_{\mathbb{R}^{dn} \times A^n} \overline{v}^{s_0}_{n,m}(t, \overline{x}, \overline{a}) \nu(dx^1, da^1) \otimes \cdots \otimes \nu(dx^n, da^n),$$

and it could be shown by following the proofs of Lemma 3.2.3, Theorem 3.2.4 and (Cosso *et al.*, 2024, Theorem A.8) that

(1). $\overline{v}^{s_0}_{n,m} \in C^{1,2}\left([0, s_0] \times (\mathbb{R}^{dn} \times A^n)\right)$ and $v^{s_0}_{n,m} \in C^{1,2}\left([0, s_0] \times \mathcal{P}_2\left(\mathbb{R}^d \times A\right)\right)$;

(2). for any $i = 1, \ldots, n$ and $(t, \overline{x}, \overline{a}) \in [0, s_0] \times \mathbb{R}^{dn} \times A^n$, it holds that

$$\left|\nabla_{x^i} \overline{v}^{s_0}_{n,m}(t, \overline{x}, \overline{a})\right| \leq \frac{C_K}{n},$$

where the constant $C_K \geq 0$ depends on d, K, T, but independent of n, m;

(3). if $t \in [0, s_0]$ and $\nu \in \mathcal{P}_q\left(\mathbb{R}^d \times A\right)$ for some $q > 2$, then

$$\lim_{n \to +\infty} \lim_{m \to +\infty} v^{s_0}_{n,m}(t, \nu) = v^{s_0}(t, \nu); \tag{3.3.28}$$

(4). the function $v^{s_0}_{n,m}(t, \nu)$ solves the following equation classically:

$$\begin{cases} \partial_t u(t, \nu) + \overline{\mathbb{E}}\Bigg[\displaystyle\sum_{i=1}^{n} \Big\{ \frac{1}{n} \widetilde{f}^i_{n,m}(t, \overline{\xi}, \mathfrak{a}^i_0) + \Big\langle \widetilde{b}^i_{n,m}(t, \overline{\xi}, \mathfrak{a}^i_0), \nabla_{x^i} \overline{v}^{s_0}_{n,m}(t, \overline{\xi}, \overline{\mathfrak{a}}_0) \Big\rangle \\ \qquad\qquad + \frac{1}{2} \mathrm{tr}\left(\left[(\sigma\sigma^\top)(t, \xi^i, \mathfrak{a}^i_0) + (\sigma^0 \sigma^{0;\top})(t, \xi^i) \right] \nabla^2_{x^i x^i} \overline{v}^{s_0}_{n,m}(t, \overline{\xi}, \overline{\mathfrak{a}}_0) \right) \Big\} \\ \qquad + \frac{1}{2} \displaystyle\sum_{i,j=1, i \neq j}^{n} \mathrm{tr}\Big[\sigma^0(t, \xi^i) \sigma^{0;\top}(t, \xi^j) \nabla^2_{x^i x^j} \overline{v}^{s_0}_{n,m}(t, \overline{\xi}, \overline{\mathfrak{a}}_0) \Big] \Bigg] = 0; \\ u(s_0, \nu) = \overline{\mathbb{E}}[u_{n,m}(s_0, \overline{\xi})], \end{cases}$$

for any $t \in [0, s_0)$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d \times A)$, where $\overline{\xi} = (\xi^1, \xi^2, \ldots, \xi^n) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{dn})$, $\overline{\mathfrak{a}}_0 = (\mathfrak{a}^1_0, \mathfrak{a}^2_0, \ldots, \mathfrak{a}^n_0) \in (\mathcal{M}_t)^n$ such that $\mathcal{L}(\overline{\xi}, \overline{\mathfrak{a}}_0) = \nu \otimes \ldots \otimes \nu$.

We now return to the hypothesis (3.3.24), with μ_0 and ν_0 mentioned therein. Let $l_0 := v^{s_0}(t_0,\nu_0) - u_2(t_0,\mu_0) > 0$. For small enough $\delta > 0$ depending on $M_2(\nu_0)$ and l_0 only, we have $v^{s_0}(t_0,\nu_0) - u_2(t_0,\mu_0) - \delta M_2(\nu_0) \geq l_0/2$ and thus $v^{s_0}_{n,m}(t_0,\nu_0) - u_2(t_0,\mu_0) - \delta M_2(\nu_0) \geq l_0/3$, where $v^{s_0}_{n,m}$ is defined in (3.3.26), and this holds for large enough n, $m \in \mathbb{N}$ depending on δ, $M_2(\nu_0)$ and l_0 only, by (3.3.28). Thus, it holds that

$$\sup_{(t,\nu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d\times A)} \breve{v}^{s_0}_{n,m}(t,\nu) - \breve{u}_2(t,\mu) - \delta M_2(\nu) \geq l_0/3, \tag{3.3.29}$$

where μ is the marginal of ν on $\mathbb{R}^d$, for large enough n, $m \in \mathbb{N}$ depending on δ, $M_2(\nu_0)$ and l_0 only. By the compactness of

$$\widetilde{U}^2_{\delta,n,m} := \Big\{(t,\nu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d\times A) : \breve{v}^{s_0}_{n,m}(t,\nu) - \breve{u}_2(t,\mu) - \delta M_2(\nu) \geq l_0/3\Big\},$$

and the upper semicontinuity of $\breve{v}^{s_0}_{n,m} - \breve{u}_2 - \delta M_2$, we argue as in Part 1A to deduce that there exists $(\widetilde{t},\widetilde{\nu}) \in [0,T]\times\widetilde{U}^2_{\delta,n,m}$ such that the maximum of $\breve{v}^{s_0}_{n,m} - \breve{u}_2 - \delta M_2$ is attained at $(\widetilde{t},\widetilde{\nu})$. We note that this maximum point depends on δ, n, m. If $\widetilde{t} = s_0 = T$, then we proceed as in Step 1B to get a contradiction. If $\widetilde{t} = s_0 < T$, then

$$\begin{aligned}(v^{s_0}_{n,m} - u_2 - \delta M_2)(t_0,\nu_0) = (\breve{v}^{s_0}_{n,m} - \breve{u}_2 - \delta M_2)(t_0,\nu_0) &\leq \breve{v}^{s_0}_{n,m}(s_0,\widetilde{\nu}) - \breve{u}_2(s_0,\widetilde{\nu}) - \delta M_2(\widetilde{\nu})\\ &\leq \breve{v}^{s_0}_{n,m}(s_0,\widetilde{\nu}) - \breve{u}_2(s_0,\widetilde{\nu}) \to 0,\end{aligned}$$

as $m \to \infty$ then $n \to \infty$, by using the definition of $\breve{v}^{s_0}_{n,m} := e^{t-t_0}v^{s_0}_{n,m}$ and that of $v^{s_0}_{n,m}$ in (3.3.26). If $\widetilde{t} < s_0$, we apply Definition 3.1.5 of supersolution and put $\varphi = \breve{v}^{s_0}_{n,m} - \delta M_2 - M_*$ in (3.3.25), with $M_* \in \mathbb{R}$ such that $\breve{v}^{s_0}_{n,m} - \breve{u}_2 - \delta M_2 - M_*$ attains the maximum with a value of 0 at $(\widetilde{t},\widetilde{\nu})$. Then we proceed as in Step 1C and utilize items (1)-(4) in the above to draw a contradiction and conclude the proof. □

3.4 Technical Proofs in Section 3.2

Proof of Lemma 3.2.2: The bounds of $f^i_{n,m}$ and $g^i_{n,m}$ in assertion (1) are obvious by the definitions in (3.2.4)-(3.2.6) and Assumption 3.1.2. For the bound of $b^i_{n,m}$ in assertion (1),

we have

$$
\begin{aligned}
|b^i_{n,m}(t,\overline{x},a)| &\le Km^{dn}\int_{\mathbb{R}^{dn}}[1+|x^i|^\rho+|y^i|^\rho]\prod_{k=1}^n\Phi(my^k)dy^k\\
&\le K\left(1+|x^i|^\rho+m^{-\rho}\int_{\mathbb{R}^d}|y^i|^\rho\Phi(y^i)dy^i\right)\\
&\le K\Big(1+C_{\Phi,\rho}m^{-\rho}+|x^i|^\rho\Big).
\end{aligned}
$$

We prove assertion (2) for g by considering

$$
\Big|g(x^i,\widehat{\mu}^{n,\overline{x}})-g^i_{n,m}(\overline{x})\Big|\le m^{dn}\int_{\mathbb{R}^{dn}}\left|g(x^i,\widehat{\mu}^{n,\overline{x}})-g\Big(x^i-y^i,\frac{1}{n}\sum_{j=1}^n\delta_{x^j-y^j}\Big)\right|\prod_{k=1}^n\Phi(my^k)dy^k.
$$

Using the Lipschitz continuity of g in Assumption 3.1.2 and the fact that

$$
\begin{aligned}
\mathcal{W}_1\Big(\widehat{\mu}^{n,\overline{x}},\frac{1}{n}\sum_{j=1}^n\delta_{x^j-y^j}\Big)&=\mathcal{W}_1\Big(\frac{1}{n}\sum_{j=1}^n\delta_{x^j},\frac{1}{n}\sum_{j=1}^n\delta_{x^j-y^j}\Big)\\
&\le\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|\left[\frac{1}{n}\sum_{j=1}^n\delta_{(x^j,x^j-y^j)}(dx,dy)\right]\\
&=\frac{1}{n}\sum_{j=1}^n|y^j|, \qquad (3.4.1)
\end{aligned}
$$

we obtain that

$$
\Big|g(x^i,\widehat{\mu}^{n,\overline{x}})-g^i_{n,m}(\overline{x})\Big|\le Km^{dn}\int_{\mathbb{R}^{dn}}\Big(|y^i|+\frac{1}{n}\sum_{j=1}^n|y^j|\Big)\prod_{k=1}^n\Phi(my^k)dy^k.
$$

We prove assertion (2) for b by considering (the proof for f is exactly the same)

$$
\begin{aligned}
&|b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)-b^i_{n,m}(t,\overline{x},a)|\\
&\le m^{dn+1}\int_{\mathbb{R}^{dn+1}}\left|b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)-b\Big(T\wedge(t-s)^+,x^i-y^i,\frac{1}{n}\sum_{j=1}^n\delta_{x^j-y^j},a\Big)\right|\phi(ms)\prod_{k=1}^n\Phi(my^k)dy^kds\\
&\le m\int_{\mathbb{R}}\left|b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)-b\left(T\wedge(t-s)^+,x^i,\widehat{\mu}^{n,\overline{x}},a\right)\right|\phi(ms)ds\\
&\quad+m^{dn+1}\int_{\mathbb{R}^{dn+1}}\left|b\left(T\wedge(t-s)^+,x^i,\widehat{\mu}^{n,\overline{x}},a\right)-b\Big(T\wedge(t-s)^+,x^i-y^i,\frac{1}{n}\sum_{j=1}^n\delta_{x^j-y^j},a\Big)\right|\cdot\\
&\qquad\qquad\phi(ms)\prod_{j=1}^n\Phi(my^j)dy^jds.
\end{aligned}
$$

The inequality in (3.4.1) and Assumption 3.1.2 imply that

$$
\begin{aligned}
&|b(t,x^i,\widehat{\mu}^{n,\overline{x}},a)-b^i_{n,m}(t,\overline{x},a)|\\
&\leq Km\int_{\mathbb{R}}\left|t-(T\wedge(t-s)^+)\right|^{\beta}\phi(ms)ds+Km^{dn}\int_{\mathbb{R}^{dn}}\left(|y^i|+\frac{1}{n}\sum_{j=1}^{n}|y^j|\right)\prod_{k=1}^{n}\Phi(my^k)dy^k.
\end{aligned}
$$

For the proof of assertion (3) for g (the proofs for f and b are exactly the same), we let $\overline{x}$, $\overline{z}\in\mathbb{R}^{dn}$ and estimate

$$
\begin{aligned}
&|g^i_{n,m}(\overline{x})-g^i_{n,m}(\overline{z})|\\
&\leq m^{dn}\int_{\mathbb{R}^{dn}}\left|g\Big(x^i-y^i,\frac{1}{n}\sum_{j=1}^{n}\delta_{x^j-y^j}\Big)-g\Big(z^i-y^i,\frac{1}{n}\sum_{j=1}^{n}\delta_{z^j-y^j}\Big)\right|\prod_{k=1}^{n}\Phi(my^k)dy^k.
\end{aligned}
$$

Then the inequality in (3.4.1) yields that

$$
\begin{aligned}
|g^i_{n,m}(\overline{x})-g^i_{n,m}(\overline{z})|&\leq Km^{dn}\int_{\mathbb{R}^{dn}}\left[|x^i-z^i|+\frac{1}{n}\sum_{j=1}^{n}|x^j-z^j|\right]\prod_{k=1}^{n}\Phi(my^k)dy^k\\
&=K\left[|x^i-z^i|+\frac{1}{n}\sum_{j=1}^{n}|x^j-z^j|\right].
\end{aligned}
$$

Finally, assertion (4) follows immediately from assertion (2). □

Proof of Lemma 3.2.3: Step 1. Lipschitz continuity of $\overline{v}_{\varepsilon,n,m}(t,\overline{x})$ in $\overline{x}$: Note that the identity (Cosso *et al.*, 2024, (A.21)) may contain potential typographical errors, so we reproduce the proof of the Lipschitz continuity property here for the case involving common noise. We aim to establish that

$$
|\overline{v}_{\varepsilon,n,m}(t,\overline{x})-\overline{v}_{\varepsilon,n,m}(t,\overline{z})|\leq\frac{C_4}{n}\,|\overline{x}-\overline{z}|\,,
$$

when the components of $\overline{x}=(x^1,\ldots,x^n)$ and $\overline{z}=(z^1,\ldots,z^n)$ are all equal, apart from one component $x^k\neq z^k$ for some $k=1,2,\ldots,n$. Recalling the definition in (3.2.8), we use the continuity in (3) of Lemma 3.2.2 to yield that

$$
\begin{aligned}
&|\overline{v}_{\varepsilon,n,m}(t,\overline{x})-\overline{v}_{\varepsilon,n,m}(t,\overline{z})|\\
&\leq 2K\sup_{\overline{\alpha}\in\overline{\mathcal{A}}^n_t}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\int_t^T\left|\overline{X}^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}}_s-\overline{X}^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}_s\right|ds+\left|\overline{X}^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}}_T-\overline{X}^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}_T\right|\right]. \quad (3.4.2)
\end{aligned}
$$

Suppose that $\overline{x}$ and $\overline{z}$ differ only for the first component $x^1 \neq z^1$. For $i = 1, 2, \ldots, n$, the $\mathbb{R}^d$-valued process $\overline{X}^i := (\overline{X}_s^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}})_{s\in[t,T]}$ solves the following equation on $[t, T]$:

$$\overline{X}_s^i = x^i + \int_t^s b_{n,m}^i(r, \overline{X}_r^1, \ldots, \overline{X}_r^n, \overline{\alpha}_r^i)dr + \int_t^s \sigma(r, \overline{X}_r^i, \overline{\alpha}_r^i)d\overline{W}_r^i + \int_t^s \sigma^0(r, \overline{X}_r^i)d\overline{W}_r^0 + \varepsilon(\overline{B}_s^i - \overline{B}_t^i).$$

As the coefficients of the above equation are regular enough and have bounded continuous derivatives by Lemma 3.2.2, the process $\Delta\overline{X}_s^i := \overline{X}_s^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}} - \overline{X}_s^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}}$ satisfies

$$\begin{aligned}\Delta\overline{X}_s^i =& (z^1 - x^1)\delta_{1i} + \int_t^s \sum_{j=1}^n \widetilde{b}_{n,m,r}^{i,j}\Delta\overline{X}_r^j dr + \int_t^s \sum_{k=1}^d \left(\Delta\overline{X}_r^i\right)_k \widetilde{\sigma}_r^{i,k} d\overline{W}_r^i, \\ &+ \int_t^s \sum_{k=1}^d \left(\Delta\overline{X}_r^i\right)_k \widetilde{\sigma}_r^{0,i,k} d\overline{W}_r^0,\end{aligned}$$

where $\widetilde{b}_{n,m,r}^{i,j} := \int_0^1 \nabla_{x^j} b_{n,m}^i(r, \overline{X}_r^1 + \theta\Delta\overline{X}_r^1, \ldots, \overline{X}_r^n + \theta\Delta\overline{X}_r^n, \overline{\alpha}_r^i)d\theta$, $\widetilde{\sigma}_r^{i,k} := \int_0^1 \partial_{x_k}\sigma(r, \overline{X}_r^i + \theta\Delta\overline{X}_r^i, \overline{\alpha}_r^i)d\theta$ and $\widetilde{\sigma}_r^{0,i,k} := \int_0^1 \partial_{x_k}\sigma^0(r, \overline{X}_r^i + \theta\Delta\overline{X}_r^i)d\theta$. The $\mathbb{R}^{dn}$-valued continuous process $\Delta\overline{X}_s := (\Delta\overline{X}_s^1, \ldots, \Delta\overline{X}_s^n)^\top$ is the unique solution to the above system of linear stochastic equations such that $\mathbb{E}\left[\sup_{s\in[t,T]} |\Delta\overline{X}_s|^2\right] < \infty$. Next, we provide an estimate of $\sup_{s\in[t,T]} \mathbb{E}\Big[\sum_{i=1}^n |\Delta\overline{X}_s^i|\Big]$ with a method akin to the proof of Tanaka's formula. Letting $\vartheta > 0$, we consider the function $u_\vartheta : \mathbb{R}^d \to \mathbb{R}$ defined by

$$u_\vartheta(y) := \sqrt{|y|^2 + \vartheta^2}.$$

Direct calculation gives

$$\nabla_y u_\vartheta(y) = \frac{y}{u_\vartheta(y)}, \quad \nabla_{yy}^2 u_\vartheta(x) = \frac{1}{u_\vartheta(y)} I_d - \frac{1}{\left[u_\vartheta(y)\right]^3} yy^\top.$$

Applying Itô's formula to $u_\vartheta\left(\Delta\overline{X}_s^i\right)$ gives

$$\begin{aligned}&du_\vartheta\left(\Delta\overline{X}_s^i\right) \\ &= \left\langle \nabla_y u_\vartheta(\Delta\overline{X}_s^i), d\Delta\overline{X}_s^i \right\rangle + \frac{1}{2}\mathrm{tr}\left\{ \left[\sum_{k=1}^d \left(\Delta\overline{X}_s^i\right)_k \widetilde{\sigma}_s^{i,k}\right]^\top \nabla_{yy}^2 u_\vartheta\left(\Delta\overline{X}_s^i\right) \left[\sum_{k=1}^d \left(\Delta\overline{X}_s^i\right)_k \widetilde{\sigma}_s^{i,k}\right] \right\} ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\mathrm{tr}\left\{\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{0,i,k}\right]^\top\nabla_{yy}^2u_\vartheta\left(\Delta\overline{X}_s^i\right)\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{0,i,k}\right]\right\}ds\\
=&\left\langle\frac{\Delta\overline{X}_s^i}{u_\vartheta(\Delta\overline{X}_s^i)},\sum_{j=1}^{n}\widetilde{b}_{n,m,s}^{i,j}\Delta\overline{X}_s^jds+\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]d\overline{W}_s^i+\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{0,i,k}\right]d\overline{W}_s^0\right\rangle\\
&+\frac{1}{2}\mathrm{tr}\left\{\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]^\top\left(\frac{1}{u_\vartheta(\Delta\overline{X}_s^i)}I_d-\frac{\left(\Delta\overline{X}_s^i\right)\left(\Delta\overline{X}_s^i\right)^\top}{\left[u_\vartheta(\Delta\overline{X}_s^i)\right]^3}\right)\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]\right\}ds\\
&+\frac{1}{2}\mathrm{tr}\left\{\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{0,i,k}\right]^\top\left(\frac{1}{u_\vartheta(\Delta\overline{X}_s^i)}I_d-\frac{\left(\Delta\overline{X}_s^i\right)\left(\Delta\overline{X}_s^i\right)^\top}{\left[u_\vartheta(\Delta\overline{X}_s^i)\right]^3}\right)\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{0,i,k}\right]\right\}ds.
\end{aligned}
$$

It is clear that

$$
\frac{\Delta\overline{X}_s^i}{u_\vartheta(\Delta\overline{X}_s^i)}\longrightarrow\frac{\Delta\overline{X}_s^i}{\left|\Delta\overline{X}_s^i\right|}\mathbb{1}_{\left\{\Delta\overline{X}_s^i\neq0\right\}},\quad\mathbb{P}\text{-a.s. as }\vartheta\to0.
$$

Moreover,

$$
\begin{aligned}
&\frac{1}{2}\mathrm{tr}\left\{\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]^\top\left(\frac{1}{u_\vartheta(\Delta\overline{X}_s^i)}I_d\right)\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]\right\}\\
\longrightarrow&\frac{1}{2}\left(\frac{1}{\left|\Delta\overline{X}_s^i\right|}\mathbb{1}_{\left\{\Delta\overline{X}_s^i\neq0\right\}}\right)\mathrm{tr}\left\{\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]^\top\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]\right\}
\end{aligned}
$$

$\mathbb{P}$-a.s. as $\vartheta\to0$. Similarly,

$$
\begin{aligned}
&\frac{1}{2}\mathrm{tr}\left\{\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]^\top\left(\frac{\Delta\overline{X}_s^i\left(\Delta\overline{X}_s^i\right)^\top}{\left[u_\vartheta(\Delta\overline{X}_s^i)\right]^3}\right)\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]\right\}\\
\longrightarrow&\frac{1}{2}\left(\frac{1}{\left|\Delta\overline{X}_s^i\right|^3}\mathbb{1}_{\left\{\Delta\overline{X}_s^i\neq0\right\}}\right)\mathrm{tr}\left\{\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]^\top\left(\Delta\overline{X}_s^i\left(\Delta\overline{X}_s^i\right)^\top\right)\left[\sum_{k=1}^{d}\left(\Delta\overline{X}_s^i\right)_k\widetilde{\sigma}_s^{i,k}\right]\right\}
\end{aligned}
$$

$\mathbb{P}$-a.s. as $\vartheta\to0$. The terms involving $\widetilde{\sigma}_s^{0,i,k}$ exhibit similar convergences as established in the preceding two results. Therefore, by taking the expectation and applying the dominated convergence theorem as $\vartheta\to0$, we conclude that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\Delta\overline{X}_s^i\right|\right]-|z^1-x^1|\delta_{1i}\\
=&\mathbb{E}\left[\int_t^s\left\langle\frac{\Delta\overline{X}_r^i}{\left|\Delta\overline{X}_r^i\right|},\sum_{j=1}^{n}\widetilde{b}_{n,m,r}^{i,j}\Delta\overline{X}_r^j\right\rangle\mathbb{1}_{\left\{\Delta\overline{X}_r^i\neq0\right\}}dr\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\mathbb{E}\left\{\int_t^s\left(\frac{1}{\left|\Delta\overline{X}^i_r\right|}\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\right)\operatorname{tr}\left\{\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]^\top\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]\right\}dr\right\}\\
&-\frac{1}{2}\mathbb{E}\left\{\int_t^s\left(\frac{1}{\left|\Delta\overline{X}^i_r\right|^3}\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\right)\operatorname{tr}\left\{\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]^\top\left(\Delta\overline{X}^i_r\left(\Delta\overline{X}^i_r\right)^\top\right)\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]\right\}dr\right\}\\
&+\frac{1}{2}\mathbb{E}\left\{\int_t^s\left(\frac{1}{\left|\Delta\overline{X}^i_r\right|}\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\right)\operatorname{tr}\left\{\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{0,i,k}_r\right]^\top\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{0,i,k}_r\right]\right\}dr\right\}\\
&-\frac{1}{2}\mathbb{E}\left\{\int_t^s\left(\frac{1}{\left|\Delta\overline{X}^i_r\right|^3}\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\right)\operatorname{tr}\left\{\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{0,i,k}_r\right]^\top\left(\Delta\overline{X}^i_r\left(\Delta\overline{X}^i_r\right)^\top\right)\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{0,i,k}_r\right]\right\}dr\right\}.
\end{aligned}
\tag{3.4.3}
$$

The term in the third line of (3.4.3) can be estimated by the Cauchy–Schwarz inequality:

$$
\begin{aligned}
&\left(\frac{1}{\left|\Delta\overline{X}^i_r\right|}\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\right)\operatorname{tr}\left\{\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]^\top\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]\right\}\\
&=\frac{1}{\left|\Delta\overline{X}^i_r\right|}\sum_{p,q=1}^d\left|\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\left(\widetilde{\sigma}^{i,k}_r\right)_{pq}\right|^2\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\\
&\leq K^2\left|\Delta\overline{X}^i_r\right|.
\end{aligned}
$$

Similarly, the term in the forth line of (3.4.3) can be estimated by

$$
\begin{aligned}
&\left(\frac{1}{\left|\Delta\overline{X}^i_r\right|^3}\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\right)\operatorname{tr}\left\{\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]^\top\left(\Delta\overline{X}^i_r\left(\Delta\overline{X}^i_r\right)^\top\right)\left[\sum_{k=1}^d\left(\Delta\overline{X}^i_r\right)_k\widetilde{\sigma}^{i,k}_r\right]\right\}\\
\leq&\frac{1}{\left|\Delta\overline{X}^i_r\right|^3}\left|\Delta\overline{X}^i_r\right|^4\sum_{k=1}^d\left|\widetilde{\sigma}^{i,k}_r\right|^2\mathbb{1}_{\left\{\Delta\overline{X}^i_r\neq 0\right\}}\\
\leq&K^2\left|\partial_{x^1_k}\overline{X}^i_r\right|.
\end{aligned}
$$

The terms involving $\widetilde{\sigma}^{0,i,k}_s$ can be estimated in a manner similar to the preceding results.

Therefore, (3.4.3) reduces to

$$
\mathbb{E}\left[\left|\Delta\overline{X}^i_s\right|\right]\leq|z^1-x^1|\delta_{1i}+\mathbb{E}\left[\int_t^s\sum_{j=1}^n\left|\widetilde{b}^{i,j}_{n,m,r}\Delta\overline{X}^j_r\right|+2K^2\left|\Delta\overline{X}^i_r\right|dr\right].
$$

Summing over $i=1,2,\ldots,n$, we have

$$
\mathbb{E}\left[\sum_{i=1}^n\left|\Delta\overline{X}^i_s\right|\right]\leq|z^1-x^1|+\int_t^s\mathbb{E}\left[\sum_{j=1}^n\left(\sum_{i=1}^n\left|\widetilde{b}^{i,j}_{n,m,r}\right|\right)\left|\Delta\overline{X}^j_r\right|+2K^2\sum_{i=1}^n\left|\Delta\overline{X}^i_r\right|\right]dr
$$

$$\leq |z^1 - x^1| + \int_t^s \mathbb{E}\left[\max_{1\leq \ell\leq n}\left(\sum_{i=1}^n \left|\widetilde{b}^{i,\ell}_{n,m,r}\right|\right)\sum_{j=1}^n \left|\Delta \overline{X}^j_r\right| + 2K^2\sum_{j=1}^n |\Delta \overline{X}^j_r|\right] dr.$$

The Lipschitz continuity estimate for $b^i_{n,m}$ in Lemma 3.2.2 deduces that

$$\begin{aligned}
\max_{1\leq \ell\leq n}\sum_{i=1}^n \left|\widetilde{b}^{i,\ell}_{n,m,s}\right| &\leq \max_{1\leq \ell\leq n}\sum_{i=1}^n \sup_{\overline{x}\in\mathbb{R}^{dn}} \left|\nabla_{x^\ell} b^i_{n,m}(s,\overline{x},\overline{\alpha}^i_s)\right| \\
&= \max_{1\leq \ell\leq n}\left(\sup_{\overline{x}\in\mathbb{R}^{dn}} \left|\nabla_{x^\ell} b^\ell_{n,m}(s,\overline{x},\overline{\alpha}^\ell_s)\right| + \sum_{i=1,i\neq\ell}^n \sup_{\overline{x}\in\mathbb{R}^{dn}} \left|\nabla_{x^\ell} b^i_{n,m}(s,\overline{x},\overline{\alpha}^i_s)\right|\right) \\
&\leq \sqrt{d}\left[K\left(1+\frac{1}{n}\right) + K\frac{n-1}{n}\right] \\
&= 2\sqrt{d}K.
\end{aligned}$$

It gives us that $\mathbb{E}\left[\sum_{i=1}^n |\Delta \overline{X}^i_s|\right] \leq |z^1 - x^1| + 2(\sqrt{d}K + K^2)\int_t^s \mathbb{E}\left[\sum_{i=1}^n \left|\Delta \overline{X}^i_r\right|\right] dr$. Grönwall's inequality yields

$$\mathbb{E}\left[\sum_{i=1}^n \left|\Delta \overline{X}^i_s\right|\right] \leq |z^1 - x^1| e^{2(\sqrt{d}K+K^2)T}, \quad \text{for every } s\in[t,T].$$

Thus, we obtain from (3.4.2) that

$$\begin{aligned}
&|\overline{v}_{\varepsilon,n,m}(t,\overline{x}) - \overline{v}_{\varepsilon,n,m}(t,\overline{z})| \\
&\leq 2K \sup_{\overline{\alpha}\in\overline{\mathcal{A}}^n_t} \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\int_t^T \left|\overline{X}^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}}_s - \overline{X}^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}_s\right| ds + \left|\overline{X}^{i,m,\varepsilon,t,\overline{x},\overline{\alpha}}_T - \overline{X}^{i,m,\varepsilon,t,\overline{z},\overline{\alpha}}_T\right|\right] \\
&\leq \frac{C_{d,K,T}}{n}|z^1 - x^1|. \qquad (3.4.4)
\end{aligned}$$

Step 2. $\overline{v}_{\varepsilon,n,m}(t,\overline{x})$ **is the unique classical solution of** (3.2.11)**:** In this step, the definition of viscosity solution is referred to the usual Crandall-Lions' definition as the equation is on $[0,T]\times\mathbb{R}^{dn}$ instead of the Wasserstein space. First note that the volatility term (the second-order term) of equation (3.2.11) can be written as $\frac{1}{2}\mathrm{tr}\big[Q\nabla^2_{\overline{x}\overline{x}} v\big]$, with

$$Q := \begin{pmatrix} \sigma\sigma^\top(t,x^1,a^1) & 0 & \dots & 0 \\ 0 & \sigma\sigma^\top(t,x^2,a^2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \sigma\sigma^\top(t,x^n,a^n) \end{pmatrix} + \varepsilon^2 I_{dn} + \begin{pmatrix} \sigma^0(t,x^1) \\ \sigma^0(t,x^2) \\ \vdots \\ \sigma^0(t,x^n) \end{pmatrix}\begin{pmatrix} \sigma^0(t,x^1) \\ \sigma^0(t,x^2) \\ \vdots \\ \sigma^0(t,x^n) \end{pmatrix}^\top.$$

It can be easily shown that Q is positive definite. Consequently, we have $Q\nabla^2_{\bar{x}\bar{x}}\overline{v}_{\varepsilon,n,m} = \Sigma\Sigma^\top\nabla^2_{\bar{x}\bar{x}}\overline{v}_{\varepsilon,n,m}$, where Σ satisfies $\Sigma\Sigma^\top \geq \varepsilon^2 I_{dn}$. Using (Yong & Zhou, 1999, Chatper 4.6, Theorem 6.2), we conclude that $\overline{v}_{\varepsilon,n,m}$ is the unique viscosity solution of the Bellman equation (3.2.11). Due to the 1/2-Hölder continuity of $\overline{v}_{\varepsilon,n,m}(t,\overline{x})$ in t, Lipschitz continuity in $\overline{x}$ as established in (3.4.4) and the boundedness of the non-homogeneous term $f^i_{n,m}$ of (3.2.11), we consider the equation (3.2.11) on $[0,T]\times B_R \subset [0,T]\times\mathbb{R}^{dn}$ with the parabolic boundary value of $\overline{v}_{\varepsilon,n,m}(t,\overline{x})$. For this localized equation, we can apply (Crandall *et al.*, 2000, Theorem 8.4) to obtain a unique strong solution $\overline{v}^\dagger_{\varepsilon,n,m}(t,\overline{x})$ which lies in $C^{\frac{1+\alpha}{2},1+\alpha}_{\text{loc}}([0,T)\times B_R)\cap C([0,T]\times\overline{B_R})$. We see that $\overline{v}^\dagger_{\varepsilon,n,m}(t,\overline{x})$ is also the viscosity solution to (3.2.11) in $[0,T]\times B_R$ under the usual Crandall-Lions' definition, by (Crandall *et al.*, 2000, Proposition 2.10). Therefore, as the viscosity solution to (3.2.11) on $[0,T]\times\mathbb{R}^{dn}$ is unique, then we see that $\overline{v}_{\varepsilon,n,m} = \overline{v}^\dagger_{\varepsilon,n,m}\mathbb{1}_{[0,T]\times B_R} + \overline{v}_{\varepsilon,n,m}\mathbb{1}_{[0,T]\times(\mathbb{R}^{dn}\setminus B_R)}$ for any $R>0$. In other words, $\overline{v}_{\varepsilon,n,m}$ is continuously differentiable with respect to $\overline{x}$ on $\mathbb{R}^{dn}\times[0,T)$. We consider another equation

$$\begin{cases} \partial_t u(t,\overline{x}) + \sup\limits_{\overline{a}\in A^n}\left\{\dfrac{1}{n}\displaystyle\sum_{i=1}^n f^i_{n,m}(t,\overline{x},a^i) + \dfrac{1}{2}\sum_{i=1}^n \operatorname{tr}\left[\Big((\sigma\sigma^\top)(t,x^i,a^i) + \sigma^0\sigma^{0,\top}(t,x^i) + \varepsilon^2 I_d\Big)\nabla^2_{x^ix^i}u(t,\overline{x})\right]\right. \\ \qquad\qquad \left. + \displaystyle\sum_{i=1}^n\left\langle b^i_{n,m}(t,\overline{x},a^i), \nabla_{x^i}\overline{v}_{\varepsilon,n,m}(t,\overline{x})\right\rangle + \frac{1}{2}\sum_{i,j=1,i\neq j}^n \operatorname{tr}\Big[\sigma^0(t,x^i)\sigma^{0,\top}(t,x^j)\nabla^2_{x^ix^j}u(t,\overline{x})\Big]\right\} \\ = 0 \text{ in } [0,T)\times\mathbb{R}^{dn}; \\ u(T,\overline{x}) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n g^i_{n,m}(\overline{x}) \quad \text{in } \mathbb{R}^{dn}. \end{cases} \tag{3.4.5}$$

In the above equation, we note the unknown u is only involved in the time differentiation and the second-order spatial differentiation, the remaining terms are all in $C^{0,\alpha}([0,T)\times B_{R'})$ for any $R'>0$ and some $\alpha>0$. As $\overline{v}_{\varepsilon,n,m}$ is the unique viscosity solution to (3.2.11) on $[0,T]\times\mathbb{R}^{dn}$ under the usual Crandall-Lions' definition, it is also a viscosity solution to (3.4.5) under the usual Crandall-Lions' definition as it is continuously differentiable on $\overline{x}$. By considering the equation in (3.4.5) in a local region with the parabolic boundary value $\overline{v}_{\varepsilon,n,m}$, we can apply (Mou, 2019, Theorem 5.2) to find a classical solution $\overline{v}^*_{\varepsilon,n,m}(t,\overline{x})$ to (3.4.5) on that

local region. As the viscosity solution of (3.4.5) on $[0,T]\times\mathbb{R}^{dn}$ is unique by (Yong & Zhou, 1999, Chatper 4.6, Theorem 6.2), we see that $\overline{v}_{\varepsilon,n,m}(t,\overline{x})=\overline{v}^{*}_{\varepsilon,n,m}(t,\overline{x})$ is actually classical.

Step 3. Bounds of derivatives of $\overline{v}_{\varepsilon,n,m}(t,\overline{x})$: Note that $\overline{v}_{\varepsilon,n,m}(t,\overline{x})\in C^{1,2}([0,T]\times\mathbb{R}^{dn})$, the boundedness of the first-order derivative with respect to $\overline{x}$ follows from (3.4.4), while the boundedness of the second-order derivative is established using (Krylov, 2008, Chapter 4.7, Theorem 4). □

Chapter 4

Feynman-Kac Formula for Time-Dependent Nonlinear Schrödinger Equations with Applications in Numerical Approximations

In this chapter, we present a novel Feynman-Kac formula and investigate learning-based methods for approximating general nonlinear time-dependent Schrödinger equations which may be high-dimensional. Our formulation integrates both the Fisk-Stratonovich and Itô integrals within the framework of backward stochastic differential equations (BSDEs). Utilizing this Feynman-Kac representation, we propose learning-based approaches for numerical approximations. To demonstrate the accuracy and effectiveness of the proposed method, we conduct numerical experiments in both low- and high-dimensional settings, complemented by a convergence analysis. These results address the open problem concerning deep-BSDE methods for numerical approximations of high-dimensional time-dependent nonlinear Schrödinger equations (cf. [Proc. Natl. Acad. Sci. 15 (2018), pp. 8505–8510] and [Frontiers Sci. Awards Math. (2024), pp. 1-14] by Han, Jentzen, and E).

This chapter is organized as follows: In Section 1, we introduce our Feynman-Kac formula for both classical and weak solutions of general Schrödinger equations and highlight some open questions arising from this Feynman-Kac representation. In Section 2, we propose a

machine learning scheme for numerically computing the BSDEs (1.2.5), covering both linear and nonlinear cases of f in both low- and high-dimensional problems. The experimental results demonstrate that the solutions of the BSDEs are in strong agreement with the explicit solutions of the Schrödinger equations. Finally, in Section 3 we present a convergence analysis regarding the proposed algorithm for simulations. Additionally, the appendix contains the estimate of certain error terms involved in the proof of Lemma 4.3.2, along with the complete proof of Lemma 4.3.2.

