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On the Distribution of Dam Content for a Dam Operating in Discrete Time

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a dissertation entitled "ON THE DISTRIBUTION OF DAM CONTENT FOR A DAM OPERATING IN DISCRETE TIME" submitted by ODUEYUNGBO, ADEFOWOPE OLORUNWA in partial fulfillment of the requirements for the degree of M.Sc IN STATISTICS.

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Abstract

This thesis considers a dam of finite capacity holding a reservoir of water. For each time period considered, the input flow into the reservoir is taken to be independent of all other time periods. We also only consider identically distributed flow volumes per time period. Distributions and moments of the reservoir heights against the dam are obtained for several dam models, one of which is original.

Simulation studies were conducted to illustrate the theoretical results. In these studies, both geometric and exponentially distributed inflows were used. As expected, the higher the capacity of the dam, the lower the probabilities of emptiness and overflow.

Acknowledgements

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction to the Control of Dams

1.1 Introduction

A dam is a barrier across flowing water that obstructs, directs or retards the flow, often creating a reservoir, lake or impoundment (*http://en.wikipedia.org/wiki/Dam*). Woodward (2005) defines a typical dam as "a wall of solid material built across a river to block the flow of water thus storing water in the lake that will form upstream of the dam as water continues to flow from the river upstream of the dam". It can be constructed from concrete or natural materials like earth and rock. Dams help in creating a permanent reservoir of water for use at a later period.

Damming water and streams has been a part of human history since recorded times (*http://www.enviroliteracy.org/article.php/59.html*). One of the first dams ever constructed was the Dujiangyan dam on the Minijiang River, a tributary of the Yangtze, in what is now the province of Sichuan.
Li Bing, considered one of the first practitioners of hydraulic engineering, directed the construction of Dujiangyan around BC 250. The Dujiangyan Irrigation System presently serves as a popular tourist attraction (*http://en.wikipedia.org/wiki/Dam*).

1.2 Uses of Dams

The main reasons why dams are built include:

- provision of water for towns, cities and mining sites
- provision of water for irrigation of crops
- provision of storage which will be filled during the wet season and used during the dry season
- provision of storage to be used in cases of exceptional rainfall to prevent/control floods
- provision of recreation areas or habitat for fish and wildlife
- generation of electricity in hydro-electric power stations

Many dams combine two or more of the uses outlined above, and are referred to as "multi-purpose dams".

1.3 Essentials of a Dam

Most dams are built solely for the purpose of creating a permanent reservoir of water for use at a later time [28]. An indispensable aspect of dam construction is therefore an "impermeable membrane", which is the watertight part of the dam that prevents water from leaking out of the dam and escaping downstream.

A dam must also be built in such a way as to withstand the water pressure in the lake upstream of the dam. This water pressure exerts a force on the dam wall tending to push it downstream. It is also desirable for the dam to have sufficient stability in the face of harsh environmental forces to which it may be subjected from time to time.

In addition to being watertight and structurally stable, a dam must have some form of outlet valve for releasing water in controlled amounts as it is needed. Depending on the purpose of the dam, water may be released into a pipeline to supply a city with water, or into an hydroelectric power station to generate electricity. Also the water may simply be released into the river bed downstream of the dam and allowed to flow naturally down the river, and eventually be pumped out and used for irrigation of crops further downstream.

A spillway/weir is a section of the dam which is very useful when the river on which the dam has been built floods. The outlet valve may be insufficient to release excess water from the storage reservoir when the river floods, and hence the need for a spillway/weir through which large volumes of water can flow around the dam without causing damage to the dam itself.

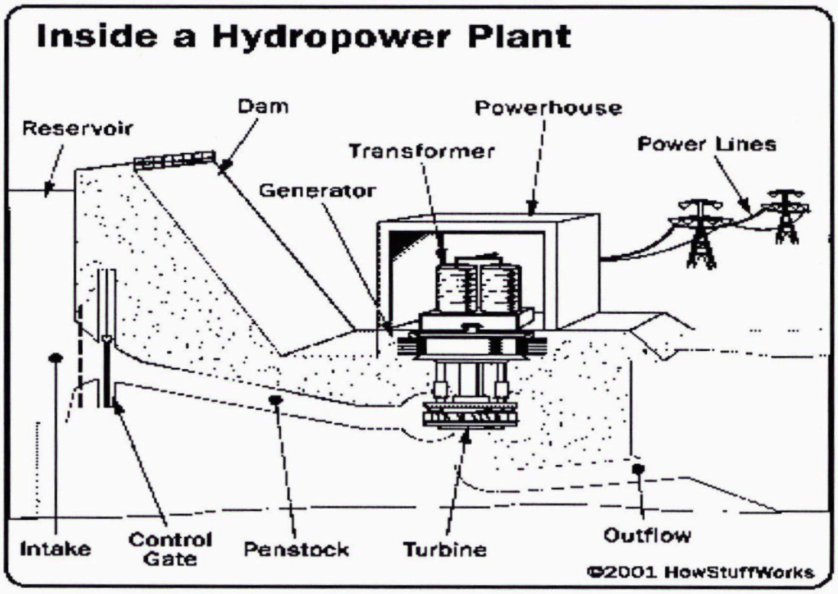


Figure 1.1: A Typical Hydroelectric Power Dam

Figure 1.1 * shows the schematic diagram of a dam built for the primary purpose of providing the necessary head for hydroelectric power.

*This diagram shows the profile of a hydroelectric dam. On the upstream side of the dam is the reservoir. The water behind the dam flows through the intake and into a pipe called a penstock. The water pushes against blades in a turbine, causing them to turn. The bigger the height difference between the intake and the turbines the greater the amount of electricity the water can produce. As the turbine is turned by the water, the energy travels to the generator and then the electrical energy goes to the transformer. Both the generator and transformer are found in the power house. From the transformer the electricity is transmitted through cables to the electrical grid. Once the water has passed through the turbines it then passes out of the dam and continues down the course of the river (Image courtesy: *http://people.howstuffworks.com/hydropower-plant1.htm*)

1.4 Choosing a Dam Site

Locating a good dam site requires finding answers to the following questions:

- Why is the water being stored?
- When is the water needed?
- How long should the water last?
- What is the volume of water needed?

