THE UNIVERSITY OF CALGARY

ANALYSIS FOR CHAOS AND FRACTALS IN NONLINEAR DYNAMICAL SYSTEMS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

DEPARTMENT OF MECHANICAL ENGINEERING

CALGARY, ALBERTA, CANADA

MARCH, 1990

ISBN 0-315-61714-4

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled,

"Analysis for Chaos and Fractals

in Nonlinear Dynamical Systems"

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ABSTRACT

This thesis is comprised of six Chapters.

In the first four Chapters, methods are developed for the analysis of mathematical models governed by nonlinear ordinary differential equation systems and initial conditions.

A general model has been outlined to analyze the chaotic behavior of nonlinear dynamical systems. The Bendixon's criterion is used to check for the possible occurrence of limit cycles. The Routh - Hurwitz criteria is used to establish the conditions for the stability of the system. The Poincare criteria is applied to determine and classify the types of singularities present in the systems.

Chapters 5 and 6 deal with strange attractors, their singularities, Lyapunov exponents, and fractal dimensions.

The behavior of trajectories is discussed for the Lorenz model, the Duffing model, the Rossler model, and the Reactor model, using methods developed in Chapter 3, for various values of parameters and initial conditions. The techniques of continuous system simulation computer languages e.g. CSSL IV

and ACSL, are employed and the numerical experiments are depicted by 2-D graphs.

Fractal dimension is a measure of the minimum number of degrees of freedom or independent variables required, to describe the dynamical behavior of a nonlinear dynamical system. Strange attractors are characterized by their fractal dimensions. Two new curves are generated and their fractal dimension is determined. Lyapunov dimension for the Lorenz model, and the Rossler model are determined for different values of the parameters.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr. K.L.Chowdhury for his valuable support and encouragement in supervising this project. His enthusiasm and dedication contributed significantly to this work.

Special thanks go to Dr. R.D.Rowe, who provided support and encouragement when they were needed the most.

Thanks also go to the administrative staff of the Department of Mechanical Engineering, especially Mrs. Maria Berry, who were more than willing to offer their assistance.

A special appreciation goes to Alka, who provided support and encouragement at all the times, and to my daughter Ambika who is waiting patiently for me finish studies and spend lots of time with her.

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LIST OF SYMBOLS

Roman Symbols

a	aspect ratio in section 5.1
a	equation parameter in section 5.2
c_p	specific heat
C	concentration
C_o	initial concentration
C_s	steady state concentration
E	activation energy
g	acceleration due to gravity
ΔH	heat of reaction
k	equation parameter
p	reaction velocity constant
R	gas constant in section 5.4
R	Raleigh number
R_c	critical Raleigh number
t	time
T	temperature
T_c	coolant temperature
T_o	initial temperature
ΔT	temperature difference
u	velocity in x-direction
U	heat transfer coefficient
v	velocity in y-direction
V	volume of reactor
w	velocity in z-direction

Greek Symbols

α	coefficient of volume expansion in section 5.1
α	equation parameter
β	equation parameter
ρ	density
ν	kinematic viscosity
ψ	stream function
σ	equation parameter
τ,ζ	time
λ	eigen value

Superscripts

$\dot{}$	first derivative with respect to time
$\ddot{}$	second derivative with respect to time

Abbreviations

ACSL	Advanced Continuous Simulation Language
CSSL IV	Continuous System Simulation Language IV
CSTR	Continuous Stirred Tank Reactor

CHAPTER 1

1.1 INTRODUCTION

Physical phenomena occurring in various research fields e.g. fluid mechanics, chemical reactors, electric oscillators, vibrating structures, etc., are governed by nonlinear ordinary differential equations. Analysis of these dynamical systems by various analytical, numerical, and graphical methods, shows that they exhibit different types of complex behavior. Researchers have called such phenomena of randomness and unpredictability as "Deterministic Chaos", "Dynamical Stochasticity", or "Hamiltonian Stochasticity". The most widely accepted term now is "Chaos".

Chaos is nothing but paradoxical combination of randomness and structure i.e. chaos is such a randomness which can be predetermined. Though there is no general definition of chaos, but it exists if the system variables do not show any repetition, when plotted against the independent variable. It has become extremely important to study the order and chaos of a physical system which is governed by a system of nonlinear differential equations. Chaos occurs more frequently than order. It is not unusual to find that chaos is a basic mode of motion in most of the natural phenomena. In many physical systems, chaos emerges when a particular set of parameters or initial condition change and complexity in behavior increases as the dimension of the reference space is increased.

Lorenz (1963) showed how a set of three simple equations with quadratic nonlinearity lead to chaotic behavior. He used this model to explain why it is not possible to forecast weather for a longer period of time. Rossler (1979), Gleick (1987), and Thompson and Stewart (1986) have studied these phenomena, but still there seems to be no general method which can solve these nonlinear dynamical systems, and analyze their behavior. The availability of simulation languages and faster computers have helped to analyze such behavior.

An attractor is a point where the trajectories converge or diverge. The attractors are classified as node, focus, saddle point, and center, depending upon how the trajectory in the neighborhood of the attractor behaves. An attractor, which is sensitive to initial conditions and normally in dimensions more than two, has been called a strange attractor by Ruelle. Strange attractors are characterized by their dimension called Hausdorff's dimension or simply the fractal dimension, which is a measure of the minimum number of degrees of freedom needed to describe the dynamic behavior of a nonlinear dynamical system.

The objective of this thesis is to develop a model for analyzing nonlinear dynamical systems represented by ordinary differential equations for chaos and fractals. The available methods based on approximations are time consuming. The general analytical model developed here gives preliminary information about the singular points, their nature, checks for the existence of limit cycles, and establishes the conditions for stability. This information is used by the numerical method to study the behavior of representative points near the singular point and also at larger distances from the singular point. The trajectories for the Lorenz model have been drawn near and far away from the singular points, and the effect of change in a system parameter while keeping all of the others unchanged has been shown. It is shown, how to use the information obtained by analytical models, and completely new trajectories are then drawn. The analytical models for the Duffing equation shows the existence of limit cycles, which have been supplemented by the plots developed by the numerical model. Similarly, a model for first order reaction in a continuously stirred tank reactor has been determined and the singular points are obtained graphically.

Fractals are irregular sets of noninteger dimensions. Various methods to determine the fractal dimension for curves are discussed. For the Lorenz model and the Rossler model the fractal dimension has been determined for different parameter values.

A summary of the contents of this thesis chapter by chapter is provided below:

In Chapter 2, a brief introduction to three approximate methods is given. The methods are the perturbation method, the averaging method, and the iteration method. The methods are useful only for small nonlinearities, as they are based on certain approximations. In the second half of the chapter topological methods known as the isocline method, the Lienard method, and the delta method are discussed. The methods are time consuming, but provide greater insight into how a representative trajectory behaves in a certain region.

In Chapter 3, a general method to analyze a nonlinear dynamical system is developed. The method consists of six steps and uses the Poincare criteria to determine the nature of singular points, the Bendixon's criteria to check for the existence of limit cycles, and the Routh-Hurwitz criteria to establish conditions of stability.

Chapter 4 is concerned with numerical methods and two simulation languages: Advanced Continuous System Language, and Continuous System Simulation Language IV. Algorithms have been discussed and a sample program with output for the Lorenz model has been constructed in appendix A.

Chapter 5 deals with the analysis of the Lorenz model, the Duffing model, the Rossler model, and the Chemical reactor model, for their chaotic behavior and conditions of stability along the lines of the general model outlined in Chapter 3. All of the results in the form of analytical solutions together with numerical methods, provide a greater insight about the behavior of each nonlinear dynamical system. The dependence on initial conditions, parameters, and nonlinearities has been discussed in detail and shown on 2D graphs.

Chapter 6 deals with fractals and their dimensions. The method to determine the Hausdorff's dimension for curves is discussed. Two curves are generated and their fractal dimensions are determined. The Lyapunov dimension for the Lorenz model and the Rossler model have been determined for different parameter values.

At the end, conclusions drawn from this work are listed and recommendations for further studies are made.

CHAPTER 2

METHODS FOR THE ANALYSIS OF DYNAMICAL SYSTEMS

A dynamical system can be analyzed by one of the following three methods:

1. Approximate Method
2. Topological Method
3. Numerical Method

A general classification of nonlinear dynamical systems is shown in Figure 2.1. Before the methods are introduced, we explain the different kinds of systems dealt with in the present work.

2.1 SYSTEM

A system is a collection of physical components which experience mutual actions and reactions under the influence of their own behavior and of the external disturbances acting on the system as a whole. The state of the system at a given point in time can fully be explained by all the internal and external variables.

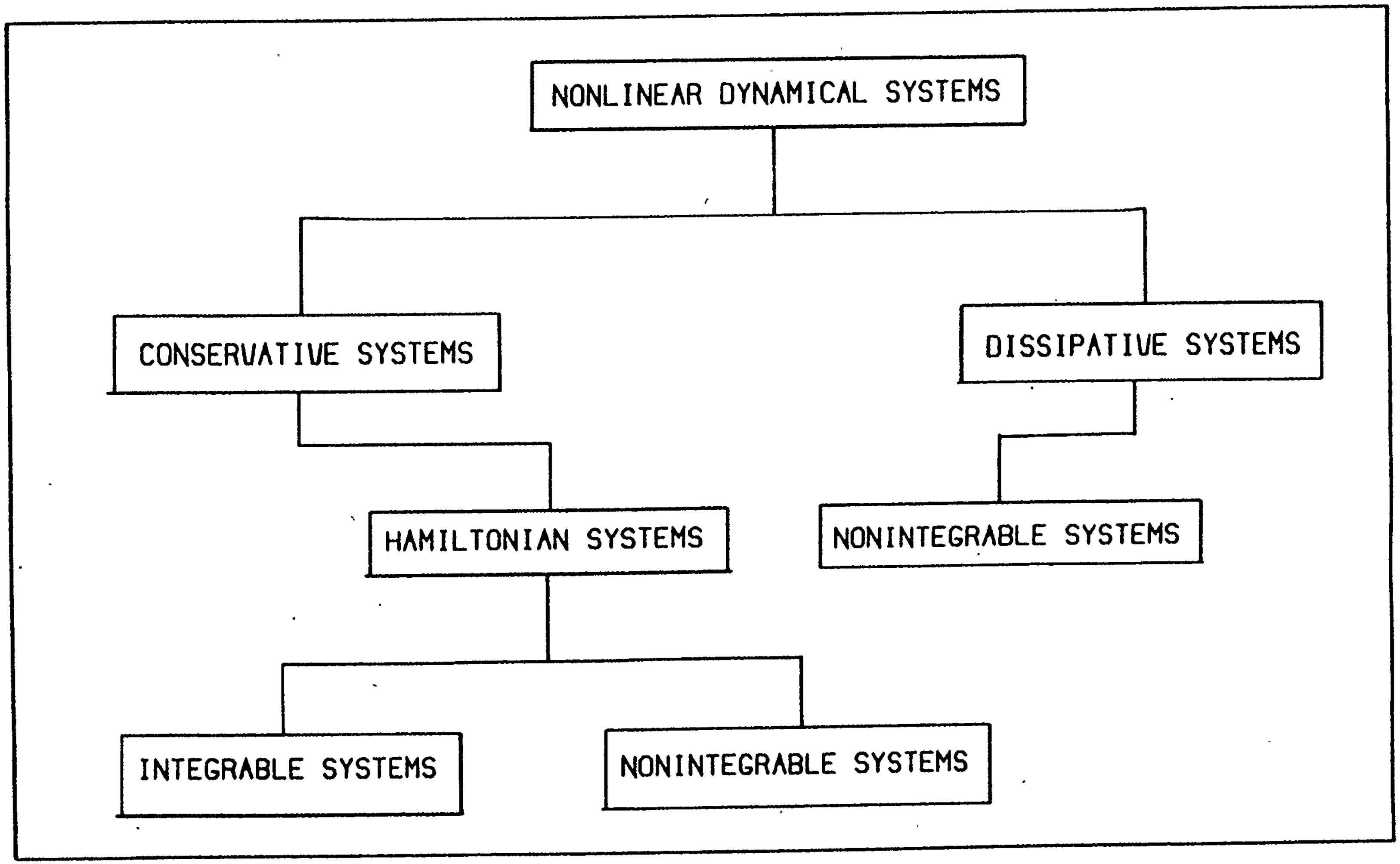


Fig 2.1 : Nonlinear Dynamical Systems

In the present work we will be limiting our study to the dynamical systems which are governed by a set of nonlinear ordinary differential equations only. Nonlinear systems are the ones which exhibit properties different from those of a linear system i.e. their behavior may not be proportional to the cause and effect. When several external forces are applied simultaneously to a nonlinear system the total effect will be different compared to the one which results in the linear case from the superposition of the individual effects.

Nonlinearities can be of two types:

1. Fundamental Nonlinearities
2. Accidental Nonlinearities

Fundamental nonlinearities are those which are knowingly introduced in the system to change its behavior as desired, where as accidental nonlinearities are the imperfections of a system with respect to its linear model and they are not intentionally introduced.

Nonlinearities can also be classified as time-dependent or time-independent. Time-dependent nonlinearities are the ones whose effects change with time.

The prime objective of analyzing the chaotic nature of a nonlinear dynamical system is to understand the behavior of the system completely from order to chaos. It helps to analyze further the stability of a nonlinear dynamical system.

The stability of a nonlinear system depends on the applied disturbances or perturbations. Often nonlinear systems exhibit self sustained oscillations, called limit oscillations or more commonly limit cycles characterized by well defined frequencies and amplitudes.

The existence of a stable limit cycle does not critically depend on the values of the system parameters. It is possible that there may exist a number of limit cycles which may be generated by disturbance of various amplitudes. The behavior of a nonlinear system and in particular stability, essentially depends on the applied initial conditions, and system parameters.

2.1.1 AUTONOMOUS SYSTEMS

Any system where time does not appear explicitly is called an autonomous system.

A system defined by a set of n-first order differential equations (2.1.1).

$$\begin{aligned}
\dot{x}_1 &= X_1(x_1,x_2,........,x_n,t) \\
\dot{x}_2 &= X_2(x_1,x_2,........,x_n,t) \\
\\
\dot{x}_n &= X_n(x_1,x_2,........,x_n,t)
\end{aligned} \tag{2.1.1}$$

where dot denotes derivative with respect to time.

In general a system can be represented in vector form as:

$$\dot{\mathbf{x}} = \bar{X}(\bar{\mathbf{x}},t) \tag{2.1.2}$$

where bar over the letter represents a vector and

$$\bar{\mathbf{x}} = (x_1,x_2,\ldots\ldots\ldots,x_n)$$

If vector functions $\bar{X}$ depend upon vector $\bar{\mathbf{x}}$ alone and not on time t, then equation (2.1.2) reduces to

$$\dot{\mathbf{x}} = \bar{X}(\bar{\mathbf{x}})$$

and the system is called autonomous. Following are examples of a two variable autonomous systems.

$$\begin{aligned} \dot{x}_1 &= 4x_1^3 + 3x_1 + 9x_2 \\ \dot{x}_2 &= e^{3x_2} + 4x_2^2 - 6x_1 \end{aligned} \tag{2.1.3}$$

Vander Pol's equation is a one variable autonomous system defined by

$$\frac{d^2x}{dt^2} - \mu(1-x)\frac{dx}{dt} + x = 0 \tag{2.1.4}$$

Lorenz equations form a three variable autonomous system and is given by

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= rx-yz-xz \\ \frac{dz}{dt} &= xy-bz \end{aligned} \tag{2.1.5}$$

2.1.2 NONAUTONOMOUS SYSTEMS

For such a system the forcing function can be a function of time with or without being a function of dependent variable and/or its derivative with respect to time e.g

$$\ddot{x} = f(x, t, \frac{dx}{dt}) \tag{2.1.6}$$

Duffing's equation is an example of this class of systems:

$$\frac{d^2x}{dt^2} + x = \mu(-\alpha x - \beta x^3 - F cost) \tag{2.1.7}$$

2.2 APPROXIMATE METHODS

If a system can be solved analytically, it is the best possible method to obtain a detailed information about a dynamical system. In this method, the analytical solutions are obtained in algebaric form without introducing the numerical values for the parameters and initial conditions involved. In general, there are no methods which can lead to exact solutions of a nonlinear dynamical system. Essentially for nonlinear equations, analytical methods are based on successive approximations, which minimize the deviation from the nonlinear behavior.

2.2.1 PERTURBATION METHOD

The method described by Hayashi (1964) is applicable to all those physical systems which have small nonlinearity. In this method, the desired quantities are expressed in powers of the small parameter multiplied by the coefficients which are functions of the independent variable and then coefficients are determined one by one usually by solving a sequence of linear equations.

Consider a dynamical system described by n-differential equations of first order:

$$\frac{d\overline{\mathbf{x}}}{dt} = \overline{X}(x,t) \tag{2.2.1}$$

Poincare' studied the systems of differential equations in which nonlinear terms were associated with a small parameter μ and in such cases function X(x,t) can be divided in two parts L(x,t) the linear part and N(x,t) the nonlinear part. Hence, Equation (2.2.1) can be written as:

$$\frac{d\overline{\mathbf{x}}}{dt} = L(x,t) + \mu N(x,t) \tag{2.2.2}$$

The solution of Equation (2.2.2) can be found in the following form provided μ is small and the series converges.

$$x = x^0(t) + \mu x^{(1)}(t) + \mu^2 x^{(2)}(t) + ... \tag{2.2.3}$$

For $\mu > 0$, let

$$x = x^0(t)$$

which is the exact solution for the linear part. The other approximations are constructed by comparing the various powers of μ. Consider a nonlinear differential equation:

$$\frac{d^2x}{dt^2} + x + \mu x^3 = 0 \qquad (2.2.4)$$

Let the initial conditions be

$$x(0) = A, \quad \frac{dx(0)}{dt} = 0 \qquad (2.2.4a)$$

Substituting the power series solution

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots \qquad (2.2.5)$$

in the differential equation (2.2.4), we obtain a power series in μ such that the coefficients of the successive powers of μ must vanish separately.

$$\ddot{x}_0 + x_0 + \mu(\ddot{x}_1 + x_1 + x_0^3) + \mu^2(x_1^3 + 3x_0 x_1^2 + 3x_0^2 x_1) = 0 \qquad (2.2.6)$$

Neglecting squares and higher powers of x_1, we get

$$(\ddot{x}_0 + x_0) + \mu(\ddot{x}_1 + x_1 + x_0^3) = 0 \qquad (2.2.7)$$

Equating the coefficients of μ to zero separately we get

$$\ddot{x}_0 + x_0 = 0 \qquad (2.2.8a)$$

$$\ddot{x}_1 + x_1 + x_0^3 = 0 \qquad (2.2.8b)$$

The solution of equation (2.2.8a) with initial conditions (2.2.4a) is

$$x_0 = A\cos t \tag{2.2.9a}$$

Substituting the value of x_0 in equation (2.2.8b) we get

$$\ddot{x}_1 + x_1 = -A^3\cos^3 t \tag{2.2.9b}$$

and we finally arrive at

$$x_1 = \frac{-3}{8}A^3 t\sin t - \frac{1}{32}A^3(\, cost - cos\,3t\,) \tag{2.2.10}$$

$$x = Acost + \mu(\frac{-3}{8}A^3 tsint - \frac{1}{32}A^3(cost - cos\,3t)) \tag{2.2.11}$$

The first term in the solution contains time t outside the trigonometric function, and is called the secular term. The solution x_0=$Acost$ is periodic with a period 2π but since μ is not equal to zero, the solution will not be periodic in 2π.

2.2.2 ITERATION METHOD

This method is based on solving a nonlinear differential equation by the process of successive iteration. The method solves the equation in the following three steps:

* Solve the equation with some terms neglected.
* Substitute the resulting solution in the equation and include the neglected terms.
* Repeat for improved accuracy.

Consider the solution of the of Duffing's equation

$$\ddot{x} + x = \mu(-\alpha x - \beta x^3 + F\cos\zeta) \tag{2.2.12}$$

where μ is a small nondimensional parameter; α, β, F are constants and dot denotes derivative with respect to ζ.

As a first step, let the solution be

$$x_0 = A_0\cos\zeta \tag{2.2.13}$$

An initial approximation to the solution is obtained by neglecting the right hand side terms of the Duffing's equation and the difference between x_0 and the exact solution x should be of the first order in μ. Substituting equation (2.2.13) in equation (2.2.12) we get

$$\ddot{x}_1 + x_1 = \mu(-\alpha A_0\cos\zeta - \beta A_0^3\cos^3\zeta + F\cos\zeta) \tag{2.2.14}$$

Finally we arrive at

$$\ddot{x}_1 + x_1 = \mu(-\alpha A_0 - \frac{3}{4}\beta A_0^3 + F)\cos\zeta - \frac{1}{4}\beta A_0^3 cos\,3\zeta \tag{2.2.15}$$

Since the above equation differs from the Duffing's equation by a term of order μ^2, we can find a second approximation correct to the first order in μ. The periodicity condition for x_1 requires that no secular terms should appear in the solution x_1. Therefore

$$\alpha A_0 + \frac{3}{4}\beta A_0^3 - F = 0 \tag{2.3.16}$$

Thus the amplitude A_0 is fixed. The equation (2.2.15) is solved for x_1 and we obtain

$$x_1 = A_1\cos\zeta + \frac{1}{36}\mu\beta A_0^3 cos\,3\zeta \qquad (2.3.17)$$

Now the amplitude A_1 differs from A_0 by a term of the first order in μ. Substituting again in Duffing's equation (2.2.12) we get

$$\ddot{x}_2 + x_2 = \mu(-\alpha A_1 - \frac{3}{4}\beta A_1^3 - \frac{3}{128}\mu\beta^2 A_0^3 A_1^2 + F)cos\,3\zeta - \frac{1}{4}\mu\beta(A^{3_1} +$$

$$\frac{1}{16}\mu(2\alpha + 3\beta A_1^2)A_0^3)cos\,3\zeta - \frac{3}{128}\mu^2\beta^2 A_0^3 A_1^2 cos\,5\zeta \qquad (2.2.18)$$

neglecting terms of the order higher than μ^2.

Finally the periodicity condition requires that secular term should be zero

$$\alpha A_1 + \frac{3}{4}\beta A_1^3 + \frac{3}{128}\mu\beta^2 A_0^3 A_1^2 - F = 0 \qquad (2.2.19)$$

Solving for A_1

$$A_1 = A_0 - \frac{3\mu\beta^2 A_0^5}{128(\alpha + \frac{3}{4}\beta A_0^2)} \qquad (2.2.20)$$

Finally we obtain

$$x_2 = \left[A_0 - \frac{3\mu\beta^2 A_0^5}{128(\alpha + \frac{3}{4}\beta A_0^2)}\right]\cos\zeta + \frac{1}{32}\mu\beta A_0^2 cos\,3\zeta \qquad (2.2.21)$$

which is the resulting solution accurate to the first order in μ. This process can be continued and final solution can be obtained to a desired order in μ.

2.2.3 AVERAGING METHOD

This method which is applicable to the study of periodic oscillations also explains the study of transient oscillations where the amplitude and the phase of the oscillations vary slowly with time.

To explain the method consider an autonomous system defined by a set of two first order differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y}+x &= \mu f(x,y) \end{aligned} \qquad (2.2.22)$$

where μ is a small nondimensional parameter and f(x,y) is a polynomial.

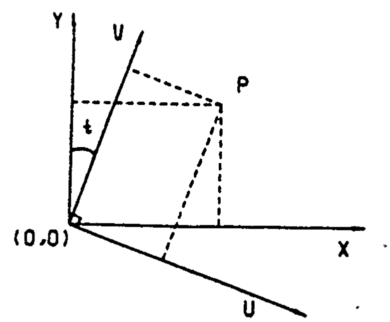
The system can be well defined by the motion of any point (x,y), which will form trajectories of the differential equations.

If $\mu = 0$ i.e. the set of differential equations are linear, the system becomes a linear oscillator with circles as trajectories and center at origin. The representative point rotates along the circle in the clockwise direction with unit angular velocity.

Consider a new coordinate system (u,v) in the phase plane which rotates together with the representative point with unit angular velocity.

Now for $\mu \neq 0$ but small as assumed, the coordinates of a point (u,v) will vary slowly with time t.

The relation between two coordinate systems xy and uv are given by



$$x = u\cos t + v\sin t \tag{2.3.23}$$

$$y = v\cos t - u\sin t \tag{2.3.24}$$

Substituting into the original set of differential equations (2.3.22) we get

$$\dot{u}\cos t + \dot{v}\sin t = 0 \tag{2.3.25}$$

$$-\dot{u}\sin t + \dot{v}\cos t = \mu f\,(u\cos t + v\sin t, v\cos t - u\sin t) \tag{2.3.26}$$

Solving the above two equations, we get

$$\dot{u} = -\mu f\left\{u\cos t + v\sin t, -u\sin t + v\cos t\right\}\sin t \tag{2.3.27}$$

$$\dot{v} = \mu f\left\{u\cos t + v\sin t, -u\sin t + v\cos t\right\}\cos t \tag{2.3.28}$$

Expanding equations (2.3.27) and (2.3.28) into Fourier series:

$$\dot{u} = \mu\left[\frac{U_0}{2} + U_1\cos t + U_1\sin t + U_2\cos 2t + U_2\sin 2t + \ldots\ldots\right] \tag{2.3.29}$$

$$\dot{v} = \mu\left[\frac{V_0}{2} + V_1\cos t + V_1\sin t + V_2\cos 2t + V_2\sin 2t + \ldots\ldots\right] \tag{2.3.30}$$

where U's and V's are Fourier functions defined as:

$$U_n = \frac{1}{\pi}\int_{o}^{\pi} f(x) cosnxdx$$

$$V_n = -\int_{0}^{\pi} f(x) cosnxdx$$

2.3 TOPOLOGICAL METHODS

These are essentially graphical methods. The behavior of any representative point on an integral or solution curve in phase space is studied with respect to singular points e.g. node, focus, or saddle point. The existence of periodic cycles called the limit cycles is also determined. The methods are applicable to differential equations with small nonlinearity. The topological methods have an advantage over approximate methods in the sense that the solution is in terms of integral curves in phase space and provides greater insight into any representative point on solution curves. The steady state solution is achieved when the representative point settles down to a singular point or on a limit cycle. The criteria established by Poincare is explained by Hayashi (1964). The main topological methods are:

1. Isocline Method
2. Lienard Method
3. Delta Method

The methods are time consuming as the entire plane has to be filled with different integral curves, as required in the first two methods. Delta method is applied to a single integral curve. With enhanced capabilities of computers and limitations of topological methods, numerical methods provide a better solution. A brief introduction of these methods is as follows.

2.3.1 ISOCLINE METHOD

Consider a system defined by two nonlinear ordinary differential equations

$$\frac{dx}{dt} = X(x,y)$$

$$\frac{dy}{dt} = Y(x,y) \tag{2.3.1}$$

Eliminating the independent variable, the integral curves can be drawn in xy plane whose slope will be

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)} \tag{2.3.2}$$

Let $F(x,y) = \dfrac{Y(x,y)}{X(x,y)} = k$

where k is a constant.

For different values of k i.e. different slopes, the integral curves are drawn which are called isoclines. The procedure is repeated to fill the whole xy plane under observation. Then the solution curve is determined in the direction of tangent determined from these curves.

Consider a second order ordinary nonlinear differential equation.

$$\frac{d^2x}{dt^2} - (1 - x^2)\frac{dx}{dt} + x = 0 \tag{2.3.3}$$

This equation may be written as system of two nonlinear ordinary differential equations as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (1 - x^2)y - x \end{aligned} \tag{2.3.4}$$

The slope is

$$\frac{dy}{dx} = \frac{(1-x^2)y-x}{y} = k \tag{2.3.5}$$

For different values of k's, integral curves are drawn to fill the phase space under observation. The existence of a limit cycle is established which is stable, as will be discussed in Chapter 3.

2.3.2 LIENARD METHOD

The method due to Lienard draws solution curves for second order differential equations. Consider the Duffing equation for which the forcing function is zero, damping term is nonlinear, and the restoring force is linear.

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}\right) + x = 0 \tag{2.3.6}$$

which may be written as

$$\dot{x} = y$$
$$\dot{y} = -f(y) - x \tag{2.3.7}$$

The slope will is found by isocline method:

$$\frac{dy}{dx} = \frac{-f(y) - x}{y} \tag{2.3.8}$$

The solution curve is plotted by the following steps:

1. Plot a curve for $\frac{dy}{dx} = 0$ *i.e.* $x = -f(y)$

2. To determine the direction of any point P(x,y), draw a line parallel to x-axis until it meets the curve x = -f(y) at u, and then draw a line perpendicular to x-axis until it meets the x-axis at V.

3. Join PV, the direction of solution curve at P will be a line perpendicular to line PV as shown in Figure 2.2.

2.3.3 DELTA METHOD

This method is used only if one solution curve is needed and is applicable to differential equations of the form:

$$\frac{d^2x}{dt^2} + f(\frac{dx}{dt}, x, t) = 0 \tag{2.3.9}$$

where $f(\frac{dx}{dt}, x, t)$ is a single valued continuous function.

Adding and subtracting $\omega^2 x$ and substituting $\frac{dx}{dt}$ = y, and $\tau = \omega t$ in equation (2.3.9). We obtain

$$\frac{dy}{dx} = \frac{-x - \delta(y, x, \tau)}{y} \tag{2.3.10}$$

where

$$\delta(y, x, \tau) = \frac{1}{\omega^2} f(\omega y, x, \frac{\tau}{\omega}) - x \tag{2.3.11}$$

Assuming $f(\omega y, x, \frac{\tau}{\omega})$ to be constant for small changes, equation (2.3.10) can be integrated to give

$$(x + \delta)^2 + y^2 = r^2 \tag{2.3.12}$$

which is an equation of a circle with radius r and origin at $(-\delta, 0)$.

The solution curve can then be plotted by taking an initial point P(x,y) on xy plane. Locate a point $Q(-\delta, 0)$ from equation (2.3.9). Then from Q draw an arc with radius equal to PQ to find a new point R, as $(\theta \rightarrow 0)$ and shown in Figure 2.3. The above step is repeated to obtain the solution curve.

