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Matrix Kundt-Newman Sequences

by

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Abstract

In this thesis we review the generalization to the Non-Abelian setting of the Kundt-Newman substitution method for solving systems of coupled wave equations in two-dimensional space, which appeared, in 1992, in the Journal of Physics in an article by L. Bombelli, W. E. Couch, and R. J. Torrence. The original contribution of this thesis is in the derivation of a particular Kundt-Newman potential sequence whose terms can be expressed explicitly as a function of its index and for which the associated wave equation has a solution in closed form. We constructed the solution by finding a method of dimensional reduction of the wave equation.

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Table of Contents

Introduction

In the presence of symmetries, many physical problems can be reduced to two dimensions. In this thesis we will discuss some of the methods for solving systems of coupled wave equations in two dimensions. One can show that the system of coupled wave equations can be represented by a partial differential equation:

$$(\partial_v J_0 \partial_u - J_1)\Psi_0 = 0$$

where the potentials J_0 and J_1 are matrix functions of coordinates u and v. One can recursively define a Kundt-Newman sequence of potentials $\{J_k\}$ and an associated sequence of wave equations $\{(\partial_v J_k \partial_u - J_{k+1})\Psi_k = 0\}$. This sequence of wave equations has the property that if a solution to one of the equation in the sequence is known the solutions of the equations preceding it can be constructed from this known solution and the potentials in the $\{J_k\}$ sequence. The recursive relation that is used to construct the Kundt-Newman sequences is described in chapter 1 of this thesis. Chapter 1 also covers the relationship between the Kundt-Newman sequences and Toda lattices.

Since the Kundt-Newman sequence is defined by a recursive equation involving the inverses of preceding potentials one often encounters two difficulties:

1. To decide whether or not a solution of the k-th equation can be expressed in closed form, it may be necessary to know the potentials J_k explicitly as functions of the index k.

2. It may not be clear, how to proceed in constructing the sequence and relating solutions if one of the matrices in the sequence $\{J_k\}$ becomes singular.

In chapter 2 we will address first of the difficulties. We will show that for a particular class of Kundt-Newman potential sequences one can find an explicit formu!a for the potentials J_k in terms of the index k.

In chapter 3 we will answer the second problem for the particular class of potentials that was derived in the chapter 2. We will show how the singularity of a matrix potential in the Kundt-Newman sequence can lead to reduction of the number of wave equations that have to be solved and how the solutions of the original and the reduced systems can be related.

In the chapter 4 we demonstrate how these methods of solving a system of wave equations can be applied to the problem of finding solutions of the Schrodinger wave equation.

In the conclusion, we present some of the open problems that may be of interest.

Chapter 1

Wave equation

The problem of solving the wave equation (i.e a hyperbolic partial differential equation) arises in many theoretical and practical settings in the investigation of physical phenomena. The properties of the wave equation and its solutions are therefore of considerable interest. In 1+1 dimensions, the wave equation can be transformed into a form that allows generation of a sequence of wave equations, called a Kundt-Newman sequence. In some particular cases the properties of this sequence can be used to generate solutions to the original wave equation. The following sections summarize some of the properties of this sequence discussed in [1] and [2], including its relation to Toda lattice motions.

1.1 The right and left normal forms of the wave equation

Let g_{ab} be the Lorentzian metric and ∇_a the induced covariant derivative on a 1+1 dimensional space-time. Then the homogeneous second-order linear wave equation can be expressed in the form:

$$(g^{ab}(x^c)\nabla_a\nabla_b + 2A^a(x^c)\nabla_a + 2M(x^c))\Psi = 0 \tag{1.1}$$

where $a, b, c = 1, 2$. The equation 1.1 can be transformed by appropriate coordinate and gauge transformations into a form (for more details see [2] and [1]):

$$(\partial_v J_0 \partial_u - J_1)\Psi_0 = 0 \tag{1.2}$$

$$(\partial_u \tilde{J}_0 \partial_v - \tilde{J}_1)\tilde{\Psi}_0 = 0 \qquad (1.3)$$

where ∂_v and ∂_u denote the usual partial derivatives with respect to coordinates v and u . Equations 1.2 and 1.3 are called the right and left normal forms of the wave equation 1.1 . As obtained from the equation 1.1, equations 1.2 and 1.3 are scalar equations. The results of this thesis shall be mainly concerned with their generalization to the case when $J_0, J_1, \tilde{J}_0$ and $\tilde{J}_1$ are matrices and Ψ_0 and $\tilde{\Psi}_0$ are either vectors or matrices. All subsequent claims in this chapter, unless otherwise indicated, are equally valid both in the commutative and non-commutative setting. The potentials $J_0, J_1, \tilde{J}_0, \tilde{J}_1$ in the right normal form and the left normal form are related by:

$$\tilde{J}_0 = J_0^{-1} \qquad (1.4)$$

$$\tilde{J}_0\tilde{J}_1 = J_0^{-1}J_1 + \partial_v J_0^{-1}\partial_u J_0 \qquad (1.5)$$

and solutions of the equations 1.2 and 1.3 by:

$$\Psi_0 = J_0^{-1}\tilde{\Psi}_0 \qquad (1.6)$$

Note that the potentials are not unique. For example the following transformation preserves the general form of the right normal wave equation:

$$J'_0 = U(u)J_0V(v) \qquad (1.7)$$

$$\Psi'_0 = V^{-1}(v)\Psi_0 \qquad (1.8)$$

More detailed discussion of the uniqueness properties of the right and left normal forms can be found in [1] and [2].

1.2 Sequence of wave equations

A general solution to the initial value problem for equation 1.2 can be obtained as a superposition of an outgoing and an incoming solution, where the incoming and outgoing solutions are defined by:

Definition 1.2.1 *Ψ is an incoming solution of the wave equation if Ψ satisfies 1.2 and the initial value condition:*

$$\partial_u \Psi|_{v=0} = 0 \tag{1.9}$$

Similarly, Ψ is an outgoing solution of the wave equation if Ψ satisfies 1.2 and the boundary condition:

$$\partial_v \Psi|_{u=0} = 0 \tag{1.10}$$

As was mentioned in the introduction to this chapter, we would like to use properties of a sequence of wave equations

$$\{(\partial_v J_k \partial_u - J_{k+1})\Psi_k = 0\}_{k=0}^{\infty} \tag{1.11}$$

to generate a solution to the equation 1.2 . How such a sequence can be constructed will become apparent from following claim.

Claim 1.2.1 *Let Ψ_{k-1} be a solution of the initial value problem:*

$$(\partial_v J_{k-1} \partial_u - J_k)\Psi_{k-1} = 0 \quad , \quad \partial_u \Psi_{k-1}|_{v=0} = 0 \tag{1.12}$$

and let J_{k+1}, Ψ_k be defined by:

$$J_k \Psi_k = \int_0^v J_k \Psi_{k-1} dv' \quad , \quad \partial_u \Psi_k|_{v=0} = 0 \tag{1.13}$$

$$J_{k+1} = J_k J_{k-1}^{-1} J_k - J_k \partial_u J_k^{-1} \partial_v J_k \tag{1.14}$$

Then Ψ_k is a solution of the initial value problem:

$$(\partial_v J_k \partial_u - J_{k+1})\Psi_k = 0 \quad , \quad \partial_u \Psi_k|_{v=0} = 0 \tag{1.15}$$

Proof: Integration of the equation 1.12 yields:

$$J_{k-1}\partial_u \Psi_{k-1} = \int_0^v J_k \Psi_{k-1} dv' \tag{1.16}$$

Then from the last equation and the definition of Ψ_k we get:

$$J_{k-1}\partial_u \Psi_{k-1} = J_k \Psi_k \tag{1.17}$$

On another hand, if we differentiate equation 1.13 with respect to v then solve for Ψ_{k-1} and differentiate again with respect to u, we obtain:

$$\partial_u \Psi_{k-1} = \partial_u J_k^{-1} \partial_v J_k \Psi_k \tag{1.18}$$

If we now combine equations 1.17 and 1.18 we get:

$$\begin{aligned}
J_k \Psi_k &= J_{k-1}\partial_u \Psi_{k-1} = J_{k-1}\partial_u J_k^{-1}\partial_v J_k \Psi_k \\
&= J_{k-1}(J_k^{-1}\partial_u \partial_v J_k \Psi_k - J_k^{-1} J_{k,u} J_k^{-1} \partial_v J_k \Psi_k) \\
&= J_{k-1} J_k^{-1}(\partial_v \partial_u J_k \Psi_k - J_{k,u}\partial_v \Psi_k - J_{k,u} J_k^{-1} J_{k,v} \Psi_k) \\
&= J_{k-1} J_k^{-1}(\partial_v[\partial_u J_k \Psi_k - J_{k,u}\Psi_k] + J_{k,uv}\Psi_k - J_{k,u} J_k^{-1} J_{k,v}\Psi_k) \\
&= J_{k-1} J_k^{-1}([\partial_v J_k \partial_u \Psi_k] + J_k[\partial_u J_k^{-1} \partial_v J_k]\Psi_k) \qquad (1.19)
\end{aligned}$$

where we have used the notation $J_{k,u} = \partial_u J_k$. The last equation can be expressed in the form:

$$\begin{aligned}
0 &= \partial_v J_k \partial_u \Psi_k - (J_k J_{k-1}^{-1} J_k - J_k \partial_u J_k^{-1} \partial_v J_k)\Psi_k \\
0 &= (\partial_v J_k \partial_u - [J_k J_{k-1}^{-1} J_k - J_k \partial_u J_k^{-1} \partial_v J_k])\Psi_k \qquad (1.20)
\end{aligned}$$

From equation 1.14 the last equation becomes:

$$(\partial_v J_k \partial_u - J_{k+1})\Psi_k = 0 \qquad (1.21)$$

□

We will refer to the sequences of wave equations, potentials and solutions satisfying the recursive relations of Claim 1.2.1 as Kundt-Newman sequences or KN sequences. A similar result to the Claim 1.2.1 also holds for the left normal form of the wave equation. Since the proofs are similar we will state it without proof.

Claim 1.2.2 *Let Φ_{k-1} be a solution of the initial value problem:*

$$(\partial_u L_{k+1} \partial_v - L_k)\Phi_{k-1} = 0 \quad , \quad \partial_v \Phi_{k-1}|_{u=0} = 0 \qquad (1.22)$$

and let L_{k-1}, Φ_k be defined by:

$$L_k \Phi_k = \int_0^u L_k \Phi_{k-1} du' \quad , \quad \partial_v \Phi_k|_{u=0} = 0 \qquad (1.23)$$

$$L_{k-1} = L_k L_{k+1}^{-1} L_k - L_k \partial_v L_k^{-1} \partial_u L_k \qquad (1.24)$$

Then Φ_k is a solution of the initial value problem:

$$(\partial_u L_k \partial_v - L_{k-1})\Phi_k = 0 \quad , \quad \partial_v \Phi_k|_{u=0} = 0 \qquad (1.25)$$

As before the recursive relations of the Claim 1.2.2 define a sequence of Kundt-Newman wave equations in left normal form, and the corresponding sequences of potentials and solutions. The reason for choosing the indexing in decreasing order will become apparent when we will discuss the relationship between wave equations and Toda lattices, in chapter 2. A useful relation between solutions of the Kundt-Newman sequence of wave equations in the right normal form can be derived by considering the equation 1.13 .