Having answered these questions as accurately as possible, the next step is to find a site that can collect and hold the required volume of water. It is required that a good dam site:

- will allow the construction of an economic and safe dam of appropriate size;
- has a catchment of adequate size to reliably fill the dam;
- will allow the safe disposal of excess water flows; and
- can meet any legal requirements.

1.4.1 Spillway needs

As mentioned earlier, it is critical for a dam site to offer support for a flood bypass system. The water yield characteristics of the catchment along with the rainfall patterns assist in determinining the size of the spillway needed. The site must be topographically suitable to enable the spillway to be constructed as an integral part of the dam. It is necessary for spillway flows to be returned to the normal drainage route before they leave the reservoir area.

1.4.2 Storage efficiency

The site with the best economic advantage and utility is where the most water is stored for the amount of soil material moved. This is known as the storage to excavation ratio. The two main types of water storage considered in the construction of dams are Gully storage and Hillside storage. They are both deployed on sloping land and which one is chosen is dictated by the shape of possible sites.

- **Gully dams**: Constructed in valleys, gully dams are more economically viable than hillside dams because they take advantage of those natural valleys or depressions suitable for water storage.

- **Hillside dams**: Hillside dams are usually selected because no suitable gully site is available. The storage to excavation ratio of hillside dams can be quite low in comparison to that of gully dams.

1.5 Types of Dams

Dams are usually classified in terms of materials and forms. The main types of dams are:

1. **Arch Dams**: Concrete is the main material used in constructing arch dams. They are curved in the shape of an arch, with the top of the arch pointing back into the water. An arch is a strong shape for resisting the pushing force of the water behind the dam. Arch dams are usually constructed in narrow, steep-sided valleys. They need good rock for their foundations, and for the sides of the valleys, to resist the forces on the dam. They are rare because

there are few sites where it is possible to build an arch dam. Arch dams are of two types, namely constant-radius arch dams and variable-radius dams. Jones Falls Dam (1830) in Ontario, Canada, is a constant radius dam.

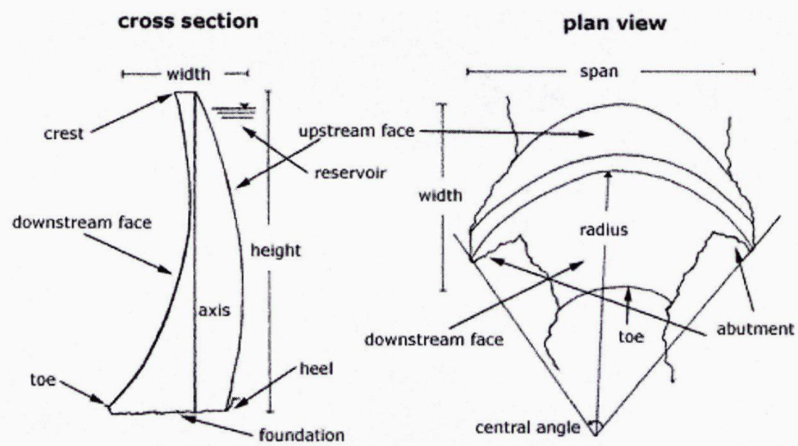


Figure 1.2: An Arch Dam

2. **Gravity Dams**: Gravity dams are solid concrete dams where the base of the dam is thicker or wider than the top. They are referred to as "gravity dams" since they rely on the force of gravity to hold them down to the ground. A cross-section (or slice) through a gravity dam will usually look roughly triangular. Gravity dams are usually constructed at sites with either wide or narrow valleys. They must also be built on solid rock foundations. Examples are the Grande Dixence Dam in Switzerland, and the well-known Aswan Dam in Egypt. Other famous gravity dams are the Grand Coulee Dam in Washington, Revelstoke Dam in British Columbia and the Three Gorges Dam in China. The Oldman dam, constructed in 1992 in response to the many droughts ex-

perienced by Southern Alberta farmers, is also a gravity dam.

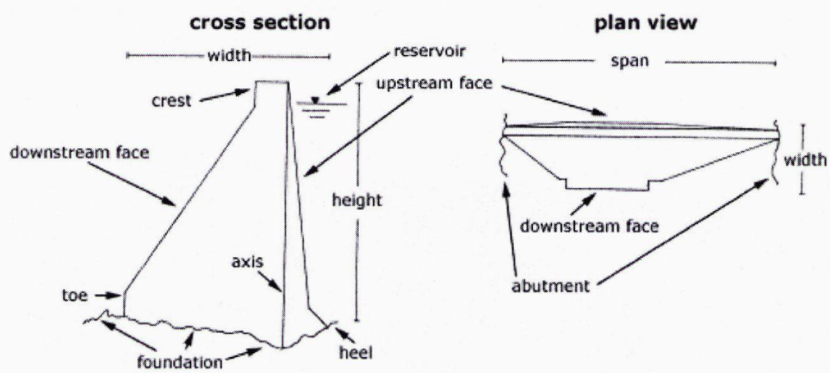


Figure 1.3: A Gravity Dam

3. **Buttress Dams**: Sometimes called hollow gravity dams; buttress dams are usually made from concrete or masonry. They have a watertight upstream side supported by triangular shaped walls, called buttresses. The buttresses are spaced at intervals on the downstream side. They resist the force of the reservoir water trying to push the dam over. The buttress dam was developed from the idea of the gravity dam, except that it uses considerably less material due to the clear spaces between the buttresses. Like gravity dams, they are suited to both narrow and wide valleys, and they must be constructed on sound rock. An example of a buttress dam is the Daniel Johnson Dam on the Manicouagan River in Quebec, Canada.

