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The Liouvillian Perturbations of the Kerr-Newman Black Hole

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Cody Holder

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a dissertation entitled "The Liouvillian Perturbations of the Kerr-Newman Black Hole" submitted by Cody Holder in partial fulfillment of the requirements for the degree of Master of Science.

Eugene Couch

Supervisor, Dr. Eugene Couch

Department of Mathematics & Statistics

Dr. Gilad Gour

Department of Mathematics & Statistics

Dr. David Hobill

Department of Physics & Astronomy

Dr. Richard Churchill

Hunter College & City University of New York

Adjunct Professor, University of Calgary

Sept. 1, 2007

Date

Abstract

The well-known Kovacic algorithm is applied to the differential equations that govern the "synthetic" radial perturbations of the Kerr-Newman black hole as a means to identify all of the Liouvillian (i.e. closed-form) solutions. The analysis includes the scalar, Dirac, electromagnetic and gravitational perturbations (i.e. spin 0,1/2,1,2 fields), the non-extreme and extreme geometries, and the perturbations of the Kerr and Schwarzschild black holes. This work presents new results that extend, and essentially complete, the analysis of the Liouvillian perturbations of the Kerr-Newman black hole initiated by this author and supervisor in a recent article.

Acknowledgements

First and foremost, I wish to thank my supervisor, Dr. Eugene Couch, for introducing me to relativity, special functions, and the Kovacic algorithm, and for his teaching, guidance and encouragement throughout my undergraduate and graduate programs. I hope to continue to benefit from his expertise as I tackle the open problems brought forth by the results of this thesis. Thanks also to Dr. Richard Churchill for his careful reading of the manuscript and for sharing with me his knowledge and experience in differential algebra, and thanks to the committee as a whole for their insightful suggestions. I also want to thank my father, Bruce Holder, for igniting my interest in math and science at a young age and for believing in me, and I am grateful to my brothers, cousins, aunts, uncles, grandparents and friends for their support.

Dedication

This thesis is dedicated in loving memory of my mother, Rhondda Mary Lamb.

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1 Introduction

The interaction of a black hole with a weak gravitational field is modelled by a linearization of the Einstein field equations about the background space-time metric of the black hole. The effects of a black hole on a non-gravitational field are described by the perturbations of the governing field equations considered on the unperturbed background geometry of the black hole. In the Newman-Penrose formalism, the first order perturbations of the black holes considered in this thesis are governed by a single wave equation [2]. The solution of this wave equation is initiated by assuming a decomposition into radial and angular parts, along with exponential time and azimuthal dependence. Namely, one assumes the solution has the form $e^{i\omega t}R(r)T(\theta)e^{im\phi}$, where (t, r, θ, ϕ) are the usual spherical space-time coordinates. This separation of variables results in two decoupled second-order linear ordinary differential equations, and introduces the complex separation constant λ, which is generally left to be determined as a non-singular eigenvalue of the angular equation. Furthermore, m is required by the angular equation to be an integer, and ω is left to be determined as the complex characteristic frequency, subject to physical boundary conditions one may wish to impose on the radial equation (e.g. quasi-normal modes [11]).

The Kerr-Newman black hole is characterized by the presence of the physical parameters mass, M, angular momentum, a, and charge, q. For the non-extreme geometries, these parameters obey the physical restriction $a^2 + q^2 < M^2$, and for the extreme geometries, these parameters obey $a^2 + q^2 = M^2$. With $q = 0$, the Kerr-Newman metric reduces to the Kerr metric, and with $a = q = 0$ to the Schwarzschild metric. Throughout this thesis, any parameter set that satisfies the restrictions

$$a^2 + q^2 \leq M^2, \;\; a, q \in \mathbb{R}, \;\; M \in \mathbb{R}^+$$

is deemed "physical", and any parameter set violating this restriction is deemed "unphysical". An additional physical restriction is imposed in the Schwarzschild case, in which the angular equation restricts the eigenvalue λ to $\lambda = \ell(\ell+1)$, where ℓ is a non-negative integer. In all cases that are not Schwarzschild we consider λ to be an arbitrary complex constant; we leave the specification of λ by the angular DE according to physical boundary conditions to a separate study.

In this thesis, we consider the radial perturbations of the Schwarzschild, Kerr, and Kerr-Newman black holes, including the interactions with scalar, Dirac, electromagnetic and gravitational fields, labelled respectively by the spin parameter $s = 0, 1/2, 1, 2$. In total, the radial equations depend on the parameter set $(\omega, \lambda, a, q, M, m, s)$, and for brevity we refer to this set as the "black hole parameters". It is common practice to scale M by an appropriate choice of units; we often choose $M = 1/2$ or $M = 1$ to simplify the analysis.

Kokkotas [2] achieves separation of variables for the perturbations of Kerr-Newman (i.e. $aq \neq 0$) by forcing either the electromagnetic or the gravitational perturbations to be zero. The resulting radial and angular equations describe what we have termed the "synthetic" perturbations of the Kerr-Newman black hole. Kokkotas shows that the equations describing the perturbations of the Kerr and Schwarzschild black holes (derived directly from the respective metrics in the Newman-Penrose formalism, e.g. [3]) are recovered by setting $q = 0$ or $a = q = 0$ in the equations for the synthetic perturbations of Kerr-Newman. Thus, Kokkotas' radial equation describes the perturbations of all of the black holes we wish to consider in this thesis. This master equation is given by:

$$\Delta^{(1-s)} \left(\Delta^{(s+1)} R_r\right)_r + BR = 0 \tag{1.1}$$

where

$$B = X^2 - \left(\lambda + a^2\omega^2 - 2am\omega\right)\Delta + is\left(-\Delta_r X + 2X_r \Delta\right)$$

with

$$X = \left(r^2 + a^2\right)\omega - am,$$

$$\Delta = r^2 - 2Mr + a^2 + q^2.$$

Here, and as will be the convention throughout this thesis, the subscript r denotes differentiation with respect to r.

To aid our analysis we introduce the quantity

$$\epsilon = \sqrt{M^2 - a^2 - q^2}/M$$

so that Δ can be written as

$$\Delta = (r - r_+)(r - r_-),$$

where

$$r_\pm = M(1 \pm \epsilon).$$

Notice that r_+ and r_- represent the finite singularities of equation (1.1).

The aim of this thesis is to identify all of the Liouvillian (i.e. closed-form) solutions to equation (1.1). To do this, we provide an extensive analysis of the application of the well-known Kovacic algorithm [1] to equation (1.1). To avoid obtaining known static solutions, we take $\omega \neq 0$.

We note that there are well-known "algebraically special" values of the frequency ω for which equation (1.1) admits exact Liouvillian solutions (see e.g. [4], [7]). These special frequencies are characterized by the property that the radial fields represent purely outgoing (or purely ingoing) radiation [4]. Chandrasekhar and others have constructed these solutions for $s = \pm 2$ and for the non-extreme Kerr and Schwarzschild black holes. The Kovacic algorithm reconstructs these known solutions, and also their analogies for the Kerr-Newman case, for $s = \pm 1/2, \pm 1$, and for the extreme geometries.

For a second-order linear homogeneous differential equation with complex rational coefficients, the Kovacic algorithm constructs a Liouvillian solution (and consequently all solutions) if and only if such a solution exists. To accomplish this, the algorithm specifies a finite set of admissible functional forms (establishes an ansatz) that exhausts all possible Liouvillian solutions. Each solution in question is associated with a linear homogeneous differential equation (with polynomial coefficients) that must admit a polynomial solution in order for a Liouvillian solution to exist.

If the coefficients of the differential equation are fixed, the possible degrees for a polynomial solution are restricted to a finite set of integer values. In this case, the Kovacic algorithm amounts to the search for polynomial solutions among a finite collection of linear differential equations. If a Liouvillian solution is found in one specific case, then the algorithm terminates.

For equation (1.1), however, the rational coefficients depend on the black hole parameters and thus the possible degrees of the polynomial solutions are not necessarily restricted by the differential equation. In most cases, the possible Liouvillian solutions are indexed by an integer n, where n is the degree of the polynomial solution in question. Thus, although we provide the analysis necessary to identify and access all of the Liouvillian solutions, it is not possible to explicitly construct all such solutions.

Furthermore, the algorithm does not decide whether Liouvillian solutions exist, rather it determines the values of the black hole parameters for which Liouvillian solutions exist. In other words, aside from physical/practical considerations, the Liouvillian solutions always exist. For this reason, the algorithm does not terminate if a Liouvillian solution is constructed in a specific case. Thus, the Kovacic analysis of equation (1.1) is necessarily divided into several independent cases and, in general, each case generates an infinite class of Liouvillian solutions that exist for constrained values of the black hole parameters.

The recurring and over-arching theme of this thesis, and the most notable original contribution, is the reduction of the number of independent cases arising in the Kovacic analysis of equation (1.1). Some of these reductions follow directly from symmetries in the distribution of the black hole parameters in equation (1.1), but the more difficult and interesting reductions require new methods that we establish in section 4.1 via the propositions 4.1, 4.2, 4.3 and the conjecture 4.1.

In section 2, we present the Kovacic algorithm in full generality and give an overview of its application to equation (1.1) for non-extreme and extreme geometries separately. The algorithm divides the possible Liouvillian solutions into three mutually exclusive types, and each type is associated with its own algorithmic ansatz; we find types 1 and 2 are the only possibilities for equation (1.1). The content of this section is essentially a review of the material presented in references [1] and [8]; there are no new results here. Rather, the purpose of this section is to introduce the reader to the Kovacic algorithm and to establish the essential details of the application of the type 1 and type 2 algorithms to equation (1.1) that will be employed and frequently referred to in subsequent sections.

In section 3, we apply the Kovacic algorithm to the perturbations of Schwarzschild and prove the non-existence of type 1 solutions up to a set of well-known algebraically special modes [7]. This proof was initiated by this author and supervisor in [8] and is taken to completion in this section. We suspect non-existence for type 2 solutions and, although we do not prove it, we present numerical evidence in support of this conjecture. The decisive character of the Schwarzschild result is lost upon introduction of the parameters (a, q, m) in the Kerr-Newman case and thus (even though the equations contained in this section can be obtained as direct specializations of those in section 4) we find it both economical, with respect to the emphasis of these unique results for Schwarzschild, and instructive, with respect to the details of the algorithm, to devote an entire section to the Schwarzschild case.

In section 4, we apply the Kovacic algorithm to the perturbations of non-extreme Kerr-Newman. The type 1 solutions are plentiful, and consist of a finite set of algebraically special [4] solutions and two distinct infinite classes of solutions existing for well-defined values of $\omega \neq 0$ and generally unknown values of $\lambda(a, \epsilon, m)$. The central result is the construction of differential operators that associate each class with a specific recursion relation. This provides a new classification of the Liouvillian perturbations of the Kerr-Newman black hole, and allows one to efficiently obtain the complete set of type 1 solutions, along with the associated parameter values, for low values of n. Furthermore, these operators reveal several new infinite classes of explicit solutions that hold for specific values of (ω, λ, a). The results of this section hold for the full perturbations of non-extreme Kerr with the charge set to $q = 0$. We describe the type 2 algorithm in terms of the Kerr-Newman parameters, but the analytical results are inconclusive; all type 2 solutions that we have obtained exist for unphysical parameter values only.

In section 5, we apply the Kovacic algorithm to the perturbations of extreme Kerr-Newman. Aside from a well defined infinite set of explicit solutions, we prove that all extreme type 1 solutions can be obtained by the specialization to $\epsilon = 0$ of an infinite class of non-extreme type 1 solutions that exist for pure imaginary ω. However, we find the separate analysis of the extreme case is necessary to obtain the functional form of these solutions. Furthermore, we illustrate through a conjectural example that the separate consideration of the extreme case may allow one to obtain a greater number of explicit solutions than in the non-extreme case.

In section 6, we summarize the results of sections 3, 4 and 5 and discuss some open problems. Most notably, we mention the extension of the methods of this thesis to the analysis of the Liouvillian perturbations of the Reissner-Nordstrom black hole initiated by this author and supervisor in [8].

2 The Kovacic Algorithm

2.1 Overview

The Kovacic algorithm is an attempt to construct a "closed-form" solution to the differential equation

$$Y_{rr} + aY_r + bY = 0, \tag{2.1}$$

where a and b are complex-valued rational functions of r. If the attempt fails, then no "closed-form" solutions exist.

The informal notion of "closed-form" we refer to is pinned down, in the language of differential algebra, by the definition of a *differential Liouvillian field.* A *differential field* is a field of characteristic zero equipped with a derivation. With respect to the solution set of the differential equation (2.1), the essential example is $\mathbb{C}(r)$, the field of complex rational functions with the ordinary derivative $\partial_r : Y \longmapsto Y_r$. $\mathbb{C}(r)$ represents the trivial Liouvillian field, and all other Liouvillian fields extend $\mathbb{C}(r)$ to include compositions and algebraic combinations of polynomials, exponentials and indefinite integrals; i.e. closed-form functions.

A differential field, F, is Liouvillian if there is a tower of differential fields:

$$\mathbb{C}(r) = F_0 \subset F_1 \subset ... \subset F_n = F,$$

satisfying for each $i = 1, ..., n$ exactly one of

$$i)\ F_i = F_{i-1}(\alpha),\ \ \alpha_r/\alpha \in F_{i-1}.$$

$$ii)\ F_i = F_{i-1}(\alpha),\ \ \alpha_r \in F_{i-1}.$$

$$iii)\ F_i \text{ algebraic over } F_{i-1}.$$

A function is Liouvillian if it is contained in a Liouvillian field. Simple examples of

Liouvillian fields (e.g. $\mathbb{C}(r)(e^{ir})$) confirm that all the usual candidates for closed-form solutions to (2.1) (e.g. polynomials, trigonometric functions, exponentials) are Liouvillian functions. Special functions (e.g. the generic hypergeometric function) are generally not Liouvillian, although there are obvious exceptions (e.g. Jacobi polynomials). For equation (2.1), Kovacic spells out precisely which types of Liouvillian solutions can occur.

First, recall the well-known factor transformation

$$Y = e^{-\frac{1}{2}\int a} y, \tag{2.2}$$

which takes equation (2.1) to the normal form

$$y_{rr} = (-b + a^2/4 + a_r/2)y. \tag{2.3}$$

It follows from (2.2) that (2.1) admits a Liouvillian solution if and only if (2.3) does. Since the coefficient of y in (2.3) is rational, in the search for Liouvillian solutions to (2.1) there is no loss of generality in applying the transformation (2.2) as a preliminary step and developing an algorithm for an equation of the form:

$$y_{rr} = \rho y, \tag{2.4}$$

where $\rho \in \mathbb{C}(r)$. This is the approach taken by Kovacic, and for the remainder of this section we refer to equation (2.4) as "the DE".

Note that for every solution to the DE a second linearly independent solution can be obtained by reduction of order:

$$y' = y \int \frac{1}{y^2}. \tag{2.5}$$

Evidently, y' is Liouvillian if and only if y is Liouvillian, and thus either all solutions to the DE are Liouvillian or none are. If all solutions are Liouvillian, the algorithm is guaranteed to find at least one Liouvillian solution.

Kovacic's Theorem The solution set of the DE is one of four types:

Type 1 : There is solution of the form $e^{\int \Omega}$, where $\Omega \in \mathbb{C}(r)$.

Type 2 : There is a solution of the form $e^{\int \varpi}$, where ϖ is quadratic over $\mathbb{C}(r)$, and type 1 does not hold.

Type 3 : All solutions are algebraic over $\mathbb{C}(r)$, and types 1 and 2 do not hold.

Type 4 : There are no Liouvillian solutions.

The proof relies on the tenets of the Picard-Vessiot theory of differential equations (also known as differential Galois theory). In a manner analogous to the classical Galois theory for polynomial equations, differential Galois theory associates every differential equation with a Galois group that contains the essential information about the algebraic properties of solutions. The details of the general theory are beyond the scope of [1] and, consequently, of this thesis. However, the construction of the Galois group corresponding to the DE is straight-forward and leads directly to the proof of Kovacic's theorem, as follows.

