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STEADY STATE TEMPERATURES IN A QUARTER PLANE

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ABSTRACT. The discontinuous boundary value problem of steady state temperatures in a quarter plane gives rise to a pair of dual integral equations which are not of Titchmarch type. These dual integral equations are considered in this paper.

KEYWORDS AND PHRASES. Harmonic boundary value problems, Dual integral equations, Heat transfer.
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1. INTRODUCTION.

We consider the problem of steady state temperatures in a quarter plane (see Fig. 1), whose edge $x=0$ is losing heat to environment at zero temperature according to Newton's Law of cooling while on the edge $y = 0$, temperature is controlled on portion of this edge, while the heat input is known on the remaining part. Typically, this problem is governed by:

Find $u = u(x,y)$ such that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad x > 0,\ y > 0; \tag{1.1a}$$

$$\frac{\partial u}{\partial x} - \alpha u = 0 \quad \text{on } x = 0 \text{ in } y > 0; \tag{1.1b}$$

and either

$$\text{(1)} \qquad u(x,0) = f_1(x) \quad \text{in} \quad 0 < x < 1 \tag{1.2a}$$

$$\text{and} \qquad u_y(x,0) = -g_1(x) \quad \text{in} \quad x > 1 \tag{1.2b}$$

or

$$\text{(2)} \qquad u_y(x,0) = -f_1(x) \quad \text{in} \quad 0 < x < 1 \tag{1.3a}$$

$$\text{and} \qquad u = u(x,0) = g_1(x) \quad \text{in} \quad x > 1. \tag{1.3b}$$

where the subscript denotes differentiation w.r.t. that variable.
Also, in each case we require that $|u|$ be bounded at infinity.

An appropriate representation for $u = u(x,y)$ in this case is

$$u(x,y) = \int_0^\infty f(t)(\alpha \sin xt + t \cos xt)\, e^{-ty}\, dt \quad \text{in } x > 0,\ y > 0. \tag{1.4}$$

where f(t) is governed by the following two cases:

Case 1: $\int_0^\infty f(t)(\alpha \sin xt + t \cos xt)\, dt = f_1(x)$ in $0 < x < 1$ (1.5a)

and $\int_0^\infty tf(t)(\alpha \sin xt + t \cos xt)\, dt = g_1(x)$ in $x > 1$ (1.5b)

or

Case 2: $\int_0^\infty tf(t)(\alpha \sin xt + t \cos xt)\, dt = f_1(x)$ in $0 < x < 1$ (1.6a)

and $\int_0^\infty f(t)(\alpha \sin xt + t \cos xt)\, dt = g_1(x)$ in $x > 1$ (1.6b)

respectively.

We propose to solve such dual integral equations for the function f(t) in this paper. We point out that these equations are not of Titchmarch type (because the kernel $k(x,t) = \alpha \sin xt + t \cos xt$ is not a Fourier Kernel) and to our knowledge, have not been considered before. While the kernel k(x,t) has been successfully inverted [1, page 70], dual integral equations involving this kernel have not been considered previously. We shall attempt only a formal solution of these dual integral equations, and shall assume throughout that the functions $f_1(x)$ and $g_1(x)$ are continuous in $0 \le x \le 1$ and in $x \ge 1$ respectively.

2. METHOD OF SOLUTION.

We shall assume that the integrals $\int_0^\infty f(t)\sin xt\, dt$, $\int_0^\infty tf(t)\sin xt\, dt$, $\int_0^\infty tf(t)\cos xt\, dt$ and $\int_0^\infty t^2 f(t)\cos xt\, dt$ exist, in which case,

$$\int_0^\infty f(t)\sin xt\, dt = F(x) \Rightarrow \int_0^\infty tf(t)\cos xt\, dt = F'(x) \tag{2.1}$$

and $$\int_0^\infty tf(t)\sin xt\, dt = G(x) \Rightarrow \int_0^\infty t^2 f(t)\cos xt\, dt = G'(x) \tag{2.2}$$

Equation (2.1) implies that $\lim_{x\to 0^+} F(x) = F(0) = 0$ and with this notation, our dual integral equations (1.5) in the first case become,

$\alpha F(x) + F'(x) = f_1(x)$ in $0 < x < 1$ (2.3a)

and $\alpha G(x) + G'(x) = g_1(x)$ in $x > 1$ (2.3b)

with the condition that $F(0) = 0$. (2.4)

In the second case (1.6), we write

$F(x) = \int_0^\infty tf(t)\sin xt\, dt,$ $0 < x < 1$ (2.5a)

and $G(x) = \int_0^\infty f(t)\sin xt\, dt,$ $x > 1$ (2.5b)

so that we again get equations (2.3) with condition (2.4).