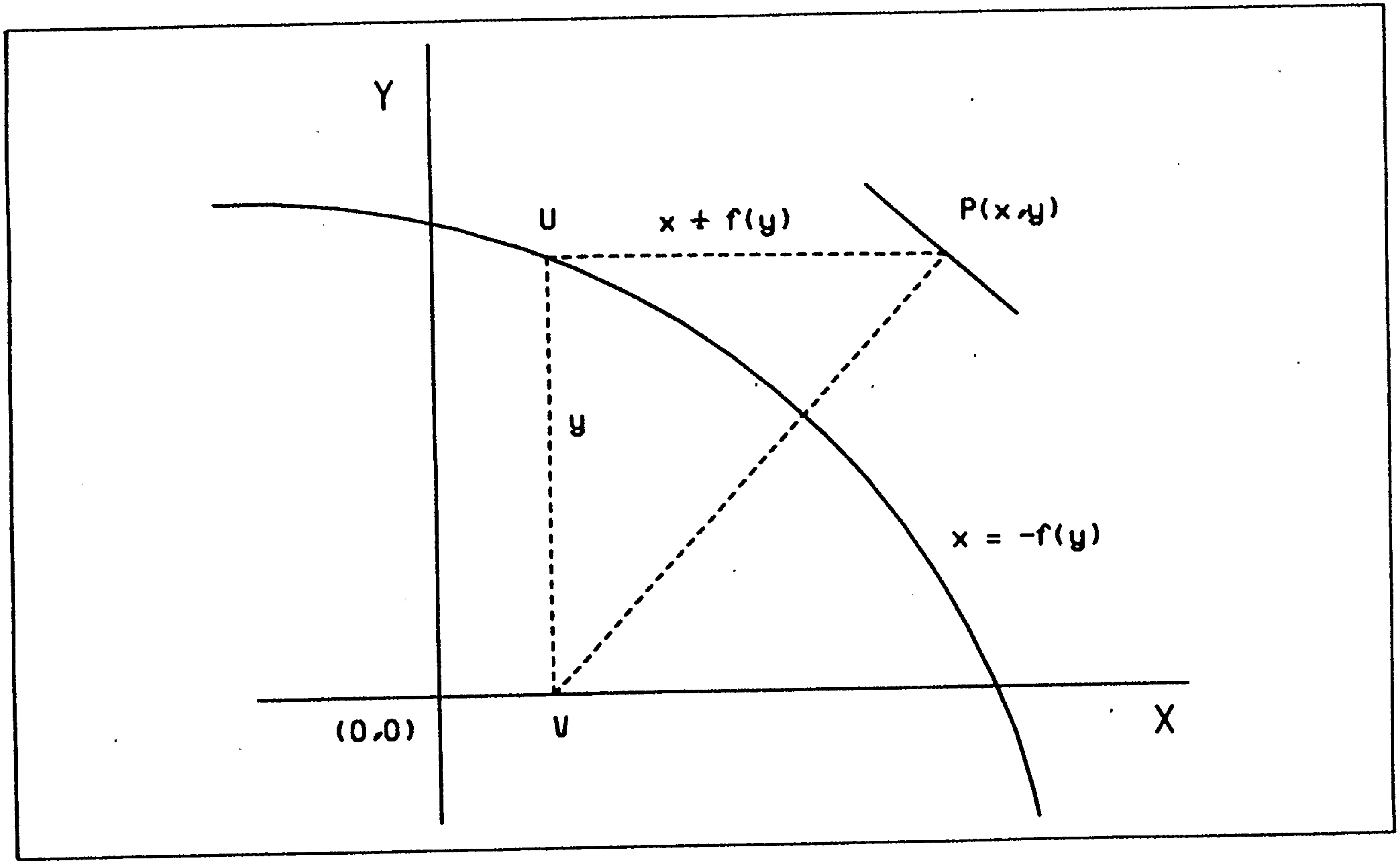


Fig 2.2 : Lienard Method

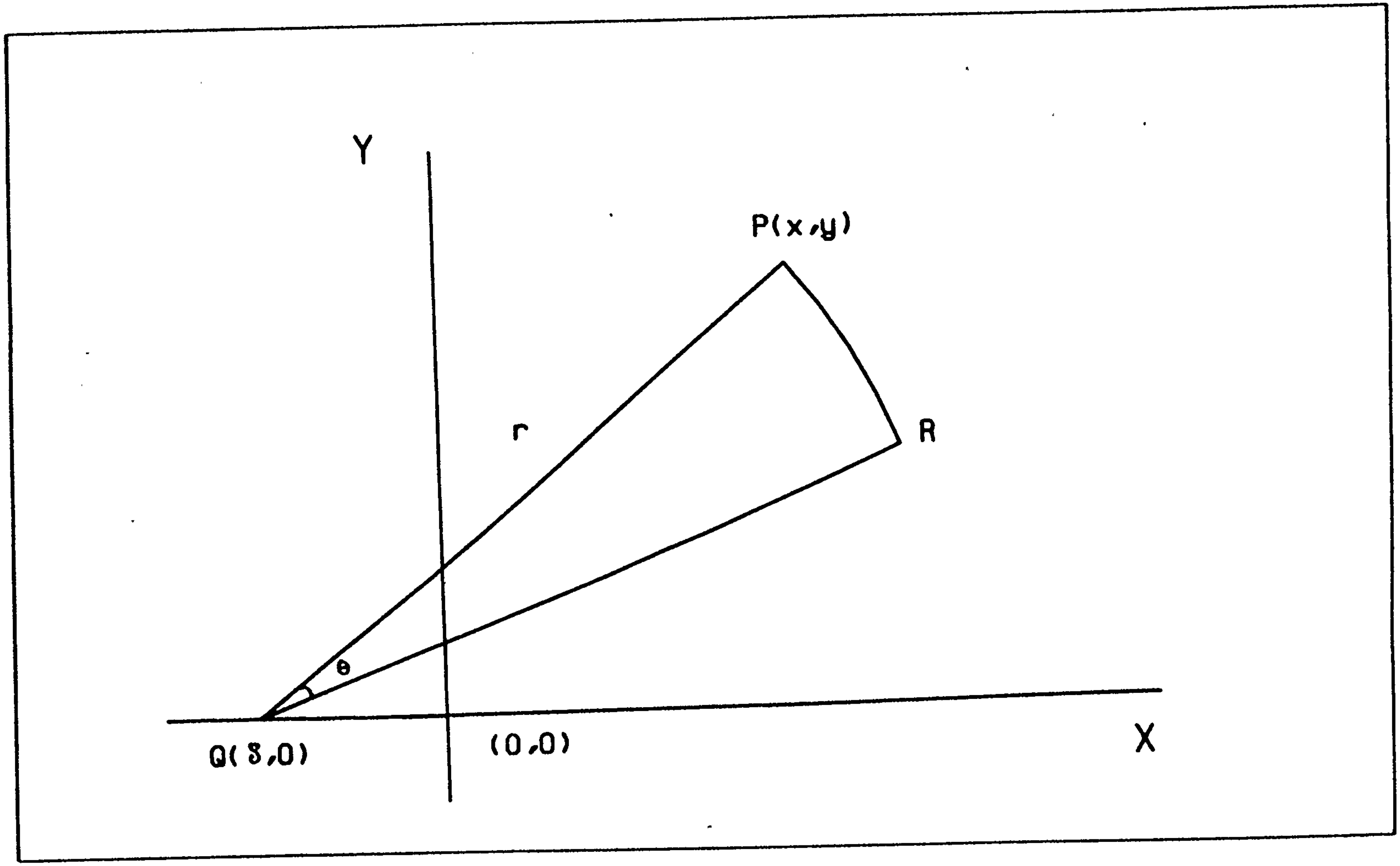


Fig 2.3 : Delta Method

CHAPTER 3

GENERAL ANALYTICAL MODEL

A general model to analyze the behavior of a nonlinear dynamical system with various initial conditions has been developed. It is necessary to check any nonlinear dynamical system for all possible initial conditions and possible parameter variations before it is actually put to practice.

3.1 METHOD

The method is written in six steps. It is essential to mention here at this point that all the steps may not be applicable to all the systems but the analysis will give a general idea about the behavior of the nonlinear dynamical system. Since the conceptual clarity and physical insight about the behavior of a nonlinear dynamical system is aimed at, only low order systems with small nonlinearities have been considered. This information coupled with that obtained by numerical method will give better understanding about the behavior of the system.

Step 1

Check the type of the system whether it is autonomous or nonautonomous system. Autonomous system is the one in which the forcing function is only a function of dependent variable and its derivatives with respect to time. For example

$$\frac{d^2x}{dt^2} + x = \mu f\left(x, \frac{dx}{dt}\right) \qquad (3.1.1)$$

While in nonautonomous systems forcing function is a function of dependent variable, its derivative with respect to time, and time. The system in equation (3.1.1) can be made nonautonomous as

$$\frac{d^2x}{dt^2} + x = \mu f\left(x, \frac{dx}{dt}\right) + g\left(x, t, \frac{dx}{dt}\right) \qquad (3.1.2)$$

where g is an arbitrary function.

Step 2

If the system is autonomous, we write it in the form

$$\frac{dx}{dt} = X(x,y) \qquad (3.2.1)$$

$$\frac{dy}{dt} = Y(x,y) \qquad (3.2.2)$$

Letting $X(x,y)=0$ and

$$Y(x,y)=0 \qquad (3.2.3)$$

we get the singular points (x,y).

Since at the singular point both $(\dot{x},\dot{y}) = (0,0)$. It represents an equilibrium point. All other points are called ordinary points. A singular point can be classified as a Node, a Saddle Point, or a Focus.

For a system of the nonautonomous form

$$\begin{aligned} \frac{dx}{dt} &= X(x,y,t) \\ \frac{dy}{dt} &= Y(x,y,t) \end{aligned} \tag{3.2.4}$$

we increase the dimension of the system by one, by putting

$$t = z, \quad ie \quad \frac{dz}{dt} = 1 \tag{3.2.5}$$

and the system now becomes

$$\begin{aligned} \frac{dx}{dt} &= X(x,y,z) \\ \frac{dy}{dt} &= Y(x,y,z) \\ \frac{dz}{dt} &= 1 \end{aligned} \tag{3.2.5}$$

and find the singular point (x_0,y_0,z_0) by putting X=Y=0, and Z=1.

Step 3

Write the Taylor series for equations (3.2.1) and (3.2.2) about the singular point (x_0,y_0):

$$\frac{dx}{dt} = X(x_0,y_0) + \left[(x-x_0)(X_x)_{(x_0,y_0)} + (y-y_0)(X_y)_{(x_0,y_0)} + \dots\dots\dots\right] \quad (3.3.1)$$

$$\frac{dy}{dt} = Y(x_0,y_0) + \left[(x-x_0)(Y_x)_{(x_0,y_0)} + (y-y_0)(Y_y)_{(x_0,y_0)} + \dots\dots\dots\right] \quad (3.3.2)$$

where

$$\begin{aligned}
\left(X_x\right)_{(x_0,y_0)} &= \left[\frac{\partial X}{\partial x}\right]_{(x_0,y_0)} \\
\left(X_y\right)_{(x_0,y_0)} &= \left[\frac{\partial X}{\partial y}\right]_{(x_0,y_0)} \\
\left(Y_x\right)_{(x_0,y_0)} &= \left[\frac{\partial Y}{\partial x}\right]_{(x_0,y_0)} \\
\left(Y_y\right)_{(x_0,y_0)} &= \left[\frac{\partial Y}{\partial y}\right]_{(x_0,y_0)}
\end{aligned} \quad (3.3.3)$$

where we have neglected the higher order differential terms.

Let a representative point (x,y) be very close to (x_0,y_0), the singular point. By transformation of coordinates, equations (3.3.2) reduce to

$$\begin{aligned}
\dot{x} &= X_x x + X_y y \\
\dot{y} &= Y_x x + Y_y y
\end{aligned} \quad (3.3.4)$$

Let

$$x = Ae^{\lambda t} \quad \text{and} \quad y = Be^{\lambda t} \quad (3.3.5)$$

then equations (3.3.4) become

$$\left(X_x-\lambda\right)A + X_y B = 0$$

$$Y_x A + \left[Y_y - \lambda \right] B = 0 \tag{3.3.6}$$

For the system of Equations (3.3.6) to have a non-trivial solution, the determinant of coefficients

$$\det \begin{bmatrix} X_x - \lambda & X_y \\ Y_x & Y_y - \lambda \end{bmatrix}_{(x_i, y_i)} = 0 \tag{3.3.7}$$

thus each singular point will give two eigen values. The eigen values are used to determine the nature of the singular points. Letting $X_x = a_1$, $X_y = a_2$, $Y_x = b_1$, $Y_y = b_2$. Equation (3.3.7) will become

$$\lambda^2 - (a_1 + b_2)\lambda - a_2 b_1 + a_1 b_2 = 0 \tag{3.3.8}$$

and the roots λ are given by

$$\lambda_{1,2} = \frac{(a_1 + b_2) \pm \sqrt{(a_1 - b_2)^2 + 4a_2 b_1}}{2} \tag{3.3.9}$$

Poincare classified the singular points according to the mapping of the integral curves near the singular points, which depend on the nature of the characteristic roots i.e. λ's. The different possibilities are:

(a) Singular point is called a node (fig 3.1a) if the characteristic roots are both real and of the same sign i.e.

$$(a_1 - b_2)^2 + 4a_2 b_1 \geq 0 \quad \text{and}$$
$$(a_1 b_2 - a_2 b_1) > 0 \tag{3.3.10}$$

(b) Singular point is a saddle point(fig 3.1b) if the roots are real and of opposite sign i.e.

$$(a_1 - b_2)^2 + 4a_2b_1 \geq 0 \quad \text{and}$$

$$(a_1b_2 - a_2b_1) < 0 \tag{3.3.11}$$

(c) Singular point is called focus (figure 3.1c) if the roots are complex conjugates i.e.

$$(a_1 - b_2)^2 + 4a_2b_1 < 0 \tag{3.3.12}$$

(d) Singular point is either a center or a focus(fig 3.1d) if the two roots are imaginary i.e.

$$(a_1 - b_2)^2 + 4a_2b_1 < 0 \quad \text{and}$$

$$(a_1 + b_2) = 0 \tag{3.3.13}$$

The singular point will be stable or unstable based upon the type of singularity i.e. the movement of the representative point on the integral or solution curve with increasing time. In the above cases the node is a stable singularity as the representative point ultimately reaches the node as time passes. The saddle point is a singularity which is terminated by four trajectories forming two distinct set of integral curves. Two of the trajectories approach the saddle point and the other two move away from the saddle point as time increases, therefore the saddle point is an unstable singularity. The other four regions surrounded by four different trajectories form continua of hyperbolically shaped integrals which do not approach the saddle point. Focal point is also an unstable singularity as trajectories start in uncertain directions and the representative point on any of these trajectories moves away from the focus.

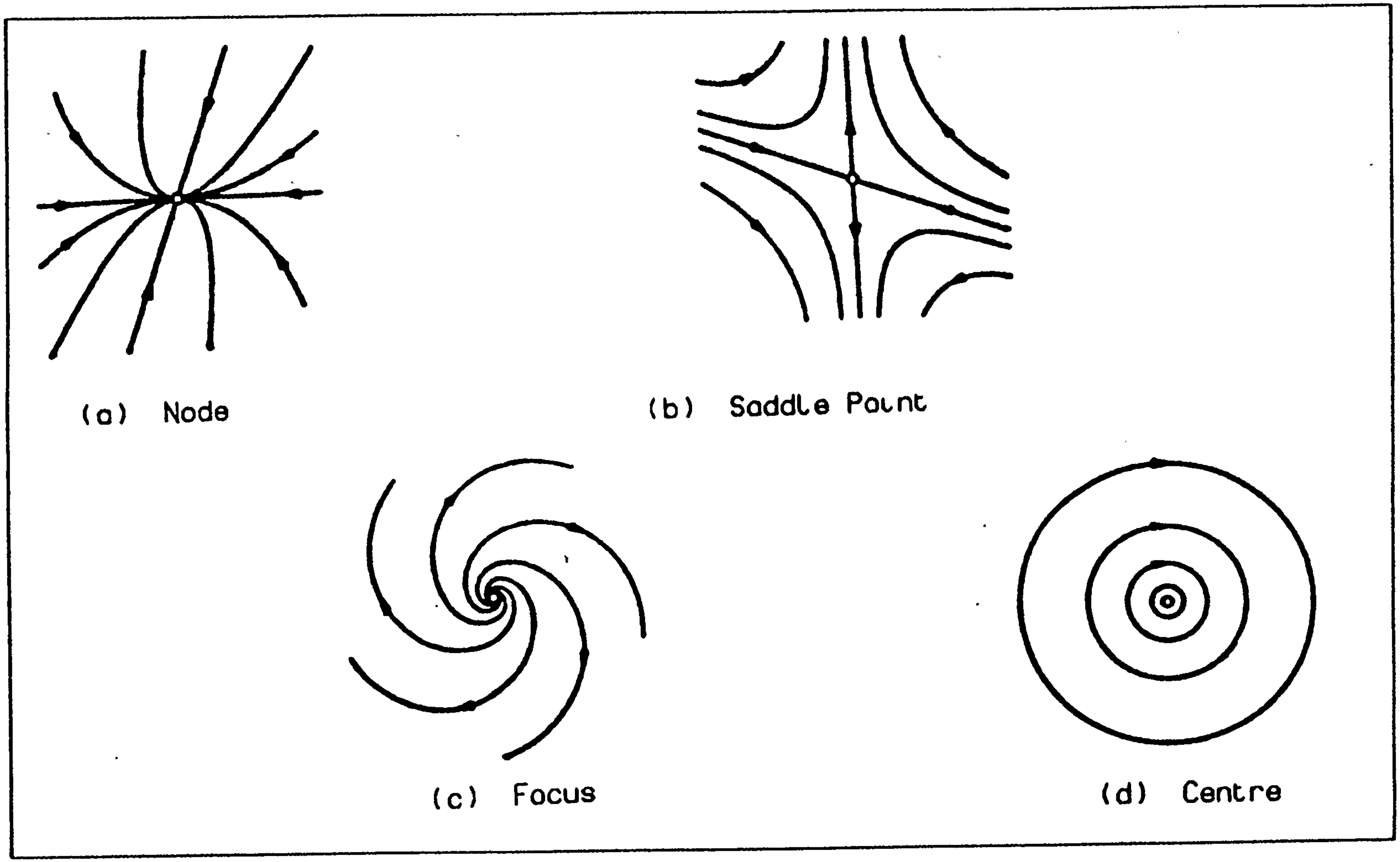


Fig 3.1 : Singular Points

Step 4

Find the eigen vectors for each eigen value

$$\begin{bmatrix} X_x-\lambda & X_y \\ Y_x & Y_y-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.4.1}$$

To establish the direction of eigen vectors, the following criterion due to Poincare will be used.

If the eigen value is negative, the trajectories tangent to the corresponding eigen vector will converge to the attractor, else they will diverge if the eigen value is positive.

Step 5

Check for the existence of limit cycles. According to Feigenbaum (1983), a limit cycle is an attractor, if for a set of initial conditions the solution of a system of ordinary differential equations settle down.

A singularity is a center if the integral curves form a continuum of concentric loops around a center such that none approaches it. Physically it occurs in conservative systems. But in nonconservative systems integral curves form closed trajectories or limit cycles towards which the neighboring trajectories spiral on both sides. In general there are three different types of limit cycles as shown in the adjoining figures.

(a) When all the trajectories converge towards the limit cycle from inside as well as from the outside, it is called a stable limit cycle as shown in figure (3.2a).

(b) Contrary to above if all the trajectories move away from the limit cycle, it is called a unstable limit cycle as shown in Figure 3.2b.

(c) If one of the set of trajectories from outside or inside of the limit cycle approach the limit cycle and the others moves away from the limit cycle, it is called a semistable limit cycle, which are shown in Figure 3.3.

It is often difficult to establish the existence of a limit cycle. The method commonly applied is Bendixon's criterion.

Bendixon's criterion is used to make sure that there are no limit cycles for a particular set of differential equations. It also establishes the condition for non-existence of closed trajectories. For a nonlinear dynamical system described by Equations (3.2.1), the Bendixon's criterion establishes that if $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}$ does not change sign in a closed region of xy plane then there are no limit cycles or closed trajectories in that region.

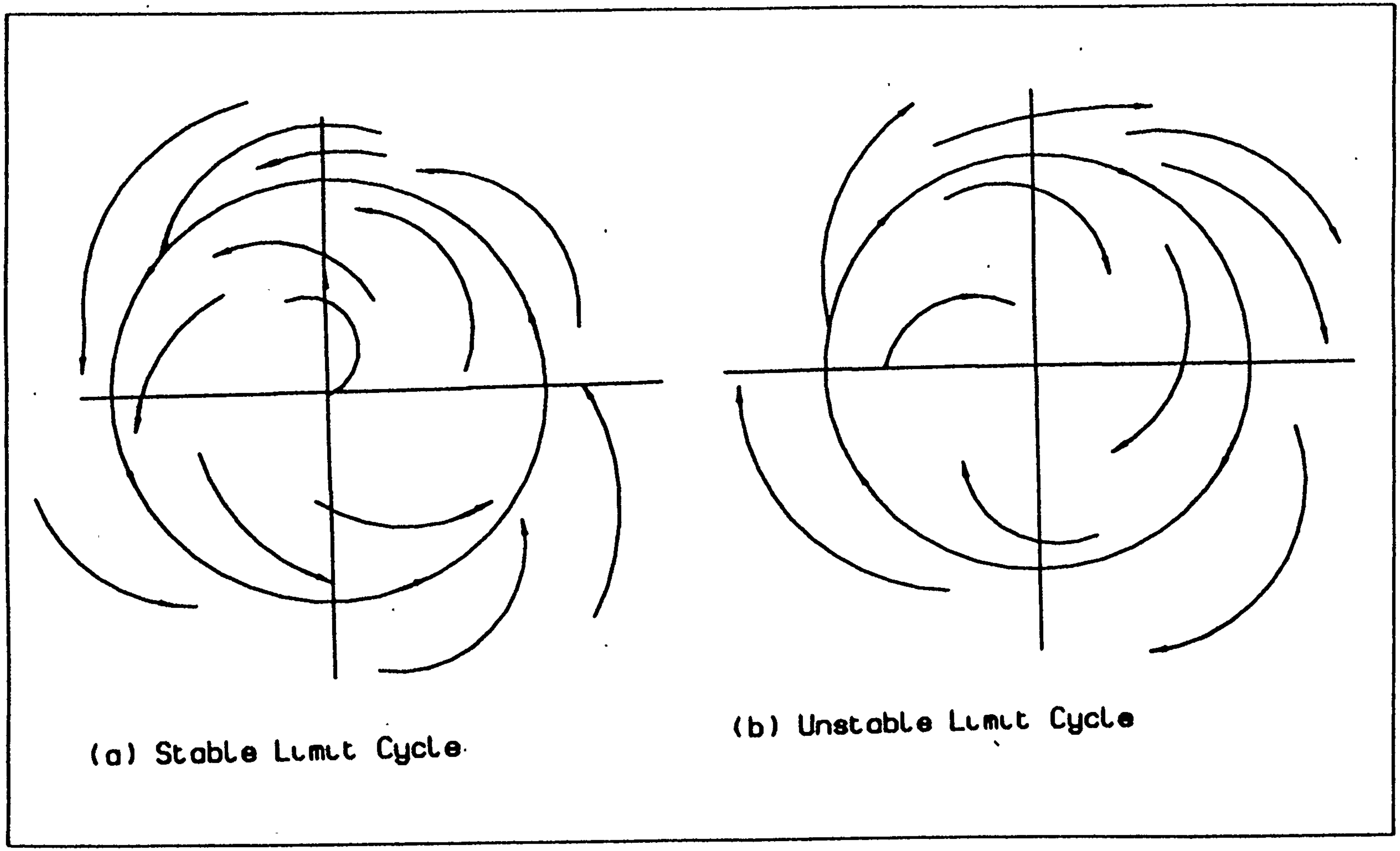


Fig 3.2 : Limit Cycles

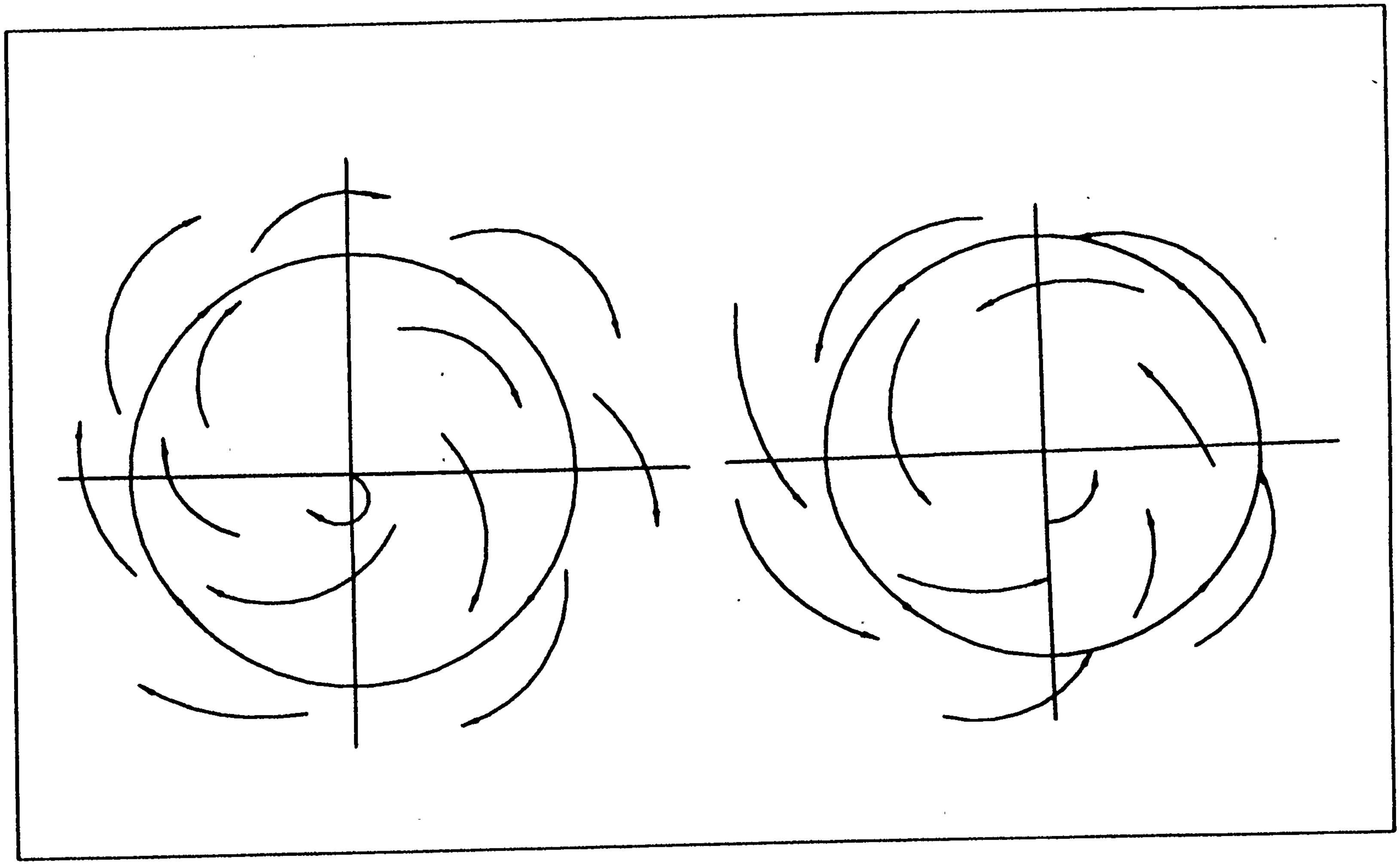


Fig 3.3 : Semistable Limit Cycles

Step 6

Once the singular points and trajectories are established, the conditions for stability become important. For this either one or both of the Routh Hurwitz Criterion or Liapunov method are used. A solution will be deemed stable if all solutions or integral curves due to a slight perturbation approaching near the solution curve remain close to it.

(a) Routh Hurwitz Criterion

The method explained by Thompson and Stewart (1986) is applicable only to those equilibrium states which are represented by singular points. That is the method is not useful for systems having limit cycles as their equilibrium state. The method is applied to systems governed by nonlinear differential equations of first order. Therefore to apply this method to a second order differential equation, the system must be transformed into first order differential equations.

The method is used for analyzing the stability of a system near the equilibrium point. A small disturbance is given to the system near the equilibrium point. If, as the time increases and tends to infinity, the point approaches to the equilibrium point, the system is then said to be stable at that point or else unstable.

Consider a system of n-first order differential equations

$$\frac{d\overline{x}_i}{dt} = \overline{X}_i(x_j) \tag{3.6.1}$$

where i,j = 1 to n, and $\overline{X}_i$'s are nonlinear functions of x_i's.

Let $(x_{1_0}, x_{2_0}, \ldots\ldots\ldots, x_{n_0})$ be a singular point. Consider a neighboring point $x_{i_0}+\xi_i$. Substituting in Equation (3.6.1), we find

$$\frac{d\overline{x}_i}{dt} = \overline{X}_i(x_{i_0}+\xi_i) \tag{3.6.2}$$

The Taylor series expansion of function on the right side of Equation (3.6.2) leads to

$$\frac{dx_{i_0}+\xi_i}{dt} = X_i(x_{j_0}) + \left[\frac{\partial X_i}{\partial x_k}\right]_0 \xi_k \tag{3.6.3}$$

Since $X_i(x_{j_0}) = 0$, x_{j_0} being the singular point, we obtain

$$\frac{d\xi_i}{dt} = \left[\frac{\partial X_i}{\partial x_k}\right]_0 \xi_k \tag{3.6.4}$$

In matrix form

$$\frac{d\overline{\xi}_j}{dt} = \overline{\overline{A}}_{ik}\overline{\xi}_k \tag{3.6.5}$$

where

$$A_{ik} = \left[\frac{\partial X_i}{\partial x_k}\right]_0 \tag{3.6.5a}$$

To solve Equation (3.6.5) for disturbances ξ_i, let

$$\xi_i = K_i e^{\lambda t} \qquad (3.6.6)$$

Substituting in Equation (3.6.5) and solving for a nontrivial solution we derive Equation (3.6.7)

$$\left| A_{ik} - \lambda E_{ik} \right| = 0 \qquad (3.6.7)$$

where E_{ik} is an identity matrix.

Equation (3.6.7) which is a characteristic polynomial of degree n of matrix $\overline{\overline{A}}$ can be written in the form

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots\ldots\ldots + a_n = 0 \qquad (3.6.8)$$

According to this criterion, for the solution $x_{i_0}+\xi_j$ to be stable, the roots of equation (3.6.8) should have the real part negative i.e. all the coefficients $a_0, a_1, \ldots, a_n$ and the associated determinants of these coefficients $\Delta_1, \Delta_2, \ldots\ldots, \Delta_n$ should be positive i.e.

$$a_0 > 0,\ a_1 > 0, \ldots\ldots\ldots\ldots,\ a_n > 0 \qquad (3.6.9)$$

$$\Delta_1 = \left| a_1 \right| > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_3 & a_2 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_5 & a_4 & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & & a_n \end{vmatrix} \tag{3.6.10}$$

Equations (3.6.9) and (3.6.10) will give 2n+1 conditions on system parameters for a stable or an unstable solution.

(b) Liapunov Method

This method is concerned with the stability of a system in a large finite region. If in some domain D about origin, there exists a Liapunov function $V(x_1,x_2,\ldots\ldots,x_n)$, then the origin is a stable point.

For the system represented by Equation (3.6.1), if origin is a singular point of the system and a positive definite function called Liapunov function can be introduced, the system is said to be stable.

Here we consider an example and discuss Bendixon's criterion and construct the Liapunov function for the differential equation

$$\frac{d^2x}{d\zeta^2} - \varepsilon(1-x^2)\frac{dx}{d\zeta} + x = 0 \tag{3.6.11}$$

Reducing Equation (3.6.11) to a system of first order differential equations

system.

$$\frac{dy}{d\zeta} = -x = Y$$

$$\frac{dx}{d\zeta} = y + \varepsilon(x - \frac{x^3}{3}) = X \qquad (3.6.12)$$

The singular point is (0,0) and the characteristic equation for this point is

$$\begin{vmatrix} \varepsilon-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \qquad (3.6.13)$$

The characteristic roots are

$$\lambda_{1,2} = \frac{\varepsilon \pm \sqrt{\varepsilon^2-4}}{2} \qquad (3.6.14)$$

i.e. λ_1 is positive since $\varepsilon > 0$ and origin is a singular point. According to Poincare's criterion origin is an unstable singularity. To find the eigen vectors, let us consider a particular case i.e. ε=2. Then for λ=1

$$\begin{bmatrix} -2-1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3.6.15)$$

which gives an equation of straight line

$$x + y = 0 \qquad (3.6.16)$$

To check for the existence of limit cycles we use Bendixon's criterion, since the expression $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \varepsilon(1-x^2)$, and this can change signs as x ranges from less than 1 to greater than 1. Hence Bendixon's criterion predicts the existence of limit cycles in a region which includes x = 1.

Since it is already established that the system has an unstable singularity at the origin, the behavior of the trajectories can be analyzed by Liapunov's Method. Consider a Liapunov function as

$$V = \frac{1}{2}(x^2 + y^2) \tag{3.6.17}$$

Differentiating with respect to t, we find

$$\dot{V} = x\dot{x} + y\dot{y}$$

Substituting values of $\dot{x}$ and $\dot{y}$ and simplifying we obtain

$$\dot{V} = \varepsilon x^2 \left(1 - \frac{x^2}{3} \right) \tag{3.6.18}$$

Therefore $\dot{V} > 0$ for $x^2 < 3$ which means that the points on the interior trajectories tend to move away from the circle $x^2 + y^2 = 3$ and $\dot{V} < 0$ for $x^2 = 3$, points on the trajectories outside the circle, move towards the circle $x^2 + y^2 = 3$. Hence the limit cycle is semistable.

3.2 ADVANTAGES

(1) The method gives a first hand information about the behavior of the physical system without the need to draw its trajectories.

(2) The method does not require any initial conditions to start with, contrary to numerical method as is described in next chapter where initial conditions must be given to start with.

3.3 DISADVANTAGES

(1) The method is applicable to all those systems with small nonlinearity, as essentially we end up neglecting nonlinearity of higher orders.

(2) The method can not solve all those systems which are governed by higher order differential equations. Even if an n^{th} order system is transformed to n first order differential equations, the behavior of the trajectories is hard to visualize and understand if n is greater than 3.

(3) The method is tedious in the sense that all steps may or may not work in establishing the liimit cycles.

CHAPTER 4

SIMULATION LANGUAGES AND ALGORITHMS

4.1 INTRODUCTION

With ever developing capabilities of computers and availability of simulation languages, numerical analysis provides powerful techniques for simulating a physical system. At present there are more than 35 simulation languages. The two languages used here are Advanced Continuous Simulation Language (ACSL) and Continuous Systems Simulation Language IV (CSSL IV). A brief summary of the two languages, is as follows :

Both simulation languages are primarily used for modeling systems described by time dependent nonlinear differential equations. Some of the systems where these two languages can be applied are chemical process representation, control system design, missile and aircraft simulation, power plant dynamics, fluid flow, and heat transfer analysis. The main features of the languages are

(1) Free form input.