Let (η, φ) be a set of fundamental solutions to the DE. The Picard-Vessiot extension of the DE is the differential field defined by:

$$G = \mathbb{C}(r)(\eta, \varphi, \eta_r, \varphi_r).$$

The Galois group of the DE, denoted by $Gal(G/\mathbb{C}(r))$, is defined as the group of all differential automorphisms of G (i.e. bijective field homomorphisms commuting with ∂_r) that leave $\mathbb{C}(r)$ pointwise invariant. The normal form (2.4) reveals an isomorphism between $Gal(G/\mathbb{C}(r))$ and a subgroup of $SL_2(\mathbb{C})$. First, for every $\sigma \in Gal(G/\mathbb{C}(r))$ we have

$$\sigma(\eta)_{rr} = \sigma(\eta_{rr}) = \sigma(\rho\eta) = \rho\sigma(\eta),$$

i.e. $\sigma(\eta)$ and $\sigma(\varphi)$ solve the DE. Since every solution is a linear combination of η

and φ we have

$$\sigma \begin{pmatrix} \eta \\ \varphi \end{pmatrix} = A_\sigma \begin{pmatrix} \eta \\ \varphi \end{pmatrix},$$

where $A_\sigma \in GL_2(\mathbb{C})$. Furthermore, since the Wronskian, $|W|$, of the DE is a complex constant (which is easily checked using the DE), it is left invariant by σ and we have

$$|W| = \sigma|W| = |\sigma W| = |A_\sigma W| = |A_\sigma|\,|W|,$$

i.e. $A(\sigma) \in SL_2(\mathbb{C})$. It is easy to verify that

$$A : Gal(G/\mathbb{C}(r)) \to SL_2(\mathbb{C}) : \sigma \mapsto A_\sigma$$

is an injective group homomorphism. This representation of the Galois group as a matrix group depends on the choice of fundamental solutions, but only up to conjugation by a matrix in $SL_2(\mathbb{C})$. One of the assertions of the fundamental theorem of Galois theory (see e.g. [10], theorem 1.27) is that the image of $Gal(G/\mathbb{C}(r))$ under A is an algebraic group. By a theorem of Kaplansky (see e.g. [10], theorem 4.29), every algebraic subgroup $g \subset SL_2(\mathbb{C})$ (in particular the image of $Gal(G/\mathbb{C}(r))$ under A), is one of four types:

Type 1 : g is triangulisable.

Type 2 : g is conjugate to a subgroup of

$$D^\dagger = \{\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} |c \in \mathbb{C},\ c \neq 0\} \cup \{\begin{pmatrix} 0 & c \\ c^{-1} & 0 \end{pmatrix} |c \in \mathbb{C},\ c \neq 0\},$$

and type 1 does not hold.

Type 3 : g is finite, and types 1 and 2 do not hold.

Type 4 : $g = SL_2(\mathbb{C})$.

A well-known corollary of the fundamental theorem (see e.g. [10], theorem 1.27) states that if $\sigma(x) = x$ for all $\sigma \in Gal(G/\mathbb{C}(r))$, then $x \in \mathbb{C}(r)$, and this is the final piece of information needed to prove Kovacic's theorem. Now, if the Galois group of the DE is type 1 (we may assume g is triangular) then $\sigma(\eta_r/\eta) = \eta_r/\eta$, and therefore $\eta_r/\eta \in \mathbb{C}(r)$. Thus $\eta = e^{\int \Omega}$ where $\Omega \in \mathbb{C}(r)$; i.e. the DE admits a type 1 solution. Next, if the Galois group is type 2 (we may assume g is a subgroup of $D^\dagger$) then $\sigma(\eta_r/\eta + \varphi_r/\varphi) = \eta_r/\eta + \varphi_r/\varphi$ and $\sigma(\eta_r\varphi_r/\eta\varphi) = \eta_r\varphi_r/\eta\varphi$, and therefore $\eta_r/\eta + \varphi_r/\varphi \in \mathbb{C}(r)$ and $\eta_r\varphi_r/\eta\varphi \in \mathbb{C}(r)$. Since $(\eta_r/\eta)^2 - [\eta_r/\eta + \varphi_r/\varphi]\eta_r/\eta + [\eta_r\varphi_r/\eta\varphi] = 0$, we have $\eta = e^{\int \varpi}$ where ϖ is quadratic over $\mathbb{C}(r)$; i.e. the DE admits a type 2 solution. For the proof of type 3 we refer the reader to [1] (subsequently, we will show that type 3 does not hold for the black hole perturbations considered in this thesis). Type 4 follows from the Lee-Kolchin theorem, which states that if G is contained in a Liouvillian field then G^0, the component of the identity of G, is solvable [10]. If the Galois group of the DE is type 4 then $G^0 = SL_2(\mathbb{C})$ is not solvable and thus G is not contained in a Liouvillian field; i.e. the DE admits no Liouvillian solutions. This completes the proof.

In addition to the theorem, Kovacic provides an algorithm, as follows, for deciding which type holds.

A type 1 solution exists if and only if there is a non-trivial *polynomial* solution to the differential equation

$$P_{rr} + 2\Omega P_r + (\Omega_r + \Omega^2 - \rho)P = 0, \tag{2.6}$$

where $\Omega \in \mathbb{C}(r)$ is one of a finite collection of functions specified algorithmically in terms of the coefficients of the partial fraction expansion of ρ. The specification of Ω is such that the order and number of the poles of $\Omega_r + \Omega^2 - \rho$ is minimal. We refer to equation (2.6) as the "type 1 DE". If a polynomial solution is found, the type 1 solution is given by

$$y = Pe^{\int \Omega}. \tag{2.7}$$

A type 2 solution exists if and only if a type 1 solution does not exist and there is a non-trivial *polynomial* solution to the differential equation

$$P_{rrr} + 3\theta P_{rr} + (3\theta_r + 3\theta^2 - 4\rho)P_r + (\theta_{rr} + 3\theta\theta_r + \theta^3 - 4\theta\rho - 2\rho_r)P = 0 \qquad (2.8)$$

where $\theta \in \mathbb{C}(r)$ is one of a finite collection of functions specified algorithmically in terms of the coefficients of the partial fraction expansion of ρ. The specification of θ is such that the order and number of the poles of $\theta_{rr} + 3\theta\theta_r + \theta^3 - 4\theta\rho - 2\rho_r$ is minimal. We refer to equation (2.8) as the "type 2 DE". If a polynomial solution is found, the type 2 solution is given by

$$y = e^{\int \varpi}, \qquad (2.9)$$

where ϖ solves the quadratic equation

$$\varpi^2 - \phi\varpi + \frac{1}{2}\phi_r + \frac{1}{2}\phi^2 - \rho = 0, \qquad (2.10)$$

with $\phi = \theta + P_r/P$.

For the differential equations considered in this thesis, we easily show there are no type 3 solutions. This follows from a set of simple conditions on the orders of poles of ρ that Kovacic proves are necessary for each type to hold:

Type 1 : Every pole must have either even order, or order 1.

Type 2 : At least one pole must have either odd order greater than 2, or order 2.

Type 3 : The order of the pole at infinity must be at least 2.

Our restriction $\omega \neq 0$ implies the order of the pole at infinity of ρ, as given by (2.16), is 0 and thus type 3 does not hold. We therefore omit discussion of the type 3 algorithm (for a list of recent improvements made to Kovacic's original type 3 algorithm see [10], pg. 128).

To implement the type 1/type 2 algorithms, we search for polynomial solutions to the type 1/type 2 DEs for every rational function Ω/θ in the specified collection. Since the coefficients of these DEs are known rational functions (specified, ultimately, in terms of the black hole parameters), constructing a polynomial solution is a straightforward application of linear algebra. The level of computation increases with the order of the DE, the order and number of poles of the rational coefficients, and the degree of the polynomial. The procedure is well known (e.g. [5], [6]), but we present the details here to establish a consistent approach.

Consider a linear homogeneous ODE, $(L)P = 0$, with rational coefficients depending on parameters. To obtain polynomial solutions, we make a polynomial ansatz:

$$P = \sum_{k=0}^{n} a_k (r - r_0)^k, \;\; a_k = 0 \text{ for } k \in)0, n(, \tag{2.11}$$

where n is a non-negative integer and where the constant r_0 is chosen to be a singularity of the DE. We re-scale $L \to \alpha(r)L$ to obtain polynomial coefficients and use the assumption (2.11) to determine the coefficient of $(r - r_0)^k$ in $\alpha(r)(L)P$. The vanishing of this coefficient is the recursion relation:

$$A_{-1}a_{k-1} + A_0 a_k + ... + A_N a_{k+N} = 0, \tag{2.12}$$

which defines a_k for all $k \in \mathbb{Z}$. The integer N increases with the degrees of the polynomial coefficients and the order of the DE. The A_i are polynomials in the index variable k and the parameters of the DE. Note that A_N satisfies $A_N|_{k=-N} = 0$.

To obtain necessary and sufficient conditions for the existence of a polynomial solution, we require the ansatz to be consistent with the recursion for all $k \in \mathbb{Z}$. For $k \in)1 - N, n + 1($ the recursion is satisfied identically, thus we only need to check $k \in [1 - N, n + 1]$.

At $k = n + 1$ the recursion requires

$$A_{-1}|_{k=n+1} = 0. \tag{2.13}$$

This condition is equivalent to the vanishing of the highest power term in $(L)\alpha(r)\,P$ and in practice is imposed prior to the computation of the recursion relation. If (2.13) happens to be independent of the DE parameters, the result is the restriction of n to a finite set of integers. Otherwise, (2.13) defines one parameter in terms of n and the other parameters, and polynomial solutions are sought for all n. We refer to (2.13) as the "integer condition".

At any particular value of n, for $k = n, ..., 1$ the recursion relation generates the coefficients $a_{n-1}, ..., a_0$ in terms of the parameters, thereby constructing the polynomial P. A rare exception occurs if $A_{-1}|_i = 0$ for some $i = n, ..., 0$, for then at $k = i$ the recursion relation is a constraint on the parameters and a_{i-1} is left to be chosen arbitrarily. It turns out that this exception does not occur for the recursion relations considered in this thesis.

Finally, for $k = 1 - N, ..., 0$ the recursion imposes N constraints on the parameters:

$$A_{N-1}|_{k=0}a_0 + A_N|_{k=0}a_1 = 0, \tag{2.14}$$

$$A_{N-2}|_{k=-1}a_0 + A_{N-1}|_{k=-1}a_1 + A_N|_{k=-1}a_2 = 0, \ldots$$

The manner in which the parameters enter these constraints depends on the generated values of the coefficients a_k at a particular value of n. With this understanding, we often refer to (2.14) simply as "the constraints" without reference to the recursion. If the integer condition (2.13) restricts n, the existence of a polynomial solution is decided by the constraints at the appropriate value of n. However, if the integer condition (2.13) depends on the parameters, the existence of polynomial solutions, and thus the decidability of the Kovacic algorithm, rests on a suitable characterization of the constraints for all n.

To clarify the nomenclature, we mention that the symbol N is introduced in a number of different contexts throughout this thesis to denote an arbitrary integer. The integer N that we use in the above discussion takes the values 1 and 3 respectively for the recursion relations that describe the type 1 and type 2 perturbations of Kerr-Newman.

In the remainder of this section, we outline the application of the Kovacic algorithm to the perturbations of the Kerr-Newman black hole. For non-extreme and extreme geometries separately, we give the algorithmic specifications of the functions Ω and θ in terms of the coefficients of the partial fraction expansion of ρ and state the form of the resulting type 1 and type 2 DEs, integer conditions and solutions. In addition, for type 1 we state the recursion relations and constraints in terms of the partial fraction expansion of ρ and indicate parameter restrictions that produce solutions for every n.

In preparation for the application of the algorithm, equation (1.1) is taken to the normal form $y_{rr} = \rho y$ by the transformation

$$R = y\Delta^{-(s+1)/2}, \tag{2.15}$$

and ρ is given by

$$\rho = \frac{s(s+1)\Delta + (s^2-1)\epsilon^2 M^2 - B}{\Delta^2}. \tag{2.16}$$

2.2 The Algorithm for Non-extreme Geometries

For non-extreme geometries ($0 < \epsilon \leq 1$), the partial fraction expansion of ρ is given by

$$\rho = -\omega^2 + \frac{b_+}{(r-r_+)^2} + \frac{b_-}{(r-r_-)^2} + \frac{b_{1+}}{r-r_+} + \frac{b_{1-}}{r-r_-}, \tag{2.17}$$

where $r_\pm = M(1 \pm \epsilon)$ and the constants $b_\pm, b_{1\pm}$ are specified in terms of the black hole parameters $(a, \epsilon, M, \omega, \lambda, m, s)$ by (2.16). In this section, we give the essential details of the type 1 and type 2 algorithms as applied to the DE

$$y_{rr} = \rho y, \tag{2.18}$$

with ρ given by (2.17).

The Type 1 Algorithm

The type 1 algorithm specifies the collection of functions Ω to be

$$\Omega = \gamma + \frac{\alpha_+}{r-r_+} + \frac{\alpha_-}{r-r_-},$$

where the "Kovacic constants" $\gamma, \alpha_\pm$ take the values

$$\gamma = i\delta_0\omega,$$
$$2\alpha_\pm = 1 + \delta_\pm\sqrt{1+4b_\pm},$$

where $\delta_i = \pm 1$ independently for $i = 0, +, -$. If $r_\pm$ has order 0 or 1 the algorithm requires $\delta_\pm = -1$ or 1 so as to obtain $\alpha_\pm = 0$ or 1, respectively. Otherwise, the collection consists of 8 different functions Ω, each labelled by a set of Kovacic constants and a corresponding set of sign choices for δ_i. We refer to a particular instance of $(\Omega,\ \gamma, \alpha_i, \delta_i)$ as a "Kovacic case" and adopt the intuitive notation $\Omega_\nu(sgn(\delta_0), sgn(\delta_+), sgn(\delta_-))$ to enumerate the cases. Every Kovacic case is considered in the search for polynomial solutions to the type 1 DE (2.6).

In terms of the Kovacic constants, the type 1 DE is

$$P_{rr} + F_1 P_r + F_2 P = 0, \tag{2.19}$$

where

$$\begin{aligned} F_1 &= 2(\gamma + \frac{\alpha_+}{r - r_+} + \frac{\alpha_-}{r - r_-}), \\ F_2 &= \frac{g_+ - b_{1+}}{r - r_+} + \frac{g_- - b_{1-}}{r - r_-}, \end{aligned}$$

with

$$g_\pm = 2\alpha_\pm(\gamma \pm \frac{\alpha_\mp}{2\epsilon M}).$$

If a polynomial solution is found, the type 1 solution is given by

$$y = e^{\gamma r}(r - r_+)^{\alpha_+}(r - r_-)^{\alpha_-} P. \tag{2.20}$$

If no polynomial solutions exist, there are no type 1 solutions.

We write

$$P = \sum_{k=0}^{n} a_k z^k, \; a_n \neq 0, \tag{2.21}$$

where $z = r - r_\pm$, then the necessary integer condition (2.13) is given by

$$n = \frac{1}{2\gamma}(b_{1+} + b_{1-}) - \alpha_+ - \alpha_-. \tag{2.22}$$

If the black hole parameters are specified such that only one pole at $r_\pm$ remains (*i.e.* $b_\mp = b_{1\mp} = 0$), the type 1 DE is given by

$$zP_{zz} + 2(\gamma z + \alpha_\pm)P_z - 2\gamma n P = 0. \tag{2.23}$$

Polynomial solutions to (2.23) are simply related to Laguerre polynomials [9], defined by:

$$P = L_n^{2\alpha_i - 1}(-2\gamma z) = \sum_{k=0}^{n} \frac{(2\gamma z)^k}{k!(n+1)_k(2\alpha)_k}, \tag{2.24}$$

where $(x)_k$ is the pochhammer symbol, defined for every $x \in \mathbb{C}$ and integer $k > 0$ by

$$(x)_k = (x)(x+1)...(x+k-1),$$

and for $k = 0$ by $(x)_0 = 1$. For this pole structure, the integer condition (2.22) is sufficient for the existence of a type 1 solution (2.20).

With both poles present, the type 1 DE is given by

$$\Delta P_{zz}+2(\gamma z^2 \pm 2\epsilon M(\gamma z+\alpha_\pm)-2(\alpha_+ + \alpha_-)z)P_z - 2(\gamma n z \pm \epsilon M(b_{1\pm}-g_\pm))P = 0. \quad (2.25)$$

Polynomial solutions to (2.25) are simply related to confluent Heun polynomials [9].

The recursion relation of equation (2.25) is

$$A_{-1}a_{k-1} + A_0 a_k + A_1 a_{k+1} = 0, \quad (2.26)$$

where

$$A_{-1} = 2\gamma(n+1-k),$$
$$A_0 = \pm 2\epsilon M(b_{1\pm} - g_\pm - 2\gamma k) + k(-2\alpha_+ - 2\alpha_- + 1 - k),$$
$$A_1 = \pm 2\epsilon M(k+1)(-2\alpha_\pm - k).$$

The corresponding constraint is

$$C = (b_{1\pm} - g_\pm)a_0 - 2\alpha_\pm a_1 = 0. \quad (2.27)$$

The recursion implies that a_k is a polynomial of degree $n-k$ in $b_{1\pm}$ (for $k = n...0$). Therefore, C is a polynomial of degree $n+1$ in $b_{1\pm}$. Given a specific integer n, $C = 0$ is necessary and sufficient for the existence of a type 1 solution (2.20).