And for both the cases, the equations (2.3) give

$F(x) = e^{-\alpha x} \int_0^x e^{\alpha t} f_1(t)\, dt,$ $0 < x < 1,$ (2.6a) and $G(x) = e^{-\alpha x} \int_1^x e^{\alpha t} g_1(t)\, dt, + Be^{-\alpha x}$ in $x > 1,$ (2.6b)

It remains to determine the constant B. We shall determine this constant by the (physically realistic) condition that the quantity u(x,o) is continuous at x = 1.

3. SOLUTION FOR THE FIRST CASE.

In this case, the dual integral equations (1.5) are reduced to dual equations

$$\int_0^\infty f(t) \sin xt\, dt = F(x) = e^{-\alpha x} \int_0^x e^{\alpha t} f_1(t)\, dt \quad \text{in} \quad 0 < x < 1 \tag{3.1a}$$

and $$\int_0^\infty t f(t) \sin xt\, dt = e^{-\alpha x} \int_1^x e^{\alpha t} g_1(t)\, dt + Be^{-\alpha x} \quad \text{in} \quad x > 1. \tag{3.1b}$$

These equations give [2]

$$f(t) = \int_0^1 u\, J_0(ut)\, f_2(u)\, du + \int_1^\infty u\, J_0(ut)\, g_2(u)\, du$$
$$+ \frac{2B}{\pi} \int_1^\infty u\, J_0(ut) \left[\int_u^\infty \frac{e^{-\alpha x}}{\sqrt{x^2 - u^2}}\, dx\right] du \tag{3.2}$$

where

$$f_2(u) = \frac{2}{\pi} \frac{d}{du} \int_0^u \frac{xF(x)}{\sqrt{u^2 - x^2}}\, dx = \frac{2}{\pi} \int_0^u \frac{F'(x)}{\sqrt{u^2 - x^2}}\, dx \tag{3.3a}$$

and $$g_2(u) = \frac{2}{\pi} \int_u^\infty \frac{e^{-\alpha x}}{\sqrt{x^2 - u^2}} \left[\int_1^x g_1(t) e^{\alpha t}\, dt\right] dx. \tag{3.3b}$$

In deriving equation (3.3a), we have used the fact that F(0) = 0.

To determine B, we now substitute this expression for f(t) in u(x,0) as given by equation (1.4) above and require that

$$\lim_{x\to 1^+} u(x,0) = \lim_{x\to 1^-} u(x,0) = \alpha F(1) + F'(1) = f_1(1). \tag{3.4}$$

Noting that [3]

$$\int_u^\infty \frac{e^{-\alpha x}}{\sqrt{x^2 - u^2}}\, dx = K_0(\alpha u),$$

where K denotes the Modified Bessel Function, we have

$$\lim_{x\to 1^+} u(x,0) = \lim_{x\to 1^+} (\alpha H(x) + H'(x)), \tag{3.5}$$

where for $x > 1$,

$$H(x) = \int_0^\infty f(t) \sin xt\, dt = \int_0^1 \frac{u f_2(u)}{\sqrt{x^2 - u^2}}\, du + \int_1^x \frac{u g_2(u)}{\sqrt{x^2 - u^2}}\, du + \frac{2B}{\pi} \int_1^x \frac{u K_0(\alpha u)}{\sqrt{x^2 - u^2}}\, du. \tag{3.6}$$

Integration by parts gives

$$H(x) = [g_2(1) - f_2(1) + \frac{2B}{\pi} K_0(\alpha)] \sqrt{x^2 - 1} + \int_0^1 f_2'(u) \sqrt{x^2 - u^2}\, du$$
$$+ \int_1^x g_2'(u) \sqrt{x^2 - u^2}\, du - \frac{2B}{\pi} \int_1^x \alpha\, K_1(\alpha u) \sqrt{x^2 - u^2}\, dx + f_2(0)x. \tag{3.7}$$