Claim 1.2.3 *Let* $\{(\partial_v J_{k-1}\partial_u - J_k)\Psi_{k-1} = 0\}_{k=1}^{\infty}$ *be a Kundt-Newman sequence* . *Then*

$$\Psi_0 = J_1^{-1}\partial_v J_1 J_2^{-1}\partial_v J_2 \dots J_{n+1}^{-1}\partial_v J_{n+1}\Psi_{n+1} \tag{1.26}$$

Proof: If we take the derivative with respect to v of the equation 1.13 we obtain:

$$\partial_v J_k \Psi_k = J_k \Psi_{k-1} \quad \text{, and hence} \quad \Psi_0 = J_1^{-1}\partial_v J_1 \Psi_1 \tag{1.27}$$

Then the claim holds for $k = 1$. If we assume:

$$\Psi_0 = J_1^{-1}\partial_v J_1 J_2^{-1}\partial_v J_2 \dots J_n^{-1}\partial_v J_n \Psi_n \tag{1.28}$$

and substitute $\Psi_n = J_{n+1}^{-1}\partial_v J_{n+1}\Psi_{n+1}$, we get:

$$\Psi_0 = J_1^{-1}\partial_v J_1 J_2^{-1}\partial_v J_2 \dots J_{n+1}^{-1}\partial_v J_{n+1}\Psi_{n+1} \tag{1.29}$$

□

As before, similar claim holds for the solution of the Kundt-Newman sequence in the left normal form.

To see how Kundt-Newman sequence can be used to construct a solution of the original wave equation 1.2 , consider the case when for some $n > 0$ we have $J_{n+1} \equiv 0$ and J_k is invertible for $0 \leq k \leq n$. Then the sequence must terminate (on the right) since equation 1.14 cannot generate J_{n+2}. Recall, that by equation 1.17 we have:

$$J_n \partial_u \Psi_n = J_{n+1}\Psi_n \tag{1.30}$$

and hence $\partial_u \Psi_n = 0$ .Then Ψ_n is an arbitrary function of v, $\Psi_n = f(v)$. Referring to the Claim 1.2.3 , the solution of the original equation can be expressed as:

$$\Psi_0 = J_1^{-1}\partial_v J_1 J_2^{-1}\partial_v J_2 \dots J_n^{-1}\partial_v J_n f(v) \tag{1.31}$$

Similarly, if there exists an $m \leq 0$ such that $L_{m-1} \equiv 0$, we have left termination and a solution to the equation 1.2 is $\Psi_0 = L_0\Phi_0$ where

$$\Phi_0 = L_1^{-1}\partial_u L_1 L_2^{-1}\partial_u L_2 \dots L_n^{-1}\partial_u L_n g(u) \tag{1.32}$$

and g is an arbitrary function of u. Since the potentials J_k, L_k and the solutions Ψ_k, Φ_k of the wave equations in right and left normal forms are related by:

$$J_k = L_k^{-1} \quad \text{and} \quad \Phi_k = J_k^{-1}\Psi_k \tag{1.33}$$

if the sequence is double terminating, the general solution of the wave equation is:

$$\Psi = \Psi_0 + L_0\Phi_0 \tag{1.34}$$

In the scalar case a Kundt-Newman sequence either terminates in a zero to the right or left or both, or it never terminates. However, in the matrix case a reduction problem naturally arises because from a given J_0 and J_1 equation 1.4 may generate a matrix J_{n+1} which is nonzero but singular with $r = \text{rank}(J_{n+1}) < \dim(J_{n+1}) = d$ so that equation 1.4 cannot be used to generate any higher indexed matrix J_k. As Dr. R.J. Torrence has pointed out, the natural question then is: Can we somehow use the singular nature of J_{n+1} to find a $d-r$ dimensional solution to the equation 1.2 and find nonsingular matrices $\tilde{J}_n$ and $\tilde{J}_{n+1}$ of dimension r such that equation 1.14 may be applied to generate $\tilde{J}_k$ for $n+2 \leq k$ from $\tilde{J}_n$ and $\tilde{J}_{n+1}$ until the next matrix of reduced rank is encountered and the process repeated? A similar reduction problem can occur on the left. To state a proper reduction procedure in the general case may prove to be difficult, but the particular Kundt-Newman sequences we find in this thesis give rise to a nice, explicit reduction which meets all the intuitive requirements which one wants to have in a solution to the reduction problem.

1.3 A scalar example of terminating Kundt-Newman sequences

To illustrate some of the previous ideas we consider as an example the wave equation:

$$(\partial_v\partial_u - \frac{n(n+1)}{(v+u)^2})\Psi = 0 \tag{1.35}$$

The equation 1.35 arises from separation of angular variables in the ordinary scalar wave equation on Minkowski space-time and in several other spherically symmetric problems (see e.g. [7, 8, 9]). Comparing 1.35 with the standard form of the wave equation1.2, we can identify the potentials in this example as:

$$J_0 \equiv 1 \quad \text{, and} \quad J_1 = \frac{n(n+1)}{(v+u)^2} = \frac{n(n+1)}{r^2} \tag{1.36}$$

where $r = v + u$. Define $p_k = \prod_{i=1-k}^{k}(n+i)$, and note that:

$$[(n+k)(n+1-k) - 2k]p_k = p_{k+1} \tag{1.37}$$

Then the Kundt-Newman sequence of potentials $\{J_k\}$ can be expressed explicitly as:

$$J_k = \frac{p_k}{r^{2k}} \quad \text{,for} \quad k \geq 0 \tag{1.38}$$

since this satisfies 1.14. Obviously the above equation holds for $k = 1$. If n is a positive integer then $p_{n+1} = 0$, $p_n \neq 0$, $J_n \neq 0$, and $J_{n+1} \equiv 0$. From the claim 1.2.3 we get that the solution to the equation 1.35 can be expressed as:

$$\begin{aligned} \Psi &= J_1^{-1} \underbrace{\partial_v J_1 J_2^{-1} \partial_v J_2 \ldots J_n^{-1} \partial_v}_{n} J_n f(v) \\ &= r^2 \underbrace{\partial_v r^2 \partial_v \ldots r^2 \partial_v}_{n} \frac{f(v)}{r^{2n}} \end{aligned} \tag{1.39}$$

where $f(v)$ is an arbitrary function of v .

1.4 Relation of Kundt-Newman sequences to Toda lattices

In this section we would like to discuss the relationship between normalized wave equations and Toda lattices [2]. A system of coupled second order differential equations:

$$\frac{d^2 y_k}{dt^2} = e^{-(y_k - y_{k-1})} - e^{-(y_{k+1} - y_k)} \quad , k \in Z \tag{1.40}$$

can be interpreted [6] as an infinite lattice of particles interacting exponentially with the closest neighbor. An equivalent form to the equation 1.40 can be obtained by a substitution:

$$n_k = -\frac{dy_k}{dt} \quad , \text{and} \quad m_k = e^{-(y_k - y_{k-1})} \tag{1.41}$$

The equation 1.40 is then transformed into:

$$\begin{aligned} \frac{dn_k}{dt} &= -\frac{d^2 y_k}{dt^2} = e^{-(y_{k+1} - y_k)} - e^{-(y_k - y_{k-1})} = m_{k+1} - m_k \\ \frac{dm_k}{dt} &= e^{-(y_k - y_{k-1})}\left(\frac{dy_{k-1}}{dt} - \frac{dy_k}{dt}\right) = m_k(n_k - n_{k-1}) \end{aligned} \tag{1.42}$$

The above set of equations can be generalized to two dimensions [3]:

$$\begin{aligned} -\partial_u n_k &= m_{k+1} - m_k \\ \partial_v m_k &= m_k(n_k - n_{k-1}) \end{aligned} \tag{1.43}$$

The last system of equations can be alternatively expressed as:

$$\begin{aligned} (\partial_t - \partial_x) n_k &= m_{k+1} - m_k \\ (\partial_t + \partial_x) m_k &= m_k(n_k - n_{k-1}) \end{aligned} \tag{1.44}$$

Under the coordinate transformations:

$$x = v + u \quad , \text{and} \quad t = v - u \tag{1.45}$$

A possible mechanical interpretation of the system 1.43 can be found in [4]. The system considered above can also be generalized into a non-commutative setting. A possible physical interpretation of such system is discussed in [5]. The generalized system can be described by the system of equations:

$$\begin{aligned}(\partial_t - \partial_x)N_k &= M_{k+1} - M_k \\ (\partial_t + \partial_x)M_k &= M_k N_{k+1} - N_k M_k \end{aligned} \tag{1.46}$$

As was shown in [2], there is a correspondence between the Kundt-Newman sequences of wave equations and Toda lattices.

Claim 1.4.1 *Let* $\{(\partial_v J_k \partial_u - J_{k+1})\Psi_k = 0\}_{k=0}^{\infty}$ *be a Kundt-Newman sequence of wave equations and* $\{J_k\}_{k=0}^{\infty}$ *the associated sequence of Kundt-Newman potentials. Then* $M_{k+1} = J_k^{-1} J_{k+1}$ *and* $N_k = J_k^{-1}\partial_v J_k$, $k \in Z$, *satisfy the generalized Toda lattice equation :*

$$\begin{aligned}(\partial_t - \partial_x)N_k &= M_{k+1} - M_k \\ (\partial_t + \partial_x)M_k &= M_k N_{k+1} - N_k M_k \end{aligned} \tag{1.47}$$

with

$$x = v + u \quad , \text{ and } \quad t = v - u \tag{1.48}$$

Proof: By the definition of the variables x and t we have:

$$\partial_v = (\partial_x + \partial_t) \tag{1.49}$$

Since $M_k = J_k^{-1} J_{k+1}$ then:

$$(\partial_x + \partial_t)M_k = \partial_v J_k^{-1} J_{k+1}$$

$$= J_k^{-1}J_{k+1}J_{k+1}^{-1}\partial_v J_{k+1} - J_k^{-1}\partial_v J_k J_k^{-1}J_{k+1}$$

$$= M_k N_{k+1} - N_k M_k \qquad (1.50)$$

$\{J_k\}$ is a Kundt-Newman sequence and hence:

$$\begin{aligned} J_k^{-1}J_{k+1} &= J_{k-1}^{-1}J_k - \partial_u J_k^{-1}\partial_v J_k \\ M_k &= M_{k-1} - \partial_u N_k \\ &= M_{k-1} + (\partial_t - \partial_x)N_k \end{aligned} \qquad (1.51)$$

The above equation can be solved for $(\partial_t - \partial_x)N_k$:

$$(\partial_t - \partial_x)N_k = M_k - M_{k-1} \qquad (1.52)$$

□

A similar claim also holds for the Kundt-Newman sequence of wave equations in the left normal form:

Claim 1.4.2 *Let $\{(\partial_u L_k \partial_v - L_{k-1})\Phi_k = 0\}_{k=0}^{\infty}$ be a Kundt-Newman sequence of wave equations and $\{L_k\}_{k=0}^{\infty}$ the associated sequence of Kundt-Newman potentials. Then $M_{k+1} = L_{k+1}^{-1}L_k$ and $N_k = L_k^{-1}\partial_v L_k$, $k \in Z$, satisfy the generalized Toda lattice equation :*

$$\begin{aligned} (\partial_t + \partial_x)N_k &= M_{k+1} - M_k \\ (\partial_t - \partial_x)M_k &= N_k M_k - M_k N_{k-1} \end{aligned} \qquad (1.53)$$

with

$$x = v + u \quad , \textit{and} \quad t = v - u \qquad (1.54)$$

In the case of antisymmetric Abelian Toda lattice motion two classes naturally arise [2]. Referring to the notation in the equation 1.40 the two classes can be characterized by following two equations:

$$y_{-k} = -y_k \quad \text{and} \quad y_{-k} = -y_{k-1} \tag{1.55}$$

Motion satisfying the first equation can be interpreted as a lattice motion with a center element and the one satisfying the second equation can be interpreted as motion without a center element. Following [2] we will refer to the first class as self-adjoint and to the second as almost self-adjoint. From the definition of n_k and m_k (see equation 1.41) and from the first of the equations 1.55 we get for the self-adjoint Toda lattice:

$$n_{-k} = -n_k \quad \text{and} \quad m_{-k} = m_{k+1} \tag{1.56}$$

Similarly, for the almost self-adjoint case we have:

$$n_{-k} = -n_{k-1} \quad \text{and} \quad m_{-k} = m_k \tag{1.57}$$

The equation 1.56 and 1.57 can be easily generalized to the non-abelian setting. If we use the definitions as in the claim 1.4.1, then in the self-adjoint non-abelian case the equations $N_{-k} = -N_k$ and $M_{-k} = M_{k+1}$ imply

$$J_{-k} = J_k^{-1} \tag{1.58}$$

and in the almost self-adjoint case the equations $N_{-k} = -N_{k-1}$ and $M_{-k} = M_k$ imply

$$J_{-k} = J_{k+1}^{-1} \tag{1.59}$$

The scalar lattice generated from equation 1.36 is an example of a self adjoint lattice.