Note that a polynomial solution with a factor of $(r - r_\pm)^{n-N}$, for some integer $N \in [0, n-1]$, exists for every n if the parameters satisfy the two conditions

$$2\alpha_\pm = -(n-N-1), \; a_{n-N-1} = 0. \quad (2.28)$$

To prove this, notice (2.28) implies the recursion at $k = n-N-1$ is $2\gamma(n-N)a_{n-N-2} = 0$, thus requiring $a_{n-N-2} = 0$. If $N = n-1$, then $a_0 = a_1 = 0$, which implies $C = 0$. Otherwise, at $k = n-N-2, n-N-3, ...1$, the recursion requires $a_{n-N-3} = ... = a_1 = a_0 = 0$. Thus $C = 0$, and the polynomial solution has a factor of $(r - r_\pm)^{n-N}$. The equation $a_{n-N-1} = 0$ is a polynomial of degree $N+1$ in $b_{1\pm}$ and therefore can be just as difficult to solve for arbitrary N as the constraint is for arbitrary n. For $N = 0$ and $2\alpha_\pm = 1 - n$, the equation $a_{n-N-1} = 0$ restricts the black hole parameters via

$$b_{1\pm} = (n+1)(\gamma \pm \frac{\alpha_\mp}{2\epsilon M}),$$

and the polynomial solution is $P = (r - r_\pm)^n$.

The Type 2 Algorithm

The type 2 algorithm specifies the collection of functions θ to be

$$\theta = \frac{\beta_+}{r - r_+} + \frac{\beta_-}{r - r_-},$$

where the Kovacic constants $\beta_\pm$ take the values

$$\beta_\pm = 1 + \delta_\pm \sqrt{1 + 4b_\pm},$$

where $\delta_i = \pm 1$ or 0 independently for $i = +, -$. If $r_\pm$ has order 0 or 1, the algorithm requires $\delta_\pm = -1$ or 1 so as to obtain $\beta_\pm = 0$ or 2, respectively. Otherwise, the collection consists of 9 different functions θ, each labelled by a set of Kovacic constants and a corresponding pair of sign choices for $\delta_\pm$. We refer to a particular instance of $(\theta, \beta_i, \delta_i)$ as a Kovacic case and adopt the intuitive notation $\theta_\nu(sgn(\delta_+), sgn(\delta_-))$ to enumerate the cases. Every Kovacic case is considered in the search for polynomial solutions to the type 2 DE (2.8).

In terms of the Kovacic constants, the type 2 DE is

$$P_{rrr} + T_2 P_{rr} + T_1 P_r + T_0 P = 0, \tag{2.29}$$

where

$$\begin{aligned} T_2 &= \frac{3\beta_+}{r-r_+} + \frac{3\beta_-}{r-r_-}, \\ T_1 &= 4\omega^2 + \frac{p_+}{(r-r_+)^2} + \frac{p_{1+}}{r-r_+} + \frac{p_-}{(r-r_-)^2} + \frac{p_{1-}}{r-r_-}, \\ T_0 &= \frac{q_+}{(r-r_+)^2} + \frac{q_{1+}}{r-r_+} + \frac{q_-}{(r-r_-)^2} + \frac{q_{1-}}{r-r_-}, \end{aligned}$$

with

$$\begin{aligned} p_\pm &= 3\beta_\pm(\beta_\pm - 1) - 4b_\pm, \\ p_{1\pm} &= \pm\frac{3\beta_+\beta_-}{2\epsilon M} - 4b_{1\pm}, \\ q_\pm &= 2(1-2\beta_\pm)b_{1\pm} \pm \frac{p_\pm\beta_\mp}{2\epsilon M}, \\ q_{1\pm} &= 4\beta_\pm\omega \mp 2\frac{\beta_+ b_{1-} + \beta_- b_{1+}}{\epsilon M} - \frac{4(\beta_+ b_- - \beta_- b_+) + 3\beta_+\beta_-(\beta_+ + \beta_-)}{4\epsilon^2 M^2}. \end{aligned}$$

If a polynomial solution is found, the type 2 solution is given by

$$y = e^{\pm\sqrt{D}/2}(r-r_+)^{\beta_+/2}(r-r_-)^{\beta_-/2}\sqrt{P}, \tag{2.30}$$

where D is the discriminant of the quadratic equation (2.10). If no polynomial solutions exist, there are no type 2 solutions.

We write

$$P = \sum_{k=0}^{n} a_k z^k, \ a_n \neq 0, \tag{2.31}$$

where $z = r - r_\pm$. Then the necessary integer condition (2.13) is given by

$$n = -\beta_+ - \beta_-. \tag{2.32}$$

Note this implies at least one choice of δ_i is non-zero and reinforces the type 2 necessary condition that at least one pole must have order two.

Claim 2.1 *There are no type 2 solutions to equation* (2.18) *with* $b_{\mp} = b_{1\mp} = 0$.

Proof. Without loss of generality, take r_+ to be the only pole of ρ. Then the type 2 DE is:

$$z^2 P_{zzz} - 3nzP_{zz} + (4\omega^2 z^2 - 4b_{1+}z + n(2n+1))P_r + (-4n\omega^2 z + 2b_{1+}(2n+1))P = 0.$$

One may verify that a polynomial solution exists if and only if

$$b_{1+} = \pm i\omega(n - 2N),$$

for some integer $N \in [0, n]$. However, if we apply the type 1 algorithm to (2.18) with b_{1+} given by the above, and choose the type 1 Kovacic constants to be $\gamma = \pm i\omega$ and $\alpha_+ = \beta_+/2$ (which is possible since (2.32) with $\beta_- = 0$ requires $\delta_+ \neq 0$), we find the type 1 integer condition,

$$n_1 = \frac{b_{1+}}{2\gamma} - \alpha_+ = n - N,$$

is satisfied , where n_1 is the degree of the type 1 polynomial. Since this is sufficient for type 1 solutions in the 1-pole case, the solution set of the DE for which the type 2 algorithm generates solutions is type 1. Hence no type 2 solutions exist. ■

This example illustrates an unavoidable difficulty in the type 2 algorithm, arising whenever the DE depends on parameters. Without parameters, the Kovacic algorithm terminates if a type 1 solution is found. This does not, however, dismiss the triviality that the type 2 algorithm may also find this solution; i.e. a polynomial solves the type 2 DE and D, the discriminant of equation (2.10), is the square of a rational function. With parameters present, this triviality is encoded in the infinite set of parameter values dictated by the type 1 and type 2 constraints. Since the family of parameter values for which type 1 solutions exist is not obtainable in explicit form, the exclusion of these values is an infeasible, not to mention impractical, approach to the type 2 algorithm. In practice, for every solution generated by the type 2 algorithm one checks whether $\sqrt{D}$ is rational to decide if the solution is a parameter specialization of a type 1 solution and should be discarded.

There are other explicit parameter restrictions that must be avoided by the type 2 algorithm. For example, the equations

$$\beta_+ = -N, \ \beta_- = N - n,$$
$$b_{1+} = \beta_+(i\omega + \frac{\beta_-}{4\epsilon M}) + 2Ni\omega,$$

where N is an integer $N \in [1, n]$, are obtained by assuming the type 2 polynomial satisfies the type 1 DE (with the obvious matching of Kovacic constants) and requiring D to be the square of a certain rational function. Although the type 1 analogue of these conditions solve the type 1 constraints for every n, they are unphysical when applied to black hole perturbations and serve only as a further example of the redundancy inherent in the type 2 algorithm.

It follows from the claim that the only pole structure that can generate type 2 solutions is the generic one, with both poles present. The appropriate recursion relation has 5 terms and generates 3 corresponding constraints, all of which are mixed polynomials in b_{1+}, b_{1-} with varying degrees that increase with n. Due to the immensity of this recursion, we do not find value in presenting it here; it can be easily generated on a computer using equation (2.29) and the procedure described in section 2.1. In later sections, we display the type 2 recursion for specific Kovacic cases in terms of the black hole parameters to support conjectures of non-existence.

2.3 The Algorithm for Extreme Geometries

For extreme geometries ($\epsilon = 0$), the partial fraction expansion of ρ is given by

$$\rho = -\omega^2 - \frac{(b_4)^2}{(r-M)^4} + \frac{2b_3b_4}{(r-M)^3} + \frac{b_2}{(r-M)^2} + \frac{b_1}{r-M}, \tag{2.33}$$

where the constants b_4, b_3, b_2, b_1 are specified in terms of the black hole parameters by (2.16). In this section, we give the essential details of the type 1 algorithm as applied to the DE

$$y_{rr} = \rho y, \tag{2.34}$$

with ρ given by (2.33).

The necessary condition for type 2 requires the pole at $r = M$ to have order 2 (i.e. $b_4 = 0$). Since we proved in section 2.2 that type 2 is empty if ρ has one pole of order two, we have the result that there are no type 2 perturbations of extreme Kerr-Newman.

The type 1 DE in the case $b_4 = 0$ is once again confluent hypergeometric, admitting the analogy of the Laguerre polynomial solutions (2.24), provided the analogous integer condition (2.22) holds. In section 5, we state these solutions explicitly in terms of n and the extreme black hole parameters. For the remainder of this section we assume $b_4 \neq 0$.

With $b_4 \neq 0$, the type 1 algorithm specifies the collection of functions Ω to be

$$\Omega = \gamma + \frac{\eta}{(r-M)^2} + \frac{\alpha}{r-M},$$

where the Kovacic constants γ, η, α take the values

$$\gamma = i\delta_0\omega,$$
$$\eta = i\delta_1 b_4,$$
$$\alpha = 1 - i\delta_1 b_3,$$

where $\delta_i = \pm 1$ independently for $i = 0, 1$. The collection consists of 4 different functions Ω, each labelled by a set of Kovacic constants and a corresponding set of sign choices for δ_i. We refer to a particular instance of $(\Omega, \gamma, \eta, \alpha, \delta_i)$ as a Kovacic case and adopt the intuitive notation $\Omega_\nu(sgn(\delta_0), sgn(\delta_1))$ to enumerate the cases. Every Kovacic case is considered in the search for polynomial solutions to the type 1 DE (2.6).

In terms of the Kovacic constants, the type 1 DE is

$$P_{rr} + F_1 P_r + F_2 P = 0, \tag{2.35}$$

where

$$F_1 = 2(\gamma + \frac{\eta}{(r-M)^2} + \frac{\alpha}{r-M}),$$
$$F_2 = \frac{g_2 - b_2}{(r-M)^2} - \frac{b_1}{r-M},$$

with

$$g_2 = \alpha(\alpha - 1) + 2\gamma\eta.$$

If a polynomial solution is found, the type 1 solution is given by

$$y = e^{\gamma r} e^{-\frac{\eta}{r-M}} (r-M)^\alpha P. \tag{2.36}$$

If no polynomial solutions exist, there are no type 1 solutions.

We write

$$P = \sum_{k=0}^{n} a_k z^k, \ a_n \neq 0, \tag{2.37}$$

where $z = r - M$, then the necessary integer condition (2.13) is given by

$$n = \frac{b_1}{2\gamma} - \alpha. \tag{2.38}$$

The type 1 DE is given by

$$z^2 P_{zz} + 2(\gamma z^2 + \alpha z + \eta)P_z - (2\gamma n z + b_2 - g_2)P = 0. \tag{2.39}$$

Polynomial solutions to (2.39) are simply related to biconfluent Heun polynomials [9].

The recursion relation of equation (2.39) is

$$2\gamma(n + 1 - k)a_{k-1} + (b_2 - g_2 - k(2\alpha + k - 1))a_k - 2\eta(k + 1)a_{k+1} = 0. \tag{2.40}$$

The corresponding constraint is

$$C = (b_2 - g_2 - 2\gamma\eta)a_0 - 2\eta a_1 = 0. \tag{2.41}$$

Note that C is a polynomial of degree $n + 1$ in b_2. Given a specific integer n, $C = 0$ is necessary and sufficient for the existence of a type 1 solution (2.36).

The implementation of the Kovacic algorithm has now been established for all of the black hole perturbations considered in this thesis. We thus commence the search for Liouvillian perturbations on the individual geometries.

3 The Liouvillian Perturbations of Schwarzschild

In this section, we identify the Liouvillian perturbations of the Schwarzschild black hole by applying the Kovacic algorithm to equation (1.1) with $a = q = 0$. We prove the new result that the only physical type 1 solutions are the algebraically special gravitational perturbations of Schwarzschild. We suspect these are the only physical Liouvillian solutions, but do not have a proof that the type 2 algorithm fails.

In section 3.1, we construct the named special perturbations by choosing a Kovacic case that leaves the integer condition independent of the black hole parameters and satisfied for specific values of (s, n). With $s = 2$, the constraint defines $\omega = \pm i\ell(\ell+1)(\ell-1)(\ell+2)/12M$, which are the known special frequencies. For one of these perturbations, we obtain a second linearly independent solution from a different Kovacic case. In all remaining cases, we construct linear operators relating the type 1 DEs that, when applied to the polynomial solutions, prove that all type 1 parameter values are decided by a single recursion relation (an exception occurs for $s = 1/2$ where an additional recursion must be considered). This recursion is rearranged to yield an estimate that proves $C \neq 0$ for every n.

In section 3.2, we rule out type 2 solutions for low n and formulate a conjecture that the (numerical) roots of the three constraints do not coalesce for higher n, suggesting there are no type 2 perturbations of Schwarzschild.

Before proceeding to the type 1 and 2 analysis, we state ρ in terms of the Schwarzschild parameters.

For convenience we take $M = 1/2$, then the partial fraction expansion of ρ is given by

$$\rho = -\omega^2 + \frac{b_+}{(r-1)^2} + \frac{b_-}{r^2} + \frac{b_{1+}}{r-1} + \frac{b_{1-}}{r},$$

where

$$1 + 4b_+ = (2i\omega + s)^2,$$

$$1 + 4b_- = s^2,$$

$$2b_{1+} = (2i\omega - s)^2 + 1 - 2s^2 + 2\lambda,$$

$$2b_{1-} = s^2 - 1 - 2\lambda.$$

The essential parameters are the complex frequency ω and the angular separation integer $\lambda = \ell(\ell + 1)$. To avoid known trivial cases, we take $\ell \geq 2$. The spin parameter s takes one of the assigned values $s = 0, \pm\frac{1}{2}, \pm 1, \pm 2$. Note that ρ is invariant under the transformation $(\omega, s) \to (-\omega, -s)$.

Notice the following simplified quantities, which prove useful for both the type 1 and 2 algorithms:

$$\sqrt{1 + 4b_+} = 2i\omega + s,$$

$$\sqrt{1 + 4b_-} = -s,$$

$$b_{1+} + b_{1-} = 4i\omega(i\omega - s).$$

3.1 Type 1 Solutions

The type 1 Kovacic cases are specified by

$$\Omega = \gamma + \frac{\alpha_+}{r-1} + \frac{\alpha_-}{r},$$

where

$$\gamma = i\delta_0\omega,$$

$$2\alpha_+ = 1 + \delta_+(2i\omega + s),$$

$$2\alpha_- = 1 - \delta_- s,$$

with $\delta_i = \pm 1$ independently for $i = 0, +, -$.

For every function $\Omega_\nu(\pm\pm\pm)$ $(\nu = I, ..., VIII)$ we seek polynomial solutions to the type 1 DE (2.25), denoted by

$$(L_\nu)P = 0.$$

If such is found, the type 1 solution is given by

$$y = e^{\gamma r}(r-1)^{\alpha_+} r^{\alpha_-} P. \tag{3.1}$$

We write

$$P = \sum_{k=0}^{n} a_k z^k, \; a_n \neq 0, \tag{3.2}$$

where $z = r - 1$, then the necessary integer condition (2.22) is

$$2(\delta_+ - \delta_0)\omega = i(2n + 2 + (2\delta_0 + \delta_+ - \delta_-)s). \tag{3.3}$$

If (3.3) holds, and with $a_0, ..., a_{n-1}$ generated by the recursion relation (2.25) of $(L_\nu)P = 0$, the corresponding constraint (2.27), $C = 0$, is necessary and sufficient for the existence of a type 1 solution. If Ω_v, n, and s are chosen such that (3.3) is satisfied identically, then for each s (as long as C depends on ω) the constraint defines $\omega(\lambda)$, and solutions are obtained with λ unrestricted. Otherwise, (3.3) defines $\omega(s,n)$ and, for every (s,n), C is a polynomial of degree $n+1$ in λ whose vanishing defines $n+1$ numerical values of λ for which type 1 solutions exist.

Note that ρ and Ω_ν, and therefore L_ν, are invariant under

$$(\omega, s, \delta_0, \delta_+, \delta_-) \rightarrow (-\omega, -s, -\delta_0, -\delta_+, -\delta_-).$$

Therefore, all type 1 solutions can be obtained by choosing one sign for s. We adopt the convention $s \geq 0$, and if a type 1 perturbation, R, is obtained, there is another perturbation for $(\omega, s) \rightarrow (-\omega, -s)$, given by $\tilde{R} = R\Delta^s$.