At this stage, we notice that unless the co-efficient of $\sqrt{x^2 - 1}$ in the expression for H(x) is zero, H′(x) will be unbounded as $x \to 1^+$, and then u(x,0) cannot be continuous at x = 1. We therefore put this co-efficient to zero to obtain

$$B = \frac{\pi}{2}\,\frac{f_2(1) - g_2(1)}{K_0(\alpha)}\,. \tag{3.8}$$

This gives the value of B in terms of the quanitites $f_2(1)$ and $g_2(1)$ which are known from the data. We shall now show that with this value of B, $u(x,0)$ is continuous at $x = 1$. We have for $x > 1$,

$$\begin{aligned}\alpha H(x) + H'(x) &= \alpha \int_0^1 f_2'(u)\sqrt{x^2-u^2}\,du + \alpha \int_1^x g_2'(u)\sqrt{x^2-u^2}\,du \\ &+ \int_0^1 \frac{xf_2'(u)}{\sqrt{x^2-u^2}}\,du + \int_1^x \frac{xg_2'(u)}{\sqrt{x^2-u^2}}\,du - \frac{2B}{\pi}\int_1^x \alpha^2K_1(\alpha u)\sqrt{x^2-u^2}\,du \\ &- \frac{2B}{\pi}\int_1^x \frac{x\alpha K_1(\alpha u)}{\sqrt{x^2-u^2}}\,du + (1+\alpha x)\,f_2(0)\end{aligned} \tag{3.9}$$

so that, after some simplification, we obtain

$$\lim_{x\to 1^+} (\alpha H(x) + H'(x)) = \alpha \int_0^1 \frac{uf_2(u)}{\sqrt{1-u^2}}\,du + \int_0^1 \frac{f_2'(u)}{\sqrt{1-u^2}}\,du + f_2(0). \tag{3.10}$$

Also $f_2(u) = \frac{2}{\pi}\int_0^u \frac{F'(x)}{\sqrt{u^2-x^2}}\,dx$

$$= \frac{2}{\pi}\,F''(0)u + \frac{2}{\pi}\int_0^u \left[\frac{F'(x)-F'(0)}{x}\right]'\sqrt{u^2-x^2}\,dx + F'(0) \tag{3.11}$$

so that

$$f_2'(u) = \frac{2}{\pi}\,F''(0) + \frac{2}{\pi}\int_0^u \left[\frac{F'(x)-F'(0)}{x}\right]'\frac{u}{\sqrt{u^2-x^2}}\,dx. \tag{3.12}$$

Substituting the values of $f_2(u)$ and $f_2'(u)$ in the expression for $\lim_{x\to 1^+}(\alpha H(x) + H'(x))$; interchanging the order of integration, and using the fact that

$$\int_x^y \frac{u\,du}{\sqrt{(u^2-x^2)(y^2-u^2)}} = \frac{\pi}{2}\,, \quad y > x > 0, \tag{3.13}$$

we obtain

$$\begin{aligned}\lim_{x\to 1^+}(\alpha H(x) + H'(x)) &= \alpha(F(1) - F(0)) + F''(0) \\ &\quad + (F'(1) - F'(0)) - F''(0) + f_2(0) \\ &= \alpha F(1) + F'(1).\end{aligned} \tag{3.14}$$

This proves the continuity of $u(x,0)$ at $x = 1$. It can also be seen that if B is given by (3.8), then under suitable restrictions on the data, $u(x,0)$ as given by equation (3.9) is bounded as $x \to \infty$.