4.1 Feynman-Kac formula for the nonlinear Schrödinger equation

A less rigorous form of Feynman-Kac formula is based on the complex time transformation, $\tau = i\,t$. By defining $U(\tau, x) = u(it, x)$, we arrive at the following *parabolic* PDE:

$$-\partial_t U(\tau, x) = \frac{\nu}{2}\Delta U(\tau, x) + f(\tau, x, U); \quad U(T, x) = G(x). \tag{4.1.1}$$

This equation may be approached using the established theory of FBSDEs (see Delarue (2002); Hu & Peng (1995); Ma *et al.* (1994); Pardoux & Peng (1992); Pardoux & Tang (1999) for instance), which suggests a representation via a $\mathbb{C}$-valued BSDE:

$$\begin{cases} -d\widetilde{Y}_s(\tau, x) = f(s, \sqrt{\nu}(W_s - W_\tau) + x, \widetilde{Y}_s(\tau, x))\,ds - \sqrt{\nu}\widetilde{Z}_s(\tau, x)\,dW_s; \\ \widetilde{Y}_T(\tau, x) = G(\sqrt{\nu}(W_T - W_\tau) + x), \end{cases} \tag{4.1.2}$$

with the relation

$$\widetilde{Y}_s(\tau, x) = U(s, \sqrt{\nu}(W_s - W_\tau) + x), \quad \widetilde{Z}_s(t, x) = \nabla U(s, \sqrt{\nu}(W_s - W_\tau) + x). \tag{4.1.3}$$

Through equation (4.1.2) and relation (4.1.3), we obtain a Feynman-Kac representation for Schrödinger equation (1.2.3). It is important to note that for all equations (1.2.3), (4.1.1), (4.1.2), and (4.1.3), the time domain lies in the *real* interval $[0, T]$. However, the complex

time transformation $\tau = i\,t$ necessitates extending the time domain to an imaginary time interval, which imposes significant restrictions on both theory and applications. A formal representation for *linear* Schrödinger equations was in fact given via so-called Feynman path integrals. However, these path integrals are not well-defined in most of their uses (cf. Albeverio *et al.* (1976); Feynman (1948); Kac (1949); Yan (1994)).

In this section, we shall present a Feynman-Kac formula that avoids both the complex time transformation and Feynman path integrals. The representation is developed for both classical and weak solutions of Schrödinger equation (1.2.5). Some further explorations of the resolution of BSDE (1.2.5) using purely probabilistic methods are left as open questions.

4.1.1 Feynman-Kac formula for classical solutions

Denote by $C^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{C})$ the space of complex-valued continuous functions $u(t,x)$ with bounded continuous derivatives $\partial_t u(t,x)$, $\partial_{x_j} u(t,x)$, and $\partial_{x_k x_l} u(t,x)$, $j,k,l = 1,\ldots,d$. In the following theorem, the Feynman-Kac formula establishes a connection between the classical solution of Schrödinger equation (1.2.3) in $C^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{C})$ and the BSDE (1.2.5) via the relation (1.2.6).

Theorem 4.1.1. *(i) Suppose that $u \in C^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{C})$ is a solution to the Schrödinger equation* (1.2.3)*. Then the pair (Y,Z) defined through* (1.2.6) *is a solution to BSDE* (1.2.5)*.*

(ii) If the pair (Y,Z) with $Y_s(t,x) = Y_s^R(t,x) + iY_s^I(t,x)$ and $Z_s(t,x) = Z_s^R(t,x) + iZ_s^I(t,x)$ for $0 \le t \le s \le T$, $x \in \mathbb{R}^d$ satisfies BSDE (1.2.5) *and there exists a function $u \in C^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{C})$ satisfying the relation* (1.2.6)*, then the function u is a solution to the Schrödinger equation* (1.2.3)*.*

Proof. First, we prove (i). Without loss of generality, we take $t = 0$. Applying Itô's formula

to u^R and u^I respectively yields that

$$du^R(s, x+\sqrt{\nu}W_s) = \left(\partial_s u^R + \frac{\nu}{2}\Delta u^R\right)(s, x+\sqrt{\nu}W_s)\,ds + \sqrt{\nu}\nabla u^R(s, x+\sqrt{\nu}W_s)\,dW_s,$$
$$du^I(s, x+\sqrt{\nu}W_s) = \left(\partial_s u^I + \frac{\nu}{2}\Delta u^I\right)(s, x+\sqrt{\nu}W_s)\,ds + \sqrt{\nu}\nabla u^I(s, x+\sqrt{\nu}W_s)\,dW_s.$$

Here, we shall only verify the real part as the imaginary part follows in a similar way. In view of (1.2.3) and (1.2.4), we have further

$$\begin{aligned} du^R(s, x+\sqrt{\nu}W_s) = &\left[\frac{\nu}{2}\Delta(u^I+u^R)(s, x+\sqrt{\nu}W_s) + f^I(s, x+\sqrt{\nu}W_s, u(s, x+\sqrt{\nu}W_s))\right] ds \\ &+ \sqrt{\nu}\nabla u^R(s, x+\sqrt{\nu}W_s)\,dW_s. \end{aligned} \tag{4.1.4}$$

Recalling the relationships between Fisk-Stratonovich integrals and Itô integrals (see (Revuz & Yor, 2013, Chapter 4, Exercise 2.18)), we have

$$\begin{aligned} \frac{\nu}{2}\Delta(u^I+u^R)(s, x+\sqrt{\nu}W_s)ds &= \sqrt{\nu}\nabla(u^I+u^R)(s, x+\sqrt{\nu}W_s)(\circ dW_s - dW_s) \\ &= \sqrt{\nu}\nabla(u^I+u^R)(s, x+\sqrt{\nu}W_s) * dW_s, \end{aligned} \tag{4.1.5}$$

which together with relation (1.2.6) substituted in (4.1.4) gives

$$dY_s^R(0,x) = \sqrt{\nu}(Z_s^R(0,x) + Z_s^I(0,x)) * dW_s + f^I(s, \sqrt{\nu}W_s + x, Y_s(0,x))\,ds + \sqrt{\nu}Z_s^R(0,x)\,dW_s.$$

Analogously, we justify the second stochastic differential equation in (1.2.5) for the imaginary part. The terminal condition is obvious and this justifies (i).

For (ii), suppose (Y, Z) and u satisfy BSDE (1.2.5) as well as the relation (1.2.6). Let φ be an arbitrary infinitely differentiable real-valued function with compact support on $\mathbb{R}^d$. Then Itô's formula implies

$$d\varphi(x+\sqrt{\nu}W_s) = \frac{\nu}{2}\Delta\varphi(x+\sqrt{\nu}W_s)\,ds + \sqrt{\nu}\nabla\varphi(x+\sqrt{\nu}W_s)\,dW_s.$$

Notice that substituting the representation (1.2.6) into the equation for $Y^R(0,x)$ in (1.2.5) gives

$$du^R(s, x+\sqrt{\nu}W_s) = \sqrt{\nu}(\nabla u^R + \nabla u^I)(s, x+\sqrt{\nu}W_s) * dW_s + f^I(s, x+\sqrt{\nu}W_s, u(s, x+\sqrt{\nu}W_s))\,ds$$

$$
\begin{aligned}
&+\sqrt{\nu}\nabla u^R(s,x+\sqrt{\nu}W_s)\,dW_s\\
&=\left[\frac{\nu}{2}\Delta(u^I+u^R)(s,x+\sqrt{\nu}W_s)+f^I(s,x+\sqrt{\nu}W_s,u(s,x+\sqrt{\nu}W_s))\right]ds\\
&+\sqrt{\nu}\nabla u^R(s,x+\sqrt{\nu}W_s)\,dW_s,
\end{aligned}
$$

where we have used the relation (4.1.5). Applying Itô's formula to the product $(u^R\varphi)(s,x+\sqrt{\nu}W_s)$ and taking integration w.r.t. (ω,x) on $\Omega\times\mathbb{R}^d$ under $\mathbb{P}\otimes dx$ (where dx denotes the Lebesgue measure), we have for each $t\in[0,T]$,

$$
\begin{aligned}
\int_{\mathbb{R}^d}u^R(t,x)\varphi(x)\,dx&=\mathbb{E}\left[\int_{\mathbb{R}^d}(u^R\varphi)(t,x+\sqrt{\nu}W_t)\,dx\right]\\
&=\int_{\mathbb{R}^d}\mathbb{E}\left[(u^R\varphi)(t,x+\sqrt{\nu}W_t)\right]dx\\
&=\int_{\mathbb{R}^d}\mathbb{E}\left[(u^R\varphi)(T,x+\sqrt{\nu}W_T)-\int_t^T\left(\frac{\nu}{2}\Delta(u^I+u^R)+f^I(\cdot,\cdot,u)\right)\varphi(s,x+\sqrt{\nu}W_s)\,ds\right.\\
&\quad-\int_t^T\nu\left((\nabla u^R)\cdot(\nabla\varphi)+\frac{1}{2}u^R\Delta\varphi\right)(s,x+\sqrt{\nu}W_s)\,ds\\
&\quad\left.-\sqrt{\nu}\int_t^T\left(\varphi\nabla u^R+u^R\nabla\varphi\right)(s,x+\sqrt{\nu}W_s)\,dW_s\right]dx\\
&=\int_{\mathbb{R}^d}\mathbb{E}\left[(u^R\varphi)(T,x+\sqrt{\nu}W_T)-\int_t^T\left(\frac{\nu}{2}\Delta(u^I+u^R)+f^I(\cdot,\cdot,u)\right)\varphi(s,x+\sqrt{\nu}W_s)\,ds\right.\\
&\quad\left.-\int_t^T\nu\left((\nabla u^R)\cdot(\nabla\varphi)+\frac{1}{2}u^R\Delta\varphi\right)(s,x+\sqrt{\nu}W_s)\,ds\right]dx\\
&=\mathbb{E}\left\{\int_{\mathbb{R}^d}\left[(u^R\varphi)(T,x+\sqrt{\nu}W_T)-\int_t^T\left(\frac{\nu}{2}\Delta(u^I+u^R)+f^I(\cdot,\cdot,u)\right)\varphi(s,x+\sqrt{\nu}W_s)\,ds\right.\right.\\
&\quad\left.\left.-\int_t^T\nu\left((\nabla u^R)\cdot(\nabla\varphi)+\frac{1}{2}u^R\Delta\varphi\right)(s,x+\sqrt{\nu}W_s)\,ds\right]dx\right\}\\
&=\int_{\mathbb{R}^d}u^R(T,x)\varphi(x)\,dx-\int_t^T\int_{\mathbb{R}^d}\left(\frac{\nu}{2}\Delta u^I(s,x)+f^I(s,x,u(s,x))\right)\varphi(x)\,dxds,
\end{aligned}
\tag{4.1.6}
$$

where we have used Fubini's theorem, the fact that $\int_{\mathbb{R}^d}g(x+\sqrt{\nu}W_s)dx=\int_{\mathbb{R}^d}g(x)dx,\ \forall\, g\in L^1(\mathbb{R}^d)$, the integration-by-parts formula

$$
\int_{\mathbb{R}^d}\left[(\nabla u^R)(s,x)\cdot(\nabla\varphi)(x)+\frac{1}{2}u^R(s,x)\Delta\varphi(x)\right]dx=\int_{\mathbb{R}^d}\left[-\Delta u^R(s,x)\varphi(x)+\frac{1}{2}\Delta u^R(s,x)\varphi(x)\right]dx
$$

$$= -\frac{1}{2}\int_{\mathbb{R}^d} \Delta u^R(s,x)\varphi(x)\,dx,$$

and the zero-expectation of the stochastic integrals since φ, u^R, $\nabla\varphi$, and ∇u^R are bounded continuous functions. Thus, by the arbitrariness of the test function φ, it follows that the function u^R satisfies the partial differential equation

$$-\partial_t u^R(t,x) = -\frac{\nu}{2}\Delta u^I(t,x) - f^I(t,x,u); \quad u^R(T,x) = G^R(x).$$

In a similar way, we may justify the imaginary part u^I satisfying (1.2.4) and thus (1.2.3). □

4.1.2 Feynman-Kac formula for weak solutions

The associated solution to (1.2.3) in Theorem 4.1.1 is confined to the space $C^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{C})$ which may be replaced by the set of functions valued in Sobolev spaces, for instance, as in Kato (1987).

For each $(k,q)\in\mathbb{N}_0\times[1,\infty)$, we define the k-th Sobolev space $(H^{k,q},\|\cdot\|_{k,q})$ on $\mathbb{R}^d$ and its dual space $(H^{-k,q'},\|\cdot\|_{-k,q'})$, where $q'=\frac{q}{q-1}$. We denote by $C_c^\infty(\mathbb{R}^d)$ (respectively, $C_c^\infty(\mathcal{O})$ for each open set $\mathcal{O}\subset\mathbb{R}^d$) the set of all infinitely differentiable functions with compact support on $\mathbb{R}^d$ (respectively, $\mathcal{O}$). As usual, when $k=0$ we also write $(L^q,\|\cdot\|_q)$ for $(H^{0,q},\|\cdot\|_{0,q})$, and by $\langle\cdot,\,\cdot\rangle$, we denote the duality between the space L^q and its dual $L^{q'}$ for $q\in(1,\infty)$. For simplicity, if a complex, vector or matrix-valued function $v=(v^{jl})_{1\le j\le N,1\le l\le n}$ has $v^{jl}\in H^{k,q}$ for $j=1,\dots,N$, $l=1,\dots,n$ with some $n,N\in\mathbb{N}^+$, we shall write $v\in(H^{k,q})^{N\times n}$ or just simply, $v\in H^{k,q}$ if there is no confusion on dimensions, with $\|v\|_{k,q}=\left(\sum_{l=1}^n\sum_{j=1}^N\left\|v^{jl}\right\|_{k,q}^q\right)^{1/q}$. Let

$$u\in C^1([0,T];H^{1,2})\cap C([0,T];H^{-1,2}) \text{ and } f(\cdot,\cdot,u)\in L^p(0,T;L^q) \text{ for some } p,q\ge 1, \quad (4.1.7)$$

and suppose that such a function $u:\,[0,T]\times\mathbb{R}^d\to\mathbb{C}$ is a weak solution to (1.2.3), i.e., for any $\varphi\in C_c^\infty(\mathbb{R}^d)$, it holds that for each $t\in[0,T]$,

$$\begin{cases} \langle u^R(t),\,\varphi\rangle = \langle G^R,\,\varphi\rangle + \displaystyle\int_t^T \langle\frac{\nu}{2}\nabla u^I(s),\,\nabla\varphi\rangle ds - \int_t^T \langle f^I(s,u),\,\varphi\rangle\,ds, \\ \langle u^I(t),\,\varphi\rangle = \langle G^I,\,\varphi\rangle - \displaystyle\int_t^T \langle\frac{\nu}{2}\nabla u^R(s),\,\nabla\varphi\rangle ds + \int_t^T \langle f^R(s,u),\,\varphi\rangle\,ds. \end{cases} \quad (4.1.8)$$

Due to the lack of continuous differentiability, one cannot apply Itô's formula to the composition $u(t, x + \sqrt{\nu}W_t)$. Instead, one may apply the Itô-Kunita-Wentzell-Krylov formula (Krylov, 2010, Theorem 1) to distribution-valued functions and it yields that

$$\begin{aligned} du^R(s, x + \sqrt{\nu}W_s) &= \Big(\frac{\nu}{2}\Delta u^I(s, x + \sqrt{\nu}W_s) + f^I(s, x + \sqrt{\nu}W_s, u(s, x + \sqrt{\nu}W_s))\Big)\, ds \\ &\quad + \frac{\nu}{2}\Delta u^R(s, x + \sqrt{\nu}W_s)\, ds + \sqrt{\nu}\nabla u^R(s, x + \sqrt{\nu}W_s)\, dW_s \\ &= \Big(f^I(\cdot, \cdot, u) + \frac{\nu}{2}(\Delta u^I + \Delta u^R)\Big)(s, x + \sqrt{\nu}W_s)\, ds + \sqrt{\nu}\nabla u^R(s, x + \sqrt{\nu}W_s)\, dW_s, \end{aligned}$$

which holds in the distributional sense as in (4.1.8).

Remark 4.1.1. *For each $g \in L^2(0, T; (L^2)^d)$, the compositions like $g(s, x + \sqrt{\nu}W_s)$ make senses $d\bar{\mathbb{P}} \otimes dt \otimes dx$-a.e. by (Barles & Lesigne, 1997, Theorem 14.3) (see also (Delbaen* et al.*, 2015, Lemma 3.1)) and the divergence $\nabla \cdot g(s, x + \sqrt{\nu}W_s)$ may be understood as $H^{-1,2}$-valued process. In view of the probabilistic interpretation for the divergence (see (Stoica, 2003, Lemma 3.1)), we have the equivalent representation for the integral $*dW_t$, i.e., for each $0 \le t \le s \le T$,*

$$\begin{aligned} \int_t^s g(\tau, x + \sqrt{\nu}W_\tau) * dW_\tau &= \frac{1}{2}\sum_{j=1}^d \Big(\int_t^s g^j(\tau, x + \sqrt{\nu}W_\tau)\, dW_\tau^j + \int_t^s g^j(\tau, x + \sqrt{\nu}W_\tau)\, \overleftarrow{dW}_\tau^j\Big) \\ &= \frac{\sqrt{\nu}}{2}\int_t^s \nabla \cdot g(\tau, x + \sqrt{\nu}W_\tau)\, d\tau, \end{aligned}$$

with the integral $\overleftarrow{dW}_\tau^j$ being the backward *stochastic integral (see Nualart & Pardoux (1988)) and dW_τ^j the standard Itô integral.*

Recalling $u \in C^1([0, T]; H^{1,2})$, we have by Remark 4.1.1 that

$$\begin{aligned} du^R(s, x + \sqrt{\nu}W_s) &= f^I(s, x + \sqrt{\nu}W_s, u(s, x + \sqrt{\nu}W_s))\, ds + \sqrt{\nu}(\nabla u^I + \nabla u^R)(s, x + \sqrt{\nu}W_s) * dW_s \\ &\quad + \sqrt{\nu}\nabla u^R(s, x + \sqrt{\nu}W_s)\, dW_s. \end{aligned}$$

By relation (1.2.6), straightforward substitution gives

$$dY_s^R(0, x) = \sqrt{\nu}(Z_s^R(0, x) + Z_s^I(0, x)) * dW_s + f^I(s, \sqrt{\nu}W_s + x, Y_s(0, x))\, ds + \sqrt{\nu}Z_s^R(0, x)\, dW_s.$$

Analogously, we justify the second stochastic differential equation in (1.2.5) for the imaginary part.

On the other hand, the converse holds in a similar way to (ii) of Theorem 4.1.1.

To sum up, we reach the following result:

Theorem 4.1.2. *(i) Suppose that u satisfying* (4.1.7) *is a weak solution to the Schrödinger equation* (1.2.3)*. Then the pair $(Y_s(t,x), Z_s(t,x))$ defined through* (1.2.6) *is a solution to BSDE* (1.2.5) *for a.e. $x \in \mathbb{R}^d$.*

(ii) If the pair (Y, Z) with $Y_s(t,x) = Y_s^R(t,x) + iY_s^I(t,x)$ and $Z_s(t,x) = Z_s^R(t,x) + iZ_s^I(t,x)$ for $0 \le t \le s \le T$, $x \in \mathbb{R}^d$ satisfies BSDE (1.2.5) *for a.e. $x \in \mathbb{R}^d$ and there exists a function u satisfying* (4.1.7) *and the relation* (1.2.6)*, then the function u is a solution to the Schrödinger equation* (1.2.3)*.*

Remark 4.1.2. *In the above theorem, a Feynman-Kac formula is established between weak solutions of the Schrödinger equation* (1.2.3) *and the associated BSDE* (1.2.5)*. The integrability of $f(\cdot,\cdot,u)$ in* (4.1.7) *helps to ensure that the corresponding integrals in* (4.1.8) *and BSDE* (1.2.5) *are well-defined, and at the same time, it allows f to be nonlinear and of polynomial growth in u (see Kato (1987)).*

4.1.3 Open questions

In Theorems 4.1.1 and 4.1.2, we presuppose the existence of the function $u(t,x)$, either as a classical or weak solution of the Schrödinger equation (1.2.3), or as a function representing the solution pair (Y, Z) of BSDE (1.2.5). In fact, the Schrödinger equation has been extensively studied, as seen in references such as Berezin & Shubin (2012); Colliander *et al.* (2008); Kato (1987), which discuss the existence and uniqueness of solutions. Using the Feynman-Kac formula in Theorems 4.1.1 and 4.1.2, the existence and uniqueness of the solution to the Schrödinger equation (1.2.3) implies the unique existence of the solution to BSDE (1.2.5).

The open question is the converse: without relying on existing well-posedness results of the associated Schrödinger equations, how can we establish the existence and uniqueness of the solution to BSDE (1.2.5), from which one may derive the unique existence of solutions to the associated Schrödinger equations?

In what follows, we explore BSDE (1.2.5) within the framework of two established solution theories for BSDEs to better understand the associated challenges.

On one hand, in comparison to the standard BSDE theory developed by Pardoux and Peng Pardoux & Peng (1990), the integral terms

$$\int_{\tau}^{T} g(s, Z_s(t,x)) * d(\sqrt{\nu} W_s), \quad t \leq \tau \leq T, \tag{4.1.9}$$

need special consideration. Indeed, such integrals are utilized for the interpretation of divergence terms of parabolic equations; see Barles & Lesigne (1997); Stoica (2003). The well-posedness of the associated BSDEs involving terms (4.1.9) hinges on the function g being Lipschitz-continuous with respect to Z. Specifically, there exists $\alpha \in (0,1)$ such that $|g(s,z_1) - g(s,z_2)| \leq \alpha |z_1 - z_2|,\ \forall\, z_1, z_2 \in \mathbb{R}^d,\ s \in [0,T]$. This condition mandates that the Lipschitz constant α is strictly less than 1, which is essential for deriving certain estimates of Z in the solution theory; refer to Stoica (2003) for more details. However, in the context of BSDE (1.2.5), we have $g(s, Z_s(t,x)) = Z_s^R(t,x) \pm Z_s^I(t,x),\ t \leq \tau \leq s \leq T$, where the function g remains Lipschitz-continuous in Z, but with a Lipschitz constant $\alpha \geq 1$. This poses a significant challenge in estimating Z.

On the other hand, BSDE (1.2.5) can be interpreted as a $\mathbb{C}$-valued second order BSDE. In fact, inspired by the work of Cheridito *et al.* (2007); Soner *et al.* (2012), we express Z as an Itô process satisfying: $dZ_s(t,x) = \sqrt{\nu}\Gamma_s(t,x)\, dW_s + A_s(t,x)\, ds,\ t \leq s \leq q$. By Remark 4.1.1, we have: for $t \leq \tau \leq T$,

$$\begin{aligned}\int_{\tau}^{T} \sqrt{\nu}(Z_s^R(t,x) \pm Z_s^I(t,x)) * dW_s &= \int_{\tau}^{T} \frac{\sqrt{\nu}}{2} \nabla \cdot (\sqrt{\nu} Z_s^R(t,x) \pm \sqrt{\nu} Z_s^I(t,x))\, ds \\ &= \int_{\tau}^{T} \frac{\nu}{2} \mathrm{tr}(\Gamma_s^R(t,x) \pm \Gamma_s^I(t,x))\, ds, \end{aligned} \tag{4.1.10}$$

which, substituted into the (1.2.5), results in a second order BSDE. In this scenario, $Y_s(t,x)$ is complex-valued, and, to the best of our knowledge, such a BSDE has not been previously studied or addressed in any existing theoretical framework.

4.2 Algorithm and Numerical Experiment

4.2.1 An algorithm

Our BSDE (1.2.5) can be rewritten as the following FBSDE:

$$\begin{cases} d\mathcal{X}_s(t,x) = \sqrt{\nu} dW_s, \quad \mathcal{X}_t(t,x) = x; \\ -dY_s^R(t,x) = -\sqrt{\nu}(Z_s^R(t,x) + Z_s^I(t,x)) * dW_s \\ \qquad\qquad - f^I\left(s, \mathcal{X}_s(t,x), Y_s(t,x)\right) ds - \sqrt{\nu} Z_s^R(t,x)\, dW_s; \\ -dY_s^I(t,x) = \sqrt{\nu}(Z_s^R(t,x) - Z_s^I(t,x)) * dW_s \\ \qquad\qquad + f^R\left(s, \mathcal{X}_s(t,x), Y_s(t,x)\right) ds - \sqrt{\nu} Z_s^I(t,x)\, dW_s; \\ Y_T(t,x) = G(\mathcal{X}_T(t,x)), \end{cases} \tag{4.2.1}$$

for all $(s,x) \in [t,T] \times \mathbb{R}^d$. It is related to the solution u of equation (1.2.3) by

$$Y_s(t,x) = u(s, \mathcal{X}_s(t,x)), \quad Z_s(t,x) = \nabla u(s, \mathcal{X}_s(t,x)), \tag{4.2.2}$$

thus the solution $u(s,x)$ would exactly be $Y_s(s,x)$. In our BSDE system (1.2.5), the presence of both the Fisk-Stratonovich integral and the Itô integral complicates the numerical computation using any conventional methods (see, for instance, Zhang (2004)). Additionally, the deep learning method proposed in Han *et al.* (2018) is not capable of handling this atypical form of BSDEs. However, the following will numerically demonstrate the effectiveness of our proposed BSDE approach in solving the Schrödinger equations (1.2.3), using a scheme inspired by Huré *et al.* (2020).

For function approximations, in the subsequent presentation of the algorithm, the execution

of numerical experiments, and the proof of convergence analysis, we employ neural network approximations for the unknown functions. This approach allows for the extension of both the algorithm and convergence analysis to high-dimensional time-dependent Schrödinger equations (cf. Isozaki (2004)); in this context, deep neural networks play a crucial role in tackling the curse of dimensionality (cf. Han *et al.* (2018)).

Let $\mathcal{N}_{d,d_1,L,m}$ denote the set of functions represented by feedforward neural networks with $L+1$ layers (where integer $L>1$), d-dimensional inputs, d_1-dimensional outputs, and m neurons in each hidden layer. Typically, we represent the parameters for each neural network function by $\theta \in \mathbb{R}^{N_m}$ where $N_m = d(1+m)+m(1+m)(L-2)+m\,(1+d_1)$ and ϱ denotes the activation function.

For any $M \in \mathbb{N}^+$, we introduce the time grid $\pi := \big\{t_0, t_1, \ldots, t_M | 0 =: t_0 < t_1 < \ldots < t_M := T\big\}$, with modulus $|\pi| = \max_{j=0,\ldots,M-1} \Delta t_j, \Delta t_j := t_{j+1} - t_j$. Without loss of generality, we assume (t,x) to be $(0,x_0)$ for some fixed $x_0 \in \mathbb{R}^d$. We employ the Euler scheme to discretize our forward process in (4.2.1) as $X_{t_{j+1}} = X_{t_j} + \sqrt{\nu}\Delta W_{t_j}, \quad j = 0,\ldots,M-1$, $X_0 = x_0$, where we set $\Delta W_{t_j} := W_{t_{j+1}} - W_{t_j}$. To alleviate notations, we omit the dependence of $X = X^\pi(0,x_0)$ on the time grid π, initial time 0 and initial data x_0 as there is no ambiguity. Define

$$
\begin{aligned}
&F^R(t,x,u^R,u^I,z^R_{t+1},z^I_{t+1},z^R_t,z^I_t,\Delta_t,\Delta_w)\\
&:= u^R + \frac{\sqrt{\nu}}{2}(z^R_{t+1}+z^I_{t+1})\Delta_w + \frac{\sqrt{\nu}}{2}(z^R_t - z^I_t)\Delta_w + f^I(t,x,u^R,u^I)\Delta_t,\\
&F^I(t,x,u^R,u^I,z^R_{t+1},z^I_{t+1},z^R_t,z^I_t,\Delta_t,\Delta_w)\\
&:= u^I - \frac{\sqrt{\nu}}{2}(z^R_{t+1}-z^I_{t+1})\Delta_w + \frac{\sqrt{\nu}}{2}(z^R_t + z^I_t)\Delta_w - f^R(t,x,u^R,u^I)\Delta_t.
\end{aligned}
$$

Here and in what follows, we do not distinguish the writings $f(t,x,u)$ and $f(t,x,u^R,u^I)$. From the equations (4.2.1) and (4.2.2), for $j=1,\ldots,M$, we have that

$$
\begin{aligned}
&u^R(t_{j+1},X_{t_{j+1}})\\
\approx &F^R\Big(t_j,X_{t_j},u^R(t_j,X_{t_j}),u^I(t_j,X_{t_j}),z^R(t_{j+1},X_{t_{j+1}}),z^I(t_{j+1},X_{t_{j+1}}),z^R(t_j,X_{t_j}),z^I(t_j,X_{t_j}),\Delta t_j,\Delta W_{t_j}\Big);
\end{aligned}
$$

$$
\begin{aligned}
&u^I(t_{j+1}, X_{t_{j+1}}) \\
\approx &F^I\Big(t_j, X_{t_j}, u^R(t_j, X_{t_j}), u^I(t_j, X_{t_j}), z^R(t_{j+1}, X_{t_{j+1}}), z^I(t_{j+1}, X_{t_{j+1}}), z^R(t_j, X_{t_j}), z^I(t_j, X_{t_j}), \Delta t_j, \Delta W_{t_j}\Big).
\end{aligned}
$$

Denote $\theta = (\xi, \eta)$. For each $j = 0, \ldots, M-1$, we set up neural networks $\mathcal{U}^R_{t_j}$, $\mathcal{U}^I_{t_j}$, $\mathcal{Z}^R_{t_j}$, $\mathcal{Z}^I_{t_j} \in \mathcal{N}_{d,d_1,L,m}$ to approximate $u^R(t_j, \cdot)$, $u^I(t_j, \cdot)$, $z^R(t_j, \cdot)$, $z^I(t_j, \cdot)$ respectively. The algorithm is stated below.

Algorithm 1 Deep Learning-based scheme for the Schrödinger equation (1.2.3)

Initialize: $\widehat{\mathcal{U}}^R_{t_M} := G^R(\cdot)$, $\widehat{\mathcal{U}}^I_{t_M} := G^I(\cdot)$, $\widehat{\mathcal{Z}}^R_{t_M} := \nabla G^R(\cdot)$, $\widehat{\mathcal{Z}}^I_{t_M} := \nabla G^I(\cdot)$ $j = M-1$ to 0

Given $\widehat{\mathcal{U}}^R_{t_{j+1}}$, $\widehat{\mathcal{U}}^I_{t_{j+1}}$, $\widehat{\mathcal{Z}}^R_{t_{j+1}}$, $\widehat{\mathcal{Z}}^I_{t_{j+1}}$, minimize

$$
\begin{cases}
\begin{aligned}
L_j(\theta) := &\mathbb{E}\Big|\widehat{\mathcal{U}}^R_{t_{j+1}}\left(X_{t_{j+1}}\right) - F^R\big(t_j, X_{t_j}, \mathcal{U}^R_{t_j}(X_{t_j};\xi), \mathcal{U}^I_{t_j}(X_{t_j};\xi), \widehat{\mathcal{Z}}^R_{t_{j+1}}(X_{t_{j+1}}), \widehat{\mathcal{Z}}^I_{t_{j+1}}(X_{t_{j+1}}), \\
&\qquad \mathcal{Z}^R_{t_j}(X_{t_j};\eta), \mathcal{Z}^I_{t_j}(X_{t_j};\eta), \Delta t_j, \Delta W_{t_j}\big)\Big|^2 \\
&+ \mathbb{E}\Big|\widehat{\mathcal{U}}^I_{t_{j+1}}\left(X_{t_{j+1}}\right) - F^I\big(t_j, X_{t_j}, \mathcal{U}^R_{t_j}(X_{t_j};\xi), \mathcal{U}^I_{t_j}(X_{t_j};\xi), \widehat{\mathcal{Z}}^R_{t_{j+1}}(X_{t_{j+1}}), \widehat{\mathcal{Z}}^I_{t_{j+1}}(X_{t_{j+1}}), \\
&\qquad \mathcal{Z}^R_{t_j}(X_{t_j};\eta), \mathcal{Z}^I_{t_j}(X_{t_j};\eta), \Delta t_j, \Delta W_{t_j}\big)\Big|^2; \\
\theta^*_j = (\xi^*_j, \eta^*_j) \in &\arg\min_{\theta \in \mathbb{R}^{N_m}} L_j(\theta).
\end{aligned}
\end{cases}
$$

Update $\widehat{\mathcal{U}}^R_{t_j} = \mathcal{U}^R_{t_j}\left(.;\xi^*_j\right)$, $\widehat{\mathcal{U}}^I_{t_j} = \mathcal{U}^I_{t_j}\left(.;\xi^*_j\right)$, $\widehat{\mathcal{Z}}^R_{t_j} = \mathcal{Z}^R_{t_j}\left(.;\eta^*_j\right)$, and $\widehat{\mathcal{Z}}^I_{t_j} = \mathcal{Z}^I_{t_j}\left(.;\eta^*_j\right)$.

Remark 4.2.1. *The above algorithm is inspired by, yet distinct from, the work of Huré, Pham, and Warin in Huré* et al. *(2020). Indeed, due to the presence of integrals with $*dW_t$ in BSDE* (1.2.5)*, at each time step $[t_j, t_{j+1}]$, the iteration functions F^R and F^I depend on the values of $Z_{t_{j+1}}$ and $\widehat{\mathcal{Z}}_{t_{j+1}}$ at time t_{j+1}, a dependence that is absent in non-Schrödinger-type parabolic cases (see, for instance, Beck* et al. *(2019); E* et al. *(2017); Han* et al. *(2018); Huré* et al. *(2020)).*

4.2.2 Numerical experiments

We conduct numerical experiments in both low-dimensional (1D) and high-dimensional (48D) settings, addressing both linear and nonlinear scenarios. Due to the oscillatory behavior of

Schrödinger equation solutions, standard activation functions such as Tanh, Sigmoid, and Softplus do not perform well in capturing these intricate patterns. Nonetheless, these activations remain important in less oscillatory regions, where the solution is less curved and rapid variations are absent. To this end, for each $j = 0, \ldots, M-1$, we implement a common neural architecture for the networks $\mathcal{U}_{t_j}^R$, $\mathcal{U}_{t_j}^I$, $\mathcal{Z}_{t_j}^R$, and $\mathcal{Z}_{t_j}^I$, purposefully designed to accommodate the specific characteristics of Schrödinger equation solutions. Each network is constructed using a gated dual-branch framework that integrates two parallel nonlinear transformation paths. One branch applies a sequence of fully connected layers with hyperbolic tangent (Tanh) activations, while the other, inspired by Sitzmann *et al.* (2020), mirrors this structure but employs sinusoidal activation functions. This configuration equips the network to capture a wide spectrum of input features, from less curved regions to highly oscillatory or periodic behaviors.

Each branch contains two hidden layers: the first linear layer maps the input of dimension d to a hidden width of either $d+h$ (in the case of $\mathcal{U}_{t_j}^R$ and $\mathcal{U}_{t_j}^I$) or $d \times h$ (for $\mathcal{Z}_{t_j}^R$ and $\mathcal{Z}_{t_j}^I$), where h is a tunable hidden expansion parameter. This is followed by a Softplus activation and a second linear layer of the same width, ending with either a Tanh or sinusoidal activation.

The outputs of the two branches are blended using a soft gating mechanism, implemented as a learnable single-neuron feedforward layer followed by a sigmoid activation. This mechanism produces a scalar gating value in $(0, 1)$ for each input, which is used to compute a convex combination of the two branch outputs. The resulting gated representation is then passed through a final fusion layer, consisting of a single linear transformation. For $\mathcal{U}_{t_j}^R$ and $\mathcal{U}_{t_j}^I$, the fusion layer maps the hidden features to a scalar output, while for $\mathcal{Z}_{t_j}^R$ and $\mathcal{Z}_{t_j}^I$, it projects to a vector in $\mathbb{R}^d$. A diagram of the architecture is provided to visually illustrate the computational structure and data flow.

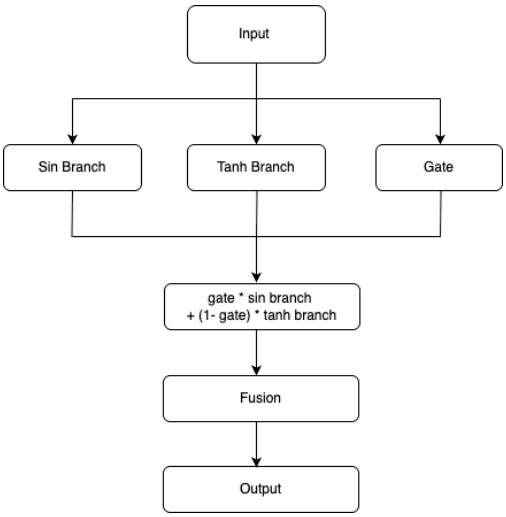


Figure 4.1: Illustration of the architecture of the proposed neural network.

Our code is available at GitHub: `https://github.com/HenryCHEUNG7373/Schrodinger_Feynman_Kac`.

Example We consider the following linear Schrodinger equation in $d = 1$:

$$i\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x), \quad t \leq T; \quad u(T,x) = e^{i\sqrt{2}x}, \tag{4.2.3}$$

which admits the solution $u(t,x) := e^{i(\sqrt{2}x-(T-t))}$. We choose $T = 0.5$, $M = 25$, batch size to be 4096 and train each networks with 30 epochs. The results are plotted in the following figures.

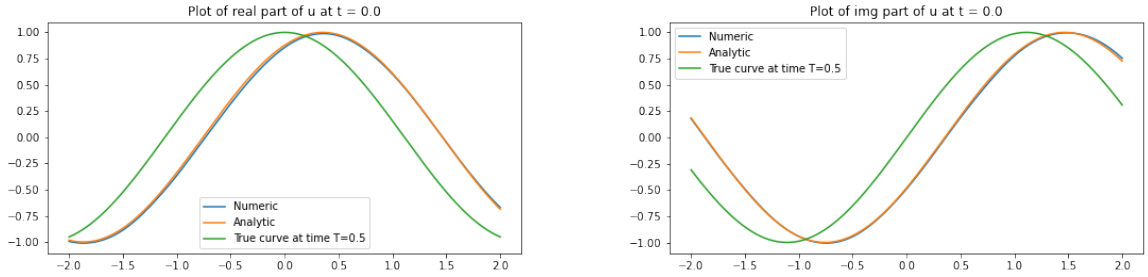


Figure 4.2: Real & imaginary parts of $u(0,\cdot)$ and its estimate.

Table 4.1: Average value and standard deviation of $u(0,0)$ over 10 independent runs.

Parts	True value	Average estimated value	Standard deviation
Real	0.8776	0.8630	0.0122
Imaginary	-0.4794	-0.4728	0.0115

□

Example Let $d = 1$ and consider the following nonlinear Schrodinger equation:

$$i\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + |u(t,x)|^2 u(t,x), \quad t \leq T; \quad u(T,x) = \operatorname{sech}(x)e^{ix}, \tag{4.2.4}$$

which has solution $u(t,x) := \operatorname{sech}(x-(T-t))e^{ix}$. Due to high nonlinearity, we take a big $M = 64$. For other parameters, we choose $T = 0.5$, batch size to be 4096 and train each networks with 10 epochs. The results are plotted in the following figures.