The integer condition (3.3) is satisfied without defining ω by choosing $\delta_+ = \delta_0$ and requiring

$$2(n+1) = -(3\delta_+ - \delta_-)s.$$

All solutions of this requirement with $s \geq 0$ are given by $((\delta_0, \delta_+, \delta_-), s, n) =$

$$((-,-,+), 1/2, 0); ((-,-,-), 1, 0); ((-,-,+), 1, 1); \tag{3.4}$$

$$((-,-,-), 2, 1); ((-,-,+), 2, 3). \tag{3.5}$$

The constraints corresponding to (3.4) are

$$4\lambda + 1 = 0; \lambda = 0; \frac{i\lambda^2}{2\omega} = 0,$$

respectively. Hence (3.4) is empty for $\lambda > 0$.

The constraints corresponding to the two cases given by (3.5) yield the algebraically special frequencies $\omega = \pm\omega^*$, where

$$\omega^* = i\frac{\lambda(\lambda - 2)}{6}.$$

For $(s,n) = (2,1)$, the constraint is $\omega - \omega^* = 0$. With $\omega = \omega^*$, the Liouvillian perturbation is given by

$$R_{S+} = \frac{(e^r(r-1))^{\frac{\lambda(\lambda-2)}{6}}}{(r-1)^2}[r - \frac{3}{\ell(\ell+1)}].$$

For $(s,n) = (2,3)$, the constraint is $\omega^2 - (\omega^*)^2 = 0$. With $\omega = -\omega^*$, the Liouvillian perturbation is given by

$$R_{S-} = \frac{((r-1)e^r)^{-\frac{\lambda(\lambda-2)}{6}}}{\Delta^2}[\lambda(\lambda-2)r^3 + 3(\lambda-2)r^2 + 9r + \frac{9}{(\lambda-2)}].$$

R_{s+} and R_{s-} are the algebraically special perturbations of Schwarzschild, and we have shown they are the only type 1 solutions among Kovacic cases with $\delta_+ = \delta_0$. To obtain the central result of this section, it remains to show this holds for $\delta_+ = -\delta_0$. Indeed, the special modes resurface in Kovacic cases with $\delta_+ = -\delta_0$, as the following example illustrates.

Choosing $\Omega(+,-,+)$, the integer condition (3.3) implies $\omega = -i\frac{n+1}{2}$, thus the Kovacic constants are given by

$$\gamma = \frac{n+1}{2}, 2\alpha_+ = -2-n, 2\alpha_- = -1,$$

and the recursion relation of the type 1 DE is

$$(n+1)(n-k)a_k + (\lambda + n^2 - 1 - k(k-1))a_{k+1} + (k+2)(n+1-k)a_{k+2} = 0.$$

We impose $\omega = -\omega^*$ by restricting n in terms of ℓ via

$$3(n+1) = (\ell-1)\ell(\ell+1)(\ell+2) = \lambda(\lambda-2),$$

i.e.

$$(\ell, n) = (2,8), (3,40), (4,120), ...$$

It follows that the recursion holds identically if

$$a_k = \frac{(n+1-k)(n+2-k)}{6k!}(\lambda(n-k) + 3k)$$

for every $k \in [0, n]$. In particular, the constraints are satisfied, thus a_k, as given above, defines a polynomial solution (3.2) to the type 1 DE. We therefore obtain the Liouvillian solution

$$R'_{S-} = \frac{((r-1)e^{-r})^{-\frac{\lambda(\lambda-2)}{6}}}{\Delta^2}P,$$

valid for $\omega = -\omega^*$ and every $\lambda = \ell(\ell+1)$.

For a given integer ℓ, R_{S-} and R'_{S-} are linearly independent solutions to the same algebraically special differential equation (1.1). The opposite sign of the exponent in these solutions indicates that the integration required for reduction of order on either solution can be carried out explicitly. We note that a solution linearly independent to the R_{S+} is not generated by the algorithm, indicating the converse for this solution. For the remainder of this section, we assume $\omega \neq \pm\omega^*$ to avoid generating any linear combinations or specializations of these algebraically special solutions.

In all remaining Kovacic cases ($\delta_+ = -\delta_0$), ω is determined by the integer condition (3.3):

$$\omega = \delta_+ \frac{i}{4}(2n + 2 - (\delta_+ + \delta_-)s).$$

The Kovacic constants $\gamma, \alpha_+, \alpha_-$ are then given in terms of n and s by

$$\gamma = \frac{1}{4}(2n + 2 - (\delta_+ + \delta_-)s),$$

$$2\alpha_+ = (3\delta_+ + \delta_-)\frac{s}{2} - n,$$

$$2\alpha_- = 1 - \delta_- s.$$

Taking into account the available choices of δ_+, δ_-, there are four independent

Kovacic cases to consider:

$$\Omega_I(+,-,+): 2\alpha_+ = -s-n,\ 2\alpha_- = 1-s,\ \gamma = \frac{1}{2}(n+1),\ \omega = -i\gamma.$$

$$\Omega_{II}(-,+,-): 2\alpha_+ = s-n,\ 2\alpha_- = 1+s,\ \gamma = \frac{1}{2}(n+1),\ \omega = i\gamma.$$

$$\Omega_{III}(+,-,-): 2\alpha_+ = -2s-n,\ 2\alpha_- = 1+s,\ \gamma = \frac{1}{2}(n+1+s),\ \omega = -i\gamma.$$

$$\Omega_{IV}(-,+,+): 2\alpha_+ = 2s-n,\ 2\alpha_- = 1-s,\ \gamma = \frac{1}{2}(n+1-s),\ \omega = i\gamma.$$

The operators, L_ν, for the type 1 DEs corresponding to these cases are:

$$L_I = \Delta\partial_{rr} + [(n+1)r(r-2) + (1-s)\Delta_r]\partial_r - (n+1)(nr+1) + s(s+n) - \lambda.$$

$$L_{II} = \Delta\partial_{rr} + [(n+1)r(r-2) + (1+s)\Delta_r]\partial_r - (n+1)(nr+1) + s(s-n) - \lambda.$$

$$L_{III} = \Delta\partial_{rr} + [(n+1+s)r(r-2) + \Delta_r - s]\partial_r - (n+1+s)(nr+1+s) - \lambda.$$

$$L_{IV} = \Delta\partial_{rr} + [(n+1-s)r(r-2) + \Delta_r + s]\partial_r - (n+1-s)(nr+1-s) - \lambda.$$

We seek polynomial solutions to $(L_\nu)P = 0$ for every integer $n \geq 0$. To avoid trivial cases in what follows, we easily rule out solutions in every case for $n \leq 2s$.

Proposition 3.1 *Let P_I be a solution to $(L_I)P_I = 0$. For $s = 0, 1/2, 1, 2$, define:*

$$\Phi = (n+1+\partial_r)^{2s}.$$

Then

$$(L_{II})(\Phi)P_I = 0.$$

Proof. We take four derivatives of $(L_I)P_I = 0$ to obtain $\partial_{(6)}P_I, \ldots, \partial_{rr}P_I$ in terms of $\partial_r P_I$ and P_I. We write

$$\Phi = \sum_{k=0}^{4} \frac{(2s-k+1)_k}{k!(n+1)^k}\partial_k$$

and obtain

$$(L_{II})(\Phi)P_I = s(2s-1)(s-1)(2s-3)(s-2)[A(r)\partial_r P + B(r)P_I],$$

where $A(r)$ and $B(r)$ denote the appropriate rational functions composed of derivatives of the coefficients of L_I and L_{II}. ∎

It follows that for every polynomial P_I satisfying $(L_I)P_I = 0$, there is a polynomial $P_{II} = (\Phi)P_I$ satisfying $(L_{II})P_{II} = 0$. To indicate this fact, we write $P_I \Rightarrow P_{II}$.

Proposition 3.2 *Let P_{II} be a solution to $(L_{II})P_{II} = 0$. For $s = 0, 1/2, 1, 2,$ define:*

$$\Phi^- = (-nr - s + \Delta\partial_r)_{2s}.$$

Then

$$L_I(\Phi^-)P_{II} = 0.$$

The method of proof is similar to that of the preceding proposition, although the computations were carried out separately for each value of s. It follows that for every polynomial P_{II} satisfying $(L_{II})P_{II} = 0$, there is a polynomial $P_I = (\Phi^-)P_{II}$ satisfying $(L_I)P_{II} = 0$. To indicate this fact, we write $P_I \Leftrightarrow P_{II}$.

Proposition 3.3 *Let P_{II} and P_{III} be solutions to $(L_{II})P_{II} = 0$ and $(L_{III})P_{III} = 0$. Then*

$$(\hat{L}_I)(r^s P_{III}) = 0,$$

$$(\hat{L}_{IV})(r^s P_{II}) = 0,$$

where $\hat{L}_I = L_I|_{n=n+s}$ and $\hat{L}_{IV} = L_{IV}|_{n=n+s}$.

The proof is a direct computation using the relevant operators; e.g. $(\hat{L}_I)(r^s P_{III}) = r^s(L_{III})P_{III}$. It follows that $P_{III} \Rightarrow P_I$ and $P_{II} \Rightarrow P_{IV}$ for $s = 0, 1, 2$.

Proposition 3.4 *Consider $s = 1/2$ and let P_{III} be a solution to $(L_{III})P_{III} = 0$. Then*

$$(\tilde{L}_{IV})(\sqrt{r}(n + 1 + \partial_r))(\sqrt{r}P_{III}) = 0,$$

where $\tilde{L}_{IV} = L_{IV}\ |_{n=n-2s}$.

This is easily proved by composing the appropriate transformations of the previous propositions. It follows that $P_{III} \Rightarrow P_{IV}$ for $s = 1/2$.

To summarize, we have established the following facts concerning the existence of polynomial solutions in cases $I - IV$:

$$s = 1/2 : P_I \Leftrightarrow P_{II} \textit{ and } P_{III} \Rightarrow P_{IV}.$$

$$s = 0, 1, 2 : P_{III} \Rightarrow P_I \Leftrightarrow P_{II} \Rightarrow P_{IV}.$$

Now, to show that the only type 1 solutions are algebraically special, it remains to prove P_I does not hold for $s = 1/2$ and P_{IV} does not hold for any s. Assuming the conditions $\omega \neq \omega^*$ and $\lambda > 0$, we obtain an estimate from the recursion relations that proves $C \neq 0$ for every n.

Theorem 3.1 *There are no polynomial solutions to* $(L_I)P_I = 0$ *for* $s = 1/2$ *given* $\lambda > 0$.

Proof. With $s = 1/2$, the recursion relation of $(L_I)P_I = 0$ is given by

$$A_{-1}a_{k-1} = (4\lambda + 4n^2 + 6n + 3 - 4k^3)a_k + 2(2n + 1 - 2k)(k + 1)a_{k+1}, \tag{3.6}$$

where

$$A_{-1} = -4(n + 1)(n + 1 - k).$$

We define $b_k = (-1)^k a_k$ and $c_{k-1} = (n + 1)b_{k-1} - kb_k$ so that (3.6) becomes:

$$c_{k-1} = \frac{1}{4(n + 1 - k)}[(4\lambda + 1)b_k + 2(2n + 1 - 2k)c_k]. \tag{3.7}$$

Without loss of generality we take $b_n = 1$, then from (3.6), we find

$$b_{n-1} = \frac{4\lambda + 6n + 3}{4(n + 1)},$$

thus

$$c_n = \lambda + \frac{n}{2} + \frac{3}{4} > 0.$$

Since $c_n, b_n > 0$, (3.7) implies $b_k, c_k > 0$ for every $k = n...0$. In particular,

$$b_0 > \frac{1}{n+1} b_1 > 0.$$

The constraint is $C = 0$ where

$$C = (4\lambda + 4n^2 + 6n + 3)b_0 - 2(2n+1)b_1 > \frac{4\lambda + 1}{n+1} b_1 > 0,$$

and hence cannot be satisfied. ■

The proof for P_{IV} is along the same lines. The algebraically special perturbation R_{S+} resurfaces in this case, and we must impose $\omega \neq i\omega^*$ to prove the $s = 2$ case. To achieve this restriction, we define the quantity $G = \lambda^2(\lambda - 2)^2 - 9(n-1)^2$ and assume $G > 0$. If $G < 0$, the proof works to show $C_n < 0$ for all n.

Theorem 3.2 *There are no polynomial solutions to $(L_{IV})P_{IV} = 0$ given $\lambda, G > 0$.*

Proof. The recursion relation of $(L_{IV})P_{IV} = 0$ is given by:

$$A_{-1}a_{k-1} = (\lambda + (n+1-s)^2 - k(k+1))a_k + (n - 2s - k)(k+1)a_{k+1}, \qquad (3.8)$$

where

$$A_{-1} = -(n+1-s)(n+1-k).$$

We define $b_k = (-1)^k a_k$ and $c_{k-1} = (n+1-s)b_{k-1} - kb_k$ so that (3.8) becomes:

$$c_{k-1} = \frac{1}{(n+1-k)}[(\lambda + (s+1)(n+1-s-k))b_k + (n-2s-k)c_k]. \qquad (3.9)$$

Note that we have ruled out solutions for every $n \leq 2s$ and thus we only need to consider $n > 2s$. Without loss of generality we take $b_n = 1$, then for each s we find from (3.8) the coefficients b_{n-2s}, b_{n-2s-1}:

$$s = 0 : b_n = 1, b_{n=1} = \frac{\lambda + n + 1}{n+1} > 0.$$
$$s = \frac{1}{2} : b_{n-1} = \frac{4\lambda + 1}{2(2n+1)}, b_{n-2} = \frac{4\lambda + 8n + 1}{4(2n+1)} b_{n-1} > 0.$$
$$s = 1 : b_{n-2} = \frac{\lambda^2}{2n^2}, b_{n-3} = \frac{\lambda + 3n - 2}{3n} b_{n-2} > 0.$$
$$s = 2 : b_{n-4} = \frac{G}{24(n-1)^4}, b_{n-5} = \frac{\lambda + 5n - 11}{5(n-1)} b_{n-4} > 0.$$

For each s we find c_{n-2s-1} :

$$s = 0 : c_{n-1} = \lambda + 1 > 0.$$
$$s = \frac{1}{2} : c_{n-2} = (4\lambda + 1) \frac{\lambda + 4}{4(2n+1)} > 0.$$
$$s = 1 : c_{n-3} = \lambda^2 \frac{\lambda + 4}{6n^2} > 0.$$
$$s = 2 : c_{n-5} = G \frac{\lambda + 9}{120(n-1)^4} > 0.$$

Since $c_{n-2s-1}, b_{n-2s-1} > 0$, (3.9) implies $c_k, b_k > 0$ for every $k = n - 2s - 1, ..., 0$. In particular,

$$b_0 > \frac{1}{n+1-s} b_1 > 0.$$

The constraint is $C = 0$ where

$$C = (\lambda + (n+1-s)^2) b_0 - (n-2s) b_1 > (\frac{\lambda}{n+1-s} + 1 + s) b_1 > 0,$$

and hence cannot be satisfied. ■

This completes the proof that the only physical type 1 Liouvillian perturbations of Schwarzschild are the known algebraically special modes. For $s = 2$ and $\omega = \pm i\omega^*$, the type 1 algorithm constructs the Liouvillian perturbations $R_{S\pm}$ and R'_{S-}, which solve equation (1.1) at $a = q = 0$ for every integer $\ell \geq 2$.

3.2 Type 2 Solutions

The type 2 Kovacic cases are specified by

$$\Omega = \gamma + \frac{\beta_+}{r - r_+} + \frac{\beta_-}{r - r_-},$$

where

$$\beta_+ = 1 + \delta_+(2i\omega + s),$$

$$\beta_- = 1 - \delta_- s,$$

with $\delta_i = \pm 1, 0$ independently for $i = +, -$.

For every function $\theta_\nu(\pm/0 \pm /0)$ $(\nu = I, ..., VIII)$ we seek polynomial solutions to the type 2 DE (2.29), denoted by

$$(L_\nu)P = 0.$$

If such is found, the type 2 solution is given by

$$y = e^{\pm\sqrt{D}/2}(r - r_+)^{\beta_+/2}(r - r_-)^{\beta_-/2}\sqrt{P}, \tag{3.10}$$

where D is the discriminant of the quadratic equation (2.10).

We write

$$P = \sum_{k=0}^{n} a_k z^k, \; a_n \neq 0, \tag{3.11}$$

where $z = r - 1$, then the necessary integer condition (2.32) is:

$$2\delta_+\omega = i(n + 2 + (\delta_+ - \delta_-)s). \tag{3.12}$$

If (3.12) holds, and with $a_0 ..., a_n$ generated by the five-term recursion relation of the type 2 DE , the corresponding three constraints are necessary and sufficient for the existence of type 2 solutions.