4. SOLUTION FOR THE SECOND CASE.

In this case, the dual equations (1.6) are reduced to

$$\int_0^\infty tf(t)\sin xt\,dt = F(x) = e^{-\alpha x}\int_0^x f_1(t)e^{\alpha t}\,dt, \quad 0 < x < 1 \tag{4.1a}$$

and $\int_0^\infty f(t)\sin xt\,dt = e^{-\alpha x}\int_1^x e^{\alpha t}g_1(t)\,dt + Ce^{-\alpha x}, \quad x > 1$

$$= h(x) + Ce^{-\alpha x}, \text{ say.} \tag{4.1b}$$

The solution $f(t)$ is now given by

$$f(t) = \frac{2}{\pi}\int_0^1 J_1(ut)\ h_1(u)\ du - \frac{2}{\pi}\int_1^\infty uJ_1(ut)\ h_2(u)\ du + \frac{2C\alpha}{\pi}\int_1^\infty uJ_1(ut)\ K_1(\alpha u)\ du \tag{4.2}$$

where $$h_1(u) = \int_0^u \frac{xF(x)}{\sqrt{u^2-x^2}}\ dx, \text{ and} \tag{4.3}$$

$$h_2(u) = \frac{d}{du}\int_u^\infty \frac{h(x)}{\sqrt{x^2-u^2}}\ dx. \tag{4.4}$$

Proceeding as in section 3, and assuming that the data $g_1(x)$ is suitably restricted so that $h_2(\infty) = 0$, we get, for $x < 1$,

$u(x,0) = \alpha H(x) + H'(x)$, where

$$H(x) = \frac{2x}{\pi}\sqrt{1-x^2}\ [h_1(1) + h_2(1) - C\alpha K_1(\alpha)]$$

$$+ \frac{2x}{\pi}\int_1^\infty\left[\frac{h_2(u) - C\alpha K_1(\alpha u)}{u}\right]'\sqrt{u^2-x^2}\ du - \frac{2x}{\pi}\int_x^1\left[\frac{h_1(u)}{u^2}\right]'\sqrt{u^2-x^2}\ du \tag{4.5}$$

and for $u(x,0)$ to be continuous at $x = 1$, we must have

$$C = \frac{h_1(1) + h_2(1)}{\alpha K_1(\alpha)}. \tag{4.6}$$

With this value of C, we have

$$\lim_{x\to 1^-}[(\alpha H(x) + H'(x)] = \frac{2(1+\alpha)}{\pi}\int_1^\infty\left[\frac{h_2(u)}{u}\right]'\sqrt{u^2-1}\ du - \frac{2}{\pi}\int_1^\infty\left[\frac{h_2(u)}{u}\right]'\frac{1}{\sqrt{u^2-1}}\ du. \tag{4.7}$$

Also,

$$h_2(u) = \frac{d}{du}\int_u^\infty \frac{h(x)}{\sqrt{x^2-u^2}}\ dx \tag{4.8}$$

$$\Rightarrow \quad h(x) = -\frac{2x}{\pi}\int_x^\infty \frac{h_2(u)}{\sqrt{u^2-x^2}}\ du. \tag{4.9}$$

Differentiating equation (4.9) and substituting in (4.7), we get

$$\lim_{x\to 1^-}[\alpha H(x) + H'(x)] = \alpha h(1) + h'(1) = g_1(1)$$

which shows that with C given by equation (4.6), $u(x,0)$ is continuous at $x = 1$.

5. THE CASE $\alpha = 0$.

The case of $\alpha = 0$ is completely different, because for <u>bounded</u> u, the representation

$$u(x,y) = \int_0^\infty f(t)(\alpha \sin xt + t \cos xt)e^{-ty}\ dt \tag{1.4}$$

is no more valid. The correct representation now is

$$u(x,y) = C_1 + \int_0^\infty tf(t)(\cos xt)e^{-ty}\ dt. \tag{5.1}$$

where C_1 is a constant.

Therefore, the dual integral equations this time are:

<u>Case 1</u>: Find C_1 and $f(t)$ such that

$$C_1 + \int_0^\infty tf(t)\cos xt\ dt = f_1(x) \quad \text{in} \quad 0 < x < 1 \tag{5.2a}$$

$$\text{and} \quad \int_0^\infty t^2 f(t)\cos xt\, dt = g_1(x) \quad \text{in} \quad x > 1. \tag{5.2b}$$