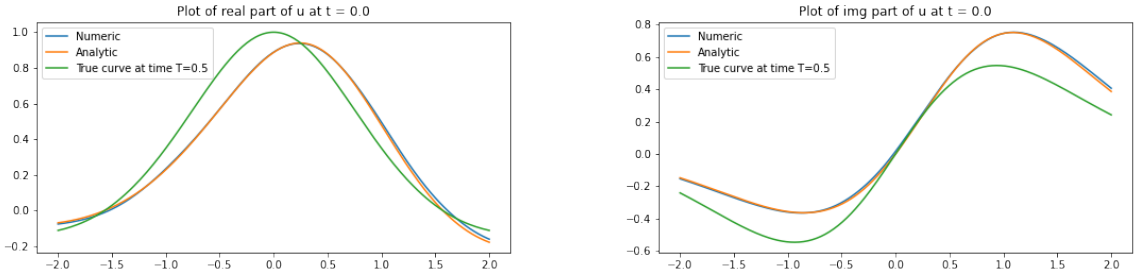


Figure 4.3: Real & imaginary parts of $u(0,\cdot)$ and its estimate.

Table 4.2: Average value and standard deviation of $u(0,0)$ over 10 independent runs.

Parts	True value	Average estimated value	Standard deviation
Real	0.8868	0.8723	0.0174
Imaginary	0	0.0164	0.0142

□

The true curves at time $T = 0.5$ correspond to the evaluations of the terminal functions, specifically $e^{i\sqrt{2}x}$ in the linear case and $\text{sech}(x)e^{ix}$ in the nonlinear case. The figures above indicate that the computed numerical solution closely approximates the analytical solution at time $t = 0$, while both remain noticeably distinct from the prescribed terminal conditions. This is further supported by Tables 4.2.2 and 4.2.2, where, combining the real and imaginary parts, the relative L^2 error, calculated as $\sqrt{\frac{(\hat{x}_{\text{real}}-x_{\text{real}})^2+(\hat{x}_{\text{img}}-x_{\text{img}})^2}{x_{\text{real}}^2+x_{\text{img}}^2}}$ for the true value $x = x_{\text{real}} + ix_{\text{img}}$ and the predicted value $\hat{x} = \hat{x}_{\text{real}} + i\hat{x}_{\text{img}}$, is approximately 1.6% in the linear case and around 2.5% in the nonlinear case.

In addition to the above one-dimensional examples, we further provide the following linear and nonlinear numerical results in high-dimensional spaces.

Example Let $d = 48$ and consider the linear Schrödinger equation

$$i\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x), \quad t \le T; \quad u(T,x) = e^{\sqrt{\frac{2}{d}} i \sum_{j=1}^{d} x_j}. \tag{4.2.5}$$

This system admits the exact solution $u(t,x) = e^{i\left(\sqrt{\frac{2}{d}}\sum_{j=1}^{d} x_j - (T-t)\right)}$. We choose $T = 0.5, M = 25$, batch size to be 245760 and train each networks with 30 epochs. The results are plotted in the following figures at $t = 0$ and $x = (x_1, 0, \cdots, 0)$.

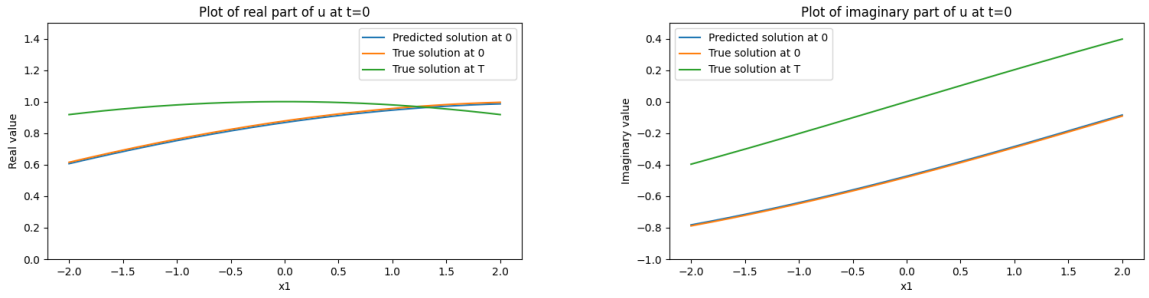


Figure 4.4: Real & imaginary parts of $u(0,x)$ and its estimate with $x = (x_1, 0, \cdots, 0)$.

Parts	True value	Average estimated value	Standard deviation
Real	0.8776	0.8649	0.0074
Imaginary	-0.4794	-0.4700	0.0023

Table 4.3: Average value and standard deviation of $u(0, \vec{0})$ over 10 independent runs

□

Example Let $d = 48$ and denote $\mathcal{S} := \frac{1}{d}\sum_{j=1}^{d} x_j$. Consider the nonlinear Schrödinger equation

$$i\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + f(t,x,u), \quad t \leq T; \quad u(T,x) = e^{i\mathcal{S}}\text{sech}(\mathcal{S}), \tag{4.2.6}$$

with the nonlinear term

$$f(t,x,u) := |u|^2 u + \left(1 + \frac{1}{d}\text{sech}^2(\mathcal{S}) + \frac{i}{d}\tanh(\mathcal{S})\right) e^{i[T-t+\mathcal{S}]}\text{sech}(\mathcal{S}) - e^{i[T-t+\mathcal{S}]}\text{sech}^3(\mathcal{S}).$$

This system admits the exact solution $u(t,x) = e^{i(T-t+\mathcal{S})}\text{sech}(\mathcal{S})$. We choose $T = 0.5, M = 50$, batch size to be 245760 and train each networks with 30 epochs. The results are plotted in the following Figure 4.5 at $t = 0$ and $x = (x_1, 0, \cdots, 0)$. We further provide Figure 4.6 at $t = 0$ and $x = (x_1, \cdots, x_1)$ to show that our algorithm is capable of approximating solutions with more curving behaviors in high-dimensional nonlinear problems.

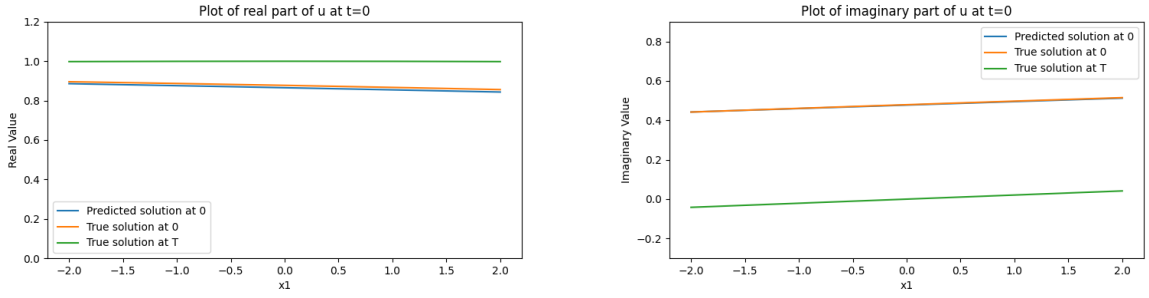


Figure 4.5: Real & imaginary parts of $u(0,x)$ and its estimate with $x = (x_1, 0, \cdots, 0)$.

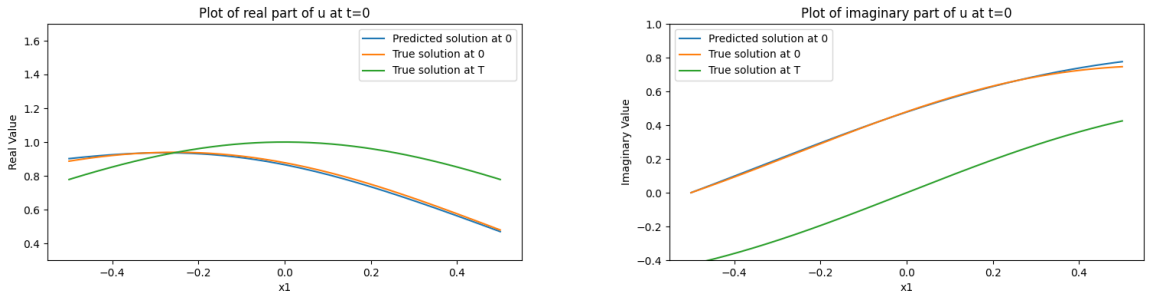


Figure 4.6: Real & imaginary parts of $u(0, x)$ and its estimate with $x = (x_1, \cdots, x_1)$.

Parts	True value	Average estimated value	Standard deviation
Real	0.8776	0.8662	8.96×10^{-4}
Imaginary	0.4794	0.4773	4.40×10^{-4}

Table 4.4: Average value and standard deviation of $u(0, \vec{0})$ over 10 independent runs.

□

From the above figures, again we observe that our algorithm fits the true solution reasonably well even in the higher dimensional cases with $d = 48$. Table 4.2.2 and 4.2.2 further supports our numerical results where the relative L^2 error can be computed in the same way as in the one-dimensional cases. The linear Example 4.2.2 exhibits a relative error around 1.58% and in the nonlinear Example 4.2.2, it is approximately 1.16%.

4.3 Convergence Analysis of Algorithm 1

4.3.1 Assumption and notations

Throughout this section, we assume the following:

(i) The function u is a *bounded* solution to the Schrödinger equation (1.2.3) with its derivatives $\partial_t u$, $\partial_x u$, $\partial_{xx} u$, $\partial_{xxx} u$, $\partial_{xxxx} u$, $\partial_t \partial_x u$, $\partial_t \partial_{xx} u$ well defined as

continuous functions.

(ii) The derivatives of u in (i) above belong to $L^\infty([0,T];L^2(\mathbb{R}^d;\mathbb{C}))$ with their norms less than K.

(iii) There exists $L>0$ such that for all $x,x'\in\mathbb{R}^d$, $y,y'\in\mathbb{C}$, $t,t'\in[0,T]$, and $\phi\in C(\mathbb{R}^d;\mathbb{C})$,

$$|f(t,x,y)-f(t,x,y')|\le L(|y-y'|),$$
$$|f(t,x,\phi(x))-f(t',x',\phi(x))|\le L\left(\sqrt{\rho(|t-t'|)}+|x-x'|\right)|\phi(x)|,$$

where $\rho:[0,\infty)\to[0,\infty)$ is a continuous increasing function with $\rho(0)=0$.

The above conditions (i)–(iii) are assumed for simplicity and to avoid cumbersome arguments. Relaxations of these conditions are feasible. Here, we assume the Lipschitz-continuity of f in y in condition (iii). In certain nonlinear cases, such as in (4.2.4), the function f takes the form $f(t,x,u^R,u^I)=|u|^2u$. When confined to bounded solutions as assumed in (i), we may truncate this function by replacing it with $f^N(u)=\Phi^N(|u|^2u)$, where $N>0$ is sufficiently large and Φ^N is a smooth, compactly supported function that satisfies $\Phi^N(x)=x$ for $|x|\le N$. Also, the solution to (4.2.4) is explicitly given as $u=\mathrm{sech}(x-(T-t))e^{ix}$ which obviously satisfies both (i) and (ii). Therefore, the example (4.2.4) actually fits in well with the above assumption.

To avoid cumbersome notations, we further assume that $d=1$, while the multi-dimensional cases follow analogously. Also, w.l.o.g., the time partition is evenly spaced, i.e., $t_{j+1}-t_j=\delta t$ for $j=0,\dots,M-1$.

Inspired by Huré *et al.* (2020), we first introduce an auxiliary system. For each j, given

$\widehat{\mathcal{U}}^R_{t_{j+1}}$, $\widehat{\mathcal{U}}^I_{t_{j+1}}$, $\widehat{\mathcal{Z}}^R_{t_{j+1}}$ and $\widehat{\mathcal{Z}}^I_{t_{j+1}}$, we define the following:

$$\begin{cases} \widehat{\mathcal{V}}^R_{t_j} :=\mathbb{E}_j\widehat{\mathcal{U}}^R_{t_{j+1}} - \dfrac{\sqrt{\nu}}{2}\mathbb{E}_j\left[\left(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}\right)\delta W_{t_j}\right] - f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t, \\ \widehat{\mathcal{V}}^I_{t_j} :=\mathbb{E}_j\widehat{\mathcal{U}}^I_{t_{j+1}} + \dfrac{\sqrt{\nu}}{2}\mathbb{E}_j\left[\left(\widehat{\mathcal{Z}}^R_{t_{j+1}} - \widehat{\mathcal{Z}}^I_{t_{j+1}}\right)\delta W_{t_j}\right] + f^R(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t, \\ \widehat{\mathcal{W}}^R_{t_j} :=\dfrac{1}{\delta t\sqrt{\nu}}\mathbb{E}_j\left[\left(\widehat{\mathcal{U}}^R_{t_{j+1}} + \widehat{\mathcal{U}}^I_{t_{j+1}}\right)\delta W_{t_j}\right] - \dfrac{1}{\delta t}\mathbb{E}_j\left[\widehat{\mathcal{Z}}^I_{t_{j+1}}(\delta W_{t_j})^2\right], \\ \widehat{\mathcal{W}}^I_{t_j} :=\dfrac{1}{\delta t\sqrt{\nu}}\mathbb{E}_j\left[\left(\widehat{\mathcal{U}}^I_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}}\right)\delta W_{t_j}\right] + \dfrac{1}{\delta t}\mathbb{E}_j\left[\widehat{\mathcal{Z}}^R_{t_{j+1}}(\delta W_{t_j})^2\right], \end{cases} \tag{4.3.1}$$

where $\mathbb{E}_j$ denotes the conditional expectation given $W_{t_1}, \ldots, W_{t_j}$. Since f is Lipschitz, by a standard fixed point argument the above system has unique solution for δt small enough. From the Markov property of the involved processes, there exist deterministic functions $\hat{v}^R_j$, $\hat{v}^I_j$, $\hat{w}^R_j$ and $\hat{w}^I_j$ such that

$$\widehat{\mathcal{V}}^R_{t_j} = \hat{v}^R_j(X_{t_j}), \quad \widehat{\mathcal{V}}^I_{t_j} = \hat{v}^I_j(X_{t_j}), \quad \widehat{\mathcal{W}}^R_{t_j} = \hat{w}^R_j(X_{t_j}), \quad \widehat{\mathcal{W}}^I_{t_j} = \hat{w}^I_j(X_{t_j}), \quad j = 0, \ldots, M-1. \tag{4.3.2}$$

Further, by martingale representation theorem there exist integrable processes $\widehat{Z}^R_t$, $\widehat{Z}^I_t$ such that

$$\begin{cases} \widehat{\mathcal{U}}^R_{t_{j+1}} =\widehat{\mathcal{V}}^R_{t_j} + f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t + \dfrac{\sqrt{\nu}}{2}(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j} - \dfrac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j} + \widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j} + \sqrt{\nu}\displaystyle\int_{t_j}^{t_{j+1}} \widehat{Z}^R_s dW_s, \\ \widehat{\mathcal{U}}^I_{t_{j+1}} =\widehat{\mathcal{V}}^I_{t_j} - f^R(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t - \dfrac{\sqrt{\nu}}{2}(\widehat{\mathcal{Z}}^R_{t_{j+1}} - \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j} + \dfrac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j} - \widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j} + \sqrt{\nu}\displaystyle\int_{t_j}^{t_{j+1}} \widehat{Z}^I_s dW_s. \end{cases} \tag{4.3.3}$$

In this way, it follows straightforwardly that

$$\widehat{\mathcal{W}}^R_{t_j} = \frac{1}{\delta t}\mathbb{E}_j\int_{t_j}^{t_{j+1}} \widehat{Z}^R_s ds, \quad \widehat{\mathcal{W}}^I_{t_j} = \frac{1}{\delta t}\mathbb{E}_j\int_{t_j}^{t_{j+1}} \widehat{Z}^I_s ds. \tag{4.3.4}$$

On the other hand, we define the L^2-projections of the processes Z^R and Z^I in the true solution to BSDE (1.2.5):

$$\overline{Z}^R_{t_j} := \frac{1}{\delta t}\mathbb{E}_j\int_{t_j}^{t_{j+1}} Z^R_s ds, \quad \overline{Z}^I_{t_j} := \frac{1}{\delta t}\mathbb{E}_j\int_{t_j}^{t_{j+1}} Z^I_s ds. \tag{4.3.5}$$

Note that from (1.2.6) and (4.3.2), the random variables $Y_{t_j}^R$, $\widehat{\mathcal{V}}_{t_j}^R$, $Z_{t_j}^R$, $\widehat{Z}_{t_j}^R$, $\widehat{\mathcal{W}}_{t_j}^R$ and their imaginary counterparts can be interpreted as deterministic functions, with $X_{t_j} = \mathcal{X}_{t_j} = x + \sqrt{\nu}W_{t_j}$ as input.

Denote $h^R := \nabla u^R(t,x)$, $h^I := \nabla u^I(t,x)$ and recall the relation (1.2.6), and the forward dynamics $(\mathcal{X}_s)_{0\le s\le T}$ and its Euler discretization X_s. Define the error terms as follows:

$$
\begin{aligned}
\mathrm{I}_{A,j}^R :=& \frac{\nu}{2}\Big[\partial_x h^R(t_j,\mathcal{X}_{t_j})\delta t - \partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2 + \partial_x h^I(t_j,\mathcal{X}_{t_j})\delta t - \partial_x h^I(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\Big]\\
\mathrm{I}_{B,j}^R :=& \sqrt{\nu}\int_{t_j}^{t_{j+1}} (Z_s^R + Z_s^I) * dW_s - \frac{\sqrt{\nu}}{2}(Z_{t_{j+1}}^R + Z_{t_{j+1}}^I - Z_{t_j}^R - Z_{t_j}^I)\delta W_{t_j} - \mathrm{I}_{A,j}^R\\
\mathrm{II}_{A,j}^R :=& \sqrt{\nu}\int_{t_j}^{t_{j+1}} Z_s^R dW_s - \sqrt{\nu}\,\overline{Z}_{t_j}^R \delta W_{t_j}\\
\mathrm{II}_{B,j}^R :=& \sqrt{\nu}\,\overline{Z}_{t_j}^R \delta W_{t_j} - \sqrt{\nu} Z_{t_j}^R \delta W_{t_j}\\
\mathrm{III}_j^R :=& \int_{t_j}^{t_{j+1}} f^I(s,\mathcal{X}_s,Y_s^R,Y_s^I)ds - f^I(t_j,\mathcal{X}_{t_j},Y_{t_j}^R,Y_{t_j}^I)\delta t\\
\mathrm{IV}_j^R :=& \sqrt{\nu}\int_{t_j}^{t_{j+1}} \widehat{Z}_s^R dW_s - \sqrt{\nu}\widehat{\mathcal{W}}_{t_j}^R \delta W_{t_j}\\
\mathrm{I}_{A,j}^I :=& -\frac{\nu}{2}\Big[\Big(\partial_x h^R(t_j,\mathcal{X}_{t_j})\delta t - \partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\Big) - \Big(\partial_x h^I(t_j,\mathcal{X}_{t_j})\delta t - \partial_x h^I(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\Big)\Big]\\
\mathrm{I}_{B,j}^I :=& -\sqrt{\nu}\int_{t_j}^{t_{j+1}} (Z_s^R - Z_s^I) * dW_s + \frac{\sqrt{\nu}}{2}(Z_{t_{j+1}}^R - Z_{t_{j+1}}^I - Z_{t_j}^R + Z_{t_j}^I)\delta W_{t_j} - \mathrm{I}_{A,j}^I\\
\mathrm{II}_{A,j}^I :=& \sqrt{\nu}\int_{t_j}^{t_{j+1}} Z_s^I dW_s - \sqrt{\nu}\,\overline{Z}_{t_j}^I \delta W_{t_j}\\
\mathrm{II}_{B,j}^I :=& \sqrt{\nu}\,\overline{Z}_{t_j}^I \delta W_{t_j} - \sqrt{\nu} Z_{t_j}^I \delta W_{t_j}\\
\mathrm{III}_j^I :=& -\int_{t_j}^{t_{j+1}} f^R(s,\mathcal{X}_s,Y_s^R,Y_s^I)ds + f^R(t_j,\mathcal{X}_{t_j},Y_{t_j}^R,Y_{t_j}^I)\delta t\\
\mathrm{IV}_j^I :=& \sqrt{\nu}\int_{t_j}^{t_{j+1}} \widehat{Z}_s^I dW_s - \sqrt{\nu}\widehat{\mathcal{W}}_{t_j}^I \delta W_{t_j}.
\end{aligned}
$$

By (1.2.5), for each $j \in \{0, \dots, M-1\}$ the true solution satisfies:

$$\begin{cases} Y^R_{t_{j+1}} - \frac{\sqrt{\nu}}{2}(Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\delta W_{t_j} - \mathrm{I}^R_{B,j} - \mathrm{II}^R_{B,j} - \mathrm{III}^R_j - f^I(t_j, \mathcal{X}_{t_j}, Y^R_{t_j}, Y^I_{t_j})\delta t \\ = Y^R_{t_j} - \frac{\sqrt{\nu}}{2}(Z^R_{t_j} + Z^I_{t_j})\delta W_{t_j} + \sqrt{\nu} Z^R_{t_j}\delta W_{t_j} + \mathrm{II}^R_{A,j} + \mathrm{I}^R_{A,j}, \\ Y^I_{t_{j+1}} + \frac{\sqrt{\nu}}{2}(Z^R_{t_{j+1}} - Z^I_{t_{j+1}})\delta W_{t_j} - \mathrm{I}^I_{B,j} - \mathrm{II}^I_{B,j} - \mathrm{III}^I_j + f^R(t_j, \mathcal{X}_{t_j}, Y^R_{t_j}, Y^I_{t_j})\delta t \\ = Y^I_{t_j} + \frac{\sqrt{\nu}}{2}(Z^R_{t_j} - Z^I_{t_j})\delta W_{t_j} + \sqrt{\nu} Z^I_{t_j}\delta W_{t_j} + \mathrm{II}^I_{A,j} + \mathrm{I}^I_{A,j}. \end{cases} \tag{4.3.6}$$

Meanwhile, (4.3.3) can be represented as

$$\begin{cases} \widehat{\mathcal{U}}^R_{t_{j+1}} - \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j} - f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t = \widehat{\mathcal{V}}^R_{t_j} - \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j} + \widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j} + \sqrt{\nu}\widehat{\mathcal{W}}^R_{t_j}\delta W_{t_j} + \mathrm{IV}^R_j, \\ \widehat{\mathcal{U}}^I_{t_{j+1}} + \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{Z}}^R_{t_{j+1}} - \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j} + f^R(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t = \widehat{\mathcal{V}}^I_{t_j} + \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j} - \widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j} + \sqrt{\nu}\widehat{\mathcal{W}}^I_{t_j}\delta W_{t_j} + \mathrm{IV}^I_j. \end{cases} \tag{4.3.7}$$

Combining equations (4.3.6) and (4.3.7) yields

$$\begin{cases} Y^R_{t_j} - \frac{\sqrt{\nu}}{2}(Z^R_{t_j} + Z^I_{t_j})\delta W_{t_j} + \sqrt{\nu} Z^R_{t_j}\delta W_{t_j} + \mathrm{I}^R_{A,j} + \mathrm{II}^R_{A,j} - \Big[\widehat{\mathcal{V}}^R_{t_j} - \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j} + \widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j} + \sqrt{\nu}\widehat{\mathcal{W}}^R_{t_j}\delta W_{t_j} + \mathrm{IV}^R_j\Big] \\ = Y^R_{t_{j+1}} - \frac{\sqrt{\nu}}{2}(Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\delta W_{t_j} - \mathrm{I}^R_{B,j} - \mathrm{II}^R_{B,j} - \mathrm{III}^R_j - f^I(t_j, \mathcal{X}_{t_j}, Y^R_{t_j}, Y^I_{t_j})\delta t \\ \quad - \Big[\widehat{\mathcal{U}}^R_{t_{j+1}} - \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j} - f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t\Big], \\ Y^I_{t_j} + \frac{\sqrt{\nu}}{2}(Z^R_{t_j} - Z^I_{t_j})\delta W_{t_j} + \sqrt{\nu} Z^I_{t_j}\delta W_{t_j} + \mathrm{I}^I_{A,j} + \mathrm{II}^I_{A,j} - \Big[\widehat{\mathcal{V}}^I_{t_j} + \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j} - \widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j} + \sqrt{\nu}\widehat{\mathcal{W}}^I_{t_j}\delta W_{t_j} + \mathrm{IV}^I_j\Big] \\ = Y^I_{t_{j+1}} + \frac{\sqrt{\nu}}{2}(Z^R_{t_{j+1}} - Z^I_{t_{j+1}})\delta W_{t_j} - \mathrm{I}^I_{B,j} - \mathrm{II}^I_{B,j} - \mathrm{III}^I_j + f^R(t_j, \mathcal{X}_{t_j}, Y^R_{t_j}, Y^I_{t_j})\delta t \\ \quad - \Big[\widehat{\mathcal{U}}^I_{t_{j+1}} + \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{Z}}^R_{t_{j+1}} - \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j} + f^R(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t\Big]. \end{cases} \tag{4.3.8}$$

4.3.2 Convergence analysis

To simplify notations, by convention we write $\int_{\mathbb{R}} g$ short for the integral $\int_{\mathbb{R}} g(x)\,dx$ in this work. Indeed, integration w.r.t. x helps us obtain estimates which are non-standard in the BSDE literature. For $\phi = Y, Z, \widehat{\mathcal{V}}, \widehat{\mathcal{W}}, \widehat{\mathcal{U}}, \widehat{\mathcal{Z}}, \widehat{Z}$, recall the notations $\phi = \phi^R + i\phi^I$ and the facts like $\mathbb{E}\int_{\mathbb{R}} |\phi^{R/I}(t_j, x + \sqrt{\nu}W_{t_j})|^2 = \int_{\mathbb{R}} |\phi^{R/I}(t_j, x + \sqrt{\nu}W_{t_j})|^2 = \int_{\mathbb{R}} |\phi^{R/I}(t_j, x)|^2$.

Define the following errors caused by the neural network approximations:

$$\varepsilon^v_j := \int_{\mathbb{R}} \inf_{\xi} \mathbb{E}|\hat{v}_{t_j}(X_{t_j}) - \mathcal{U}_{t_j}(X_{t_j}; \xi)|^2, \quad \varepsilon^{\mathcal{W}}_j := \int_{\mathbb{R}} \inf_{\eta} \mathbb{E}|\widehat{\mathcal{W}}_{t_j}(X_{t_j}) - \mathcal{Z}_{t_j}(X_{t_j}; \eta)|^2.$$

For Algorithm 1, we have the following convergence analysis whose proof follows straightforwardly by combining Lemmas 4.3.2 and 4.3.2.

Theorem 4.3.1. *There exists a constant $C > 0$ depending on K, L, T, ν such that the following holds:*

$$\max_{j=0,\dots,M-1} \int_{\mathbb{R}} \left[|Y_{t_j} - \widehat{\mathcal{U}}_{t_j}|^2 + \frac{(\delta t)\nu}{2} |\overline{Z}_{t_j} - \widehat{\mathcal{Z}}_{t_j}|^2 \right] \leq \frac{C}{\delta t} \sum_{j=0}^{M-1} \varepsilon_j^v + C\nu \sum_{j=0}^{M-1} \varepsilon_j^{\mathcal{W}} + C[\delta t + \rho(\delta t)]. \tag{4.3.9}$$

Remark 4.3.1. *The right-hand side of* (4.3.9) *is the error contributions for Algorithm 1, and it consists of four terms. The first two correspond to the error caused by neural network approximations; the better the neural networks are able to approximate or learn the functions v and w for each $j = 0, \cdots, M-1$, the smaller these terms are in the error estimation. From the universal approximation (Hornik* et al.*, 1989, theorem* (I)*), the neural network error terms can be made arbitrarily small for sufficiently large number of neurons and sufficiently many hidden layers. The other two terms are due to the regularity of Y, Z and f. On the other hand, Theorem 4.3.1, mainly addresses the error w.r.t. Y as δt tends to zero. This is consisting with Schrödinger equation's wave-like feature, which can be illustrated as follows. Itô formula gives*

$$\begin{aligned} d(Y_s^R)^2 &= 2Y_s^R \left[\sqrt{\nu}(Z_s^R + Z_s^I) * dW_s + f_s^I \, ds + \sqrt{\nu} Z_s^R \, dW_s \right] + \nu |Z_s^R|^2 \, ds, \\ d(Y_s^I)^2 &= 2Y_s^I \left[-\sqrt{\nu}(Z_s^R - Z_s^I) * dW_s - f_s^R \, ds + \sqrt{\nu} Z_s^I \, dW_s \right] + \nu |Z_s^R|^2 \, ds. \end{aligned}$$

Using integration by parts and the relation (1.2.6)*, we obtain*

$$\begin{aligned} \int_{\mathbb{R}} \mathbb{E}|Y_T^R|^2 - \int_{\mathbb{R}} \mathbb{E}|Y_t^R|^2 &= -\nu \mathbb{E} \int_{\mathbb{R}} \int_t^T Z_s^R Z_s^I \, ds + 2\mathbb{E} \int_{\mathbb{R}} \int_t^T Y_s^R f_s^I \, ds, \\ \int_{\mathbb{R}} \mathbb{E}|Y_T^I|^2 - \int_{\mathbb{R}} \mathbb{E}|Y_t^I|^2 &= \nu \mathbb{E} \int_{\mathbb{R}} \int_t^T Z_s^R Z_s^I \, ds - 2\mathbb{E} \int_{\mathbb{R}} \int_t^T Y_s^I f_s^R \, ds. \end{aligned}$$

Summing these equations leads to the cancellation of the Z-terms, highlighting the challenges in deriving non-trivial estimates for Z_s^R and Z_s^I.

For the neural network approximations, we have the following estimate whose proof is more or less standard and postponed to Appendix 4.4.1. There exists a constant $C > 0$ depending

on L, ν such that

$$\int_{\mathbb{R}} |\widehat{\mathcal{V}}_{t_j} - \widehat{\mathcal{U}}_{t_j}|^2 + \frac{\nu\delta t}{2} \int_{\mathbb{R}} \Big[|\widehat{\mathcal{W}}_{t_j} - \widehat{\mathcal{Z}}_{t_j}|^2 \Big] \leq C \Big[\varepsilon_j^v + (\delta t)\varepsilon_j^{\mathcal{W}} \Big].$$

The following result measures the distance between Y_{t_j} and $\widehat{\mathcal{U}}_{t_j}$. There exists a constant $C > 0$ depending on K, L, T, ν such that the following holds:

$$\begin{aligned}
&\max_{j=0,\dots,M-1} \int_{\mathbb{R}} \Big[|Y_{t_j} - \widehat{\mathcal{U}}_{t_j}|^2 + \frac{(\delta t)\nu}{2} \Big(|Z_{t_j} - \widehat{\mathcal{Z}}_{t_j}|^2 \Big) \Big] \\
&\leq \frac{C}{\delta t} \sum_{j=0}^{M-1} \left[\int_{\mathbb{R}} \Big[|\widehat{\mathcal{U}}_{t_j} - \widehat{\mathcal{V}}_{t_j}|^2 \Big] + C\nu \sum_{j=0}^{M-1} \left[\int_{\mathbb{R}} \Big(|\widehat{\mathcal{W}}_{t_j} - \widehat{\mathcal{Z}}_{t_j}|^2 \Big) \right] \right] + C \left[\delta t + \rho(\delta t) \right]. \qquad (4.3.10)
\end{aligned}$$

Proof. **Step 1.** Notice that both sides of (4.3.10) are deterministic. First, we examine the left-hand side of the real part of (4.3.8). Using the L^2-projections (4.3.4) and (4.3.5), we arrive at

$$\begin{aligned}
&\mathbb{E}_j \Bigg(\Big(Y_{t_j}^R - \frac{\sqrt{\nu}}{2} (Z_{t_j}^R + Z_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} Z_{t_j}^R \delta W_{t_j} \\
&\qquad\qquad - \Big[\widehat{\mathcal{V}}_{t_j}^R - \frac{\sqrt{\nu}}{2} (\widehat{\mathcal{W}}_{t_j}^R + \widehat{\mathcal{W}}_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} \widehat{\mathcal{W}}_{t_j}^R \delta W_{t_j} \Big] \Big) + (\mathrm{I}_{A,j}^R + \mathrm{II}_{A,j}^R - \mathrm{IV}_j^R) \Bigg)^2 \\
=&\mathbb{E}_j \left(Y_{t_j}^R - \frac{\sqrt{\nu}}{2} (Z_{t_j}^R + Z_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} Z_{t_j}^R \delta W_{t_j} - \Big[\widehat{\mathcal{V}}_{t_j}^R - \frac{\sqrt{\nu}}{2} (\widehat{\mathcal{W}}_{t_j}^R + \widehat{\mathcal{W}}_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} \widehat{\mathcal{W}}_{t_j}^R \delta W_{t_j} \Big] \right)^2 \\
&+ \mathbb{E}_j \left(\mathrm{I}_{A,j}^R + \mathrm{II}_{A,j}^R - \mathrm{IV}_j^R \right)^2 \\
\geq&\mathbb{E}_j \left(Y_{t_j}^R - \frac{\sqrt{\nu}}{2} (Z_{t_j}^R + Z_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} Z_{t_j}^R \delta W_{t_j} - \Big[\widehat{\mathcal{V}}_{t_j}^R - \frac{\sqrt{\nu}}{2} (\widehat{\mathcal{W}}_{t_j}^R + \widehat{\mathcal{W}}_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} \widehat{\mathcal{W}}_{t_j}^R \delta W_{t_j} \Big] \right)^2 \\
=&\mathbb{E}_j \Bigg\{ \Big[(Y_{t_j}^R - \widehat{\mathcal{V}}_{t_j}^R) + \Big(- \frac{\sqrt{\nu}}{2} (Z_{t_j}^R + Z_{t_j}^I) \delta W_{t_j} + \frac{\sqrt{\nu}}{2} (\widehat{\mathcal{W}}_{t_j}^R + \widehat{\mathcal{W}}_{t_j}^I) \delta W_{t_j} \Big) \Big]^2 \\
&+ \nu \Big(\widehat{\mathcal{W}}_{t_j}^R + \widehat{\mathcal{W}}_{t_j}^I - Z_{t_j}^R - Z_{t_j}^I \Big) (\delta W_{t_j})^2 (Z_{t_j}^R - \widehat{\mathcal{W}}_{t_j}^R) + \nu (Z_{t_j}^R - \widehat{\mathcal{W}}_{t_j}^R)^2 (\delta W_{t_j})^2 \Bigg\}. \qquad (4.3.11)
\end{aligned}$$

Meanwhile, with similar calculation the left-hand side of the imaginary part yields

$$\begin{aligned}
&\mathbb{E}_j \Bigg(Y_{t_j}^I + \frac{\sqrt{\nu}}{2} (Z_{t_j}^R - Z_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} Z_{t_j}^I \delta W_{t_j} \\
&\qquad\qquad - \Big[\widehat{\mathcal{V}}_{t_j}^I + \frac{\sqrt{\nu}}{2} (\widehat{\mathcal{W}}_{t_j}^R - \widehat{\mathcal{W}}_{t_j}^I) \delta W_{t_j} + \sqrt{\nu} \widehat{\mathcal{W}}_{t_j}^I \delta W_{t_j} \Big] + \Big(\mathrm{I}_{A,j}^I + \mathrm{II}_{A,j}^I - \mathrm{IV}_j^I \Big) \Bigg)^2
\end{aligned}$$

$$\geq \mathbb{E}_j\Bigg\{\Big[(Y^I_{t_j}-\widehat{\mathcal{V}}^I_{t_j})+\Big(\frac{\sqrt{\nu}}{2}(Z^R_{t_j}-Z^I_{t_j})\delta W_{t_j}-\frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j}-\widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j}\Big)\Big]^2$$

$$+\nu\Big(-\widehat{\mathcal{W}}^R_{t_j}+\widehat{\mathcal{W}}^I_{t_j}+Z^R_{t_j}-Z^I_{t_j}\Big)(\delta W_{t_j})^2(Z^I_{t_j}-\widehat{\mathcal{W}}^I_{t_j})+\nu(Z^I_{t_j}-\widehat{\mathcal{W}}^I_{t_j})^2(\delta W_{t_j})^2\Bigg\}. \qquad (4.3.12)$$

Then adding (4.3.11) and (4.3.12) leads to

$$\mathbb{E}_j\Bigg(\Big(Y^R_{t_j}-\frac{\sqrt{\nu}}{2}(Z^R_{t_j}+Z^I_{t_j})\delta W_{t_j}+\sqrt{\nu}Z^R_{t_j}\delta W_{t_j}$$

$$-\Big[\widehat{\mathcal{V}}^R_{t_j}-\frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j}+\widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j}+\sqrt{\nu}\widehat{\mathcal{W}}^R_{t_j}\delta W_{t_j}\Big]\Big)+(\mathrm{I}^R_{A,j}+\mathrm{II}^R_{A,j}-\mathrm{IV}^R_j)\Bigg)^2$$

$$+\mathbb{E}_j\Bigg(\Big(Y^I_{t_j}+\frac{\sqrt{\nu}}{2}(Z^R_{t_j}-Z^I_{t_j})\delta W_{t_j}+\sqrt{\nu}Z^I_{t_j}\delta W_{t_j}$$

$$-\Big[\widehat{\mathcal{V}}^I_{t_j}+\frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j}-\widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j}+\sqrt{\nu}\widehat{\mathcal{W}}^I_{t_j}\delta W_{t_j}\Big]\Big)+(\mathrm{I}^I_{A,j}+\mathrm{II}^I_{A,j}-\mathrm{IV}^I_j)\Bigg)^2$$

$$\geq\mathbb{E}_j\Big[(Y^R_{t_j}-\widehat{\mathcal{V}}^R_{t_j})+\Big(-\frac{\sqrt{\nu}}{2}(Z^R_{t_j}+Z^I_{t_j})\delta W_{t_j}+\frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j}+\widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j}\Big)\Big]^2$$

$$+\mathbb{E}_j\Big[(Y^I_{t_j}-\widehat{\mathcal{V}}^I_{t_j})+\Big(\frac{\sqrt{\nu}}{2}(Z^R_{t_j}-Z^I_{t_j})\delta W_{t_j}-\frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j}-\widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j}\Big)\Big]^2.$$

Finally, integrating both sides under the Lebesgue measure yields that

$$\begin{aligned}
&\mathbb{E}_j\int_{\mathbb{R}}\Big[(Y^R_{t_j}-\widehat{\mathcal{V}}^R_{t_j})+\Big(-\frac{\sqrt{\nu}}{2}(Z^R_{t_j}+Z^I_{t_j})\delta W_{t_j}+\frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j}+\widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j}\Big)\Big]^2\\
&+\mathbb{E}_j\int_{\mathbb{R}}\Big[(Y^I_{t_j}-\widehat{\mathcal{V}}^I_{t_j})+\Big(\frac{\sqrt{\nu}}{2}(Z^R_{t_j}-Z^I_{t_j})\delta W_{t_j}-\frac{\sqrt{\nu}}{2}(\widehat{\mathcal{W}}^R_{t_j}-\widehat{\mathcal{W}}^I_{t_j})\delta W_{t_j}\Big)\Big]^2\\
=&\mathbb{E}_j\int_{\mathbb{R}}\Big[(Y^R_{t_j}-\widehat{\mathcal{V}}^R_{t_j})^2+\frac{(\delta t)\nu}{4}\Big((Z^R_{t_j}+Z^I_{t_j})-(\widehat{\mathcal{W}}^R_{t_j}+\widehat{\mathcal{W}}^I_{t_j})\Big)^2\Big]\\
&+\mathbb{E}_j\int_{\mathbb{R}}\Big[(Y^I_{t_j}-\widehat{\mathcal{V}}^I_{t_j})^2+\frac{(\delta t)\nu}{4}\Big((Z^R_{t_j}-Z^I_{t_j})-(\widehat{\mathcal{W}}^R_{t_j}-\widehat{\mathcal{W}}^I_{t_j})\Big)^2\Big].
\end{aligned}$$

Step 2. We examine the right-hand side of (4.3.8). Using the inequality $(a+b)^2\leq(1+\gamma\delta t)a^2+\left(1+\frac{1}{\gamma\delta t}\right)b^2$, for some constant $\gamma>0$, we obtain

$$\mathbb{E}_j\int_{\mathbb{R}}|\mathbf{RHS}|^2$$

$$:=\mathbb{E}_j\int_{\mathbb{R}}\Bigg(Y^R_{t_{j+1}}-\frac{\sqrt{\nu}}{2}(Z^R_{t_{j+1}}+Z^I_{t_{j+1}})\delta W_{t_j}-\mathrm{I}^R_{B,j}-\mathrm{II}^R_{B,j}-\mathrm{III}^R_j-f^I(t_j,\mathcal{X}_{t_j},Y^R_{t_j},Y^I_{t_j})\delta t$$

$$
\begin{aligned}
&- \Big[\widehat{\mathcal{U}}^R_{t_{j+1}} - \frac{\sqrt{\nu}}{2}(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j} - f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t\Big]\Big)^2 \\
&\le (1+\gamma\delta t)\mathbb{E}_j \int_{\mathbb{R}} \Big(\Big[Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}}\Big] + \frac{\sqrt{\nu}}{2}\delta W_{t_j}\Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\Big]\Big)^2 \\
&\quad + 4\Big(1 + \frac{1}{\gamma\delta t}\Big)\mathbb{E}_j \int_{\mathbb{R}} \Big((\mathrm{I}^R_{B,j})^2 + (\mathrm{II}^R_{B,j})^2 + (\mathrm{III}^R_j)^2 + \big(f^I(t_j, \mathcal{X}_{t_j}, Y^R_{t_j}, Y^I_{t_j})\delta t - f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t\big)^2\Big).
\end{aligned}
$$

Further, in view of the Lipschitz-continuity of function f in condition (iii), we have

$$
\begin{aligned}
&\mathbb{E}_j \int_{\mathbb{R}} |\mathbf{RHS}|^2 \\
&\le (1+\gamma\delta t)\mathbb{E}_j \int_{\mathbb{R}} \Big(\Big[Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}}\Big] + \frac{\sqrt{\nu}}{2}\delta W_{t_j}\Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\Big]\Big)^2 \\
&\quad + \frac{8}{\gamma}\Big(\gamma\delta t + 1\Big)L^2(\delta t)\Big(\mathbb{E}_j \int_{\mathbb{R}} |Y^R_{t_j} - \widehat{\mathcal{V}}^R_{t_j}|^2 + \mathbb{E}_j \int_{\mathbb{R}} |Y^I_{t_j} - \widehat{\mathcal{V}}^I_{t_j}|^2\Big)\Big) \\
&\quad + 4\Big(1 + \frac{1}{\gamma\delta t}\Big)\mathbb{E}_j \int_{\mathbb{R}} \Big((\mathrm{I}^R_{B,j})^2 + (\mathrm{II}^R_{B,j})^2 + (\mathrm{III}^R_j)^2\Big) \\
&= (1+\gamma\delta t)\Big(\mathbb{E}_j \int_{\mathbb{R}} \Big[Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}}\Big]^2 + \frac{(\delta t)\nu}{4}\Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\Big]^2\Big) \\
&\quad + \frac{8}{\gamma}\Big(\gamma\delta t + 1\Big)L^2(\delta t)\Big(\mathbb{E}_j \int_{\mathbb{R}} |Y^R_{t_j} - \widehat{\mathcal{V}}^R_{t_j}|^2 + \mathbb{E}_j \int_{\mathbb{R}} |Y^I_{t_j} - \widehat{\mathcal{V}}^I_{t_j}|^2\Big)\Big) \\
&\quad + 4\Big(1 + \frac{1}{\gamma\delta t}\Big)\mathbb{E}_j \int_{\mathbb{R}} \Big((\mathrm{I}^R_{B,j})^2 + (\mathrm{II}^R_{B,j})^2 + (\mathrm{III}^R_j)^2\Big),
\end{aligned}
$$

where in the last equality we have used the facts

$$
\begin{aligned}
&\mathbb{E}_j \int_{\mathbb{R}} \Big[Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}}\Big]\delta W_{t_j}\Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\Big] \\
&= \mathbb{E}_j \delta W_{t_j} \int_{\mathbb{R}} \Big[Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}}\Big]\Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\Big] \\
&= 0, \qquad (4.3.13)
\end{aligned}
$$

and analogously,

$$
\mathbb{E}_j \int_{\mathbb{R}} (\delta W_{t_j})^2 \Big|(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\Big|^2 = \delta t \mathbb{E}_j \int_{\mathbb{R}} \Big|(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}})\Big|^2.
$$

Step 3. We combine the separate estimates for the left-hand and right-hand sides of (4.3.8).