Note that ρ and θ_ν, and therefore L_ν, are invariant under

$$(\omega, s, \delta_+, \delta_-) \to (-\omega, -s, -\delta_+, \delta_-).$$

Therefore, all type 2 solutions can be obtained by choosing one sign for s. We adopt the convention $s \geq 0$, and if a type 2 perturbation, R, is obtained, there is another perturbation for $(\omega, s) = (-\omega, -s)$, given by $\tilde{R} = R\Delta^s$.

The integer condition (3.12) is satisfied identically by choosing $\theta(0,+)$ and $(s,n) = (2,0)$. However, the constraints cannot be satisfied for any values of (ω,λ) in this case.

In all remaining Kovacic cases ($\delta_+ \neq 0$), ω is determined by the integer condition (3.12) to be

$$\omega = \delta_+ \frac{i}{2}(n + 2 + (\delta_+ - \delta_-)s).$$

Thus, the Kovacic constants β_+, β_- are given in terms of n and s by

$$\beta_+ = -1 - n + \delta_- s,$$
$$\beta_- = 1 - \delta_- s.$$

Taking into account the available choices of δ_+, δ_-, there are six independent Kovacic cases to consider. In each case, we seek polynomial solutions (3.11) to the corresponding type 2 DE at every integer $n \geq 0$. We find no solutions to the constraints in any of the cases with $n \leq 7$, and we suspect there are no solutions for any n. To support this conjecture, we discuss the $s = 0$ case, for which all Kovacic cases coalesce to one case. We find the qualitative behaviour of the constraints in the $s = 0$ case are representative of all Kovacic cases with $s \neq 0$ and $n \leq 7$, the only exception being for $s = 2$, where the algebraically special solutions are again generated by the type 2 algorithm and must be discarded. We now display the type 2 recursion and constraints for $s = 0$ and present our conjecture.

The recursion relation of the type 2 DE at $s = 0$ is given by

$$A_{-1}a_{k-1} + A_0 a_k + A_1 a_{k+1} + A_2 a_{k+2} + A_3 a_{k+3} = 0,$$

where

$$A_{-1} = (n+1-k)(n+2)^2,$$
$$A_0 = 2(2n+1-2k)(n+2)^2,$$
$$A_1 = (n-k)(4\lambda + 5n^2 + 24n + 25 + 2nk - k^2)),$$
$$A_2 = (2n-1-2k)(2\lambda + n^2 + 9n + 8 + 2nk + k - k^2),$$
$$A_3 = (n-1-k)(2n+1-k)(k+3).$$

The constraints corresponding to this recursion are easiest to state if P is expanded about $r = 0$, instead of $r = 1$. Let b_k be the coefficient of r^k in P, then the constraints take the form:

$$C_1 = (2\lambda + n + 2)b_0 + b_1 = 0,$$
$$C_2 = (4\lambda(n+1) + 3n^2 + 10n + 8)b_0 + (6\lambda + 7n + 8)b_1 + 8b_2 = 0,$$
$$C_3 = (n+2)^3 b_0 + (4\lambda n + 6n^2 + 13n + 8)b_1 + (10\lambda + 19n + 8)b_2 + 27b_3 = 0.$$

For $n = 2N$, $N \in \mathbb{N}$, C_1 and C_2 are polynomials of degree $N+1$ in λ, and C_3 is a polynomial of degree N in λ. For simplicity, we consider $n = 2N+1$, $N \in \mathbb{N}$, in which case all three constraints are polynomials of degree $N+1$ in λ. We conjecture C_1 and C_2 have no common roots for every n.

For $N \leq 30$, we find C_1 and C_2 each have N distinct positive real roots, $\{\lambda_1^i\}_{i=1\ldots N}$ and $\{\lambda_2^i\}_{i=1\ldots N}$, ordered as follows:

$$\lambda_2^1 < \lambda_1^1 < \lambda_2^2 < \lambda_1^2 < \ldots < \lambda_2^N < \lambda_1^N.$$

The i_{th} roots occur in paired intervals (or bundles), $B^i = (\lambda_2^i, \lambda_1^i)$, with the following properties:

$$|B^{i-1}| < |B^i| << d(B^i, B^{i-1}) < d(B^{i+1}, B^i),$$

In other words, as λ increases, the rate of expansion of the size of each bundle is much less than the rate of expansion of the distance between bundles. When the value of N increases to $N+1$, the size and separation of the first N bundles contracts, but the size of the first bundle remains non-zero and the size and separation of the extra $N+1_{th}$ bundle is magnified. To illustrate this separation of roots numerically, we find the following approximate roots for $N=3$ and $N=4$:

$$N = 3 : 4.86 < 6.67 < 41.09 < 43.55 < 281.79 < 292.92$$

$$N = 4 : \; 3.87 < 5.02 < 28.62 < 30.11 < 123.40 < 127.72 < 682.61 < 698.51$$

From this behaviour for $n \leq 60$, we conjecture that the bundles containing the root pairings for larger n are non-zero and disjoint, and therefore the roots of the constraints will never coalesce. We also conjecture that this behaviour carries over to all of the Kovacic cases with $s \neq 0$, and therefore claim that there are no type 2 perturbations of Schwarzschild. Should this hold true, then we have the result that the algebraically special modes are the only Liouvillian perturbations of Schwarzschild.

4 The Liouvillian Perturbations of Non-Extreme Kerr-Newman

In this section, we identify the Liouvillian perturbations of the non-extreme Kerr-Newman black hole by applying the Kovacic algorithm to equation (1.1) with $a \neq 0$. The results specialize to non-extreme Kerr with the charge set to $q = 0$. We develop a new approach to the Kovacic type 1 analysis that enables one to generate all type 1 solutions (of a certain class), and the parameter values for which they exist, from a single recursion. The results for type 2 are not definitive; the only certainty is that the type 2 constraints lead to type 1 solutions and/or unphysical parameter values for low n.

In section 4.1 we consider the type 1 solutions. Analogous to Schwarzschild, there is a Kovacic case that leaves the integer condition satisfied with specific choices of (s, n), and in this case the constraint defines a finite set of values of ω for which type 1 solutions exist. These solutions hold with all other parameters unrestricted; in particular, λ is free to be determined as the non-singular eigenvalue of the angular equation. With $q = 0$ and $s = 2$ these solutions include the algebraically special modes of Kerr. In the remaining type 1 cases, ω is defined by the integer condition in terms of (a, ϵ, m, n) and at every n the constraint defines $n + 1$ values of $\lambda(a, \epsilon, m)$ for which type 1 solutions exist. The central results of this section depend on the construction of differential operators relating the type 1 DEs in the various Kovacic cases. We use these operators to prove (up to conjecture 4.1) that all non-algebraically special type 1 solutions are exhausted by two exclusive recursion relations. For one of these recursions, the appropriate operator enables us to express complicated polynomial solutions of high degree in terms of simpler, lower degree potentials. Furthermore, with $m = 0$ and with special physical values of $\lambda(\epsilon, n)$,

$a\,(\epsilon, n)$, this operator reveals large classes of explicit solutions that were not identified in [5], [6], [8].

In section 4.2 we consider the type 2 solutions. We find the type 2 DEs do not admit any physical solutions at low values of n and are difficult to characterize for higher n. Although type 2 solutions are shown to exist in every case and for every n, in any case where we can solve the constraints explicitly, the resulting parameter values are unphysical by virtue of $a^2 < 0$. We suspect this is the case for all n, however, unlike the Schwarzschild case, we do not have sufficient evidence to formulate a conjecture. For two specific Kovacic cases, we do conjecture that all of the solutions are type 1, and we state the five-term recursion relation from which a proof could be developed.

Before proceeding to the type 1 and 2 analysis, we state the partial fraction expansion of ρ in terms of the non-extreme Kerr-Newman parameters.

For convenience we take $M = 1$, then the partial fraction expansion of ρ is given by

$$\rho = -\omega^2 + \frac{b_+}{(r-r_+)^2} + \frac{b_-}{(r-r_-)^2} + \frac{b_{1+}}{r-r_+} + \frac{b_{1-}}{r-r_-},$$

where

$$1 + 4b_\pm = -(\frac{\omega\sigma_\pm - am \mp i\epsilon s}{2\epsilon})^2,$$

$$\begin{aligned} 4\epsilon^3 b_{1\pm} = \mp(\omega^2[(\pm 3\epsilon - 1)r_\pm^3 - a^2(a^2+2)] \\ + 2\omega[2i\epsilon^2 s r_\pm + am(\epsilon^2+1)] - a^2m^2 + \epsilon^2[s^2 - 1 - 2\lambda]). \end{aligned}$$

with

$$\sigma_\pm = (1 \pm \epsilon)^2 + a^2.$$

The essential parameters $(\omega, \lambda, a, \epsilon, m)$ range over $(\mathbb{C}, \mathbb{C}, (0,1), (-1,1), \mathbb{Z})$ and the spin parameter s takes the assigned values $s = 0, \pm\frac{1}{2}, \pm 1, \pm 2$.

Note that ρ is invariant under the mappings

$$(\omega, s, m) \to (-\omega, -s, -m) : (b_\pm, b_{1\pm}) \mapsto (b_\pm, b_{1\pm})$$

and

$$\epsilon \to -\epsilon : (b_\pm, b_{1\pm}, r_\pm) \mapsto (b_\mp, b_{1\mp}, r_\mp).$$

Notice the following simplified quantities, which prove useful for both the type 1 and 2 algorithms::

$$\sqrt{1 + 4b_\pm} = \frac{i[\omega\sigma_\pm - am] \pm \epsilon s}{2\epsilon},$$
$$b_{1+} + b_{1-} = 2i\omega(2i\omega - s).$$

To simplify the presentation of the recursions and constraints, we often refer to the quantity:

$$\Lambda = \lambda + a^2\omega^2 - 2am\omega.$$

Notice Λ is invariant under $(\omega, m) \to (-\omega, -m)$.

4.1 Type 1 Solutions

The type 1 Kovacic cases are specified by

$$\Omega = \gamma + \frac{\alpha_+}{r - r_+} + \frac{\alpha_-}{r - r_-},$$

where

$$\gamma = i\delta_0\omega,$$
$$2\alpha_\pm = 1 + \frac{i\delta_\pm}{\epsilon}(\omega\sigma_\pm - am \mp i\epsilon s),$$

with $\delta_i = \pm 1$ independently for $i = 0, +, -$.

For every function $\Omega_\nu(\pm\pm\pm)$ $(\nu = I, ..., VIII)$ we seek polynomial solutions to the type 1 DE (2.25), denoted by:

$$(L_\nu)P = 0.$$

If such is found, the type 1 solution is given by

$$y = e^{\gamma r}(r - r_+)^{\alpha_+}(r - r_-)^{\alpha_-}P. \tag{4.1}$$

We write

$$P = \sum_{k=0}^{n} a_k z^k, \; a_n \neq 0, \tag{4.2}$$

where $z = r - r_+$, then the necessary integer condition (2.22) is:

$$(\delta_+\sigma_+ + \delta_-\sigma_- - 4\delta_0\epsilon)\omega = (\delta_+ + \delta_-)am + i\epsilon(2n + 2 + s(\delta_+ - \delta_- + 2\delta_0)). \tag{4.3}$$

If (4.3) holds, and with $a_0, ..., a_{n-1}$ generated by the recursion relation (2.26) of L_ν, the corresponding constraint (2.27), $C = 0$, is necessary and sufficient for the existence of a type 1 solution. If Ω_v, n, and s are chosen such that (4.3) is satisfied identically, then the constraint defines $\omega(\lambda, a, \epsilon, m, s)$, and for each s, solutions are obtained with $(\lambda, a, \epsilon, m)$ unrestricted. Otherwise, (4.3) defines $\omega(a, \epsilon, m, n, s)$ and for every (s, n), C is a polynomial of degree $n+1$ in λ which defines $n+1$ values of $\lambda(a, \epsilon, m)$ for which type 1 solutions exist.

Note that ρ and Ω_ν, and therefore L_ν, are invariant under

$$(\omega, m, s, \delta_0, \delta_+, \delta_-) \to (-\omega, -m, -s, -\delta_0, -\delta_+, -\delta_-).$$

Therefore, all type 1 solutions can be obtained by choosing one sign for s. We adopt the convention $s \geq 0$ and if a type 1 perturbation, R, is obtained, there is another perturbation for $(\omega, m, s) = (-\omega, -m, -s)$, given by $\tilde{R} = R\Delta^s$.

In addition, note that L_ν is invariant under

$$(\epsilon, \delta_0, \delta_+, \delta_-) \to (-\epsilon, \delta_0, -\delta_-, -\delta_+).$$

Thus, we may eliminate any one of two cases that are invariant under $(\delta_0, \delta_+, \delta_-) \to (\delta_0, -\delta_-, -\delta_+)$ by allowing $\epsilon \to -\epsilon$. Note that for the sign choices $(--+)$ and $(++-)$, the integer condition leaves ω arbitrary for appropriate values of (s, n), whereas for the transformed sign choices $(-+-)$ and $(+-+)$, the integer condition restricts ω. Hence, these four cases are not invariant and must be considered separately (the case $(++-)$ is easily eliminated with $s \geq 0$). The transformation $\epsilon \to -\epsilon$ will be imposed to coalesce the four remaining cases, noting that this process reverses the values of the constants $\alpha_\pm$ and the orders of the poles at $r_\pm$.

We take the 5 independent cases to be:

$$\Omega_0(--+), \Omega_I(+-+), \Omega_{II}(-+-), \Omega_{III}(+--), \Omega_{IV}(-++).$$

In the first case, the integer condition requires $n = 2s - 1$, and there are 2^{2s} values of ω that satisfy the termination constraint. We give the corresponding Liouvillian perturbations, which hold with all other parameters unrestricted. For brevity, we refer to the finite set of solutions and parameter values dictated by Ω_0 as algebraically special.

In cases I and II, we construct differential operators relating the type 1 DEs that we use to show all type 1 parameter values and solutions in these two cases are generated by a single recursion relation. These operators, and the proof of their existence, are generalizations to the Kerr-Newman case of the operators Φ and Φ^- for Schwarzschild. Non-algebraically special solutions exist for each n with $4i\omega = \pm(n+1)$ and with the $n+1$ values of λ defined by the appropriate constraint.

In cases III and IV, we construct new differential operators relating the type 1 DEs that we use to show all non-algebraically special type 1 parameter values and solutions in these two cases are again generated by a single recursion. Non-algebraically special solutions exist for each n with $\sigma_\pm\omega = am \pm i\epsilon(n+1+s)$ and with the $n+1$ values of λ defined by the corresponding constraint. The solutions and parameter values also hold for $\epsilon \rightarrow -\epsilon$ in these cases. To conclude this section, we apply one of these operators to obtain the complete set of solutions and parameter values for $n = 0, 1$, as well as a large class of explicit solutions that hold for special values of $\lambda(\epsilon, n)$, $a\,(\epsilon, n)$.

For $\Omega_0(--+)$ we have $n = 2s-1$, and the recursion relation of $(L_0)P_0 = 0$ is given by

$$A_{-1}a_{k-1} + A_0a_k + A_1a_{k+1} = 0,$$

where

$$A_{-1} = 2i\omega(2s-k),$$

$$A_0 = \Lambda - 2i\omega r_\pm(2s-1-k) - (s-k)(s-1-k),$$

$$A_1 = (k+1)(2i\omega\sigma_\pm - iam \pm \epsilon(s-1-k)).$$

The constraint is

$$G_s = (\Lambda - 2i\omega r_\pm(2s-1) - s(s-1))a_0 + 2(i\omega\sigma_\pm - iam \pm \epsilon(s-1))a_1 = 0.$$

For $(s,n) = (1/2, 0)$ we have $P_{1/2} = 1$, thus the constraint is

$$G_{1/2} = \Lambda + 1/4 = 0.$$

This equation is quadratic in ω, and we solve it to obtain

$$a\omega^*_{1/2} = m \pm \sqrt{m^2 - \lambda - 1/4}.$$

For $(s,n) = (1,1)$, we have $P_1 = 2i\omega r + \Lambda$, thus the constraint is

$$G_1 = \Lambda^2 - 4a\omega(a\omega - m) = 0.$$

For $(s,n) = (2,3)$, the recursion generates the polynomial

$$P_2 = 48\omega^3 r^3 - 24i\omega^2(\Lambda - 2)r^2 - 6\omega(\Lambda(\Lambda-2) + 12\omega(i + am - a^2\omega))r$$
$$+i[\Lambda^2(\Lambda-2) + 4\omega\Lambda(3i + 7am - 7a^2\omega) - 16\omega(\omega(a^2+3q^2) + 2am)].$$

and the constraint is given by

$$G_2 = \Lambda^2(\Lambda-2)^2 + 144\epsilon^2\omega^2 + 48(1-\epsilon^2)\omega^2(2\Lambda - 1)$$
$$+8a\omega(a\omega - m)[18a\omega(a\omega - m) - (5\Lambda - 4)(\Lambda - 2)] = 0.$$

With $a = q = 0$ and $s = 2$, the equation $G_2 = 0$ recovers the special frequencies of Schwarzschild. With $a \neq 0, q = 0, s = 2$ we obtain the algebraically special frequencies of Kerr [4], and with $aq \neq 0$, we have the analogue for Kerr-Newman.