Chapter 2

Kac and Kundt-Newman Sequences

An example of a Kundt-Newman one step terminating sequence of symmetric matrices was given by Couch, Torrence and Bombelli in [2]. We would like to construct an example of Kundt-Newman matrix sequence, which is not necessarily symmetric, and terminates in more than a single step. To this end we would like to take advantage of some of the properties of a Kac matrix sequence $\ldots, V_0, J_1, V_1, J_2, V_2, J_3, \ldots$. It is known that the Kac sequence consists of two coupled Kundt-Newman sequences $\{V_k\}$ and $\{J_k\}$. In our approach, we will choose a simple sequence of Kundt-Newman potentials $\{V_k\}$ and show how to use these to obtain a more complex and interesting sequence of potentials $\{J_k\}$. Depending on the choice of certain parameters, the sequence of potentials $\{J_k\}$ can then be either terminated or reduced to a sequence of lower dimensional matrices. We will make use of the one dimensional Kac sequence and thus we take J_k and V_k to be matrix functions of a single variable t.

2.1 Non-Abelian Kac Sequence

Definition 2.1.1 *A matrix sequence $\{V_k, J_k\}$ is a Kac sequence if the matrices J_k and V_k satisfy following recursive equations:*

$$J_{k+1} = \partial_t V_k + V_k J_k^{-1} V_k \tag{2.1}$$

$$\partial_t J_{k+1} = J_{k+1} V_k^{-1} J_{k+1} - V_{k+1} \tag{2.2}$$

Claim 2.1.1 *Subsequences $\{V_k\}$ and $\{J_k\}$ satisfying the recursive equations 2.1 and 2.2 are Kundt-Newman sequences.*

Proof: Recall, from section 1.2, that a matrix sequence $\{M_k\}$ is a Kundt-Newman sequence if it satisfies the equation:

$$M_k^{-1}M_{k+1} = M_{k-1}^{-1}M_k - \partial_t M_k^{-1}\partial_t M_k \tag{2.3}$$

To show that $\{V_k\}$ satisfies the above definition, take the partial derivative of both sides of the equation 2.1 to obtain:

$$\begin{aligned} \partial_t J_{k+1} &= \partial_{tt}V_k + \partial_t(V_k)J_k^{-1}V_k - V_kJ_k^{-1}\partial_t(J_k)J_k^{-1}V_k \\ &\quad +V_kJ_k^{-1}\partial_t(V_k) \end{aligned} \tag{2.4}$$

Solve equation 2.2 for V_{k+1} and substitute for $\partial_t J_{k+1}$ from the equation 2.4 .

$$\begin{aligned} V_{k+1} &= J_{k+1}V_k^{-1}J_{k+1} - \partial_t J_{k+1} \\ &= J_{k+1}V_k^{-1}J_{k+1} - \partial_{tt}V_k - \partial_t(V_k)J_k^{-1}V_k + V_kJ_k^{-1}\partial_t(J_k)J_k^{-1}V_k \\ &\quad - V_kJ_k^{-1}\partial_t(V_k) \\ &= (\partial_t V_k + V_kJ_k^{-1}V_k)V_k^{-1}(\partial_t V_k + V_kJ_k^{-1}V_k) - \partial_{tt}V_k - \partial_t(V_k)J_k^{-1}V_k \\ &\quad + V_kJ_k^{-1}\partial_t(J_k)J_k^{-1}V_k - V_kJ_k^{-1}\partial_t(V_k) \\ &= -\partial_{tt}V_k + (\partial_t V_k)V_k^{-1}\partial_t V_k + V_kJ_k^{-1}V_kJ_k^{-1}V_k + V_kJ_k^{-1}\partial_t(J_k)J_k^{-1}V_k \\ &= -\partial_{tt}V_k + (\partial_t V_k)V_k^{-1}\partial_t V_k + V_kJ_k^{-1}V_kJ_k^{-1}V_k \\ &\quad + V_kJ_k^{-1}(J_kV_{k-1}^{-1}J_k - V_k)J_k^{-1}V_k \\ &= -\partial_{tt}V_k + (\partial_t V_k)V_k^{-1}\partial_t V_k + V_kV_{k-1}^{-1}V_k \end{aligned} \tag{2.5}$$

Multiply equation 2.5 on both sides by V_k^{-1} . We obtain:

$$V_k^{-1}V_{k+1} = -V_k^{-1}\partial_{tt}V_k + V_k^{-1}(\partial_t V_k)V_k^{-1}\partial_t V_k + V_{k-1}^{-1}V_k$$

$$= V_{k-1}^{-1}V_k - ((\partial_t V_k^{-1})\partial_t V_k + V_k^{-1}\partial_{tt}V_k)$$

$$= V_{k-1}^{-1}V_k - \partial_t(V_k^{-1}\partial_t V_k) \quad (2.6)$$

Hence the matrix sequence $\{V_k\}$ is a Kundt-Newman sequence. The same result can be obtained for the sequence of matrices $\{J_k\}$. First, solve equation 2.2 for V_{k+1} and differentiate both sides with respect to t, to get:

$$\partial_t V_{k+1} = -\partial_{tt}J_{k+1} + (\partial_t J_{k+1})V_k^{-1}J_{k+1} - J_{k+1}V_k^{-1}(\partial_t V_k)V_k^{-1}J_{k+1} + J_{k+1}V_k^{-1}(\partial_t J_{k+1}) \quad (2.7)$$

If we now substitute for $\partial_t V_k$ in equation 2.1, we obtain:

$$\begin{aligned}
J_{k+1} &= -\partial_{tt}J_k + (\partial_t J_k)V_{k-1}^{-1}J_k - J_kV_{k-1}^{-1}(\partial_t V_{k-1})V_{k-1}^{-1}J_k \\
&\quad + J_kV_{k-1}^{-1}(\partial_t J_k) + V_kJ_k^{-1}V_k \\
&= -\partial_{tt}J_k + (\partial_t J_k)V_{k-1}^{-1}J_k + J_kV_{k-1}^{-1}(\partial_t J_k) \\
&\quad + (J_kV_{k-1}^{-1}J_k - \partial_t J_k)J_k^{-1}(J_kV_{k-1}^{-1}J_k - \partial_t J_k) \\
&\quad - J_kV_{k-1}^{-1}(J_k - V_{k-1}J_{k-1}^{-1}V_{k-1})V_{k-1}^{-1}J_k \\
&= -\partial_{tt}J_k + (\partial_t J_k)J_k^{-1}(\partial_t J_k) + J_kV_{k-1}^{-1}J_kV_{k-1}^{-1}J_k \\
&\quad - J_kV_{k-1}^{-1}(J_k - V_{k-1}J_{k-1}^{-1}V_{k-1})V_{k-1}^{-1}J_k \\
&= -\partial_{tt}J_k + (\partial_t J_k)J_k^{-1}(\partial_t J_k) + J_kV_{k-1}J_kV_{k-1}J_k \\
&\quad - J_kV_{k-1}^{-1}J_kV_{k-1}^{-1}J_k + J_kJ_{k-1}^{-1}J_k \\
&= -\partial_{tt}J_k + (\partial_t J_k)J_k^{-1}(\partial_t J_k) + J_kJ_{k-1}^{-1}J_k \quad (2.8)
\end{aligned}$$

As we did before in the case of the matrix sequence V_k , we can now multiply both sides of equation 2.8 by J_k^{-1} .

$$J_k^{-1}J_{k+1} = -J_k^{-1}\partial_{tt}J_k + J_k^{-1}(\partial_t J_k)J_k^{-1}(\partial_t J_k) + J_k^{-1}J_kJ_{k-1}^{-1}J_k$$

$$= J_{k-1}^{-1} J_k - \partial_t J_k^{-1} \partial_t J_k \tag{2.9}$$

Equation 2.9 defines Kundt-Newman matrix sequence J_k . Hence, the Kac sequence consists of two coupled Kundt-Newman sequences V_k and J_k .

□

As the starting point in our construction of a new, interesting family of Toda lattices $\{J_k\}$ we choose $n \times n$ matrices V_0 and V_1 to be:

$$V_0 = I, \text{ and } V_1 = \begin{bmatrix} l_1(l_1+1) & 0 & \dots & 0 \\ 0 & l_2(l_2+1) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & l_n(l_n+1) \end{bmatrix} \frac{1}{t^2} \tag{2.10}$$

where I is the $n \times n$ identity matrix, and the l_i are real valued parameters which satisfy $0 < l_n \leq l_{n-1} \leq \dots \leq l_1$. With this choice of the matrices V_0 and V_1 the form of the k-th matrix in the sequence $\{V_k\}$ can be determined exactly.

Claim 2.1.2 *Let $\{V_k\}$ be Kundt-Newman sequence of $n \times n$ matrices such that:*

$$V_0 = I \quad and \quad V_1 = \frac{1}{t^2} diag(l_1(l_1+1), l_2(l_2+1), \dots, l_n(l_n+1)) \tag{2.11}$$

Where I is the $n \times n$ identity matrix. Then V_k can be expressed as:

$$V_k = \frac{P_k}{t^{2k}} \quad , where \quad P_k = diag\Big(\prod_{i=1-k}^{k} (l_1+i), \prod_{i=1-k}^{k} (l_2+i), \dots, \prod_{i=1-k}^{k} (l_n+i)\Big) \tag{2.12}$$

Proof: For $k = 1$ the matrix given by the equation 2.12 is obviously equal to V_1 . Assume that V_k and V_{k-1} can be described by equation 2.12. Since

$$-2k(2k+1) + 4k^2 + (l_j+k)(l_j+1-k) = (l_j+k+1)(l_j-k) \tag{2.13}$$

then:

$$-2k(2k+1)I + 4k^2I + P_kP_{k-1}^{-1} = P_{k+1}P_k^{-1} \tag{2.14}$$

$\{V_k\}$ is a Kundt-Newman sequence and hence:

$$\begin{aligned} V_{k+1} &= -\partial_t^2 V_k + (\partial_t V_k)V_k^{-1}(\partial_t V_k) + V_kV_{k-1}^{-1}V_k \\ &= [-2k(2k+1)I + 4k^2I + P_kP_{k-1}^{-1}]\frac{P_k}{t^{2k+2}} \\ &= \frac{P_{k+1}}{t^{2k+2}} \end{aligned} \tag{2.15}$$

□

Thus $\{V_k\}$ is a trivial, diagonal matrix generalization of the scalar lattice 1.38 having n such uncoupled lattices strung along the diagonal. We will use this $\{V_k\}$ to find a non-trivial, non-diagonal sequence of matrix potentials $\{J_k\}$ satisfying equations 2.1, 2.2 and 2.9 and expressed in closed form in terms of k, t, l_i, and arbitrary constant parameters of integration.