Let $\gamma = L + 1$. Together with the imaginary part, for δt small enough, and a constant C which is possibly changing line by line, we have

$$
\begin{aligned}
&\mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^R_{t_j} - \widehat{\mathcal{V}}^R_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} + Z^I_{t_j}) - (\widehat{\mathcal{W}}^R_{t_j} + \widehat{\mathcal{W}}^I_{t_j}) \Big)^2 \Big] \\
&+ \mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^I_{t_j} - \widehat{\mathcal{V}}^I_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} - Z^I_{t_j}) - (\widehat{\mathcal{W}}^R_{t_j} - \widehat{\mathcal{W}}^I_{t_j}) \Big)^2 \Big] \\
\leq &(1 + C\delta t) \bigg(\mathbb{E}_j \int_{\mathbb{R}} \Big[Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}} \Big]^2 + \frac{(\delta t)\nu}{4} \Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}}) \Big]^2 \bigg) \\
&+ (1 + C\delta t) \bigg(\mathbb{E}_j \int_{\mathbb{R}} \Big[Y^I_{t_{j+1}} - \widehat{\mathcal{U}}^I_{t_{j+1}} \Big]^2 + \frac{(\delta t)\nu}{4} \Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} - \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} - Z^I_{t_{j+1}}) \Big]^2 \bigg) \\
&+ 8 \Big(1 + \frac{1}{\gamma \delta t} \Big) \mathbb{E}_j \int_{\mathbb{R}} (\mathrm{I}^R_{B,j})^2 + (\mathrm{II}^R_{B,j})^2 + (\mathrm{III}^R_j)^2 + (\mathrm{I}^I_{B,j})^2 + (\mathrm{II}^I_{B,j})^2 + (\mathrm{III}^I_j)^2.
\end{aligned}
$$

Using the inequality $(a+b)^2 \geq (1-\delta t)a^2 - \frac{1}{\delta t} b^2$, we obtain

$$
\begin{aligned}
&\mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^R_{t_j} - \widehat{\mathcal{V}}^R_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} + Z^I_{t_j}) - (\widehat{\mathcal{W}}^R_{t_j} + \widehat{\mathcal{W}}^I_{t_j}) \Big)^2 \Big] \\
&+ \mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^I_{t_j} - \widehat{\mathcal{V}}^I_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} - Z^I_{t_j}) - (\widehat{\mathcal{W}}^R_{t_j} - \widehat{\mathcal{W}}^I_{t_j}) \Big)^2 \Big] \\
\geq &(1 - \delta t) \bigg\{ \mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^R_{t_j} - \widehat{\mathcal{U}}^R_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} + Z^I_{t_j}) - (\widehat{\mathcal{Z}}^R_{t_j} + \widehat{\mathcal{Z}}^I_{t_j}) \Big)^2 \Big] \\
&+ \mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^I_{t_j} - \widehat{\mathcal{U}}^I_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} - Z^I_{t_j}) - (\widehat{\mathcal{Z}}^R_{t_j} - \widehat{\mathcal{Z}}^I_{t_j}) \Big)^2 \Big] dx \bigg\} \\
&- \frac{1}{\delta t} \bigg\{ \mathbb{E}_j \int_{\mathbb{R}} \Big[(\widehat{\mathcal{U}}^R_{t_j} - \widehat{\mathcal{V}}^R_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((\widehat{\mathcal{W}}^R_{t_j} + \widehat{\mathcal{W}}^I_{t_j}) - (\widehat{\mathcal{Z}}^R_{t_j} + \widehat{\mathcal{Z}}^I_{t_j}) \Big)^2 \Big] \\
&+ \mathbb{E}_j \int_{\mathbb{R}} \Big[(\widehat{\mathcal{U}}^I_{t_j} - \widehat{\mathcal{V}}^I_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((\widehat{\mathcal{W}}^R_{t_j} - \widehat{\mathcal{W}}^I_{t_j}) - (\widehat{\mathcal{Z}}^R_{t_j} - \widehat{\mathcal{Z}}^I_{t_j}) \Big)^2 \Big] \bigg\}.
\end{aligned}
$$

Thus for δt small enough, it follows that

$$
\begin{aligned}
&\mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^R_{t_j} - \widehat{\mathcal{U}}^R_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} + Z^I_{t_j}) - (\widehat{\mathcal{Z}}^R_{t_j} + \widehat{\mathcal{Z}}^I_{t_j}) \Big)^2 \Big] \\
&+ \mathbb{E}_j \int_{\mathbb{R}} \Big[(Y^I_{t_j} - \widehat{\mathcal{U}}^I_{t_j})^2 + \frac{(\delta t)\nu}{4} \Big((Z^R_{t_j} - Z^I_{t_j}) - (\widehat{\mathcal{Z}}^R_{t_j} - \widehat{\mathcal{Z}}^I_{t_j}) \Big)^2 \Big] \\
\leq &(1 + C\delta t) \bigg(\mathbb{E}_j \int_{\mathbb{R}} \Big[Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}} \Big]^2 + \frac{(\delta t)\nu}{4} \Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} + Z^I_{t_{j+1}}) \Big]^2 \bigg) \\
&+ (1 + C\delta t) \bigg(\mathbb{E}_j \int_{\mathbb{R}} \Big[Y^I_{t_{j+1}} - \widehat{\mathcal{U}}^I_{t_{j+1}} \Big]^2 + \frac{(\delta t)\nu}{4} \Big[(\widehat{\mathcal{Z}}^R_{t_{j+1}} - \widehat{\mathcal{Z}}^I_{t_{j+1}}) - (Z^R_{t_{j+1}} - Z^I_{t_{j+1}}) \Big]^2 \bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ C\Big(1+\frac{1}{\gamma\delta t}\Big)\mathbb{E}_j\int_{\mathbb{R}}(\mathrm{I}^R_{B,j})^2+(\mathrm{II}^R_{B,j})^2+(\mathrm{III}^R_j)^2+(\mathrm{I}^I_{B,j})^2+(\mathrm{II}^I_{B,j})^2+(\mathrm{III}^I_j)^2\\
&+\frac{C}{\delta t}\Bigg\{\mathbb{E}_j\int_{\mathbb{R}}\Big[(\widehat{\mathcal{U}}^R_{t_j}-\widehat{\mathcal{V}}^R_{t_j})^2+\frac{(\delta t)\nu}{4}\Big((\widehat{\mathcal{W}}^R_{t_j}+\widehat{\mathcal{W}}^I_{t_j})-(\widehat{\mathcal{Z}}^R_{t_j}+\widehat{\mathcal{Z}}^I_{t_j})\Big)^2\Big]dx\\
&+\mathbb{E}_j\int_{\mathbb{R}}\Big[(\widehat{\mathcal{U}}^I_{t_j}-\widehat{\mathcal{V}}^I_{t_j})^2+\frac{(\delta t)\nu}{4}\Big((\widehat{\mathcal{W}}^R_{t_j}-\widehat{\mathcal{W}}^I_{t_j})-(\widehat{\mathcal{Z}}^R_{t_j}-\widehat{\mathcal{Z}}^I_{t_j})\Big)^2\Big]\Bigg\}.
\end{aligned}
$$

Further simplification gives us

$$
\begin{aligned}
&\mathbb{E}_j\int_{\mathbb{R}}\Big[|Y_{t_j}-\widehat{\mathcal{U}}_{t_j}|^2+\frac{(\delta t)\nu}{2}|Z_{t_j}-\widehat{\mathcal{Z}}_{t_j}|^2\Big]\\
\le&(1+C\delta t)\left(\mathbb{E}_j\int_{\mathbb{R}}\Big|Y_{t_{j+1}}-\widehat{\mathcal{U}}_{t_{j+1}}\Big|^2+\frac{(\delta t)\nu}{2}|\widehat{\mathcal{Z}}_{t_{j+1}}-Z_{t_{j+1}}|^2\right)+\frac{C}{\delta t}\mathbb{E}_j\int_{\mathbb{R}}\Big[|\widehat{\mathcal{U}}_{t_j}-\widehat{\mathcal{V}}_{t_j}|^2+\frac{(\delta t)\nu}{2}|\widehat{\mathcal{W}}_{t_j}-\widehat{\mathcal{Z}}_{t_j}|^2\Big]\\
&+16\Big(1+\frac{1}{\gamma\delta t}\Big)\mathbb{E}_j\int_{\mathbb{R}}(\mathrm{I}^R_{B,j})^2+(\mathrm{II}^R_{B,j})^2+(\mathrm{III}^R_j)^2+(\mathrm{I}^I_{B,j})^2+(\mathrm{II}^I_{B,j})^2+(\mathrm{III}^I_j)^2.
\end{aligned}
$$

Recalling $\mathcal{X}_T = X_T$, taking expectations, and using the discrete Grönwall's inequality, we obtain

$$
\begin{aligned}
&\max_{j=0,\dots,M-1}\int_{\mathbb{R}}\Big[|Y_{t_j}-\widehat{\mathcal{U}}_{t_j}|^2+\frac{(\delta t)\nu}{2}|Z_{t_j}-\widehat{\mathcal{Z}}_{t_j}|^2\Big]\\
&\le\frac{C}{\delta t}\sum_{j=0}^{M-1}\int_{\mathbb{R}}|\widehat{\mathcal{U}}_{t_j}-\widehat{\mathcal{V}}_{t_j}|^2+C\nu\sum_{j=0}^{M-1}\int_{\mathbb{R}}|\widehat{\mathcal{W}}_{t_j}-\widehat{\mathcal{Z}}_{t_j}|^2\\
&\quad+\frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left(|\mathrm{I}^R_{B,j}|^2+|\mathrm{II}^R_{B,j}|^2+|\mathrm{III}^R_j|^2+|\mathrm{I}^I_{B,j}|^2+|\mathrm{II}^I_{B,j}|^2+|\mathrm{III}^I_j|^2\right).
\end{aligned}
\tag{4.3.14}
$$

Then the proof may be completed by appropriately estimating the Roman numeral error terms. To avoid cumbersome arguments, the estimate is postponed to Appendix 4.4.2. □

Remark 4.3.2. *The proofs of Lemma 4.3.2 and Lemma 4.3.2 are inspired by, but distinct from, the work of Huré, Pham, and Warin in Huré* et al. *(2020). The integrals with $*dW_t$ in BSDE* (1.2.5)*, in each time step $[t_j, t_{j+1}]$, generate terms involving the backward processes such as $Z_{t_{j+1}}$, $\widehat{\mathcal{Z}}_{t_{j+1}}$ at time t_{j+1}, which are multiplied by the increment of the Brownian motion (see terms of the form $\mathbb{E}_j[(\widehat{\mathcal{Z}}^R_{t_{j+1}}+\widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j}]$ in* (4.3.1) *and its counterpart in* (4.3.6)*). This is in contrast to (Huré* et al.*, 2020, Equation 4.5), where no*

such terms appear. The difficulty arises in the estimation of these terms due to the randomness $x + \sqrt{\nu}W_{t_{j+1}}$ *introduced in* $Z_{t_{j+1}}$, $\widehat{\mathcal{Z}}_{t_{j+1}}$ *which are not* $\mathscr{F}_{t_j}$*-measurable. However, by taking integration over* x *under the Lebesgue measure, and applying Fubini's Theorem, we can recover key properties such as* $\int_{\mathbb{R}^d} \mathbb{E}_j[(Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}})(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}})\delta W_{t_j}] = 0$ *and* $\int_{\mathbb{R}^d} \mathbb{E}_j[(Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}})(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}})(\delta W_{t_j})^2] = (\delta t)\int_{\mathbb{R}}(Y^R_{t_{j+1}} - \widehat{\mathcal{U}}^R_{t_{j+1}})(\widehat{\mathcal{Z}}^R_{t_{j+1}} + \widehat{\mathcal{Z}}^I_{t_{j+1}})$*, which are crucial for our analysis. Another significant challenge in our problem arises from the structure of the complex-valued Schrödinger equation. This requires us to carefully split certain terms, such as* $\mathrm{I}^R_{A,j}$ *and* $\mathrm{I}^R_{B,j}$*, allowing us to carry out error calculations through cancellations, measurability arguments, and conditional expectations.*

4.4 Appendix

4.4.1 Proof of Lemma 4.3.2

Proof of Lemma 4.3.2. Recalling equation (4.3.7), the L^2 projection (4.3.4), and the real part of cost functional $L^R_j(\theta)$ in Algorithm 1, for all neural network parameters $\theta = (\xi, \eta)$ we have

$$\begin{aligned}
L^R_j(\theta) =&\mathbb{E}\Big|\widehat{\mathcal{V}}^R_{t_j} - \mathcal{U}^R_{t_j}(X_{t_j};\xi) + f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t - f^I(t_j, X_{t_j}, \mathcal{U}^R_{t_j}(X_{t_j};\xi), \mathcal{U}^I_{t_j}(X_{t_j};\xi))\delta t \\
&+ \frac{\sqrt{\nu}}{2}\big(\widehat{\mathcal{W}}^R_{t_j} - \mathcal{Z}^R_{t_j}(X_{t_j};\eta)\big)\delta W_{t_j} - \frac{\sqrt{\nu}}{2}\big(\widehat{\mathcal{W}}^I_{t_j} - \mathcal{Z}^I_{t_j}(X_{t_j};\eta)\big)\delta W_{t_j}\Big|^2 + \nu\mathbb{E}\int_{t_j}^{t_{j+1}} |\widehat{Z}^R_s - \widehat{\mathcal{W}}^R_{t_j}|^2 ds \\
=&\mathbb{E}\Big|\widehat{\mathcal{V}}^R_{t_j} - \mathcal{U}^R_{t_j}(X_{t_j};\xi) + f^I(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t - f^I(t_j, X_{t_j}, \mathcal{U}^R_{t_j}(X_{t_j};\xi), \mathcal{U}^I_{t_j}(X_{t_j};\xi))\delta t\Big|^2 \\
&+ \mathbb{E}\Big|\frac{\sqrt{\nu}}{2}\big(\widehat{\mathcal{W}}^R_{t_j} - \mathcal{Z}^R_{t_j}(X_{t_j};\eta)\big)\delta W_{t_j} - \frac{\sqrt{\nu}}{2}\big(\widehat{\mathcal{W}}^I_{t_j} - \mathcal{Z}^I_{t_j}(X_{t_j};\eta)\big)\delta W_{t_j}\Big|^2 + \nu\mathbb{E}\int_{t_j}^{t_{j+1}} |\widehat{Z}^R_s - \widehat{\mathcal{W}}^R_{t_j}|^2 ds \\
=:&\widetilde{L}^R_j(\theta) + \nu\mathbb{E}\int_{t_j}^{t_{j+1}} |\widehat{Z}^R_s - \widehat{\mathcal{W}}^R_{t_j}|^2 ds.
\end{aligned}$$

Note that the last integral term is a constant independent of the parameters, hence the minimization only involves $\widetilde{L}^R_j$. Similarly we derive the $\widetilde{L}^I_j$ of the imaginary part to be

$$\widetilde{L}^I_j(\theta) := \mathbb{E}\Big|\widehat{\mathcal{V}}^I_{t_j} - \mathcal{U}^I_{t_j}(X_{t_j};\xi) - f^R(t_j, X_{t_j}, \widehat{\mathcal{V}}^R_{t_j}, \widehat{\mathcal{V}}^I_{t_j})\delta t + f^R(t_j, X_{t_j}, \mathcal{U}^R_{t_j}(X_{t_j};\xi), \mathcal{U}^I_{t_j}(X_{t_j};\xi))\delta t\Big|^2$$

$$+\mathbb{E}\Big|\frac{\sqrt{\nu}}{2}\big(\widehat{\mathcal{W}}^R_{t_j}-\mathcal{Z}^R_{t_j}(X_{t_j};\eta)\big)\delta W_{t_j}+\frac{\sqrt{\nu}}{2}\big(\widehat{\mathcal{W}}^I_{t_j}-\mathcal{Z}^I_{t_j}(X_{t_j};\eta)\big)\delta W_{t_j}\Big|^2.$$

Adding $\widetilde{L}^R_j$ and $\widetilde{L}^I_j$ together and using the Lipschitz property of f and the parallelogram law yield

$$\widetilde{L}^R_j(\theta)+\widetilde{L}^I_j(\theta)\le\big(2(1+L\delta t)^2+2L^2(\delta t)^2\big)\,|\widehat{\mathcal{V}}_{t_j}-\mathcal{U}_{t_j}(X_{t_j};\xi)|^2+\frac{\nu(\delta t)}{2}\mathbb{E}|\widehat{\mathcal{W}}_{t_j}-\mathcal{Z}_{t_j}(X_{t_j};\eta)|^2.$$

Using the inequality $(a+b)^2\ge(1-\delta t)a^2-\frac{1}{\delta t}b^2$, we derive that

$$\widetilde{L}^R_j(\theta)+\widetilde{L}^I_j(\theta)\ge(1-C\delta t)\Big[\mathbb{E}|\widehat{\mathcal{V}}_{t_j}-\mathcal{U}_{t_j}(X_{t_j};\xi)|^2\Big]+\frac{\nu\delta t}{2}\mathbb{E}\Big[|\widehat{\mathcal{W}}_{t_j}-\mathcal{Z}_{t_j}(X_{t_j};\eta)|^2\Big].$$

Thus taking $\theta^*_j=(\xi^*_j,\eta^*_j)\in\arg\min_{\theta\in\mathbb{R}^{N_m}}L_j(\theta)$ such that $\widehat{\mathcal{U}}^R_{t_j}=\mathcal{U}^R_{t_j}(X_{t_j};\xi^*_j)$, $\widehat{\mathcal{U}}^I_{t_j}=\mathcal{U}^I_{t_j}(X_{t_j};\xi^*_j)$, $\widehat{\mathcal{Z}}^R_{t_j}=\mathcal{Z}^R_{t_j}(X_{t_j};\eta^*_j)$ and $\widehat{\mathcal{Z}}^I_{t_j}=\mathcal{Z}^I_{t_j}(X_{t_j};\eta^*_j)$, we have

$$\begin{aligned}(1-C\delta t)\mathbb{E}|\widehat{\mathcal{V}}_{t_j}-\widehat{\mathcal{U}}_{t_j}|^2+\frac{\nu\delta t}{2}\mathbb{E}|\widehat{\mathcal{W}}_{t_j}-\widehat{\mathcal{Z}}_{t_j}|^2&\le\widetilde{L}^R_j(\theta^*_j)+\widetilde{L}^I_j(\theta^*_j)\le\widetilde{L}^R_j(\theta)+\widetilde{L}^I_j(\theta)\\&\le\big(2(1+L\delta t)^2+2L^2(\delta t)^2\big)\,\mathbb{E}\Big[|\widehat{\mathcal{V}}_{t_j}-\mathcal{U}_{t_j}(X_{t_j};\xi)|^2\Big]+\frac{\nu(\delta t)}{2}\mathbb{E}|\widehat{\mathcal{W}}_{t_j}-\mathcal{Z}_{t_j}(X_{t_j};\eta)|^2.\end{aligned}$$

Minimizing w.r.t. θ and integrating w.r.t. x yield the desired estimate when δt is small enough. □

4.4.2 Estimate of the Roman-numeral error terms in the proof of Lemma 4.3.2

Complete the proof of Lemma 4.3.2. We complete the proof of Lemma 4.3.2, by appropriately estimating the error terms associated with Roman numerals in (4.3.14). We focus on the real parts, as the imaginary components can be treated similarly. First, there holds

$$\begin{aligned}\frac{1}{\nu\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}|\mathrm{II}^R_{B,j}|^2&=\sum_{j=0}^{M-1}\frac{1}{\delta t}\int_{\mathbb{R}}\mathbb{E}\Big|(\overline{Z}^R_{t_j})\delta W_{t_j}-(Z^R_{t_j})\delta W_{t_j}\Big|^2\\&=\sum_{j=0}^{M-1}\frac{1}{(\delta t)^3}\int_{\mathbb{R}}\mathbb{E}\left[\left(\mathbb{E}_j\int_{t_j}^{t_{j+1}}(Z^R_s-Z^R_{t_j})ds\right)^2\delta W^2_{t_j}\right].\end{aligned}$$

Recalling $h^R = \nabla u^R(t,x)$ and the relation (1.2.6), we have by Itô's formula

$$
\begin{aligned}
&\sum_{j=0}^{M-1} \frac{1}{(\delta t)^3} \int_{\mathbb{R}} \mathbb{E}\left[\left(\mathbb{E}_j \int_{t_j}^{t_{j+1}} (Z_s^R - Z_{t_j}^R) ds \right)^2 \delta W_{t_j}^2 \right] \\
&= \sum_{j=0}^{M-1} \frac{1}{\delta t^3} \int_{\mathbb{R}} \mathbb{E}\left[\left(\mathbb{E}_j \int_{t_j}^{t_{j+1}} \Big(\int_{t_j}^{s} \partial_t h^R(k, \mathcal{X}_k) dk + \sqrt{\nu} \int_{t_j}^{s} \partial_x h^R(k, \mathcal{X}_k) dW_k + \frac{\nu}{2} \int_{t_j}^{s} \partial_{xx} h^R(k, \mathcal{X}_k) dk \Big) ds \right)^2 (\delta W_{t_j})^2 \right] \\
&= \sum_{j=0}^{M-1} \frac{1}{\delta t^3} \int_{\mathbb{R}} \mathbb{E}\left[\left(\mathbb{E}_j \int_{t_j}^{t_{j+1}} \Big(\int_{t_j}^{s} \partial_t h^R(k, \mathcal{X}_k) dk + \frac{\nu}{2} \int_{t_j}^{s} \partial_{xx} h^R(k, \mathcal{X}_k) dk \Big) ds \right)^2 (\delta W_{t_j})^2 \right] \\
&\le \sum_{j=0}^{M-1} \frac{1}{\delta t} \int_{\mathbb{R}} \mathbb{E}\left[\left(\mathbb{E}_j \Big(\int_{t_j}^{t_{j+1}} |\partial_t h^R(k, \mathcal{X}_k)| dk + \frac{\nu}{2} \int_{t_j}^{t_{j+1}} |\partial_{xx} h^R(k, \mathcal{X}_k)| dk \Big) \right)^2 (\delta W_{t_j})^2 \right] \\
&\le \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} \left[\left(\mathbb{E}_j \Big(\int_{t_j}^{t_{j+1}} 2\,|\partial_t h^R(k, \mathcal{X}_k)|^2 dk + \frac{\nu^2}{2} \int_{t_j}^{t_{j+1}} |\partial_{xx} h^R(k, \mathcal{X}_k)|^2 dk \Big) \right) (\delta W_{t_j})^2 \right] \\
&= (\delta t) \sum_{j=0}^{M-1} \left(\int_{\mathbb{R}} \int_{t_j}^{t_{j+1}} 2 |\partial_t h^R(k,x)|^2 dk + \int_{\mathbb{R}} \frac{\nu^2}{2} \int_{t_j}^{t_{j+1}} |\partial_{xx} h^R(k,x)|^2 dk \right) \\
&= (\delta t) \left(\int_{\mathbb{R}} \int_0^T 2 |\partial_t h^R(k,x)|^2 dk + \int_{\mathbb{R}} \frac{\nu^2}{2} \int_0^T |\partial_{xx} h^R(k,x)|^2 dk \right).
\end{aligned}
$$

Next, the continuity of function f in condition (iii) implies that

$$
\begin{aligned}
\frac{1}{\delta t} \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} (\mathrm{III}_j^R)^2 &= \frac{1}{\delta t} \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} \left(\int_{t_j}^{t_{j+1}} f^I(s, \mathcal{X}_s, Y_s^R, Y_s^I) ds - f^I(t_j, \mathcal{X}_{t_j}, Y_{t_j}^R, Y_{t_j}^I) \delta t \right)^2 \\
&\le \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} \int_{t_j}^{t_{j+1}} |f^I(s, \mathcal{X}_s, Y_s^R, Y_s^I) - f^I(t_j, \mathcal{X}_{t_j}, Y_{t_j}^R, Y_{t_j}^I)|^2 ds \\
&\le C \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} \int_{t_j}^{t_{j+1}} \left[|Y_s^R - Y_{t_j}^R|^2 + |Y_s^I - Y_{t_j}^I|^2 + (\rho(s - t_j) + |\mathcal{X}_s - \mathcal{X}_{t_j}|^2) |Y_{t_j}|^2 \right] ds \\
&\le C\,[\rho(\delta t) + \delta t] + C \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} \int_{t_j}^{t_{j+1}} \left[|Y_s^R - Y_{t_j}^R|^2 + |Y_s^I - Y_{t_j}^I|^2 \right] ds.
\end{aligned}
$$

We will only look at the real part as the imaginary part is concluded similarly.

$$
\begin{aligned}
&C \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} \int_{t_j}^{t_{j+1}} |Y_s^R - Y_{t_j}^R|^2 ds \\
&\le C \sum_{j=0}^{M-1} \mathbb{E} \int_{\mathbb{R}} \int_{t_j}^{t_{j+1}} \left[\left| \int_{t_j}^{s} \partial_t u(k, \mathcal{X}_k) dk \right|^2 + \left| \int_{t_j}^{s} \partial_x u(k, \mathcal{X}_k) dW_k \right|^2 + \left| \int_{t_j}^{s} \partial_{xx} u(k, \mathcal{X}_k) dk \right|^2 \right] ds.
\end{aligned}
$$

We concentrate only on the middle term as this term gives us the worst decay, and the other terms are handled similarly. Itô isometry and straightforward calculations yield that

$$C\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\int_{t_j}^{t_{j+1}}\left|\int_{t_j}^{s}\partial_x u(k,\mathcal{X}_k)dW_k\right|^2\leq(\delta t)C\int_{\mathbb{R}}\int_0^T|\partial_x u(k,\mathcal{X}_k)|^2dk.$$

Thus, $\frac{1}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}(\mathrm{III}_j^R)^2\leq C\left\{\delta t+\rho(\delta t)\right\}$.

Now we examine

$$\begin{aligned}\frac{1}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}(\mathrm{I}_{B,j}^R)^2=\frac{1}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\Bigg(&\sqrt{\nu}\int_{t_j}^{t_{j+1}}(Z_s^R+Z_s^I)*dW_s-\frac{\sqrt{\nu}}{2}(Z_{t_{j+1}}^R+Z_{t_{j+1}}^I-Z_{t_j}^R-Z_{t_j}^I)\delta W_{t_j}\\&-\frac{\nu}{2}\Big[\partial_x h^R(t_j,\mathcal{X}_{t_j})\delta t-\partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2+\partial_x h^I(t_j,\mathcal{X}_{t_j})\delta t-\partial_x h^I(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\Big]\Bigg)^2.\end{aligned}$$

As the part involving Z^I is similar, we only focus on

$$\begin{aligned}&\frac{1}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left(\sqrt{\nu}\int_{t_j}^{t_{j+1}}Z_s^R*dW_s-\frac{\sqrt{\nu}}{2}(Z_{t_{j+1}}^R-Z_{t_j}^R)\delta W_{t_j}-\frac{\nu}{2}\Big[\partial_x h^R(t_j,\mathcal{X}_{t_j})\delta t-\partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\Big]\right)^2\\
\leq&\frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\nu}{2}\int_{t_j}^{t_{j+1}}\partial_x h^R(s,\mathcal{X}_s)ds-\frac{\nu}{2}\partial_x h^R(t_j,\mathcal{X}_{t_j})\delta t\right]^2\\
&+\frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\sqrt{\nu}}{2}(Z_{t_{j+1}}^R-Z_{t_j}^R)\delta W_{t_j}-\frac{\nu}{2}\partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\right]^2\\
=:&J_1+J_2.\end{aligned}$$

For J_1, applying Itô's formula yields that

$$\begin{aligned}J_1=&\frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\nu}{2}\int_{t_j}^{t_{j+1}}\partial_x h^R(s,\mathcal{X}_s)ds-\frac{\nu}{2}\partial_x h^R(t_j,x+\sqrt{\nu}W_{t_j})\delta t\right]^2\\
\leq&C\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\int_{t_j}^{t_{j+1}}\left|\partial_x h^R(s,\mathcal{X}_s)ds-\partial_x h^R(t_j,x+\sqrt{\nu}W_{t_j})\right|^2ds\\
=&C\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\int_{t_j}^{t_{j+1}}\Big|\int_{t_j}^{s}\partial_t\partial_x h^R(k,\mathcal{X}_k)dk\Big|^2+\Big|\int_{t_j}^{s}\partial_{xx}h^R(k,\mathcal{X}_k)dW_k\Big|^2+\Big|\int_{t_j}^{s}\partial_{xxx}h^R(k,\mathcal{X}_k)dk\Big|^2ds.\end{aligned}$$

The middle term has the worst decay rate, but it is still enough for our purpose, as

$$C\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\int_{t_j}^{t_{j+1}}\Big|\int_{t_j}^{s}\partial_{xx}h^R(k,\mathcal{X}_k)dW_k\Big|^2ds\leq C\delta t\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\int_{t_j}^{t_{j+1}}|\partial_{xx}h^R(k,\mathcal{X}_k)|^2dk$$

$$= C\delta t\mathbb{E}\int_{\mathbb{R}}\int_0^T |\partial_{xx}h^R(k,x)|^2 dk.$$

For J_2, we have

$$\begin{aligned}
J_2 =& \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\sqrt{\nu}}{2}(h^R(t_{j+1},\mathcal{X}_{t_{j+1}})-h^R(t_j,\mathcal{X}_{t_j}))\delta W_{t_j} - \frac{\nu}{2}\partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\right]^2\\
\le& \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\sqrt{\nu}}{2}(h^R(t_{j+1},\mathcal{X}_{t_{j+1}})-h^R(t_{j+1},\mathcal{X}_{t_j}))\delta W_{t_j} - \frac{\nu}{2}\partial_x h^R(t_{j+1},\mathcal{X}_{t_j})(\delta W_{t_j})^2\right]^2\\
&+\frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\sqrt{\nu}}{2}(h^R(t_{j+1},\mathcal{X}_{t_j})-h^R(t_j,\mathcal{X}_{t_j}))\delta W_{t_j}\right]^2\\
&+\frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\nu}{2}\partial_x h^R(t_{j+1},\mathcal{X}_{t_j})(\delta W_{t_j})^2 - \frac{\nu}{2}\partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\right]^2\\
=:& J_2^1 + J_2^2 + J_2^3.
\end{aligned}$$

For J_2^2, it holds that

$$\begin{aligned}
J_2^2 &= \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\Big(h^R(t_{j+1},x+\sqrt{\nu}W_{t_j})-h^R(t_j,x+\sqrt{\nu}W_{t_j})\Big)\delta W_{t_j}\right]^2\\
&= \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\Big(\int_{t_j}^{t_{j+1}}\partial_t h^R(k,x+\sqrt{\nu}W_{t_j})dk\Big)^2(\delta W_{t_j})^2\right]\\
&\le C\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\Big(\int_{t_j}^{t_{j+1}}|\partial_t h^R(k,x+\sqrt{\nu}W_{t_j})|^2dk\Big)(\delta t)\right]\\
&= C\delta t\int_{\mathbb{R}}\int_0^T|\partial_t h^R(k,x)|^2dk.
\end{aligned}$$

For J_2^3, applying Fubini theorem and Hölder's inequality gives

$$\begin{aligned}
J_2^3 =& \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\nu}{2}\partial_x h^R(t_{j+1},\mathcal{X}_{t_j})(\delta W_{t_j})^2 - \frac{\nu}{2}\partial_x h^R(t_j,\mathcal{X}_{t_j})(\delta W_{t_j})^2\right]^2\\
=& \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\int_{t_j}^{t_{j+1}}\partial_t\partial_x h^R(k,x+\sqrt{\nu}W_{t_j})dk\right]^2(\delta W_{t_j})^4\\
\le& C\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\int_{t_j}^{t_{j+1}}|\partial_t\partial_x h^R(k,x+\sqrt{\nu}W_{t_j})|^2dk(\delta W_{t_j})^4\\
\le& C|\delta t|^2\int_{\mathbb{R}}\int_0^T|\partial_t\partial_x h^R(k,x)|^2dk.
\end{aligned}$$

For J_2^1, simply integrating by parts gives

$$
\begin{aligned}
J_2^1 &= \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\sqrt{\nu}}{2}(h^R(t_{j+1},\mathcal{X}_{t_{j+1}})-h^R(t_{j+1},\mathcal{X}_{t_j}))\delta W_{t_j}-\frac{\nu}{2}\partial_x h^R(t_{j+1},\mathcal{X}_{t_j})(\delta W_{t_j})^2\right]^2\\
&= \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\left[\frac{\sqrt{\nu}}{2}\left(\partial_x h^R(t_{j+1},x+\sqrt{\nu}W_{t_j})\sqrt{\nu}(\delta W_{t_j})^2\right.\right.\\
&\quad\left.\left.+\int_0^1\partial_{xx}h^R\big(t_{j+1},x+\sqrt{\nu}W_{t_j}+\theta(\sqrt{\nu}W_{t_{j+1}}-\sqrt{\nu}W_{t_j})\big)(1-\theta)d\theta\nu(\delta W_{t_j})^3\right)-\frac{\nu}{2}\partial_x h^R(t_{j+1},\mathcal{X}_{t_j})(\delta W_{t_j})^2\right]^2\\
&\le \frac{C}{\delta t}\sum_{j=0}^{M-1}\mathbb{E}\int_{\mathbb{R}}\int_0^1|\partial_{xx}h^R\big(t_{j+1},x+\sqrt{\nu}W_{t_j}+\theta(\sqrt{\nu}W_{t_{j+1}}-\sqrt{\nu}W_{t_j})\big)|^2d\theta(\delta W_{t_j})^6\\
&\le C|\delta t|^2\int_{\mathbb{R}}\sum_{j=0}^{M-1}|\partial_{xx}h^R(t_{j+1},x)|^2.
\end{aligned}
$$

Summing up the above estimates yields the desired result. □

Chapter 5

Future Work: A Viscosity Solution Theory for Stochastic Hamilton-Jacobi-Bellman Equations in Wasserstein Space

This chapter treats the stochastic-coefficient case: we rigorously prove the dynamic programming principle and subsequently show existence of a viscosity solution; we do not address uniqueness here.