4. **Embankment Dams**: Embankment Dams are normally made of clay, stones and rock. The two main types are earthfill dams and rockfill dams. Earthfill dams are made up mostly from compacted earth, while rockfill dams are made

up mainly from dumped and compacted rockfill. The materials are usually excavated or quarried from nearby sites, preferably within the reservoir basin. A cross-section through an embankment dam shows that it is shaped like a bank, or hill. Most embankment dams have a central section, called the core, made from an impermeable material to stop water passing through the dam. Clayey soils, concrete or asphaltic concrete can be used for the core. Embankment dams are suited for sites with wide valleys. They can be built on hard rock or softer soils, as they do not exert too much pressure on their foundations. An example of an embankment dam is the Tarbela Dam on the Indus river in Pakistan. Mica dam, built on the Upper Columbia River in British Columbia, Canada, is an earthfill embankment dam.

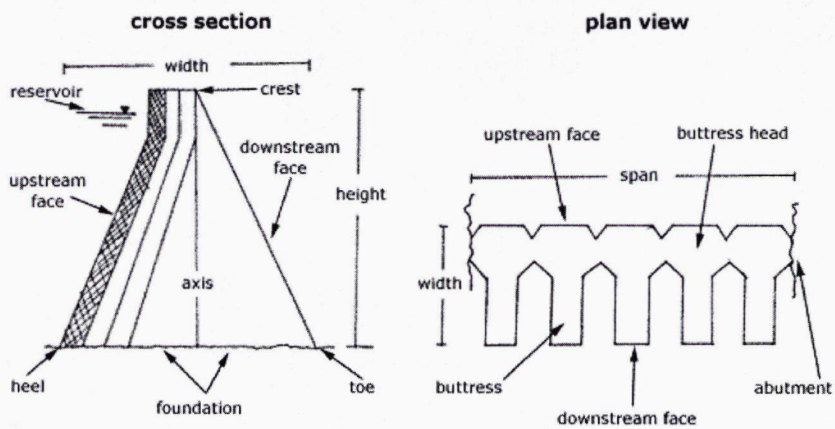


Figure 1.4: A Buttress Dam

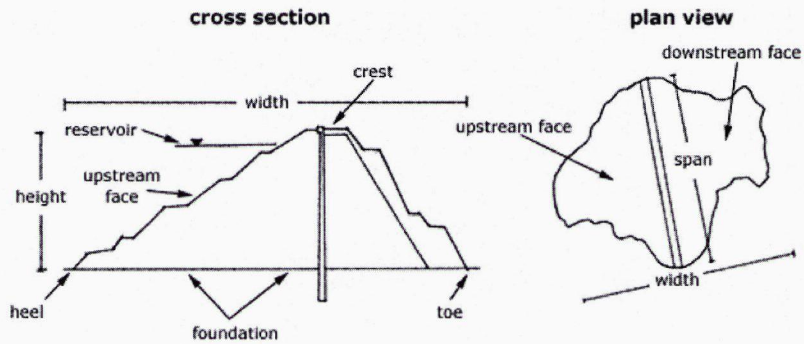


Figure 1.5: An Embankment Dam

1.6 Advantages and Disadvantages of Dams

The benefits of dams include:

- production of hydroelectric energy (e.g the Kainji, Shiroro and Jebba dams in Nigeria)
- the efficiency of a modern hydropower plant exceeds 90 percent, which is more than twice the efficiency of a thermal plant [1]
  - nearly one-fifth of the world's electricity is generated by hydro power [1]
- dams are cost effective in terms of maintenance
- dams retain water in the reservoirs and control the amount that is released to the downstream, helping to prevent floods
- dams are used for irrigation, tap water and industrial purposes

- the lake created behind the dam can be used for water sports.

Problems caused by dams include:

- destruction of lives and properties in the event of dam failure e.g the Teton Dam, 44 miles northeast of Idaho Falls in southeastern Idaho, failed abruptly on June 5, 1976. It released nearly 300,000 acre feet of water, then flooded farmland and towns downstream with eventual loss of 14 lives, directly or indirectly, and with a cost estimated to be nearly one billion dollars [30]
- dam reservoirs may constitute breeding grounds for disease-causing organisms such as mosquitoes; which are the leading cause of Malaria in many African countries
- dam construction creates a barrier to the migration of fish and other aquatic life
- pesticides from farmland and toxic materials from industrial land can pollute the water, which can cause serious health problems for residents, as their tap water comes from the reservoir
- historical sites can be lost due to the construction of reservoirs
- when a river valley is filled with water, animals are forced to leave the area and plants and trees are killed; rare species can also be affected
- sometimes huge number of people have to be evicted from their lands and homes to make way for the construction of reservoirs e.g. the Kainji dam in Nigeria, completed in 1968, caused the displacement of 44,000 people [17].

- excessive silt accumulation in the reservoir can alter upstream hydrological regimes and can lead to flooding, swamp formation, alteration of watertables and reduction of nutrient levels downstream.

Given the pros and cons itemized above, a decision has to be made by governing authorities on which of the disadvantages they're willing to accept in order to enjoy the benefits of dam construction. Of great importance is the controversy concerning the construction of China's Three Gorges Dam, considered by many to be China's biggest construction project since the Great Wall. Projected to be completed in 2009, an estimated 1.2 million people will be resettled due to its construction. Many observers from within and outside China have referred to the project as "the most environmentally and socially destructive project in the world". Historians have also questioned the dam's impact on attempts to preserve some aspects of China's long and illustrious history. Archaeologists and historians have estimated that nearly 1,300 important sites will disappear under the reservoir's waters. The Chinese government in response to criticisms, sees the project as an important future source of energy for China's growing electrical consumption. China's Three Gorges Dam present a typical example of a case in which the pros have to be carefully weighed against the cons, in order to arrive at a decision on dam construction (CNN Interactive: *http://edition.cnn.com/SPECIALS/1999/china.50/asian.superpower/three.gorges/*).