For $k \geq 0$, V_k given by 2.15 is well defined at all points of the parameter space, it is a solution of the matrix Toda lattice equations 2.6 at all points except those having an integer value for some l_i. At integer values some of the $V_k's$ for $k > 0$ are not invertible and the corresponding equations from 2.1, 2.2, and 2.6 are not defined. We shall show that the sequence $\{J_k\}$ has the same property. Generation of the complete sequences by equations 2.1 and 2.2 or 2.6 and 2.9 starting with $k = 0$ and $k = 1$ would not be possible in the integer case. In this sense $\{V_k\}$ and $\{J_k\}$ provide an extension of sequences to real values of $l's$ from integer values and extend the generation by the Kac and Toda equations from that possible with integer values. In the next chapter we will use our extension to show how to obtain a reduction of the Toda lattice $\{J_k\}$ and the wave equation 1.2 in the integer case.

When every l_i is an integer, for $k > l_n$ V_k and J_k have decreasing rank as k increases and become zero matrices for $k \geq k_1 + 1$. In that case our reduction of the wave equation provides an n dimensional solution of 1.2. Thus we obtain a generalization of 1.31.

To find the solution for J_k, we first consider following linearization of the equation 2.2. Let $J_k = -V_{k-1}(\partial_t \Phi_k)\Phi_k^{-1}$, then it follows from the equation 2.2 that:

$$\begin{aligned} J_k V_{k-1}^{-1} J_k - V_k &= -\partial_t V_{k-1}(\partial_t \Phi_k)\Phi_k^{-1} \\ &= -(\partial_t V_{k-1}(\partial_t \Phi_k))\Phi_k^{-1} + V_{k-1}(\partial_t \Phi_k)\Phi_k^{-1}(\partial_t \Phi_k)\Phi_k^{-1} \\ &= -(\partial_t V_{k-1}(\partial_t \Phi_k))\Phi_k^{-1} + J_k V_{k-1}^{-1} J_k \end{aligned} \tag{2.16}$$

and hence, we obtain a linear differential equation for Φ_k:

$$\partial_t V_{k-1} \partial_t \Phi_k = V_k \Phi_k \tag{2.17}$$

We point out that this ansatz for J_k transforms equation 2.2 into the Kundt-Newman sequence of wave equations with coefficients V_k. In our calculations we found it more convenient to use the linearization

$$J_k = -V_{k-1}\Psi_k^{-1}\partial_t \Psi_k \tag{2.18}$$

Since $\partial_t J_k = J_k V_{k-1}^{-1} J_k - V_k$ then

$$\begin{aligned} J_k V_{k-1}^{-1} J_k - V_k &= -\partial_t V_{k-1}\Psi_k^{-1}\partial_t \Psi_k \\ &= -(\partial_t V_{k-1}\Psi_k^{-1}V_{k-1}^{-1})V_{k-1}\partial_t \Psi_k - V_{k-1}\Psi_k^{-1}V_{k-1}^{-1}\partial_t V_{k-1}\partial_t \Psi_k \\ &= J_k V_{k-1}^{-1} J_k - V_{k-1}\Psi_k^{-1}V_{k-1}^{-1}\partial_t V_{k-1}\partial_t \Psi_k \end{aligned} \tag{2.19}$$

and hence we get a linear differential equation for Ψ_k :

$$\partial_t V_{k-1}\partial_t \Psi_k = V_{k-1}\Psi_k V_{k-1}^{-1} V_k \tag{2.20}$$

Since we chose the matrices V_k to be diagonal, the system of differential equations 2.20 is not coupled and the solutions can be found explicitly as we will show in the following claim.

Claim 2.1.3 *Let $\{V_k, J_k\}$ be a Kac sequence where V_k is defined as :*

$$V_k = \frac{P_k}{t^{2k}} \qquad P_k = diag(\prod_{i=1-k}^{k}(l_1+i), \ldots, \prod_{i=1-k}^{k}(l_n+i)) \tag{2.21}$$

$detV_{k-1} \neq 0$ and

$$J_k = -V_{k-1}\Psi_k^{-1}\partial_t\Psi_k \tag{2.22}$$

then Ψ_k has the form:

$$\Psi_k = t^k(A_kT + \frac{1}{t}B_kT^{-1}) \quad T = diag(t^{l_1}, \ldots, t^{l_n}) \tag{2.23}$$

A_k and B_k are arbitrary constant matrices.

Proof: The proof is a simple matter of showing that Ψ_k is a solution of equation 2.20 by direct substitution.

□

The matrices Ψ_k and Ψ_{k-1} can be related by following claim.

Claim 2.1.4 *If J_k satisfies 2.1, 2.2, and 2.20 then $\Psi_k = t^2\partial_t\Psi_{k-1}$.*

Proof: By 2.21 and 2.22 we have:

$$J_k = \frac{P_{k-1}}{t^{2k-2}}\Psi_k^{-1}\partial_t\Psi_k \tag{2.24}$$

Also J_k satisfies:

$$J_k = \partial_t V_{k-1} + V_{k-1}J_{k-1}^{-1}V_{k-1} \tag{2.25}$$

Let $\Psi_k = t^{2k-2}\Phi_k$. If we substitute for Ψ_k in the equation 2.24, we obtain:

$$\begin{aligned} J_k &= -\frac{(2k-2)P_{k-1}}{t^{2k-1}} - \frac{P_{k-1}}{t^{2k-2}}\Phi_k^{-1}\partial_t\Phi_k \\ &= \partial_t V_{k-1} - V_{k-1}\Phi_k^{-1}\partial_t\Phi_k \end{aligned} \quad (2.26)$$

If we compare equations 2.25 and 2.26, we get:

$$\Phi_k^{-1}\partial_t\Phi_k = -J_{k-1}^{-1}V_{k-1} \quad (2.27)$$

Since J_{k-1} can be expressed as:

$$J_{k-1} = \frac{P_{k-2}}{t^{2k-4}}\Psi_{k-1}^{-1}\partial_t\Psi_{k-1} \quad (2.28)$$

then by substitution into equation 2.27 for J_{k-1}^{-1} we get:

$$\Phi_k^{-1}\partial_t\Phi_k = \frac{1}{t^2}(\partial_t\Psi_{k-1})^{-1}\Psi_{k-1}P_{k-2}^{-1}P_{k-1} \quad (2.29)$$

It can be easily shown by substitution that $\Phi_k = \frac{N_k}{t^{2k-4}}\partial_t\Psi_{k-1}$ is a solution of the last equation for any invertible constant matrix N_k. If we now go back to how we defined Φ_k we get:

$$\Psi_k = t^{2k-2}\Phi_k = \frac{t^{2k-2}}{t^{2k-4}}N_k\partial_t\Psi_{k-1} = t^2 N_k\partial_t\Psi_{k-1} \quad (2.30)$$

Without loss of generality, we can take $N_k = I$ since by equation 2.18 J_k does not depend on N_k.

□

It remains to satisfy the Kac equation 2.1 with $\{V_k\}$ given by 2.15 and $\{J_k\}$ by 2.22 and 2.23. It is clear from the form of J_k established thus far that imposing equation 2.1 will relate the constant matrices (A_{k+1}, B_{k+1}) to (A_k, B_k) and hence establish

(A_k, B_k) as functions of (A_1, B_1) for all $k \geq 2$. Since 2.1 involves J_k^{-1} it appears that the relationship of (A_{k+1}, B_{k+1}) to (A_k, B_k) will be complicated and iteration from (A_1, B_1) to (A_k, B_k) will compound the complications. However, we shall now show that (A_k, B_k) is simply related to (A_1, B_1) and derive a simple, nice formula for J_k as a function of t, A_1, B_1 and k. To avoid confusion we will denote J_k considered as function of A_k and B_k by G_k, so if Ψ_k and V_k are defined as in the claim 2.1.3 then:

$$G_k = -V_{k-1}\Psi_k^{-1}\partial_t\Psi_k \quad , \text{and} \qquad J_k = G_k(A_k, B_k) \tag{2.31}$$

Claim 2.1.5 *Let D be an $n \times n$ matrix, $D = diag(l_1, l_2, \ldots, l_n)$, and let P_k, V_k, Ψ_k, J_k be as defined in the claim 2.1.3 . If $G_k(A_k, B_k) = -V_{k-1}\Psi_k^{-1}(A_k, B_k)\partial_t\Psi_k(A_k, B_k)$ and $J_k = G_k(A_k, B_k)$ then*

$$J_{k+1} = \frac{P_k P_{k-1}^{-1}}{t^2}(G_k(\tilde{A}_k, \tilde{B}_k) - \frac{P_{k-1}}{t^{2k-1}}) \tag{2.32}$$

where $\tilde{A}_k, \tilde{B}_k$ are $n \times n$ constant matrices:

$$\tilde{A}_k = A_k(kI + D) \quad , \textit{and} \qquad \tilde{B}_k = B_k(kI - D - I) \tag{2.33}$$

Proof: By claim 2.1.3, we have $\Psi_k(A, B) = t^k(AT + \frac{1}{t}BT^{-1})$. Where T is defined as $T = \text{diag}(t^{l_1}, t^{l_2}, \ldots, t^{l_n})$. Since $\partial_t T = \frac{1}{t}DT$, then :

$$\partial_t\Psi_k(A_k, B_k) = \frac{t^k}{t}(A_k(kI + D)T + B_k(kI - D - I)T^{-1}) = \frac{1}{t}\Psi_k(\tilde{A}_k, \tilde{B}_k) \tag{2.34}$$

By the claim 2.1.4, we have $\Psi_{k+1} = t^2\partial_t\Psi_k(A_k, B_k) = t\Psi_k(\tilde{A}_k, \tilde{B}_k)$. Since,

$$J_{k+1} = -\frac{P_k}{t^{2k}}\Psi_{k+1}^{-1}\partial_t\Psi_{k+1} \tag{2.35}$$

then:

$$J_{k+1} \;=\; -\frac{P_k}{t^{2k}}(t\Psi_k(\tilde{A}_k, \tilde{B}_k))^{-1}\partial_t(t\Psi_k(\tilde{A}_k, \tilde{B}_k))$$

$$\begin{aligned}
&= -\frac{P_k}{t^{2k}}(t\Psi_k(\tilde{A}_k,\tilde{B}_k))^{-1}[t\partial_t\Psi_k(\tilde{A}_k,\tilde{B}_k)+\Psi_k(\tilde{A}_k,\tilde{B}_k)] \\
&= \frac{P_kP_{k-1}^{-1}}{t^2}[-\frac{P_{k-1}}{t^{2k-2}}\Psi_k^{-1}(\tilde{A}_k,\tilde{B}_k)\partial_t\Psi_k(\tilde{A}_k,\tilde{B}_k)-\frac{P_{k-1}}{t^{2k-1}}] \\
&= \frac{P_kP_{k-1}^{-1}}{t^2}(G_k(\tilde{A}_k,\tilde{B}_k)-\frac{P_{k-1}}{t^{2k-1}}) \qquad (2.36)
\end{aligned}$$

□

To see how the equation 2.32 can be used to compute J_{k+1}, let $\tilde{A}_k = A_k^1$ and $\tilde{B}_k = B_k^1$. Applying the equation 2.32 recursively we obtain:

$$G_k(A_k^1,B_k^1) = \frac{P_{k-1}P_{k-2}^{-1}}{t^2}(G_{k-1}(A_k^2,B_k^2)-\frac{P_{k-2}}{t^{2k-3}}) \qquad (2.37)$$