And,

Case 2: Find C_1 and $f(t)$, such that

$$\int_0^\infty t^2 f(t)\cos xt\, dt = f_1(x) \quad \text{in} \quad x < 1 \tag{5.3a}$$

$$\text{and} \quad C_1 + \int_0^\infty tf(t)\cos xt\, dt = g_1(x) \quad \text{in} \quad x > 1. \tag{5.3b}$$

In case 2, C_1 is that constant, if any, for which $|g_1(x) - C_1| \to 0$ as $x \to \infty$. Let us consider dual equations (5.2) in Case 1. We shall again determine C_1, by the requirement that $u(x,0)$ is continuous at $x = 1$. We have from (5.2)

$$\int_0^\infty tf(t)\cos xt\, dt = f_1(x) - C_1, \quad 0 < x < 1 \tag{5.4a}$$

$$\text{and} \quad \int_0^\infty tf(t)\sin xt\, dt = -\int_x^\infty g_1(x)dx = g(x), \text{ say.} \tag{5.4b}$$

This gives

$$f(t) = \frac{2}{\pi}\int_0^1 uJ_0(ut)\, F_1(u)\, du + \frac{2}{\pi}\int_1^\infty uJ_0(ut)\, G_1(u)\, du - C_1 \int_0^1 uJ_0(ut)\, du \tag{5.5}$$

where

$$F_1(u) = \int_0^u \frac{f_1(x)\, dx}{\sqrt{u^2-x^2}} \text{ and } G_1(u) = \int_u^\infty \frac{g(x)}{\sqrt{x^2-u^2}}\, dx. \tag{5.6}$$

For $x > 1$, we have $u(x,0) - C_1 = \frac{d}{dx}\int_0^\infty f(t)\, \sin xt\, dt = H'(x)$ say, where after substituting the value of $f(t)$ from (5,5) and simplifying, we obtain

$$H(x) = \int_0^\infty f(t)\sin xt\, dt = \frac{2}{\pi}\sqrt{x^2-1}\,[G_1(1) - F_1(1) + \frac{\pi}{2} C_1] + \frac{2}{\pi} F_1(0)x - C_1 x$$
$$+ \frac{2}{\pi}\int_0^1 F_1'(u)\sqrt{x^2-u^2}\, du + \frac{2}{\pi}\int_1^x G_1'(u)\sqrt{x^2-u^2}\, du. \tag{5.7}$$

And in order for $u(x,0)$ to be continuous at $x = 1$, we must have

$$C_1 = \frac{2}{\pi}[F_1(1) - G_1(1)]. \tag{5.8}$$

With this value of C_1, it is easy to see that

$$\lim_{x\to 1^+} u(x,0) = \lim_{x\to 1^+} H'(x) + C_1$$
$$= \frac{2}{\pi} F_1(0) + \frac{2}{\pi}\int_0^1 \frac{F_1'(u)}{\sqrt{1-u^2}}\, du \tag{5.9}$$

Now from above,

$$F_1(u) = \int_0^u \frac{f_1(x)}{\sqrt{u^2-x^2}}\, dx,$$

$$\Rightarrow \quad f_1(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{uF_1(u)}{\sqrt{x^2-u^2}}\, du$$

$$= \frac{2}{\pi} F_1(0) + \frac{2}{\pi} x \int_0^x \frac{F_1'(u)}{\sqrt{x^2 - u^2}} du.$$

Hence from (5.9),

$$\lim_{x \to 1^+} u(x,0) = f_1(1)$$

which implies continuity of u(x,0) at x = 1.

Once again, it can be seen from (5.7) that if $g_1(x)$ is suitably restricted then u(x,o) is bounded as $x \to \infty$.

For Case 2, the solution is given by (4.2) in the limit as $\alpha \to 0^+$.

It should be pointed out that the problem posed by equations (5.2) has been considered by Sneddon [4, page 99]. Sneddon considers the problem (5.2) with $C_1 = 0$ and $g_1(x) = 0$. He then imposes the condition that the heat input on y = 0 must remain finite as $x \to 1^-$ and arrives at the conclusion that we must have $F_1(1) = 0$. All this, however, is a special case of our equation (5.8) wherein if $C_1 = 0$ and $G_1(1) = 0$, we get $F_1(1) = 0$. It would appear therefore that this problem ought to be posed as we have done it.