1.7 Dam Control

As noted earlier, the main reason a dam is constructed is to create a permanent reservoir of water for use at a later time. In this thesis, we will be concerned mainly with the use of reservoirs for storage provision and not with the specific usage of the stored resource. The release of water from the dam can be regulated through an outlet valve but the flow of water into the dam cannot be determined with certainty.

Reservoir inflow occurs randomly and hence the need to treat dam control problems as stochastic rather than deterministic problems. Reservoir inflow has a direct effect on the dam content at any specific point in time, and consequently determines the amount to be released in order to keep the dam operating at an optimal level.

A dry dam is not desirable since demands for water may not be met, and thus costly consequences like loss of revenue (in hydro-electricity) may be incurred. On the other hand, a dam which is always full may not be desirable as well, since an adequate storage space is required to catch expected/unexpected floods in case of heavy flooding upstream. If an adequate storage space is not available in the reservoir, an overflow occurs resulting in a flood downstream which can lead to loss of lives and properties. Hence the need to maintain an "optimum" amount of water in the reservoir to provide for varying demands and also to provide an adequate storage space to mitigate floods.

In this thesis, we are concerned with evaluating the distribution of dam content. We are also concerned with determining the effect of dam capacity on the probabilities of dam emptiness and overflow.

1.8 Thesis Arrangement

Figure 1.6 gives a flow-chart for the arrangement of the work in this thesis.

In Chapter 2, the theory of dam control as it relates to probability modeling is discussed within the purview of the pioneering and groundbreaking research work conducted by various statisticians.

Chapter 3 gives a simple formulation of the problem being discussed using Moran's theory. Expressions are derived for the stationary distribution of dam content and the first two moments of the distribution of dam content are given. A solution for discrete reservoir inflow is proposed in Chapter 4 due to the non-analytic nature of the expressions derived in Chapter 3. In Chapter 5, a continuous approximation is provided to the discrete approximation presented in Chapter 4 by a direct limiting process.

Results of simulation studies are presented in Chapter 6.

Results of simulation studies are discussed, recommendations provided and conclusions given in Chapter 7.

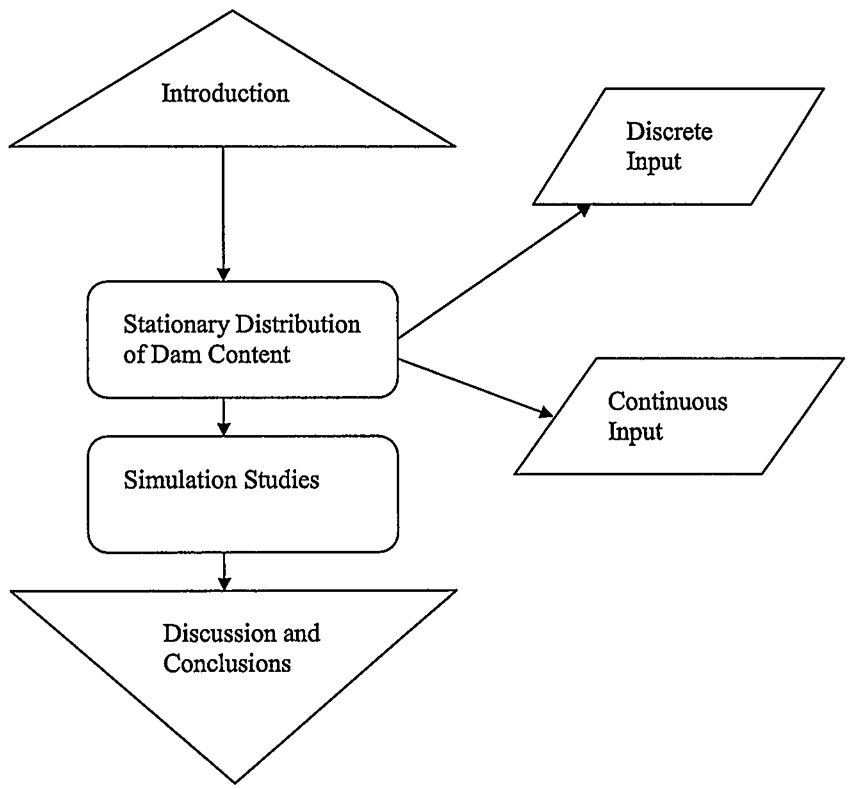


Figure 1.6: Thesis Arrangement

Chapter 2

Theory of Dam Control

The stochastic study of dam theory was initiated as a result of the pioneering work of Patrick Alfred Pierce Moran(1917-1988). In a ground breaking 1954 paper, Moran describes a dam of finite capacity with serially independent and identically distributed inflow of water[18]. He assumes that the distribution of reservoir inflow has an underlying continuous distribution.

Based on a simple operating rule, Moran(1954) derived an integral equation from which the stationary distribution of dam content can be derived. We have extended this by deriving expressions for the first two moments of the distribution of dam content using the Laplace transformation. These expressions are difficult to solve analytically, and so numerical approximations have been given. We have also obtained the expectation for the water level in the reservoir in the case of a discrete input distribution and discrete water release rate. Probability generating functions are also presented.

Since Moran's ground breaking work of 1954, many statisticians/mathematicians and engineers have obtained other important results. Finite dams with compound Poison inputs and variable release rates have been studied by Yeo(1975), Phatarfod(1969), Brockwell & Chung(1975) and Lee & Ahn (1998).
Stadje(1993), Glynn(1989), and Brockwell & Pacheco-Satiago(1980) present a Markovian treatment of the theory of dam control.
Weesakul(1961) obtained explicit expressions for the probabilities of emptiness in a

finite discrete dam for geometric inputs. Dam emptiness has also been studied by Brockwell & Chung(1975), Prabhu(1958), Ghosal(1960) etc.