5.1 Dynamic Programming Principle of the McKean Vlasov Control Problem

5.1.1 Setup of the Probability Space

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form $(\Omega^0 \times \Omega^1, \mathcal{F}^0 \otimes \mathcal{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$. The space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ supports a d-dimensional Brownian motion W^0 which we regard as the common noise. The space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ is further decomposed as $(\tilde{\Omega}^1 \times \hat{\Omega}^1, \mathcal{G} \otimes \hat{\mathcal{F}}^1, \tilde{\mathbb{P}}^1 \otimes \hat{\mathbb{P}}^1)$. On $(\hat{\Omega}^1, \hat{\mathcal{F}}^1, \hat{\mathbb{P}}^1)$, there lives a d-dimensional Brownian motion W which is independent of W^0 and is regarded as the idiosyncratic noise. The space $(\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1)$ is where the initial

random variables lives. Without loss of generality, we assume that this space is rich enough to support all probability laws on $\mathbb{R}^d$.

We write an element $\omega \in \Omega$ as $\omega = (\omega^0, \omega^1)$, and identify the Brownian motions $W(\omega) = W(\omega^1)$, $W^0(\omega) = W^0(\omega^0)$. Expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$, while $\mathbb{E}^0$ (resp. $\mathbb{E}^1$) refers to expectation under $\mathbb{P}^0$ (resp. $\mathbb{P}^1$). Also, we define the following filtrations:

- $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0} := (\sigma(W^0_r)_{0\leq r\leq s} \vee \sigma(W_r)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$;
- $\mathbb{F}^t = (\mathcal{F}^t_s)_{s\geq 0} := (\sigma(W^0_r)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$;
- $\mathbb{F}^0 = (\mathcal{F}^0_s)_{s\geq 0} := (\sigma(W^0_r)_{0\leq r\leq s})_{s\geq 0}$, $\mathbb{F}^1 = (\mathcal{F}^1_s)_{s\geq 0} := (\sigma(W_r)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$;
- $\mathbb{F}^{1;t} = (\mathcal{F}^{1;t}_s)_{s\geq 0} := (\sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$, $\hat{\mathbb{F}}^{1;t} = (\hat{\mathcal{F}}^{1;t}_s)_{s\geq 0} := (\sigma(W_{r\vee t} - W_t)_{0\leq r\leq s})_{s\geq 0}$;
- $\bar{\mathbb{F}}^t = (\bar{\mathcal{F}}^t_s)_{s\geq 0} := (\sigma(W^0_{r\vee t} - W^0_t)_{0\leq r\leq s} \vee \sigma(W_{r\vee t} - W_t)_{0\leq r\leq s} \vee \mathcal{G})_{s\geq 0}$, $\bar{\mathbb{F}}^{0;t} = (\bar{\mathcal{F}}^{0;t}_s)_{s\geq 0} := (\sigma(W^0_{r\vee t} - W^0_t)_{0\leq r\leq s})_{s\geq 0}$.

Without loss of generality, we assume they are $\mathbb{P}$-complete.

5.1.2 Sense of Differentiability in the Wasserstein Space

Definition 5.1.1. *We introduce over $\mathbb{R}^d$ the space of probability measures $\mathcal{P}(\mathbb{R}^d)$ and its subset $\mathcal{P}_p(\mathbb{R}^d)$ of those with finite p-th moment for $p \geq 1$. For $q \in [1, \infty)$, the q-Wasserstein distance is defined by*

$$\mathcal{W}_q(\mu, \nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^q \pi(dx, dy) \right)^{\frac{1}{q}}, \quad \text{for } \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d) \text{ and } p \geq q \quad (5.1.1)$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second marginal ν.

We call $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p)$ the p-th Wasserstein space over $\mathbb{R}^d$, and it is a Polish space. Finally, we denote $\mathrm{Supp}(\mu)$ *the support of $\mu \in \mathcal{P}(\mathbb{R}^d)$.*

For a map $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we adopt the notion of L-derivative (see Cardaliaguet *et al.* (2019) for instance) which is recalled as follows:

Definition 5.1.2. *The function $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be first order L-differentiable if its lifting $F : [0,T] \times L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d) \to \mathbb{R}$; $F(t,\xi) := f(t, \mathcal{L}(\xi))$ is Fréchet differentiable in $L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$ for each $t \in [0,T]$.*

Remark 5.1.1. *By (Carmona & Delarue, 2018c, Proposition 5.25), the Fréchet derivative $D_\xi F$ can be represented by a measurable function, denoted by $\partial_\mu f(t,\mu)(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, such that $D_\xi F(t,\xi) = \partial_\mu f(t,\mu)(\xi)$ for any $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\xi \in L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$ with $\mathcal{L}(\xi) = \mu$. We say that $\partial_\mu f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is the first order L-derivative of f.*

Definition 5.1.3. *The function $f : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be second order L-differentiable if f is first order L-differentiable and for any $(t,x) \in [0,T] \times \mathbb{R}^d$, the function $\mu \mapsto \partial_\mu f(t,\mu)(x)$ is L-differentiable. Similar to Remark 5.1.1, we say that $\partial^2_\mu f = \partial_\mu(\partial_\mu f) : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is the second order L-derivative of f.*

5.1.3 The Set of Admissible Controls

Let $A \subset \mathbb{R}^d$ be a compact subset equipped with the distance d_A. Let $t \in [0,T]$, we denote $\mathcal{A}$ (resp. $\mathcal{A}_t$) the set of A-valued $\mathbb{F}$-progressively measurable processes (resp. $\mathbb{F}^t$-progressively measurable processes). The spaces $\mathcal{A}$ and $\mathcal{A}_t$, endowed with the respective Krylov distances $\Delta(\alpha,\beta) := \mathbb{E}\big[\int_0^T d_A(\alpha_r, \beta_r) dr\big]$ and $\Delta_t(\alpha,\beta) := \mathbb{E}\big[\int_t^T d_A(\alpha_r, \beta_r) dr\big]$, are separable metric spaces. Denote by $\mathcal{B}_\mathcal{A}$ (resp. $\mathcal{B}_{\mathcal{A}_t}$) the Borel σ-algebra of $\mathcal{A}$ (resp. $\mathcal{A}_t$).

Without loss of generality, we assume that $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ and $(\hat{\Omega}^1, \hat{\mathcal{F}}^1, \hat{\mathbb{P}}^1)$ are the canonical spaces, i.e., $\Omega^0 = C(\mathbb{R}_+, \mathbb{R}^d)$ and $\hat{\Omega}^1 = C(\mathbb{R}_+, \mathbb{R}^d)$. We introduce the class of shifted control processes constructed by concatenation of paths: for $\alpha \in \mathcal{A}$ and $(r, \bar{\omega}^0, \bar{\tilde{\omega}}^1) \in [0,T] \times \Omega^0 \times \hat{\Omega}^1$, set

$$\alpha_s^{r,\bar{\omega}^0,\bar{\tilde{\omega}}^1}(\omega^0, \hat{\omega}^1, \tilde{\omega}^1) := \alpha_s(\bar{\omega}^0 \otimes_r \omega^0, \bar{\tilde{\omega}}^1 \otimes_r \hat{\omega}^1, \tilde{\omega}^1), \text{ for } (s, \omega^0, \hat{\omega}^1, \tilde{\omega}^1) \in [0,T] \times \Omega^0 \times \hat{\Omega}^1 \times \tilde{\Omega}^1,$$

where

$$\bar{\omega}^0 \otimes_r \omega^0(s) := \bar{\omega}^0(s)\mathbb{1}_{\{s<r\}} + \big(\bar{\omega}^0(r) + \omega^0(s) - \omega^0(r)\big)\mathbb{1}_{\{s\geq r\}},$$

and similarly for $\bar{\hat{\omega}}^1 \otimes_r \hat{\omega}^1$. For $\alpha \in \mathcal{A}$, and $(r, \bar{\omega}^0) \in [0,T] \times \Omega^0$, we set

$$\alpha_s^{r,\bar{\omega}^0}(\omega^0, \hat{\omega}^1, \tilde{\omega}^1) := \alpha_s(\bar{\omega}^0 \otimes_r \omega^0, \hat{\omega}^1, \tilde{\omega}^1), \text{ for } (s, \omega^0, \hat{\omega}^1, \tilde{\omega}^1) \in [0,T] \times \Omega^0 \times \hat{\Omega}^1 \times \tilde{\Omega}^1.$$

Fix $t \in [0,T]$, the concatenations $\alpha^{r,\bar{\omega}^0,\bar{\hat{\omega}}^1}$ and $\alpha^{r,\bar{\omega}^0}$ for $\alpha \in \mathcal{A}_t$ and $(r, \bar{\omega}^0, \bar{\hat{\omega}}^1) \in [t,T] \times \Omega^0 \times \hat{\Omega}^1$ are defined in the same way. For $s \in [0,T]$, we define α^s by

$$\alpha^s(\omega^0, \omega^1) := \alpha^{s,\omega^0,\omega^1}(\omega^0, \omega^1), \text{ for } (\omega^0, \omega^1) \in \Omega^0 \times \Omega^1.$$

Finally, we define $\bar{\mathcal{A}}_t$ as the set of A-valued $\bar{\mathbb{F}}^t$-progressively measurable processes, with the Krylov distance $\Delta_t(\alpha, \beta)$. Denote by $\mathcal{B}_{\bar{\mathcal{A}}_t}$ the Borel σ-algebra of $\bar{\mathcal{A}}_t$.

Lemma 5.1.4. *Let $t \in [0,T]$ and $\alpha \in \mathcal{A}_t$, the map $(\Omega^0, \mathcal{F}_t^0) \ni \omega^0 \mapsto \alpha^{t,\omega^0} \in (\bar{\mathcal{A}}_t, \mathcal{B}_{\bar{\mathcal{A}}_t})$ is measurable.*

We postpone the proof of the above lemma to Appendix 5.3.

5.1.4 Function Spaces and Notations

Notations We denote by $\mathscr{P}$ the σ-algebra of predictable sets on $\Omega^0 \times [0,T]$ associated with $\{\mathcal{F}_t^0\}_{t\geq 0}$. Let $Prog(\mathbb{F}^0)$ denote the σ-algebra on $[0,T] \times \Omega^0$ of all $\mathbb{F}^0$-progressive sets. Furthermore, let $\mathcal{T}_{t,T}^0$ be the set of $\mathcal{F}^0$-stopping times taking values in $[t,T]$, and denote by $\mathcal{M}_t$ the set of $\mathcal{F}_t^t$-measurable random variables.

As we will be working with both measures and measure-valued random variables, we use bold letters to distinguish between them. For instance, we denote a measure by μ and a measure-valued random variable by $\boldsymbol{\mu}$. Since we will not consider any deterministic flow of measures, μ_s (without the use of bold letters) will refer to a measure-valued stochastic process.

For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\varphi \in L^2((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu); \mathbb{R}^d)$, $\mu \otimes \mu$, $\psi \in L^2(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d), \mu \otimes \mu)$, we define the notation,

$$\mu(\varphi) := \int_{\mathbb{R}^d} \varphi(x)\mu(dx), \quad \mu \otimes \mu(\psi) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, x')\mu(dx)\mu(dx').$$

Definition 5.1.5. *(Function Spaces). Let $\mathbb{B}$ be a Banach space equipped with the norm $\|\cdot\|_{\mathbb{B}}$. For each $t \in [0, T]$, denote by $L^0((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{B})$ the space of $\mathbb{B}$-valued $\mathcal{F}_t$-measurable random variables. For $p \geq 1$, denote by $L^p((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{B})$ the Banach space of p-integrable $\mathbb{B}$-valued $\mathcal{F}_t$-measurable random variables equipped with the norm:*

$$\|X\|_p := (\mathbb{E}\|X\|_{\mathbb{B}}^p)^{1/p}.$$

For $p \in [1, \infty]$, $\mathcal{S}^p(\mathbb{B})$ is set of all the $\mathbb{B}$-valued, $\mathscr{P}$-measurable continuous processes $\{\mathcal{X}_t\}_{t\in[0,T]}$ such that

$$\|\mathcal{X}\|_{\mathcal{S}^p(\mathbb{B})} := \Big\| \sup_{t\in[0,T]} \|\mathcal{X}_t\|_{\mathbb{B}} \Big\|_{L^p(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)} < \infty.$$

Denote by $\mathcal{L}^p(\mathbb{B})$ totality of all the $\mathbb{B}$-valued, $\mathscr{P}$-measurable processes $\{\mathcal{X}_t\}_{t\in[0,T]}$ such that

$$\|\mathcal{X}\|_{\mathcal{L}^p(\mathbb{B})} := \Big\| \Big(\int_0^T \|\mathcal{X}_t\|_{\mathbb{B}}^p dt \Big)^{1/p} \Big\|_{L^p(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)} < \infty.$$

Obviously $(\mathcal{S}^p(\mathbb{B}), \|\cdot\|_{\mathcal{S}^p(\mathbb{B})})$ and $(\mathcal{L}^p(\mathbb{B}), \|\cdot\|_{\mathcal{L}^p(\mathbb{B})})$ are Banach spaces. Now, we introduce various spaces of continuous functions. Let (X, d) be a metric space, define

$$C(X; \mathbb{B}) := \Big\{ f : X \to \mathbb{B} \,\Big|\, \sup_{x\in X} \|f(x)\|_{\mathbb{B}} < \infty \Big\}.$$

We equip this space with the usual supremum norm, i.e., $\|f\|_{C(X;\mathbb{B})} := \sup_{x\in X} \|f(x)\|_{\mathbb{B}}$. We consider the following subspaces of $C(X; \mathbb{B})$,

$$C_0(X; \mathbb{B}) := \left\{ f \in C(X; \mathbb{B}) \;\middle|\; \begin{array}{l} \textit{for all } \varepsilon > 0, \textit{ there exists a} \\ \\ \textit{compact subset } J \subseteq X \textit{ such that } \sup\limits_{x\in X\setminus J} \|f\|_{C(X;\mathbb{B})} < \varepsilon \end{array} \right\},$$

$C^{\mathrm{Lip}(K)}(X;\mathbb{B}) := \Big\{ f \in C(X;\mathbb{B}) \,\Big|\, \|f(x)-f(y)\|_{\mathbb{B}} \le K d(x,y) \text{ for all } x,y \in X \Big\}$, *where* $K>0$.

We write $C(X)$, $C_0(X)$, *and* $C^{\mathrm{Lip}(K)}(X)$ *if* $\mathbb{B}=\mathbb{R}$. *Whenever a function from these classes allows for a version with enhanced properties, we consistently choose the version with superior attributes.*

5.1.5 The Problem

Let the following functions

$$b : [0,T]\times\Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}^d, \quad \sigma,\, \sigma^0 : [0,T]\times\Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}^{d\times d},$$

$$f : [0,T]\times\Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\to\mathbb{R}, \quad g : \Omega^0\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}.$$

For any $t\in[0,T]$, $\xi\in L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)$ and $\alpha\in\mathcal{A}_t$, we consider the process $X^{t,\xi,\alpha}$ evolving as follows:

$$\begin{cases} dX_s^{t,\xi,\alpha} &= b_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + \sigma_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s \\ &\quad +\sigma_s^0(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s^0, \quad \text{for } s\in[t,T]; \\ X_t^{t,\xi,\alpha} &= \xi. \end{cases} \tag{5.1.2}$$

The cost functional J on $[0,T]\times L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)\times\mathcal{A}_t\times\Omega^0$ is defined by

$$J(t,\xi,\alpha) := \mathbb{E}\Big[\int_t^T f_s\Big(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s\Big)ds + g\Big(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}\Big)\,\Big|\,\mathcal{F}_t^0\Big]. \tag{5.1.3}$$

We study the corresponding value function v on $[0,T]\times L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)\times\Omega^0$:

$$v(t,\xi) := \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_t} J(t,\xi,\alpha). \tag{5.1.4}$$

The following assumptions will be used throughout our work.

Assumption $(\mathcal{A}1)$**.** *The coefficient functions* g *and* $h=f$, b, σ *or* σ^0 *satisfy the following:*

(1). For all $a\in A$, $\big(h_t(\cdot,\cdot,a)\big)_{0\le t\le T}$ *is* $(\mathcal{F}_t^0)_{t\ge0}$*-adapted. (or progressively measurable)*

(2). For $\mathcal{L}^1\otimes\mathbb{P}^0$*-a.e.* $(t,\omega^0)\in[0,T]\times\Omega^0$, $h_t(\cdot,\cdot,\cdot)$ *is continuous on* $\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A$.

(3). There exists $K > 0$ such that for all x, $x' \in \mathbb{R}^d$, $t \in [0,T]$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ such that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\omega^0 \in \Omega^0} |g(x,\mu)| + \operatorname*{ess\,sup}_{\omega^0 \in \Omega^0} \Big(\sup_{a \in A} |h_t(x,\mu,a)| \Big) \leq K \quad \text{and} \\
&\operatorname*{ess\,sup}_{\omega^0 \in \Omega^0} |g(x,\mu) - g(x',\mu')| + \operatorname*{ess\,sup}_{\omega^0 \in \Omega^0} \Big(\sup_{a \in A} |h_t(x,\mu,a) - h_t(x',\mu',a)| \Big) \\
&\leq K(|x - x'| + \mathcal{W}_2(\mu,\mu')).
\end{aligned}$$

Assumption ($\mathcal{A}1^*$)**.** *This is a strengthening assumption of ($\mathcal{A}1$). In addition to ($\mathcal{A}1$), we further assume that there exists $K > 0$ such that for all x, $x' \in \mathbb{R}^d$, $t \in [0,T]$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, it holds that*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\omega^0 \in \Omega^0} |g(x,\mu) - g(x',\mu')| + \operatorname*{ess\,sup}_{\omega^0 \in \Omega^0} \Big(\sup_{a \in A} |h_t(x,\mu,a) - h_t(x',\mu',a)| \Big) \\
&\leq K(|x - x'| + \mathcal{W}_1(\mu,\mu'))
\end{aligned}$$

Assumption ($\mathcal{A}3$)**.** *We need the following assumption in order to ensure the approximation results like Lemma 5.1.6.*

(1). The map $g : (\Omega^0, \mathcal{F}_T^0, \mathbb{P}^0) \to C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is Bochner measurable.

(2). For all $t \in [0,T]$, the maps $\big(\Omega^0, \mathcal{F}_t^0, \mathbb{P}^0\big) \ni \omega^0 \mapsto b_t(\omega^0, \cdot, \cdot, \cdot) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$, $\big(\Omega^0, \mathcal{F}_t^0, \mathbb{P}^0\big) \ni \omega^0 \mapsto f_t(\omega^0, \cdot, \cdot, \cdot) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$ and $\big(\Omega^0, \mathcal{F}_t^0, \mathbb{P}^0\big) \ni \omega^0 \mapsto \sigma_t(\omega^0, \cdot, \cdot) \in C(\mathbb{R}^d \times A)$ are Bochner measurable.

(3). For $h = f$, b, σ, $(\Omega^0 \times [0,T], \mathcal{F}^0 \otimes \mathcal{B}[0,T], \mathbb{P}^0 \otimes ds) \ni (\omega^0, t) \mapsto h_t(\omega^0, \cdot, \cdot, \cdot) \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$ is progressively measurable.

The below approximation result follows from Assumption ($\mathcal{A}3$), with the proof deferred to the Appendix 5.3.

Lemma 5.1.6. *For each $\varepsilon > 0$, there exists a partition $0 = t_0 < t_1 < \cdot < t_{N-1} < t_N = T$ for some $N > 3$ and functions*

$$(g^N, f^N, b^N) \in C\Big(\mathcal{P}_2(\mathbb{R}^d); C^3(\mathbb{R}^{d \times N} \times \mathbb{R}^d)\Big) \times C\Big(\mathcal{P}_2(\mathbb{R}^d) \times A; C^3([0,T] \times \mathbb{R}^{d \times N} \times \mathbb{R}^d)\Big) \times$$

$$\times C\Big(\mathcal{P}_2(\mathbb{R}^d)\times A; C^3([0,T]\times\mathbb{R}^{d\times N}\times\mathbb{R}^d)\Big)$$

such that

$$\begin{aligned}
g^\varepsilon &:= \operatorname*{ess\,sup}_{(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)} \Big|g^N(W^0_{t_1},\dots,W^0_{t_N},x,\mu)-g(x,\mu)\Big|,\\
f^\varepsilon_t &:= \operatorname*{ess\,sup}_{(x,\mu,a)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A} \Big|f^N_t(W^0_{t_1\wedge t},\dots,W^0_{t_N\wedge t},x,\mu,a)-f_t(x,\mu,a)\Big|,\\
b^\varepsilon_t &:= \operatorname*{ess\,sup}_{(x,\mu,a)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A} \Big|b^N_t(W^0_{t_1\wedge t},\dots,W^0_{t_N\wedge t},x,\mu,a)-b_t(x,\mu,a)\Big|,
\end{aligned}$$

are $\mathcal{F}^0_t$-adapted with

$$\|g^\varepsilon\|_{L^2((\Omega^0,\mathcal{F}^0_T,\mathbb{P}^0);\mathbb{R})}+\|f^\varepsilon\|_{\mathcal{L}^2(\mathbb{R})}+\|b^\varepsilon\|_{\mathcal{L}^2(\mathbb{R})}<\varepsilon,$$

and g^N, f^N and b^N are uniformly Lipschitz continuous in the variables x, μ with an identical Lipschitz constant K independent of N and ε. Moreover, for fixed $(t,x,\mu,a)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A$, the functions $h=g^N,f^N,b^N$ have compact support on $\mathbb{R}^{d\times N}$ when considering $h_t(\cdot,\dots,\cdot,x,\mu,a)$.

Below are some properties of strong solutions to the SDE (5.1.2).

Lemma 5.1.7. *Suppose that ($\mathcal{A}1$) holds. For any $p\in[2,\infty)$, there exists a constant $C>0$ depending only on K, p and T such that for any $0\le r\le t\le s\le T$, $\alpha\in\mathcal{A}_r$, $\xi\in L^p((\Omega,\mathcal{F}_r,\mathbb{P});\mathbb{R}^d)$ and strong solution $X^{r,\xi,\alpha}$ to the SDE (5.1.2), there following statements are true.*

(1). The two processes $(X^{r,\xi,\alpha}_s)_{t\le s\le T}$ and $(X^{t,X^{r,\xi,\alpha}_t,\alpha}_s)_{t\le s\le T}$ are $\mathbb{P}$-indistinguishable.

(2). $\mathbb{E}_{\mathcal{F}^0_r}\sup\limits_{r\le l\le T}|X^{r,\xi,\alpha}_l|^p\le C\big(1+\mathbb{E}_{\mathcal{F}^0_r}|\xi|^p\big)$, $\mathbb{P}^0$-a.s.;

(3). $\mathbb{E}_{\mathcal{F}^0_r}|X^{r,\xi,\alpha}_s-X^{r,\xi,\alpha}_t|^p\le C\big(1+\mathbb{E}_{\mathcal{F}^0_r}|\xi|^p\big)(s-t)^{p/2}$, $\mathbb{P}^0$-a.s.;

(4). Given another $\eta\in L^p((\Omega,\mathcal{F}_r,\mathbb{P});\mathbb{R}^d)$, it holds that $\mathbb{E}_{\mathcal{F}^0_r}\max\limits_{r\le l\le T}|X^{r,\xi,\alpha}_l-X^{r,\eta,\alpha}_l|^p\le C\,\mathbb{E}_{\mathcal{F}^0_r}|\xi-\eta|^p$, $\mathbb{P}^0$-a.s..

The cost functional J and the value function v exhibit certain regularity properties.

Lemma 5.1.8. *Suppose that (A1) holds. For each $t \in [0,T]$, $\alpha \in \mathcal{A}_t$ and $\eta, \xi \in L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{R}^d)$, the following are true:*

(1). The process $\big(\mathbb{E}[J(s, X_s^{t,\xi,\alpha}, \alpha)|\mathcal{F}_s^0]\big)_{t\le s\le T}$ is continuous;

(2). The process $(v(s, X_s^{t,\xi,\alpha}))_{t\le s\le T}$ is continuous;

(3). There exists $C > 0$ depending on K and T such that

$$|v(t,\xi) - v(t,\eta)| + |J(t,\xi,\alpha) - J(t,\eta,\alpha)| \le C\Big(\mathbb{E}_{\mathcal{F}_r^0}|\xi-\eta|^2\Big)^{1/2}, \quad \mathbb{P}^0\text{-a.s.}$$

(4). $v(\cdot,\xi)$ and $J(\cdot,\xi,\alpha)$ are continuous on $[t,T]$ with probability 1 and

$$\operatorname*{ess\,sup}_{(t,\xi)\in[0,T]\times L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)} \max\{|v(t,\xi)|, |J(t,\xi,\alpha)|\} \le K(T+1), \quad \mathbb{P}^0\text{-a.s.;}$$

(5). When the control is restricted on $\bar{\mathcal{A}}_t$, $J(t,\xi,\cdot)$ is continuous in $\bar{\mathcal{A}}_t$ with probability 1.

Proof. **Proofs of (2):** For each $\beta \in \mathcal{A}_t$, define the nonempty set

$$\mathscr{J}(t,\xi,\beta) := \left\{J(t,\xi,\beta^\dagger) : J(t,\xi,\beta^\dagger) \le J(t,\xi,\beta) \text{ for } \beta^\dagger \in \mathcal{A}_t\right\}.$$

For any $J(t,\xi,\beta^1), J(t,\xi,\beta^2) \in \mathscr{J}(t,\xi,\beta)$, we define

$$\gamma_s := \beta_s \mathbb{1}_{[0,t)}(s) + \left(\beta_s^1 \mathbb{1}_{\{J(t,\xi,\beta^1)\le J(t,\xi,\beta^2)\}} + \beta_s^2 \mathbb{1}_{\{J(t,\xi,\beta^1)> J(t,\xi,\beta^2)\}}\right) \mathbb{1}_{[t,T]}(s).$$

One has $\gamma \in \mathcal{A}_t$ and

$$J(t,\xi,\beta^1) \wedge J(t,\xi,\beta^2) = J(t,\xi,\gamma) \in \mathscr{J}(t,\xi,\beta).$$

Hence, by (Karatzas *et al.*, 1998, Theorem A.3), there exists $\{\vartheta^n\}_{n\in\mathbb{N}} \subset \mathcal{A}_t$ such that $J(t,\xi,\vartheta^n)$ converges decreasingly to $v(t,\xi)$ as $n \to \infty$, with probability 1.

Thus, for $0 \le t \le t_2 \le T$, there is $\{\bar{\vartheta}^n\}_{n\in\mathbb{N}} \subset \mathcal{A}_t$ such that $J(t_2, X_{t_2}^{t,\xi,\alpha}, \bar{\vartheta}^n)$ converges decreasingly to $v(t_2, X_{t_2}^{t,\xi,\alpha})$ as $n \to \infty$, with probability 1. Therefore, for $0 \le t \le t_1 \le t_2 \le T$, dominated convergence theorem guarantees

$$\mathbb{E}_{\mathcal{F}_{t_1}^0} v(t_2, X_{t_2}^{t,\xi,\alpha}) = \lim_{n\to\infty} \mathbb{E}_{\mathcal{F}_{t_1}^0} J(t_2, X_{t_2}^{t,\xi,\alpha}, \bar{\vartheta}^n)$$

Let $F_s(t,\xi,\alpha) := f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)$ and $G(t,\xi,\alpha) := g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}})$. Hence, for $0 \le t \le t_1 \le t_2 \le T$, we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}^0_{t_1}} v(t_2, X_{t_2}^{t,\xi,\alpha}) + \mathbb{E}_{\mathcal{F}^0_{t_1}} \int_{t_1}^{t_2} F_s(t,\xi,\alpha)\, ds \\
&= \lim_{n\to\infty} \mathbb{E}_{\mathcal{F}^0_{t_1}} \left[\int_{t_2}^{T} F_s\left(t_2, X_{t_2}^{t,\xi,\alpha}, \bar{\vartheta}^n\right) ds + \int_{t_1}^{t_2} F_s(t,\xi,\alpha)\, ds + G\left(t_2, X_{t_2}^{t,\xi,\alpha}, \bar{\vartheta}^n\right) \right] \\
&\ge v(t_1, X_{t_1}^{t,\xi,\alpha}),
\end{aligned}$$

$\mathbb{P}$-a.s.. Consequently, Assumption $(\mathcal{A}1)$ implies

$$\begin{aligned}
I :=& \operatorname*{ess\,inf}_{\vartheta\in\mathcal{A}_t} \mathbb{E}_{\mathcal{F}^0_{t_1}} \left[\int_{t_1}^{T} F_s\left(t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta\right) ds + G\left(t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta\right) \right] \\
&- \mathbb{E}_{\mathcal{F}^0_{t_1}} \left\{ \operatorname*{ess\,inf}_{\vartheta\in\mathcal{A}_t} \mathbb{E}_{\mathcal{F}^0_{t_2}} \left[\int_{t_2}^{T} F_s\left(t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta\right) ds + G\left(t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta\right) \right] \right\} \\
\le& \mathbb{E}_{\mathcal{F}^0_{t_1}} \int_{t_1}^{t_2} F_s(t,\xi,\alpha)\, ds \\
\le& K(t_2 - t_1) \qquad (5.1.5)
\end{aligned}$$

For the reverse inequality sign, we interchange the infimum and expectation to yield the inequality

$$\begin{aligned}
I \ge \operatorname*{ess\,inf}_{\vartheta\in\mathcal{A}_t} \mathbb{E}_{\mathcal{F}^0_{t_1}} \Big\{ &\int_{t_1}^{t_2} F_s\left(t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta\right) ds + \int_{t_2}^{T} \left[F_s\left(t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta\right) - F_s\left(t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta\right) \right] ds \\
&+ G\left(t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta\right) - G\left(t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta\right) \Big\}
\end{aligned}$$

Noting that Assumption $(\mathcal{A}1)$ and Lemma 5.1.7 deduce

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}^0_{t_1}} \int_{t_2}^{T} \left| f_s\left(X_s^{t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta}, \mathbb{P}^{W^0}_{X_s^{t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta}}, \vartheta_s \right) - f_s\left(X_s^{t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta}, \mathbb{P}^{W^0}_{X_s^{t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta}}, \vartheta_s \right) \right| ds \\
&\le K \mathbb{E}_{\mathcal{F}^0_{t_1}} \Big[\int_{t_2}^{T} \left| X_s^{t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta} - X_s^{t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta} \right| + \left(\mathbb{E} \left| X_s^{t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta} - X_s^{t_2, X_{t_2}^{t,\xi,\alpha}, \vartheta} \right|^2 \right)^{1/2} ds \Big] \\
&\le C_{K,T} \mathbb{E}_{\mathcal{F}^0_{t_1}} \left[\mathbb{E}_{\mathcal{F}^0_{t_2}} \left| X_{t_2}^{t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta} - X_{t_2}^{t,\xi,\alpha} \right| + \left(\mathbb{E} \left| X_{t_2}^{t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta} - X_{t_2}^{t,\xi,\alpha} \right|^2 \right)^{1/2} \right] \\
&\le C_{K,T} \mathbb{E}_{\mathcal{F}^0_{t_1}} \left[\left(\mathbb{E}_{\mathcal{F}^0_{t_2}} \left| X_{t_2}^{t_1, X_{t_1}^{t,\xi,\alpha}, \vartheta} - X_{t_1}^{t,\xi,\alpha} \right|^2 \right)^{1/2} + \left(\mathbb{E}_{\mathcal{F}^0_{t_2}} \left| X_{t_1}^{t,\xi,\alpha} - X_{t_2}^{t,\xi,\alpha} \right|^2 \right)^{1/2} \right]
\end{aligned}$$

$$
\begin{aligned}
&+ C_{K,T}\mathbb{E}_{\mathcal{F}^0_{t_1}}\left[\left(\mathbb{E}\left|X^{t_1,X^{t,\xi,\alpha}_{t_1},\vartheta}_{t_2} - X^{t,\xi,\alpha}_{t_1}\right|^2\right)^{1/2} + \left(\mathbb{E}\left|X^{t,\xi,\alpha}_{t_1} - X^{t,\xi,\alpha}_{t_2}\right|^2\right)^{1/2}\right] \\
&\leq C_{K,T}\mathbb{E}_{\mathcal{F}^0_{t_1}}\left[\left(1+\mathbb{E}_{\mathcal{F}^0_{t_2}}|X^{t,\xi,\alpha}_{t_1}|^2\right)^{1/2}(t_2-t_1)^{1/2} + \left(1+\mathbb{E}_{\mathcal{F}^0_{t_2}}|\xi|^2\right)^{1/2}(t_2-t_1)^{1/2}\right] \\
&\quad + C_{K,T}\mathbb{E}_{\mathcal{F}^0_{t_1}}\left[\left(1+\mathbb{E}|X^{t,\xi,\alpha}_{t_1}|^2\right)^{1/2}(t_2-t_1)^{1/2} + \left(1+\mathbb{E}|\xi|^2\right)^{1/2}(t_2-t_1)^{1/2}\right].
\end{aligned}
$$

where the constant $C_{K,T} > 0$ may have varying values from line to line, but depending only on K and T. Similar inequality holds for $G\left(t_1, X^{t,\xi,\alpha}_{t_1}, \vartheta\right) - G\left(t_2, X^{t,\xi,\alpha}_{t_2}, \vartheta\right)$. Hence, we have

$$
\begin{aligned}
I \geq &-C_{K,T}\mathbb{E}_{\mathcal{F}^0_{t_1}}\left[(t_2-t_1)^{1/2} + \left(1+\mathbb{E}_{\mathcal{F}^0_{t_2}}|X^{t,\xi,\alpha}_{t_1}|^2\right)^{1/2} + \left(1+\mathbb{E}_{\mathcal{F}^0_{t_2}}|\xi|^2\right)^{1/2}\right](t_2-t_1)^{1/2} \\
&- C_{K,T}\mathbb{E}_{\mathcal{F}^0_{t_1}}\left[(t_2-t_1)^{1/2} + \left(1+\mathbb{E}|X^{t,\xi,\alpha}_{t_1}|^2\right)^{1/2} + \left(1+\mathbb{E}|\xi|^2\right)^{1/2}\right](t_2-t_1)^{1/2} \qquad (5.1.6)
\end{aligned}
$$

Together with (5.1.5), we conclude that $I \to 0$ as $|t_2 - t_1| \to 0^+$.

Following the same derivations of (5.1.5) and (5.1.6), one proves the continuity of $\mathbb{E}v(s, X^{t,\xi,\alpha}_s)$ in $s \in [t,T]$. Thus, by Doob's regularization theorem and the supermartingale nature of $v(s, X^{t,\xi,\alpha}_s)$, it holds that $v(s, X^{t,\xi,\alpha}_s)$ is right continuous in $s \in [t,T]$. Furthermore, by the BSDE theory, $\mathbb{E}_{\mathcal{F}_s}v(t_2, X^{t,\xi,\alpha}_{t_2})$ is continuous in $s \in [t,t_2]$. Together with the calculations of (5.1.5) and (5.1.6), it implies the left continuity of $v(s, X^{t,\xi,\alpha}_s)$ in $s \in [t,T]$. Hence, $v(s, X^{t,\xi,\alpha}_s)$ is a continuous process.

Proof of (1): Following a similar arguments as in (5.1.5), we have

$$
\left|\mathbb{E}_{\mathcal{F}^0_{t_1}}\left[J(t_2, X^{t,\xi,\alpha}_{t_2}, \alpha) - J(t_1, X^{t,\xi,\alpha}_{t_1}, \alpha)\right]\right| \leq C_K(t_2 - t_1)
$$

for $0 \leq t \leq t_1 \leq t_2 \leq T$. Thus, the desired continuity can be argued similarly as in those after (5.1.6).

Proof of (5): For $\mathbb{P}^0$-a.s. ω^0, fixed $(t,\xi) \in [0,T] \times L^2((\Omega^1, \mathcal{F}^1_t, \mathbb{R}^1); \mathbb{R}^d)$, consider $\alpha \in \bar{\mathcal{A}}_t$, $\alpha^n \in \bar{\mathcal{A}}_t$, $n \in \mathbb{N}$, and $\Delta_t(\alpha^n, \alpha) \to 0$ as $n \to \infty$. Without loss of generality, we may assume that $\alpha^n \to \alpha$ $dt \otimes \mathbb{P}$-a.s.. Let X^n be the solution of the SDE (5.1.2) subject to the control α^n, and X is subject to the control α. We have the following trivial inequality:

$$
\mathbb{E}^0\left[\sup_{t\leq r\leq s} \mathcal{W}_2^2(\mathbb{P}^{W^0}_{X^n_r}, \mathbb{P}^{W^0}_{X_r})\Big|\mathcal{F}^0_t\right] \leq \mathbb{E}\left[\sup_{t\leq r\leq s}|X^n_r - X_r|^2\Big|\mathcal{F}^0_t\right].
$$

By applying the BDG inequality, Gronwall's inequality, and the above relation, we can identify a constant $C_{K,T}$, independent of n but dependent on the Lipschitz constant K of the coefficients and T, such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le s\le T}|X_s^n - X_s|^2\Big|\mathcal{F}_t^0\right] \le C_{K,T}\mathbb{E}\Bigg[\int_t^T &|b_r(X_r,\mathbb{P}_{X_r}^{W^0},\alpha_r^n) - b_r(X_r,\mathbb{P}_{X_r}^{W^0},\alpha_r)|^2 dr\\ &+|\sigma_r(X_r,\mathbb{P}_{X_r}^{W^0},\alpha_r^n) - \sigma_r(X_r,\mathbb{P}_{X_r}^{W^0},\alpha_r)|^2\\ &+|\sigma_r^0(X_r,\mathbb{P}_{X_r}^{W^0},\alpha_r^n)dr - \sigma_r^0(X_r,\mathbb{P}_{X_r}^{W^0},\alpha_r)|^2 dr\Big|\mathcal{F}_t^0\Bigg] \to 0,\end{aligned}$$

as $n\to\infty$, by the dominated convergence theorem. Now, for some constant $C_{K,T}$ not necessarily the same as the one above, but independent on n and depends only on K, T, we have

$$\begin{aligned}&|J(t,\xi,\alpha^n) - J(t,\xi,\alpha)|\\ \le&\mathbb{E}\left[\int_t^T |f_s(X_s^n,\mathbb{P}_{X_s^n}^{W^0},\alpha_s^n) - f_s(X_s,\mathbb{P}_{X_s}^{W^0},\alpha_s)| + |g(X_T^n,\mathbb{P}_{X_T^n}^{W^0}) - g(X_T,\mathbb{P}_{X_T}^{W^0})|\Big|\mathcal{F}_t^0\right]\\ \le&C_{K,T}\mathbb{E}\left[\sup_{t\le s\le T}|X_s^n - X_s| + \int_t^T |f_s(X_s,\mathbb{P}_{X_s}^{W^0},\alpha_s^n) - f_s(X_s,\mathbb{P}_{X_s}^{W^0},\alpha_s)|\Big|\mathcal{F}_t^0\right]\to 0,\end{aligned}$$

again by the dominated convergence theorem. □

We now define the function $\bar{J}$ on $[0,T]\times L^2((\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1);\mathbb{R}^d)\times\bar{\mathcal{A}}_t\times\Omega^0$ by

$$\bar{J}(t,\xi,\alpha) := J(t,\xi,\alpha).$$

We have the following lemma relating J and $\bar{J}$.