For $s = 1/2, 1, 2$ and with the values of ω satisfying $G_s = 0$ we have the type 1 solutions

$$y = e^{-i\omega r}(r - r_+)^{\alpha_+}(r - r_-)^{\alpha_-} P_s, \tag{4.4}$$

where

$$2\epsilon\alpha_{\pm} = i[\omega\sigma_{\pm} - am] \pm \epsilon(s-1).$$

For the remainder of this section, we impose $G_s \neq 0$ to avoid generating any solutions that can be obtained by a specialization of (4.4) or a linearly independent solution thereof.

In cases I and II, the Kovacic constants are given by:

$$\Omega_I(+-+) : \gamma = \frac{n+1}{4},\ 2\alpha_{\pm} = \pm\frac{4[iam \pm \epsilon(1-s)] - (n+1)\sigma_{\pm}}{4\epsilon}.$$

$$\Omega_{II}(-+-) : \gamma = \frac{n+1}{4}, 2\alpha_{\pm} = \mp\frac{4[iam \mp \epsilon(1+s)] + (n+1)\sigma_{\pm}}{4\epsilon}.$$

Solutions exist for every n with $\omega = -i\delta_0(n+1)/4$, and with $n+1$ values of λ defined by the appropriate constraint. The only physical parameter values that achieve pole reduction and the truncated solutions, as described in section 2.2, occur for $m = 0$ in case I and are found to be algebraically special, thus ruling out any obvious characterizations of the constraints. Because the sign of ω is reversed in the two cases, the overall parameter specifications are mutually exclusive (i.e. the cases generate solutions to different perturbation equations). However, we show that these specifications are equivalent up to the sign of ω, and that the polynomial solutions in the two cases are related by simple differential operators.

The operators for the type 1 DEs corresponding to these cases are:

$$\begin{aligned} L_I &= 4\Delta\partial_{rr} + 2[(n+1)(r^2 - a^2 - 2\epsilon^2 - 2) - 4((n+s)(r-1) - iam)]\partial_r \\ &\quad -4\Lambda - 2n(n+1)r - 4(n+1)(1-iam) + 4s(s+n) + (n+1)^2q^2. \end{aligned}$$

$$\begin{aligned} L_{II} &= 4\Delta\partial_{rr} + 2[(n+1)(r^2 - a^2 - 2\epsilon^2 - 2) - 4((n-s)(r-1) + iam)]\partial_r \\ &\quad -4\Lambda - 2n(n+1)r - 4(n+1)(1+iam) + 4s(s-n) + (n+1)^2q^2. \end{aligned}$$

In a manner analogous to the construction of Φ for Schwarzschild, we obtain the following result:

Proposition 4.1 *Let P_I be a solution to $(L_I)P_I = 0$. For $s = 0, 1/2, 1, 2$, define:*

$$\Theta = (n + 1 + 2\partial_r)^{2s}.$$

Then

$$(\hat{L}_{II})(\Theta)P_I = 0,$$

where $\hat{L}_{II} = L_{II}|_{m \to -m}$.

Now, supposing the constraint of L_I holds at some integer n for $\lambda = \lambda_0(a, \epsilon, m)$, then we have a polynomial P_I satisfying $(L_I)P_I = 0$. It follows from the proposition that $P_{II} = (\Theta)P_I|_{m \to -m}$ is a polynomial satisfying $(L_{II})P_{II} = 0$; i.e. the constraint of L_{II} holds with $\lambda = \lambda_0(a, \epsilon, -m)$. Since λ_0 is a valid parameter specification for *every* integer m, the transformation $m \to -m$ preserves the spectrum of values specified by λ_0. Therefore, the entire set of λs for which type 1 solutions exist in case I is a subset of the λs for which the solutions exist in case II. The next proposition proves these sets are equivalent.

In a manner analogous to the construction of Φ^{-1} for Schwarzschild, we obtain the following result:

Proposition 4.2 *Let P_{II} be a solution to $(L_{II})P_{II} = 0$. For $s = 0, 1/2, 1, 2$, define:*

$$\Theta^- = (-nr - s + 2\Delta\partial_r)_{2s}.$$

Then

$$(\hat{L}_I)(\Theta^-)P_{II} = 0,$$

where $\hat{L}_I = L_I|_{m \to -m}$.

Since Θ^- has polynomial coefficients, the analogy of the arguments that follow the construction of Θ apply here to prove the equivalence of the constraints in cases

I and II. Thus, the recursion relation for case I dictates all type 1 parameter values and polynomial solutions in both case I and II as follows.

The recursion for case I is

$$A_{-1}a_{k-1} + A_0 a_k + A_1 a_{k+1} = 0, \tag{4.5}$$

where

$$A_{-1} = 2(n+1)(n+1-k),$$
$$A_0 = 4\Lambda + \sigma_+(n+1)^2 - 2(n+1)(\epsilon + 2\epsilon k + 2iam - 1) - 4(s-k)(n+s-k),$$
$$A_1 = 2(k+1)[(n+1)\sigma_- + 4\epsilon(n+s-k) - 4iam].$$

The constraint is

$$C = [4\Lambda + (n+1)^2\sigma_+ - 2(n+1)(\epsilon - 1 + 2iam) - 4s(n+s)]a_0 + [2(n+1)\sigma_- + 8(\epsilon(n+s) - iam)]a_1 = 0.$$

The type 1 solutions exist for every n, with ω defined by

$$\omega = -\delta in/4,$$

where $\delta = \delta_0 = \pm 1$. The solutions are given by

$$y = e^{nr/4}(r - r_+)^{\alpha_+}(r - r_-)^{\alpha_-}P_\delta, \tag{4.6}$$

where

$$\alpha_\pm = \pm\frac{4[\delta iam + \epsilon(1-\delta s)] - (n+1)\sigma_\pm}{4\epsilon},$$

and where P_1 is generated by the recursion (4.5) and $P_{-1} = (\Theta)P_1|_{m=-m}$. The solutions hold for each $s = 0, 1/2, 1, 2$ with $n+1$ values of λ defined by the constraint.

Equations (4.5) and (4.6) account for all non-algebraically special type 1 Liouvillian perturbations of Kerr-Newman for which ω is purely imaginary (for $m \neq 0$). For illustrative purposes, we construct the solutions now for $s = 2$ and $n = 0, 1$.

For $n = 0$ the constraint requires

$$16\lambda = -\delta 8iam - 4\epsilon^2 - 3a^2 + 52,$$

and the solutions are given by (4.6) where $P_\delta = 1$.

For $n = 1$ the constraint requires

$$\lambda = 2 - a^2 - \epsilon^2 + \delta 2iam \pm \sqrt{5 + a^2 + 2\epsilon^2 - \delta 2iam}.$$

With

$$P = r - (\lambda - \delta iam + \epsilon^2 + 3a^2/4 + 1),$$

the solutions are given by (4.6) where $P_1 = P|_{\delta=1}$ and $P_{-1} = (\Theta)P|_{\delta=-1}$.

In cases III and IV, the Kovacic constants are given by

$$\Omega_{III}(+--) : \gamma\sigma_+ = iam + \epsilon(n + 1 + s), 2\alpha_+ = -2s - n,\ 2\alpha_- = 4\gamma - n.$$

$$\Omega_{IV}(-++) : \gamma\sigma_+ = -iam + \epsilon(n + 1 - s),\ 2\alpha_+ = 2s - n,\ 2\alpha_- = 4\gamma - n.$$

Solutions exist for every n with $\omega = -i\delta_0\gamma$, and with $n + 1$ values of λ defined by the appropriate constraints. Here, the transformation $\epsilon \longrightarrow -\epsilon$ accounts for two additional classes of solutions. Since all solutions we obtain hold for arbitrary ϵ, this duality of parameter values and solutions under the sign of ϵ is taken to be implicit unless further mention is made. Again, the sign of the imaginary part of ω is reversed and the overall parameter specifications of the two cases are exclusive. We construct new differential operators to show the cases are equivalent up to the sign of ω, given the restriction to $G_s \neq 0$.

We first note that all solutions in case IV with $n < 2s$ are algebraically special, whereas the solutions in case III are not. In fact, at $n = 2s - 1$, the constraint of case IV is $G_s|_{\omega=-i\gamma} = 0$. Since $2\alpha_+ = -(n-2s)$, it follows from (2.28) that for every $n \geq 2s$, the constraint has a factor of a_{n-2s}. We easily verify $a_{n-2s} = G_s|_{\omega=-i\gamma}$ for each $s = 1/2, 1, 2$, and thus the constraint of case IV has an algebraically special factor for every n. Recall that G_s is a polynomial of degree $2s$ in λ. Hence for every $n = 2s + N$ there are $N + 1$ values of λ that solve the constraint of case IV which are, in general, not algebraically special. We then notice that imposition of $n \to n + 2s$ and $m \to -m$ in case IV aligns the value of γ and the degree of the non-algebraically special part of the constraint with that of case III. Based on our analysis up to this point, the existence of operators that prove the equivalence of the sets of non-algebraically special values of λ specified by the two cases is suggested.

The operators for the type 1 DEs corresponding to these cases are:

$$L_{III} = \Delta\partial_{rr} + 2[(r-1-\epsilon)(r+3+\epsilon)\gamma - 2(n+s)(r-1) - 2\epsilon s]\partial_r + \\ + 4(r-1-\epsilon)\gamma^2 - 2[(2n+s)(r+1) + 3s(\epsilon+2)]\gamma + 2n(n+2s) - 2\epsilon b_{1+}.$$

$$L_{IV} = \Delta\partial_{rr} + 2[(r-1-\epsilon)(r+3+\epsilon)\gamma - 2(n-s)(r-1) + 2\epsilon s]\partial_r + \\ + 4(r-1-\epsilon)\gamma^2 - 2[(2n-3s)(r+1) - s(\epsilon+2)]\gamma + 2n(n-2s) - 2\epsilon b_{1+}.$$

It is important to note that γ takes the value assigned by the appropriate Kovacic case, and represents a different quantity in each equation. As mentioned, it is precisely when these quantities coincide that we find a relation between the equations.

For clarity, from this point on we define the quantity γ to be

$$\gamma\sigma_+ = iam + \epsilon(n + 1 + s).$$

Proposition 4.3 *Let P_{III} be a solution to $(L_{III})P_{III} = 0$. For $s = 0, 1/2, 1, 2$, define*

$$\Upsilon = \sum_{k=0}^{2s} \frac{\binom{2s}{k}(r-r_-)^k(2\gamma+\partial_r)^k}{(4\gamma-n-k)_k}.$$

Then

$$(\hat{L}_{IV})(\Upsilon)P_{III} = 0,$$

where $\hat{L}_{IV} = L_{IV}|_{(m,n)\to(-m,n+2s)}$.

Again, the method of proof is that of brute force: by writing Υ explicitly for the individual values of s and using the hypothesis to solve for higher derivatives of P_{III}, one obtains the result by direct substitution.

It follows that if P_{III} is a polynomial of degree n satisfying $(L_{III})P_{III} = 0$, then $P_{IV} = (\Upsilon)P_{III}|_{m=-m}$ is a polynomial of degree $n+2s$ satisfying $(L_{IV}|_{n\to n+2s})P_{IV} = 0$.

Conjecture 4.1 *Let P_{IV} be a polynomial solution to $(L_{IV})P_{IV} = 0$ and assume $G_s \neq 0$. For $s = 0, 1/2, 1, 2$, define*

$$\Upsilon^- = (-n + (r - r_+)\partial_r)_{2s}.$$

Then

$$(\hat{L}_{III})(\Upsilon^-)P_{IV} = 0,$$

where $\hat{L}_{III} = L_{III}|_{(m,n)\to(-m,n-2s)}$.

Although we have not developed a full proof, we have verified this for each s and every $n \leq 5$. The proof is left to a later work, but for now we assume this conjecture is true.

It follows that if P_{IV} is a polynomial of degree n satisfying $(L_{IV})P_{IV} = 0$, then $P_{III} = (\Upsilon^-)P_{IV}|_{m=-m}$ is a polynomial of degree $n-2s$ satisfying $(L_{III}|_{n\to n-2s})P_{III} = 0$.

By analogy to the arguments given for cases I and II, it is clear that Υ provides a one-to-one correspondence between the non-algebraically special values of λ and the corresponding polynomial solutions in the two cases. Thus, all non-algebraically special parameter values that allow type 1 solutions are dictated by the recursion relation for case III, and the polynomials are generated either directly from this recursion or with the aid of the operator Υ, as follows.

The recursion for case III is

$$A_{-1}a_{k-1} + A_0 a_k + A_1 a_{k+1} = 0, \tag{4.7}$$

where

$$A_{-1} = 2\gamma(n+1-k),$$
$$A_0 = \Lambda + 2\gamma r_+(2n+2s+1-2k) - (n+s-k)(n+1+s-k),$$
$$A_1 = 2\epsilon(k+1)(n+2s-k).$$

The constraint is

$$[\Lambda + 2\gamma r_+(2n+2s+1) - (n+s)(n+1-s)]a_0 + 2\epsilon(n+2s)a_1 = 0.$$

The type 1 solutions exist for every n, with ω defined by

$$\omega = (am - \delta i\epsilon(n+1+s))/\sigma_+,$$

where $\delta = \delta_0 = \pm 1$. The solutions are given by

$$y = e^{\delta i\omega r}(r-r_+)^{[-n-(1-\delta)s]/2}(r-r_-)^{[4\delta i\omega - n-(1+\delta)s]/2}P_\delta, \tag{4.8}$$

where P_1 is generated by the recursion (4.7) and $P_{-1} = (\Upsilon)P_1|_{m=-m}$. The solutions hold for each $s = 0, 1/2, 1, 2$ and for the $n+1$ non-algebraically special values of λ defined by the constraint For generic values of a and ϵ, the transformation $\epsilon \to -\epsilon$ provides additional solutions.

Equations (4.7) and (4.8) account for all type 1 Liouvillian perturbations of Kerr-Newman for which ω has a non-zero real part. We construct the complete solutions now for $n = 0, 1$.

For $n = 0$, the constraint defines λ via

$$\Lambda = s(s+1) - 2\delta i\omega r_+(2s+1).$$

The type 1 solutions are then given by (4.8) where $P_1 = 1$ and

$$P_{-1} = \Upsilon(P_1) = \sum_{k=0}^{2s} \frac{\binom{2s}{k}(2\gamma(r-r_-))^k}{(4\gamma - k)_k}.$$

Note that $\delta = -1$ corresponds to $\alpha_+ = 0$ in case IV and thus the requirement on λ is equivalent to $b_{1+} = 0$ in this case. Evidently, P_{-1} is the Laguerre polynomial $L_{2s}^{2\alpha_- -1}(r - r_-)$ one expects to find when the pole at r_+ is eliminated in case IV via $n = 2s$.

For $n = 1$, the constraint defines λ via:

$$\Lambda = (s+1)^2 - 4\delta i\omega r_+(s+1) \pm \sqrt{(2\delta i\omega r_+ - s - 1)^2 + 4\epsilon\delta i\omega(2s+1)}.$$

With

$$P = r - (2s+1)r_+ - (\Lambda - s(s+1))/2\delta i\omega,$$

the type 1 solutions are given by (4.8) where $P_1 = P|_{\delta=1}$ and $P_{-1} = (\Upsilon)P|_{\delta=-1}$. For example, at $s = 2$ and with the appropriate values of λ as given above, we have the Liouvillian perturbations.

$$R_+ = e^{i\omega_+ r}(r - r_+)^{-2}(r - r_-)^{2i\omega_+ - 4}(r - 5r_+ - (\Lambda - 6)/2i\omega_+),$$

and

$$R_- = e^{-i\omega_- r}(r - r_+)^{-4}(r - r_-)^{-2i\omega_- - 2}[(4\gamma - 5 + 2\gamma(r - r_-))(r - 5r_+ + (\Lambda - 6)/2i\omega_-) + 4(r - r_-)],$$

where

$$\omega_{\pm} = (am \mp 4i\epsilon)/\sigma_{+}.$$

These examples illustrate how all of the type 1 parameter values and solutions existing in cases III and IV are determined for higher n by the recursion (4.7), together with the operator Υ. In the sense that the smaller degree polynomials, P_1, are potential functions for the larger degree polynomials, P_{-1}, the computing power provided by Υ is accentuated compared with Θ in cases I and II.

We conclude this section with a consideration of explicit solutions arising in the $m = 0$ case. The analysis is simplified since the sign of m is not essential to the operator Υ and thus the constraints of the two cases are independent of an index δ. We find several functions $\lambda(\epsilon, n)$ and $a(\epsilon, n)$, that solve the constraints for arbitrary n, and the application of Υ to the resulting explicit solutions produces additional solutions that have not been identified in [5], [6], [8].