The theory of dam control has also been related to the theory of queues in Lund(1994), Blomqvist(1973) and Lee & Kinateder(2000).

Here we limit our consideration to Moran's dam because it provides the basic framework for the treatment of dam control problems using the theory of probability.

Chapter 3

Problem Formulation

3.1 A Simple Operating Rule: Moran's Dam

We consider a single dam of finite capacity K. Let X_t be the total quantity of water which has flowed into the dam during period $t\,(=1,2,3,\ldots)$. X_t is assumed to be a random variable with continuous probability distribution on the interval $(0,\infty)$.

A crucial assumption in developing the probability theory of storage is that the X_t for different values of t are identically and independently distributed. Moran (1959) has noted that "In the case of yearly storage in which the unit of time is one year this will probably hold reasonably well in practice if, for example, the time at which release occurs is the late summer, and the input is the result of winter rains and also possible melting of snow which has fallen during the winter".

Let Z_t be the quantity of water in the dam before the amount X_t arrives. Then at the end of time period t, the amount of water U_t in the dam is

$$U_t = X_t + Z_t \tag{3.1}$$

The effects of evaporation and other environmental factors have been omitted to simplify the problem.

If $U_t > K$, then an overflow occurs in the dam. If W_t is the amount of the overflow, then

$$W_t = max\,(X_t + Z_t - K, 0)$$

The quantity of water remaining in the dam is the lesser of the two amounts K and $X_t + Z_t$. A quantity Y_t of water is then released according to some pre-specified operating rule. Here, we shall assume that the quantity Y_t released at each t is a fixed value $M \in (0, K)$.

A simple operating rule for a dam can be formulated as follows:

$$Y_t = min\,(M, X_t + Z_t)$$

where M is a fixed quantity. The above operating rule corresponds to releasing a fixed amount M if the amount of water in the dam is greater than M. If the quantity of water in the dam is less than M, then whatever is in the dam is the quantity released. After the release, the quantity remaining in the dam is Z_{t+1} and the input-output cycle is again repeated. Hence

$$Z_{t+1} = min\,[Z_t + X_t, K] - min\,[Z_t + X_t, M]\,, \quad t = 0, 1, 2, \ldots.$$

Graphically, the release rule described above can be represented as follows:

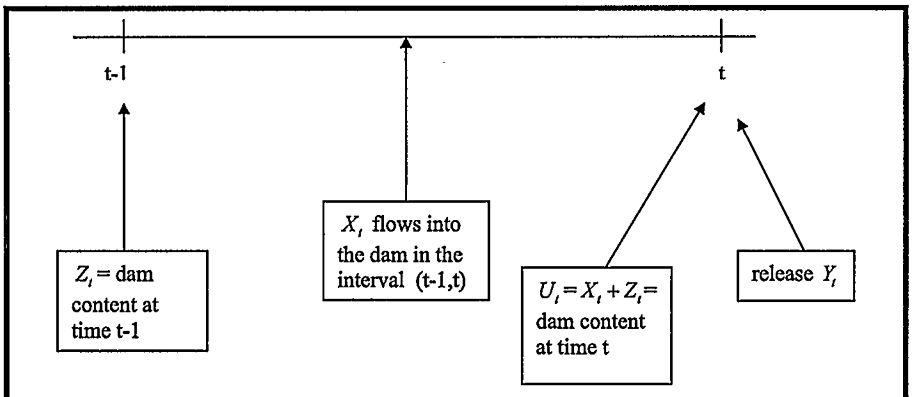


Figure 3.1: A Simple Dam in Operation (a)

The release rule described in this section will be assumed throughout this dissertation unless otherwise stated.

3.2 Probability Distribution of Z_t using Moran's Theory

The main problem lies in determining the probability distribution of Z_t given that the random process defined by the system described in Section 3.1 is in a stationary state, independent of the initial assumptions.

A random process $\{Z_t, t \geq 0\}$ is said to be in a stationary state if for all $n, s, t_1, \ldots, t_n$ the random vectors $Z(t_1), \ldots, Z(t_n)$ and $Z(t_1 + s), \ldots, Z(t_n + s)$ have the same joint distribution. In other words the process Z_t is stationary if, in choosing any fixed point s as the origin, the ensuing process has the same probability law. We can also express $\{Z_t, t \geq 0\}$ as a Markovian process since the conditional joint distribution of $Z_{t_1}, \ldots, Z_{t_n}$, given the value of Z_{t0}, where $t_0 < t_1, \ldots, t_0 < t_n$, is independent of all values of Z_t for $t < t_0$.

To obtain the distribution of Z_t we equate it to the distribution of Z_{t+1} which can be expressed in terms of the distribution of Z_t and X_t. Note that $Z_t \in \{0, K - M\}$ and it will be found that its distribution consists of nonzero probabilities at the points

$$Z_t = 0 \quad \text{and} \quad Z_t = K - M,$$

together with a continuous probability distribution in the region between these two points. Some difficulties arise if one attempts to obtain the distribution of Z_t using direct methods. Alternatively, we consider obtaining the distribution of the quantity $U_t = X_t + Z_t$.