Lemma 5.1.9. *Let (A1) hold. For $\xi\in L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)$, $\alpha\in\mathcal{A}_t$, we have*

$$J(t,\xi,\alpha)(\omega^0) = \bar{J}(t,\xi(\omega^0),\alpha^{t,\omega^0})(\omega^0),\ \ \mathbb{P}^0\,a.s..$$

Proof. First of all we look at ξ, α (in the interpretation of Lemma 5.1.4) being simple functions. Note that ξ could be written as $\xi=\sum_{j=1}^m \xi_j(\omega^1)\mathbb{1}_{B_j^0}(\omega^0)$, where for $j=1,\dots,m$, $B_j^0\in\mathcal{F}_t^0$, $\{B_j^0\}_{j=1}^m$ is a disjoint partition of Ω^0, $\xi_j\in L^2((\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1);\mathbb{R}^d)$. Without loss

of generality, we assume $\alpha(t,\omega^0,\omega^1) := \sum_{j=1}^m \alpha_j(t,\omega^0,\omega^1)\mathbb{1}_{B_j^0}(\omega^0)$, $\alpha_j \in \bar{\mathcal{A}}_t$. Solving for $j = 1,\dots,m$,

$$\begin{cases} dX_s^{t,\xi_j,\alpha_j} = & b_s(X_s^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_s^{t,\xi_j,\alpha_j}}, \alpha_{j;s})ds + \sigma_s(X_s^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_s^{t,\xi_j,\alpha_j}}, \alpha_{j;s})dW_s \\ & +\sigma_s^0(X_s^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_s^{t,\xi_j,\alpha_j}}, \alpha_{j;s})dW_s^0, \ \ s \in [t,T], \\ X_t^{t,\xi_j,\alpha_j} = & \xi_j, \end{cases} \tag{5.1.7}$$

we have

$$\begin{aligned} \sum_{j=1}^m X_s^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}(\omega^0) =& \sum_{j=1}^m \xi_j \mathbb{1}_{B_j^0}(\omega^0) + \sum_{j=1}^m \left(\int_t^s b_r(X_r^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_r^{t,\xi_j,\alpha_j}}, \alpha_{j;r})dr \right) \mathbb{1}_{B_j^0}(\omega^0) \\ & + \sum_{j=1}^m \left(\int_t^s \sigma_r(X_r^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_r^{t,\xi_j,\alpha_j}}, \alpha_{j;r})dW_r \right) \mathbb{1}_{B_j^0}(\omega^0) \\ & + \sum_{j=1}^m \left(\int_t^s \sigma_r^0(X_r^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_r^{t,\xi_j,\alpha_j}}, \alpha_{j;r})dW_r^0 \right) \mathbb{1}_{B_j^0}(\omega^0) \\ =& \sum_{j=1}^m \xi_j \mathbb{1}_{B_j^0}(\omega^0) + \int_t^s b_r\Big(\sum_{j=1}^m X_r^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}, \mathbb{P}^{W^0}_{\sum_{j=1}^m X_r^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}}, \alpha_r \Big)dr \\ & + \int_t^s \sigma_r\Big(\sum_{j=1}^m X_r^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}, \mathbb{P}^{W^0}_{\sum_{j=1}^m X_r^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}}, \alpha_r \Big)dW_r \\ & + \int_t^s \sigma_r^0\Big(\sum_{j=1}^m X_r^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}, \mathbb{P}^{W^0}_{\sum_{j=1}^m X_r^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}}, \alpha_r \Big)dW_r^0. \end{aligned}$$

Thus by uniqueness of solution we have

$$\sum_{j=1}^m X_s^{t,\xi_j,\alpha}\mathbb{1}_{B_j^0}(\omega^0)$$

solves

$$\begin{cases} dX_s^{t,\xi,\alpha} = & b_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + \sigma_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s \\ & +\sigma_s^0(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s^0, \ \ s \in [t,T], \\ X_t^{t,\xi,\alpha} = & \xi. \end{cases} \tag{5.1.8}$$

Therefore it holds that

$$J(t,\xi,\alpha)(\omega^0) = \mathbb{E}\left[\int_t^T f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \Big| \mathcal{F}_t^0 \right](\omega^0)$$

$$
\begin{aligned}
=&\mathbb{E}\bigg[\int_t^T f_s(\sum_{j=1}^m X_s^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}, \mathbb{P}^{W^0}_{\sum_{j=1}^m X_s^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}}, \alpha_s)ds \\
&+ g(\sum_{j=1}^m X_T^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}, \mathbb{P}^{W^0}_{\sum_{j=1}^m X_T^{t,\xi_j,\alpha_j}\mathbb{1}_{B_j^0}})\Big|\mathcal{F}_t^0\bigg](\omega^0) \\
=&\sum_{j=1}^m \mathbb{E}\bigg[\int_t^T f_s(X_s^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_s^{t,\xi_j,\alpha_j}}, \alpha_{j;s})ds + g(X_T^{t,\xi_j,\alpha_j}, \mathbb{P}^{W^0}_{X_T^{t,\xi_j,\alpha_j}})\Big|\mathcal{F}_t^0\bigg]\mathbb{1}_{B_j^0}(\omega^0) \\
=&\sum_{j=1}^m \bar{J}(t,\xi_j,\alpha_j)\mathbb{1}_{B_j^0}(\omega^0) \\
=&\bar{J}(t,\xi(\omega^0),\alpha^{t,\omega^0})(\omega^0), \quad \mathbb{P}^0\text{-a.s. } \omega^0.
\end{aligned}
$$

Now, for general $\xi \in L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)$ and $\alpha \in L^0((\Omega^0,\mathcal{F}_t^0,\mathbb{P}^0);\bar{\mathcal{A}}_t)$, there exists a sequence of simple functions $\{\xi^n\}$ and $\{\alpha^n\}$ that converge pointwise to ξ and α, respectively, and ξ^n, α^n can be written in the form $\xi^n = \sum_{j=1}^{m_n}\xi_j^n(\omega^1)\mathbb{1}_{B_j^{n;0}}(\omega^0)$, $\alpha^n = \sum_{j=1}^{m_n}\alpha_j^n\mathbb{1}_{B_j^{n;0}}(\omega^0)$. By Lemma 5.1.8 we have

$$
\begin{aligned}
&\mathbb{E}^0\Big[|J(t,\xi,\alpha) - \bar{J}(t,\xi(\omega^0),\alpha^{t,\omega^0})|^2\Big] \\
\le& C\,\mathbb{E}^0\Big[|J(t,\xi,\alpha) - J(t,\xi^n,\alpha^n)|^2\Big] + C\,\mathbb{E}^0\Big[|J(t,\xi^n,\alpha^n) - \bar{J}(t,\xi(\omega^0),\alpha^{t,\omega^0})|^2\Big] \\
\le& C\,\mathbb{E}^0\Big[|J(t,\xi,\alpha) - J(t,\xi^n,\alpha^n)|^2\Big] + C\,\mathbb{E}^0\Big[|\bar{J}(t,\xi^n,\alpha^n) - \bar{J}(t,\xi(\omega^0),\alpha^n)|^2\Big] \\
&+ C\,\mathbb{E}^0\Big[|\bar{J}(t,\xi,\alpha^n) - \bar{J}(t,\xi(\omega^0),\alpha^{t,\omega^0})|^2\Big],
\end{aligned}
$$

taking $n \to \infty$ concludes the proof. □

Define the function $\bar{v}$ on $[0,T]\times L^2((\Omega^1,\mathcal{F}_t^1,\mathbb{P}^1);\mathbb{R}^d)\times\Omega^0$ by

$$
\bar{v}(t,\xi) := \operatorname*{ess\,inf}_{\alpha\in\bar{\mathcal{A}}_t} \bar{J}(t,\xi,\alpha),
$$

we have a similar relation between v and $\bar{v}$ as J and $\bar{J}$.

Lemma 5.1.10. *Let (A1) hold. Let $\xi \in L^2((\Omega,\mathcal{F}_t,\mathbb{P});\mathbb{R}^d)$, we have that*

$$
v(t,\xi)(\omega^0) = \bar{v}(t,\xi(\omega^0))(\omega^0), \ \ \mathbb{P}^0\text{-a.s. } \omega^0.
$$

Proof. Let $\alpha \in \mathcal{A}_t$ be arbitrary. Again, we start with simple functions. As in the beginning of the proof of Lemma 5.1.9 we can assume that $\xi(\omega^0, \omega^1) = \sum_{j=1}^m \xi_j(\omega^1)\mathbb{1}_{B_j^0}(\omega^0)$, where $m \in \mathbb{N}$, and for $j = 1, \dots, m$, $B_j^0 \in \mathcal{F}_t^0$, $\{B_j^0\}_{j=1}^m$ is a disjoint partition of Ω^0, $\xi_j \in L^2((\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1); \mathbb{R}^d)$. It holds that

$$\sum_{j=1}^m X_s^{t,\xi_j,\alpha}\mathbb{1}_{B_j^0}(\omega^0)$$

solves the SDE

$$\begin{cases} dX_s^{t,\xi,\alpha} &= b_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + \sigma_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s \\ &\quad +\sigma_s^0(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)dW_s^0, \quad s \in [t,T], \\ X_t^{t,\xi,\alpha} &= \xi. \end{cases} \tag{5.1.9}$$

Thus,

$$\begin{aligned} &J(t,\xi,\alpha)(\omega^0) \\ &= J(t,\xi,\alpha^{t,\omega^0})(\omega^0) \\ &= \mathbb{E}\left[\int_t^T f_s(X_s^{t,\xi,\alpha^{t,\omega^0}}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha^{t,\omega^0}}}, \alpha_s^{t,\omega^0})ds + g(X_T^{t,\xi,\alpha^{t,\omega^0}}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha^{t,\omega^0}}})\Big|\mathcal{F}_t^0\right](\omega^0) \\ &= \sum_{j=1}^m \mathbb{E}\left[\int_t^T f_s(X_s^{t,\xi_j,\alpha^{t,\omega^0}}, \mathbb{P}^{W^0}_{X_s^{t,\xi_j,\alpha^{t,\omega^0}}}, \alpha_s^{t,\omega^0})ds + g(X_T^{t,\xi_j,\alpha^{t,\omega^0}}, \mathbb{P}^{W^0}_{X_T^{t,\xi_j,\alpha^{t,\omega^0}}})\Big|\mathcal{F}_t^0\right]\mathbb{1}_{B_j^0}(\omega^0) \\ &\geq \sum_{j=1}^m \bar{v}(t,\xi_j)\mathbb{1}_{B_j^0}(\omega^0) \\ &= \bar{v}(t, \sum_{j=1}^m \xi_j\mathbb{1}_{B_j^0}(\omega^0)), \quad \mathbb{P}^0\text{-a.s..} \end{aligned}$$

By arbitrariness of α we have

$$v(t,\xi) \geq \bar{v}(t, \sum_{j=1}^m \xi_j\mathbb{1}_{B_j^0}(\omega^0)), \quad \mathbb{P}^0\text{-a.s..}$$

On the other hand, for each $j = 1, \dots, m$, there exists $\alpha_{ji} \in \mathcal{A}_t$, $i \in \mathbb{N}$ such that

$$\lim_{i\to\infty} J(t,\xi_j,\alpha_{ji}) = \bar{v}(t,\xi_j), \quad \mathbb{P}^0\text{-}a.s..$$

Define $\hat{\alpha}_i := \sum_{j=1}^m \alpha_{ji} \mathbb{1}_{B_j^0}$, then $\hat{\alpha}_i \in \mathcal{A}_t$. Thus

$$v(t,\xi) \le J(t,\xi,\hat{\alpha}_i) = \sum_{j=1}^m \mathbb{E}\bigg[\int_t^T f_s(X_s^{t,\xi_j,\alpha_{ji}}, \mathbb{P}^{W^0}_{X_s^{t,\xi_j,\alpha_{ji}}}, \alpha_{ji,s})ds + g(X_T^{t,\xi_j,\alpha_{ji}}, \mathbb{P}^{W^0}_{X_T^{t,\xi_j,\alpha_{ji}}}) \Big| \mathcal{F}_t^0 \bigg] \mathbb{1}_{B_j^0}(\omega^0)$$
$$\to \sum_{j=1}^m \bar{v}(t,\xi_j)\mathbb{1}_{B_j^0}, \quad \mathbb{P}^0\text{-a.s.}, \quad \text{as } i \to \infty.$$

Thus we have

$$v(t,\xi) \le \bar{v}(t, \sum_{j=1}^m \xi_j \mathbb{1}_{B_j^0}(\omega^0)), \quad \mathbb{P}^0\text{-a.s..}$$

Finally, for general $\xi \in L^2((\Omega,\mathcal{F},\mathbb{P});\mathbb{R}^d)$, the proof is essentially the same as the last part in Lemma 5.1.9. □

Remark 5.1.2. *For $0 \le t \le k$ and $\alpha \in \mathcal{A}_t$, α is not necessarily in $\mathcal{A}_k$. However, for every fixed $(\omega^0,\hat{\omega}^1) \in \Omega^0 \times \hat{\Omega}^1$, $\alpha^{k,\omega^0,\hat{\omega}^1} \in \mathcal{A}_k$. Thus, for $\xi \in \mathcal{L}^2((\Omega,\mathcal{F}_k,\mathbb{P});\mathbb{R}^d)$, $J(k,\xi,\alpha)$ can be regard as a function on $\Omega^0 \times \hat{\Omega}^1$, i.e.,*

$$J(k,\xi,\alpha)(\omega^0,\hat{\omega}^1) = J(k,\xi,\alpha^{k,\omega^0,\hat{\omega}^1})(\omega^0).$$

We actually have the following lemma about the explicit form of $J(k,\xi,\alpha)$.

Lemma 5.1.11. *Let (A1) holds. Let $0 \le t \le k \le T$, $\xi \in L^2((\Omega,\mathcal{F}_k,\mathbb{P});\mathbb{R}^d)$, $\alpha \in \mathcal{A}_t$. We have*

$$J(k,\xi,\alpha) = \mathbb{E}\bigg[\int_k^T f_s(X_s^{k,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{k,\xi,\alpha}}, \alpha_s)ds + g(X_T^{k,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{k,\xi,\alpha}}) \Big| \mathcal{F}_k^0 \vee \hat{\mathcal{F}}_k^{1;t} \bigg], \quad \mathbb{P}^0 \times \mathbb{P}^1\text{-a.s.},$$

and hence

$$\mathbb{E}[J(k,\xi,\alpha)|\mathcal{F}_k^0] = \mathbb{E}\bigg[\int_k^T f_s(X_s^{k,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{k,\xi,\alpha}}, \alpha_s)ds + g(X_T^{k,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{k,\xi,\alpha}}) \Big| \mathcal{F}_k^0 \bigg], \quad \mathbb{P}^0\text{-a.s..}$$

Proof. Let $(u^0,\hat{u}^1)$ be elements of $\Omega^0 \times \hat{\Omega}^1$. We denote $\mathbb{E}^{\omega^0}$ as the expectation with respect to $\mathbb{P}^0(d\omega^0)$, similarly for $\mathbb{E}^{\tilde{\omega}^1}$ and $\mathbb{E}^{\hat{\omega}^1}$. Writing explicitly the arguments we have:

$$\mathbb{E}\bigg[\int_k^T f_s(X_s^{k,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{k,\xi,\alpha}}, \alpha_s)ds + g(X_T^{k,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{k,\xi,\alpha}}) \Big| \mathcal{F}_k^0 \vee \hat{\mathcal{F}}_k^{1;t} \bigg](u^0,\hat{u}^1)$$

$$
\begin{aligned}
=&\mathbb{E}\bigg[\int_k^T f_s\Big(\omega^0, X_s^{k,\xi,\alpha}(\omega^0,\hat{\omega}^1,\tilde{\omega}^1), \mathbb{P}^{W^0}_{X_s^{k,\xi,\alpha}(\omega^0)}, \alpha_s(\omega^0,\hat{\omega}^1,\tilde{\omega}^1)\Big)ds \\
&+ g\Big(X_T^{k,\xi,\alpha}(\omega^0,\hat{\omega}^1,\tilde{\omega}^1), \mathbb{P}^{W^0}_{X_T^{k,\xi,\alpha}(\omega^0)}\Big)\Big|\mathcal{F}_k^0\vee\hat{\mathcal{F}}_k^{1;t}\bigg](u^0,\hat{u}^1) \\
=&\mathbb{E}^{\tilde{\omega}^1}\mathbb{E}^{\hat{\omega}^1}\mathbb{E}^{\omega^0}\bigg[\int_k^T f_s\Big(u^0\otimes_k\omega^0, X_s^{k,\xi,\alpha}(u^0\otimes_k\omega^0,\hat{u}^1\otimes_k\hat{\omega}^1,\tilde{\omega}^1), \\
&\qquad\qquad \mathbb{P}^{W^0}_{X_s^{k,\xi,\alpha}(u^0\otimes_k\omega^0)}, \alpha_s(u^0\otimes_k\omega^0,\hat{u}^1\otimes_k\hat{\omega}^1,\tilde{\omega}^1)\Big)ds \\
&+ g\Big(X_T^{k,\xi,\alpha}(u^0\otimes_k\omega^0,\hat{u}^1\otimes_k\hat{\omega}^1,\tilde{\omega}^1), \mathbb{P}^{W^0}_{X_T^{k,\xi,\alpha}(u^0\otimes_k\omega^0)}\Big)\bigg] \\
=&\mathbb{E}\bigg[\int_k^T f_s(X_s^{k,\xi,\alpha^{k,u^0,\hat{u}^1}}, \mathbb{P}^{W^0}_{X_s^{k,\xi,\alpha^{k,u^0,\hat{u}^1}}}, \alpha_s^{k,u^0,\hat{u}^1})ds + g(X_T^{k,\xi,\alpha^{k,u^0,\hat{u}^1}}, \mathbb{P}^{W^0}_{X_T^{k,\xi,\alpha^{k,u^0,\hat{u}^1}}})\Big|\mathcal{F}_k^0\bigg](u^0) \\
=&J(k,\xi,\alpha^{k,u^0,\hat{u}^1})(u^0) \\
=&J(k,\xi,\alpha)(u^0,\hat{u}^1).
\end{aligned}
$$

□

Lemma 5.1.12. *Let (A1) holds. Letting $\theta\in\mathcal{T}_{t,T}^0$, for all $\xi\in L^2((\Omega,\mathcal{F}_\theta,\mathbb{P});\mathbb{R}^d)$, we have*

$$
\mathbb{E}\Big[J(\theta,X_\theta^{t,\xi,\alpha},\alpha^\theta)\,\Big|\,\mathcal{F}_\theta^0\Big] = \mathbb{E}\bigg[\int_\theta^T f_s(X_s^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}},\alpha_s)ds + g(X_T^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}})\,\Big|\,\mathcal{F}_\theta^0\bigg],\ \mathbb{P}^0\text{-}a.s..
\tag{5.1.10}
$$

Proof. Recall that from the uniqueness of SDE we have the flow property:

$$
X_s^{t,\xi,\alpha}(\omega^0,\hat{\omega}^1,\tilde{\omega}^1) = X_s^{\theta,X_\theta^{t,\xi,\alpha},\alpha^\theta}(\omega^0,\omega^1).
$$

Assume $\theta=\sum_{i=1}^n t_i\mathbb{1}_{B_i^0}$, where for $i=1,\dots,n$, $t\le t_i\le T$, and $\{B_i^0\}_{i=1}^n$ is a disjoint partition of Ω^0, thus $B_i^0=\{\theta=t_i\}$, and $B_i^0\in\mathcal{F}_{t_i}^0$. Let $A\in\mathcal{F}_\theta^0$. Then

$$
\begin{aligned}
&\int_A\mathbb{E}\bigg[\int_\theta^T f_s(X_s^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}},\alpha_s)ds + g(X_T^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}})\,\Big|\,\mathcal{F}_\theta^0\bigg]d\mathbb{P}^0 \\
=&\int_A\bigg(\int_\theta^T f_s(X_s^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}},\alpha_s)ds + g(X_T^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}})\bigg)\sum_{i=1}^n\mathbb{1}_{\{\theta=t_i\}}d\mathbb{P}^0 \\
=&\int_{\Omega^0}\bigg(\int_\theta^T f_s(X_s^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}},\alpha_s)ds + g(X_T^{t,\xi,\alpha},\mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}})\bigg)\sum_{i=1}^n\mathbb{1}_{\{\theta=t_i\}\cap A}d\mathbb{P}^0
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_{\Omega^0} \mathbb{E}\left[\int_{\theta}^{T} f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\middle|\, \mathcal{F}^0_{t_i} \right] \mathbb{1}_{\{\theta = t_i\} \cap A} d\mathbb{P}^0 \\
&= \sum_{i=1}^{n} \int_{\Omega^0} \mathbb{E}\left[\int_{t_i}^{T} f_s(X_s^{t_i, X_{t_i}^{t,\xi,\alpha}, \alpha^{t_i}}, \mathbb{P}^{W^0}_{X_s^{t_i, X_{t_i}^{t,\xi,\alpha}, \alpha^{t_i}}}, \alpha_s^{t_i}) ds + g(X_T^{t_i, X_{t_i}^{t,\xi,\alpha}, \alpha^{t_i}}, \mathbb{P}^{W^0}_{X_T^{t_i, X_{t_i}^{t,\xi,\alpha}, \alpha^{t_i}}}) \,\middle|\, \mathcal{F}^0_{t_i} \right] \mathbb{1}_{\{\theta = t_i\} \cap A} d\mathbb{P}^0 \\
&= \sum_{i=1}^{n} \int_{\Omega^0} \mathbb{E}\Big[J(t_i, X_{t_i}^{t,\xi,\alpha}, \alpha^{t_i}) \,\Big|\, \mathcal{F}^0_{t_i} \Big] \mathbb{1}_{\{\theta = t_i\} \cap A} d\mathbb{P}^0 \text{ (by Lemma 5.1.11)} \\
&= \int_A \mathbb{E}\Big[\sum_{i=1}^{n} J(t_i, X_{t_i}^{t,\xi,\alpha}, \alpha^{t_i}) \mathbb{1}_{\{\theta = t_i\}} \,\Big|\, \mathcal{F}^0_{t_i} \Big] d\mathbb{P}^0 \\
&= \int_A \mathbb{E}[J(\theta, X_\theta^{t,\xi,\alpha}, \alpha^\theta) | \mathcal{F}^0_\theta] d\mathbb{P}^0.
\end{aligned}
$$

For the general case, let θ_i be an increasing sequence of discrete stopping times such that $\lim_i \theta_i = \theta$. By Theorem 4.6.10 in Durrett (2019) and Chapter 1, Exercises 24 and 27 in Protter (2005), it holds that

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{\theta_i}^{T} f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\Big|\, \mathcal{F}^0_{\theta_i} \Big] \\
&\quad \to \mathbb{E}\Big[\int_{\theta}^{T} f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\Big|\, \mathcal{F}^0_{\theta} \Big] \quad \mathbb{P}^0\text{-}a.s.,
\end{aligned}
$$

which, together with 1 of Lemma 5.1.8 , implies that

$$
\begin{aligned}
&\Big| \mathbb{E}\Big[J(\theta, X_\theta^{t,\xi,\alpha}, \alpha^\theta) \,\Big|\, \mathcal{F}^0_\theta \Big] - \mathbb{E}\Big[\int_{\theta}^{T} f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\Big|\, \mathcal{F}^0_\theta \Big] \Big| \\
&= \Big| \mathbb{E}\Big[J(\theta, X_\theta^{t,\xi,\alpha}, \alpha^\theta) \,\Big|\, \mathcal{F}^0_\theta \Big] - \mathbb{E}\Big[J(\theta_i, X_{\theta_i}^{t,\xi,\alpha}, \alpha^{\theta_i}) \,\Big|\, \mathcal{F}_{\theta_i^0} \Big] \\
&\quad + \mathbb{E}\Big[J(\theta_i, X_{\theta_i}^{t,\xi,\alpha}, \alpha^{\theta_i}) \,\Big|\, \mathcal{F}_{\theta_i^0} \Big] - \mathbb{E}\Big[\int_{\theta}^{T} f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\Big|\, \mathcal{F}^0_\theta \Big] \Big| \\
&= \Big| \mathbb{E}\Big[J(\theta, X_\theta^{t,\xi,\alpha}, \alpha^\theta) \,\Big|\, \mathcal{F}^0_\theta \Big] - \mathbb{E}\Big[J(\theta_i, X_{\theta_i}^{t,\xi,\alpha}, \alpha^{\theta_i}) \,\Big|\, \mathcal{F}_{\theta_i^0} \Big] \\
&\quad + \mathbb{E}\Big[\int_{\theta_i}^{T} f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\Big|\, \mathcal{F}^0_{\theta_i} \Big] \\
&\quad - \mathbb{E}\Big[\int_{\theta}^{T} f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\Big|\, \mathcal{F}^0_\theta \Big] \Big| \\
&\to 0 \ \mathbb{P}^0 \, a.s. \text{ as } i \to \infty.
\end{aligned}
$$

□

Theorem 5.1.13. *(Dynamic Programming Principle). Let (A1) hold. We have for all* $(t,\xi) \in [0,T] \times L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{R}^d)$,

$$\begin{aligned} v(t,\xi) =& \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_t} \operatorname*{ess\,inf}_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}\Big[\int_t^\theta f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + v(\theta, X_\theta^{t,\xi,\alpha}) \,\Big|\, \mathcal{F}_t^0 \Big] \\ =& \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_t} \operatorname*{ess\,sup}_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}\Big[\int_t^\theta f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + v(\theta, X_\theta^{t,\xi,\alpha}) \,\Big|\, \mathcal{F}_t^0 \Big]. \end{aligned}$$

Proof. Let $(t,\xi,\alpha) \in [0,T] \times L^2((\Omega;\mathcal{F}_t,\mathbb{P});\mathbb{R}^d) \times \mathcal{A}_t$. From (5.1.10) it follows that

$$\begin{aligned} J(t,\xi,\alpha) =& \mathbb{E}\Big\{ \int_t^\theta f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + \mathbb{E}\Big[\int_\theta^T f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + g(X_T^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}}) \,\Big|\, \mathcal{F}_\theta^0 \Big] \,\Big|\, \mathcal{F}_t^0 \Big\} \\ =& \mathbb{E}\Big[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + J(\theta, X_\theta^{t,\xi,\alpha}, \alpha^\theta) \,\Big|\, \mathcal{F}_t^0 \Big]. \end{aligned}$$

Now assume that $\theta = \sum_{i=1}^n t_i \mathbb{1}_{B_i^0}$, where for $i = 1,\dots,n$, $t \le t_i \le T$, and $\{B_i^0\}_{i=1}^n$ is a disjoint partition of Ω^0, then

$$\begin{aligned} &\mathbb{E}\Big[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + J(\theta, X_\theta^{t,\xi,\alpha}, \alpha^\theta) \,\Big|\, \mathcal{F}_t^0 \Big] \\ &= \sum_{i=1}^n \mathbb{E}\Big[\Big[\int_t^{t_i} f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + J(t_i, X_{t_i}^{t,\xi,\alpha}, \alpha^{t_i}) \Big] \mathbb{1}_{B_i^0} \,\Big|\, \mathcal{F}_t^0 \Big] \\ &\ge \sum_{i=1}^n \mathbb{E}\Big[\Big[\int_t^{t_i} f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(t_i, X_{t_i}^{t,\xi,\alpha}) \Big] \mathbb{1}_{B_i^0} \,\Big|\, \mathcal{F}_t^0 \Big] \\ &= \mathbb{E}\Big[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta, X_\theta^{t,\xi,\alpha}) \,\Big|\, \mathcal{F}_t^0 \Big]. \end{aligned}$$

Thus if θ is simple,

$$J(t,\xi,\alpha) \ge \mathbb{E}\Big[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta, X_\theta^{t,\xi,\alpha}) \,\Big|\, \mathcal{F}_t^0 \Big].$$

Now, let $\theta \in \mathcal{T}^0_{t,T}$ and $\theta_i \to \theta$ a simple increasing approximation of stopping times of θ, then we have

$$J(t,\xi,\alpha) \ge \mathbb{E}\Big[\int_t^{\theta_i} f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta_i, X_{\theta_i}^{t,\xi,\alpha}) \,\Big|\, \mathcal{F}_t^0 \Big], \text{ for every } i \in \mathbb{N}.$$

Take limit on the right hand side and from Lemma 5.1.8 2 we have

$$J(t,\xi,\alpha) \ge \mathbb{E}\Big[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta, X_\theta^{t,\xi,\alpha}) \,\Big|\, \mathcal{F}_t^0 \Big].$$

Since $\theta \in \mathcal{T}_{t,T}^0$ and $\alpha \in \mathcal{A}_t$ are arbitrary, we have

$$v(t,\xi) \geq \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_t} \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t,T}^0} \mathbb{E}\bigg[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta, X_\theta^{t,\xi,\alpha}) \,\bigg|\, \mathcal{F}_t^0 \bigg].$$

For the other side of inequality, again assume that $\theta = \sum_{i=1}^n t_i \mathbb{1}_{B_i^0}$, where for $i = 1, \ldots, n$, $t \leq t_i \leq T$, and $\{B_i^0\}_{i=1}^n$ is a disjoint partition of Ω^0. Let $\alpha \in \mathcal{A}_t$ be arbitrary. For each t_i, there exists $\alpha_{ij} \in \mathcal{A}_{t_i}$, $j \in \mathbb{N}$ such that

$$\lim_{j\to\infty} J(t_i, X_{t_i}^{t,\xi,\alpha}, \alpha_{ij}) = v(t_i, X_{t_i}^{t,\xi,\alpha}), \quad \mathbb{P}^0\text{-}a.s..$$

Define $\alpha_{j;s}^* := \alpha_s \mathbb{1}_{s<\theta} + \sum_{i=1}^n \alpha_{ij;s} \mathbb{1}_{s\geq t_i} \mathbb{1}_{B_i^0}$. Then we have

$$\begin{aligned}
v(t,\xi) \leq J(t,\xi,\alpha_j^*) =& \sum_{i=1}^n \mathbb{E}\bigg[\bigg[\int_t^{t_i} f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + J(t_i, X_{t_i}^{t,\xi,\alpha}, \alpha_{ij}) \bigg] \mathbb{1}_{B_i^0} \,\bigg|\, \mathcal{F}_t^0 \bigg] \\
\to& \sum_{i=1}^n \mathbb{E}\bigg[\bigg[\int_t^{t_i} f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(t_i, X_{t_i}^{t,\xi,\alpha}) \bigg] \mathbb{1}_{B_i^0} \,\bigg|\, \mathcal{F}_t^0 \bigg] \\
=& \mathbb{E}\bigg[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta, X_\theta^{t,\xi,\alpha}) \,\bigg|\, \mathcal{F}_t^0 \bigg].
\end{aligned}$$

Again, using increasing simple function approximation to stopping times and from Lemma 5.1.8, we have for any $\theta \in \mathcal{T}_{t,T}^0$,

$$v(t,\xi) \leq \mathbb{E}\bigg[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta, X_\theta^{t,\xi,\alpha}) \,\bigg|\, \mathcal{F}_t^0 \bigg],$$

and by arbitrariness of $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^0$, we have

$$v(t,\xi) \leq \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_t} \operatorname*{ess\,inf}_{\theta\in\mathcal{T}_{t,T}^0} \mathbb{E}\bigg[\int_t^\theta f(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) + v(\theta, X_\theta^{t,\xi,\alpha}) \,\bigg|\, \mathcal{F}_t^0 \bigg],$$

and we conclude the required equality. □

5.1.6 Dynamic Programming Principle

5.1.7 Conditional Law Invariant Interpretation

Theorem 5.1.14. *Let (A1) hold. Let $t \in [0,T]$, ξ, $\eta \in L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{R}^d)$. If for $\mathbb{P}^0$-a.s. ω^0, $\mathcal{L}(\xi(\omega^0, \cdot)) = \mathcal{L}(\eta(\omega^0, \cdot))$, then*

$$v(t,\xi) = v(t,\eta) \quad \mathbb{P}^0\text{-}a.s..$$

Proof. The proof is inspired by (Cosso *et al.*, 2023, Theorem 3.6). First of all, we suppose that both ξ, η are discrete. By Lemma 5.3.6, there exists U_ξ and U_η, which are $\mathcal{F}_t$ measurable, and when given $\mathcal{F}_t^0$ (which is characterized by the random variable ζ in Definition (5.3.3)), they are independent of ξ and η respectively, having the uniform distribution. By Theorem 5.3.5, there exists $b : [0,T]\times\Omega^0\times\Omega^1\times\mathbb{R}^d\times[0,1] \to A$, measurable with respect to $Prog((\mathcal{F}^0\vee \hat{\mathcal{F}}^{1;t}))\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}([0,1])$ such that $(b_s(\xi,U_\xi))_{s\in[0,T]}$ is $\mathbb{F}^t$ progressively measurable, and

$$\begin{aligned}&\mathcal{L}\Big(\xi,(\alpha_s)_{s\in[0,T]},(W^0_s)_{s\in[0,T]},(W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta\\ =&\mathcal{L}\Big(\xi,(b_s(\xi,U_\xi))_{s\in[0,T]},(W^0_s)_{s\in[0,T]},(W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta \quad \mathbb{P}^0\text{-}a.s..\end{aligned}$$

Note that when ζ is given,

$$\begin{aligned}&\mathcal{L}\Big(\xi,(b_s(\xi,U_\xi))_{s\in[0,T]},(W^0_s)_{s\in[0,T]},(W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta\\ =&\mathcal{L}\Big(\xi,(b_s(\xi,U_\xi))_{s\in[0,T]},(W^0_s)_{s\in[0,t]},(W^0_s-W^0_t)_{s\in[t,T]},(W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta\\ =&\mathcal{L}\Big(\eta,(b_s(\eta,U_\eta))_{s\in[0,T]},(W^0_s)_{s\in[0,t]},(W^0_s-W^0_t)_{s\in[t,T]},(W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta\\ =&\mathcal{L}\Big(\eta,(\beta_s)_{s\in[0,T]},(W^0_s)_{s\in[0,t]},(W^0_s-W^0_t)_{s\in[t,T]},(W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta,\end{aligned}$$

where $\beta_s := b_s(\eta,U_\eta)$, and the second last equality follows from the fact that ξ, U_ξ are $\mathcal{F}_t$-measurable, and therefore independent of $W^0_s-W^0_t$, W_s-W_t, $s\geq t$ when given $\mathcal{F}^0_t$. By Lemma 5.3.7, we have

$$\mathcal{L}\Big((X^{t,\xi,\alpha}_s)_{s\in[t,T]},(\alpha_s)_{s\in[0,T]},(W^0_s)_{s\in[0,T]}\Big)\Big|_\zeta = \mathcal{L}\Big((X^{t,\eta,\beta}_s)_{s\in[t,T]},(\beta_s)_{s\in[0,T]},(W^0_s)_{s\in[0,T]}\Big)\Big|_\zeta \ \mathbb{P}^0\text{-}a.s..$$

Note that this also implies $\mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_s} = \mathbb{P}^{W^0}_{X^{t,\eta,\beta}_s}$ $\mathbb{P}^0$-a.s., $\forall s\in[t,T]$. Therefore we have

$$J(t,\xi,\alpha) = J(t,\eta,\beta) \quad \mathbb{P}^0\text{-}a.s..$$

As a consequence we have that $v(t,\xi) = v(t,\eta)$. The general case follows from standard continuity arguments and the fact that for $\mathbb{P}^0$-a.s. ω^0, $v(t,\cdot)$ is continuous. □

As we have assumed that $\mathcal{G}$ is rich enough to support all laws in $\mathcal{P}_2(\mathbb{R}^d)$, by leveraging the results obtained in the above theorem and Lemma 5.1.10, albeit with a slight abuse of notation, we can define the following function

$$v : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \Omega^0 \to \mathbb{R},$$
$$v(t,\mu) := v(t,\xi),$$

for all $\xi \in L^2((\Omega^1, \mathcal{F}_t^1, \mathbb{P}^1); \mathbb{R}^d)$ such that $\mathcal{L}(\xi) := \mu$. Immediately we have the following corollary from Theorem 5.1.13:

Corollary 5.1.14.1. *(Dynamic Programming Principle). Let (A1) hold. For all* $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned} v(t,\mu) &= \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_t} \operatorname*{ess\,inf}_{\theta \in \mathcal{T}_{t,T}^0} \mathbb{E}\left[\int_t^\theta f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + v(\theta, \mathbb{P}^{W^0}_{X_\theta^{t,\xi,\alpha}}) \,\middle|\, \mathcal{F}_t^0 \right] \\ &= \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_t} \operatorname*{ess\,sup}_{\theta \in \mathcal{T}_{t,T}^0} \mathbb{E}\left[\int_t^\theta f_s(X_s^{t,\xi,\alpha}, \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s) ds + v(\theta, \mathbb{P}^{W^0}_{X_\theta^{t,\xi,\alpha}})) \,\middle|\, \mathcal{F}_t^0 \right], \end{aligned}$$

for all $\xi \in L^2((\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1); \mathbb{R}^d)$, $\mathcal{L}(\xi) = \mu$.