(2) Functions up to three variables can be generated.

(3) Ability to control error independently on each integrator.

(4) Sorting of the order of continuous model equations automatically unlike other programming languages like Fortran, Pascal where execution of model critically depends on the statement order.

The statements supplied by the user are in two distinct groups. One group defines the model i.e. the structure of the system being simulated is generally a set of differential equations. The other group exercises this model i.e. starts execution, runs, changes parameters, applies controls which draw plots.

The languages consist of a set of arithmetic operators, standard functions, a set of special statements, and a macro capability. Integration is the special operator which is written as INTEG or INTVC, as

$$Y = INTEG\ (X, YZERO) \tag{4.1}$$

which means $$Y = YZERO + \int_0^T X dt$$

This is the main operator which is used in the solution of nonlinear differential equations. The model is written in different blocks like Initial Block, Dynamic Block, Discrete Block, Derivative Block, Terminal Block, all embedded in two statements Program and End.

After the model is defined and the run time commands are written, the model is compiled. Once the compilation is done, the model is ready to be executed. This can be done interactively or in the batch mode with run time commands.

One of the important difference between ACSL and Fortran is that, in ACSL all the variables are assumed to be real unless explicitly specified unlike Fortran where variable names starting from i to n are assumed to be integers. All functions and subroutines available in Fortran are also available in these simulation languages. Both simulation languages have macro capability that allows extension of the special language statements. Once the model is complete and compiled the program flows as follows:

First the program is activated with a run time command START. Then the program proceeds sequentially through the initial section. Here the calculations are performed before the dynamic model begins. Generally all those variables whose values do not change during run are calculated in this initial section.

Once the control is transferred from the initial section to the dynamic section integration routine is initialized. This procedure involves transferring all initial conditions into the corresponding states and evaluating the code in the derivative and discrete sections.

The stop flag is reset after initialization of the derivative and the discrete code. Then the program executes the code within the dynamic block. All state variables have values assigned, so the intermediate calculations in the derivative and discrete sections are done in advance.

After the dynamic section has been evaluated, the stop flag is tested. If it is true, the control is transferred to the terminal section. This stop flag is set by terminating condition. On the contrary if the stop flag is false, the program writes out the values of all the variables specified in the output command and prints the list to an output file or on the terminal if working interactively. The prepar list output goes to a scratch file for later plotting and/or printing.

The above step is repeated over and over again through each step of communication interval until a terminating condition is reached. In both the simulation languages it is possible to transfer control between some sections using GO TO statements and statement labels. It is however not possible to transfer to dynamic section again as integration initialization need to be prescribed again, but transfer to initial or terminal section is possible from dynamic section.

4.2 NUMERICAL METHODS

The following two methods used for the integration of differential equations are summarized though there are eight different methods available on ACSL.

(a) Gear's Stiff method.

(b) Runge-Kutta fourth order Method.

The methods are represented in the model definition program and are called by an ACSL statement as

$$ALGORITHM \quad ILAG = n \tag{4.2}$$

where n is an integer. Here, we take n = 2 for Gear's Stiff method and n = 5 for Runge-Kutta fourth order method.

The Gear's Stiff method is chosen because it can take variable step since in the analysis of nonlinear dynamical systems it has been found that at times there is vast difference between the position of a representative point for the same communication interval. One can also specify the maximum error and thus the step size is controlled automatically.

The Runge-Kutta fourth order method means that truncation error per step h is of the order h^5. The method works by evaluating four function values per step. Analytically for any initial value problem

$$\dot{y} = f(x,y) \tag{4.3}$$

with initial condition as

$$y(x_0) = y_0$$

The method evaluates different y's as

$$y_{n+1} = y_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n) \tag{4.4}$$

where A_n, B_n, C_n, D_n are defined as follows

$$A_n = hf(x_n, y_n) \tag{4.5}$$

$$B_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}A_n) \tag{4.6}$$

$$C_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}B_n) \tag{4.7}$$

$$D_n = hf(x_{n+1}, y_n + C_n) \tag{4.8}$$

It is obvious that smaller the step size h, the better will be the accuracy but computations will take more computer time.

A sample program in ACSL and its output is given in Appendix A, as run on Cyber860 with Network Operating System / Virtual Environment.

CHAPTER 5

MODELS OF SOME NONLINEAR DYNAMICAL SYSTEMS

In this Chapter some of the models have been analyzed analytically and numerically to show how trajectories behave and to establish the conditions for the chaotic behavior of the system.

5.1 LORENZ MODEL

5.1.1 INTRODUCTION

This model is due to E. N. Lorenz (1963) and it consists of a set of three ordinary differential equations dependent on three initial conditions and three positive parameters. This well known example is explained in detail and forms the basis for analyzing a set of chaotic ordinary differential equations. This is the first model which predicted a very complicated behavior from three simple nonlinear ordinary differential equations.

Frequently this model is also called Lorenz weather model as it is used to explain why it is not possible to accurately forecast weather over a long period of time.

The model was first described in a paper by Lorenz (1963) which subsequently has drawn attention of many researchers e.g. Saltzman (1981), Schuster (1988), and Sparrow (1982). The equations were derived from a model of convection in a fluid cell entrapped in two plates kept at a constant temperature difference. The equations are

$$\frac{dx}{dt} = \sigma(y-x)$$

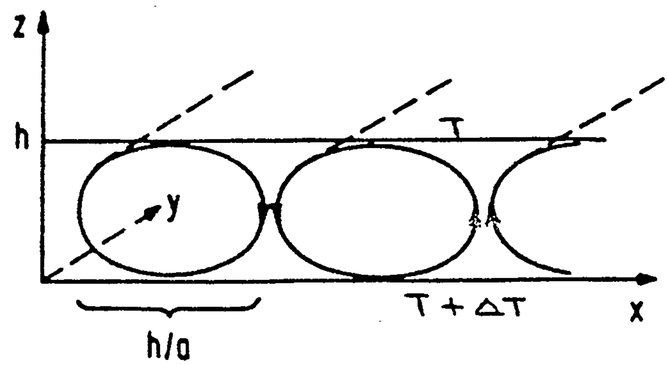
$$\frac{dy}{dt} = rx-y-xz$$

$$\frac{dz}{dt} = xy-bz \qquad (5.1.1)$$

where σ, r, and b are real positive parameters.

5.1.2 DERIVATION

In the interest of completeness, we derive the basic equations of this model which describes the convective motion of fluid cells warmed to a constant temperature from a plate under a colder plate held at a constant temperature difference of ΔT_0. In equations (5.1.1), x is proportional to the velocity of fluid flow circulation, y is proportional to the temperature difference between the ascending and descending fluid flow elements, and z is proportional to the variation of vertical temperature from linearity. The parameter σ is proportional to the Prandtl number which is equal to the ratio between kinematic viscosity and the thermal diffusivity, the parameter r is proportional to the Rayleigh number which is equal to the ratio between the driving buoyancy force and the dissipative viscous forces and is proportional to ΔT, and the parameter b is proportional to the aspect ratio which is equal to the ratio between the lateral dimension of the cell and its height.



Convective Model

In the system shown above basic equations are:

a. Navier-Stokes Equation

$$\rho \frac{d\vec{v}}{dt} = \vec{F} - \vec{\nabla} P + \mu \vec{\nabla}^2 \vec{v} \qquad (5.1.2)$$

$$\frac{d\vec{v}}{dt} = (\vec{v}.\vec{\nabla})\vec{v} + \frac{\partial \vec{v}}{\partial t}$$

$$\vec{F} = \rho g \vec{e}_z$$

b. Heat conduction equation

$$\frac{dT}{dt} = k\vec{\nabla}^2 T \tag{5.1.3}$$

c. Continuity equation

$$\frac{\partial \rho}{dt} + div(\rho\vec{v}) = 0 \tag{5.1.4}$$

d. Boundary conditions are

$$\begin{aligned} T(x,y,z=0,t) &= T_0 + \Delta T \\ T(x,y,z=h,t) &= T_0 \end{aligned} \tag{5.1.5}$$

The assumptions made are:

1. Boussinesq approximation is applied i.e. difference in temperature ΔT affects the density of the fluid linearly.

2. System is transitionally invariant towards Y-axis and extends to infinity i.e. the development of convective rolls is limited to X and Z direction, which means that the velocity v along Y-axis is equal to zero.

3. The plates are at a constant temperature difference Δ T with colder plate on top. i.e. $\Delta T = T(0) - T(h)$

Now by Boussinesq approximation $\frac{\partial \rho}{\partial t} = 0$, therefore, equation (5.1.4) becomes

$$div(\rho\vec{v}) = 0$$

which means

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5.1.6}$$

Let ψ (x,z,t) be the stream function such that

$$u = -\frac{\partial \psi}{\partial z}$$

$$w = \frac{\partial \psi}{\partial x} \tag{5.1.7}$$

Equation (5.1.2) can be expanded in X and Z directions, as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} - \nu\nabla^2 u = 0 \tag{5.1.8}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} - F_z - \nu\nabla^2 w = 0 \tag{5.1.9}$$

where F = g α T, and α being the coefficient of volume expansion.

Equation (5.1.3) can be written as

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} - k\nabla^2 T = 0 \tag{5.1.10}$$

Saltzman (1981) representation for T is

$$T(x,z,t) = \overline{T}(x,z,t) + T'(x,z,t) \tag{5.1.11}$$

$$\overline{T}(x,z,t) = \left[T(0,t) - \frac{\Delta T}{h}z\right] + T''(z,t) \tag{5.1.12}$$

where $\overline{T}$ is the average value along the horizontal, T' is the departure, and T'' is the departure from the linear variation between lower and upper boundary.

Substituting equation (5.1.11) in (5.1.12), we get

$$T(x,z,t) = \left[T(0,t) - \frac{\Delta T}{h} z \right] + \theta \tag{5.1.13}$$

where θ is the departure of the temperature from the state of no-convection and is equal to $T'(x,z,t) + T''(z,t)$.

Since the temperature at the boundaries is constant, therefore

$$\frac{\partial T(0,t)}{\partial t} = \frac{\partial T(h,t)}{\partial t} = 0$$

Differentiating (5.1.8) w.r.t. z and (5.1.9) w.r.t. x and subtracting, we obtain

$$\frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial \psi}{\partial z}\frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial z} - g\alpha \frac{\partial \theta}{\partial x} - \nu \nabla^4 \psi = 0 \tag{5.1.14}$$

Substituting T from equation (5.1.13) in (5.1.10) we obtain

$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} - \frac{\Delta T_0}{h}\frac{\partial \psi}{\partial x} - k\nabla^2 \theta = 0 \tag{5.1.15}$$

Equations (5.1.14) and (5.1.15) can be written with the help of Jacobian operator, as

$$\frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial(\psi, \nabla^2 \psi)}{\partial(z,x)} - g\alpha \frac{\partial \theta}{\partial x} - \nu \nabla^4 \psi = 0 \tag{5.1.16}$$

$$\frac{\partial \theta}{\partial t} - \frac{\partial(\psi, \theta)}{\partial(z,x)} - \frac{\Delta T_0}{h}\frac{\partial \psi}{\partial x} - k\nabla^2 \theta = 0 \tag{5.1.17}$$

where

$$\frac{\partial(a,b)}{\partial(x,z)} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial z} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial z} \qquad \text{and}$$

$$\nabla^4 \quad = \quad \nabla^2\,\nabla^2 \quad = \quad \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2 \partial z^2}$$

To solve the above two equations, Lorenz assumed upper and lower boundary to be free i.e.

$$\theta(0,0,t) \quad = \quad \theta(0,h,t) \quad = \quad \psi(0,0,t) \quad = \quad \psi(0,h,t) \quad =$$
$$\nabla^2\psi(0,0,t) \quad = \quad \nabla^2\psi(0,h,t) \quad = \quad 0 \tag{5.1.18}$$

Saltzman (1981) derived a set of ordinary differential equations by writing the Fourier series representation of functions ψ and θ retaining only low order terms as

$$\frac{a}{1+a^2}\frac{1}{k}\psi \quad = \quad \sqrt{2}X(t)\sin\left[\frac{\pi a x}{h}\right]\sin\left[\frac{\pi z}{h}\right] \tag{5.1.19}$$

$$\frac{\pi R}{R_c \Delta T}\theta \quad = \quad \sqrt{2}Y(t)\cos\left[\frac{\pi a x}{h}\right]\sin\left[\frac{\pi z}{h}\right] - Z(t)\sin\left[\frac{2\pi z}{h}\right] \tag{5.1.20}$$

where $R = \dfrac{g\,\alpha h^3}{k\nu}$ (Rayleigh number) and $R_c = \pi^4 a^{-2}\left[1+a^2\right]^3$ is critical Rayleigh number.

Substituting equations (5.1.19) and (5.1.20) in equations (5.1.16) and (5.1.17) we obtain the Lorenz system

$$\frac{dx}{dt} \quad = \quad \sigma\,(y-x)$$

$$\frac{dy}{dt} \quad = \quad rx-y-xz$$

$$\frac{dz}{dt} \quad = \quad xy-bz$$

where $\sigma = \frac{\nu}{k}$, $b = 4(1+a^2)^{-1}$, $r = \frac{R}{R_c}$ and the normalized time $t = \frac{\pi^2(1+a^2)kt}{h^2}$.

5.1.3 ANALYSIS

Now we analyze the Lorenz model according to the steps described in chapter 2.

Step 1

The system is autonomous with three variables x, y, and z dependent on time t. The equations (5.1.1) can be written in the matrix form by separating the nonlinear terms from linear terms as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -xz \\ xy \end{bmatrix} \tag{5.2.21}$$

The forcing functions are

$$\begin{aligned} X &= -\sigma(x-y) \\ Y &= rx-y-xz \\ Z &= -bz+xy \end{aligned} \tag{5.2.22}$$

Step 2

The equilibrium points are obtained by putting all the forcing functions equal to zero i.e.

$$-\sigma(x-y) = 0 \tag{5.2.23}$$

$$-xz+rx-y = 0 \tag{5.2.24}$$

$$xy-bz = 0 \tag{5.2.25}$$

From equation (5.2.23), we find x = y since $\sigma \neq 0$.

From equation (5.2.24), either x = 0 or z = r-1 (provided r > 1).

From equation (5.2.25) z $= \frac{x^2}{b}$.

Therefore z = 0 if x = 0. If z = r-1 then x = y $= \pm\sqrt{b(r-1)}$.

For r < 1 there is only one equilibrium point (0,0,0), but for r > 1 there are three equilibrium points (0,0,0), $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$.

The physical impact of r < 1 will be that, the heat transfer between the upper and lower boundaries is due to pure heat conduction with no convection. While for r > 1, the heat transfer is due to convection only with no heat conduction.

Step 3

The characteristic equation for the linearized system at (0,0,0) is given by

$$\begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + (\sigma+b+1)\lambda^2 + (r+\sigma)b\lambda + 2\sigma b(1-r) = 0 \qquad (5.2.26)$$

Solving the above equation we obtain

$$\lambda = -b, \frac{-(\sigma+1)\pm\sqrt{(\sigma+1)^2-4\sigma(1-r)}}{2}$$

The roots of equation (5.2.26) are one real negative and two complex conjugates if the product of coefficient of λ^2 and λ is equal to the constant term i.e.

$$(\sigma+b+1)\,(r+\sigma)b = 2\sigma b(r-1)$$

which leads to

$$r = \frac{\sigma(\sigma+b+3)}{(\sigma-b-1)} \qquad (5.2.27)$$

This will give the critical value of r for an unstable convection solution. If $\sigma < b+1$, the value of r will be negative and there will be no solution except (0,0,0). On the other hand if $\sigma > b+1$ the values of r will be positive giving three equilibrium points provided $r > 1$, else there will only be one equilibrium point.

Saltzman(1962) assumed the value of square of aspect ratio as $\frac{1}{2}$ and $\sigma = 10$. Therefore by relation (5.2.27) $r = \frac{470}{19}$. Lorenz considered the value of r higher than this critical value as 28. The equilibrium points then are (0,0,0), $(6\sqrt{2},6\sqrt{2},27)$, $(-6\sqrt{2},-6\sqrt{2},27)$. The first equilibrium point represents a state of no convection while other two represent the state of steady convection. Similarly for a value of r = 55, there will be three equilibrium points as (12,12,54), (-12,-12,54), and (0,0,0).

Step 4

Once the eigen values are determined for a particular set of parameters, the eigen vectors can be constructed. Consider a set of parameter values as $\sigma = 10$, r $= 28$, b $= \frac{8}{3}$.

The characteristic equation can be solved to get three eigen values $\lambda_1 = 11.82$, $\lambda_2 = -2.67$, $\lambda_3 = -22.82$, which for the singular point (0,0,0) gives eigen vectors whose directions are represented by the following three equations :

$$21.82x - 10y = 0$$

$$28x - 12.82y = 0$$

$$z = 0$$

Step 5

To establish the existence of limit cycles, we consider Bendixon's criterion. It is easily found that

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = -(\sigma + b + 1)$$

i.e. $\nabla \cdot V = -(\sigma + b + 1)$.

Since parameters b, r, and σ are real positive numbers, $\nabla \cdot V < 0$ i.e the phase space volume will contract and also the system is dissipative. According to

Bendixon's criterion since $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$ does not change the sign, therefore there are no limit cycles.

Step 6

To examine the stability conditions for this system, we consider Routh-Hurwitz criterion for equation (5.2.26). The conditions for the system to be stable are:

1. $1 > 0$
2. $\sigma + b + 1 > 0$
3. $\sigma(1-r) + b(1+\sigma) > 0$
4. $\sigma b(1-r) > 0$

The first two conditions are automatically satisfied as parameters are real positive numbers.

Condition 4 is satisfied only if $r < 1$. Hence the system is globally stable if $r < 1$ else it is unstable.

Condition 3 gives an inequality $\sigma(r-1) < b(1+\sigma)$. This condition will help us to find how the value of third parameter behaves for a stable or unstable solution, while the other two parameters are fixed.

Now for a further insight as to how the trajectories behave, numerical analysis is done as explained in the chapter IV for general numerical model. The simulation language program is given in Appendix A along with data for the first two figures.

In the following pages, the figures have been drawn for a time interval of 0 to 20 seconds, using either Runge-Kutta fourth order method or the Gear's stiff method. The communication interval is 0.02 seconds and the representative points plotted are 0.08 seconds apart.

Many cases have been discussed for different set of parameter values. In the following pages various cases have been analyzed for different parameters and initial conditions which to the best of our knowledge have not been mapped before in the literature. First to prove the validity of our computer simulation a few mappings have been repeated.

Figure 5.1.1 is a two dimensional mapping of Lorenz model in XZ plane, as has been seen in many books. The parameters chosen are $\sigma = 10.0$, $r = 28.0$, and $b = \frac{8}{3}$. Figure 5.1.2a to 5.1.2c show how the dependent variables x,y, and z vary with independent variable time. It is observed that x and y behave almost identically with respect to time, while behavior of dependent variable z with respect to time t is altogether different. Hence subsequent figures have been drawn between x and z for different parameters and initial conditions. Where ever necessary the mappings between x and y have also been shown.

Figures 5.1.3 and 5.1.4 have been drawn to show the behavior of trajectories for variations in b and r, keeping all the other parameters same. The figures are a set of 10 trajectories starting from (5,5) and (-5,5). The parameter b has values 0.333, 2.0, 3.0, 4.0, 5.0, and r has values 0.01, 1.0, 5.0, 10.0, 15.0.

It has been observed that r is the most critical parameter determining the behavior of the system, as the condition for stability shows that the system is globally stable for $r < 1$ else unstable. This has been shown in figures 5.1.5a and 5.1.5b. In the first figure six trajectories have been drawn from six initial values (20,20), (20,-20), (-20,-20), (-20,20), (-20,0), and (0,20) with the parameter $r = 0.001$. As predicted for $r < 1$ all the trajectories converge to the origin, which is a stable equilibrium point. It can also be seen that the trajectories starting at (20,20) and (20,-20) form the mirror image of the trajectories starting at (-20,20) and (-20,-20). This is due to the fact that the two equilibrium points for $r > 1$ are $(\sqrt{b(1-r)}, sqrtb(1-r), (1-r)), (\sqrt{b(1-r)}, \sqrt{b(1-r)}, (1-r))$. Also the transition to the stable origin is very fast from the initial points as can be noticed from the breaks in the lines, e.g. the trajectory starting at (-20,-20) is plotted for 251 points starting from time $t = 0$ to $t = 20$ seconds with an interval of 0.08 seconds. It can be seen that the representative point reaches from point A to F in just 0.48 seconds and then ultimately approaches origin. Figure 5.1.5b has been drawn for $r = 100$ and as predicted by Routh-Hurwitz criteria all trajectories show an unstable solution and are chaotic.

Figures 5.1.6a to 5.1.6c have been drawn to show the behavior of any two dependent variables.

Figures 5.1.7a and 5.1.7b have been shown for different σ's i.e. for σ = 1.0 and σ = 100.0. Though as seen chaoticness increases as σ is increased from 0 to 100 but one thing is common in both the curves that all the trajectories converge to a plane z = r-1, as the equilibrium points found analytically are (0,0,0), $(\sqrt{b(1-r)},\sqrt{b(1-r)},(1-r)),(\sqrt{b(1-r)},\sqrt{b(1-r)},(1-r))$ i.e. (0,0,0), (8.5,8.5,27), and (-8.5,-8.5,27) and the equilibrium points are independent of σ.

Figure 5.1.8 is a combination of six trajectories starting from six different initial conditions for parameters σ = 10.0, b = $\frac{8}{3}$, and r = $\frac{490}{17}$ = 24.74 which is the critical value of r = $\frac{\sigma(\sigma+b+3)}{(\sigma-b-1)}$. The equilibrium points for these parameters are (0,0,0), $6\sqrt{2},6\sqrt{2},27),\text{and}(-6\sqrt{2},-6\sqrt{2},27)$.

Figures 5.1.9a and 5.1.9b have been mapped for very large r equal to 20,000. The curves seemed to form a hyperbolloid.