For the particular case of $g_1(x) = 0$, the problem posed by equations (5.3) has also been considered by Sneddon [5, page 26]. For this particular case, our solution coincides with his.

We shall now consider some special cases.

6. SOME SPECIAL CASES

We consider the dual integral equations

$$\int_0^\infty f(t)(\alpha \sin xt + t \cos xt)\, dt = f_1(x), \quad 0 < x < 1 \tag{6.1a}$$

and
$$\int_0^\infty t f(t)(\alpha \sin xt + t \cos xt)\, dt = 0, \quad x > 1. \tag{6.1b}$$

with the (additional) requirement that the quantity $\int_0^\infty f(t)\ (\alpha \sin xt + t \cos xt)\, dt$ is continuous at x = 1.

We give results for various special cases:

1. $f_1(x) = 1$ in $0 < x < 1$

In this case

$$f(t) = \int_0^1 u J_0(ut)\, f_2(u)\, du + \frac{f_2(1)}{K_0(\alpha)} \int_1^\infty u J_0(ut)\, K_0(\alpha u)\, du \tag{6.2a}$$

where
$$f_2(u) = \frac{2}{\pi} \int_0^u \frac{e^{-\alpha x}}{\sqrt{u^2 - x^2}} dx. \tag{6.2b}$$

2. $f_1(x) = 1 + \alpha x$ in $0 < x < 1$.

In this case

$$f(t) = \frac{J_1(t)}{t} + \frac{1}{K_0(\alpha)} \int_1^\infty u\, J_0(ut)\, K_0(\alpha u)\, du \tag{6.3}$$

and $u(x,0) = \int_0^\infty f(t)\,(\alpha \sin xt + t\cos xt)\,dt$ is given by

$$u(x,0) = 1 + \alpha x, \qquad x \le 1$$
$$= 1 + \alpha x - \frac{\alpha^2}{K_0(\alpha)} \int_1^x K_1(\alpha u)\sqrt{x^2-u^2}\,du$$
$$- \frac{\alpha x}{K_0(\alpha)} \int_1^x \frac{K_1(\alpha u)}{\sqrt{x^2-u^2}}\,du, \quad x \ge 1. \tag{6.4}$$

For numerical calculations, it is more convenient to write

$$u(x,0) = 1 + \alpha x - \frac{K_1(\alpha)}{K_0(\alpha)}\,\alpha x\sqrt{x^2-1}$$
$$+ \frac{\alpha^2}{K_0(\alpha)} \int_1^x \left[\frac{x}{u} K_2(\alpha u) - K_1(\alpha u)\right]\sqrt{x^2-u^2}\,du, \qquad x \ge 1. \tag{6.5}$$

For $\alpha = 0$, we get $u(x,0) = 1$, $x \ge 1$ which is correct. For $\alpha > 0$, the graphs of $u(x,0)/(1+\alpha)$ for various values of α are given in Figure 2.

3. For $f_1(x) = \alpha x^2 + 2x$, we get

$$f(t) = \frac{4}{\pi}\int_0^1 u^2 J_0(ut)\,du + \frac{4}{\pi K_0(\alpha)}\int_1^\infty uJ_0(ut)\,K_0(\alpha u)\,du \tag{6.6}$$

and so on. It is easy to obtain $f(t)$ for $f_1(x) = \alpha x^p + px^{p-1}$, $p \ge 1$, and then by superposition, for any analytic function $f_1(x)$.

As a final example, we take $\alpha = 0$ and take $f_1(x) = x^p$, $p > 0$, and $g_1(x) = 0$ in (5.2). The resulting problem is: Find C_1 and $f(t)$ such that

$$C_1 + \int_0^\infty tf(t)\cos xt\,dt = x^p \quad \text{in} \quad 0 < x < 1 \tag{6.7a}$$

and $$\int_0^\infty t^2 f(t)\cos xt\,dt = 0 \quad \text{in} \quad x > 1. \tag{6.7b}$$

We find

$$C_1 = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma[(p+1)/2]}{\Gamma[(p+2)/2]} \tag{6.8}$$

and $$f(t) = C_1\int_0^1 u^{p+1} J_0(ut)\,du - C_1\int_0^1 uJ_0(ut)\,du \tag{6.9}$$

and then

$$u(x,0) = C_1 + \int_0^\infty tf(t)\cos xt\,dt = x^p \quad \text{in } 0 \le x \le 1$$
$$= C_1\int_0^1 \frac{pu^{p-1}x}{\sqrt{x^2-u^2}}\,du \quad \text{in } x \ge 1. \tag{6.10}$$

For $p = 0$, we get $f(t) = 0$, $C_1 = 1$, which is correct.