Moreover, for all $(t,\xi,\alpha) \in [0,T] \times L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{R}^d) \times \mathcal{A}_t$, by solving the dynamics equation (5.1.2), we obtain a process $(X_s^{t,\xi,\alpha})_{t \le s \le T}$. By conditioning on $\mathcal{F}^0$, this process can actually be interpreted as a $\mathcal{P}_2(\mathbb{R}^d)$-valued process, and shares the following property from Pham & Wei (2017):

Lemma 5.1.15. *Let (A1) hold. For any* $t \in [0,T], \mu \in \mathcal{P}_2(\mathbb{R}^d), \alpha \in \mathcal{A}_t$, *the relation given by*

$$\rho_s^{t,\mu,\alpha} := \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \quad t \le s \le T, \text{ for } \xi \in L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{R}^d) \text{ s.t. } \mathbb{P}^{W^0}_\xi = \mu,$$

defines a square integrable $\mathbb{F}^{W^0}$*-progressive continuous process in* $\mathcal{P}_2(\mathbb{R}^d)$. *Moreover, the map* $(s,t,\omega^0,\mu,\alpha) \in [0,T] \times [0,T] \times \Omega^0 \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{A}_t \to \rho_s^{t,\mu,\alpha}(\omega^0) \in \mathcal{P}_2(\mathbb{R}^d)$ *(with the convention that* $\rho_s^{t,\mu,\alpha} = \mu$ *for* $s \le t$*) is measurable.*

Proof. See Pham & Wei (2017) Lemma 3.1. □

5.2 A Theory of Viscosity Solution

5.2.1 The Space $\mathcal{K}^{1,2}(\mathcal{P}_2(\mathbb{R}^d))$ and Itô-Wentzell Formula

Definition 5.2.1. *(The Space $\mathcal{K}^{1,2}(\mathcal{P}_2(\mathbb{R}^d))$). In the context of the settings outlined in Section 5.1.4, we say that $u \in \mathcal{K}^{1,2}(\mathcal{P}_2(\mathbb{R}^d))$ if*

(1). $u \in \mathcal{S}^2(C(\mathcal{P}_2(\mathbb{R}^d)))$.

(2). There exists $(\eth_t u, \eth_w u) \in \mathcal{L}^2(C(\mathcal{P}_2(\mathbb{R}^d))) \times \mathcal{L}^2(C(\mathcal{P}_2(\mathbb{R}^d)))$ such that for $\mathbb{P}^0$-a.s. ω^0,

$$u(t,\mu) = u(T,\mu) - \int_t^T \eth_s u(s,\mu) ds - \int_t^T \eth_w u(s,\mu) dW_s^0 \quad \forall (t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d).$$

(3). For any $t \in [0,T]$ the map $\mu \mapsto u(t,\mu)$ is L-differentiable $\mathbb{P}^0$-a.s. at every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and $\partial_\mu u \in \mathcal{S}^\infty(C(\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d))$.

(4). For any $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ the map $x \mapsto \partial_\mu u(\mu, x)$ is $\mathbb{R}^d$ differentiable $\mathbb{P}^0$-a.s. at every $x \in Supp(\mu)$, and $\partial_x \partial_\mu u \in \mathcal{S}^\infty(C(\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d))$.

(5). For any $(t,x) \in [0,T] \times Supp(\mu)$, the map $\mu \mapsto \partial_\mu u(t,\mu,x)$ is L-differentiable $\mathbb{P}^0$-a.s. at every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and $\partial_\mu^2 u \in \mathcal{S}^\infty(C(\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d))$.

(6). For any $t \in [0,T]$, the map $\mu \mapsto \eth_w u(\mu)$ is L-differentiable $\mathbb{P}^0$-a.s. at every point $\mu \in \mathbb{P}_2(\mathbb{R}^d)$, and $\partial_\mu \eth_w u \in \mathcal{S}^\infty(C(\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d))$.

Theorem 5.2.2. *(Itô-Wentzell Formula). Let $u \in \mathcal{K}^{1,2}(\mathcal{P}_2(\mathbb{R}^d))$, and $(X_t)_{0 \le t \le T}$ solves (5.1.2). For $\mathbb{P}^0$ -a.s. ω^0 take $(\rho_t)_{t\in[0,T]} := \mathcal{L}(X_t(\omega^0, \cdot))_{t\in[0,T]}$. Then $(u_t(\rho_t))_{t\in[0,T]}$ is an Itô process $\mathbb{P}^0$-a.s. satisfying the expansion*

$$\begin{aligned} u_t(\rho_T) - u_0(\rho_0) = &\int_0^T \eth_s u_s(\rho_s) ds + \int_0^T \eth_w u_s(\rho_s) dW_s^0 \\ &+ \int_0^T \tilde{\mathbb{E}}^1 \Big[\partial_\mu u_s(\rho_s)(\tilde{X}_s) \cdot \tilde{b}_s \Big] ds + \int_0^T \tilde{\mathbb{E}}^1 \Big[\tilde{\sigma}_s^{0;\intercal} \partial_\mu u_s(\rho_s)(\tilde{X}_s) \Big] \cdot dW_s^0 \\ &+ \int_0^T \frac{1}{2} \tilde{\mathbb{E}}^1 \Big[\mathrm{tr}\{ \partial_x \partial_\mu u_s(\rho_s)(\tilde{X}_s)(\tilde{\sigma}_s^0 \tilde{\sigma}_s^{0;\intercal} + \tilde{\sigma}_s \tilde{\sigma}_s^\intercal) \} \Big] ds \end{aligned}$$

$$+\int_0^T \frac{1}{2}\hat{\mathbb{E}}^1\Big[\tilde{\mathbb{E}}^1\Big[\mathrm{tr}\{\partial_\mu^2 u_s(\rho_s)(\tilde{X}_s,\hat{X}_s)\tilde{\sigma}_s^0\hat{\sigma}_s^{0;T}\}\Big]\Big]ds$$
$$+\int_0^T \tilde{\mathbb{E}}^1\Big[\mathrm{tr}\{\partial_\mu \eth_w u_s(\rho_s)(\tilde{X}_s)\sigma_s^{0;\intercal}\}\Big]ds,$$

where the formula above $\tilde{\mathbb{E}}$ and $\hat{\mathbb{E}}$ denote the expectation acting on the model twin spaces $(\tilde{\Omega},\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ and $(\hat{\Omega},\hat{\mathbb{F}},\hat{\mathbb{P}})$ respectively, and let the processes $(\tilde{X}_t,\tilde{b}_t,\tilde{\sigma}_t,\tilde{\sigma}_t^0)_{t\in[0,T]}$ and $(\hat{X}_t,\hat{b}_t,\hat{\sigma}_t,\hat{\sigma}_t^0)_{t\in[0,T]}$ be the independent twin processes of $(X_t,b_t,\sigma_t,\sigma_t^0)_{t\in[0,T]}$ respectively living within.

Proof. We refer readers to Dos Reis & Platonov (2022). □

5.2.2 Test Functions, Definition of Viscosity solution

We are concerned with the following BSPDE: for $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $v:\Omega^0\times[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ satisfies

$$\begin{cases} -\eth_t v(t,\mu) = \Bigg(\int_{\mathbb{R}^d}\operatorname*{ess\,inf}_{a\in A}\Bigg\{f_t(x,\mu,a)+\Big\langle b_t(x,\mu,a),\partial_\mu v(t,\mu)(x)\Big\rangle+\frac{1}{2}\mathrm{tr}\Big\{(\sigma_t\sigma_t^\intercal(x,\mu,a) \\ +\sigma_t^0\sigma_t^{0;\intercal}(x))\partial_x\partial_\mu v(t,\mu)(x)\Big\}+\int_{\mathbb{R}^d}\frac{1}{2}\mathrm{tr}\Big\{\sigma_t^0(x)\sigma_t^{0;\intercal}(x')\partial_\mu^2 v(t,\mu)(x,x')\Big\}\mu(dx') \\ +\mathrm{tr}\Big\{\sigma_t^{0;\intercal}(x)\partial_\mu\eth_w v(t,\mu)(x)\Big\}\Bigg\}\mu(dx)\Bigg)dt, \\ v(T,\mu)=\int_{\mathbb{R}^d}g(x,\mu)\mu(dx). \end{cases} \tag{5.2.1}$$

For ease of notation, we define the following function:

$$\mathbb{H}(t,\mu,P,Q,R,S) := \int_{\mathbb{R}^d}\operatorname*{ess\,inf}_{a\in A}\Bigg\{f_t(x,\mu,a)+\Big\langle b_t(x,\mu,a),P\Big\rangle+\frac{1}{2}\mathrm{tr}\Big\{\Big(\sigma_t\sigma_t^\intercal(x,\mu,a)+\sigma_t^0\sigma_t^{0;\intercal}(x)\Big)Q\Big\}$$
$$+\int_{\mathbb{R}^d}\frac{1}{2}\mathrm{tr}\Big\{\sigma_t^0(x)\sigma_t^{0;\intercal}(x')R\Big\}\mu(dx')+\mathrm{tr}\Big\{\sigma_t^{0;\intercal}(x)S\Big\}\Bigg\}\mu(dx),$$
$$\mathbb{J}(t,\nu,P,Q,R,S) := \int_{\mathbb{R}^d\times A}\Bigg\{f_t(x,\mu,a)+\Big\langle b_t(x,\mu,a),P\Big\rangle+\frac{1}{2}\mathrm{tr}\Big\{\Big(\sigma_t\sigma_t^\intercal(x,\mu,a)+\sigma_t^0\sigma_t^{0;\intercal}(x)\Big)Q\Big\}$$
$$+\int_{\mathbb{R}^d}\frac{1}{2}\mathrm{tr}\Big\{\sigma_t^0(x)\sigma_t^{0;\intercal}(x')R\Big\}\mu(dx')+\mathrm{tr}\Big\{\sigma_t^{0;\intercal}(x)S\Big\}\Bigg\}\nu(dx,da),$$

where μ is the $\mathbb{R}^d$ marginal distribution of ν, $\mathbb{H} : \Omega^0 \times [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{L}^2(L^2((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu); \mathbb{R}^d)) \times \mathcal{L}^2(L^2((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu); \mathbb{R}^{d\times d}) \times \mathcal{L}^2(L^2((\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d), \mu \otimes \mu); \mathbb{R}^{d\times d})) \times \mathcal{L}^2(L^2((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu); \mathbb{R}^{d\times d})) \to \mathbb{R}$,

and $\mathbb{J} : \Omega^0 \times [0,T] \times \mathcal{P}_2(\mathbb{R}^d \times A) \times \mathcal{L}^2(L^2((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu); \mathbb{R}^d)) \times \mathcal{L}^2(L^2((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu); \mathbb{R}^{d\times d}) \times \mathcal{L}^2(L^2((\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d), \nu \otimes \nu); \mathbb{R}^{d\times d})) \times \mathcal{L}^2(L^2((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu); \mathbb{R}^{d\times d})) \to \mathbb{R}$.

Definition 5.2.3. *(The set of test functions $\mathscr{S}$). We say that $u \in \mathscr{S}$ if*

(1). $u \in \mathcal{K}^{1,2}(\mathcal{P}_2(\mathbb{R}^d))$.

(2). There exists a constant $\beta \in (0,1)$ such that there exists a constant $L_{u,\beta} > 0$ such that for $\mathbb{P}^0$ a.s. $\omega^0 \in \Omega^0$,

(a). for all $t \in [0,T]$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|\partial_t u(t,\mu) - \partial_t u(t,\mu')| \le L_{u,\beta} \mathcal{W}_2^\beta(\mu,\mu'),$$

(b). for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, $h = \partial_\mu u, \partial_x \partial_\mu u, \partial_\mu \partial_w u$,

$$|h(t,\mu)(x) - h(t,\mu')(x')| \le L_{u,\beta}\Big(\mathcal{W}_2^\beta(\mu,\mu') + |x-x'|^\beta\Big),$$

(c). for all $t \in [0,T]$, $x, x', y, y' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|\partial_\mu^2 u(t,\mu)(x,y) - \partial_\mu^2 u(t,\mu')(x',y')| \le L_{u,\beta}\Big(\mathcal{W}_2^\beta(\mu,\mu') + |x-x'|^\beta + |y-y'|^\beta\Big).$$

Remark 5.2.1. *As a consequence that $u \in \mathcal{K}^{1,2}(\mathcal{P}_2(\mathbb{R}^d))$, there exists a constant $L_u > 0$ such that for $\mathbb{P}^0$ a.s. ω^0, for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,*

$$|\partial_\mu u(t,\mu)(x)| + |\partial_x \partial_\mu u(t,\mu)(x)| + |\partial_\mu^2 u(t,\mu)(x,x')| + |\partial_\mu \partial_w u(t,\mu)(x)| < L_u.$$

Each $u \in \mathscr{S}$ can be thought of as an Itô process and thus a semi-martingale parameterized by $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. The Doob-Meyer decomposition theorem ensures the uniqueness of the integrable pair $(\partial_t u, \partial_w u)$.

Recall for each $\mathcal{F}^0$-stopping time $\tau \leq T$, we denote by $\mathcal{T}^0_{t,T}$ the set of $\mathcal{F}^0$-stopping times τ valued in $[t,T]$ and by $\mathcal{T}^0_{t+}$ the subset of $\mathcal{T}^0_{t,T}$ such that $\tau > t$ for any $\tau \in \mathcal{T}^0_{t,T}$. When we are referring to $\mathcal{P}_2(\mathbb{R}^d)$-valued random variables, we will use bold fonts to avoid potential confusions. For example, we refer to $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ while $\boldsymbol{\mu}$ refers to a $\mathcal{P}_2(\mathbb{R}^d)$-valued random variable.

We are ready to define our test functions for the purpose of viscosity solutions.

Definition 5.2.4. *(Set of test functions). Let $\tau \in \mathcal{T}^0_{0,T}$, $\hat{\tau} \in \mathcal{T}^0_{\tau+}$, $\Omega^0_\tau \in \mathcal{F}^0_\tau$ with $\mathbb{P}^0(\Omega^0_\tau) > 0$, $\boldsymbol{\rho} \in L^2((\Omega^0_\tau, \mathcal{F}^0_\tau, \mathbb{P}^0); \mathcal{P}_2(\mathbb{R}^d))$, we define the following set of test functions:*

$$\begin{aligned}\overline{\mathcal{G}}u(\tau, \boldsymbol{\rho}, \hat{\tau}; \Omega^0_\tau) :=& \Bigg\{ \phi \in \mathscr{S} \,\Big|\, (\phi - u)(\tau, \boldsymbol{\rho}) \mathbb{1}_{\Omega^0_\tau} = 0 \\ & = \operatorname*{ess\,sup}_{\bar{\tau} \in \mathcal{T}^0_{\tau,T}} \mathbb{E}_{\mathcal{F}^0_\tau}\Big[\operatorname*{ess\,sup}_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} (\phi - u)(\bar{\tau} \wedge \hat{\tau}, \mu)\Big] \mathbb{1}_{\Omega^0_\tau} \quad \mathbb{P}^0\text{-}a.s. \Bigg\}.\end{aligned}$$

Let $\tau \in \mathcal{T}^0_{0,T}$, $\hat{\tau} \in \mathcal{T}^0_{\tau+}$, $\Omega^0_\tau \in \mathcal{F}^0_\tau$ with $\mathbb{P}^0(\Omega^0_\tau) > 0$, $\boldsymbol{\nu} \in L^2((\Omega^0_\tau, \mathcal{F}^0_\tau, \mathbb{P}^0); \mathcal{P}_2(\mathbb{R}^d \times A))$, we define the following set of test functions:

$$\begin{aligned}\underline{\mathcal{G}}u(\tau, \boldsymbol{\nu}, \hat{\tau}; \Omega^0_\tau) :=& \Bigg\{ \phi \in \mathscr{S} \,\Big|\, (\phi - u)(\tau, \boldsymbol{\nu}) \mathbb{1}_{\Omega^0_\tau} = 0 \\ & = \operatorname*{ess\,inf}_{\bar{\tau} \in \mathcal{T}^0_{\tau,T}} \mathbb{E}_{\mathcal{F}^0_\tau}\Big[\operatorname*{ess\,inf}_{\eta \in \mathcal{P}_2(\mathbb{R}^d \times A)} (\phi - u)(\bar{\tau} \wedge \hat{\tau}, \eta)\Big] \mathbb{1}_{\Omega^0_\tau} \quad \mathbb{P}^0\text{-}a.s. \Bigg\}.\end{aligned}$$

Definition 5.2.5. *(Viscosity supersolution). We say $u \in \mathcal{S}^2(C(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_1))$ is a viscosity supersolution of the BSPDE (5.2.1) if*

(1). For the terminal condition, we have for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$u(T, \mu) \geq \int_{\mathbb{R}^d} g(x, \mu) \mu(dx) \;\; \mathbb{P}^0 \text{ - } a.s..$$

(2). For any $\tau \in \mathcal{T}^0_{0,T}$, $\hat{\tau} \in \mathcal{T}^0_{\tau+}$, $\Omega^0_\tau \in \mathcal{F}^0_\tau$ with $\mathbb{P}(\Omega^0_\tau) > 0$, $\boldsymbol{\rho} \in L^2((\Omega^0_\tau, \mathcal{F}^0_\tau, \mathbb{P}^0); \mathcal{P}_2(\mathbb{R}^d))$, and any $\phi \in \overline{\mathcal{G}}u(\tau, \boldsymbol{\rho}, \hat{\tau}; \Omega^0_\tau)$, there holds

$$\operatorname*{ess\,limsup}_{(s,\mu) \to (\tau^+, \boldsymbol{\rho})} \mathbb{E}_{\mathcal{F}^0_\tau}\Bigg[\mathbb{1}_{s < \hat{\tau}}\Big[- \partial_s \phi(s, \mu)$$

$$- \mathbb{H}\Big(s,\mu,\partial_\mu\phi(s,\mu)(\cdot),\partial_x\partial_\mu\phi(s,\mu)(\cdot),\partial^2_\mu\phi(s,\mu)(\cdot,\cdot),\partial_\mu\eth_w\phi(s,\mu)(\cdot)\Big)\Big]\Big] \geq 0,$$

for almost all $\omega^0 \in \Omega^0_\tau$.

(Viscosity subsolution). We say $u \in \mathcal{S}^2(C(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_1))$ *is a viscosity subsolution of the BSPDE (5.2.1) if*

(1). For the terminal condition, we have for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$u(T,\mu) \leq \int_{\mathbb{R}^d} g(x,\mu)\mu(dx) \ \ \mathbb{P}^0 \text{ - } a.s..$$

(2). For any $\tau \in \mathcal{T}^0_{0,T}$, $\hat{\tau} \in \mathcal{T}^0_{\tau+}$, $\Omega^0_\tau \in \mathcal{F}^0_\tau$ *with* $\mathbb{P}(\Omega^0_\tau) > 0$, $\boldsymbol{\nu} \in L^2((\Omega^0_\tau,\mathcal{F}^0_\tau,\mathbb{P}^0);\mathcal{P}_2(\mathbb{R}^d \times A))$ *and any* $\phi \in \underline{\mathcal{G}}u(\tau,\boldsymbol{\nu},\hat{\tau};\Omega^0_\tau)$, *there holds*

$$\operatorname*{ess\,liminf}_{(s,\eta)\to(\tau^+,\boldsymbol{\nu})} \mathbb{E}_{\mathcal{F}^0_\tau}\Bigg[\mathbb{1}_{s<\hat{\tau}}\Big[-\eth_s\phi(s,\eta)$$
$$- \mathbb{J}\Big(s,\eta,\partial_\mu\phi(s,\eta)(\cdot),\partial_x\partial_\mu\phi(s,\eta)(\cdot),\partial^2_\mu\phi(s,\eta)(\cdot,\cdot),\partial_\mu\eth_w\phi(s,\eta)(\cdot)\Big)\Big]\Bigg] \leq 0,$$

for almost all $\omega^0 \in \Omega^0_\tau$.

We say u *is a viscosity solution of the BSPDE (5.2.1) if it is both viscosity subsolution and supersolution. Our definition is a simple extension of the case of* $\mathbb{R}^d$, *in which it is typical that the classical solution* u *may not be differentiable in the time variable* t, *and* $(\eth_t u,\eth_w u)$ *may not be time-continuous but just measurable in* t, *this fact motivating us to use essential limits in the above.*

Remark 5.2.2. *For a progressively measurable function* $H : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\Omega^0 \to \mathbb{R}$, *when we refer to* $\operatorname*{ess\,liminf}_{(s,\mu)\to(\tau^+,\boldsymbol{\rho})}$ $\Big($*resp.* $\operatorname*{ess\,limsup}_{(s,\eta)\to(\tau^+,\boldsymbol{\nu})}\Big)$, *we are taking the following limit for stopping times* s, $s \geq \tau$ *and* $\mathcal{F}_s$*-measurable* μ*:*

$$\operatorname*{ess\,liminf}_{(s,\mu)\to(\tau^+,\boldsymbol{\rho})} H(s,\mu) :=_{\delta\to 0} \left(\operatorname*{ess\,inf}_{\substack{s\in B_\delta(\tau),s\geq\tau\\ \mu\in B_\delta(\boldsymbol{\rho})}} H(s,\mu)\right),$$

$$(\textit{resp.}\ \operatorname*{ess\,limsup}_{(s,\eta)\to(\tau^+,\boldsymbol{\nu})} H(s,\eta) :=_{\delta\to 0} \left(\operatorname*{ess\,sup}_{\substack{s\in B_\delta(\tau),s\geq\tau\\ \eta\in B_\delta(\boldsymbol{\nu})}} H(s,\eta)\right)).$$

5.2.3 Existence of the viscosity solution

We introduce the following notation which will be useful in the proof of existence of viscosity solution. Write

$$Q_r^+(t,\rho) := \big\{(s,\mu) \in [t, t+r^2) \times B_r(\rho)\big\}.$$

Also, for convenience of notation, for $\phi \in \mathscr{S}$, $t \in [0,T]$, $\alpha \in \mathcal{A}$ (or some $\mathcal{A}_k$, $k \geq 0$), $\xi \in (\Omega, \mathcal{F}_t, \mathbb{P})$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ or $\mathcal{P}_2(\mathbb{R}^d \times A)$, we denote

$$\begin{aligned}
&\mathscr{L}^\alpha \phi(t,\xi,\mu) \\
:=&\partial_t \phi(t,\mu) + \mathbb{E}^1\Big\langle b_t(\xi,\mu,\alpha), \partial_\mu \phi(t,\mu)(\xi)\Big\rangle + \frac{1}{2}\mathbb{E}^1 \mathrm{tr}\Big\{(\sigma_t \sigma_t^\intercal(\xi,\mu,\alpha) + \sigma_t^0 \sigma_t^{0;\intercal}(\xi))\partial_x \partial_\mu \phi(t,\mu)(\xi)\Big\} \\
&+ \frac{1}{2}\mathbb{E}^1\mathbb{E}^{1'} \mathrm{tr}\Big\{\sigma_t^0(\xi)\sigma_t^{0;\intercal}(\xi')\partial_\mu^2 \phi(t,\mu)(\xi,\xi')\Big\} + \mathbb{E}^1 \mathrm{tr}\Big\{\sigma_t^{0;\intercal}(\xi)\partial_\mu \partial_w \phi(t,\mu)(\xi)\Big\},
\end{aligned}$$

with ξ' being an independent copy of ξ. We have the following existence theorem:

Theorem 5.2.6. *Let (A1), (A2) hold. The value function v is a viscosity solution of the stochastic HJB equation (5.2.1).*

Proof. **Step 1**. ($\mathcal{W}_1$ continuity of v). The proof follows the same argument as that used to estimate
$|\overline{v}_{\varepsilon,n,m}(t,\overline{x}) - \overline{v}_{\varepsilon,n,m}(t,\overline{z})|$ in (Cheung *et al.*, 2025, Lemma 5.5), and is therefore omitted here.

Step 2. (v is a subsolution). Let $\tau \in \mathcal{T}_{0,T}^0$, $\hat{\tau} \in \mathcal{T}_{\tau+}^0$, $\Omega_\tau^0 \in \mathcal{F}_\tau^0$ with $\mathbb{P}^0(\Omega_\tau^0) > 0$, $\boldsymbol{\nu} \in L^2((\Omega_\tau^0, \mathcal{F}_\tau^0, \mathbb{P}^0); \mathcal{P}_2(\mathbb{R}^d \times A))$, $\phi \in \underline{\mathcal{G}} v(\tau, \boldsymbol{\nu}, \hat{\tau}; \Omega_\tau^0)$. By Lemma 5.3.2, there exists ξ, α' on $L^2((\Omega_\tau^0 \times \Omega^1, \mathcal{F}_\tau^0 \otimes \mathcal{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1); \mathbb{R}^d)$ such that $\mathcal{L}\Big(\xi(\omega^0,\cdot), \alpha'(\omega^0,\cdot)\Big) = \boldsymbol{\nu}(\omega^0)$. Pick $a \in A$ arbitrary, define $\alpha_s := a\mathbb{1}_{[0,\tau)}(s) + \alpha' \mathbb{1}_{[\tau,T]}(s)$. Let $(X_s^{\tau,\xi,\alpha})_{\tau \leq s \leq T}$ be the solution of the dynamics solves (5.1.2) with initial time τ, and initial data ξ, control α. By Lemma 5.1.15, we can now examine the $\mathcal{P}_2(\mathbb{R}^d)$-valued process $(\nu_s^{\tau,\boldsymbol{\nu}})_{\tau \leq s \leq T} := \mathcal{L}(X_s^{\tau,\xi,\alpha}(\omega^0,\cdot), \alpha_s(\omega^0,\cdot))_{\tau \leq s \leq T}$, $(\rho_s^{\tau,\boldsymbol{\nu}})_{\tau \leq s \leq T} := \mathcal{L}(X_s^{\tau,\xi,\alpha}(\omega^0,\cdot))_{\tau \leq s \leq T}$. Define $\tau^\alpha := \inf\{s > \tau : \rho_s^{\tau,\boldsymbol{\nu}} \notin B_{\tilde{\delta}/4}(\boldsymbol{\rho})\}$, by Lemma

5.1.7 we have the estimate:

$$\begin{aligned}\mathbb{E}_{\mathcal{F}_\tau^0}[\mathbb{1}_{\tau+h>\tau^\alpha}] =&\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\mathbb{1}_{\max_{\tau\le s\le\tau+h}\mathcal{W}_2(\rho_s^{\tau,\boldsymbol{\nu}},\boldsymbol{\rho})>\tilde{\delta}/4}\Big]\\ \le& C_{\tilde{\delta},T,K}\mathbb{E}_{\mathcal{F}_\tau^0}\max_{\tau\le s\le\tau+h}\mathcal{W}_2^2(\rho_s^{\tau,\boldsymbol{\nu}},\boldsymbol{\rho})\\ \le& C_{\tilde{\delta},T,K}(1+\mathbb{E}_{\mathcal{F}_\tau^0}|\xi|^2)h,\end{aligned}\tag{5.2.2}$$

where $C_{\tilde{\delta},T,K}$ is a constant depending only on $\tilde{\delta}$, T and K and independent of the control α.

By the dynamic programming principle and the Itô-Wentzell formula, for any $h\in(0,\delta)$, h small enough and $\mathbb{P}^0$-a.s. $\omega^0\in\Omega_\tau^0$, we have

$$\begin{aligned}0\ge&\frac{1}{h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[(\phi-v)(\tau,\boldsymbol{\nu})-(\phi-v)\Big((\tau+h)\wedge\hat{\tau},\nu^{\tau,\boldsymbol{\nu}}_{(\tau+h)\wedge\hat{\tau}}\Big)\Big]\\ \ge&\frac{1}{h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\phi(\tau,\boldsymbol{\nu})-\phi((\tau+h)\wedge\hat{\tau},\nu^{\tau,\boldsymbol{\nu}}_{(\tau+h)\wedge\hat{\tau}})-\int_\tau^{(\tau+h)\wedge\hat{\tau}}\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\nu}},\alpha_s)ds\Big]\\ =&\frac{1}{h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\int_\tau^{(\tau+h)\wedge\hat{\tau}}-\mathscr{L}^\alpha\phi(s,X_s^{\tau,\xi,\alpha},\nu_s^{\tau,\boldsymbol{\nu}})-\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\rho}},\alpha_s)ds\Big]\\ =&\frac{1}{h}\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\mathbb{1}_{s<\hat{\tau}}\Big[-\mathscr{L}^\alpha\phi(s,X_s^{\tau,\xi,\alpha},\nu_s^{\tau,\boldsymbol{\nu}})-\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\rho}},\alpha_s)\Big]\Big]ds\\ =&\frac{1}{h}\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\mathbb{1}_{\tau+h\le\tau^\alpha}\mathbb{1}_{s<\hat{\tau}}\Big[-\mathscr{L}^\alpha\phi(s,X_s^{\tau,\xi,\alpha},\nu_s^{\tau,\boldsymbol{\nu}})-\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\rho}},\alpha_s)\Big]\Big]ds\\ &-\frac{1}{h}\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\mathbb{1}_{\tau+h>\tau^\alpha}\Big|\mathscr{L}^\alpha\phi(s,X_s^{\tau,\xi,\alpha},\nu_s^{\tau,\boldsymbol{\nu}})+\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\rho}},\alpha_s)\Big|\Big]ds\\ \ge&\frac{1}{h}\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\mathbb{1}_{\tau+h\le\tau^\alpha}\mathbb{1}_{s<\hat{\tau}}\Big[-\mathscr{L}^\alpha\phi(s,X_s^{\tau,\xi,\alpha},\nu_s^{\tau,\boldsymbol{\nu}})-\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\rho}},\alpha_s)\Big]\Big]ds\\ &-\frac{1}{h}\sqrt{\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big|\mathscr{L}^\alpha\phi(s,X_s^{\tau,\xi,\alpha},\nu_s^{\tau,\boldsymbol{\nu}})+\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\rho}},\alpha_s)\Big|^2ds}\sqrt{\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\mathbb{1}_{\tau+h>\tau^\alpha}ds}\\ \ge&\operatorname*{ess\,inf}_{(s,\eta)\in Q_\delta^+(\tau,\boldsymbol{\nu})}\mathbb{E}_{\mathcal{F}_\tau^0}\Big[\mathbb{1}_{s<\hat{\tau}}\Big[-\eth_s\phi(s,\eta)\\ &\qquad-\mathbb{J}\Big(s,\eta,\partial_\mu\phi(s,\eta)(\cdot),\partial_x\partial_\mu\phi(s,\eta)(\cdot),\partial_\mu^2\phi(s,\eta)(\cdot,\cdot),\partial_\mu\eth_w\phi(s,\eta)(\cdot)\Big)\Big]\Big]\\ &-\frac{1}{h}\sqrt{\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\Big|\mathscr{L}^\alpha\phi(s,X_s^{\tau,\xi,\alpha},\nu_s^{\tau,\boldsymbol{\nu}})+\mathbb{E}^1 f_s(X_s^{\tau,\xi,\alpha},\rho_s^{\tau,\boldsymbol{\rho}},\alpha_s)\Big|^2ds}\sqrt{\int_\tau^{\tau+h}\mathbb{E}_{\mathcal{F}_\tau^0}\mathbb{1}_{\tau+h>\tau^\alpha}ds}\end{aligned}$$

taking limit on $\delta \to 0$, with the help of (5.2.2) we get the desired conclusion.

Step 3. (v is a supersolution). Now we prove that v is a viscosity supersolution. Suppose to the contrary, that there exists $\tau \in \mathcal{T}^0_{0,T}$, $\hat{\tau} \in \mathcal{T}^0_{\tau+}$, $\boldsymbol{\rho} \in L^2((\Omega^0_\tau, \mathcal{F}^0_\tau, \mathbb{P}^0); \mathcal{P}_2(\mathbb{R}^d))$, $\Omega^0_\tau \in \mathcal{F}^0_\tau$, $\phi \in \overline{\mathcal{G}}v(\tau, \boldsymbol{\rho}, \hat{\tau}; \Omega^0_\tau)$ such that there exists $\varepsilon, \tilde{\delta} > 0$, and $\Omega^{0'}_\tau \in \mathcal{F}^0_\tau$, $\Omega^{0'}_\tau \subseteq \Omega^0_\tau$, $\mathbb{P}^0(\Omega^{0'}_\tau) > 0$, with

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(s,\boldsymbol{\mu}) \in Q^+_{\tilde{\delta}}(\tau,\boldsymbol{\rho})} \mathbb{E}_{\mathcal{F}^0_\tau}\Bigg[\mathbb{1}_{s<\hat{\tau}}\Big[- \eth_s \phi(s, \boldsymbol{\mu}) \\
&- \mathbb{H}\Big(s, \boldsymbol{\mu}, \partial_\mu \phi(s, \boldsymbol{\mu})(\cdot), \partial_x \partial_\mu \phi(s, \boldsymbol{\mu})(\cdot), \partial^2_\mu \phi(s, \boldsymbol{\mu})(\cdot, \cdot), \partial_\mu \eth_w \phi(s, \boldsymbol{\mu})(\cdot)\Big)\Big]\Bigg] \leq -\varepsilon,
\end{aligned}$$

$\mathbb{P}^0$-a.s. ω^0 in $\Omega^{0'}_\tau$. As in Step 1, there exists ξ on $L^2((\Omega^0_\tau \times \Omega^1, \mathcal{F}^0_\tau \otimes \mathcal{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1); \mathbb{R}^d)$ such that $\mathcal{L}(\xi(\omega^0, \cdot)) = \boldsymbol{\rho}(\omega^0)$. For each $\alpha \in \mathcal{A}_\tau$, let $(X^{\tau,\xi,\alpha}_s)_{\tau \leq s \leq T}$ solves (5.1.2) with inital time τ, initial data ξ and control α. By Lemma 5.1.15, we can now examine the $\mathcal{P}_2(\mathbb{R}^d)$-valued process $(\rho^{\tau,\boldsymbol{\rho},\alpha}_s)_{\tau \leq s \leq T} := \mathcal{L}(X^{\tau,\xi,\alpha}_s(\omega^0, \cdot))_{\tau \leq s \leq T}$. Define $\tau^\alpha := \inf\{s > \tau : \rho^{\tau,\boldsymbol{\rho},\alpha}_s \notin B_{\tilde{\delta}/4}(\boldsymbol{\rho})\}$, by Lemma 5.1.7 we have the estimate:

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}^0_\tau}[\mathbb{1}_{\tau+h>\tau^\alpha}] =& \mathbb{E}_{\mathcal{F}^0_\tau}\Big[\mathbb{1}_{\max_{\tau \leq s \leq \tau+h} \mathcal{W}_2(\rho^{\tau,\boldsymbol{\rho},\alpha}_s, \boldsymbol{\rho}) > \tilde{\delta}/4}\Big] \\
\leq& C_{\tilde{\delta},T,K} \mathbb{E}_{\mathcal{F}^0_\tau} \max_{\tau \leq s \leq \tau+h} \mathcal{W}^2_2(\rho^{\tau,\boldsymbol{\rho},\alpha}_s, \boldsymbol{\rho}) \\
\leq& C_{\tilde{\delta},T,K}(1 + \mathbb{E}_{\mathcal{F}^0_\tau}|\xi|^2)h, \qquad (5.2.3)
\end{aligned}$$

where $C_{\tilde{\delta},T,K}$ is a constant depending only on $\tilde{\delta}$, T and K and independent of the control α. For $h \in (0, \tilde{\delta}^2/4)$ and h small enough, by the dynamic programming principle and the Ito-Wentzell formula, for $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^{0'}_\tau$, we have

$$\begin{aligned}
0 =& \frac{1}{h}\Big(v(\tau, \boldsymbol{\rho}) - \phi(\tau, \boldsymbol{\rho})\Big) \\
=& \frac{1}{h} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_\tau} \mathbb{E}_{\mathcal{F}^0_\tau}\left[\int_\tau^{\hat{\tau} \wedge (\tau+h)} \mathbb{E}^1 f_s(X^{\tau,\boldsymbol{\rho},\alpha}_s, \rho^{\tau,\boldsymbol{\rho},\alpha}_s, \alpha_s) ds + v(\hat{\tau} \wedge (\tau+h), \rho^{\tau,\boldsymbol{\rho},\alpha}_{\hat{\tau} \wedge (\tau+h)}) - \phi(\tau, \boldsymbol{\rho})\right] \\
\geq& \frac{1}{h} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_\tau} \mathbb{E}_{\mathcal{F}^0_\tau}\left[\int_\tau^{\hat{\tau} \wedge (\tau+h)} \mathbb{E}^1 f_s(X^{\tau,\boldsymbol{\rho},\alpha}_s, \rho^{\tau,\boldsymbol{\rho},\alpha}_s, \alpha_s) ds + \phi(\hat{\tau} \wedge (\tau+h), \rho^{\tau,\boldsymbol{\rho},\alpha}_{\hat{\tau} \wedge (\tau+h)}) - \phi(\tau, \boldsymbol{\rho})\right]
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{h}\operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_\tau}\mathbb{E}_{\mathcal{F}_\tau^0}\left[\int_\tau^{\tau+h}\Big(\mathbb{E}^1 f_s(X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha},\alpha_s)+\mathscr{L}^\alpha\phi(s,X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha})\Big)\mathbb{1}_{s\le\hat\tau}ds\right]\\
\ge&\frac{1}{h}\operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_\tau}\mathbb{E}_{\mathcal{F}_\tau^0}\left[\int_\tau^{\tau+h}\Big(\mathbb{E}^1 f_s(X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\alpha_{s\wedge\tau^\alpha})+\mathscr{L}^\alpha\phi(s,X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha})\Big)\mathbb{1}_{s\le\hat\tau}\mathbb{1}_{\tau+h\le\tau^\alpha}ds\right]\\
&-\frac{1}{h}\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_\tau}\mathbb{E}_{\mathcal{F}_\tau^0}\left[\int_\tau^{\hat\tau\wedge(\tau+h)}\Big|\mathbb{E}^1 f_s(X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha},\alpha_s)+\mathscr{L}^\alpha\phi(s,X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha})\Big|\mathbb{1}_{\tau+h>\tau^\alpha}ds\right]\\
\ge&\frac{1}{h}\operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_\tau}\mathbb{E}_{\mathcal{F}_\tau^0}\left[\int_\tau^{\tau+h}\Big(\mathbb{E}^1 f_s(X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\alpha_{s\wedge\tau^\alpha})+\mathscr{L}^\alpha\phi(s,X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha})\Big)\mathbb{1}_{s\le\hat\tau}ds\right]\\
&-\frac{1}{h}\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_\tau}\mathbb{E}_{\mathcal{F}_\tau^0}\left[\int_\tau^{\tau+h}\Big|\mathbb{E}^1 f_s(X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\alpha_{s\wedge\tau^\alpha})+\mathscr{L}^\alpha\phi(s,X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha})\Big|\mathbb{1}_{s\le\hat\tau}\mathbb{1}_{\tau+h>\tau^\alpha}ds\right]\\
&-\frac{1}{h}\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_\tau}\mathbb{E}_{\mathcal{F}_\tau^0}\left[\int_\tau^{\hat\tau\wedge(\tau+h)}\Big|\mathbb{E}^1 f_s(X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha},\alpha_s)+\mathscr{L}^\alpha\phi(s,X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha})\Big|\mathbb{1}_{\tau+h>\tau^\alpha}ds\right]\\
\ge&\varepsilon-\frac{1}{h}\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_\tau}\sqrt{\mathbb{E}_{\mathcal{F}_\tau^0}\int_\tau^{\tau+h}\Big|\mathbb{E}^1 f_s(X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha},\alpha_s)+\mathscr{L}^\alpha\phi(s,X_s^{\tau,\boldsymbol{\rho},\alpha},\rho_s^{\tau,\boldsymbol{\rho},\alpha})\Big|^2ds}\sqrt{\mathbb{E}_{\mathcal{F}_\tau^0}\int_\tau^{\tau+h}\Big|\mathbb{1}_{\tau+h>\tau^\alpha}\Big|^2ds}\\
&-\frac{1}{h}\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_\tau}\sqrt{\mathbb{E}_{\mathcal{F}_\tau^0}\int_\tau^{\tau+h}\Big|\mathbb{E}^1 f_s(X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\alpha_{s\wedge\tau^\alpha})+\mathscr{L}^\alpha\phi(s,X_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha},\rho_{s\wedge\tau^\alpha}^{\tau,\boldsymbol{\rho},\alpha})\Big|^2ds}\sqrt{\mathbb{E}_{\mathcal{F}_\tau^0}\int_\tau^{\tau+h}\Big|\mathbb{1}_{\tau+h>\tau^\alpha}\Big|^2d}\\
\to&\varepsilon,\ \text{as } h\to 0,\ \text{with the help of (5.2.3).}
\end{aligned}
$$

The above draws a contradiction and hence the value function v is a viscosity supersolution.