The distribution of U_t has been obtained by Moran (1959) as follows:

Let $f(x)$ and $g(x)$ be the probability density functions of X_t and U_t respectively. We write

$$P(U_t < u) = \begin{cases} 0 & \text{if } u \le 0 \\ G(u) & \text{if } u > 0 \end{cases}$$

From the simple operating rule discussed in Section 3.1 above, we obtain

$$Z_t = \begin{cases} 0 & \text{if } Z_{t-1} + X_{t-1} < M \\ Z_{t-1} + X_{t-1} - M & \text{if } M < Z_{t-1} + X_{t-1} < K \\ K - M & \text{if } Z_{t-1} + X_{t-1} > K \end{cases}$$

The above three conditions on the value of Z_t can be combined to give an expression for the probability distribution of U_t as follows:

$$P(U_t < u) = P(U_t < u, Z_t = 0) + P(U_t < u, Z_t = K - M) + P(U_t < u, 0 < Z_t < K - M). \tag{3.2}$$

where

$$\begin{aligned} P(U_t < u, Z_t = 0) &= P(Z_t + X_t < u, Z_t = 0) \\ &= P(X_t < u, Z_t = 0) \\ &= P(X_t < u, 0 < U_{t-1} < M) \end{aligned}$$

but X_t is independent of U_{t-1}, and U_{t-1} is identically distributed as U_t (stationarity), hence

$$P\left(U_t < u, Z_t = 0\right) = \int_0^u f\left(x\right)\left(\int_0^M g\left(v\right)dv\right)dx \tag{3.3}$$

When $Z_t = K - M$,

$$\begin{aligned} P\left(U_t < u, Z_t = K - M\right) &= P\left(X_t + Z_t < u, Z_t = K - M\right) \\ &= P\left(X_t + K - M < u, Z_t = K - M\right) \\ &= P\left(X_t + K - M < u, U_{t-1} > K\right) \\ &= P\left(X_t < u + M - K, U_{t-1} > K\right) \\ &= \int_0^{u+M-K} f\left(x\right)\left(\int_K^{+\infty} g\left(v\right)dv\right)dx \end{aligned} \tag{3.4}$$

and for $Z_t = X_{t-1} + Z_{t-1} - M$

$$P\left(U_t < u, 0 < Z_t < K - M\right) = P\left(X_t + Z_t < u, M < U_{t-1} < K\right)$$

but

$Z_t = Z_{t-1} + X_{t-1} - M = U_{t-1} - M$

$$\begin{aligned} \Rightarrow P\left(U_t < u, 0 < Z_t < K - M\right) &= P\left(X_t + U_{t-1} - M < u, M < U_{t-1} \leq K\right) \\ &= P\left(X_t < u + M - U_{t-1}, M < U_{t-1} \leq K\right) \end{aligned}$$

$$= \int_M^K P\left(X_t < M + u - U_{t-1} \backslash U_{t-1} = v\right) p_v\left(v\right)dv$$

Hence

$$P\left(U_t < u, 0 < Z_t < K - M\right) = \int_M^K P\left(X_t < M + u - v\right) p_v\left(v\right) dv$$
$$= \int_M^K \left(\int_0^{M+u-v} f\left(x\right) dx\right) g\left(v\right) dv \qquad (3.5)$$

Combining (3.3), (3.4) and (3.5) we get

$$P\left(U_t < u\right) = \int_0^u f\left(x\right) \left(\int_0^M g\left(v\right) dv\right) dx + \int_0^{u+M-K} f\left(x\right) \left(\int_K^{+\infty} g\left(v\right) dv\right) dx$$
$$+ \int_M^K \left(\int_0^{M+u-v} f\left(x\right) dx\right) g\left(v\right) dv \qquad (3.6)$$

Since M and K are fixed, and U_t is a random variable,

$$\frac{d}{du} P\left(U_t < u\right) = f\left(u\right) \int_0^M g\left(v\right) dv + f\left(u + M - K\right) \int_K^{+\infty} g\left(v\right) dv$$
$$+ \int_M^K f\left(M + u - v\right) g\left(v\right) dv$$

$$\Rightarrow g\left(x\right) = f\left(x\right) \int_0^M g\left(u\right) du + f\left(x + M - K\right) \int_K^{+\infty} g\left(u\right) du$$
$$+ \int_M^K f\left(x + M - u\right) g\left(u\right) du \qquad (3.7)$$

The three terms in (3.7) arise as a result of

- $0 \leq U_t \leq M$(when the dam runs dry),

- $K < U_t$(when there is an overflow of amount $U_t - K$) and
- $M < U_t \leq K$(when there is no overflow nor does the dam run dry).

From (3.7) we can derive the distribution of Z_t which clearly has a concentration p_0 at $Z_t = 0$, a continuous probability density function in the interval $0 < Z_t < K - M$ and a concentration p_1 at $Z_t = K - M$.
Here

$$p_0 = \int_0^M g(u)\,du, \quad p_1 = \int_K^{+\infty} g(u)\,du$$

It is also of interest to derive the corresponding equation for the characteristic function of the distribution of U_t. Since U_t is non-negative, it is convenient to use the Laplace transform, and we write

$$\widetilde{f}(\theta) = \int_0^{+\infty} e^{-\theta u} f(u)\,du, \quad \widetilde{g}(\theta) = \int_0^{+\infty} e^{-\theta u} g(u)\,du$$

3.2.1 Moments of U_t : Our Extension

It is quite easy to obtain the first two moments of U_t which we have derived using the Laplace transform method. As far as we know, these moments have not been obtained in any previous work using the method we have proposed. Multiplying (3.7) by $e^{-\theta x}$ and integrating over the range $(0, +\infty)$, we get

$$\int_0^{+\infty} e^{-\theta x} g(x)\,dx = \int_0^M \left(\int_0^{+\infty} f(x)\, e^{-\theta x} dx\right) g(u)\,du + \int_0^{+\infty} e^{-\theta x} f(x + M - K)\,dx$$
$$\int_K^{+\infty} g(u)\,du + \int_M^K \left(\int_0^{+\infty} e^{-\theta x} f(x + M - u)\,dx\right) g(u)\,du$$

$$= \widetilde{f}(\theta) \int_0^M g(u)\,du + \int_0^{+\infty} e^{-\theta x} f(x+M-K)\,dx \int_K^{+\infty} g(u)\,du$$
$$+ \int_0^{+\infty} \int_M^K e^{-\theta x} f(x+M-u)\,g(u)\,du\,dx \qquad (3.8)$$

Let $v = x + M - K$ and $w = x + M - u$. Thus we can rewrite (3.8) as

$$\widetilde{g}(\theta) = \widetilde{f}(\theta) \int_0^M g(u)\,du + \int_0^{+\infty} e^{-\theta(v-M+K)} f(v)\,dv \int_K^{+\infty} g(u)\,du$$
$$+ \int_0^{+\infty} \int_M^K e^{-\theta(w-M+u)} f(w)\,g(u)\,du\,dw$$
$$= \widetilde{f}(\theta) \left\{ \int_0^M g(u)\,du + e^{\theta(M-K)} \int_K^{+\infty} g(u)\,du + e^{\theta M} \int_M^K e^{-\theta u} g(u)\,du \right\} \quad (3.9)$$

after some manipulations.