Some other interesting cases are:

$$p = 1 \Rightarrow u(x,0) = x \quad \text{in} \quad 0 \le x \le 1$$
$$= \frac{2}{\pi}\,x\sin^{-1}\left[\frac{1}{x}\right] \quad \text{in} \quad x \ge 1,$$
$$p = 2 \Rightarrow u(x,0) = x^2 \quad \text{in} \quad 0 \le x \le 1$$

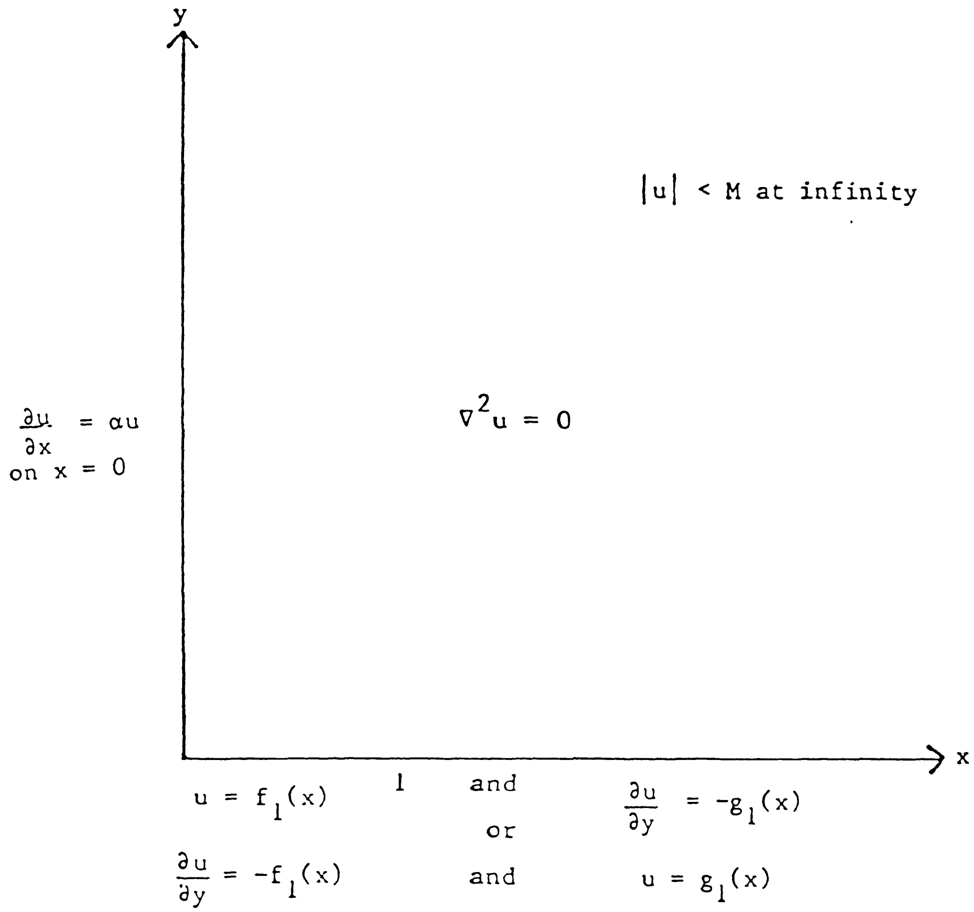
$$= x^2 - x\sqrt{x^2-1} \qquad \text{in} \qquad x \geq 1,$$

$$p = 3 \quad \Rightarrow \quad u(x,0) = x^3 \qquad \text{in} \qquad 0 \leq x \leq 1$$