□

5.3 Measurable Selection Theorem and some technical lemmas

First of all, we provide a proof of Lemma 5.1.6.

Proof of Lemma 5.1.6. We focus exclusively on f, given that the remaining cases are analogous. First of all we assume that for $\mathbb{P}^0$-a.s. ω^0, $t\mapsto f(\omega^0,t,\cdot,\cdot,\cdot)\in C(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A)$ is continuous. We approximate f by random fields of the form:

$$
\bar f^l(\omega^0,t,\cdot,\cdot,\cdot)=\phi_1(\omega^0,\cdot,\cdot,\cdot)\mathbb{1}_{[0,t_1]}(t)+\sum_{j=2}^{l}\phi_j(\omega^0,\cdot,\cdot,\cdot)\mathbb{1}_{(t_{j-1},t_j]}(t),
$$

where $0 = t_0 < t_1 < \ldots < t_l < T$, and for $j = 1, \ldots, l$, $\phi_j = f(t_{j-1}, \cdot, \cdot, \cdot) \in L^2\Big((\Omega^0, \mathcal{F}^0_{t_{j-1}}, \mathbb{P}^0); C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)\Big)$. By the dominated convergence theorem and the continuity, we have

$$\mathbb{E}\int_0^T \|\bar{f}^l - f\|^2_{C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)} \to 0, \text{ as } \max_{j=1,\ldots,l} |t_j - t_{j-1}| \to 0.$$

By the Bochner measurability, ϕ_j can be approximated pointwisely (in $C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$) by functions of the form

$$\sum_{i=1}^{l_j} \mathbb{1}_{A_i^j}(\omega^0) h_i^j(x, \mu, a), \quad \text{with } h_i^j \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A), \quad A_i^j \in \mathcal{F}^0_{t_{j-1}}, \quad i = 1, \ldots, l_j. \tag{5.3.1}$$

In fact we can take

$$h_i^j \in C\left(\mathcal{P}_2(\mathbb{R}^d) \times A; C^\infty\left(\mathbb{R}^d\right)\right), \quad i = 1, \ldots, l_j.$$

This is because for each fixed $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $a \in A$, we consider its mollification on $\mathbb{R}^d$, given by

$$h_i^{j;\varepsilon}(x, \mu, a) := \int_{\mathbb{R}^d} \eta_\varepsilon(x - y) h_i^j(y, \mu, a) dy,$$

where η is the standard mollifier given by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

the constant $C > 0$ is selected so that $\int_{\mathbb{R}^d} \eta dx = 1$, and $\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right)$ for each $\varepsilon > 0$. Note that

$$\begin{aligned} |h_i^{j;\varepsilon}(x, \mu, a) - h_i^j(x, \mu, a)| \leq & \int_{\mathbb{R}^d} \eta_\varepsilon(x - y) |h_i^j(y, \mu, a) - h_i^j(x, \mu, a)| dy \\ \leq & K \int_{\mathbb{R}^d} \eta_\varepsilon(x - y)|x - y| dy \to 0, \text{ as } \varepsilon \to 0, \end{aligned}$$

uniformly on $(x, \mu, a) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$, and K is the constant in the Assumption $(\mathcal{A}1)$, thus $\|h_i^{j;\varepsilon} - h_i^j\|_{C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)} \to 0$ as $\varepsilon \to 0$. Note that $|h_i^{j;\varepsilon}(x, \mu, a) - h_i^{j;\varepsilon}(x', \mu', a)| \leq K[|x - x'| +$

$\mathcal{W}_2(\mu,\mu')]$ (or $K[|x-x'|+\mathcal{W}_1(\mu,\mu')]$ under Assumption $(\mathcal{A}1^*)$). (Øksendal & Øksendal, 2003, Lemma 4.3.1) implies that each $\mathbb{1}_{A_i^j}$ may be approximated in $L^2(\Omega^0,\mathcal{F}^0_{t_{j-1}},\mathbb{P}^0)$ by functions in the following set

$$\left\{g\left(W^0_{\tilde{t}_1},\ldots,W^0_{\tilde{t}_{l_i^j}}\right):\tilde{t}_r\in[0,t_{j-1}],g\in C_0^\infty\left(\mathbb{R}^{d\times l_i^j}\right),\text{ i.e., smooth functions with compact support}\right\}.$$

In addition, each $\mathbb{1}_{(t_{j-1},t_j]}$ can be increasingly approximated by compactly-supported non-negative functions $\varphi_j\in C^\infty\left((t_{j-1},T];\mathbb{R}\right)$ in $L^2([0,T],\mathcal{B}([0,T]),ds)$. To sum up, f can be approximated in $\mathcal{L}^2\left(C\left(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A\right)\right)$ by the following random fields:

$$f^N\left(W_{\bar{t}_1\wedge t},\ldots,W_{\bar{t}_N\wedge t},t,x,\mu,a\right)=\sum_{j=1}^{l}\sum_{i=1}^{l_j}g_i^j\left(W^0_{\bar{t}_1\wedge t_{j-1}},\ldots,W^0_{\bar{t}_N\wedge t_{j-1}}\right)h_i^j(x,\mu,a)\varphi_j(t),$$

where $0=\bar{t}_0<\bar{t}_1<\ldots<\bar{t}_{N-1}<\bar{t}_N=T$, the functions g_i^j have compact support, and the functions h_i^j are Lipschitz continuous with respect to x and μ, with Lipschitz constant K.

Now we consider the case for $t\mapsto f(\omega^0,t,\cdot,\cdot,\cdot)\in C(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A)$ not necessarily continuous. We define the continuous approximation $\xi^n:\left(\Omega^0\times[0,T],Prog(\mathbb{F}^0),\mathbb{P}^0\otimes ds\right)\to C(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A)$ by

$$\xi^n(\omega^0,t):=\int_0^t\psi_n(s-t)f(\omega^0,s,\cdot,\cdot,\cdot)ds,$$

where $\psi_n(s)=n\psi(ns)$, and $\psi(s)$ is any nonnegative bounded continuous function on $\mathbb{R}$ with support in $[-1,0]$ and such that $\int_{\mathbb{R}}\psi(s)ds=1$. We now show that for $\mathbb{P}^0$-a.s. ω^0, $t\mapsto\xi^n(\omega^0,t)$ is continuous. We have

$$\begin{aligned}
&\|\xi^n(\omega^0,t+h)-\xi^n(\omega^0,t)\|_{C(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A)}\\
=&\left\|\int_0^{t+h}\psi_n(s-(t+h))f(\omega^0,s,\cdot,\cdot,\cdot)ds-\int_0^t\psi_n(s-t)f(\omega^0,s,\cdot,\cdot,\cdot)ds\right\|_{C(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A)}\\
\leq&\left\|\int_0^t\Big[\psi_n(s-(t+h))-\psi_n(s-t)\Big]f(\omega^0,s,\cdot,\cdot,\cdot)ds\right\|_{C(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A)}\\
&+\left\|\int_t^{t+h}\psi_n(s-(t+h))f(\omega^0,s,\cdot,\cdot,\cdot)ds\right\|_{C(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times A)},
\end{aligned}$$

the second term goes to zero by dominated convergence theorem. For the first term, we have

$$\begin{aligned}
&\left\| \int_0^t \Big[\psi_n(s-(t+h)) - \psi_n(s-t)\Big] f(\omega^0, s, \cdot, \cdot, \cdot) ds \right\|_{C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)} \\
\leq & \sqrt{\int_0^t |\psi_n(s-(t+h)) - \psi_n(s-t)|^2 ds} \sqrt{\int_0^t \|f(\omega^0, s, \cdot, \cdot, \cdot)\|^2_{C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)} ds} \\
\to & 0, \text{ as } h \to 0 \text{ by dominated convergence theorem.}
\end{aligned}$$

By (Hytönen *et al.*, 2016, Proposition 1.2.32), for $\mathbb{P}^0$-a.s. ω^0, $\int_0^T \|\xi_n(\omega^0, s) - f(\omega^0, s)\|^2_{C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)} ds \to 0$ as $n \to \infty$. By the dominated convergence theorem, $\mathbb{E} \int_0^T \|\xi_n(\omega^0, s) - f(\omega^0, s)\|^2_{C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)} ds \to 0$. Note that from our assumption, for all $t \in [0, T]$, $(\omega^0, s) \in \Omega^0 \times [0, t]$, $f(\omega^0, s)$ can be written as the pointwise limit of simple functions of the form $\sum_{i=1}^k \mathbb{1}_{A_i}(s) \mathbb{1}_{B_i}(\omega^0) h_i$, $A_i \in \mathcal{B}([0, t])$, $B_i \in \mathcal{F}_t^0$, $h_i \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A)$. Hence for each fixed $t \in [0, T]$, ξ^n can be written as the pointwise limit of functions of the form

$$\sum_{i=1}^k \mathbb{1}_{B_i}(\omega^0) \left[\int_0^t \psi_n(s-t) \mathbb{1}_{A_i}(s) ds \right] h_i,$$

which is still of the form in (5.3.1). □

We now provide a proof of Lemma 5.1.4.

Proof of Lemma 5.1.4. As in the proof of Lemma 5.1.6 in Appendix 5.3, for any $\alpha \in \mathcal{A}_t$, there exists a sequence of progressively measurable, continuous $\xi^n : (\Omega \times [t, T], Prog(\mathbb{F}^0), \mathbb{P} \otimes ds) \to A$ such that $\mathbb{E}[\int_t^T |\xi^n - \alpha|^2 | \mathcal{F}_t^0] \to 0$ as $n \to \infty$. Therefore, it is without loss of generality that we assume that for $\mathbb{P}$-a.s. $(\omega^0, \omega^1) \in \Omega^0 \times \Omega^1$, $[t, T] \ni s \mapsto \alpha_s(\omega^0, \omega^1)$ is continuous. We consider approximation of the form

$$\alpha_s^n(\omega^0, \omega^1) := \alpha_t(\omega^0, \omega^1) \mathbb{1}_{\{t\}}(s) + \sum_{i=0}^{n-1} \alpha_{t_i}(\omega^0, \omega^1) \mathbb{1}_{(t_i, t_{i+1}]}(s)$$

where $t =: t_0 < \ldots < t_{n-1} < t_n := T$. Note that for all (s, ω^0, ω^1), $\alpha^n(s, \omega^0, \omega^1) \to \alpha(s, \omega^0, \omega^1)$ due to the continuity, as $\max_{i=0,\ldots,n-1} |t_{i+1} - t_i| \to 0$. For each t_i, $\alpha_{t_i}(\omega^0, \omega^1) = \alpha_{t_i}(\omega^0 \otimes_t \omega^0, \omega^1)$, hence α_{t_i} can be interpreted as $(\Omega^0 \times \Omega^0 \times \Omega^1, \mathcal{F}_t^0 \otimes \bar{\mathcal{F}}_{t_i}^{0;t} \otimes \mathcal{F}_{t_i}^{1;t}) \to A$.

Therefore, each $\alpha_{t_i}(\omega^0,\omega^1)$ can be approximated by function of the form

$$\sum_{j=1}^{n_i} a_j^i \mathbb{1}_{A_j^i}(\omega^0)\mathbb{1}_{B_j^i}(\omega^0)\mathbb{1}_{C_j^i}(\omega^1),\ \ a_j^i \in A, A_j^i \in \mathcal{F}_t^0, B_j^i \in \bar{\mathcal{F}}_{t_i}^{0;t}, C_j^i \in \mathcal{F}_{t_i}^{1;t}.$$

Note that this approximation is also pointwise. Hence, α^n can be approximated by functions of the form

$$\phi^m(s,\omega^0,\omega^1) := \sum_{j=1}^{n_0} a_j^0 \mathbb{1}_{A_j^0}(\omega^0)\mathbb{1}_{C_j^0}(\omega^1)\mathbb{1}_{\{t\}}(s) + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} a_j^i \mathbb{1}_{A_j^i}(\omega^0)\mathbb{1}_{B_j^i}(\omega^0)\mathbb{1}_{C_j^i}(\omega^1)\mathbb{1}_{(t_i,t_{i+1}]}(s).$$

Rearranging gives the representation

$$\begin{aligned}\phi^m(s,\omega^0,\omega^1) =& \sum_{j=1}^{n_0} a_j^0 \mathbb{1}_{A_j^0}(\omega^0)\mathbb{1}_{C_j^0}(\omega^1)\mathbb{1}_{\{t\}}(s) \\ &+ \sum_{j_{n-1}=1}^{n_{n-1}} \cdots \sum_{j_0=1}^{n_0} \mathbb{1}_{A_{j_0}^0 \cap A_{j_1}^1 \cap \ldots \cap A_{j_{n-1}}^{n-1}}(\omega^0) \underbrace{\left(\sum_{i=0}^{n-1} a_{j_i}^i \mathbb{1}_{B_{j_i}^i}(\omega^0)\mathbb{1}_{C_{j_i}^i}(\omega^1)\mathbb{1}_{(t_i,t_{i+1}]}(s) \right)}_{\in \bar{\mathcal{A}}_t}.\end{aligned}$$

Moreover, because of the pointwise approximation, and the fact that A is a compact subset of the Euclidean space, by the dominated convergence theorem for conditional expectation, we conclude that

$$\mathbb{E}\left[\int_t^T |\phi^m(s,\omega^0,\omega^1) - \alpha(s,\omega^0,\omega^1)|^2 \Big| \mathcal{F}_t^0 \right] \to 0,$$

which leads to our desired conclusion. □

Recall the measurable selection theorem in Wagner (1977):

Theorem 5.3.1. *Let $(\Lambda, \mathscr{M})$ be a measurable space equipped with a nonnegative measure μ, and let $(\mathcal{O}, \mathcal{B}(\mathcal{O}))$ be a Polish space. Suppose F is a set-valued function from Λ to $\mathcal{B}(\mathcal{O})$ satisfying the following:*

(1). for μ-a.s. $\lambda \in \Lambda$, $F(\lambda)$ is a closed nonempty subset of $\mathcal{O}$,

(2). for any open set $O \subset \mathcal{O}$, $\{\lambda : F(\lambda) \cap O \neq \phi\} \in \mathscr{M}$.

Then there exists a measurable function $f : (\Lambda, \mathscr{M}) \to (\mathcal{O}, \mathcal{B}(\mathcal{O}))$ such that for μ-a.s. $\lambda \in \Lambda$, $f(\lambda) \in F(\lambda)$.

Note that the above sets are non-empty due to the compactness of $\mathcal{J}$ in $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_1)$ and upper-semicontinuouity of the concerned functions in $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_1)$. By invoking the regularity conditions for the concerned functions, we can apply the measurable selection theorem.

Lemma 5.3.2. *Let $t \geq 0$, $p \geq 1$, let $\boldsymbol{\mu} : (\Omega^0, \mathcal{F}_t^0, \mathbb{P}^0) \to (\mathcal{P}_p(\mathbb{R}^d), \mathcal{B}(\mathcal{P}_p(\mathbb{R}^d)))$ be a random variable, where $\mathcal{B}(\mathcal{P}_p(\mathbb{R}^d))$ is the Borel σ-algebra of $\mathcal{P}_p(\mathbb{R}^d)$, and $\int_{\Omega^0} \int |x|^p \boldsymbol{\mu}(\omega^0, dx) d\mathbb{P}^0(\omega^0) < \infty$. Then there exists a $\xi \in L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ such that $\mathcal{L}(\xi(\omega^0, \cdot)) = \boldsymbol{\mu}(\omega^0)$ for $\mathbb{P}^0$-a.s. ω^0.*

Proof. It is without loss of generality that we may assume $\boldsymbol{\mu} : (\Omega^0, \mathcal{F}_t^0, \mathbb{P}^0) \to \big(\mathcal{P}_p([0,1]), \mathcal{B}(\mathcal{P}_p([0,1]))\big)$, as $\mathbb{R}^d$ is a Borel space. We claim that $\forall a \in [0,1]$,

$$\omega^0 \mapsto \boldsymbol{\mu}(\omega^0, [0, a]) \tag{5.3.2}$$

is measurable. Define the map $\phi_a : \mathcal{P}_p(\mathbb{R}^d) \to \mathbb{R}$, $\phi_a(\mu) := \mu([0,a])$. This map is upper semi-continuous since if $\mu_n \to \mu$ in $\mathcal{W}_p$, then μ_n converges weakly μ, thus $\limsup \mu_n([0,a]) \leq \mu([0,a])$, i.e., $\limsup \phi_a(\mu_n) \leq \phi_a(\mu)$. Therefore ϕ_a is measurable, and the mapping in (5.3.2) is simply $\omega^0 \mapsto \phi_a(\boldsymbol{\mu}(\omega^0))$ is therefore measurable. Define

$$f(\omega^0, t) := \sup\{a \in [0,1] : \boldsymbol{\mu}(\omega^0, [0,a]) < t\}, \quad \omega^0 \in \Omega^0, t \in [0,1].$$

f is product measurable on $\Omega^0 \times [0,1]$, since the set $\{(\omega^0, t);\, \boldsymbol{\mu}(\omega^0, [0,a]) < t\}$ is measurable for each a, and the supremum could be restricted to rational a. Let $U \in L^p(\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1)$, $U \sim U(0,1)$, the uniform distribution on $[0,1]$, then

$$\tilde{\mathbb{P}}^1\{f(\omega^0, U) \leq a\} = \tilde{\mathbb{P}}^1\{U \leq \boldsymbol{\mu}(\omega^0, [0,a])\} = \boldsymbol{\mu}(\omega^0, [0,a]),$$

so $f(\omega^0, U)$ has distribution $\boldsymbol{\mu}(\omega^0, \cdot)$. Moreover,

$$\mathbb{E}^0\mathbb{E}^1 |f(\omega^0, U(\omega^1))|^p = \int_{\Omega^0} \int |x|^p \boldsymbol{\mu}(\omega^0, dx) d\mathbb{P}^0(\omega^0) < \infty.$$

□

5.3.1 Conditional Law Invariance

The below proof is inspired by (Cosso *et al.*, 2023, Appendix B). First of all we recall the definition of probability kernel.

Definition 5.3.3. *(Probability kernel). Given two measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$, a mapping $\mu : S \times \mathcal{T} \to \overline{\mathbb{R}}_+$ is called a (probability) kernel from S to T if the function $\mu_s B = \mu(s, B)$ is $\mathcal{S}$-measurable in $s \in S$ for fixed $B \in \mathcal{T}$ and a (probability) measure in $B \in \mathcal{T}$ for fixed $s \in S$.*

Before we proceed, we recall the readers the notion of conditional independence. Consider sub σ-algebras $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{G} \subset \mathcal{F}$. We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditional independent given $\mathcal{G}$ if

$$\mathbb{P}(B_1 \cap B_2|\mathcal{G}) = \mathbb{P}(B_1|\mathcal{G})\mathbb{P}(B_2|\mathcal{G}) \text{ a.s., } B_1 \in \mathcal{F}_1, B_2 \in \mathcal{F}_2.$$

We denote the above by $\mathcal{F}_1 \perp\!\!\!\perp_{\mathcal{G}} \mathcal{F}_2$. This notation is generalised to the case of random variable by considering their induced σ-algebras.

Theorem 5.3.4. *Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, measurable spaces $(K, \mathcal{K})$, $(S, \mathcal{S})$ and a Borel space $(T, \mathcal{B}(T))$. Define the random variables:*

$$\begin{aligned} \zeta &: (\Omega, \mathcal{F}, \mathbb{P}) \to (K, \mathcal{K}), \\ \xi, \tilde{\xi} &: (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S}), \\ \eta &: (\Omega, \mathcal{F}, \mathbb{P}) \to (T, \mathcal{B}(T)). \end{aligned}$$

Assume that

(1). For $\mathbb{P}$-a.s. ω, $\forall B \in \mathcal{S}$, $\mathbb{P}(\xi \in B|\zeta) = \mathbb{P}(\tilde{\xi} \in B|\zeta)$,

(2). there exists a random variable θ such that $\theta \perp\!\!\!\perp_{\zeta} \tilde{\xi}, \xi$ and for $\mathbb{P}$-a.s. ω, $\mathcal{L}(\theta|\zeta) = U(0,1)$, where $U(0,1)$ is the law of a Uniform(0,1) distribution.

Then, there exists a measurable mapping $f : K \times S \times [0,1] \to T$ *such that if* $\tilde{\eta} := f(\zeta, \tilde{\xi}, \theta)$, *then for* $\mathbb{P}$ *a.s.* ω,

$$\mathbb{E}[g(\tilde{\xi}, \tilde{\eta})|\zeta](\omega) = \mathbb{E}[g(\xi, \eta)|\zeta](\omega), \text{ for any bounded measurable } g : S \times T \to \mathbb{R}.$$

Proof. By (Kallenberg, 2006, Theorem 6.3), there exists a probability kernel μ from $K \times S \to T$ satisfying for $\mathbb{P}$-a.s. ω, $\forall B \in \mathcal{B}(T)$,

$$\mu(\zeta, \xi, B) = \mathbb{P}[\eta \in B|\zeta, \xi].$$

By (Kallenberg, 2006, Lemma 3.22), we choose a measurable function $f : K \times S \times [0,1] \to T$ such that when given ζ, $f(k, s, \theta)$ has the distribution $\mu(k, s, \cdot)$ for every $(k, s) \in K \times S$. Define $\tilde{\eta} := f(\zeta, \tilde{\xi}, \theta)$. We have $\mathbb{P}$-a.s. ω for all bounded measurable g,

$$\begin{aligned}
\mathbb{E}[g(\tilde{\xi}, \tilde{\eta})|\zeta] =& \mathbb{E}\Big[g(\tilde{\xi}, f(\zeta, \tilde{\xi}, \theta))\Big|\zeta\Big] \\
=& \mathbb{E}\Big[g(\xi, f(\zeta, \xi, \theta))\Big|\zeta\Big] \text{ (due to (i) and (ii))} \\
=& \mathbb{E}\Big[\mathbb{E}\big[g(\xi, f(\zeta, \xi, \theta))|\zeta, \xi\big]|\zeta\Big] \\
=& \mathbb{E}\Big[\mathbb{E}\big[g(\xi, \eta)|\zeta, \xi\big]|\zeta\Big] \\
=& \mathbb{E}[g(\xi, \eta)|\zeta].
\end{aligned}$$

□

We introduce the setting for the next theorem. Let $t \in [0, T]$ and

$$\overline{\Omega} = [0, T] \times \Omega, \quad \overline{\mathcal{F}} = \mathcal{B}([0, T]) \otimes \mathcal{F}_T^0 \otimes \hat{\mathcal{F}}_T^{1;t} \otimes \mathcal{G}, \quad \overline{\mathbb{P}} = \lambda_{[0,T]} \otimes \mathbb{P},$$

with $\lambda_{[0,T]}$ being the uniform distribution on $([0, T], \mathcal{B}([0, T]))$. Let $Prog((\mathbb{F}^0 \vee \hat{\mathbb{F}}^{1;t}))$ denote the progressive σ-algebra w.r.t. $\mathbb{F}^0 \vee \hat{\mathbb{F}}^{1;t}$, and $\zeta : (\overline{\Omega}, \overline{\mathcal{F}}) \to (\Omega^0, \mathcal{F}_t^0)$ be the following projection map,

$$\zeta(s, \omega^0, \omega^1) := \omega^0_{\cdot \wedge t}. \tag{5.3.3}$$

Theorem 5.3.5. *Let ($\mathcal{A}1$) hold. Let $\alpha \in \mathcal{A}_t$, $\xi \in L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathbb{R}^d)$. Suppose that there exists an $\mathcal{F}_t$-measurable random variable U_ξ, $U_\xi \perp\!\!\!\perp_\zeta \xi$ and for $\mathbb{P}$-a.s. ω, $\mathcal{L}(U_\xi|\zeta) = U(0,1)$. Then, there exists a function $b : [0,T] \times \Omega^0 \times \hat{\Omega}^1 \times \mathbb{R}^d \times [0,1] \to A$, measurable with respect to $Prog((\mathbb{F}^0 \vee \hat{\mathbb{F}}^{1;t})) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}([0,1])$ such that $(b_s(\xi, U_\xi))_{s\in[0,T]}$ is $\mathbb{F}^t$-progressively measurable, and*

$$\begin{aligned}&\mathcal{L}\Big(\xi, (b_s(\xi, U_\xi))_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t} - W_t)_{s\in[0,T]}\Big)\Big|_\zeta \\ =&\mathcal{L}\Big(\xi, (\alpha_s)_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t} - W_t)_{s\in[0,T]}\Big)\Big|_\zeta \quad \mathbb{P}^0\text{-a.s.}.\end{aligned}$$

Proof. Consider the canonical extension of U_ξ, ξ to $\overline{\Omega}$, denoted by $\overline{U}_\xi$, $\overline{\xi}$. Let $(\bar{E}, \bar{\mathscr{E}})$ be the measurable space defined as $\bar{E} = [0,T] \times \Omega^0 \times \hat{\Omega}^1$ and $\bar{\mathscr{E}} = Prog((\mathbb{F}^0 \vee \hat{\mathbb{F}}^{1;t}))$. Let $\pi : (\overline{\Omega}, \overline{\mathcal{F}}) \to (\bar{E}, \bar{\mathscr{E}})$ be the following projection map, $\pi : [0,T] \times \Omega^0 \times \hat{\Omega}^1 \times \tilde{\Omega}^1 \to [0,T] \times \Omega^0 \times \hat{\Omega}^1$,

$$\pi(s, \omega^0, \hat{\omega}^1, \tilde{\omega}^1) := (s, \omega^0, \hat{\omega}^1).$$

Define $\Gamma := (\pi, \overline{\xi})$. One readily verifies that when given ζ, $\overline{U}_\xi$ is independent of Γ. Γ takes values in the measurable space $(E, \mathscr{E})$, with $E = [0,T] \times \Omega^0 \times \hat{\Omega}^1 \times \mathbb{R}^d$, $\mathscr{E} = Prog((\mathbb{F}^0 \vee \hat{\mathbb{F}}^{1;t})) \otimes \mathcal{B}(\mathbb{R}^d)$. Working under the conditional expectation on $\mathcal{F}^0_t$ is equivalent to conditioning on the random variable ζ. From above Theorem 5.3.4, there exists a function $a : \Omega^0 \times [0,T] \times \Omega^0 \times \hat{\Omega}^1 \times \mathbb{R}^d \times [0,1] \to A$, measurable with respect to the σ-algebras $\mathcal{F}^0_t \otimes Prog((\mathbb{F}^0 \vee \hat{\mathbb{F}}^{1;t})) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}([0,1])$, such that we have

$$\mathcal{L}\big(\Gamma, (a_s(\zeta, \Gamma, \overline{U}_\xi))_{s\in[0,T]}\big)\Big|_\zeta = \mathcal{L}\big(\Gamma, (\alpha_s)_{s\in[0,T]}\big)\Big|_\zeta \quad \overline{\mathbb{P}}\text{-a.s.},$$

from which we deduce that

$$\begin{aligned}&\mathcal{L}\Big(\overline{\xi}, (a_s(\zeta, \Gamma, \overline{U}_\xi))_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t} - W_t)_{s\in[0,T]}\Big)\Big|_\zeta \\ =&\mathcal{L}\Big(\overline{\xi}, (\alpha_s)_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t} - W_t)_{s\in[0,T]}\Big)\Big|_\zeta \quad \mathbb{P}^0\text{-a.s.}.\end{aligned}$$

Now we define $b : [0,T] \times \Omega^0 \times \Omega^1 \times \mathbb{R}^d \times [0,1] \to A$ by

$$b(s, \omega^0, \omega^1, x, u) := \alpha_s(\omega^0, \omega^1)\mathbb{1}_{[0,t)}(s) + a_s(\omega^0_{\cdot\wedge t}, s, \omega^0, \hat{\omega}^1, x, u)\mathbb{1}_{[t,T]},$$

then $(b_s(\xi, U_\xi))_{s\in[0,T]}$ is $\mathbb{F}^t$-progressively measurable, and

$$\begin{aligned}&\mathcal{L}\Big(\xi, (b_s(\xi,U_\xi))_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta\\ =&\mathcal{L}\Big(\xi, (\alpha_s)_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta \ \ \mathbb{P}^0\text{-}a.s..\end{aligned}$$

□

Lemma 5.3.6. *Assume ($\mathcal{A}1$) holds and let ξ be $\mathcal{F}_t$-measurable. Suppose that*

$$\xi = \sum_{i=1}^n a_i \mathbb{1}_{B_i^0}(\omega^0)\mathbb{1}_{B_i^1}(\omega^1),$$

where $\{B_i^0\times B_i^1\}_{i=1,\dots,n}$ is a (disjoint) partition of $\Omega^0\times\Omega^1$. Then there exists an $\mathcal{F}_t$-measurable random variable U_ξ, for $\mathbb{P}$-a.s. ω, $\mathcal{L}(\theta|\zeta) = U(0,1)$ and $U_\xi \perp\!\!\!\perp_\zeta \xi$, where ζ is defined in (5.3.3).

Proof. Note that ξ can also be written as

$$\xi = \sum_{i=1}^m \Xi_i(\omega^1)\mathbb{1}_{C_i^0}(\omega^0),$$

where Ξ_i is discrete value random variable on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, and $\{C_i^0\}_{i=1,\dots,m}$ is a disjoint partition of Ω^0. For each Ξ_i, apply Lemma B.3 in Cosso *et al.* (2023), we get U_{Ξ_i} of uniform distribution on $[0,1]$ and independent of Ξ_i. Then

$$U_\xi := \sum_{i=1}^m U_{\Xi_i}(\omega^1)\mathbb{1}_{C_i^0}(\omega^0),$$

is $\mathcal{F}_t$ measurable, having uniform distribution on $[0,1]$, and independent of ξ, when given ζ. □

Lemma 5.3.7. *Assume ($\mathcal{A}1$) and ($\mathcal{A}3$) hold. If*

$$\begin{aligned}&\mathcal{L}\Big(\xi, (\alpha_s)_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta\\ &= \mathcal{L}\Big(\eta, (\beta_s)_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}, (W_{s\vee t}-W_t)_{s\in[0,T]}\Big)\Big|_\zeta \ \ \mathbb{P}^0\text{-}a.s.,\end{aligned}$$

then

$$\mathcal{L}\Big((X^{t,\xi,\alpha}_s)_{s\in[t,T]}, (\alpha_s)_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}\Big)\Big|_\zeta = \mathcal{L}\Big((X^{t,\eta,\beta}_s)_{s\in[t,T]}, (\beta_s)_{s\in[0,T]}, (W^0_s)_{s\in[0,T]}\Big)\Big|_\zeta \ \ \mathbb{P}^0\text{-}a.s.$$

Proof. We establish this result by examining the finite-dimensional distributions. We extract a subsequence such that the Picard iteration $\sup_{t\le k\le s}|X_k^{(n),t,\xi,\alpha}-X_k^{t,\xi,\alpha}|\to 0$ $\mathbb{P}$-a.s. and $\sup_{t\le k\le s}|X_k^{(n),t,\eta,\beta}-X_k^{t,\eta,\beta}|$ $\mathbb{P}$-a.s.. By Lemma 5.1.6, for b, σ, there exists b^N, σ^N such that b^N is of the form $b_t^N(W^0_{t_1\wedge t},\dots,W^0_{t_N\wedge t},x,\mu,a)$ (and similarly for σ^N), and $b_t^\varepsilon\to 0$ as $N\to\infty$ (and similarly for σ^ε).

Let h be an arbitrary bounded continuous function (it is without loss of generality by truncation and mollification argument). For $k_1,\dots,k_q\in[t,T]$, $q\in\mathbb{N}$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[h(X_{k_1}^{(1),t,\xi,\alpha},\alpha_{k_1},W^0_{k_1},\dots,X_{k_q}^{(1),t,\xi,\alpha},\alpha_{k_q},W^0_{k_q})|\zeta\Big]\\
=&\mathbb{E}\Big[h\Big(\xi+\int_t^{k_1}b_r(0,\delta_0,\alpha_r)dr+\int_t^{k_1}\sigma_r(0,\delta_0,\alpha_r)dW_r+\int_t^{k_1}\sigma_r^0(0,\delta_0,\alpha_r)dW_r^0,\alpha_{k_1},W^0_{k_1},\dots,\\
&\qquad\xi+\int_t^{k_q}b_r(0,\delta_0,\alpha_r)dr+\int_t^{k_q}\sigma_r(0,\delta_0,\alpha_r)dW_r+\int_t^{k_q}\sigma_r^0(0,\delta_0,\alpha_r)dW_r^0,\alpha_{k_q},W^0_{k_q}\Big)|\zeta\Big]\\
=&\lim_N\mathbb{E}\Big[h\Big(\xi+\int_t^{k_1}b_r^N(0,\delta_0,\alpha_r)dr+\int_t^{k_1}\sigma_r^N(0,\delta_0,\alpha_r)dW_r+\int_t^{k_1}\sigma_r^{0;N}(0,\delta_0,\alpha_r)dW_r^0,\alpha_{k_1},W^0_{k_1},\dots,\\
&\qquad\xi+\int_t^{k_q}b_r^N(0,\delta_0,\alpha_r)dr+\int_t^{k_q}\sigma_r^N(0,\delta_0,\alpha_r)dW_r+\int_t^{k_q}\sigma_r^{0;N}(0,\delta_0,\alpha_r)dW_r^0,\alpha_{k_q},W^0_{k_q}\Big)|\zeta\Big]\\
=&\lim_N\mathbb{E}\Big[h\Big(\eta+\int_t^{k_1}b_r^N(0,\delta_0,\beta_r)dr+\int_t^{k_1}\sigma_r^N(0,\delta_0,\beta_r)dW_r+\int_t^{k_1}\sigma_r^{0;N}(0,\delta_0,\beta_r)dW_r^0,\beta_{k_1},W^0_{k_1},\dots,\\
&\qquad\eta+\int_t^{k_q}b_r^N(0,\delta_0,\beta_r)dr+\int_t^{k_q}\sigma_r^N(0,\delta_0,\beta_r)dW_r+\int_t^{k_q}\sigma_r^{0;N}(0,\delta_0,\beta_r)dW_r^0,\beta_{k_q},W^0_{k_q}\Big)|\zeta\Big]\ \text{(by our assu}\\
=&\mathbb{E}\Big[h\Big(\eta+\int_t^{k_1}b_r(0,\delta_0,\beta_r)dr+\int_t^{k_1}\sigma_r(0,\delta_0,\beta_r)dW_r+\int_t^{k_1}\sigma_r^0(0,\delta_0,\beta_r)dW_r^0,\beta_{k_1},W^0_{k_1},\dots,\\
&\qquad\eta+\int_t^{k_q}b_r(0,\delta_0,\beta_r)dr+\int_t^{k_q}\sigma_r(0,\delta_0,\beta_r)dW_r+\int_t^{k_q}\sigma_r^0(0,\delta_0,\beta_r)dW_r^0,\beta_{k_q},W^0_{k_q}\Big)|\zeta\Big]\\
=&\mathbb{E}\Big[h(X_{k_1}^{(1),t,\eta,\beta},\beta_{k_1},W^0_{k_1},\dots,X_{k_q}^{(1),t,\eta,\beta},\beta_{k_q},W^0_{k_q})|\zeta\Big],
\end{aligned}
$$

hence

$$\mathcal{L}\Big((X_k^{(1),t,\xi,\alpha})_{k\in[t,T]},(\alpha_k)_{k\in[t,T]},(W_k^0)_{k\in[t,T]}\Big)\Big|_\zeta=\mathcal{L}\Big((X_k^{(1),t,\eta,\beta})_{k\in[t,T]},(\beta_k)_{k\in[t,T]},(W_k^0)_{k\in[t,T]}\Big)\Big|_\zeta\quad\mathbb{P}^0\text{-}a.s..$$

Inductively, we have $\forall n\in\mathbb{N}$,

$$\mathcal{L}\Big((X_k^{(n),t,\xi,\alpha})_{k\in[t,T]},(\alpha_k)_{k\in[t,T]},(W_k^0)_{k\in[t,T]}\Big)\Big|_\zeta=\mathcal{L}\Big((X_k^{(n),t,\eta,\beta})_{k\in[t,T]},(\beta_k)_{k\in[t,T]},(W_k^0)_{k\in[t,T]}\Big)\Big|_\zeta\quad\mathbb{P}^0\text{-}a.s..$$

Finally, let h be an arbitrary bounded continuous function,

$$\begin{aligned}
&\mathbb{E}\Big[h(X_{k_1}^{t,\xi,\alpha}, \alpha_{k_1}, W_{k_1}^0, \ldots, X_{k_q}^{t,\xi,\alpha}, \alpha_{k_q}, W_{k_q}^0)|\zeta\Big] \\
=&\lim_n \mathbb{E}\Big[h(X_{k_1}^{(n),t,\xi,\alpha}, \alpha_{k_1}, W_{k_1}^0, \ldots, X_{k_q}^{(n),t,\xi,\alpha}, \alpha_{k_q}, W_{k_q}^0)|\zeta\Big] \\
=&\lim_n \mathbb{E}\Big[h(X_{k_1}^{(n),t,\eta,\beta}, \alpha_{k_1}, W_{k_1}^0, \ldots, X_{k_q}^{(n),t,\eta,\beta}, \alpha_{k_q}, W_{k_q}^0)|\zeta\Big] \\
=&\mathbb{E}\Big[h(X_{k_1}^{t,\eta,\beta}, \alpha_{k_1}, W_{k_1}^0, \ldots, X_{k_q}^{t,\eta,\beta}, \alpha_{k_q}, W_{k_q}^0)|\zeta\Big].
\end{aligned}$$

□

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