Defining A_{k-i+1}^i and B_{k-i+1}^i by:

$$\begin{aligned}
A_{k-i+1}^i &= A_{k-i+1}\prod_{j=k+1-i}^{k}(jI+D) \\
B_{k-i+1}^i &= B_{k-i+1}\prod_{j=k-i}^{k-1}(jI-D) \qquad (2.38)
\end{aligned}$$

we find J_{k+1} to be:

$$\begin{aligned}
J_{k+1} &= \frac{P_kP_{k-1}^{-1}}{t^2}(G_k(A_k^1,B_k^1)-\frac{P_{k-1}}{t^{2k-1}}) \\
&= \frac{P_kP_{k-2}^{-1}}{t^4}(G_{k-1}(A_{k-1}^2,B_{k-1}^2)-\frac{2P_{k-2}}{t^{2k-3}}) \\
&\vdots \\
&= \frac{P_kP_{k-i}^{-1}}{t^{2i}}(G_{k-i+1}(A_{k-i+1}^i,B_{k-i+1}^i)-\frac{iP_{k-i}}{t^{2k-2i+1}}) \\
&\vdots \\
&= \frac{P_k}{t^{2k}}(G_1(A_1^k,B_1^k)-\frac{kI}{t}) \qquad (2.39) \\
J_{k+1} &= \frac{P_k}{t^{2k}}(J_1(t,A_1^k,B_1^k)-\frac{kI}{t}) \qquad (2.40)
\end{aligned}$$

where the matrices A_1^k and B_1^k are:

$$A_1^k = A_1 \prod_{j=1}^{k} (jI + D)$$
$$B_1^k = B_1 \prod_{j=0}^{k-1} (jI - D) \tag{2.41}$$

and $J_1(t, A_1, B_1)$ is given by:

$$J_1(t, A_1, B_1) = -\frac{1}{t}(A_1 T + \frac{1}{t} B_1 T^{-1})^{-1}(A_1(D + I)T + \frac{1}{t} B_1(I - D)T^{-1}) \tag{2.42}$$

Equation 2.40 is our simple formula for J_k and is the main result of this chapter. To compute the matrices in the sequence $\{J_k\}$ we merely need to substitute into the function J_1 given by 2.42.

It is clear from 2.1 and 2.2 that if V_0 and V_1 are given then all the matrices V_k and J_k can be generated from 2.1 and 2.2 (as long as the required inverses exist), once the single differential equation 2.2 with $k = 0$ is solved for J_1. Hence, the only arbitrary constant parameters on which J_k depends are the constants in V_0 and V_1 and the integration constants in J_1. We see that the expression 2.40 for J_k agrees with this fact. However, generation of J_k via 2.1 and 2.2 from V_0, V_1 and J_1 yields such highly complicated expressions for J_k that it can only be carried out for small values of k , whereas our method of solving all the differential equations 2.2 produces the simple and useful formulas 2.23 and 2.40 valid for all $k \geq 0$. The set of $J_k's$ given by equation 2.40 can be extended to a full sequence $\{J_k\}$ including negative values of k by noting that the equation 2.1 with $k = 0$ implies $J_0 = J_1^{-1}$ and then 2.9 implies:

$$J_{-k} = J_{k+1}^{-1} \tag{2.43}$$

for all $k \in Z$ which makes $\{J_k\}$ be a almost self-adjoint matrix Toda lattice as defined in section 1.4. For the sequence $\{V_k\}$ equation 2.6 implies:

$$V_{-k} = V_k^{-1} \tag{2.44}$$

for all k which makes $\{V_k\}$ be a self-adjoint matrix Toda lattice [4]. Of course, since V_k is diagonal , $\{V_k\}$ is a trivial generalization of self-adjoint scalar lattices. $\{J_k\}$, however, is a nontrivial matrix generalization of almost self-adjoint scalar lattices such as:

$$J_k = \frac{c_k}{t^{2k-1}} = J_{-k+1}^{-1} \tag{2.45}$$

The method of this section could easily be adapted to find a nontrivial self-adjoint matrix Toda lattice . We need only choose J_k to be a diagonal matrix whose elements are almost self-adjoint scalar Toda lattices of the form 2.45. Then in analogy to 2.22 the ansatz:

$$V_k = J_k \Phi_{k+1}^{-1} \partial_t \Phi_{k+1} \tag{2.46}$$

yields a solvable linear differential equation for Φ_{k+1} thus determining V_k. We do not carry out the calculation in this thesis. In this case V_k will satisfy $V_{-k} = V_k^{-1}$ for all $k \neq 0$, but we must take the solution of 2.1 with $V_0 \neq I$ in order to have V_k non-diagonal. Thus, in using the Kac equations in generalizing the definition of self-adjoint lattices to the matrix case we must drop the requirement that $V_0 = I$.

2.2 An example of an explicit Kac potential sequence

In this section we would like to give an example how the claim 2.1.5 can be applied to derive an explicit formula for the Kac sequence of 2×2 matrix potentials. Let us first consider the general form of the potential J_1. From the claim 2.1.3 we have:

$$J_1 = -\Psi_1^{-1}\partial_t\Psi_1 = -\frac{\text{adj }(\Psi_1)}{\det(\Psi_1)}\partial_t\Psi_1 \quad \text{, where} \quad \Psi_1 = t(AT + \frac{1}{t}BT^{-1}) \tag{2.47}$$

Since a determinant of a matrix is a multi-linear function of its columns, then $\det(\Psi_1)$ can be expressed as:

$$\det(\Psi_1) = d_0 t^{l_1+l_2+2} + d_1 t^{l_1-l_2+1} + d_2 t^{l_2-l_1+1} + d_3 t^{-l_2-l_1} \tag{2.48}$$

$$\begin{aligned} d_0 &= d_0(A,B) = \det([a_1, a_2]) \\ d_1 &= d_1(A,B) = \det([a_1, b_2]) \\ d_2 &= d_2(A,B) = \det([b_1, a_2]) \\ d_3 &= d_3(A,B) = \det([b_1, b_2]) \\ d_4 &= d_4(A,B) = \det([a_1, b_1]) \\ d_5 &= d_5(A,B) = \det([a_2, b_2]) \end{aligned} \tag{2.49}$$

where a_1, a_2 and b_1, b_2 are the columns of matrices A and B, respectively. Next consider the product $\text{adj}\Psi_1\partial_t\Psi_1$. We have:

$$[\text{adj}\Psi_1\partial_t\Psi_1]_{i,j} = \det(M(i,j)) \tag{2.50}$$

where $M(i,j)$ is the matrix Ψ_1 with the i-th column replaced by j-th column of the matrix $\partial_t\Psi_1$. Since the matrices Ψ_1 and $\partial_t\Psi_1$ are of the form

$$\Psi_1 = \begin{bmatrix} a_{11}t^{l_1+1} + b_{11}t^{-l_1}; & a_{12}t^{l_2+1} + b_{12}t^{-l_2} \\ a_{21}t^{l_1+1} + b_{21}t^{-l_1}; & a_{22}t^{l_2+1} + b_{22}t^{-l_2} \end{bmatrix} \tag{2.51}$$

$$\partial_t\Psi_1 = \begin{bmatrix} (1+l_1)a_{11}t^{l_1} - l_1 b_{11}t^{-l_1-1}; & (1+l_2)a_{12}t^{l_2} - l_2 b_{12}t^{-l_2-1} \\ (1+l_1)a_{21}t^{l_1} - l_1 b_{21}t^{-l_1-1}; & (1+l_2)a_{22}t^{l_2} - l_2 b_{22}t^{-l_2-1} \end{bmatrix} \tag{2.52}$$

then the elements of $\mathrm{adj}\Psi_1\partial_t\Psi_1$ can be expressed as:

$$\begin{aligned}(\Psi_1\partial_t\Psi_1)_{1,1} &= \det\begin{bmatrix}(1+l_1)a_{11}t^{l_1}-l_1b_{11}t^{-l_1-1}; & a_{12}t^{l_2+1}+b_{12}t^{-l_2}\\ (1+l_1)a_{21}t^{l_1}-l_1b_{21}t^{-l_1-1}; & a_{22}t^{l_2+1}+b_{22}t^{-l_2}\end{bmatrix}\\ &= (1+l_1)d_0t^{l_1+l_2+1}+(1+l_1)d_1t^{l_1-l_2}-l_1d_2t^{l_2-l_1}-l_1d_3t^{-l_1-l_2-1}\\ &= t^{-l_1-l_2-1}[(1+l_1)d_0t^{2l_1+2l_2+2}+(1+l_1)d_1t^{2l_1+1}-l_1d_2t^{2l_2+1}-l_1d_3]\end{aligned} \tag{2.53}$$

$$\begin{aligned}(\Psi_1\partial_t\Psi_1)_{1,2} &= \det\begin{bmatrix}(1+l_2)a_{12}t^{l_2}-l_2b_{12}t^{-l_2-1}; & a_{12}t^{l_2+1}+b_{12}t^{-l_2}\\ (1+l_2)a_{22}t^{l_2}-l_2b_{22}t^{-l_2-1}; & a_{22}t^{l_2+1}+b_{22}t^{-l_2}\end{bmatrix}\\ &= (1+2l_2)d_5\end{aligned} \tag{2.54}$$

$$\begin{aligned}(\Psi_1\partial_t\Psi_1)_{2,1} &= \det\begin{bmatrix}a_{11}t^{l_1+1}+b_{11}t^{-l_1}; & (1+l_1)a_{11}t^{l_1}-b_{11}t^{-l_1-1}\\ a_{21}t^{l_1+1}+b_{21}t^{-l_1}; & (1+l_1)a_{21}t^{l_1}-b_{21}t^{-l_1-1}\end{bmatrix}\\ &= (1+2l_1)d_6\end{aligned} \tag{2.55}$$

$$\begin{aligned}(\Psi_1\partial_t\Psi_1)_{2,2} &= \begin{bmatrix}a_{11}t^{l_1+1}+b_{11}t^{-l_1}; & (1+l_2)a_{12}t^{l_2}-l_2b_{12}t^{-l_2-1}\\ a_{21}t^{l_1+1}+b_{21}t^{-l_1}; & (1+l_2)a_{22}t^{l_2}-l_2b_{22}t^{-l_2-1}\end{bmatrix}\\ &= (1+l_2)d_0t^{l_1+l_2+1}-l_2d_1t^{l_1-l_2}+(1+l_2)d_2t^{l_2-l_1}-l_2d_3t^{-l_1-l_2-1}\\ &= t^{-l_1-l_2-1}[(1+l_2)d_0t^{2l_1+2l_2+2}-l_2d_1t^{2l_1+1}+(1+l_2)d_2t^{2l_2+1}-l_2d_3]\end{aligned} \tag{2.56}$$

The elements of J_1 are then:

$$(J_1)_{11} = -\frac{1}{t}\frac{(1+l_1)d_0t^{2l_1+2l_2+2}+(1+l_1)d_1t^{2l_1+1}-l_1d_2t^{2l_2+1}-l_1d_3}{d_0t^{2l_1+2l_2+2}+d_1t^{2l_1+1}+d_2t^{2l_2+1}+d_3} \tag{2.57}$$

$$(J_1)_{12} = \frac{-(1+2l_2)d_5t^{l_1+l_2}}{d_0t^{2l_1+2l_2+2}+d_1t^{2l_1+1}+d_2t^{2l_2+1}+d_3} \tag{2.58}$$

$$(J_1)_{21} = \frac{(1+2l_1)d_4 t^{l_1+l_2}}{d_0 t^{2l_1+2l_2+2} + d_1 t^{2l_1+1} + d_2 t^{2l_2+1} + d_3} \tag{2.59}$$

$$(J_1)_{22} = -\frac{1}{t}\frac{(1+l_2)d_0 t^{2l_1+2l_2+2} - l_2 d_1 t^{2l_1+1} + (1+l_2)d_2 t^{2l_2+1} - l_2 d_3}{d_0 t^{2l_1+2l_2+2} + d_1 t^{2l_1+1} + d_2 t^{2l_2+1} + d_3} \tag{2.60}$$