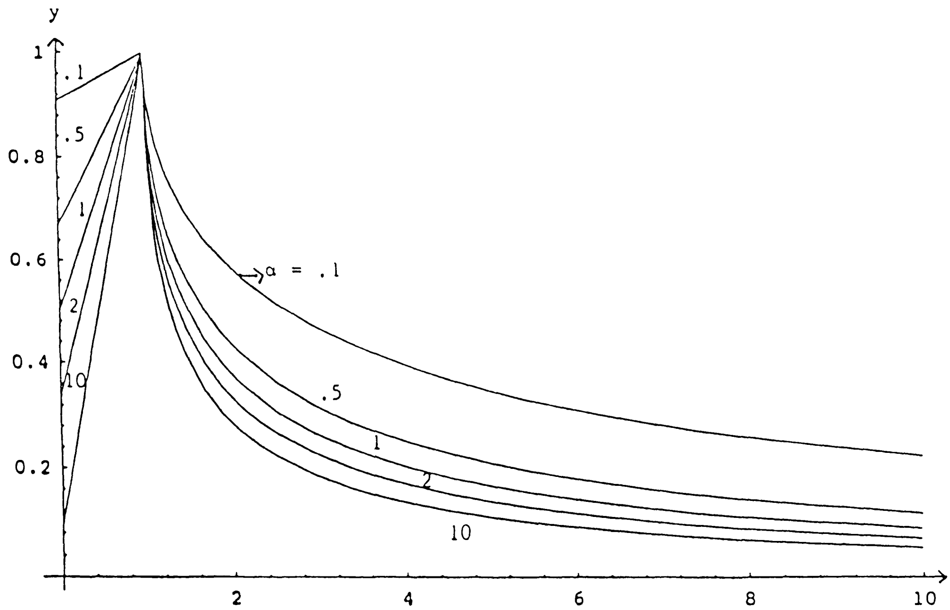
$$= \frac{2}{\pi}\left[x^3 \sin^{-1}\left(\frac{1}{x}\right) - x\sqrt{x^2-1}\,\right] \qquad \text{in} \qquad x \geq 1,$$

and so on.

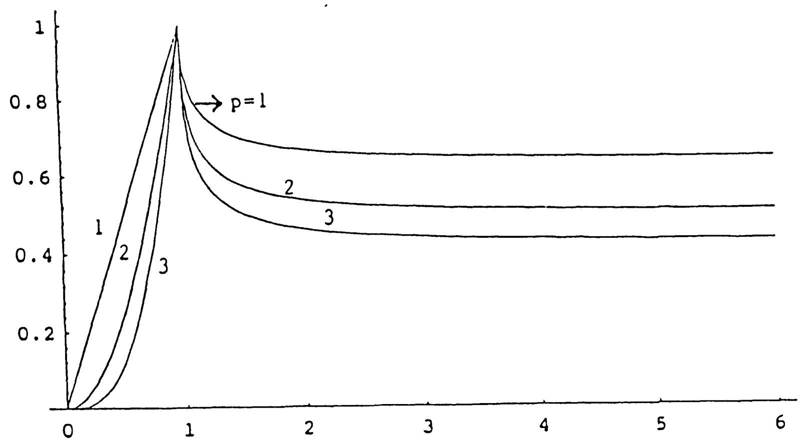
The graphs of u(x,0) for several values of p are given in Fig. 3.



F I G. 1 - The Problem



F I G. 2
Values of y = u(x,0)/(1 + α), equation (6.5), for several values of α.



F I G. 3
Values of u(x,0), equation (6.10), for several values of p.

REFERENCES

1. SNEDDON, I.N., The use of Integral Transforms, McGraw Hill Book Company (1972).

2. NASIM, C., and AGGARWALA, B.D. On some dual integral equation. Indian. J. Pure Appl. Math. 15 (3), 323–340, 1984.

3. ERDELYI et al, Tables of Integral Transforms, Vol. I, Bateman Manuscript Project, McGraw Hill Book Co. Inc., New York, (1954).

4. SNEDDON, I.N., Mixed Boundary Value Problems in Potential Theory, John Wiley and Sons Inc., New York, (1966).

5. SNEDDON, I.N., Crack Problems in the Classical Theory of Elasticity, John Wiley and Sons, Inc., New York, (1969).