Next we take the derivative of (3.9) with respect to θ to get

$$\widetilde{g}'(\theta) = \widetilde{f}'(\theta) \left\{ \int_0^M g(u)\,du + e^{\theta(M-K)} \int_K^{+\infty} g(u)\,du + e^{\theta M} \int_M^K e^{-\theta u} g(u)\,du \right\}$$
$$+ \widetilde{f}(\theta) \left\{ 0 + \int_M^K (M-u)\,e^{\theta M} e^{-\theta u} g(u)\,du + (M-K)\,e^{(M-K)\theta} \int_K^{+\infty} g(u)\,du \right\}$$
$$(3.10)$$

Evaluating (3.10) at $\theta = 0$ we obtain

$$\widetilde{g}'(0) = \widetilde{f}'(0) \left\{ \int_0^M g(u)\,du + \int_K^{+\infty} g(u)\,du + \int_M^K g(u)\,du \right\}$$
$$+ \widetilde{f}(0) \left\{ \int_M^K (M-u)\,g(u)\,du + (M-K) \int_K^{+\infty} g(u)\,du \right\} \qquad (3.11)$$

Let $G(M) = \int_0^M g(u)\,du, \quad G(K) - G(M) = \int_M^K g(u)\,du \quad \text{with} \quad \widetilde{f}(0) = 1$

$$\widetilde{g}'(0) = \widetilde{f}'(0)\{G(M) + G(K) - G(M) + 1 - G(K)\} + M\int_M^K g(u)\,du - \int_M^K ug(u)\,du$$
$$+ (M - K)(1 - G(K))$$

or equivalently,

$$\widetilde{g}'(0) = \widetilde{f}'(0) + M\{1 - G(M)\} - K\{1 - G(K)\} - \int_M^K ug(u)\,du \qquad (3.12)$$

Note that

$$E(U_t) = -\widetilde{g}'(0) \text{ and } E(X_t) = -\widetilde{f}'(0)$$

Hence the first moment of U is

$$E(U_t) = E(X_t) - M\{1 - G(M)\} + K\{1 - G(K)\} + \int_M^K ug(u)\,du \qquad (3.13)$$

Recall that

$$E(U_t) = E(X_t) + E(Z_t)$$

Thus from (3.13)

$$E(Z_t) = \int_M^K ug(u)\,du + K\{1 - G(K)\} - M\{1 - G(M)\}$$

which is the expected value of dam content in the long run. From expression (3.13), one realizes that the relationship between the first moment of X_t and those of U_t and Z_t is not a simple one. To obtain $E(Z_t)$ analytically, one needs to know the theoretical form of the distribution of U_t. Since this distribution is unknown, (3.13) may not be open to a simple analytic solution. One way of solving the problem is to assume a distributional form for U_t, irrespective of the distribution of X_t. Doing this leaves much to be desired as the distribution of X_t plays a very important role in determining that of Z_t.

Next we take the second derivative of (3.9) and evaluate at $\theta = 0$ to obtain

$$\begin{aligned} \widetilde{g}''(0) = \widetilde{f}'(0)\left\{MG(K) - MG(M) - I_1 + (M-K)\overline{G}(K)\right\} \\ +\widetilde{f}''(0) + \widetilde{f}(0)\left\{M^2G(K) - M^2G(M) + I_2 - 2MI_1 + (M-K)^2\overline{G}(K)\right\} \\ +\widetilde{f}'(0)\left\{M\left\{G(K) - G(M)\right\} - I_1 + (M-K)\overline{G}(K)\right\} \end{aligned} \quad (3.14)$$

where $I_1 = \int_M^K ug(u)\,du$, $I_2 = \int_M^K u^2 g(u)\,du$ and $\overline{G}(K) = 1 - G(K)$.

Note that $\widetilde{f}''(0) = E(X^2)$ and $\widetilde{g}''(0) = E(U^2)$. Hence

$$\begin{aligned} E\left(U^2\right) = 2\cdot M\cdot E(X)\cdot G(M) + 2I_1E(X) - 2\cdot M\cdot E(X)\cdot G(K) \\ +2\cdot K\cdot \overline{G}(K)\cdot E(X) - 2\cdot M\cdot \overline{G}(K)\cdot E(X) + M^2G(K) \\ -M^2G(M) + I_2 - 2MI_1 + M^2\overline{G}(K) + K^2\overline{G}(K) - 2\cdot M\cdot K\cdot \overline{G}(K) + E\left(X^2\right) \end{aligned} \quad (3.15)$$

From (3.13)

$$\begin{aligned} (E(U))^2 = (E(X))^2 + 2I_1E(X) + 2\cdot K\cdot \overline{G}(K)\cdot E(X) - 2\cdot M\cdot \overline{G}(M)\cdot E(X) \\ +I_1^2 + 2I_1K\overline{G}(K) - 2MI_1\overline{G}(M) - 2K\cdot M\cdot \overline{G}(K)\cdot \overline{G}(M) \\ +K^2\left(\overline{G}(K)\right)^2 + M^2\left(\overline{G}(M)\right)^2 \end{aligned} \quad (3.16)$$

Since we know that

$$Var(U) = E(U^2) - (E(U))^2,$$

the variance of U_t can be expressed as

$$\begin{aligned} Var(U) = Var(X) + M^2\overline{G}(M)G(M) + K^2\overline{G}(K) \cdot G(K) - 2MK \cdot \overline{G}(K) \cdot G(M) \\ - 2I_1 \left\{ MG(M) + K\overline{G}(K) \right\} + I_2 - I_1^2 \qquad (3.17) \end{aligned}$$

Thus we have derived expressions for the first two moments of U_t and as a consequence, the first two moments of the distribution of Z_t can be determined. As shown by (3.13) and (3.17), no simple relations hold between the moments of X_t and those of Z_t and U_t. Obtaining analytical solutions for specific distributional forms of X_t may be quite a difficult task. For this reason, numerical approximations have been proposed as a way of overcoming the shortcoming of the integral equation (3.7). This is the focus of the discussion presented in Chapter 4.