We first impose $b_- = b_{1-} = 0$ to eliminate the pole at r_-. Since the algorithm requires $\alpha_- = 4\gamma - n = 0$ and since a, ϵ are real we must have $m = 0$. Then the values of a and λ that achieve this requirement are:

$$a^2 = -(1-\epsilon)^2 + 4\epsilon(s+1)/n,$$
$$\lambda = s(s+1) - \frac{n}{16}[nr_-^2 - 4(1+3s)r_- - 4(1+s)],$$

and ω is given by $\omega = -\delta in/4$ where $\delta = \pm 1$. For each $s = 0, 1/2, 1, 2$ and for every ϵ sufficiently close to 1, the above restriction on a is physical (i.e. $a^2 + q^2 < M^2$) for a finite number of values of a and n. The transformation $\epsilon \to -\epsilon$ is unphysical (i.e. $a^2 < 0$) and is therefore not imposed. For a valid parameter set, the type 1 solutions are given by

$$y = e^{nr/4}(r - r_+)^{-(n+(\delta+1)s)/2}(r - r_-)^{(\delta-1)s/2}P_\delta,$$

where

$$P_1 = L_n^{-n-2s-1}(-(n+1)(r-r_+)/2),$$

and

$$P_{-1} = \sum_{k=0}^{2s} \frac{1}{k!}\binom{2s}{k}(-(r-r_-))^k(n+1+2\partial_r)^k P_1.$$

Finally, we obtain truncated polynomial solutions with P expanded about r_-. The condition $2\alpha_- = -(n-1-N)$, where $N = 0...n-1$, is achieved by $m = 0$ and the parameter restriction

$$a^2 = -(1-\epsilon)^2 + 4\epsilon(n-N+s)/(N+1).$$

In this case ω is given by $\omega = -\delta iN/4$. For each $s = 0, 1/2, 1, 2$ and for every ϵ sufficiently close to 1, the above restriction on a is physical (i.e. $a^2 + q^2 < M^2$) for a finite number of values of a, n and N. Again, $\epsilon \to -\epsilon$ is unphysical and is not imposed. For a valid parameter set and with λ defined by $a_{n-N-1} = 0$, the type 1 solutions are given by

$$y = e^{\delta i\omega r}(r-r_+)^{-(n+(1-\delta)s)/2}(r-r_-)^{-(n-N+(1+\delta)s)/2}P_\delta,$$

where P_1 is the truncated polynomial generated by the recursion (4.7) and $P_{-1} = \Upsilon(P_1)$.

For $N = 0$, λ is defined by

$$16(\lambda - s^2) = -(\epsilon+3)^2 - 12\epsilon(n+s)$$

and we have $P_1 = (r-r_-)^n$ and $P_{-1} = \Upsilon(P_1)$.

For $N = 1$, λ is defined by

$$4(\lambda - s^2 - 1) = -(\epsilon+3)^2 - 6\epsilon(n+1+s) \pm \sqrt{(\epsilon+s)^2 + 2\epsilon n},$$

and we have

$$P_1 = (r - r_-)^{n-1}[r - 1 - s \mp \sqrt{(\epsilon + s)^2 + 2\epsilon n}],$$

and $P_{-1} = \Upsilon(P_1)$. The polynomials and parameter values for $N = 2, ..., n-1$ are obtained in analogous fashion.

We note that the explicit solutions P_{-1} found above do not arise naturally from the case IV recursion. The operator Υ uncovers solutions that are otherwise hidden and expresses these seemingly complicated polynomials in a compact explicit form in terms of the potential P_1. The direct search for explicit solutions in case IV leads to physical conditions with $\epsilon \to -\epsilon$, but the parameter values are found to be algebraically special. This guarantees there are no analogies of explicit solutions arising in case IV that reveal solutions in case III via Υ^-.

In summary, the Kerr-Newman type 1 perturbations lie in three distinctive classes: the explicit and persistent set of algebraically special solutions, the solutions existing for every $4i\omega \in \mathbb{Z} - \{0\}$, generated by the recursion (4.5), and the solutions existing for every n with $\omega = (am \pm \delta i\epsilon(n + 1 + s))/\sigma_\pm$, generated by the recursion (4.7). Within the latter two non-algebraically special infinite classes, the operators Θ and Υ provide access to all of the available solutions, and at every n the set of λs for which the solutions exist is uniform over the respective class. An efficient procedure for generating these solutions at every n was developed and demonstrated through examples that provided the complete solutions at $n = 0, 1$. We note that conjecture 4.1 remains to be proved in order to verify these results for all n. For the third class with $m = 0$, we employed these results to identify and construct several classes of explicit solutions that hold for every n with restricted physical values of the black hole parameter a.

4.2 Type 2 Solutions

The type 2 Kovacic cases are specified by

$$\theta = \frac{\beta_+}{r - r_+} + \frac{\beta_-}{r - r_-},$$

where

$$\beta_\pm = 1 + \frac{i\delta_\pm}{\epsilon}(\omega\sigma_\pm - am \mp i\epsilon s),$$

with $\delta_i = \pm 1, 0$ independently for $i = +, -$.

For every function θ_ν ($\nu = I, ..., VIII$) we seek polynomial solutions to the type 2 DE (2.29), denoted by:

$$(L_\nu)P = 0.$$

If such is found, the type 2 solution is given by

$$y = e^{\pm\sqrt{D}/2}(r - r_+)^{\beta_+/2}(r - r_-)^{\beta_-/2}\sqrt{P}, \tag{4.9}$$

where D is the discriminant of equation (2.10).

We write

$$P = \sum_{k=0}^{n} a_k z^k, \ a_n \neq 0, \tag{4.10}$$

where $z = r - r_+$, then the necessary integer condition (2.32) is:

$$(\delta_+\sigma_+ + \delta_-\sigma_-)\omega = (\delta_+ + \delta_-)am + i\epsilon(n + 2 + s(\delta_+ - \delta_-)). \tag{4.11}$$

Since no valid choices of $\delta_\pm$ allow the coefficient of ω in (4.11) to vanish, in all cases this condition defines $\omega(a, \epsilon, m, s, n)$. Then for every n, $a_0, ..., a_{n-1}$ are generated by the five-term recursion relation of L_ν, and the three corresponding constraints define the values of $\lambda(m, s)$, $a(m, s)$, $\epsilon(m, s)$ for which type 2 solutions exist.

In some cases, there are admissible values of s for which one constraint is satisfied identically at a low value of n, leaving ϵ arbitrary. However, all such solutions are type 1 (and unphysical) and are discarded.

Once λ is specified according to one of the constraints, the two remaining constraints are real mixed polynomials in a,ϵ, s and im. Since a, ϵ, m, s are real, for $m \neq 0$ these constraints form four generally independent restrictions on a, ϵ, m, s. Furthermore, for Kerr geometries with $q = 0$ and with a fixed value of s, the constraints are comprised of two equations defining $\lambda(a, m)$ and $a(m)$, and two equations that must be satisfied by a single choice of the integer m. These strong restrictions imply the $m \neq 0$ case is not well suited to produce type 2 solutions, all physical considerations aside. Thus we restrict our attention to $m = 0$, and furthermore we assume $q \neq 0$ so that in all cases the algorithm defines a set of numerical values of $(\omega, \lambda, a, \epsilon)$ for which type 2 solutions exist. If a physical parameter set is found for Kerr-Newman, it is valid for Kerr only if $\epsilon^2 + a^2 = 1$. These solutions exist in every Kovacic case and for every n (although in several instances the solutions are type 1), however we suspect the required numerical values of a are unphysical.

Note that ρ and θ_ν, and therefore L_ν, are invariant under

$$(\omega, m, s, \delta_+, \delta_-) \rightarrow (-\omega, -m, -s, -\delta_+, -\delta_-).$$

Therefore, all type 2 solutions can be obtained by choosing one sign for s. We adopt the convention $s \geq 0$, and if a type 2 perturbation, R, is obtained, there is another perturbation for $(\omega, m, s) = (-\omega, -m, -s)$, given by $\tilde{R} = R\Delta^s$ (of course for our purposes the sign of m is not essential to this transformation).

Also, L_ν is invariant under

$$(\epsilon, \delta_+, \delta_-) \rightarrow (-\epsilon, -\delta_-, -\delta_+).$$

Thus, we may eliminate any one of two cases that are invariant under $(\delta_+, \delta_-) \rightarrow (-\delta_-, -\delta_+)$ by allowing $\epsilon \rightarrow -\epsilon$. Note that this map reverses the values of the constants $\alpha_\pm$ and the orders of the poles at $r_\pm$.

We take the 5 independent cases to be as follows, where we have indicated the values of ω and $\beta_\pm$ required by each case:

$$\theta_0(++) : 2i\omega = -\epsilon(n+2)/(1+a^2+\epsilon^2),\ 2\beta_\pm = \pm 4i\omega - n \pm 2s$$

$$\theta_I(+-) : 4i\omega = -(n+2+2s),\ \epsilon\beta_\pm = \pm i(\omega\sigma_\pm \mp i\epsilon s)$$

$$\theta_{II}(-+) : 4i\omega = n+2-2s,\ \epsilon\beta_\pm = \mp i(\omega\sigma_\pm \mp i\epsilon s)$$

$$\theta_{III}(+\ 0) : i\omega = -\epsilon(n+2+s)/\sigma_+,\ \beta_+ = -1-n,\ \beta_- = 1$$

$$\theta_{IV}(-\ 0) : i\omega = \epsilon(n+2-s)/\sigma_+,\ \beta_+ = -1-n,\ \beta_- = 1$$

The character of the recursions is similar among the first three cases, and we find type 2 solutions at low n for unphysical values of the parameters. We give the numerical values of (λ, a, ϵ) that allow polynomial solutions to $(L_I)P_\nu = 0$ for $n = 0, 1, 2$.

All solutions in cases III and IV reduce to type 1 solutions for $n = 0, 1, 2$, and we suspect this holds for all n. Although we do not prove this, we state the recursion relation from which a proof could be developed. Note that the transformation $\epsilon \rightarrow -\epsilon$ accounts for additional solutions and parameter values in cases 0, III and IV.

For $n = 0$, the constraints of cases θ_0, θ_I, and θ_{II} are solved respectively by

$$\Lambda = 7s^2 \pm 7s + 5/4, \epsilon = -2\frac{1 \pm 4s}{1 \pm 2s}, a^2 = -\frac{68s^2 \pm 68s + 13}{(2s+1)^2}$$

$$\Lambda = 7s^2 + 8s + 2, \epsilon = \pm 2\frac{1+2s}{1+s}, a^2 = -\frac{17s^2 + 22s + 7}{(s+1)^2}$$

$$\Lambda = 7s^2 - 8s + 2, \epsilon = \pm 2\frac{1-2s}{1-s}, a^2 = -\frac{17s^2 - 22s + 7}{(s-1)^2}$$

These parameter values give rise to authentic type 2 solutions, but are clearly unphysical. At $n = 1$, the degrees of a and ϵ in the constraints increase to the extent that we are not able to obtain analytic expressions for the values of these parameters. There are some special cases arising for particular values of s, but as mentioned they are all type 1.

In the case θ_0, with $s = 0$ and $n = 1, 2$ the type 2 solutions exist for the following approximate real values of (a^2, ϵ) :

$$n = 1 : (-31.19, 4.20).$$

$$n = 2 : \ (-3.80, 0.81), \ (-9.46, 0.95).$$

We obtain similar values for $s = 1/2, 1, 2$ and in the other cases. This example may suggest that the real values of a^2 required for for type 2 solutions are negative for all n, however the calculational evidence for this is limited by the complexity of the type 2 recursion (i.e. for $n \geq 3$ a computer cannot easily approximate the roots of the system of equations defined by the constraints). We thus find the existence of physical Liouvillian solutions is not settled by the Kovacic algorithm alone in the first three cases. Our inclination is that all type 2 solutions of non-extreme Kerr-Newman are unphysical.

The recursion relation for cases III and IV is given by

$$A_{-1}a_{k-1} + A_0 a_k + A_1 a_{k+1} + A_2 a_{k+2} + A_3 a_{k+3} = 0,$$

where

$$A_{-1} = -4\omega^2(n+1-k),$$

$$A_0 = 4i\omega(2n+1-2k)(2\gamma r_+ + s),$$

$$A_1 = (n-k)[4\Lambda - 16\omega^2 r_+^2 + \delta 8 i\epsilon\omega(n+2+4\delta s) - (n+1-k)(n-1-k)],$$

$$A_2 = 2\epsilon(2n-1-2k)[2\Lambda + 16\omega^2 r_+ - 2\delta s(n+2+\delta s) - (n-k)(n-1-k)],$$

$$A_3 = 4\epsilon^2(k+3)(2n+1-k)(n-1-k),$$

with

$$\omega = \delta i(n+2+2\delta s)/4,$$

where $\delta = 1$ for case III and $\delta = -1$ for case IV. The three constraints are obtained from this recursion evaluated at $k = 0, -1, -2$. These equations determine all of the type 2 solutions and parameter values existing in cases III and IV. As mentioned, all solutions at low values of n are type 1, and we conjecture this holds for all n. The agreement of these values of ω with those required by cases III and IV in type 1 (with $m = 0$ and integer values of the constants $\alpha_{\pm}$) suggests the existence of a linear operator relating the type 2 DEs to the type 1 DEs, indicating a possible direction for the proof of this conjecture, as well as the proof of the type 2 conjecture for Schwarzschild.

5 The Liouvillian Perturbations of Extreme Kerr-Newman

In this section, we apply the Kovacic algorithm to equation (1.1) with $a^2 + q^2 = M^2$ (*i.e.* $\epsilon = 0$). The results hold for extreme Kerr geometries with the charge set to $q = 0$. Aside from an infinite set of explicit solutions existing for real $\omega = am/(a^2 + M^2)$ (i.e. no damping), we show that all extreme Liouvillian solutions are specializations to $\epsilon = 0$ of either the algebraically special non-extreme solutions or the solutions generated by the non-extreme recursion (4.5), the only difference being the expansion of the relevant functions Ω, and hence the form of the solutions, in the limit $\epsilon \longrightarrow 0$. In other words, aside from the explicit case, the type 1 polynomials and parameter values (ω, λ) in the non-extreme cases $0, I$ and II, evaluated at $\epsilon = 0$, account for all of the extreme solutions. The individual analysis of the extreme case is, however, required to obtain the explicit case and the proper functional form of the solutions. Recall that type 2 solutions do not exist for extreme geometries.

Before proceeding to the type 1 analysis, we state ρ in terms of the extreme Kerr-Newman parameters.

The partial fraction expansion of ρ is given by

$$\rho = -\omega^2 + \frac{b_1}{r - M} + \frac{b_2}{(r - M)^2} + \frac{b_3 b_4}{(r - M)^3} - \frac{(b_4)^2}{(r - M)^4},$$

where

$$b_1 = -2\omega(2M\omega + is),$$

$$b_2 = \lambda - \omega^2(a^2 + 6M^2),$$

$$b_3 = -(2M\omega - is),$$

$$b_4 = (a^2 + M^2)\omega - am.$$

The essential parameters (ω, λ, a, m) range over $(\mathbb{C}, \mathbb{C}, (0, M), \mathbb{Z})$ and the spin parameter s takes the assigned values $s = 0, \pm\frac{1}{2}, \pm 1, \pm 2$.

In the case $b_4 = 0$, the type 1 Kovacic cases are specified by

$$\Omega = \gamma + \frac{\alpha}{r - M},$$

where

$$\gamma = i\delta_0\omega,$$
$$2\alpha = 1 + \delta_1\sqrt{1 + 4b_2},$$

with $\delta_i = \pm 1$ independently for $i = 0, 1$. The condition $b_4 = 0$ requires ω to be

$$\omega = \frac{am}{a^2 + M^2},$$

and the necessary integer condition, $n = b_1/2\gamma + \alpha$, defines λ to be

$$\lambda = -(a^2 + 6M^2)\gamma^2 - \alpha(\alpha - 1) - 1/2,$$

with

$$2\alpha = 2M\gamma - n - \delta_0 s.$$

For these values of ω and λ, the type 1 solutions are given by

$$y = e^{\gamma r}(r - M)^\alpha \sum_{k=0}^{n} \frac{(2\gamma(r - M))^k}{k!(n + 1)_k(2\alpha)_k}, \tag{5.1}$$

valid for $\delta_0 = \pm 1$, each $s = 0, \pm 1/2, \pm 1, \pm 2$ and every n. These are all the Liouvillian solutions in the case $b_4 = 0$.