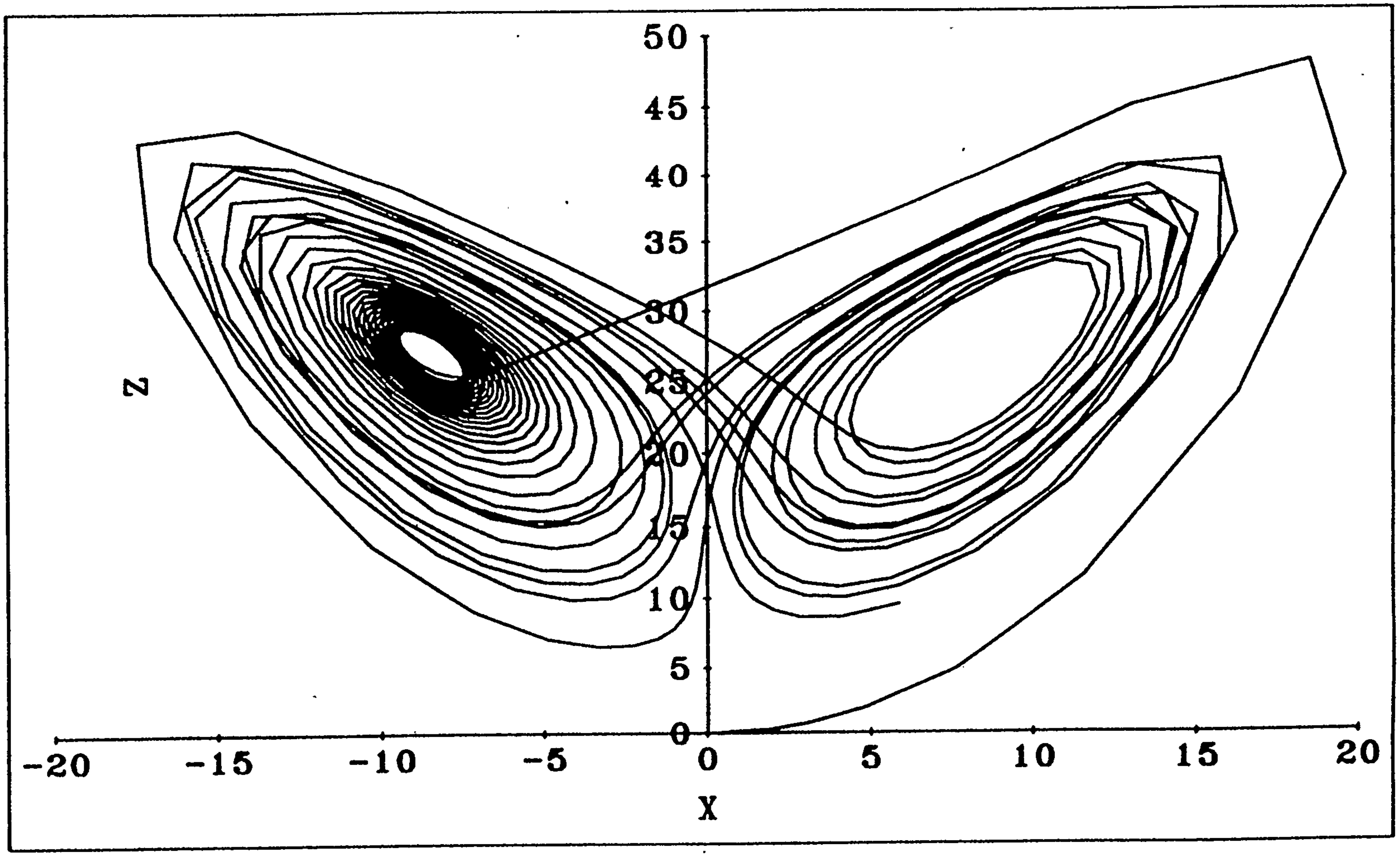


Fig 5.1.1 : Simulation of Lorenz Model
Sigma=10.0, r=28.0, b=8/3

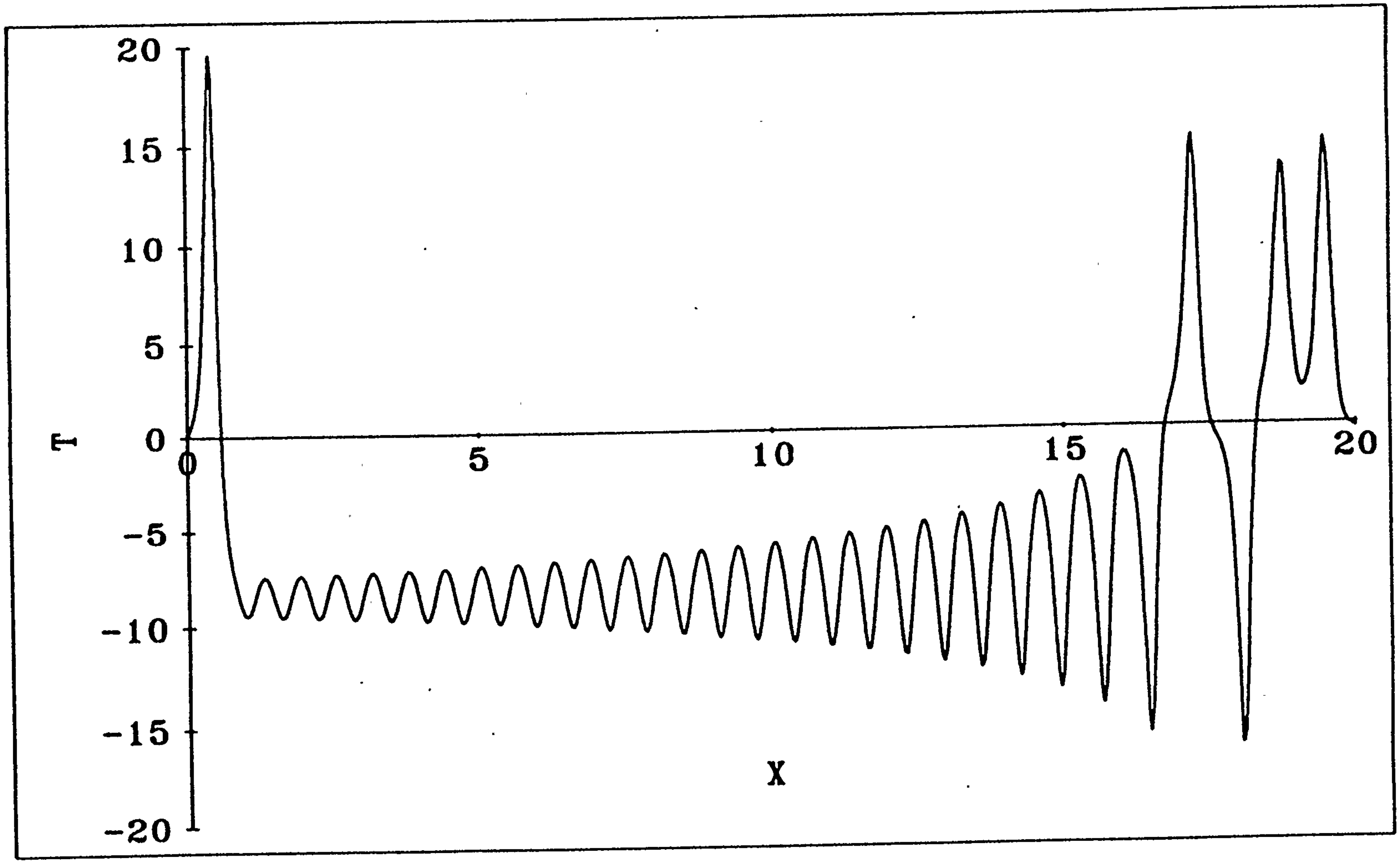


Fig 5.1.2a: Sigma=10.0, r=28.0, b=8/3

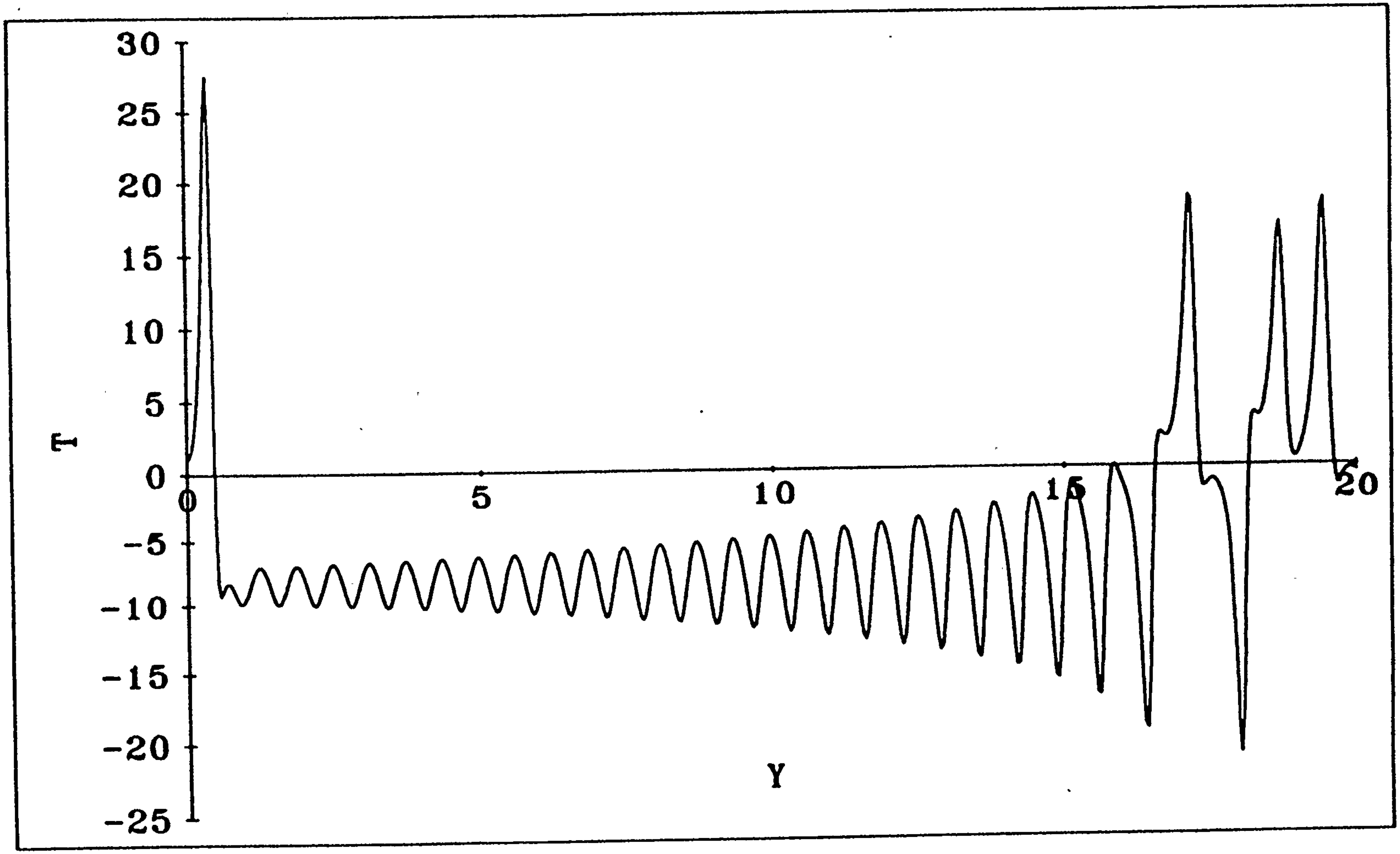


Fig 5.1.2b: Sigma=10.0, r=28.0, b=8/3

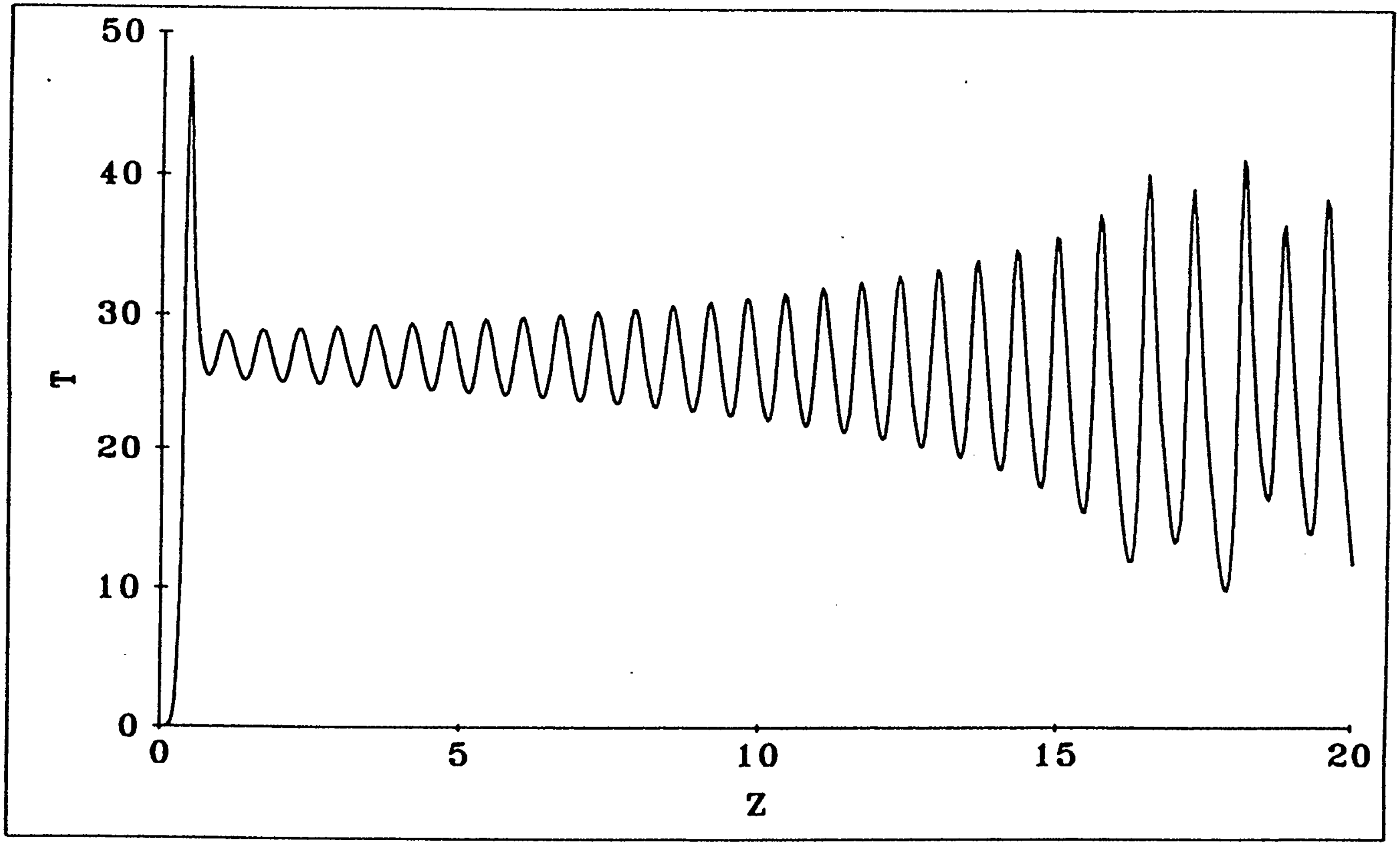


Fig 5.1.2c: Sigma=10.0, r=28.0, b=8/3

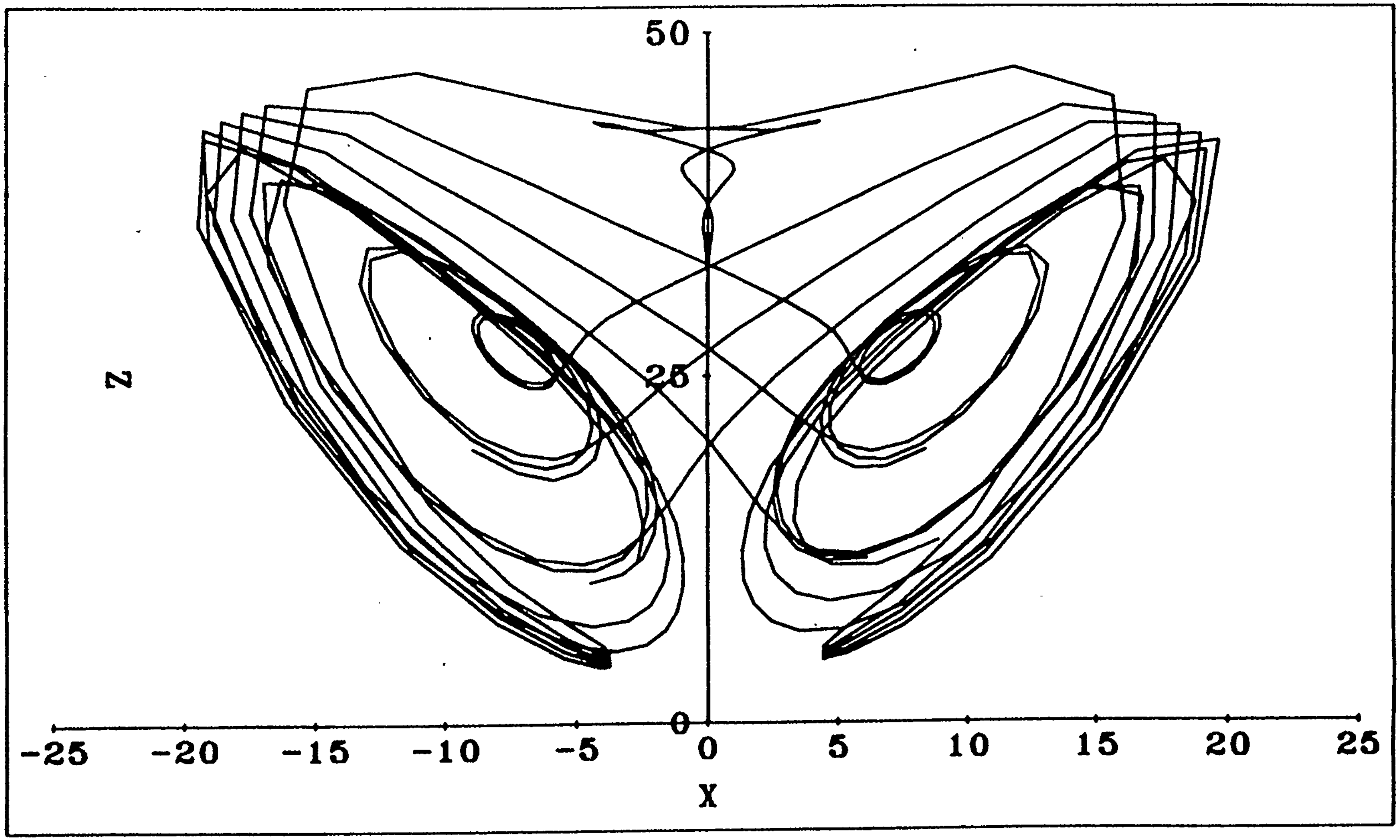


Fig 5.1.3 : Behavior of Trajectories
with variation in b

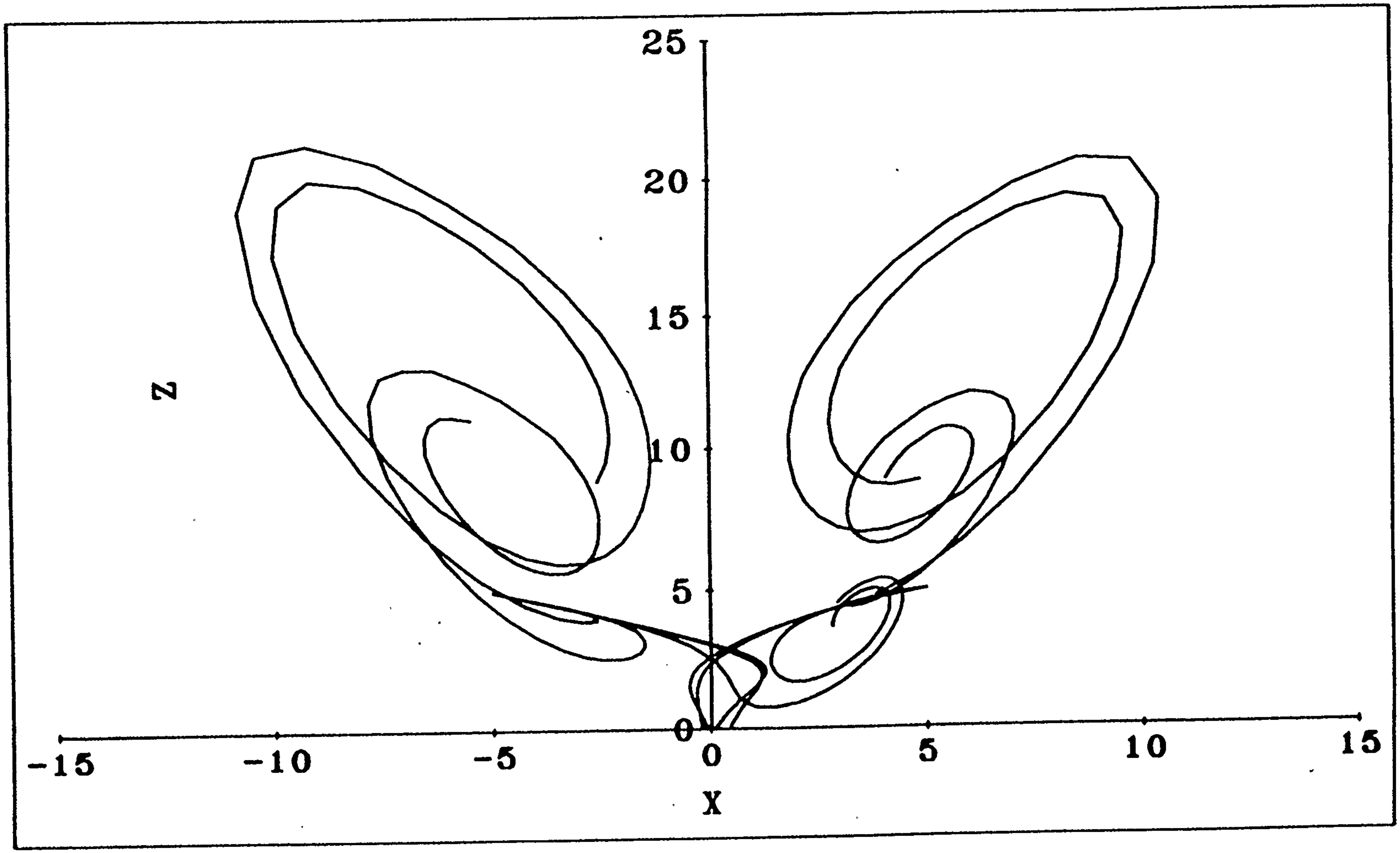


Fig 5.1.4 : Behavior of Trajectories with variation in r

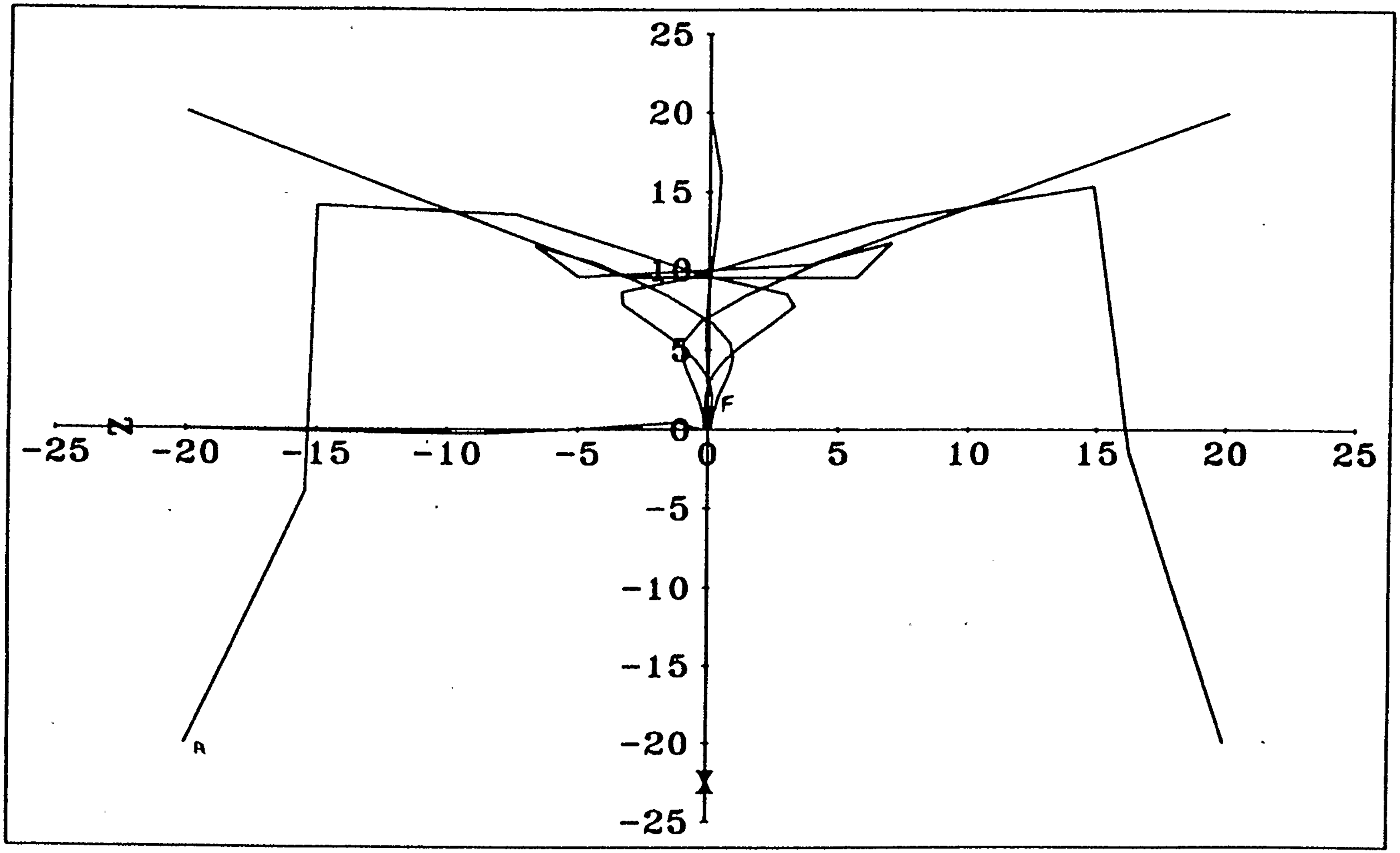


Fig 5.1.5a: Sigma=10.0, r=0.001, b=8/3

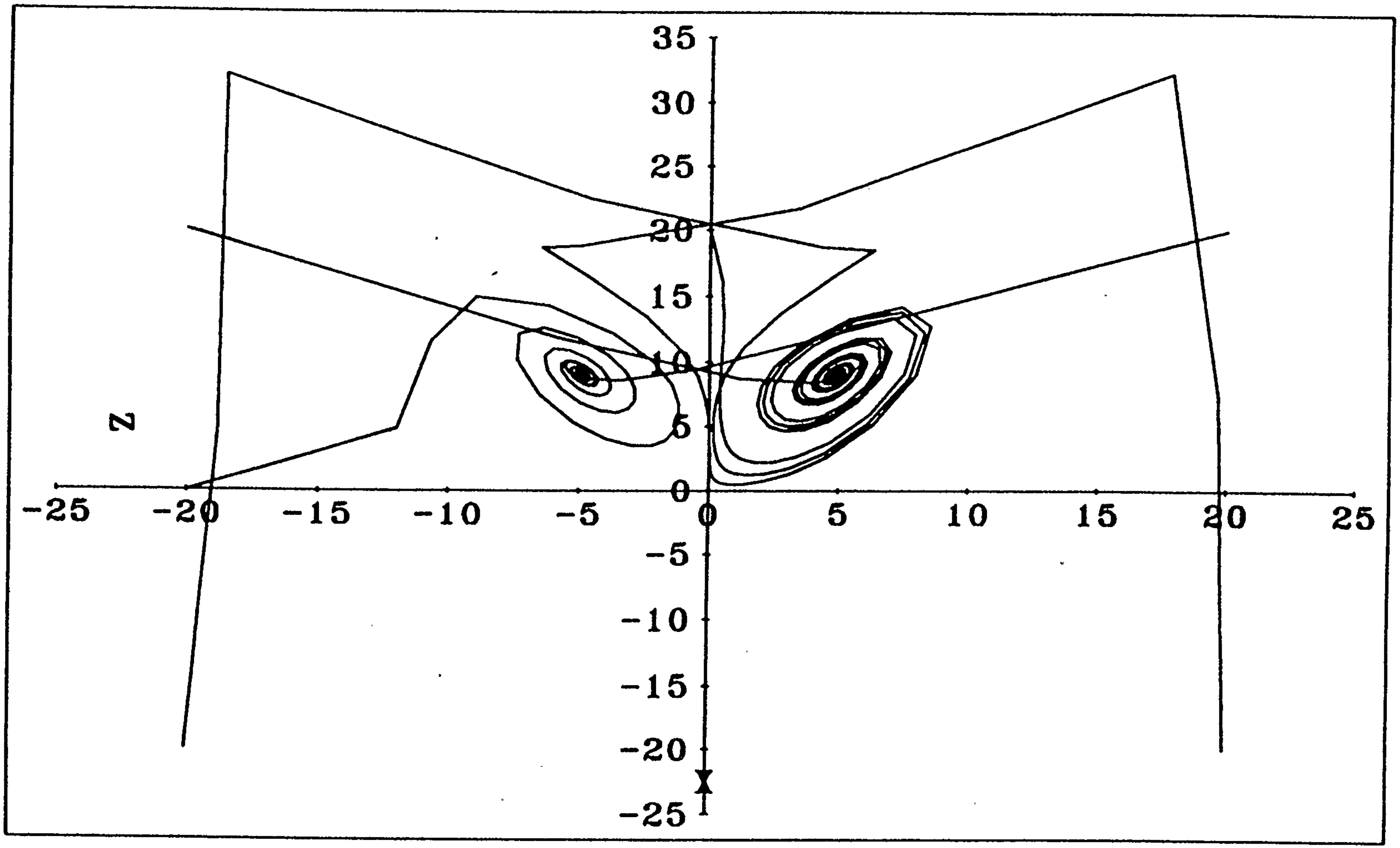


Fig 5.1.5b: Sigma=10.0, r=10.0, b=8/3

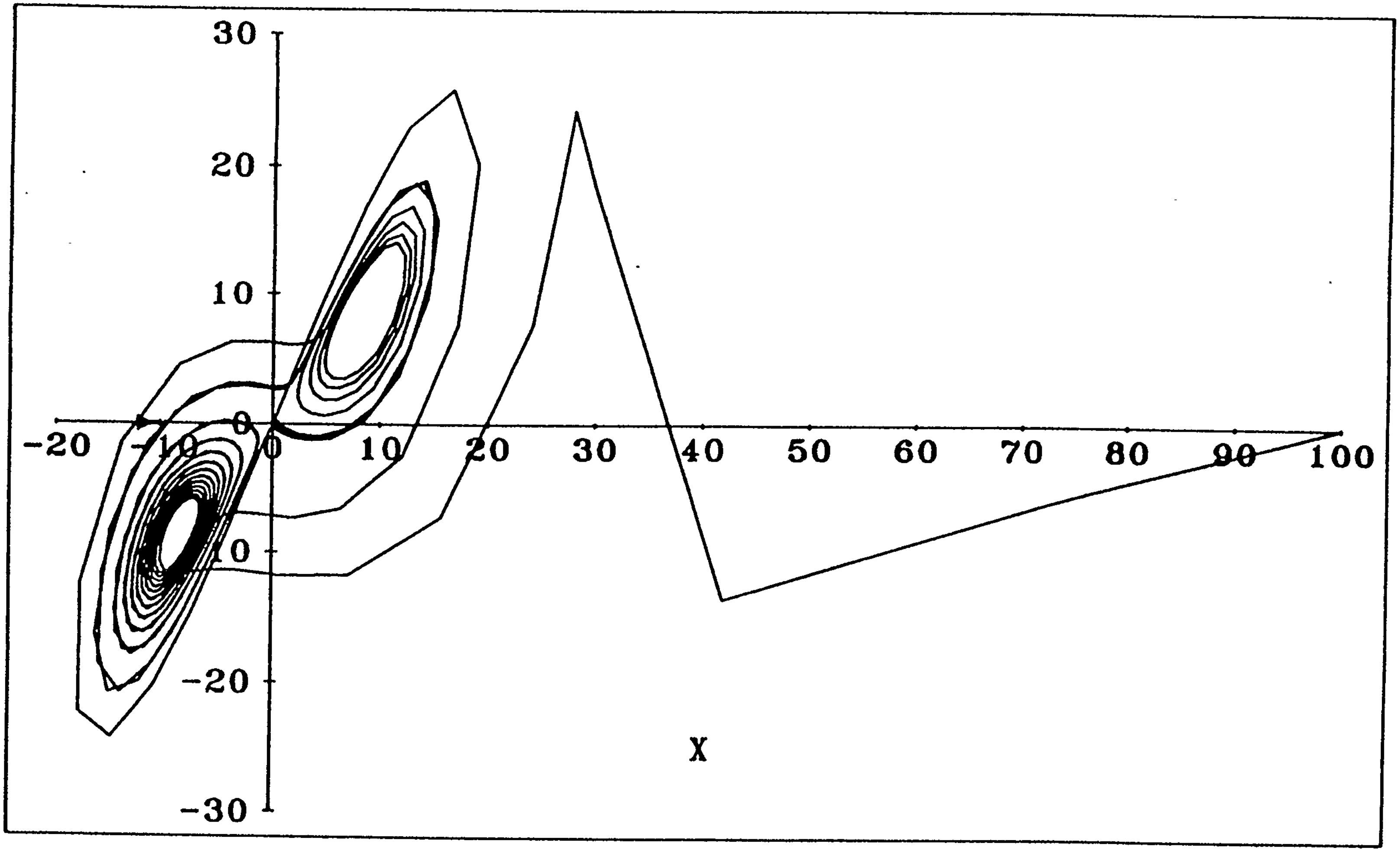


Fig 5.1.6a: Sigma=10.0, r=29.720, b=8/3

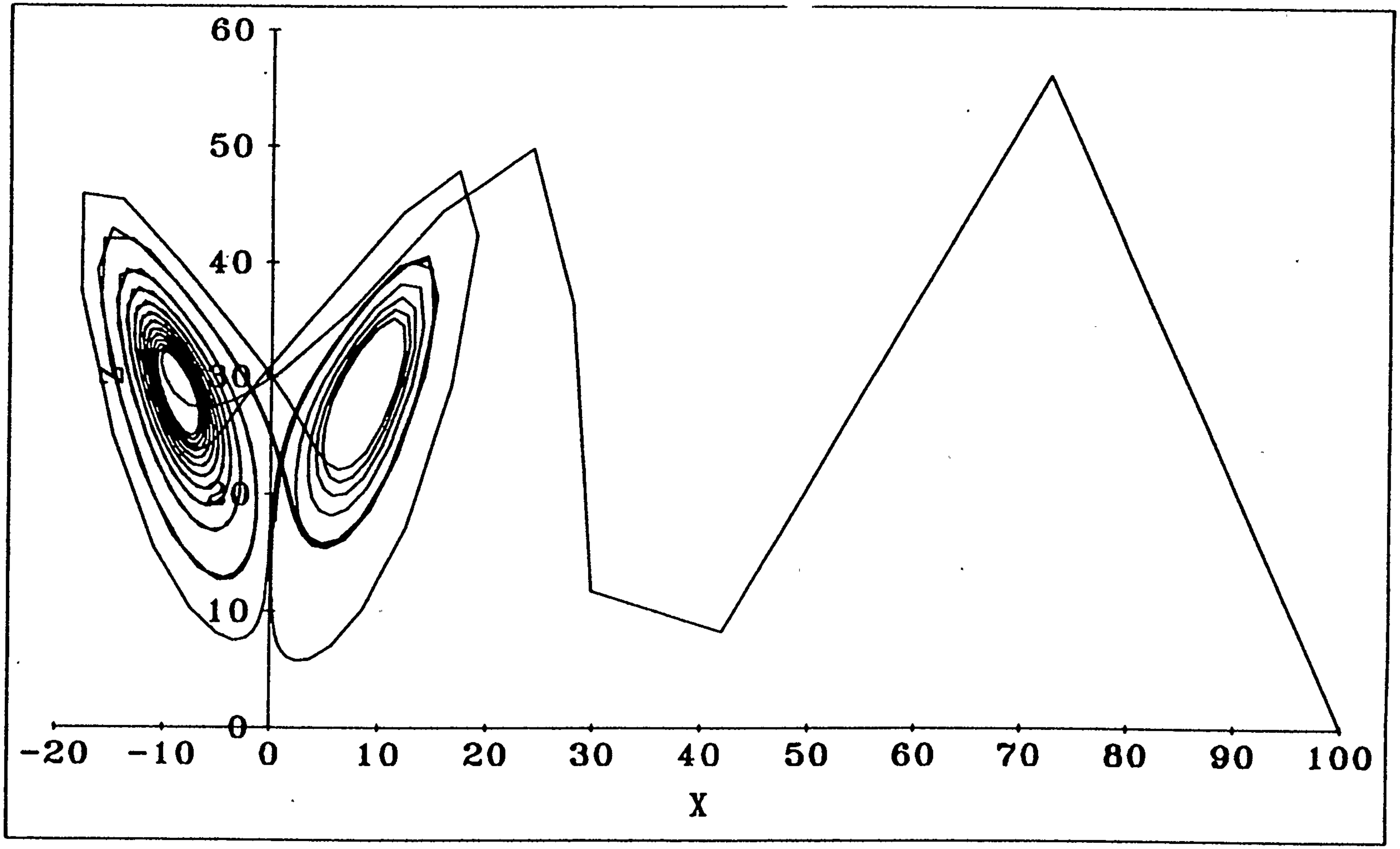


Fig 5.1.6b: Sigm. 10.0, r=29.720, b=8/3

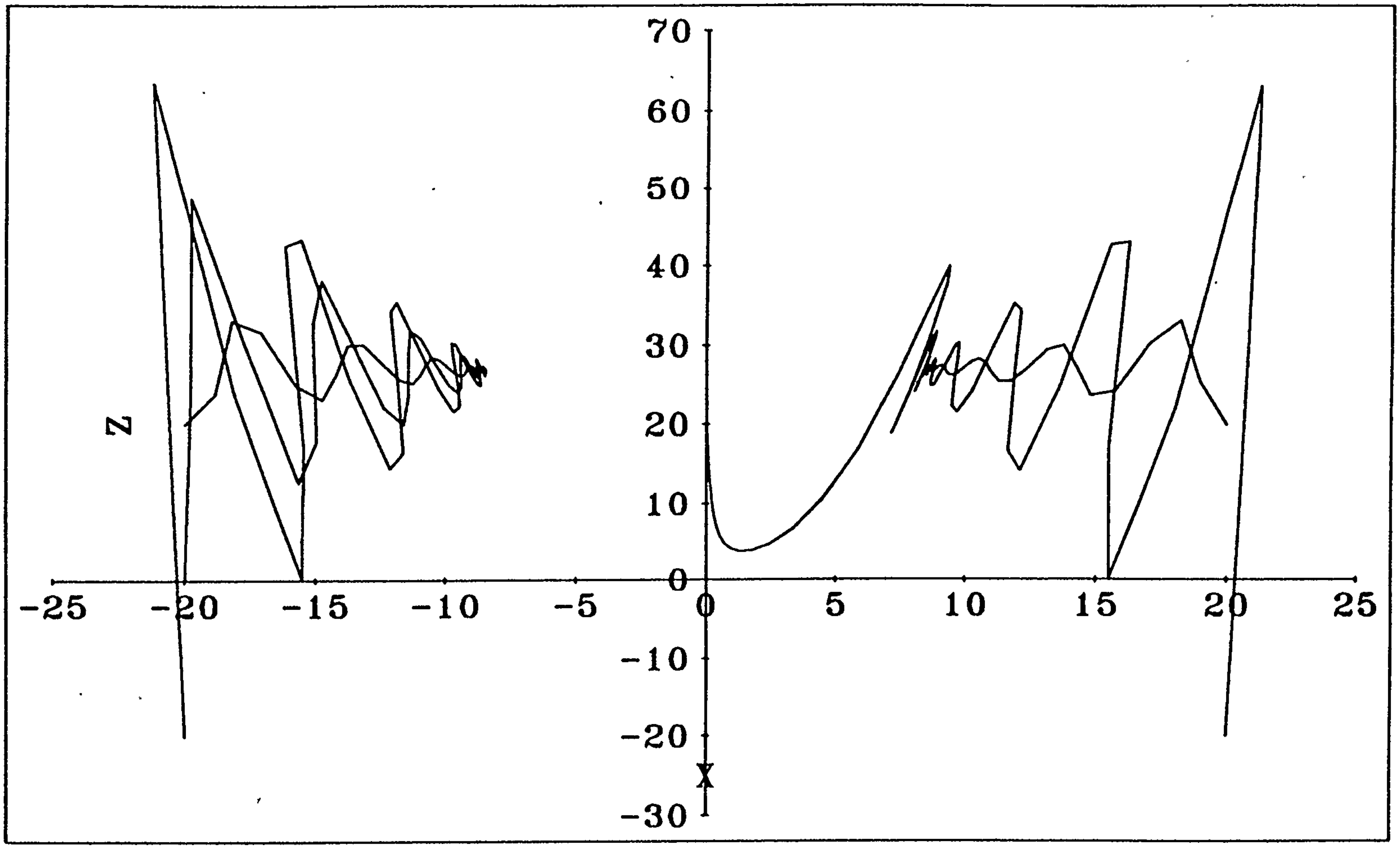


Fig 5.1.7a: Sigma=1.0, r=28.0, b=8/3

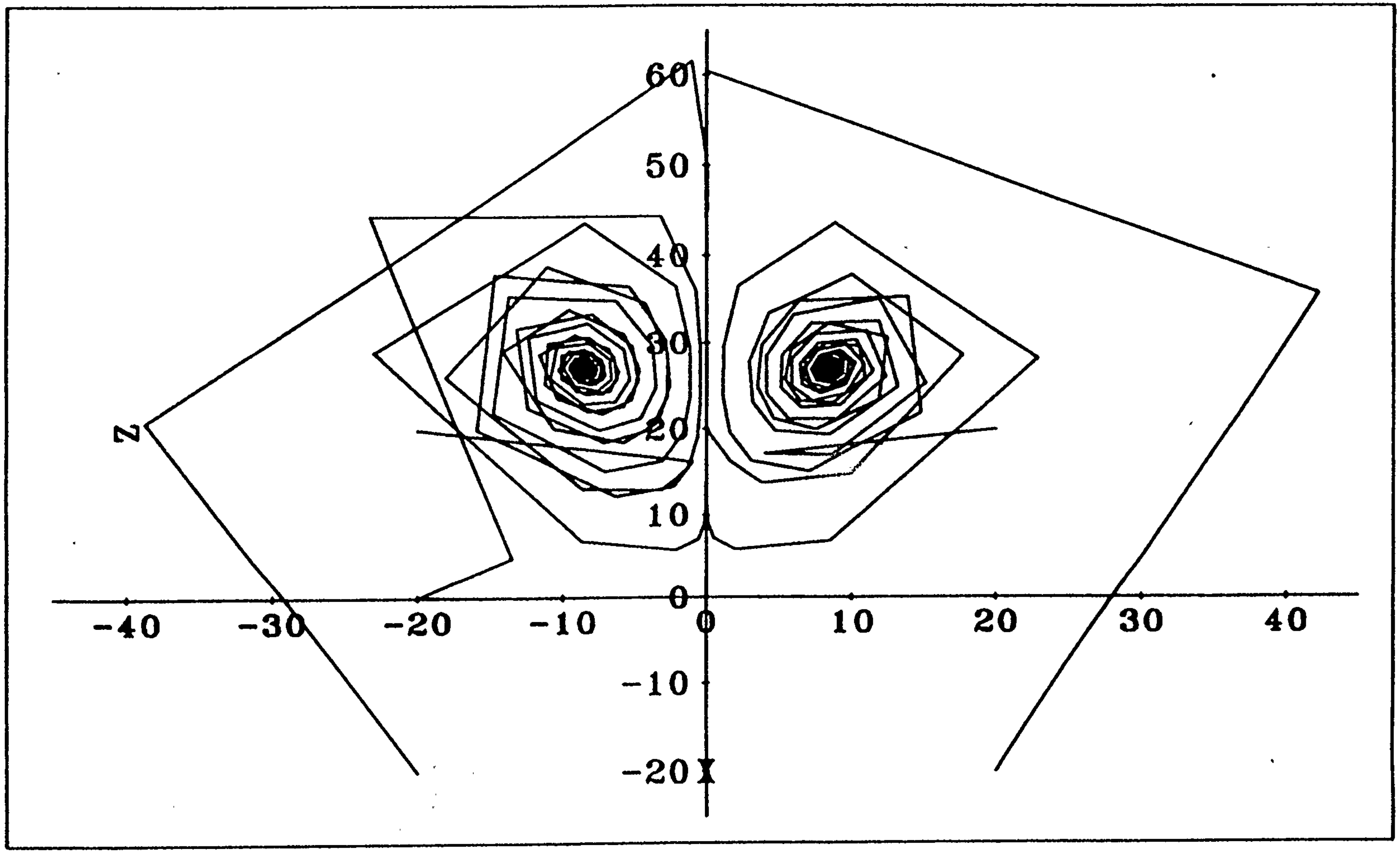


Fig 5.1.7b: Sigma=100.0, r=28.0, b=8/3

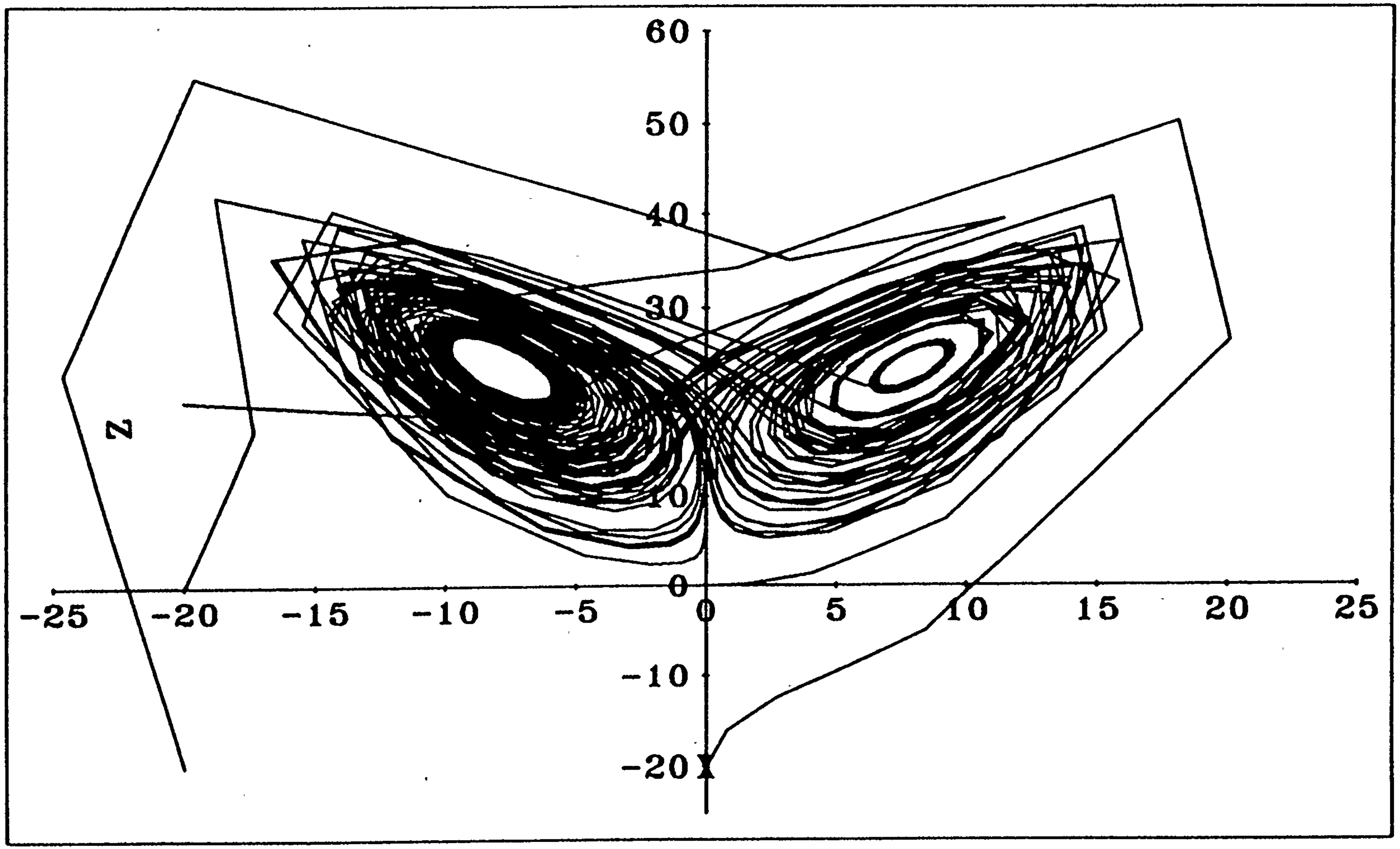


Fig 5.1.8: Sigma=100.0, r=24.74, b=8/3

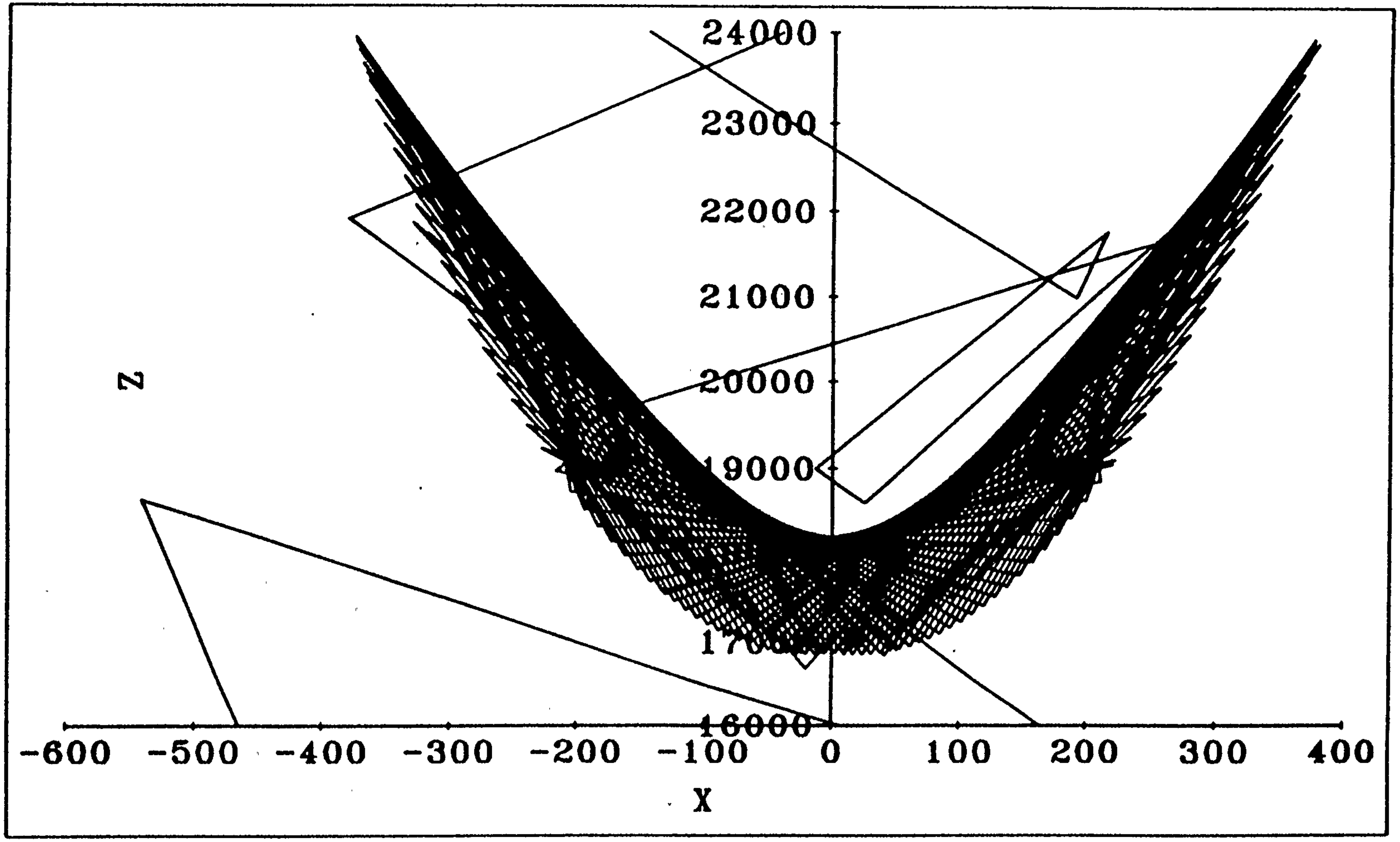


Fig 5.1.9a: Sigma=10.0, r=2e4, b=8/3

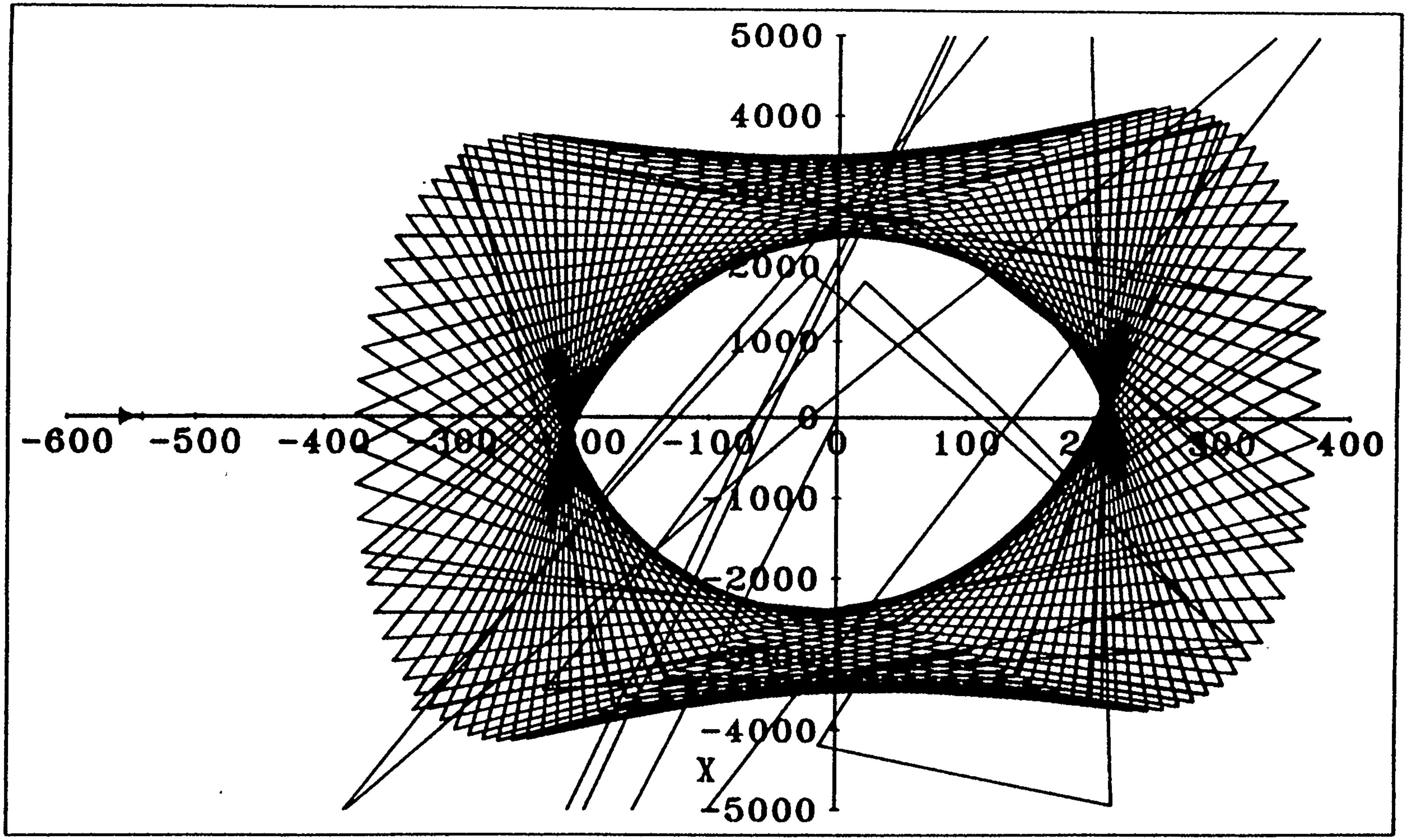


Fig 5.1.9b: Sigma=10.0, r=2e4, b=8/3

5.2 DUFFING MODEL

5.2.1 INTRODUCTION

Duffing's model is a second order differential equation with a cubic non-linearity and harmonic forcing function. In general form it is represented as

$$\ddot{x} + k\dot{x} + \alpha x + \beta x^3 = f(t) \tag{5.2.1}$$

It is a classical model for studying nonlinear effects in elastic mechanical systems e.g. a mass on a spring with nonlinear stiffness. The equation also describes certain electrical circuits for k = f(t) = 0, in this case, it reduces to,

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{5.2.2}$$

Depending upon the different values of parameters k, α, β, and the forcing function f(t), various authors e.g. Ueda (1979), and Guckenheimer and Holmes (1983) have explained different phenomenon. Moon and Holmes (1979) studied the Duffing equation in the form :

$$\ddot{x} + k\dot{x} - x + x^3 = b\cos\omega t \tag{5.2.3}$$

which corresponds to forced vibrations of a cantiliver beam in the presence of a uniform magnetic field generated by two permanent magnets. A sketch of the experimental setup as used by Moon and Holmes (1979) is shown in Figure 5.2.1.

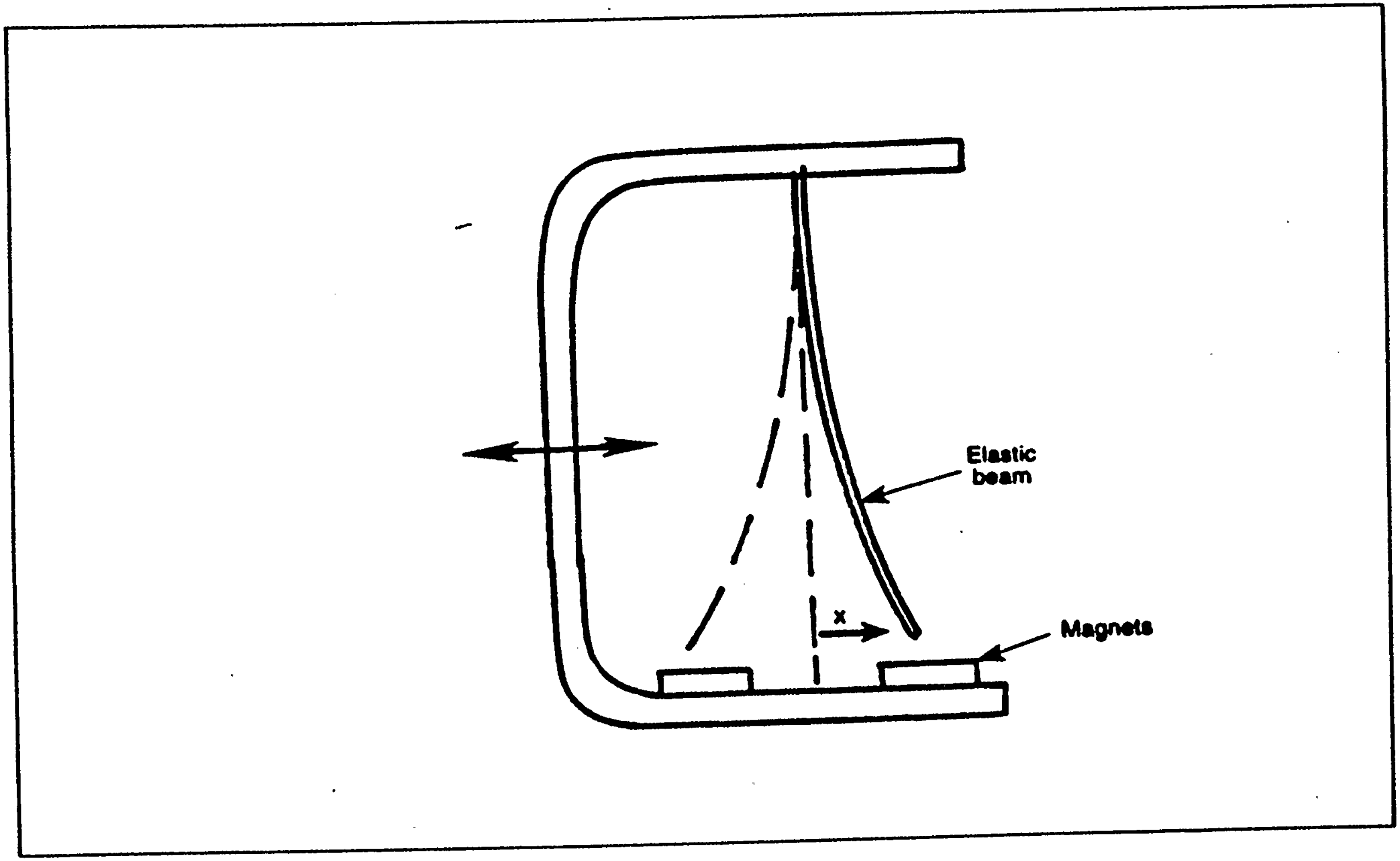


Fig 5.2.1 : Moon's Experimental Model

5.2.2 ANALYSIS

Consider the following form of Duffing equation in which the forcing function is zero

$$\ddot{x} + k\dot{x} + \alpha x - \beta x^3 = 0 \qquad (5.2.4)$$

which corresponds to a mass hanging with nonlinear stiffness and damping coefficient is proportional to the velocity. The system is transformed to two first order equations as :

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \beta x^3 - \alpha x - ky \end{aligned} \qquad (5.2.5)$$

which can be written in matrix form by separating the linear and the non-linear part, as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & -k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \beta x^3 \end{bmatrix} \qquad (5.2.6)$$

The forcing functions are :

$$X = y \qquad (5.2.7a)$$

$$Y = \beta x^3 - \alpha x - ky \qquad (5.2.7b)$$

Equating the forcing functions to zero for singular points, we obtain