Since we now have a general expression for the potential $J_1(A, B)$, we can calculate the potential $J_1(A_1^k, B_1^k)$. Observe, that the only parts of the matrix J_1 depending on the matrices A and B are the functions $d_1(A, B), d_2(A, B), \dots, d_5(A, B)$. Define functions $e_0(k), e_1(k), \dots, e_{13}(k)$ as:

$$e_0(k) = \prod_{j=1}^{k}(j+l_1)(j+l_2) \;\; ; \;\; e_1(k) = \prod_{j=1}^{k}(j+l_1)\prod_{j=0}^{k-1}(j-l_2) \tag{2.61}$$

$$e_2(k) = \prod_{j=0}^{k-1}(j-l_1)\prod_{j=1}^{k}(j+l_2) \;\; ; \;\; e_3(k) = \prod_{j=0}^{k-1}(j-l_1)\prod_{j=0}^{k-1}(j-l_2) \tag{2.62}$$

$$e_4(k) = e_0(k)\prod_{j=1-k}^{k+1}(l_1+j) \;\; ; \;\; e_5(k) = e_1(k)\prod_{j=1-k}^{k+1}(l_1+j) \tag{2.63}$$

$$e_6(k) = -e_2(k)\prod_{j=-k}^{k}(l_1+j) \;\; ; \;\; e_7(k) = -e_3(k)\prod_{j=-k}^{k}(l_1+j) \tag{2.64}$$

$$e_8(k) = e_0(k)\prod_{j=1-k}^{k+1}(l_2+j) \;\; ; \;\; e_9(k) = -e_1(k)\prod_{j=-k}^{k}(l_2+j) \tag{2.65}$$

$$e_{10}(k) = e_2(k)\prod_{j=1-k}^{k+1}(l_2+j) \;\; ; \;\; e_{11}(k) = -e_3(k)\prod_{j=-k}^{k}(l_2+j) \tag{2.66}$$

$$e_{12}(k) = \prod_{j=1-k}^{k}(l_2+j)\prod_{j=1}^{k}(j+l_1)\prod_{j=0}^{k-1}(j-l_1) \tag{2.67}$$

$$e_{13}(k) = \prod_{j=1-k}^{k}(l_1+j)\prod_{j=1}^{k}(j+l_2)\prod_{j=0}^{k-1}(j-l_2) \tag{2.68}$$

Using the above definitions and the equation 2.40 we can express the elements of J_{k+1} as:

$$
\begin{aligned}
(J_{k+1})_{11} &= -\frac{1}{t^{2k+1}}\frac{e_4(k)d_0t^{2l_1+2l_2+2}+e_5(k)d_1t^{2l_1+1}+e_6(k)d_2t^{2l_2+1}+e_7(k)d_3}{e_0(k)d_0t^{2l_1+2l_2+2}+e_1(k)d_1t^{2l_1+1}+e_2(k)d_2t^{2l_2+1}+e_3(k)d_3} \\
(J_{k+1})_{12} &= \frac{-(1+2l_2)e_{13}(k)d_5t^{l_1+l_2-2k}}{e_0(k)d_0t^{2l_1+2l_2+2}+e_1(k)d_1t^{2l_1+1}+e_2(k)d_2t^{2l_2+1}+e_3(k)d_3} \\
(J_{k+1})_{21} &= \frac{(1+2l_1)e_{12}(k)d_4t^{l_1+l_2-2k}}{e_0(k)d_0t^{2l_1+2l_2+2}+e_1(k)d_1t^{2l_1+1}+e_2(k)d_2t^{2l_2+1}+e_3(k)d_3} \\
(J_{k+1})_{22} &= -\frac{1}{t^{2k+1}}\frac{e_8(k)d_0t^{2l_1+2l_2+2}+e_9(k)d_1t^{2l_1+1}+e_{10}(k)d_2t^{2l_2+1}+e_{11}(k)d_3}{e_0(k)d_0t^{2l_1+2l_2+2}+e_1(k)d_1t^{2l_1+1}+e_2(k)d_2t^{2l_2+1}+e_3(k)d_3}
\end{aligned}
\tag{2.69}
$$

Chapter 3

Reduction of the wave equation and Kundt-Newman sequences

3.1 Reduction

In this section, we would like to show how the problem of finding a solution to the wave equation involving $n \times n$ dimensional matrix potentials can be reduced to a problem involving potentials of lesser dimensions, when some of the $l_i's$ are integers. In order to do this we first need to discuss some of the properties of the Kac sequences we found in chapter 2. To simplify the notation, we will assume that in the definition of the matrices V_i the $l's$ satisfy $0 < l_n < l_{n-1} \leq \ldots \leq l_1$, where l_n is assumed to be an integer. All the arguments, we will present, can be easily extended to the case when $0 < l_n \leq l_{n-1} \leq \ldots \leq l_1$. Recall that if l_n is a positive integer then the matrix V_{l_n+1} is singular and matrices J_k are invertible only up to $k+1 = l_n+2$. The matrix J_{k+1} can be written in the block diagonal form as:

$$J_{k+1} = \begin{bmatrix} j_{k+1} & 0 \\ 0 & 0 \end{bmatrix}, \quad k \geq l_n + 1 \tag{3.1}$$

where j_{k+1} is a $s \times s$ matrix $(0 \leq s < n)$. In our approach, we would like to lower the dimension of the problem by relating solutions of the equations:

$$(\partial_t J_k \partial_t - J_{k+1})\Psi_k = 0 \tag{3.2}$$

$$(\partial_t j_{k+1} \partial_t - j_{k+2})\psi_{k+1} = 0 \tag{3.3}$$

for some appropriate $s \times s$ matrix potential j_{k+2}. To show how the dimension reduction can be accomplished we will need the results of following claims.

Claim 3.1.1 *Let $l_1, l_2, \ldots, l_n$ be such that, $0 < l_n < l_{n-1} \leq \ldots \leq l_1$. Then*

$$G_k(A_k(kI + D), B_k(kI - D - I)) = \begin{bmatrix} j_k & \overline{0} \\ \overline{a}^T & b \end{bmatrix} \quad , \textit{for} \quad k = l_n + 1 \tag{3.4}$$

where j_k is an $(n-1) \times (n-1)$ matrix and $\overline{0}$ is an $n-1$ dimensional zero column vector.

<u>Proof</u>: Let $A = A_k(kI + D)$ and $B = B_k(kI - D - I)$ then

$$G_k(A, B) = -V_k \Psi_k^{-1}(A, B)\partial_t \Psi_k(A, B) \tag{3.5}$$

where $\Psi_k = t^k(AT + \frac{1}{t}BT^{-1})$. To see what form $G_k(A, B)$ takes, consider the product

$$\Psi_k^{-1}(A, B)\partial_t \Psi_k(A, B) \tag{3.6}$$

in the definition of $G_k(A, B)$. If $k = l_n + 1$ then the last column of the matrix $kI - D - I$ is a zero column. But then, the last column of the matrix Ψ_k and the matrix $\partial_t \Psi_k$ are scalar multiples of each other. It follows that the last column of $\Psi_k^{-1}\partial_t \Psi_k$ is a multiple of the last column of I. Hence by 3.5 $G_k(A, B)$ is of the form :

$$G_k(A, B) = \begin{bmatrix} j_k & \overline{0} \\ \overline{a}^T & b \end{bmatrix}$$

□

Up to this point, we have considered sequences of $n \times n$ matrices $\{J_k\}$ and $\{V_k\}$. Now, we show how to define lower dimensional $(n-1) \times (n-1)$ Kac sequences

$\{v_k\}, \{j_k\}$ which can be used to derive the solution to the original wave equation with $n \times n$ matrix potentials when $\{J_k\}$ becomes singular. In general to determine the Kac sequence $\{v_k, j_k\}$ satisfying the equations:

$$j_{k+1} = \partial_t v_k + v_k j_k^{-1} v_k \tag{3.7}$$

$$\partial_t j_{k+1} = j_{k+1} v_k^{-1} j_{k+1} - v_{k+1} \tag{3.8}$$

we need one pair of matrix potentials j_k and one pair of the potentials v_k. If l_n is a positive integer then:

$$J_{l_n+2} = \begin{bmatrix} M & \bar{0} \\ \bar{0}^T & 0 \end{bmatrix} \tag{3.9}$$

where M is a $(n-1) \times (n-1)$ matrix. A natural choice for one of the $j_k'^s$ is then $j_{l_n+2} = M$. The other j necessary to determine the sequence can be defined in reference to the potential J_{l_n+1}. Let the potential J_{l_n+1} have a form :

$$J_{l_n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.10}$$

where a is an $(n-1) \times (n-1)$ matrix. Define the $(n-1) \times (n-1)$ matrix potential as :

$$j_{l_n+1} = (a - bd^{-1}c) \tag{3.11}$$

The $\{v_k\}$ sequence can be chosen to be the sequence of the upper $(n-1) \times (n-1)$ blocks of the matrices V_k. Next we would like to show that our choices for the starting points of the Kac sequence v_k, j_k satisfy the equations 3.7 and 3.8. Let us first show that

$$j_{l_n+2} = \partial_t v_{l_n+1} + v_{l_n+1} j_{l_n+1}^{-1} v_{l_n+1} \tag{3.12}$$

The higher dimensional potentials satisfy:

$$\begin{bmatrix} j_{l_n+2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \partial_t v_{l_n+1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} v_{l_n+1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} v_{l_n+1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.13}$$

Consider $J_{l_n+1}^{-1}$ to have the form:

$$J_{l_n+1}^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \tag{3.14}$$

The upper $(n-1)\times(n-1)$ block of the equation 3.13 then satisfies:

$$j_{l_n+2} = \partial_t v_{l_n+1} + v_{l_n+1} e v_{l_n+1} \tag{3.15}$$

Since $e = (a - bd^{-1}c)^{-1} = j_{l_n+1}^{-1}$ then the equation 3.12 follows immediately. To prove the second equation we cannot use the Kac equation for higher dimensional matrices, since it is not well defined. We can, however, use the alternative equation derived in the claim 2.1.5. To simplify the notation denote $\tilde{J}_{l_n+1} = G_{l_n+1}(\tilde{A}_{l_n+1}, \tilde{B}_{l_n+1})$ The higher dimensional matrices satisfy the equations:

$$J_{l_n+2} = \begin{bmatrix} j_{l_n+2} & 0 \\ 0 & 0 \end{bmatrix} = \frac{P_{l_n+1}P_{l_n}^{-1}}{t^2}\tilde{J}_{l_n+1} - \frac{V_{l_n+1}}{t} \tag{3.16}$$

$$\partial_t \tilde{J}_{l_n+1} = \tilde{J}_{l_n+1} V_{l_n}^{-1} \tilde{J}_{l_n+1} - V_{l_n+1} \tag{3.17}$$

From the claim 3.1.1, we have that $\tilde{J}_{l_n+1}$ is of the form:

$$\tilde{J}_{l_n+1} = \begin{bmatrix} \tilde{j}_{l_n+1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.18}$$

where $\tilde{j}_{l_n+1}$ is an $(n-1)\times(n-1)$ matrix. If we consider the upper $(n-1)\times(n-1)$ block part of the equation 3.17, we get:

$$\partial_t \tilde{j}_{l_n+1} = \tilde{j}_{l_n+1} v_{l_n}^{-1} \tilde{j}_{l_n+1} - v_{l_n+1} \tag{3.19}$$

Take the derivative of both sides of the equation 3.16:

$$\begin{aligned} \partial_t j_{l_n+2} &= \partial_t\left(\frac{p_{l_n+1}p_{l_n}^{-1}}{t^2}\dot{j}_{l_n+1} - \frac{v_{l_n+1}}{t}\right) \\ &= \dot{j}_{l_n+2}v_{l_n+1}^{-1}\dot{j}_{l_n+2} - v_{l_n+2} \end{aligned} \tag{3.20}$$

Hence the reduced dimension matrices satisfy both Kac equations. Next we would like to show how to construct a solution to the higher dimensional problem from a known solution of the lower dimensional wave equation. To simplify the index notation let $m = l_n + 1$. Consider first following claim.