In the generic case, $b_4 \neq 0$, the type 1 Kovacic cases are specified by

$$\Omega = \gamma + \frac{\alpha}{r - M} + \frac{\eta}{(r - M)^2},$$

where

$$\gamma = i\delta_0\omega,$$

$$\alpha = 1 + \delta_1(2Mi\omega + s),$$

$$\eta = i\delta_1[(a^2 + M^2)\omega - am],$$

with $\delta_i = \pm 1$ independently for $i = 0, 1$.

For every function $\Omega_\nu(\pm\pm)$ ($\nu = I, ..., IV$) we seek polynomial solutions to the type 1 DE (2.39), denoted by:

$$(L_\nu)P = 0.$$

If such is found, the type 1 solution is given by

$$y = e^{\gamma r} e^{-\frac{\eta}{r-M}} (r - r_-)^\alpha P. \tag{5.2}$$

We write

$$P = \sum_{k=0}^{n} a_k z^k, \; a_n \neq 0, \tag{5.3}$$

where $z = r - M$, then the necessary integer condition (2.38) is

$$2\left(\delta_0\delta_1 - 1\right) M\omega = i\left[\delta_0\left(n+1\right) + s\left(\delta_0\delta_1 + 1\right)\right]. \tag{5.4}$$

If (5.4) holds, and with $a_0, ..., a_{n-1}$ generated by the recursion relation (2.40) of L_ν, the corresponding constraint (2.41), $C = 0$, is necessary and sufficient for the existence of a type 1 solution. If Ω_v, n, and s are chosen such that (5.4) is satisfied identically, then the constraint defines $\omega(\lambda, a, m)$, and solutions are obtained with λ unrestricted. Otherwise, (5.4) defines $\omega(a, m, s, n)$ and, for every n, C is a polynomial of degree $n + 1$ in λ that defines $n + 1$ values of $\lambda(a, m, s)$ for which type 1 solutions exist.

The case $\delta_0 = \delta_1 = 1$ is easily ruled out for $s \geq 0$, and thus there are 3 independent cases to consider:

$$\Omega_0(--): \gamma = -i\omega,\ \alpha = 1 - (2Mi\omega + s),\ \eta = -i[(a^2 + M^2)\omega - am],\ n = 2s - 1$$

$$\Omega_I(+-): \gamma = (n+1)/4M,\ 2\alpha = -n - 1 + 2(1 - s),\ 4\eta = -(a^2 + M^2)(n+1) + 4iam$$

$$\Omega_{II}(-+): \gamma = (n+1)/4M,\ 2\alpha = -n - 1 + 2(1 + s),\ 4\eta = -(a^2 + M^2)(n+1) - 4iam$$

In the first case, ω is arbitrary and in cases I and II, ω is given by $\omega = -\delta_0 i\gamma$.

It is a simple matter to verify that these functions are identical to the functions $\Omega_0, \Omega_I, \Omega_{II}$ in the non-extreme case, evaluated at $\epsilon = 0$. The non-extreme function,

$$\Omega = \gamma + \frac{\alpha_+}{r - r_+} + \frac{\alpha_-}{r - r_-},$$

simplifies to

$$\Omega = \frac{-\gamma\epsilon^2 + (\alpha_+ - \alpha_-)\epsilon + \gamma(r - M)^2 + (\alpha_+ + \alpha_-)(r - M)}{\Delta}.$$

Taking the general expressions for the non-extreme constants $\alpha_\pm$, we choose $\delta_+ = -\delta_-$ to obtain

$$\alpha_+ - \alpha_- = \delta_+[M^2\omega\epsilon^2 + ((a^2 + M^2)\omega - am)]/\epsilon,$$
$$\alpha_+ + \alpha_- = 1 + \delta_+(2M^2\omega - is).$$

Substituting these values into the above expression for Ω, simplifying, and setting $\epsilon = 0$, we obtain the partial fraction expansion of Ω :

$$\Omega = \gamma + \frac{1 + \delta_+(2M\omega - is)}{r - M} + \frac{\delta_+((a^2 + M^2)\omega - am)}{(r - M)^2},$$

which corresponds to the functions Ω required by the algorithm for extreme geometries. It follows that the extreme type 1 DEs are simply the non-extreme type 1 DEs evaluated at $\epsilon = 0$. Thus, all of the polynomial solutions and the values of λ for which they exist are evaluations at $\epsilon = 0$ of the solutions obtained in section 4.1 in the cases Ω_0, Ω_I, Ω_{II}.

In particular, for the case Ω_0 we have the extreme version of the algebraically special solutions. The algebraically special values of ω are defined by $G_s|_{\epsilon=0} = 0$, the polynomials are defined by $P_s \longrightarrow P_s|_{\epsilon=0}$, and the solutions are given by (5.2) with the appropriate extreme Kovacic constants α, η.

Also, for the extreme cases Ω_I and Ω_{II} the polynomial solutions are generated by the single recursion (4.5) evaluated at $\epsilon = 0$. These solutions exist for every n with $\omega = \pm i(n+1)/4$ and with $n+1$ values of λ defined by the corresponding constraint evaluated at $\epsilon = 0$. The recursion generates the polynomials for $\omega = -i(n+1)/4$ and the operator Θ, evaluated at $\epsilon = 0$, provides access to the polynomials for $\omega = i(n+1)/4$. Hence, further analysis of the extreme case is somewhat redundant. However, the separate study of the recursion (4.5) evaluated at $\epsilon = 0$ can uncover additional results, as the following example illustrates.

In the $s = 0, \epsilon = 0$ case, we find the constraint (4.5) has a factor of $a_{n/2}$ for every $n = 0, 2, 4, 6, 8$; we conjecture this is the case for all even n but do not have a proof. Since the constraint factors, we can find explicit expressions for the solutions and parameter values for $n = 0, 1, 2, 4$. Note that with $s = 0$ the operator Θ is the identity map, thus the polynomials that solve the DEs in cases I and II are identical under $m \longrightarrow -m$ for each n.

For $n = 0$ the constraint requires

$$16M^2\lambda = -(3a^2 + 12M^2 - \delta 8Miam),$$

and the polynomial is $P = 1$. For $n = 1$ the constraint requires

$$4M^2(\lambda \pm \sqrt{a^2 + M^2 + 2\delta Miam}) = -3a^2 - 8M^2 + 4M\delta iam,$$

and the polynomial is

$$P = z \pm \sqrt{a^2 + M^2 + 2\delta Miam}.$$

For $n = 2$ the constraint requires

$$16M^2\lambda = -3(9a^2 + 20M^2 - \delta 8Miam),$$

$$8M^2(2\lambda \pm \sqrt{36a^2 + 37M^2 - 48\delta Miam}) = -68M^2 - 27a^2 + 24\delta Miam),$$

and the corresponding polynomials are

$$P = r(r - 2M) - a^2 + 4\delta Miam/3,$$

$$P = 3z^2 + z(M^2 \mp \sqrt{36a^2 + 37M^2 - 48\delta Miam}) + 3(a^2 + M^2) - 4\delta Miam,$$

Finally, for $n = 4$ the constraint is solved by $a_2 = 0$, which defines λ to be

$$8M^2(2\lambda \pm \sqrt{100a^2 + 109M^2 - 80\delta Miam}) = -164M^2 - 75a^2 + 40\delta Miam),$$

The corresponding polynomial is

$$P = [5z^2 - (5a^2 + 5M^2 - 4\delta Miam)][5z^2 + 5a^2 + M^2 - 4\delta Miam + z(3M \mp \sqrt{100a^2 + 109M^2 - 80\delta Miam})]$$

For each of the above 16 sets of (λ, P), and for $4\omega = -\delta i(n+1)$ we obtain the type 1 perturbations:

$$R = e^{(n+1)r/4} e^{-\frac{(a^2+M^2)(n+1)/4 - \delta iam}{r-M}} (r - r_-)^{2n+3} P,$$

valid for $\delta = \pm 1$. We leave the proof of our conjecture (that the constraint corresponding to the recursion (4.5), evaluated at $\epsilon = 0$ and $s = 0$, has a factor of $a_{n/2}$ for every even n) as an open problem. The above example illustrates how the parameter values and polynomial solutions in the extreme case are generated by the single non-extreme recursion (4.5), evaluated at $\epsilon = 0$, for higher values of n and the other values of $s = 1/2, 1, 2$.

In summary, the only additional solutions found by a consideration of the extreme case are the explicit Laguerre polynomial solutions (5.1), existing for every n with $\omega = am/(M^2 + a^2)$ and with a special set of values of $\lambda(n)$. Otherwise, we have the result that the algebraically special solutions of non-extreme Kerr-Newman, and the infinite class of solutions non-algebraically special solutions existing in the non-extreme case I, evaluated at $\epsilon = 0$, together with the form of the solution (5.2), the appropriate constants α, η, and the operator Θ, account for all of the Liouvillian solutions of extreme Kerr-Newman.

6 Conclusions

For the Schwarzschild black hole, we proved for the first time that the only type 1 Liouvillian perturbations are the well-known algebraically special gravitational modes, which we constructed explicitly. The proof of this important result relied on the construction of differential operators that reduced all of the Kovacic cases down to one specific case. The recursion relation corresponding to this case was then rearranged to prove the constraint is not satisfied for any n, given the restriction to non-algebraically special parameter values and the physical restriction $\lambda = \ell(\ell+1) > 0$.

We also proved there are no type 2 perturbations of Schwarzschild for $n \leq 60$, and developed a conjecture that there are no type 2 solutions for any n. This conjecture was based on a pattern that was detected in the separation of the numerical values of λ required by two of the three constraints. We illustrated this conjecture in the $s = 0$ case, and we identified the same patterns for $s = 1/2, 1, 2$ in all of the available Kovacic cases. The occurrence of these patterns leads us to believe there are no type 2 perturbations of Schwarzschild.

Combining the results for the type 1 and type 2 perturbations of Schwarzschild, the application of the Kovacic algorithm was completely resolved: the only Liouvillian perturbations are algebraically special. No immediate further work on this problem is anticipated. Although one may be able to develop a proof that type 2 is empty by alternative methods (e.g. differential operators, reduction to type 1, finding a non-zero linear combination of the constraints, or a direct proof of the conjecture as stated), it is likely that our approach to the problem is convincing enough to make this pursuit somewhat of a triviality. The main advantage of pursuing this avenue of research is that any results one may obtain could point towards methods that one could apply to obtain additional information concerning the type 2 solutions of the more general Kerr-Newman case.

For the non-extreme Kerr-Newman black hole, we developed a new classification scheme of the type 1 solutions. We proved (up to the conjecture 4.1) that all type 1 solutions belong to either the finite set of algebraically special perturbations, or one of two distinctive infinite classes of type1 perturbations, distinguished by either pure imaginary values of ω or values of ω that have a real part. Aside from the algebraically special perturbations (which we constructed explicitly for each $s = 1/2, 1, 2$), the type 1 solutions exist for every n and are generated by one of two specific recursion relations. The proof of this result relied on the construction of new differential operators that showed the equivalence of the constraints among several of the Kovacic cases. This classification greatly simplifies the Kovacic analysis, provides an extremely efficient way to access all of the available type 1 solutions, and opens several avenues for future research involving the application of our methods to the perturbations of different kinds of black holes (e.g. Reissner-Nordstrom) We used these operators to uncover the complete set of the non-algebraically special type 1 solutions for low values of n, a task that prior to the introduction of our methods would have involved significantly more computational work and a consideration of a significant number of additional cases; our results have made this a simple task. In addition to these solutions, we identified new infinite classes of explicit solutions, valid for every n and with special values of $(a(n), \lambda(n))$, and contained within the class of solutions for which ω has a real part. The application of one of our operators to these explicit solutions uncovered more explicit solutions that have never been previously identified (furthermore it is not clear how these solutions would have been identified or characterized without the use of these operators).

The first notable direction for future research concerning the type 1 perturbations of Kerr-Newman involves working towards a more general proof of the propositions

4.1, 4.2, and 4.3. In this thesis, we have proved all of these propositions by brute force and direct substitution. This method is quite inefficient and, furthermore, fails for the conjecture 4.1. A proof of this conjecture may follow if one works towards a more general characterization of these differential operators as inherent properties of the type 1 operators (for example, it is likely that the propositions hold for every half-integer s). One suggestion is to seek a new operator that directly transforms the two type 1 DEs into each other, without the need to solve for higher derivative terms. This idea is similar to proposition 3.4 for Schwarzschild, where the operators were simply factor transformations that directly related the DEs. The challenge is to find new differential operators with this property. Another (possibly related) suggestion for the reformulation of the proofs is to survey the literature on differential transformations of elementary differential equations to see if the relationships we have found can be characterized as a subset of known relationships between more general classes of DEs.

We obtained no type 2 perturbations of non-extreme Kerr-Newman at low values of n, and we suspect it is difficult to prove this is the case for higher n. A starting place for future work on the type 2 solutions of Kerr-Newman is a development of a proof of our conjecture that two of the Kovacic cases produce type 1 solutions for every n, or as previously mentioned, working towards a proof of the non-existence of type 2 solutions in the Schwarzschild case.

For the perturbations of extreme Kerr-Newman, we identified an explicit set of solutions existing for real values of ω, and we proved the new result that all remaining extreme solutions are direct specializations to $\epsilon = 0$ of the infinite class of non-extreme solutions that exist for pure imaginary ω. An interesting phenomenon arises for $s = 0$, in which case we conjectured that the constraint has a factor of $a_{n/2}$ for every even n. The proof of this conjecture, which is likely an inherent property of the appropriate recursion relation, is left to future work.

In the recent article [8], the Kovacic algorithm is applied to the black hole perturbations considered in this thesis, and also to the full gravitational perturbations of the non-extreme and extreme Reissner-Nordstrom black holes (as derived by Chandrasekhar in [3]). For Reissner-Nordstrom, there is an additional pole and one less parameter ($a = 0$). The type 1 recursion relations are 4-term and impose 2 independent constraints on the parameters, and there are several more Kovacic cases to consider. We expect the type 1 DEs in the various cases to be related by differential operators that provide the extension of the results of this thesis to the Reissner-Nordstrom case. The construction of such operators will extend the Kovacic analysis initiated in [8] by establishing a new classification of the type 1 perturbations of Reissner-Nordstrom. These operators will potentially uncover new explicit solutions and may allow us to prove the non-existence of type 1 solutions in certain cases by the analogy of the methods used in theorems 3.1 and 3.2 in this thesis (e.g. coalesced poles [8]). Furthermore, [8] presents several conjectures concerning the existence of explicit solutions and the form of the recursion constraints, and these operators may be useful in developing the proofs for these conjectures. Such future work is anticipated to complete, as far as possible, the analysis of the Liouvillian perturbations of the Reissner-Nordstrom black hole initiated in [8].

We note that the methods developed in this thesis may be useful in the application of the Kovacic algorithm to perturbations of more exotic geometries, such as NUT space [12] and higher dimensional black holes [13], including asymptotically deSitter geometries [14].

Suggestions for the practical applications of the plethora of available Liouvillian perturbations of black holes include:

- The consideration of the solutions at the quasi-normal mode boundaries.
- The testing and comparison of numerical schemes for approximating quasi-normal modes.
- The application of the Kovacic algorithm to the angular perturbation equations, and the search for a physical correlation between the resulting constrained values of the black hole parameters and the parameter values derived in this thesis.

One or more of these avenues will likely be pursued in future works; this thesis has only intended to identify the Liouvillian perturbations of the Kerr-Newman black hole. To our knowledge, the application of these solutions to physical problems has not been initiated and is in need of further study. This author has received NSERC funding for a Ph.D. program in support of these pursuits.

References

[1] J. J. Kovacic, J. Symbolic Computation **2**, 3 (1986).

[2] K. D. Kokkotas, Nuovo Cimento **108B**, 991 (1993).

[3] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).

[4] S. Chandrasekhar, Proc. R. Soc. Lond. **A 392**, 1 (1984).

[5] W. E. Couch, J. Math. Phys. **22**, 1457 (1981).

[6] W. E. Couch, J. Math. Phys. **26**, 2286 (1985).

[7] W. E. Couch and E. T. Newman, J. Math. Phys. **14**, 285 (1973).

[8] W.E. Couch and C.L. Holder, "Liouvillian Perturbations of Black Holes", J.Math. Phys, to appear.

[9] S. Slavyanov and W. Lay, *Special Functions: A Unified Theory Based on Singularities* (Oxford, New York, 2000).

[10] M. van der Put and M. F. Singer, *Galois Theory of Linear Differential Equations* (Springer-Verlag, Berlin, 2003).

[11] K. D. Kokkotos and B. G. Schmidt, Living Reviews in Relativity, **2**(2), lrr-1999-2, (1999).

[12] D. Bini and C. Cherubini, Class. Quantum Grav. **21**, 4523 (2004).

[13] H.K. Kunduri, J. Lucietti, and H.S. Reall, Phys. Rev. D **74**, 084021 (2006).

[14] G.T. Horowitz and V.E. Hubeny, Phys. Rev. D **62**, 024027 (2000).