$$y=0$$

Substituting in equation (5.2.7b) we find

$$x = 0, \pm\sqrt{\frac{\alpha}{\beta}}.$$

Hence there are three singular points $(-\sqrt{\frac{\alpha}{\beta}},0)$, $(0,0)$, $(\sqrt{\frac{\alpha}{\beta}},0)$.

The characteristic equation for (0,0) will be

$$\begin{vmatrix} -\lambda & 1 \\ -\alpha & -k-\lambda \end{vmatrix} = 0 \qquad (5.2.8)$$

$$\textit{which yields}\ \lambda = \frac{-k \pm \sqrt{k^2 - 4\alpha}}{2}$$

With each eigen value at each singular point, we can obtain eigen vectors to establish the direction of any representative point.

We apply Bendixon's criterion to check for the possible existence of limit cycles.

$$\nabla.V = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x} = -k$$

Since k is a real positive parameter, the sign of $\nabla.V$ does not change and hence there are no limit cycles.

Using Routh-Hurwitz criterion the conditions of stability can be determined from equation (5.2.8), which are :

$$1 > 0, \quad k > 0, \quad \text{and} \quad \alpha > 0$$

From equation (5.2.8) it can be inferred that if $k^2 - 4\alpha < 0$ then the equilibrium point will be a node, and for $k^2 - 4\alpha < 0$ and k = 0, the equilibrium point will be a center. Using this information the trajectories starting at 12 different points, which are near the equilibrium points are drawn. Figure (5.2.2) satisfies the condition for an equilibrium point to be a node. Figure (5.2.3) is drawn for k = 0 and it can be observed that the equilibrium point is center, surrounded by a family of concentric curves. This condition can also be derived from the Bendixon's criterion.

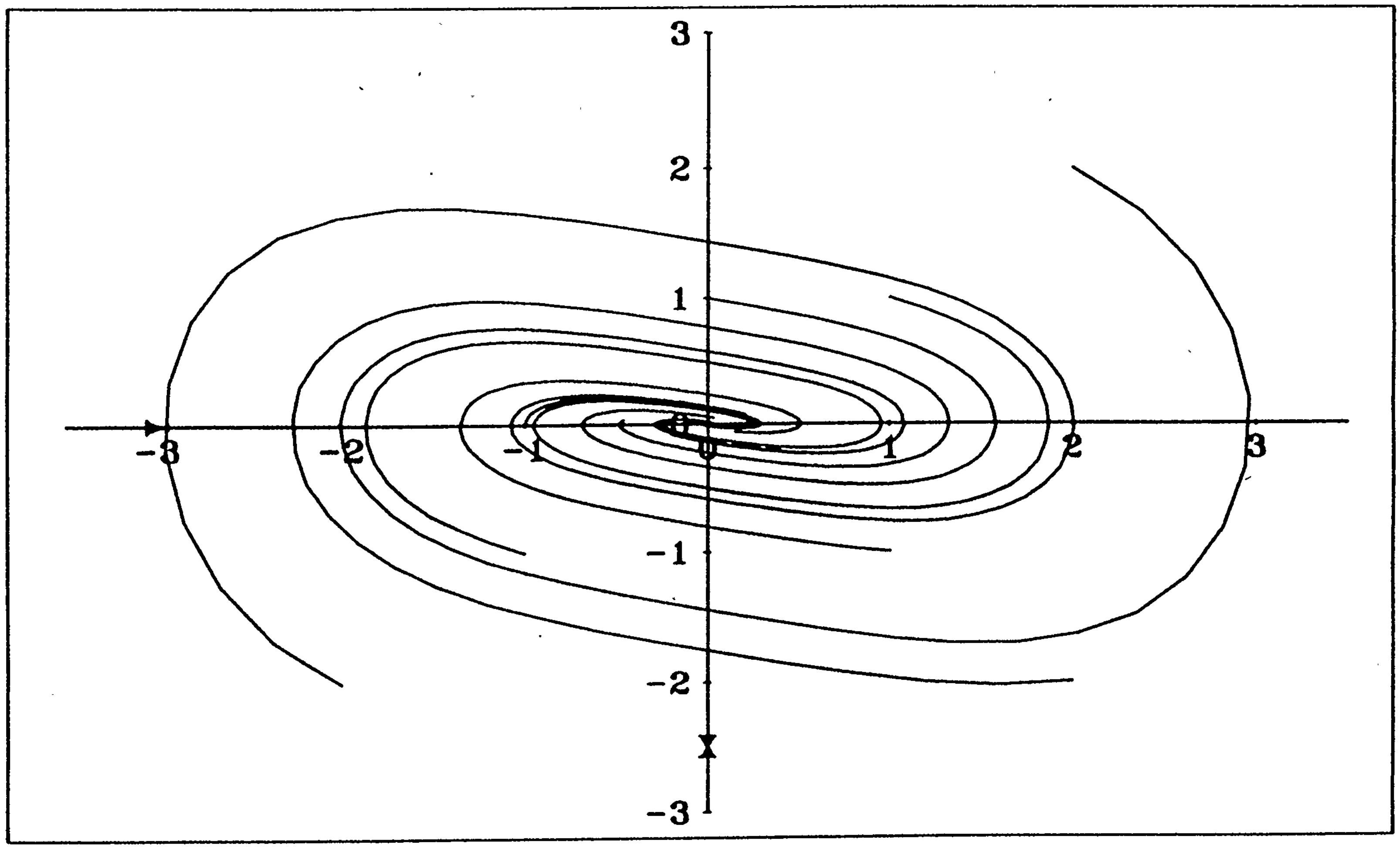


Fig 5.2.2 : k=0.25, α= 0.05, β= 0.1

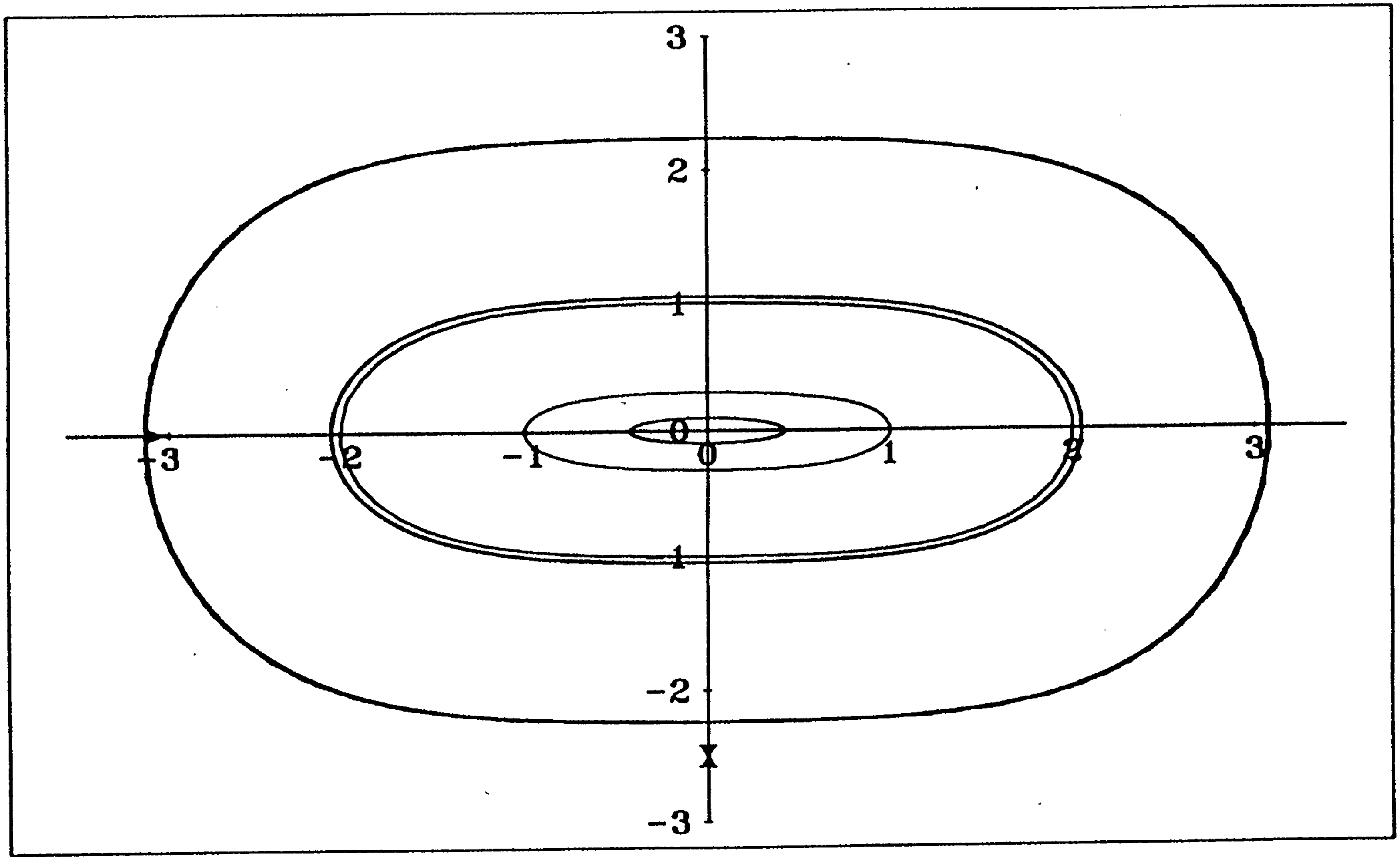


Fig 5.2.3 : k=0.00, α = 0.05, β =0.1

Consider another form of Duffing equation in which the forcing function is equal to a constant term plus a trigonometric term and with only cubic nonlinearity.

$$\ddot{x} + k\dot{x} + x^3 = b_0 + b_1 cost \tag{5.2.9}$$

The system is nonautonomous and to transform it to an autonomous system, we introduce another variable z which is a function of time t. The system then can be written in the form of three first order differential equations as :

$$\begin{aligned} \dot{x} &= y \\ \dot{z} &= 1 \\ \dot{y} &= -ky - x^3 + b_0 + b_1 \cos z \end{aligned} \tag{5.2.10}$$

Letting the forcing functions be zero for singular points, we obtain

$$\begin{aligned} y &= 0 \\ -ky - x^3 + b_0 + b_1 \cos z &= 0 \end{aligned} \tag{5.2.11}$$

which will give n singular points corresponding to n values of time t, i.e. there will be a singular point $((b_0+b_1\cos z)^{\frac{1}{3}},0)$ on each of planes z = t.

The eigen values are determined from the characteristic equation and are given by

$$\lambda_{1,2,3} = 0, \frac{-k \pm \sqrt{k^2 - 12x^2}}{2}$$

From Routh-Hurwitz criterion one finds that for stability

$$k > 0, \quad x^2 > 0$$

Following the above guidelines the model, figures are drawn for a number of different parameter values. It can be observed from the plots that an increase in the value of parameter k increases the density of trajectories in the phase plane. Figure (5.2.7) is a set of two trajectories starting from (0,0) and (10,10). The trajectory starting from (10,10) approaches the origin while the other trajectory starting from (0,0) diverges away from the origin.