Claim 3.1.2 *Let $\phi_{1,m+1}$ be a solution of the equation*

$$(\partial_t j_{m+1}\partial_t - j_{m+2})\phi_{1,m+1} = 0 \tag{3.21}$$

and let

$$\phi_{1,m} = \int j_m^{-1} j_{m+1}\phi_{1,m+1}dt \tag{3.22}$$

$$\phi_{2,m} = -\int d^{-1}cj_m^{-1}j_{m+1}\phi_{1,m+1}dt \tag{3.23}$$

If $\phi_{2,m+1}$ is an arbitrary function of t and

$$\Psi_m = \begin{bmatrix} \phi_{1,m} \\ \phi_{2,m} \end{bmatrix} \quad , \textit{ and } \quad \Psi_{m+1} = \begin{bmatrix} \phi_{1,m+1} \\ \phi_{2,m+1} \end{bmatrix} \tag{3.24}$$

then Ψ_m and Ψ_{m+1} are a solution of the equation $J_m\partial_t\Psi_m = J_{m+1}\Psi_{m+1}$.

Proof: Consider following equalities:

$$J_m\partial_t\Psi_m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} j_m^{-1}j_{m+1}\phi_{1,m+1} \\ d^{-1}cj_m^{-1}j_{m+1}\phi_{1,m+1} \end{bmatrix}$$

$$= \begin{bmatrix} (a - bd^{-1}c)j_m^{-1}j_{m+1}\phi_{1,m+1} \\ (c - dd^{-1}c)j_m^{-1}j_{m+1}\phi_{1,m+1} \end{bmatrix}$$

$$= \begin{bmatrix} j_{m+1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{1,m+1} \\ \phi_{2,m+1} \end{bmatrix} = J_{m+1}\Psi_{m+1} \tag{3.25}$$

□

Claim 3.1.3 *If Ψ_m, Ψ_{m+1} are defined as in the previous claim, then they also satisfy the equation:*

$$\partial_t J_{m+1}\Psi_{m+1} = J_{m+1}\Psi_m \tag{3.26}$$

Proof: Since j_k is a Kundt-Newman sequence then

$$j_{m+2} = -\partial_t^2 j_{m+1} + (\partial_t j_{m+1})j_{m+1}^{-1}(\partial_t j_{m+1}) + j_{m+1}j_m^{-1}j_{m+1} \tag{3.27}$$

and hence

$$\begin{aligned} j_m^{-1}j_{m+1}\phi_{1,m+1} &= j_{m+1}^{-1}j_{m+2}\phi_{1,m+1} + j_{m+1}^{-1}(\partial_t^2 j_{m+1})\phi_{1,m+1} \\ &\quad - j_{m+1}^{-1}(\partial_t j_{m+1})j_{m+1}^{-1}(\partial_t j_{m+1})\phi_{1,m+1} \end{aligned} \tag{3.28}$$

Since $\phi_{1,m+1}$ is a solution of the equation 3.21 then

$$j_{m+2}\phi_{1,m+1} = \partial_t j_{m+1}\partial_t \phi_{1,m+1} \tag{3.29}$$

and hence the equation 3.27 will become:

$$\begin{aligned} j_m^{-1}j_{m+1}\phi_{1,m+1} &= j_{m+1}^{-1}(\partial_t j_{m+1}\partial_t\phi_{1,m+1}) + j_{m+1}^{-1}(\partial_t^2 j_{m+1})\phi_{1,m+1} \\ &\quad - j_{m+1}^{-1}(\partial_t j_{m+1})j_{m+1}^{-1}(\partial_t j_{m+1})\phi_{1,m+1} \\ &= \partial_t j_{m+1}^{-1}\partial_t j_{m+1}\phi_{1,m+1} \end{aligned} \tag{3.30}$$

But, then :

$$\partial_t j_{m+1}\phi_{1,m+1} = j_{m+1}\int j_m^{-1} j_{m+1}\phi_{1,m+1}dt = j_{m+1}\phi_{1,m} \tag{3.31}$$

Hence we have for the higher dimensional potentials:

$$\partial_t J_{m+1}\Psi_{m+1} = J_{m+1}\Psi_m \tag{3.32}$$

□

If we, now, combine the results of the claims 3.1.2 and 3.1.3 it immediately follows that Ψ_m is a solution of the wave equation:

$$(\partial_t J_m \partial_t - J_{m+1})\Psi_m = 0 \tag{3.33}$$

3.2 An example of reduction

As an example of the methods discussed in the previous chapter let us consider the 2×2 case. We will first generate the sequences of potentials $\{V_k\}$ and $\{v_k\}$. For simplicity we would like the potential sequences to terminate in few steps and hence we choose low integer values for the parameters l_1 and l_2. Let $l_1 = 2$ and $l_2 = 1$. Then equation 2.12 gives:

$$\{V_k\}_{k=0}^3 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{6}{t^2} & 0 \\ 0 & \frac{2}{t^2} \end{bmatrix}, \begin{bmatrix} \frac{24}{t^4} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \tag{3.34}$$

The matrix V_3 is well defined in 2.12 but we could not have generated it from V_0 and V_1 via Kac equations since V_2 is singular. We can now identify the first three elements in the lower dimensional sequence of potentials as:

$$v_0 = 1 \quad , v_1 = \frac{6}{t^2} \quad , \text{and} \quad v_2 = \frac{24}{t^4} \tag{3.35}$$

Given potentials v_1 and v_2, we can use either the equation 2.5 or the equation 2.12 to generate v_3. In either case, we will find that the lower dimensional Kac sequence terminates i.e. $v_3 = 0$. Next we can consider the sequence of potentials $\{J_k\}$ and $\{j_k\}$. In the equation 2.23, we can make following choices for the constant matrices:

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{, and} \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \tag{3.36}$$

If we now use the equation 2.22 to generate J_1. We obtain:

$$J_1 = \begin{bmatrix} -\frac{3t^8-3t^5-6t^3+2}{t^9-t^6+3t^4-t} & \frac{6t^3}{t^8-t^5+3t^3-1} \\ -\frac{5t^3}{t^8-t^5+3t^3-1} & -\frac{2t^8+t^5+6t^3+1}{t^9-t^6+3t^4-t} \end{bmatrix} \tag{3.37}$$

To determine the next two potentials we can use either the equation 2.40 or the equation 2.1. In either case, we obtain:

$$J_2 = \begin{bmatrix} -\frac{144t^8+72t^5+72t^3+12}{6t^{11}+3t^8-12t^6-2t^3} & -\frac{72t}{6t^8+3t^5-12t^3-2} \\ \frac{60t}{6t^8+3t^5-12t^3-2} & -\frac{36t^5-72}{6t^8+3t^5-12t^3-2} \end{bmatrix} \quad \text{, and} \quad J_3 = \begin{bmatrix} -\frac{240}{2t^5+1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.38}$$

Using the equations 3.9 and 3.7 we can identify by examination of the potentials J_2 and J_3 the lower dimensional potentials j_2 and j_3 as:

$$j_2 = -\frac{24t^5+12}{t^8-2t^3} \quad \text{, and} \quad j_3 = -\frac{240}{2t^5+1} \tag{3.39}$$

If compute the next potential j_4, we will find that the sequence $\{j_k\}$ terminates i.e. $j_4 = 0$. But then the equation

$$(\partial_t j_3 \partial_t - j_4)\psi_3 = 0 \tag{3.40}$$

can be easily integrated to find the solution for ψ_3 :

$$\psi_3 = c_2 - c_1\left(t^6 + 3t\right) \tag{3.41}$$

where c_1 and c_2 are arbitrary constants. The solution of the equation 3.40 can be then used to construct a solution of the problem involving higher dimensional potentials:

$$(\partial_t J_2 \partial_t - J_3)\Psi_2 = 0 \tag{3.42}$$

by taking advantage of the relationships given by equations 3.22 and 3.23. If we let $m = 2$, $\phi_{1,m+1} = \psi_3$ and substitute into 3.22, 3.23 we will get:

$$\begin{aligned} \Psi_2 &= \begin{bmatrix} -\int \frac{c_1 t^{14} + c_1 t^9 - c_2 t^8 - 6 c_1 t^4 + 2 c_2 t^3}{4 t^{10} + 4 t^5 + 1} dt \\ -\int \frac{5 c_1 t^{10} + 15 c_1 t^5 - 5 c_2 t^4}{12 t^{10} + 12 t^5 + 3} dt \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2 c_2 t^4 + 25 c_1}{4 t^5 + 2} - c_1 t^5 + \frac{19 c_1}{2} \\ \frac{25 c_1 t - c_2}{6 t^5 + 3} - \frac{25 c_1 t}{3} - c_3 \end{bmatrix} \end{aligned} \tag{3.43}$$

We have previously shown that the sequence $\{J_k\}$ is a Kundt-Newman sequence and hence using the equation 1.26 we get:

$$\Psi_1 = J_2^{-1} \partial_t J_2 \Psi_2 \tag{3.44}$$

If we substitute in the equation 3.47, we get:

$$\Psi_1 = \begin{bmatrix} -\frac{2 c_1 t^{13} + c_1 t^{10} + 12 c_1 t^8 - 2 c_2 t^7 - 6 c_1 t^5 - (2 c_3 + c_2) t^4 + 18 c_1 t^3 + 3 c_1}{6 t^9 + 3 t^6 - 12 t^4 - 2 t} \\ \frac{10 c_1 t^8 + 3 c_3 t^7 - 5 c_1 t^5 + 5 c_1 t^3 - (6 c_3 + 2 c_2) t^2 + \frac{10}{3} c_1}{6 t^8 + 3 t^5 - 12 t^3 - 2} \end{bmatrix} \tag{3.45}$$

where Ψ_1 is the solution of wave equation:

$$(\partial_t J_1 \partial_t - J_2)\Psi_1 = 0 \tag{3.46}$$

The calculation could be continued further to determine Ψ_0 using the equation:

$$\Psi_0 = J_1^{-1} \partial_t J_1 \Psi_2 \tag{3.47}$$

Chapter 4

Multichannel scattering and Kundt-Newman sequences

Consider the matrix Schrodinger wave equation:

$$\partial_t^2 \Psi - V\Psi = -K^2 \Psi \tag{4.1}$$

where the wave number matrix $K = \text{diag}(k_1, k_2, \ldots, k_n)$ is such that $k_1^2 = k_j^2 + \delta_j^2$ and the potential matrix V is symmetric. This type of matrix Schrodinger equation arises in problems of multichannel scattering of particles without spin. In this chapter we consider a special case of the equation 4.1 for which the wave matrix K is such that K^2 is a multiple of the identity and hence commutes with any matrix. It was suggested in [2] that the method of terminating Kundt-Newman sequences could be used to derive solutions of the equation 4.1. A class of potentials V for which there are known exact solutions was presented in [10]. We would like to examine a simple potential from this class to determine whether or not it can be related to a class of one step terminating symmetric Kundt-Newman potentials and compare the corresponding solution sets.