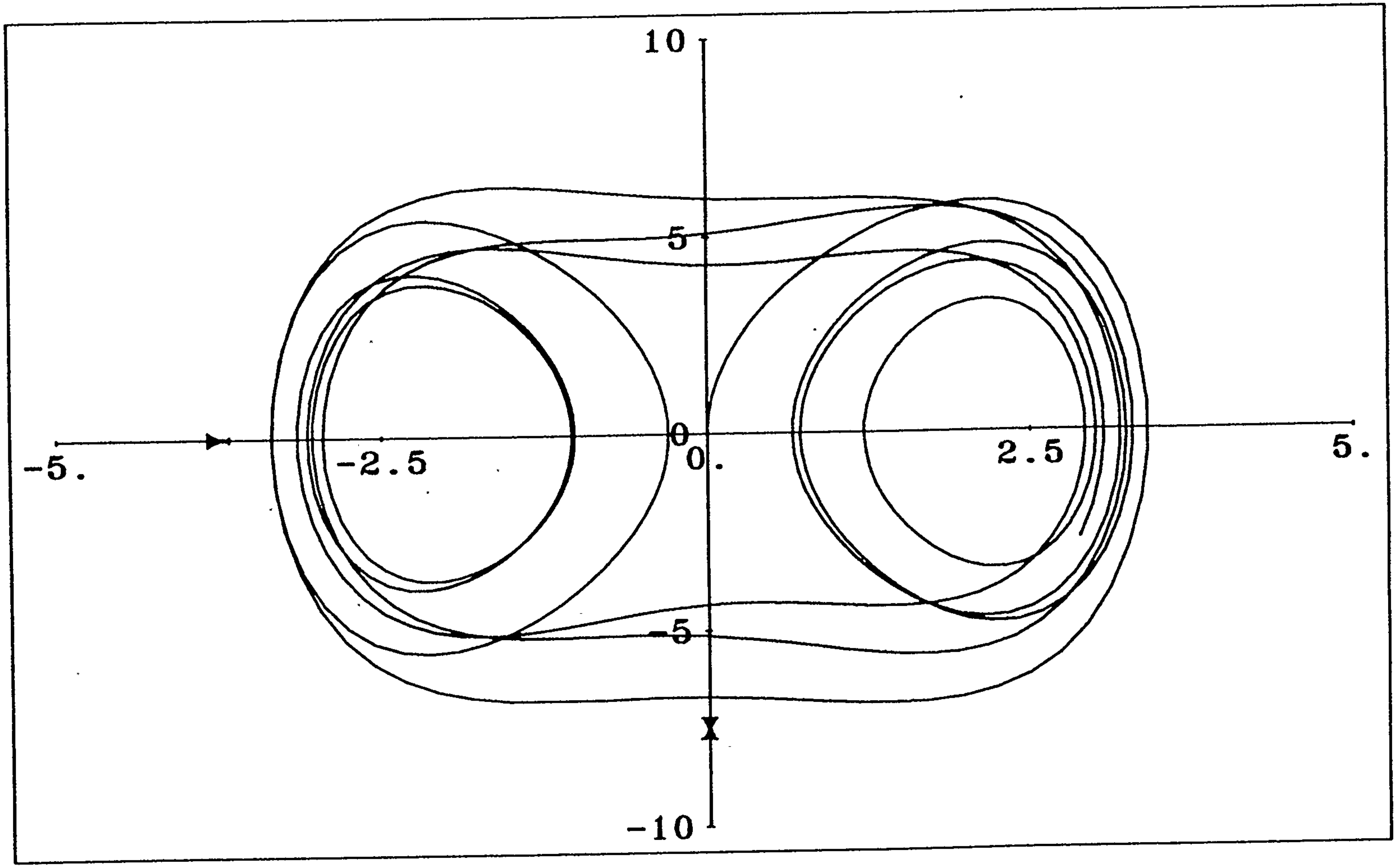


Fig 5.2.4 : Simulation of Duffing Model
k=0.05, b0 = 0.0, b1 = 12.0

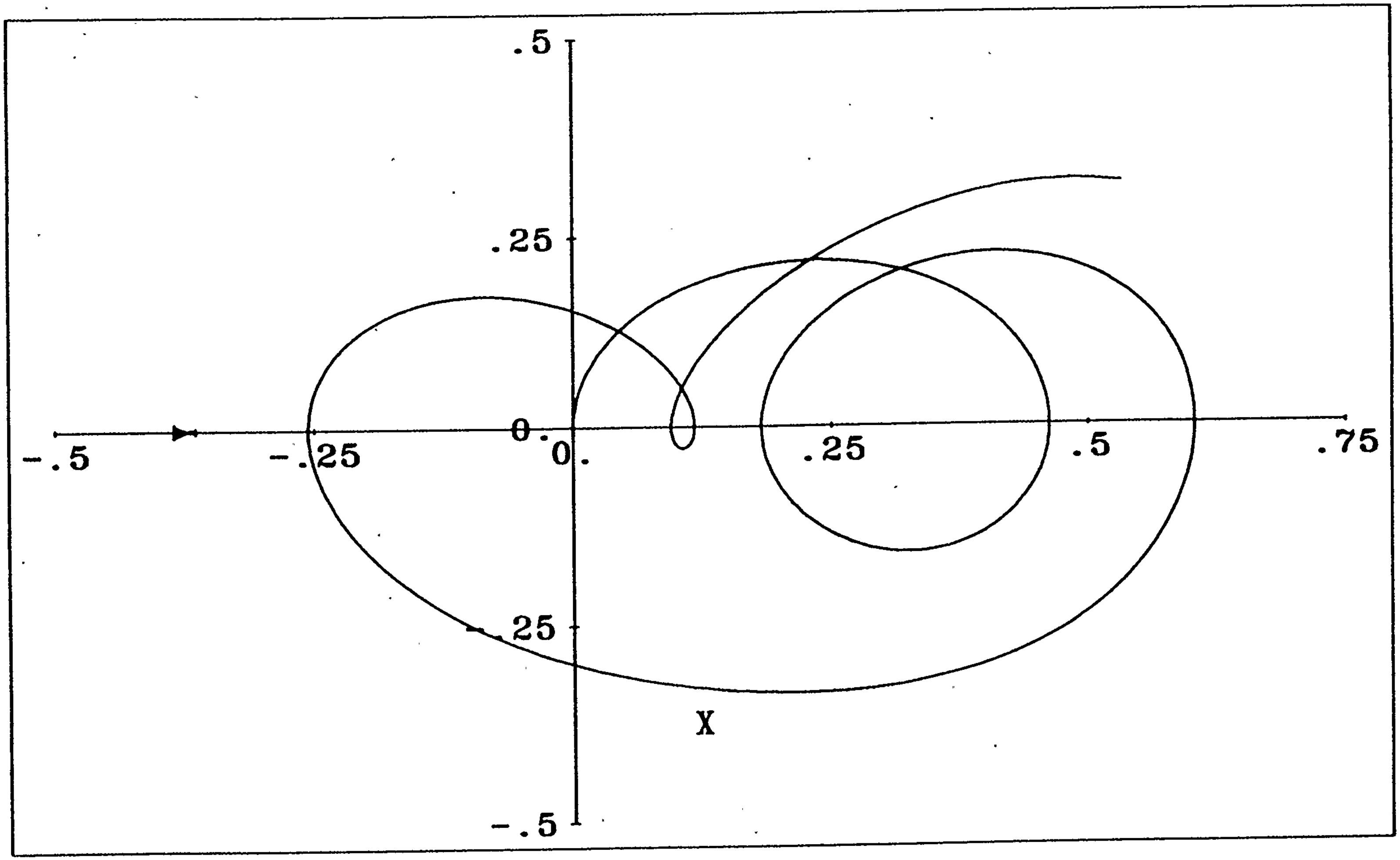


Fig 5.2.5 : k=0.05, b0=0.045, b1=0.16

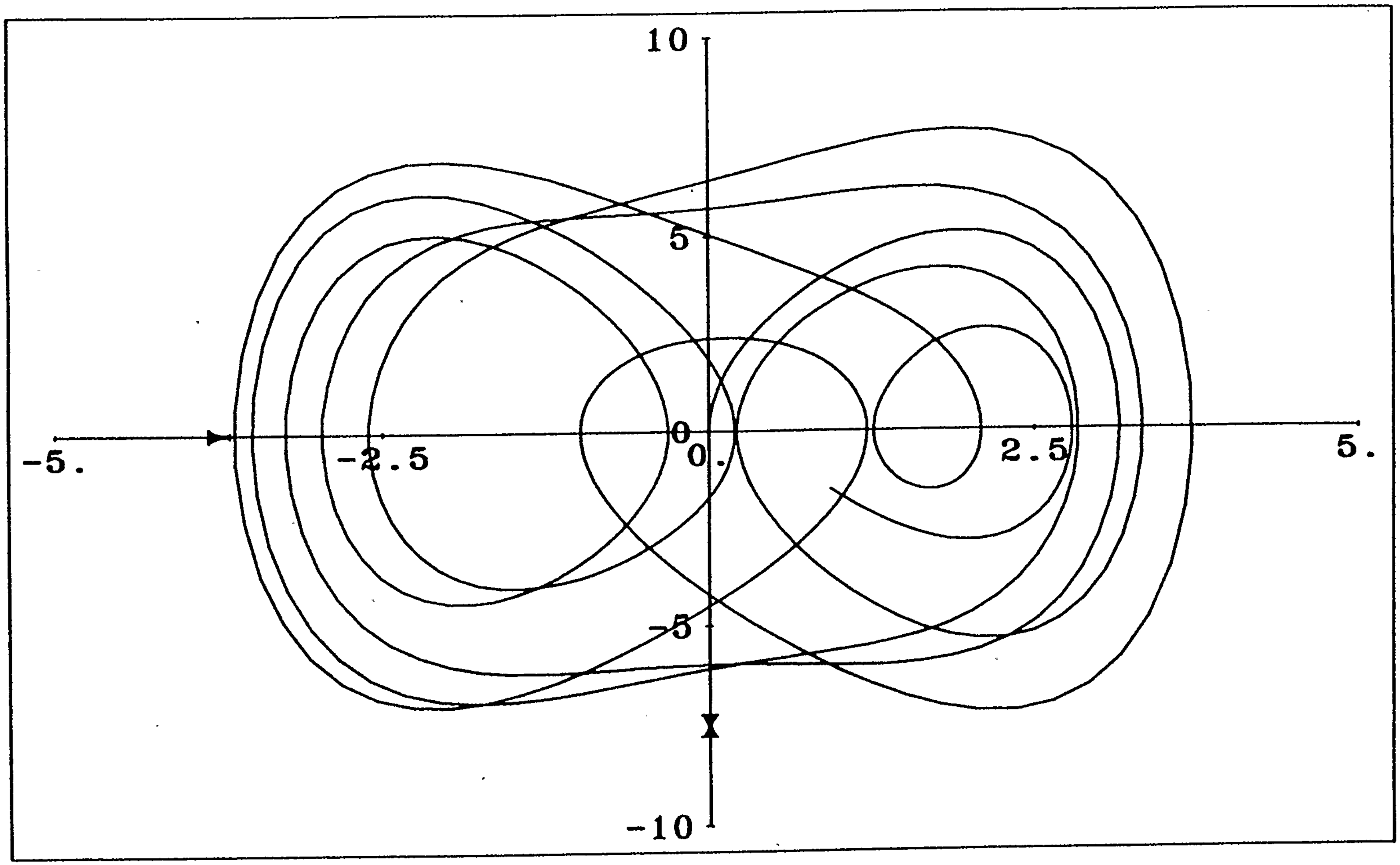


Fig 5.2.6 : k=0.1, b0=0.0, b1=10.0

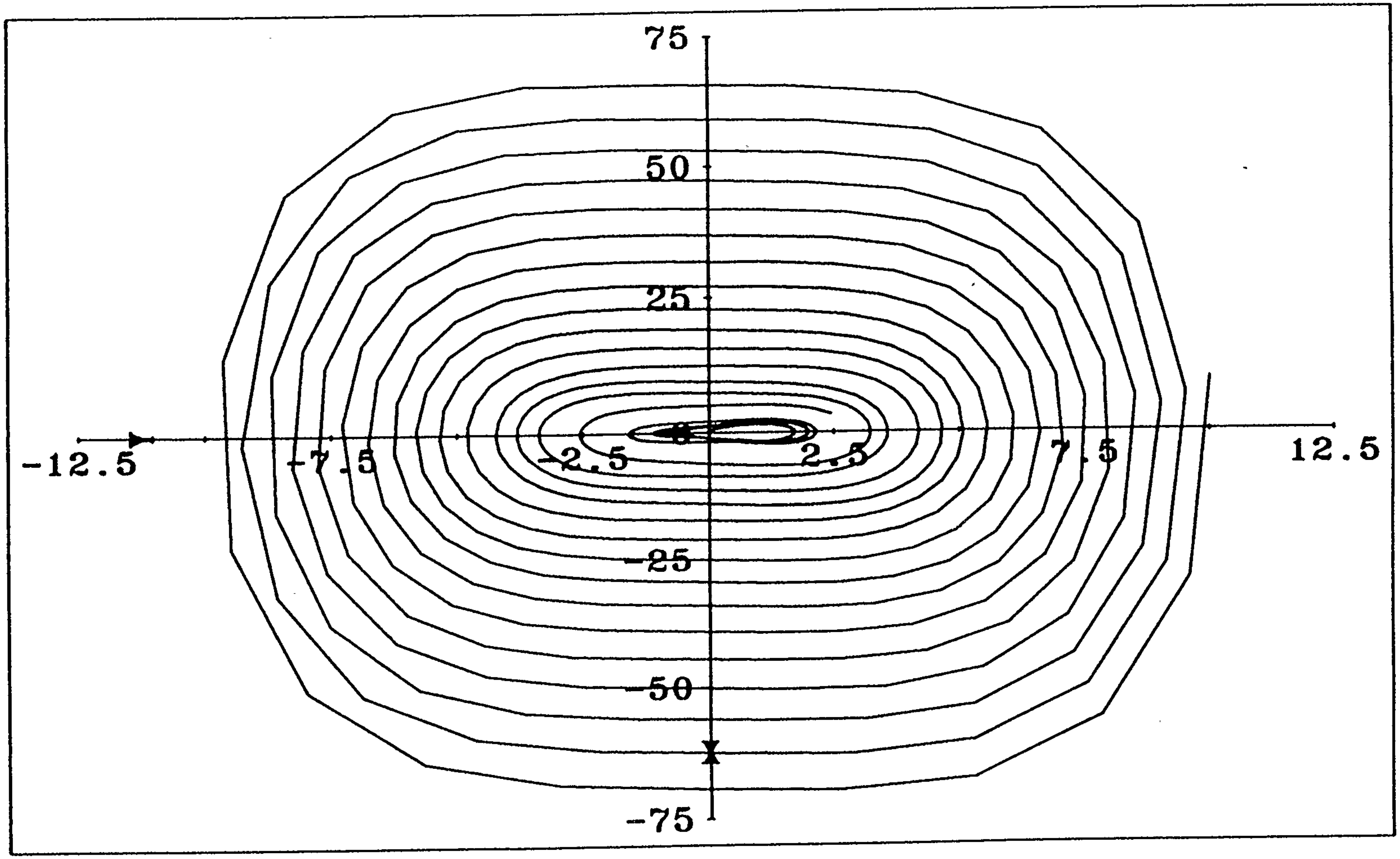


Fig 5.2.7 : k=0.2, b0=0.85, b1=1.2

5.3 ROSSLER MODEL

5.3.1 INTRODUCTION

There are several Rossler models available in the literature. The model considered here is a three dimensional flow system in which only one quadratic nonlinearity is present.

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b - cz + xz \end{aligned} \tag{5.3.1}$$

which is represented in matrix form as :

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b+xz \end{bmatrix} \tag{5.3.2}$$

5.3.2 ANALYSIS

The singular points are calculated to be :

$$\left(\frac{-c \pm \sqrt{c^2+4ab}}{2}, \frac{-(-c \pm \sqrt{c^2+4ab})}{2a}, \frac{-c \pm \sqrt{c^2+4ab}}{2a} \right) \tag{5.3.3}$$

The characteristic equation is

$$\begin{vmatrix} -\lambda & -1 & -1 \\ 1 & a-\lambda & 0 \\ z & 0 & -c+x-\lambda \end{vmatrix} = 0 \tag{5.3.4}$$

Consider a set of parametric values as a = 0.343, b = 1.82, and c = 9.75, the singular points will be (9.81,28.61,-28.61), and (-9.81,-28.61,28.61). For the first singular point the eigen values are determined by using IMSL subroutine EIGRF are found as

$$\lambda_1 = -17.97,\ \lambda_2 = -0.62+0.39i,\ \lambda_3 = -0.62-0.39i$$

The eigen vectors are determined as

$$\begin{bmatrix} -1.00 \\ 0.54 \\ 18.02 \end{bmatrix} + i \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 1.00 \\ 0.88 \\ 1.51 \end{bmatrix} + i \begin{bmatrix} 0.0 \\ -0.36 \\ -0.31 \end{bmatrix}, \begin{bmatrix} 1.00 \\ -0.88 \\ 1.51 \end{bmatrix} + i \begin{bmatrix} 0.0 \\ 0.36 \\ 0.31 \end{bmatrix}$$

The existence of limit cycles is established by Bendixon's criterion i.e. $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} = a-c$. If c > a, there will be no limit cycles but if a ≥ c, there may exist limit cycles.

The stability conditions according to Routh-Hurwitz criterion, from equation (5.3.4) are

1. 1 > 0
2. c - a > 0
3. 1 - ac > 0
4. c > 0
5. c^2 - ac - 1 > 0

Conditions 1 and 4 are automatically satisfied as parameters are real positive numbers. From condition 2, $c > a$, and from condition 3, $ac < 1$ for the solution to be stable.

The behavior of the system has been shown for two sets of parameters, for trajectories starting from (0,0), (10,10), and (-10,-10). Both the solutions are unstable as all the conditions of stability are not satisfied. The system is sensitive to initial conditions as the behavior of trajectories starting from different initial conditions is altogether different, though system parameters remain unchanged.

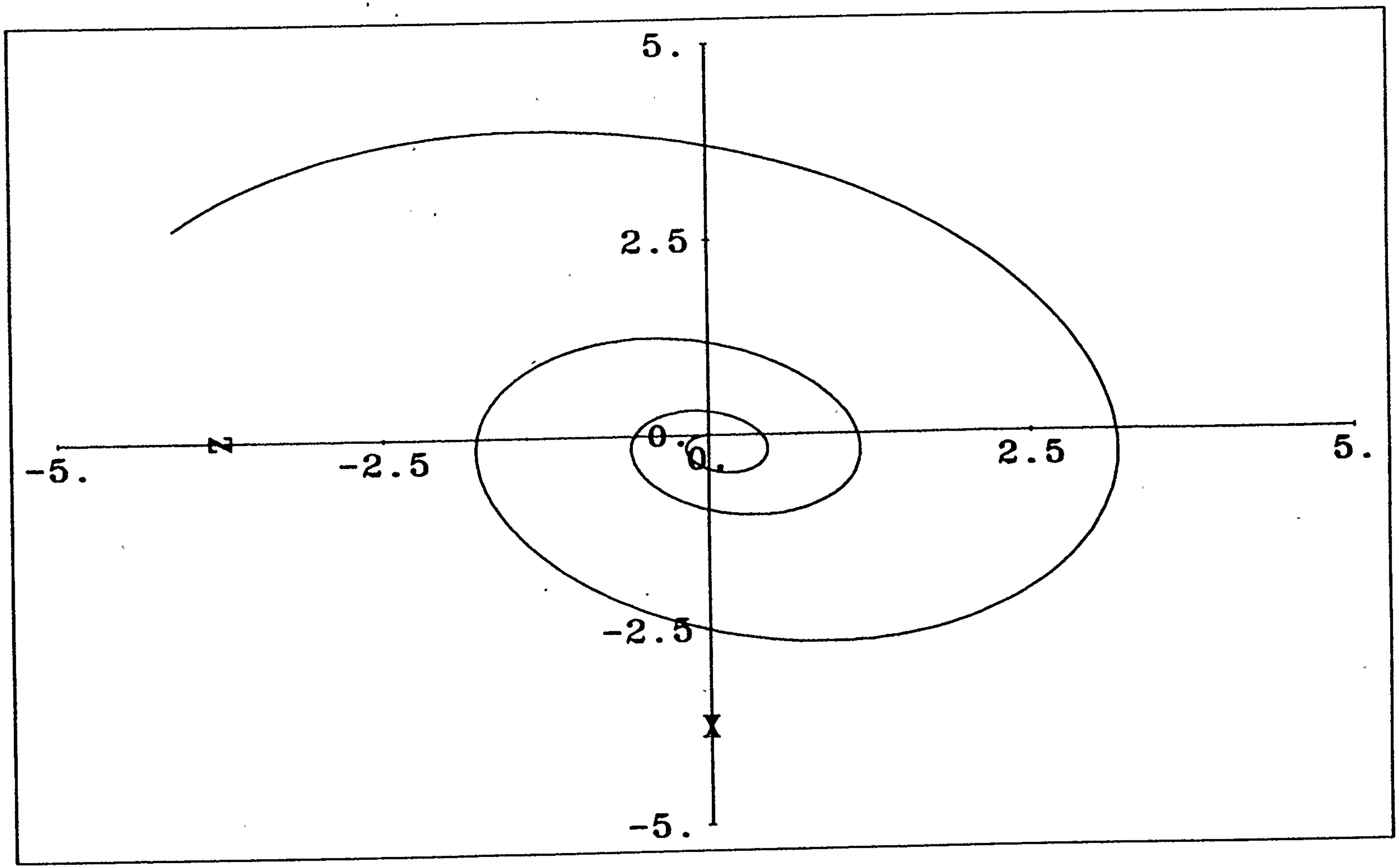


Fig 5.3.1a:Simulation of Rossler Model
a=0.343, b=1.82, c=9.75

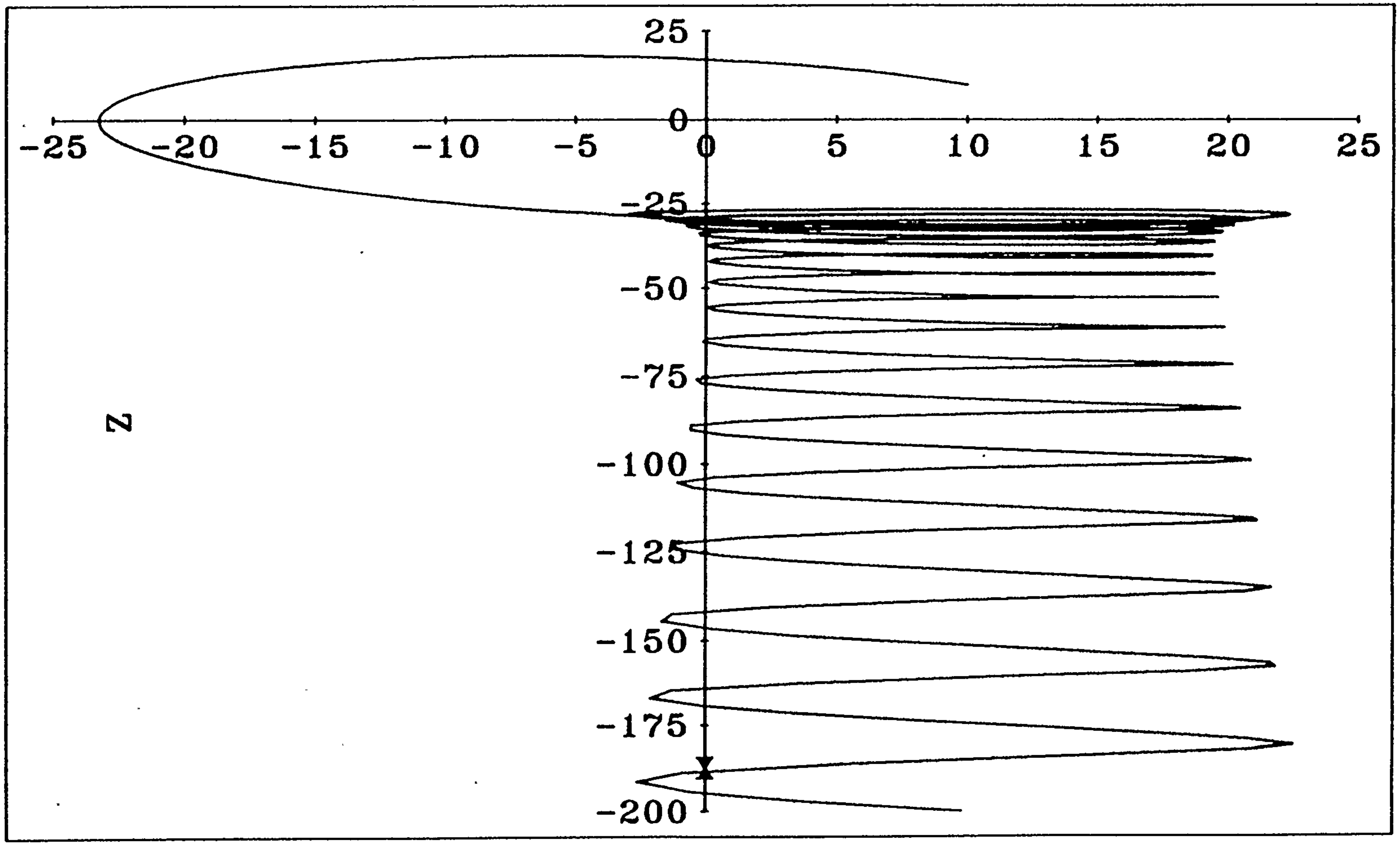


Fig 5.3.1b : Trajectory from 10,10
a=0.343, b=1.82, c=9.75

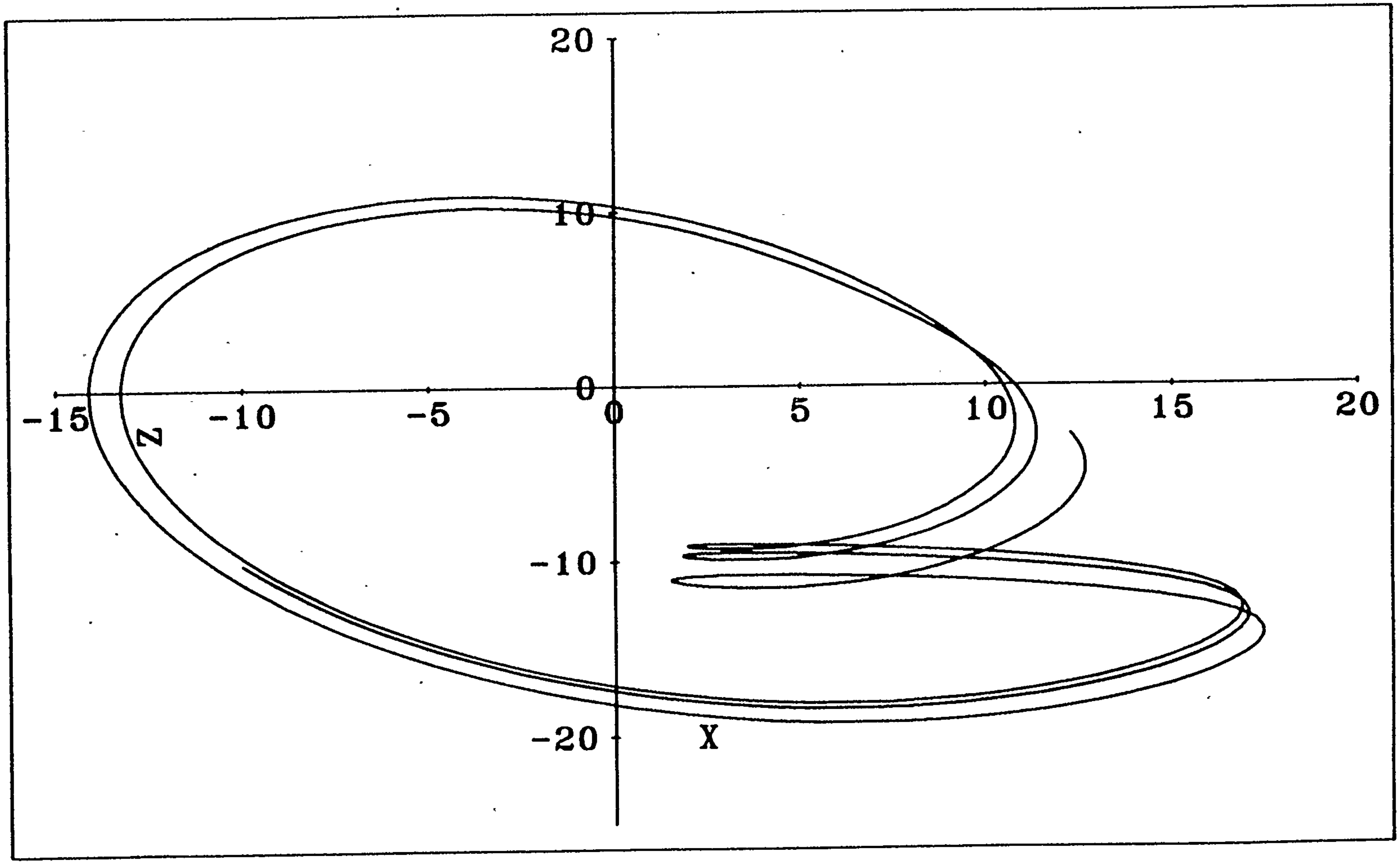


Fig 5.3.1c : Trajectory from -10,-10
a=0.343, b=1.82, c=9.75

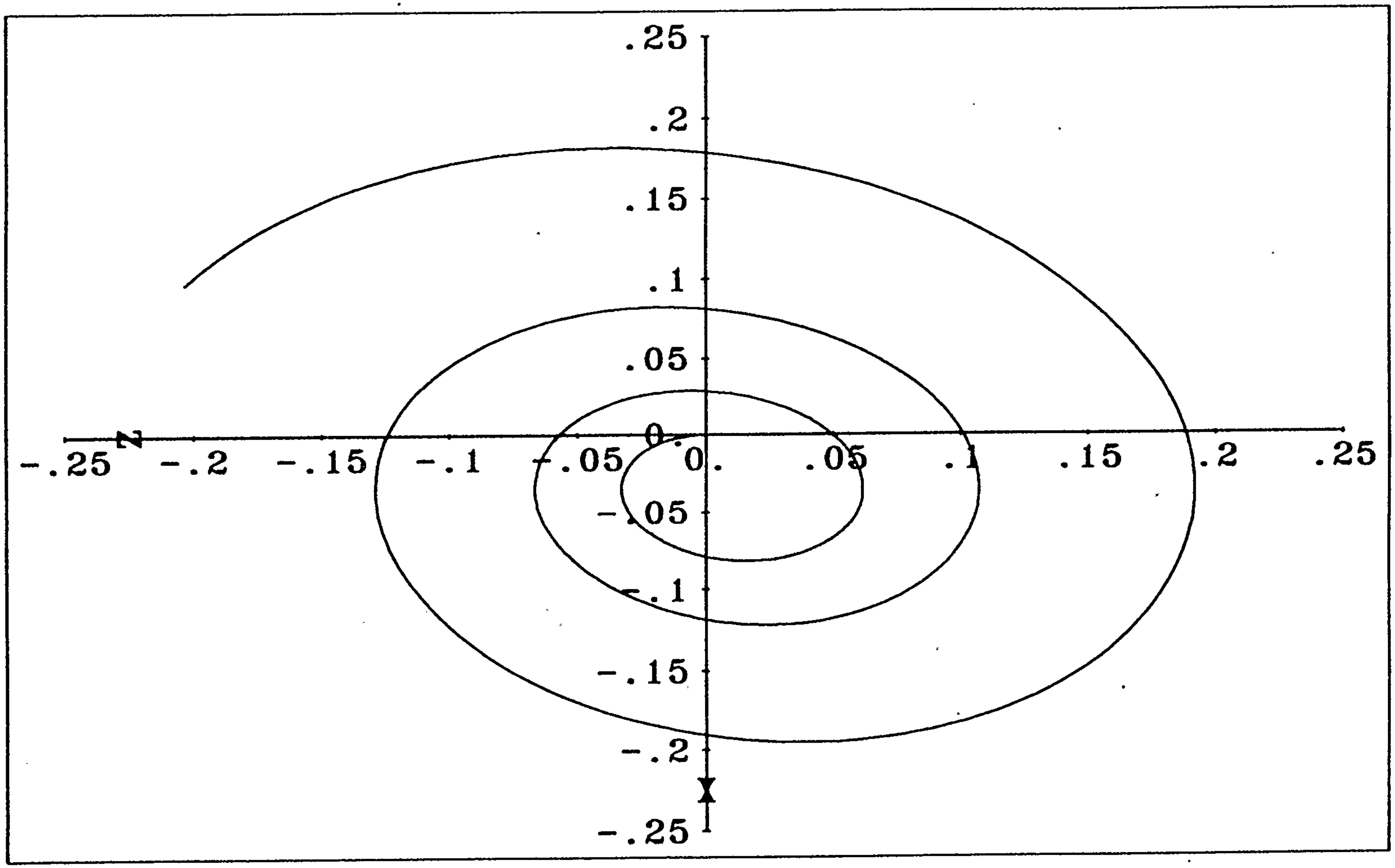


Fig 5.3.2a : Trajectory from 0,0
a=0.2, b=0.2, c=5.7

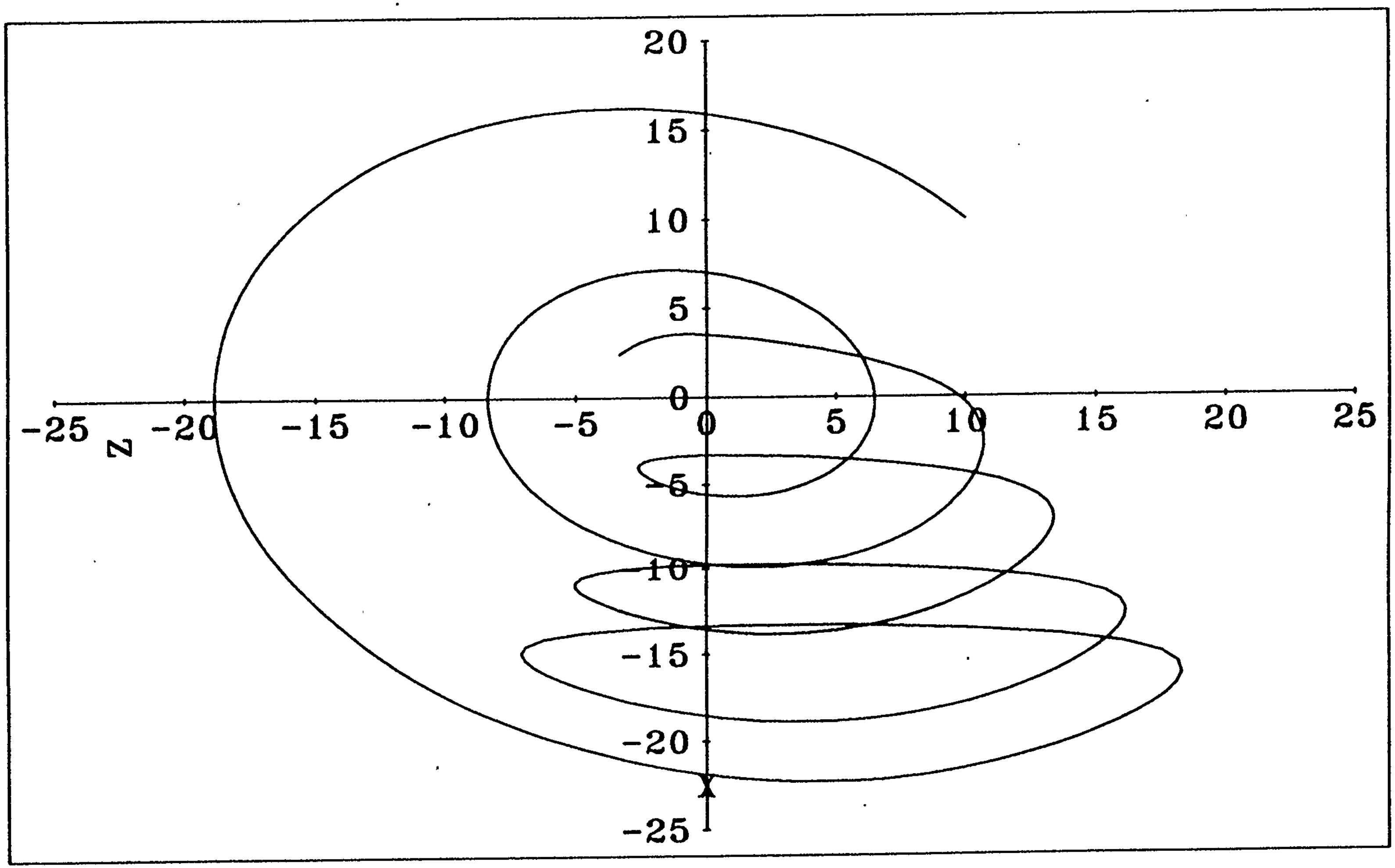


Fig 5.3.2b : Trajectory from 10,10
a=0.2, b=0.2, c=5.7

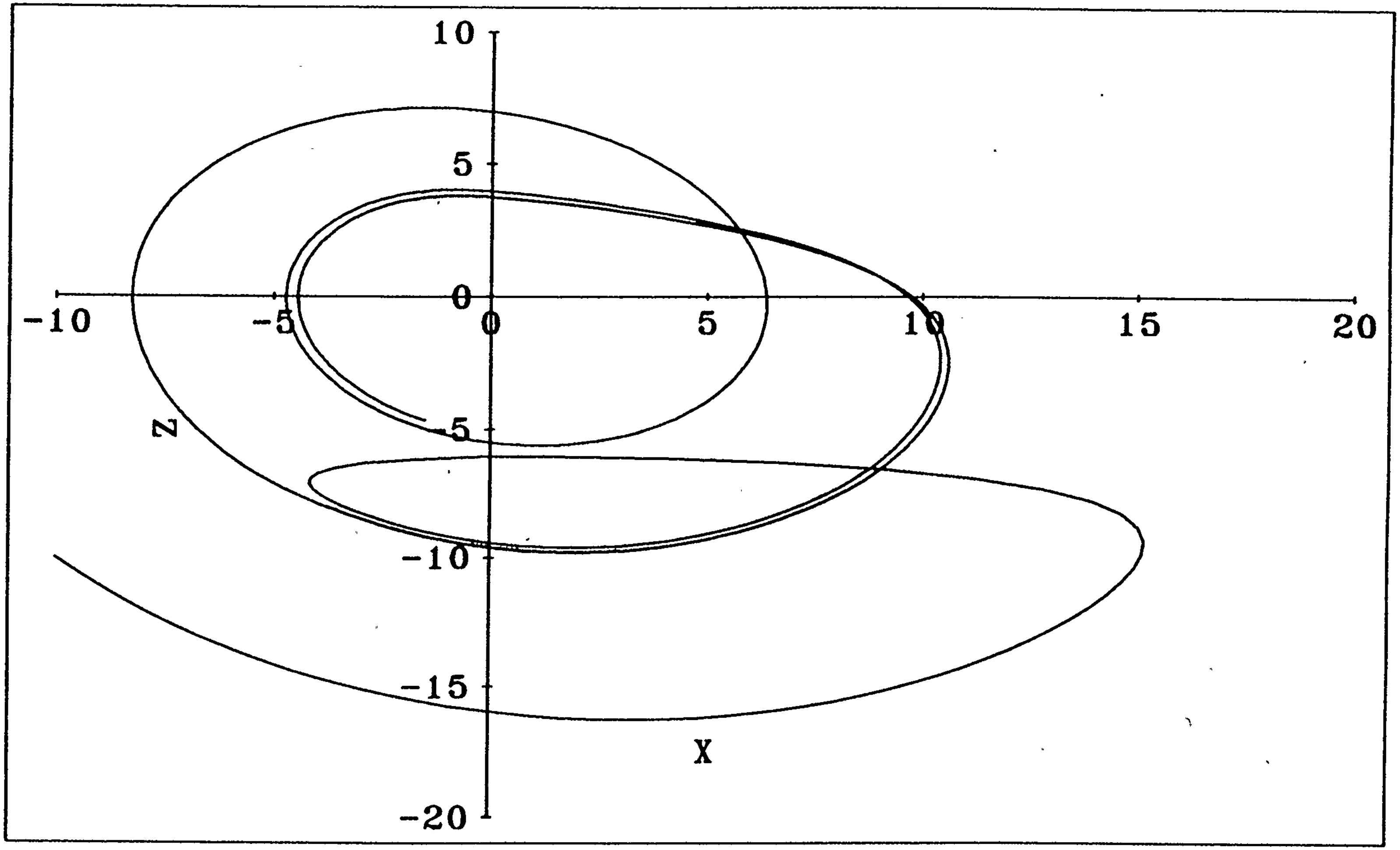


Fig 5.3.2c : a=0.2, b=0.2, c=5.7

5.4 REACTOR STABILITY MODEL

5.4.1 Introduction

The idea of stability in a chemical reactor revolves around the operating conditions inside a chemical reactor. Following two possibilities arise :

1. If the reaction is exothermic, the rate of heat removal should balance the rate of heat generation at steady state, so that there are no hot spots which are regions of excessive local temperature in the system. Hot spot can change the kinetics of a chemical reaction and can be damaging to the reactor equipment and catalyst. Conversely if the heat removal rate is more than the heat generation rate, it will result in a slow down of the chemical reaction and the catalysts in the system may loose their activity.

2. If the reaction is endothermic, constant rate of heat supply has to be maintained so that the kinetics is not affected. Also the catalysts may get deactivated thermally, which is called sintering.

For example if temperature is not controlled in case of conversion of Napthalene to Phthalic Anhydride, the end products of the chemical reaction may be altogether different e.g. Carbon-dioxide, and water.

In our analysis it is assumed that the model is already in existence, that is we have a set of ordinary differential equations for heat balance, mass balance, and kinetics of chemical reaction. Essentially there are three types of chemical reactors

1. Batch Reactor
2. Tubular Reactor
3. Continuous Stirred Tank Reactor

In batch reactor all the reactants are put in a reactor and mixed. There is no continuous flow of reactants into the reactor or of products out of the reactor.

In a tubular reactor there is a steady movement of reactants in a particular direction and they come out as products. No mixing is done by external forces.

As the name signifies, a continuously stirred tank reactor (CSTR) is the one in which there is continuous flow of reactants and products in and out of the reactor respectively. As soon as the reactants enter the reactor, they are immediately mixed with the material inside the reactor to form a homogeneous mixture, and this is the most important characteristic of CSTR.

The model considered here is for a CSTR for which the basic equations as shown by Aris (1979) and Permutter (1972) are as follows.

The mass balance can be represented as :

$$\begin{bmatrix} \textit{Rate of} \\ \textit{accumulation} \end{bmatrix} = \begin{bmatrix} \textit{Rate of} \\ \textit{infeed} \end{bmatrix} - \begin{bmatrix} \textit{Rate of} \\ \textit{outfeed} \end{bmatrix} - \begin{bmatrix} \textit{Rate of consumption} \\ \textit{or generation} \end{bmatrix}$$

$$V\frac{dC}{d\tau} \;=\; q(C_o - C) - VR \tag{5.4.1}$$

The energy balance can be represented as :

$$\begin{bmatrix} \textit{Rate of heat} \\ \textit{accumulation} \end{bmatrix} = \begin{bmatrix} \textit{Heat of} \\ \textit{reactants} \end{bmatrix} - \begin{bmatrix} \textit{Heat of} \\ \textit{products} \end{bmatrix} +$$

$$\begin{bmatrix} \textit{Rate of Heat generation} \\ \textit{or consumption} \\ \textit{by products} \end{bmatrix} - \begin{bmatrix} \textit{Rate of Heat removed} \\ \textit{by coolant} \end{bmatrix}$$

$$Vc_p\rho\frac{dT}{d\tau} \;=\; qc_p\rho(T_o - T) + \Delta HVR - U(T - T_c) \tag{5.4.2}$$

Equations (5.4.1) and (5.4.2) with initial conditions will give concentration and temperature in the reactor at any instant of time. The steady state will be achieved when

$$\frac{dC}{d\tau} \;=\; 0$$

$$\frac{dT}{d\tau} \;=\; 0$$

If T_s is the steady state temperature and C_s being the steady state concentration, the steady state equations will be :

$$q(C_o - C_s) - VR \;=\; 0$$

$$qc_{p_s}\rho(T_o - T_s) + \Delta HVR - U(T_s - T_c) \;=\; 0 \tag{5.4.3}$$

5.4.2 ANALYSIS

The stability can be analyzed by the transient equations for the mass balance and the energy balance as given by Aris (1979).

$$V\frac{dC}{d\tau} = qC_o - qC - VCpe^{-\frac{E}{RT}} \tag{5.4.4}$$

$$Vc_p\rho\frac{dT}{d\tau} = qc_p\rho(T_o-T) - pe^{-\frac{E}{RT}}CV(\Delta H) - U(T-T_c) \tag{5.4.5}$$

Equations (5.4.4) and (5.4.5) are nonlinear in nature due to the exponential term. Since no direct method is available to find a straight solution, the perturbation method is used. Let c and t be the perturbations in concentration and temperature from the steady state solutions C_s and T_s respectively. Then at any time if the concentration and temperature in the reactor are C and T, then

$$\begin{aligned} c &= C_s - C \\ t &= T_s - T \end{aligned} \tag{5.4.6}$$

Equations (5.4.4) and (5.4.5) are expanded by Taylor series and higher order terms of perturbations are neglected. We obtain simplified equations as

$$\frac{dc}{d\tau} = -a_{11}c - a_{12}t \tag{5.4.7}$$

$$\frac{dt}{d\tau} = -b_{21}c - b_{22}t \tag{5.4.8}$$

where

$$a_{11} = \frac{q}{V} + pe^{-\frac{E}{RT_s}}$$

$$a_{12} = C_s\frac{E}{RT_s^2}pe^{-\frac{E}{RT_s}}$$

$$b_{12} = \frac{(\Delta H)}{c_p \rho} pe^{-\frac{E}{RT_s}}$$

$$b_{22} = \frac{q}{V} + \frac{(\Delta H)}{c_p \rho} pe^{-\frac{E}{RT_s}} + \frac{U}{Vc_p \rho} \qquad (5.4.9)$$

The characteristic equation is obtained as

$$\lambda^2 + (a_{11}+b_{22})\lambda + a_{11}b_{22} - a_{12}b_{21} = 0 \qquad (5.4.10)$$

By Routh-Hurwitz criteria, the system will be stable if the following inequalities are satisfied.

$$\frac{2q}{V} + pe^{-\frac{E}{RT_s}} + \frac{U}{Vc_p \rho} + \left[\frac{-\Delta H}{c_p \rho}\right] pe^{-\frac{E}{RT_s}} \left[C_s \frac{E}{RT_s^2}\right] > 0 \qquad (5.4.12)$$

$$\frac{q}{V} - \left[\frac{-\Delta H}{c_p \rho}\right] C_s \frac{E}{RT_s^2} pe^{-\frac{E}{RT_s}} + pe^{-\frac{E}{RT_s}} + \left[\frac{V}{qc_p \rho}\right] pe^{-\frac{E}{RT_s}} > 0 \qquad (5.4.13)$$

Consider a first order irreversible exothermic reaction in a CSTR with parametric value fixed as:

$$q = 1\frac{cm^3}{\text{sec}}, \quad V = 2000 cm^3, \quad U = 0.007\frac{cal}{c \text{ sec}}$$

$$-\Delta H = 10{,}000\frac{cal}{mole}, \quad E = 22{,}500\frac{cal}{c}\text{sec}, \quad T_o = 300^o K$$

$$p = 7.86e\,12\frac{1}{\text{sec}}, \quad \rho = 0.005\frac{gm}{cm^3}, \quad c_p = 0.025\frac{cal}{gm^o C}$$

A graph between rate of heat generation and rate of heat taken out is shown in Figure (5.4.1), for three different coolant temperatures ($282^o C$, $292^o C$, $306^o C$) keeping all other parameters same.

It is observed that the cooling curve for $T_c = 292^o C$ intersects the heat generation curve at three different points. The three equilibrium points, also called as steady states are found and tabulated.

Position	T_s ($^o C$)	C_s ($\frac{moles}{litre}$)
A	295	$3.75 X 10^{-3}$
B	323.5	$4.59 X 10^{-4}$
C	348.6	$4.04 X 10^{-5}$

When the values of the steady states are substituted in the inequalities (5.4.12) and (5.4.13), it is found that the values at A and C satisfy both the inequalities, while values at B does not. Hence A and C are stable nodes while B is a saddle point. Slight perturbations to either side will lead to an equilibrium position. A sketch of trajectories around equilibrium point is drawn and shown in Figure (5.4.2).

Since the system is a second order, it doesn't exhibit chaos. But for higher order systems when two or more second order reactions are involved, the heat generation curve will be complex and there can be five or more singular points. Such a system under certain parameter values may exhibit chaos.

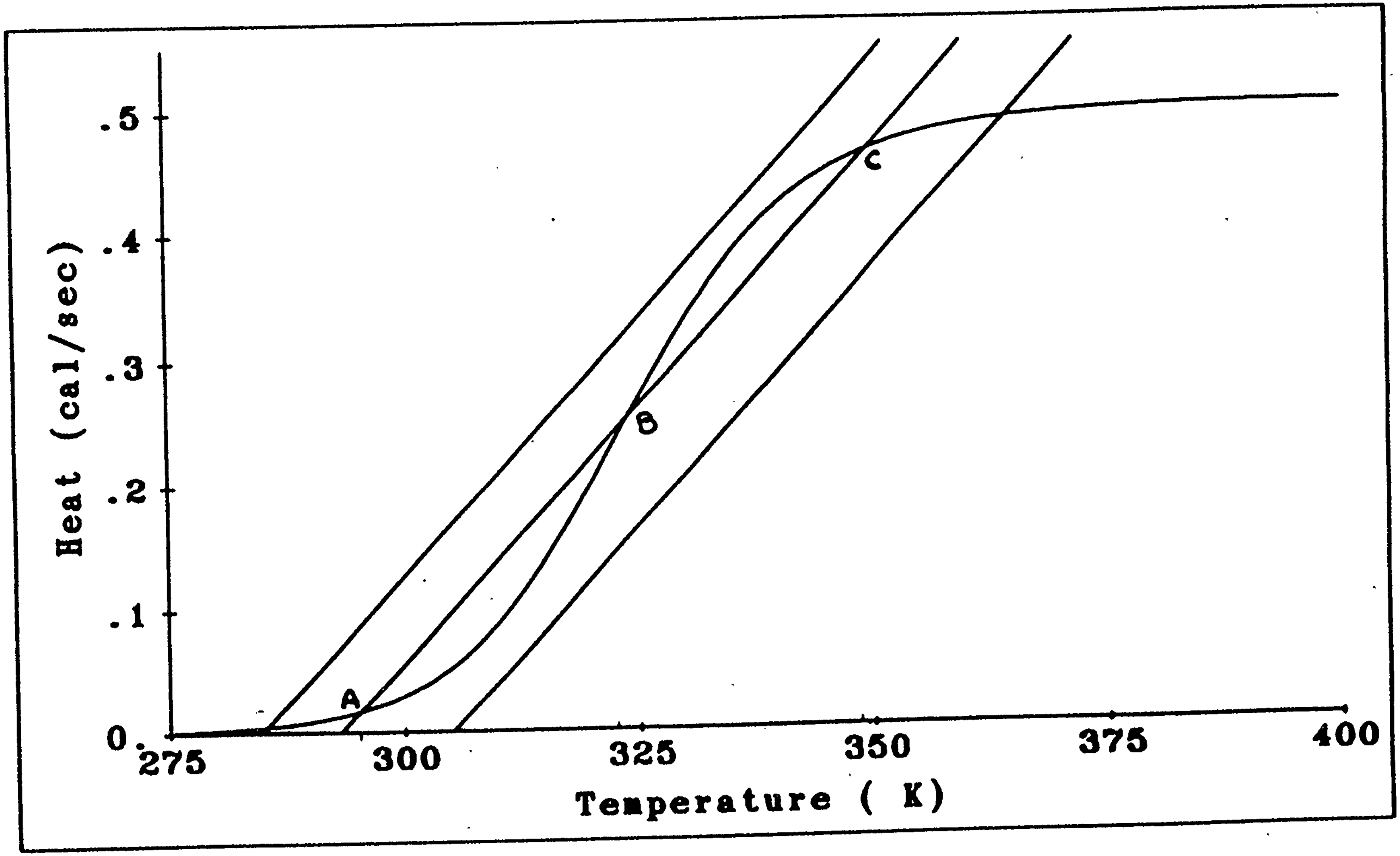


Fig 5.4.1 : Chemical Reactor Model

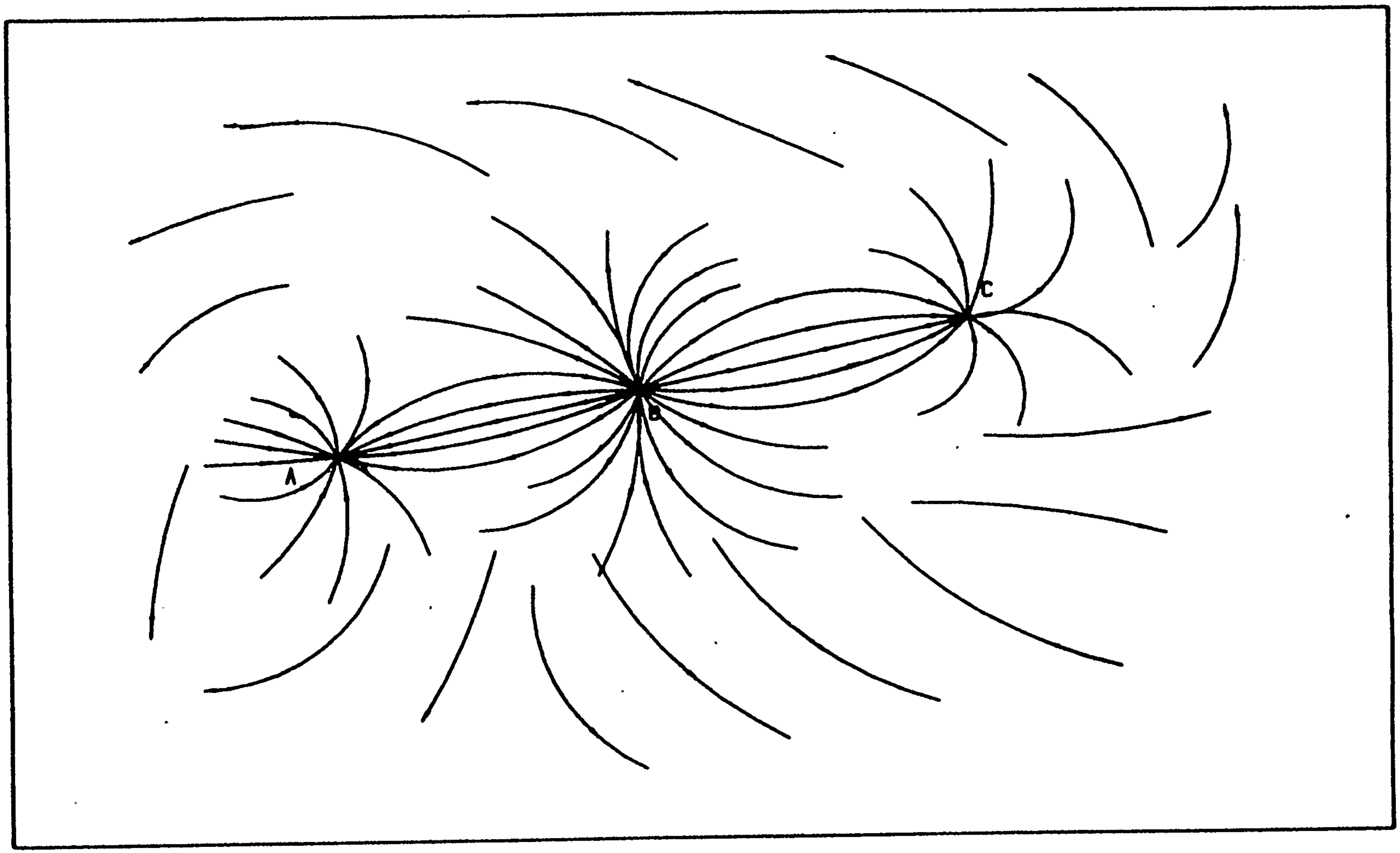


Fig 5.4.2 : Behavior of Trajectories

CHAPTER 6

FRACTALS

The term Fractals, was coined by Mandelbrot (1977) and means a set of irregular, and fragmented shapes. He referred a set as a fractal if its Hausdorff's dimension exceeds topological dimension. Lindquist (1988) called all those strange attaractors as fractals which correspond to a non-integral dimension. Fractals can be sets of points or curves or surfaces embedded in space. Mandelbrot observed the natural phenomenon in a different way and writes, " clouds are not spheres, mountains are not cones, and bark is not smooth, nor does lightening travel in a straight line ". Many natural phenomenon will look different when viewed from different distances at the macroscopic or microscopic level.

Farmer (1983) classified a number of dimensions depending upon their metric and probabilistic properties. For any chaotic attractor fractal dimension will give an idea as how much information one needs to know about the position of any representative point on the strange attractor. For example a curve similar to a line will have a fractal dimension one, while a random scatter of lines filling a plane will have fractal dimension close to two.

6.1 HAUSDORFF'S DIMENSION

Hausdorff's dimension is a metric dimension, since it is a measure of distance i.e. how the points are spread over an area or volume. For simple attractors like a stationary point Hausdorff's dimension is zero, where as it is one for limit cycles. The concept of Hausdorff's dimension which is also called as fractal dimension is applicable only to strange attractors, as their structure is complex, chaotic and irregular.

Consider a set of discrete points belonging to a curve. Let

$$N_o \quad = \quad Number \;\; of \;\; points.$$

$$N(\varepsilon) \quad = \quad Minimum \;\; number \;\; of \;\; boxes/cubes \;\; which \;\; enclose \;\; all \;\; the \;\; points.$$

$$\varepsilon \quad = \quad Dimension \;\; of \;\; each \;\; box$$

For a two dimensional plane the number of points $N_o \;\alpha\; L^2$, where L being the length of the square. The minimum number of squares needed to fill the bigger square will be inversely proportional to the square of the dimension of the smaller square i.e.

$$N(\varepsilon) \;\; \alpha \;\; \frac{1}{\varepsilon^2}$$

Similarly for a three dimensional space, the minimum number of cubes required to cover the whole space will be inversely proportional to the dimension of the smaller cube i.e.

$$N(\varepsilon) \;\; \alpha \;\; \frac{1}{\varepsilon^3}$$

In general if in an n-dimensional space, a set of points N_o be covered by $N(\varepsilon)$ cubes of side ε, then for $N_o \rightarrow \infty$ and $\varepsilon \rightarrow 0$, the number of cubes will be

$$N(\varepsilon) \;\; \alpha \;\; \frac{1}{\varepsilon^d}$$

$$d = \lim_{\varepsilon \to 0} \left[\frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})} \right] \qquad (6.1.1)$$

where d is called the Hausdorff's dimension or the fractal dimension. Fractal dimension has its origin from Latin word fractus i.e. a fraction as it is a noninteger number.

Let us consider following four sets of curves, whose fractal dimension is to be determined. The first two curves are found in literature but the last two curves to the author's knowledge are new. Peterson (1988) commented that these curves are so easy to generate but are very complex.

6.2 CANTOR SET

Introduced by Cantor in 1883, the set obtained by dividing a straight line into shorter and shorter segments is called a Cantor set as shown in figure (6.1).

It can be constructed by dividing a straight line into three equal parts and rejecting the middle one. It is repeated again and again until we obtain finally a a set of points, which is called the Cantor set. Its fractal dimension is determined as follows: After n iterations

$$\text{Number of segments} = 2^n$$

$$\text{Length of each segment} = \frac{1}{3^n}$$

$$d = \lim_{n\to\infty} \frac{\ln 2^n}{\ln 3^n} = 0.63092 \qquad (6.2.2)$$

When the above procedure is extended to three dimensions, we obtain what Mandelbrot called as Cantor dust.

6.3 KOCH CURVE

Koch or Snow flake curve is a closed curve having a finite area but infinite length. It is constructed by dividing a segment into three parts and then replacing the middle part by two equal parts of length equal to the replaced part and joining at the tip. It will result in four equal segments of one third the original length. Repeating the above procedure again and again, we obtain Koch curve resembling the shape of a snow flake. After n iterations

$$\text{Number of segments} = 4^n$$

$$\text{Length of each segment} = \frac{1}{3^n}$$

$$d = \lim_{n\to\infty} \frac{\ln 4^n}{\ln 3^n} = 1.26185 \qquad (6.3.1)$$

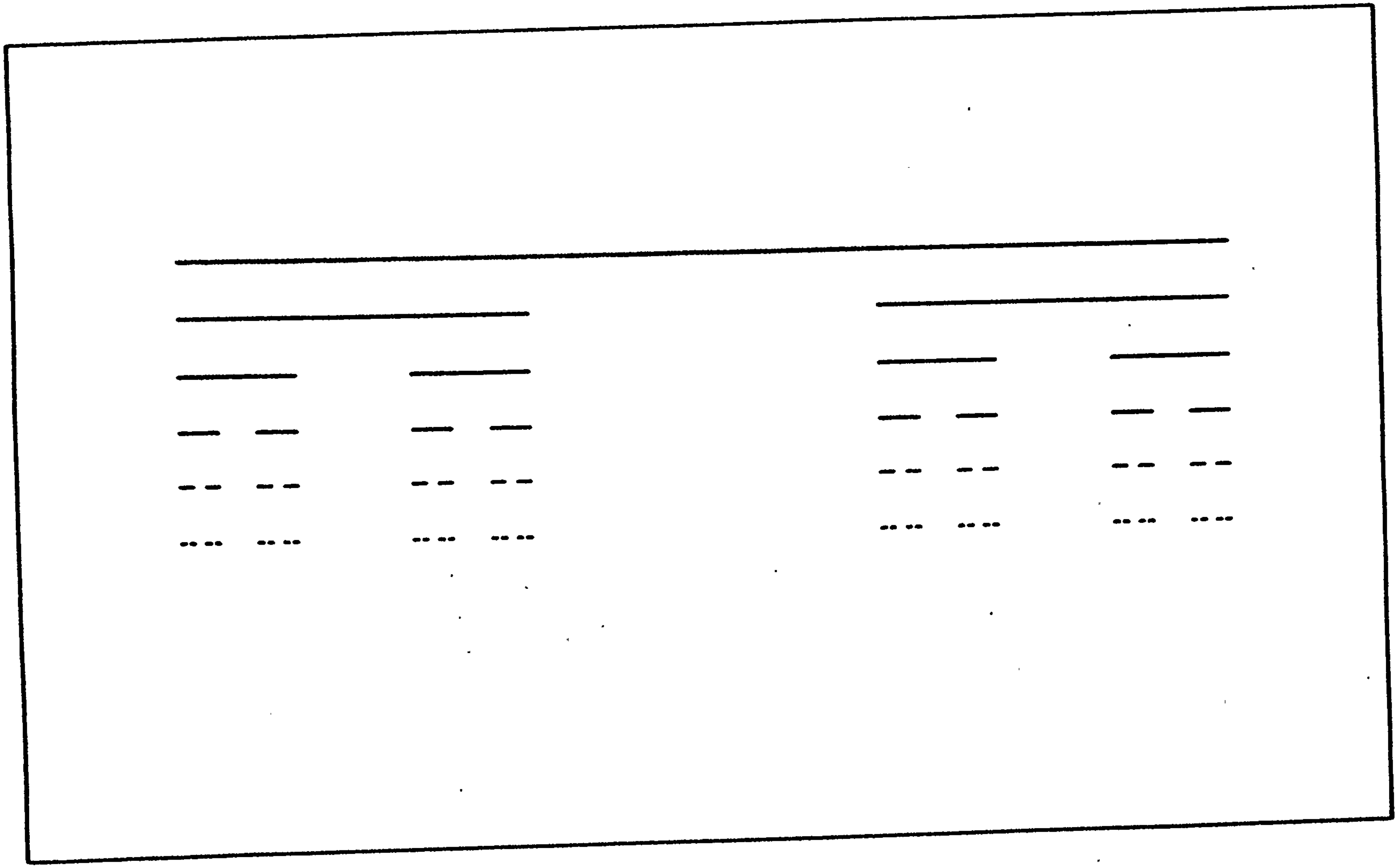


Fig 6.1 : Cantor Set

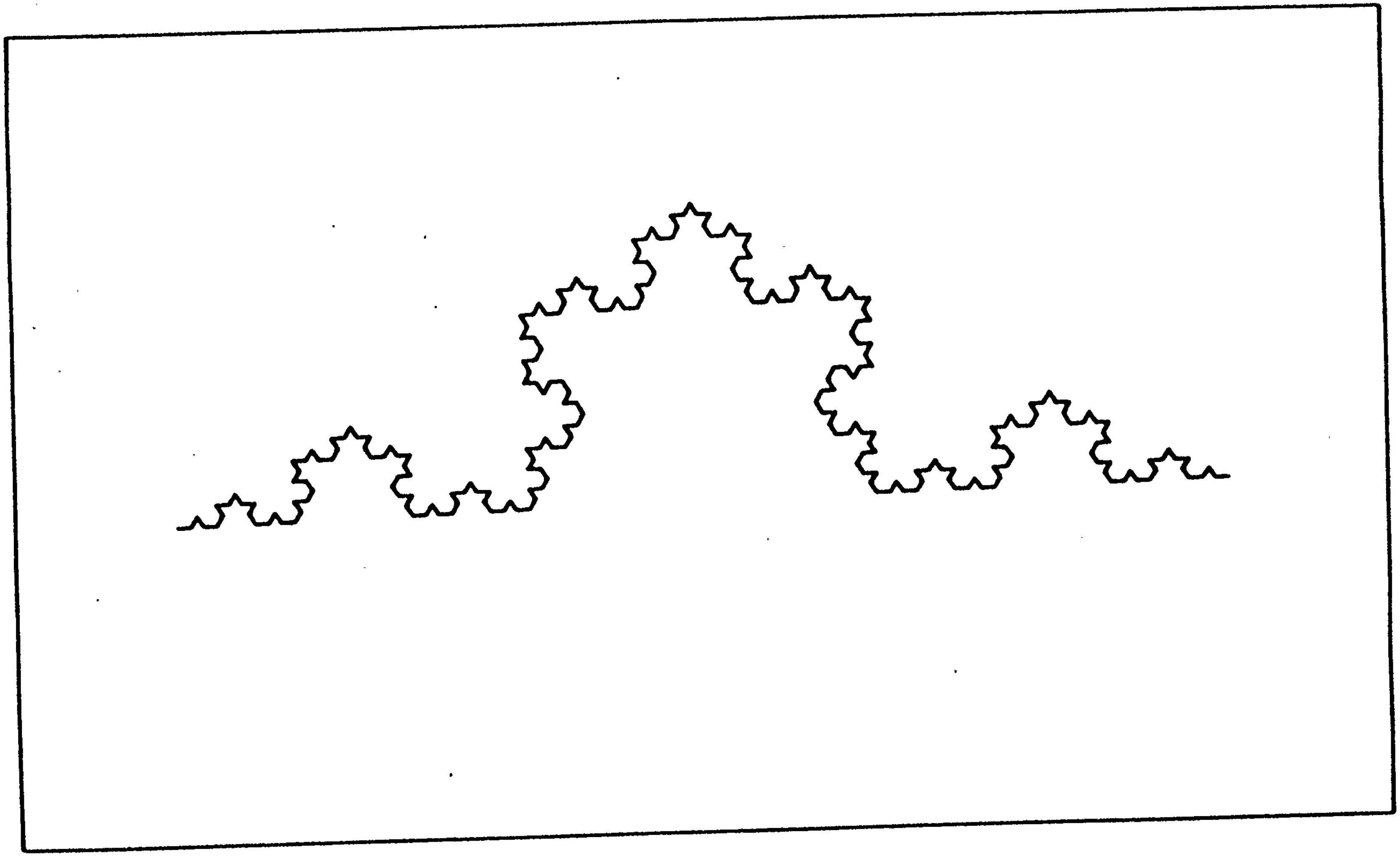


Fig 6.2 : Koch Curve

6.4 Similar Curves

Following two curves are drawn which have fractal dimension more than the fractal dimension of Koch curve. It also demonstrates how the complexity of the curve is increased resulting in a higher and higher fractal dimension.