4.1 One step terminating Kundt-Newman potentials and the Multi-channel problem

In this section we will summarize some of the properties of self-adjoint one step terminating Kundt-Newman sequences presented in [2]. In general the potentials in a self-adjoint Kundt-Newman sequence are related by $J_{-k} = J_k^{-1}$ which immediately implies that $J_0 = I$, where I denotes the $n \times n$ identity matrix. The sequence of matrix potentials $\{J_k\}$ terminates in one step if $J_2 = 0$. Hence one step termination implies that the equation 1.14 has, for $k = 1$, the following form:

$$J_1^2 - J_1 \partial_r J_1^{-1} \partial_r J_1 = 0 \tag{4.2}$$

where J_1 is considered to be a symmetric matrix function of a single coordinate r. Multiplying both sides of the equation 4.2 we obtain a termination condition on the matrix potential J_1:

$$J_1 = \partial_r J_1^{-1} \partial_r J_1 \tag{4.3}$$

Since J_1 in the above equation is symmetric then:

$$\partial_r(J_1^{-1}\partial_r J_1) - \partial_r((\partial_r J_1)J_1^{-1}) = J_1 - J_1^T = 0 \tag{4.4}$$

Integrating both sides of the equation 4.4 gives us another termination condition on J_1:

$$J_1^{-1}(\partial_r J_1) - (\partial_r J_1)J_1^{-1} = J_1 - J_1^T = C \tag{4.5}$$

where C is a constant antisymmetric $n \times n$ matrix. Note that the matrix C is invariant under similarity transformations of potential J_1 by constant matrices.

The coordinates u,v in the equation 1.14 and the coordinates r,t in the equation 4.2 are related by:

$$u + v = 2r \quad \text{and} \quad u - v = 2t \tag{4.6}$$

Then the partial differential operators satisfy:

$$\partial_u = \partial_r + \partial_t \quad \partial_v = \partial_r - \partial_t \quad \text{and} \quad \partial_u\partial_v = \partial_r^2 - \partial_t^2 \tag{4.7}$$

When u, v and J_0 are defined as above, for $k = 0$ the wave equation 1.12 becomes:

$$(\partial_r^2 - \partial_t^2 - J_1)\Psi_0 = 0 \tag{4.8}$$

Since the sequence $\{J_k\}$ is terminating the solution Ψ_0 can be derived by the methods described in previous sections. We have:

$$\Psi_0 = J_1^{-1}\partial_r J_1 A(r+t) \tag{4.9}$$

Let $A(r+t) = e^{B(r+t)}$ and let $E = -\frac{1}{2}e^{-B(r+t)}$, for any constant $n \times n$ matrix B. From the equation 4.8 it follows that for any t, Ψ_0 satisfies:

$$\partial_r^2\Psi_0 = J_1\Psi_0 + \Psi_0 B^2 \tag{4.10}$$

Further more, the potential J_1 can be expressed as:

$$J_1 = -2\partial_r\Psi_0 E \tag{4.11}$$

The following claim is a matrix generalization of some of the results in [11].

Claim 4.1.1 *Let G be a matrix solution of the differential equation*

$$\partial_r^2 G = VG + GB^2 \tag{4.12}$$

where V is a matrix function of r such that

$$V = -2\partial_r G E \quad and \quad \partial_r E = -BE \tag{4.13}$$

and B is a constant matrix. Then $F = [I + iG(K+iB)^{-1}E]e^{iKr}$ is a solution of the differential equation:

$$\partial_r^2 F - VF = -FK^2 \tag{4.14}$$

<u>Proof</u>: Proof is a simple matter of substituting for F in the equation 4.14

□

Since in our case K^2 commutes with the matrix F, then F is also the solution for the Schrodinger differential equation 4.1. Hence if the potential V in the equation 4.1 is one step terminating Kundt-Newman potential then the solution to the Schrodinger equation 4.1 can be found using the claim above and the properties of Kundt-Newman sequences. Note that in the case when K^2 is a multiple of identity the Kundt-Newman potentials are independent of the matrix B and the matrix B is independent of the wave number matrix K. In the next section we will show that this is not true for Cox's potentials.

4.2 Cox's matrix potentials

In this section we describe class of potentials V, presented in [10], with known exact solutions. We will show that if K^2 is a multiple of identity then these potentials are contained in the class of one step terminating Kundt-Newman potentials . Cox's potentials involve a parameter q. It is our conjecture that this parameter determines in how many steps the potential terminates. Since we are considering only one step terminating potentials we assume for rest of this section that $q = 1$. Then Cox's matrix potential can be expressed as:

$$V = 2EY^{-T}(AB + BA)Y^{-1}E \tag{4.15}$$

where A is an $n \times n$ invertible constant matrix. The matrix B is diagonal such that if $B = \text{diag}(b_1, b_2, \ldots, b_n)$ then $b_1^2 = b_i^2 - \delta_i^2$ for $i = 1, 2, \ldots, n$. The constants δ_i are same as in the definition of the wave number matrix K in the equation 4.1. Since the matrix K^2 is a multiple of of identity then $\delta_i^2 = 0$ for $i = 1, \ldots, n$ and the matrix B^2 is then also a multiple of the identity matrix. The matrices E and Y are defined as:

$$E = e^{-Br} \quad \text{and} \quad Y = I - \frac{1}{2}B^{-1}e^{-2Br}A \tag{4.16}$$

Since the definition of V involves Y^{-1} then the constant matrices A and B have to be such that Y is non-singular. Let $G = EAY^{-1}$, then

$$\Psi = [I + iG(K + iB)^{-1}E]e^{iKr} \tag{4.17}$$

is Cox's solution of equation 4.1. Since B^2 commutes with any matrix, it can be easily shown that the above G satisfies equations 4.12 and 4.13 in the claim 4.1.1 and the potential V also satisfies the termination condition 4.3. It was determined by direct calculation that in the case of general 2×2 Cox's potential the matrix C in equation 4.5 is zero. Since an example of one step Kundt-Newman terminating potential with $C \neq 0$ was presented in [2], Cox's potential is not the most general one step terminating potential.

Conclusion

In the conclusion to this thesis, we would like to present some of the open problems that we encountered and which may pose interesting topics for further research.

One of the main results of this thesis was the construction of a non-trivial Kundt-Newman matrix potential sequence $\{J_k\}$ which we were able to explicitly describe in terms of the variable t and the index k. To construct the $\{J_k\}$ sequence we used a trivial generalization of terminating scalar Kundt-Newman potential sequence to diagonal matrix sequence $\{V_k\}$ and the Kac coupling of two Kundt-Newman potential sequences. The diagonal potential sequence, $\{V_k\}$ we used, was self-adjoint. It may be of interest to carry out the construction based on 2.35 mentioned in chapter 2 using a diagonal almost self-adjoint potential sequence.

One can also consider whether or not it is possible to use the Kac equations and the non-trivial Kundt-Newman sequence $\{J_k\}$ again to derive further examples of Kundt-Newman sequences which could be determined explicitly in terms of the index. One candidate for this is 2.10 with t replaced by $\cosh t$. With this modification the diagonal elements of V_1 are still terminating when values of l_i's are integers.

Of particular interest is also the reduction method of the chapter 3. It would be interesting to find further examples of Kundt-Newman sequences for which one could apply this method of generating solutions to wave equations.

In the chapter 4, we were considering application of the Kundt-Newman substitution method to the Schrodinger wave equation. Our attention was restricted to the case when all channels have the same energy i.e. the square of the wave number matrix was a multiple of identity. At present, it is not quite clear if one can use the same approach to derive the solutions in the case of an arbitrary wave number matrix.

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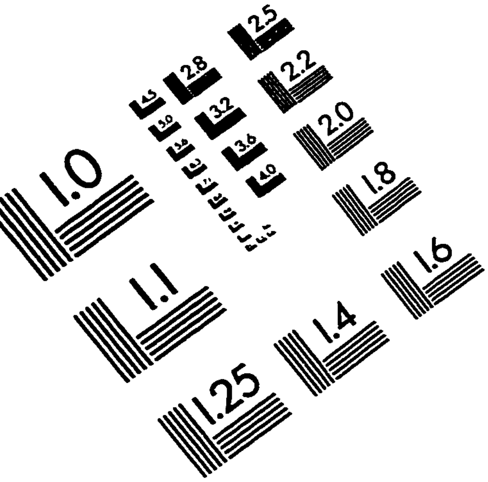

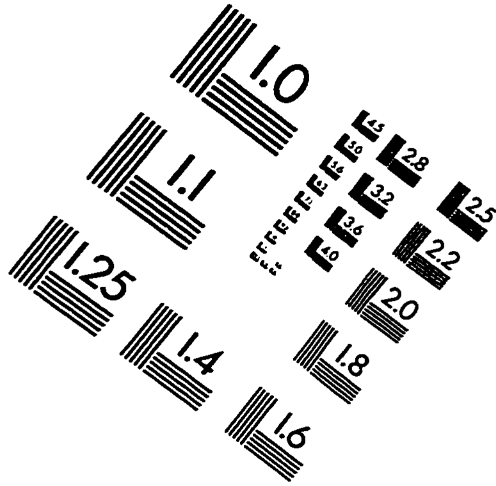

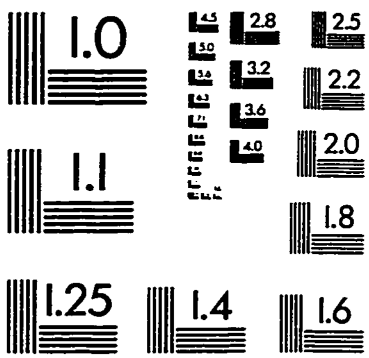

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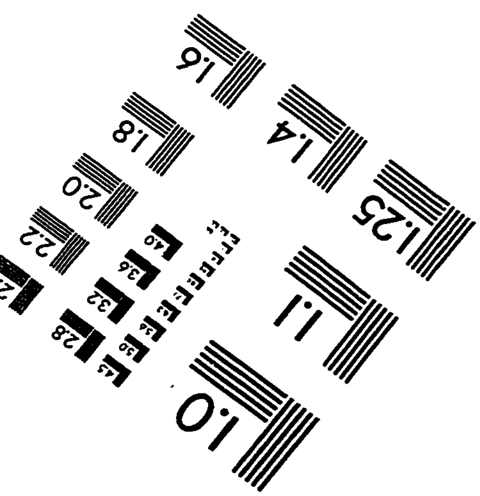

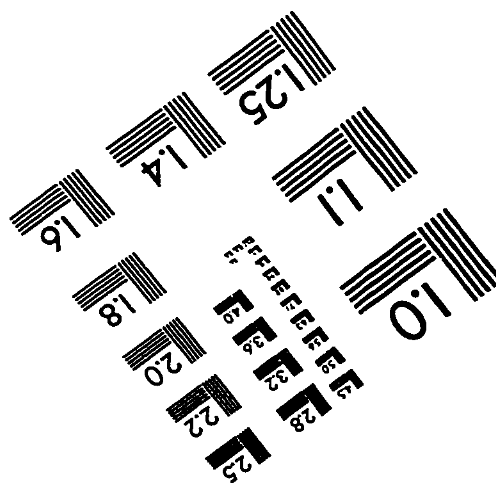

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