[a] The curve as shown in figure (6.3) is obtained by dividing a straight line into three equal segments. Then add two segments to the middle segment to obtain five equal segments each equal to one third of the original segment. Its fractal dimension can be determined after n iterations as :

$$d = \lim_{n\to\infty} \frac{\ln 5^n}{\ln 3^n} = 1.46497 \tag{6.4.1}$$

[b] The curve as shown in figure (6.4) is obtained by dividing a straight line into three equal segments. Then replace middle segment by four equal segment to the middle segment two on each side to form a rombus to obtain six equal segments each equal to one third of the original segment. Its fractal dimension can be determined after n iterations as :

$$d = \lim_{n\to\infty} \frac{\ln 6^n}{\ln 3^n} = 1.63093 \tag{6.4.2}$$

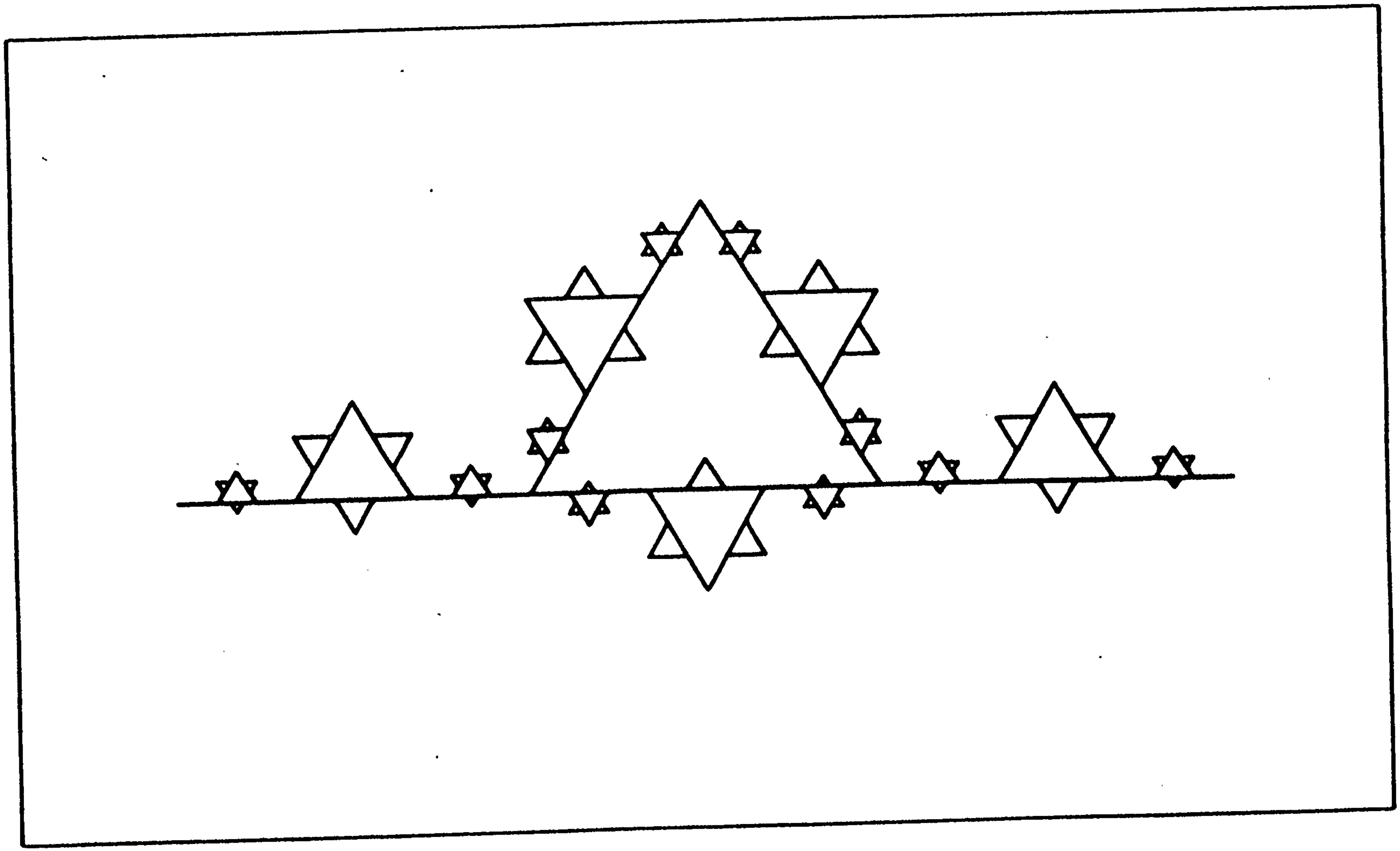


Fig 6.3 : Fractal Dimension 1.46497

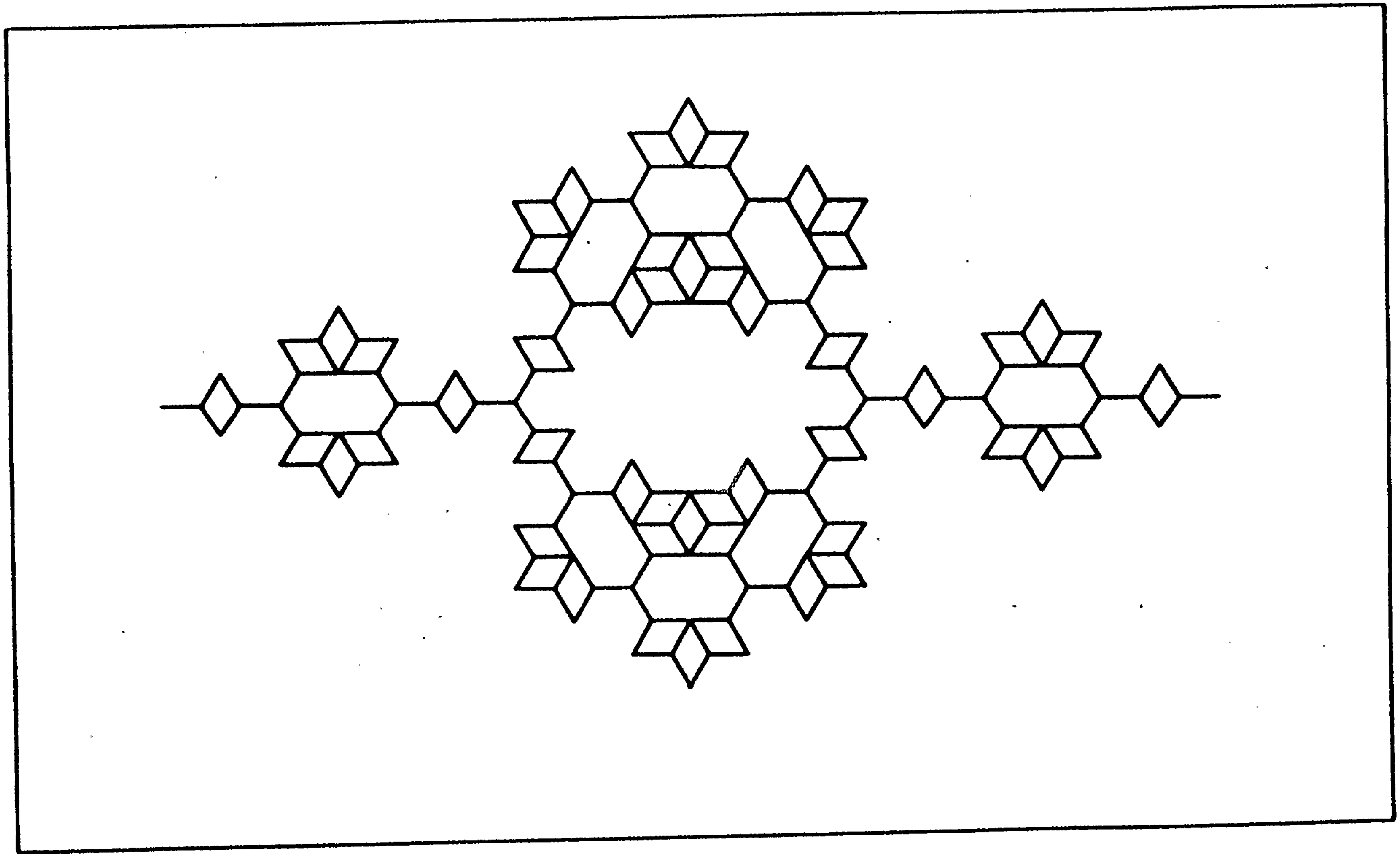


Fig 6.4 : Fractal Dimension 1.63093

6.5 LYAPUNOV DIMENSION

It is a metric dimension and does not depend on the probability of finding a representative point near or far from attractor. It gives the minimum number of state variables or degrees of freedom required to describe the steady state behavior of the system. For a strange attractor this dimension will be a noninteger. Lyapunov dimension for a strange attractor is calculated from Lyapunov exponents, which according to Parker (1989) are the eigen values at the equilibrium point. Benettin (1980) formulated a method to determine Lyapunov exponents from the a matrix obtained by multiplying characteristic matrices of all the vectors. Lyapunov exponents will thus depend upon on the parametric values and initial conditions of the nonlinear dynamical system. They will be same for a basin of attraction, as all the representative points from the basin will converge to or diverge from an attractor as time approaches infinity. Hence it can be inferred that the Lyapunov exponents and Lyapunov dimension are essentially for the equilibrium point and do not depend on the behavior of the trajectories in the transient state, so long they are converging at or diverging from the same attractor.

According to Parker (1989) $\hat{\lambda}_i$ are the eigen values at the equilibrium point, then Lyapunov exponents are equal to the real parts of the eigen values at the equilibrium point.

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} Re(\hat{\lambda}_i)t = Re(\hat{\lambda}_i) \tag{6.5.1}$$

If Lyapunov exponent is less than zero, it gives a measure of the rate of contraction, else the rate of expansion if greater than zero. For a non-chaotic attracting set, none of the Lyapunov exponent is greater than zero i.e. all trajectories converge to an attractor as time approaches infinity. But for all chaotic attractors at least one of the Lyapunov exponents must be greater than zero. In terms of Lyapunov exponents, an attractor will be called a strange attractor if sum of all Lyapunov exponents is negative, one of the exponents is zero, and other is positive i.e.

$$| \lambda_1 | > \lambda_2 = \lambda_3 \tag{6.5.2}$$

$$\lambda_1 < 0 \tag{6.5.3}$$

For a strange attractor at least one of the Lyapunov exponent is zero, since it has to be bounded, the negative exponent represents the contraction, while positive exponents represents stretching. The overall sum of all the exponents should be negative. Hence the strange attractors are possible to exist only in systems having three or more state variables. The attractor will be strange if both the conditions represented by equations (6.5.2) and (6.5.3) are met, which will mean that contraction is more than expansion. Wolf et al. (1985) determined the fractal dimension for the Lorenz model as 2.07 for the parameter values as $\sigma = 16.0$, $r = 45.92$, and $b = 8/3$, indicating an almost planer attractor. The fractal dimnsion for the Rossler model was determined to be 2.01 for $a = 0.15$, $b = 0.20$, and $c = 10.0$. In figure (6.5), the Lyapunov dimension is determined for the Lorenz model and the Rossler model for various values of parameters. The Lyapunov exponents has been taken as the real part of the eigen values at the equilibrium point. The Lyapunov dimension is determined as

$$D = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda|_{max}} \tag{6.5.4}$$

where D = Lyapunov dimension

λ = Lyapunov exponent

The Lyapunov dimension lies between 2 and 3, which means we need minimum three state variables to describe the dynamical models completely.

LORENZ MODEL			
Parameters			Lyapunov
σ	r	b	Dimension
1.0	28.0	2.667	2.25
10.0	28.0	2.667	2.40
100.0	28.0	2.667	2.15
10.0	10.0	2.667	2.16
10.0	29.72	2.667	2.41
10.0	100.5	2.667	2.63
10.0	147.5	2.667	2.69
10.0	197.5	2.667	2.72
10.0	20000	2.667	2.96

ROSSLER MODEL			
Parameters			Lyapunov
a	b	c	Dimension
0.343	1.82	9.75	2.08
0.2	0.2	5.7	2.36

Fig 6.5 : Lyapunov Dimension

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

There are a number of conclusions which can be drawn from this analysis. First, it is important and possible to a reasonable accuracy, to study the behavior of representative points and trajectories of a physical system represented by a system of nonlinear ordinary differential equations. The trajectories may lead to a number of singularities like focus, node, saddle point, or limit cycles. Often existence of strange attractors is noticed. The affect of system parameters on this behavior has been studied. This analysis is important to determine the conditions under which a nonlinear system will be stable. The sensitivity of initial conditions and the affect of quadratic and cubic nonlinearities on the behavior of trajectories has been shown by 2D graphs.

Though it is not possible to eliminate chaos, but it can be dampened by changing system parameters or initial conditions. It is possible to shift the chaotic regions to another zone, if the singularities are dependent on system parameters.

Recommendations

It is recommended that all nonlinear dynamical systems be studied for chaos before they are put to practice. Chaotic analysis should form an integral part to design safe nonlinear dynamical systems.

A lot more work has still to be done in this area. Even the validity of simulation languages can be questioned, as they involve certain approximations, e.g. in ACSL Runge-Kutta method fourth order, higher order terms are neglected. When the systems are sensitive to initial conditions these approximations can lead to errors. Lyapunov exponents give the rate of convergence or divergence, and can be used as a precaution against errors if the exponents are positive.

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APPENDIX A

In this appendix the syntax code in Advanced Continuous Simulation Language for the chaotic analysis of Lorenz model is given as has been used on Cyber computer system at University of Calgary. At the end of the program in section b, a sample output for all the independent and dependent variables has been presented.

A Model Definition of Lorenz Model

```
PROGRAM CHAOS_IN_LORENZ

"************************************************"
"*                                              *"
"* ACSL SYNTAX AND STRUCTURE USED IN DESCRIBING *"
"* THE MODEL OF LORENZ WEATHER MODEL            *"
"*                                              *"
"*      CODED BY SANJEEV  KHANNA                *"
"*                                              *"
"************************************************"

  INITIAL

    "---<DEFINE PRESET  ARIABLES>---"
    "---< DEFINE SYSTEM PARAMETERS >---"
    CONSTANT    SIGMA = 10.0,  B   = 2.667,  R =  28.0

    "---< SPECIFY INITIAL CONDITIONS >---"
```

```
CONSTANT    XIC  =  0.0,  YIC=  1.0,     ZIC = 0.0

"---< DEFINE MAXIMUM TIME FOR SIMULATION RUN >---"
CONSTANT    TSTOP = 20.0

"---< SPECIFY THE ALGORITHM TO BE USED >---"

"---< 5 IS FOR RUNGE-KUTTA METHOD OF FOURTH ORDER >---"
ALGORITHM   IALG  = 2

"--<DEFINE MAXIMUM AND MINIMUM INTERVAL FOR THE STEP SIZE>--"
MAXTERVAL   MAXT  = 0.05
MINTERVAL   MINT  = 1.0E-4
VARIABLE    TM=0.0

END $" INITIAL"


DYNAMIC

 CINTERVAL CINT=0.02

 DERIVATIVE

   "---< REDUCING COMMUNICATION INTERVAL AT START >---"
   IF (TM .LE. 10.0) CINT=0.01

   "---< WRITING THE SYSTEM OF ODE >---"
      DX  =  SIGMA*(Y-X)
      DY  =  R*X-X*Z-Y
      DZ  =  X*Y-B*Z

   "---< PERFORMING INTEGRATION >---"
      X  =  INTEG(DX,XIC)
      Y  =  INTEG(DY,YIC)
      Z  =  INTEG(DZ,ZIC)


 END $" DERIVATIVE"

 "---<SPECIFY TERMINATION CONDITION>---"
 TERMT(TM.GE.TSTOP)
```

```
  END $" DYNAMIC"

  TERMINAL

  END $" TERMINAL"

END $" OF PROGRAM "
```

B Sample Output

```
ACSL RUN-TIME EXEC VERSIN 1 LEVEL 8R1   1989-12-01 12:35:26


Title = 'CHAOS IN LORENZ MODEL'

Start

COUNT OF TIMES STATE CONTROLLED STEP SIZE
MINUS (-) REL ERR ALWAYS BELOW ABS ERR
        Z PC FAIL   0  ERR CONTROL   422
        X PC FAIL   0  ERR CONTROL   413
        Y PC FAIL   0  ERR CONTROL  1298

  NUMBER OF JACOBIAN EVALUATIONS WAS    1
    NUMBER OF LU DECOMPOSITIONS WAS   13
PRINT 'NCIPRN'=10,TM,X,Y,Z


LINE        TM          X           Y           Z
  0    0.          0.          1.0000000   0.
 10    0.1000000   0.9120543   2.0919670   0.0624538
 20    0.2000000   3.0548013   6.6384246   0.7671564
 30    0.3000000   9.4628724   19.370414   7.5870310
 40    0.4000000   19.594713   23.402338   39.849566
 50    0.5000000   9.8197140  -6.6200336   41.602092
 60    0.6000000  -2.0438180  -8.9410857   29.854116
```

70	0.7000000	-6.0972037	-8.3420913	26.229412
80	0.8000000	-7.7620953	-9.1224942	25.459967
90	0.9000000	-8.9554625	-9.9111188	26.601680
100	1.0000000	-9.4448077	-9.3794113	28.340660
110	1.1000000	-8.8607422	-7.8909561	28.679757
120	1.2000000	-7.8760859	-7.0546302	27.349527
130	1.3000000	-7.4511898	-7.4720369	25.722862
140	1.4000000	-7.9027502	-8.7273763	25.115484
150	1.5000000	-8.8859385	-9.8716172	26.169259
160	1.6000000	-9.5213235	-9.6562150	28.120104
170	1.7000000	-9.0732287	-8.1361215	28.874228
180	1.8000000	-8.0111305	-7.0268600	27.702630
190	1.9000000	-7.3995591	-7.2364079	25.919302
200	2.0000000	-7.7096565	-8.4519061	24.989895
210	2.1000000	-8.7009759	-9.7966218	25.767037
220	2.2000000	-9.5341573	-9.9202416	27.818695
230	2.3000000	-9.2840153	-8.4423180	29.013011
240	2.4000000	-8.1854452	-7.0534418	28.081275
250	2.5000000	-7.3818674	-7.0142977	26.181388
260	2.6000000	-7.5217685	-8.1458229	24.928420
270	2.7000000	-8.4817129	-9.6489714	25.369885
280	2.8000000	-9.4944249	-10.144209	27.425952
290	2.9000000	-9.4869105	-8.8044698	29.070751
300	3.0000000	-8.4012986	-7.1462496	28.473392
310	3.1000000	-7.4058802	-6.8152427	26.510780
320	3.2000000	-7.3471410	-7.8165911	24.941788
330	3.3000000	-8.2333220	-9.4269026	24.998061
340	3.4000000	-9.3948943	-10.305458	26.948319
350	3.5000000	-9.6673403	-9.2129598	29.022623
360	3.6000000	-8.6570299	-7.3184522	28.862389
370	3.7000000	-7.4792691	-6.6507234	26.907337
380	3.8000000	-7.1946949	-7.4731100	25.039581
390	3.9000000	-7.9630820	-9.1326832	24.673106
400	4.0000000	-9.2309616	-10.381420	26.400521
410	4.1000000	-9.8078571	-9.6497440	28.843885
420	4.2000000	-8.9474642	-7.5833434	29.225001
430	4.3000000	-7.6099290	-6.5348160	27.368611
440	4.4000000	-7.0744525	-7.1254842	25.230414
450	4.5000000	-7.6801522	-8.7723025	24.416813
460	4.6000000	-9.0014264	-10.352126	25.805765
470	4.7000000	-9.8889980	-10.086885	28.513237
480	4.8000000	-9.2624848	-7.9523814	29.529550
490	4.9000000	-7.8055139	-6.4851369	27.888646
500	5.0000000	-6.9977842	-6.7852569	25.521910

510	5.1000000	-7.3954925	-8.3551159	24.250429
520	5.2000000	-8.7091499	-10.202786	25.195218
530	5.3000000	-9.8907861	-10.486145	28.017534
540	5.4000000	-9.5853833	-8.4316913	29.734467
550	5.5000000	-8.0725032	-6.5239048	28.455954
560	5.6000000	-6.9777320	-6.4660299	25.920601
570	5.7000000	-7.1218764	-7.8933005	24.194103
580	5.8000000	-8.3614221	-9.9260265	24.606466
590	5.9000000	-9.7949715	-10.800681	27.357377
600	6.0000000	-9.8913047	-9.0165600	29.788067
610	6.1000000	-8.4144992	-6.6787942	29.050243
620	6.2000000	-7.0293505	-6.1846705	26.431561
630	6.3000000	-6.8740932	-7.4012970	24.266656
640	6.4000000	-7.9700108	-9.5233004	24.081186
650	6.5000000	-9.5878513	-10.979555	26.552500
660	6.6000000	-10.146386	-9.6839321	29.631227
670	6.7000000	-8.8293472	-6.9827950	29.637360
680	6.8000000	-7.1699073	-5.9633814	27.057446
690	6.9000000	-6.6694490	-6.8954813	24.485671
700	7.0000000	-7.5509594	-9.0051562	23.662466
710	7.1000000	-9.2632520	-10.975065	25.645269
720	7.2000000	-10.308543	-10.384133	29.205173
730	7.3000000	-9.3046432	-7.4714322	30.162122
740	7.4000000	-7.4185818	-5.8329720	27.796263
750	7.5000000	-6.5286836	-6.3944138	24.867837
760	7.6000000	-7.1243247	-8.3902966	23.392395
770	7.7000000	-8.8250549	-10.751749	24.700556
780	7.8000000	-10.331072	-11.035139	28.466252
790	7.9000000	-9.8114340	-8.1743978	30.539667
800	8.0000000	-7.7949078	-5.8375898	28.636627
810	8.1000000	-6.4774001	-5.9202170	25.429330
820	8.2000000	-6.7141624	-7.7037226	23.310484
830	8.3000000	-8.2886300	-10.294729	23.801103
840	8.4000000	-10.169921	-11.524966	27.407204
850	8.5000000	-10.296832	-9.0974377	30.648738
860	8.6000000	-8.3144993	-6.0405380	29.548065
870	8.7000000	-6.5480096	-5.5020448	26.185769
880	8.8000000	-6.3492205	-6.9745843	23.453388
890	8.9000000	-7.6808437	-9.6146086	23.039102
900	9.0000000	-9.7942570	-11.727951	26.079526
910	9.1000000	-10.678369	-10.191437	30.335959
920	9.2000000	-8.9794457	-6.5284361	30.462325
930	9.3000000	-6.7817785	-5.1833701	27.150513
940	9.4000000	-6.0649684	-6.2347502	23.856052

950	9.5000000	-7.0388410	-8.7467619	22.506583
960	9.6000000	-9.1972139	-11.535305	24.604605
970	9.7000000	-10.846773	-11.316955	29.445024
980	9.8000000	-9.7628800	-7.4056110	31.247101
990	9.9000000	-7.2280742	-5.0301793	28.331768
1000	10.000000	-5.9029822	-5.5134350	24.551344
1010	10.100000	-6.4039068	-7.7417673	22.279629
1020	10.200000	-8.4038616	-10.895661	23.160772
1030	10.300000	-10.683971	-12.217345	27.892354
1040	10.400000	-10.571979	-8.7566122	31.643146
1050	10.500000	-7.9462783	-5.1732650	29.708624
1060	10.600000	-5.9331551	-4.8596484	25.586628
1070	10.700000	-5.8348626	-6.6635137	22.443275
1080	10.800000	-7.4720921	-9.8275441	21.970459
1090	10.900000	-10.096775	-12.554775	25.762559
1100	11.000000	-11.215797	-10.533426	31.246915
1110	11.100000	-8.9735292	-5.8282821	31.176368
1120	11.200000	-6.2518328	-4.3488805	27.009503
1130	11.300000	-5.4038967	-5.5779884	23.070171
1140	11.400000	-6.4900087	-8.4350960	21.245493
1150	11.500000	-9.0757919	-12.047753	23.391550
1160	11.600000	-11.396366	-12.360377	29.605212
1170	11.700000	-10.255724	-7.3090397	32.410589
1180	11.800000	-6.9974869	-4.1397650	28.861789
1190	11.900000	-5.2153237	-4.5516787	24.240944
1200	12.000000	-5.5660800	-6.8712707	21.159877
1210	12.100000	-7.7244885	-10.641944	21.305046
1220	12.200000	-10.800982	-13.429032	26.577092
1230	12.300000	-11.498684	-9.8518249	32.623952
1240	12.400000	-8.3348626	-4.5979036	31.105491
1250	12.500000	-5.4421685	-3.6764323	26.064058
1260	12.600000	-4.8317877	-5.2916448	21.853256
1270	12.700000	-6.2427456	-8.5762627	20.034479
1280	12.800000	-9.3179179	-12.854803	22.816151
1290	12.900000	-12.009294	-12.931588	30.515286
1300	13.000000	-10.303521	-6.4906480	33.293026
1310	13.100000	-6.3910009	-3.1854144	28.690015
1320	13.200000	-4.4789692	-3.8210912	23.474616
1330	13.300000	-4.8757265	-6.2610248	19.953634
1340	13.400000	-7.2142852	-10.478811	19.693884
1350	13.500000	-10.922841	-14.437297	25.475023
1360	13.600000	-12.269486	-10.685426	33.557730
1370	13.700000	-8.5293200	-3.8831070	32.178522
1380	13.800000	-4.8885841	-2.5987176	26.275042

1390	13.900000	-3.8988249	-4.0615976	21.296730
1400	14.000000	-5.0374167	-7.1460056	18.435059
1410	14.100000	-8.1526680	-12.204240	19.747434
1420	14.200000	-12.190586	-15.155161	28.253003
1430	14.300000	-11.870311	-8.1080755	35.099040
1440	14.400000	-6.9434085	-2.2589539	30.728279
1450	14.500000	-3.7870607	-2.1656064	24.394762
1460	14.600000	-3.3516627	-3.9427942	19.565728
1470	14.700000	-4.8714792	-7.3725795	16.967054
1480	14.800000	-8.4888088	-13.159413	19.063295
1490	14.900000	-12.978493	-15.896576	29.545884
1500	15.000000	-11.798232	-6.7261244	36.142930
1510	15.100000	-6.0374281	-1.0897936	30.230476
1520	15.200000	-2.8771812	-1.3400378	23.510704
1530	15.300000	-2.4360182	-2.9004827	18.456986
1540	15.400000	-3.6994197	-5.8333556	15.269960
1550	15.500000	-7.0078552	-11.640102	15.768808
1560	15.600000	-12.495826	-17.758198	25.509193
1570	15.700000	-13.678148	-9.5639576	37.704470
1580	15.800000	-7.0185798	-0.2395446	32.714857
1590	15.900000	-2.3369290	0.3136179	24.934311
1600	16.000000	-0.9651289	-0.5111925	19.104057
1610	16.100000	-0.9989130	-1.4368024	14.709239
1620	16.200000	-1.8673030	-3.2656997	11.550356
1630	16.300000	-4.1826535	-7.7121326	10.261473
1640	16.400000	-9.5666151	-16.874512	15.335005
1650	16.500000	-16.124211	-18.549779	35.152436
1660	16.600000	-10.784297	-0.4772747	38.918372
1670	16.700000	-2.2339934	3.4470427	28.715025
1680	16.800000	1.2435717	3.2551981	21.953164
1690	16.900000	2.9382761	4.7116555	17.553780
1700	17.000000	5.3916847	8.7708894	15.874433
1710	17.100000	9.9152779	15.329785	20.457381
1720	17.200000	13.963706	14.756235	33.493640
1730	17.300000	10.213393	3.4972528	35.696049
1740	17.400000	4.3321615	0.1813036	28.134701
1750	17.500000	1.9335033	0.9407691	21.655651
1760	17.600000	1.7797953	2.3154027	16.829025
1770	17.700000	2.9723866	4.9243618	13.616693
1780	17.800000	6.0812844	10.599119	13.461873
1790	17.900000	12.010120	18.575299	22.479417
1800	18.000000	14.879674	11.757449	38.505097
1810	18.100000	7.6596717	-0.5320532	34.407978
1820	18.200000	1.8254764	-1.6233990	25.829751

1830	18.300000	-0.1817265	-1.2480317	19.707153
1840	18.400000	-0.9990820	-1.7882762	15.176556
1850	18.500000	-2.1094968	-3.7184425	11.989418
1860	18.600000	-4.6910263	-8.5601412	10.968290
1870	18.700000	-10.386765	-17.772711	17.280424
1880	18.800000	-16.046610	-16.580301	36.927375
1890	18.900000	-9.6795165	0.2620103	37.544501
1900	19.000000	-1.9157348	3.0286625	27.773534
1910	19.100000	1.0933818	2.8483388	21.246111
1920	19.200000	2.6170046	4.2611525	16.856431
1930	19.300000	4.9457505	8.2143158	14.945118
1940	19.400000	9.4896087	15.179530	18.988799
1950	19.500000	14.242547	15.983374	32.891707
1960	19.600000	10.814299	3.6661815	36.682938
1970	19.700000	4.3072756	-0.4130798	28.748581
1980	19.800000	1.4932920	0.1847028	21.964280
1990	19.900000	0.9912014	1.0824022	16.882814
2000	20.000000	1.5081282	2.4732144	13.116198

STOP

2 WORDS TABLE SPACE USED

Job is $0860_0458_ABZ_6531. User is SKKHANNA. Account is GENME.
Project is GENME.

APPENDIX B

DEFINITIONS

In this section some definitions are provided in alphabetic order for a quick reference.

Attractor

For dimensions more than two in the reference phase space, when a representative point approaches periodically to some past locus repeatedly, the asymptotic orbit is called an attractor. In this case orbital stability is lost and future behaviour becomes extremely chaotic.

Bifurcation

The dictionary meaning is to bifurcate i.e. a break in a regular or continuous behaviour. It signifies a change in behaviour of a long time dynamical motion when a parameter or a set of parameters is varied.

Chaos

The phenomenon of randomness and unpredictability in nonlinear dynamical systems.

Chaotic Motion

The dynamical motion which is sensitive to any variation in initial conditions. It can also be characterized by a dynamical motion with positive Lyapunov exponent. For such kind of motions, a slight difference in initial conditions results in trajectories to diverge apart exponentially.

Focus

An unstable singular point from which all the neighbouring trajectories diverge as time increases.

Fractal

An irregular or fragmented set with noninteger dimension. A measure for strangeness of strange attractors.

Fractal Dimension

It is a quantitative property of a set of points in an n-dimensional space which measures the extent to which the points fill a subspace as the number of points become very large.

Limit Cycle

A limit cycle is said to exist when in two dimensions an orbit of oscillation asymptotically forms a closed curve. In this case point stability is taken over by orbital stability.

Lyapunov Exponent

A set of numbers which measure the exponential convergence or divergence with time for any two adjacent trajectories in the phase space starting from different initial conditions.

Node

A stable singular point to which all the neighbouring trajectories converge.

Period Doubling

It is a sequence of periodic variations in which the period doubles as some parameter in the system is changed.

Phase Space

It is an abstract mathematical space whose coordinates are generalized coordinates and generalized momenta. In dynamical systems governed by a set of differential equations, the coordinates are the state variables or the components of state vector.

Saddle Point

It is a singular point formed by the intersection of four trajectories, two of which converge while the other two diverge from the point of intersection. It is an unstable singular point.

Singular Point

The point at which the systems behaviour becomes independent of time. Essentially it represents steady state solution. All other points are called ordinary points.

Strange Attractor

It is an attracting set in the phase space on which the chaotic orbits move.. It is an attractor in phase space with fractal dimension. An attractor is called a strange attracter if, it is sensitive to initial condition and behaves in a strange manner.

Trajectory

It is the path traversed by a representative points for $-\infty < t <$.