Chapter 4

Discrete Inputs

4.1 Stationary Distribution of Dam Content

One method of solving (3.7) is to replace it by an approximating set of linear equations. These equations are also useful in obtaining particular solutions of (3.7) by a limiting process as discussed in Chapter 5. The initial model introduced by Moran(1954) is a dam of finite capacity K into which serially independent inputs $X_t(t = 0, 1, 2, \ldots)$ flow during intervals of time $(t, t+1)$. In this chapter, we consider the discrete case where the input X_t is assumed to take values $0, 1, \ldots$ with probabilities $p_0, p_1, \ldots$ respectively .i.e

$$p_i = P\left[X_t = i\right]$$

for $i = 0, 1, \ldots$.

The assumption of discrete inputs greatly simplifies the problem. Assuming K and M are also integral, and $K - M \geq M$, we easily see that at times just after the release of M quantities of water, the sequence of dam contents $\{Z_t, t \geq 0\}$ with possible values $0, 1, \ldots, K - M$, is a Markov chain i.e

$$P\left\{Z_{t+1} \backslash Z_0, Z_1, \ldots, Z_t\right\} = P\left\{Z_{t+1} \backslash Z_t\right\}.$$

The stochastic matrix $\mathbf{P}$ of transition probabilities is of the form

$$\mathbf{P} = \left(P_{ij}\right)_{K-M \times K-M}$$

where

$$P_{ij}\left(t\right) = P_{ij} = P\left\{Z_{t+1} = j \backslash Z_t = i\right\}.$$

$\mathbf{P}$ can also be written in matrix form as:

$$\mathbf{P} = \begin{pmatrix} q_M & p_{M+1} & p_{M+2} & \cdots & p_{K-M} & \cdots & p_{K-1} & (1-q_{K-1}) \\ q_{M-1} & p_M & p_{M+1} & \cdots & p_{K-M-1} & \cdots & p_{K-2} & (1-q_{K-2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_0 & p_1 & p_2 & \cdots & p_{K-2M} & \cdots & p_{K-M-1} & (1-q_{K-M-1}) \\ 0 & p_0 & p_1 & \cdots & p_{K-2M-1} & \cdots & p_{K-M-2} & (1-q_{K-M-2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p_0 & \cdots & p_{M-1} & (1-q_{M-1}) \end{pmatrix}. \quad (4.1)$$

where

$$q_j = \sum_{i=0}^{j} p_i$$

For instance,

$$P_{00} = q_M$$

is the probability that the system starts in state 0 and returns to 0 after a release of M units of water. This occurs only if the dam content is zero with an input of less than or equal to M units.

Our goal is to derive the stationary distribution of the dam content Z_t given a sequence of inputs $\{X_t\}$ with a certain probability distribution. Let

$$\mathbf{V} = (V_0, \ldots, V_{K-M})$$

denote the row vector of stationary probabilities for Z_t at time t, and

$$\mathbf{V}' = (V_0', \ldots, V_{K-M}')$$

the row vector of stationary probabilities at time $t+1$. Clearly, $\mathbf{V}'$ is the solution of the matrix equation

$$\mathbf{V}' = \mathbf{VP} \tag{4.2}$$

where

$$\sum_{i=0}^{K-M} V_i = 1.$$

(4.2) can be written as a system of equations as follows:

$$V_0' = V_0 q_M + V_1 q_{M-1} + \ldots\ldots + V_M p_0,$$

$$V_1' = V_0 p_{M+1} + V_1 p_M + \ldots\ldots + V_{M+1} p_0,$$

$$\vdots$$

$$V_{K-M}' = V_0(p_K + \ldots) + V_1(p_{K-1} + \ldots) + \ldots\ldots + V_{K-M}(p_M + \ldots) \tag{4.3}$$

(4.3) gives the stationary conditions for a finite Markov chain.

Clearly for the release rule described in Section 3.1,

$$V_{K-M+s} = 0 \text{ for } s > 0.$$

Given an initial distribution of Z_t at time t, the distribution of Z_{t+1}, Z_{t+2} and so on can be found from the above formulation. Hence with

$$p_0 > 0, p_1 > 0, \ldots p_k > 0,$$

the probability distribution of Z_{t+n} will converge, as n increases, to a stable distribution independent of the initial conditions. This distribution is known as the "stationary distribution" of the input Z_t whose formal definition can be found in Section 3.1.

To find the stationary distribution of Z_t, we let

$$V_i' = V_i \quad \text{for } i = 0, \ldots, K - M$$

and then solve the set of equations in (4.3). Using an additional condition that

$$V_1 + V_2 + \ldots + V_{K-M} = 1,$$

the final equation of (4.3) can be replaced by

$$\sum_{i=1}^{K-M} V_i = 1.$$

This is possible because the final equation in (4.3) is a consequence of the first $K-M$ equations. We thus end up with a set of non-homogeneous equations which can be solved using a number of methods. (4.3) can be written as a matrix equation such that

$$(\mathbf{V}')^T = \mathbf{P}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}.$$

If $\mathbf{V}^{\mathbf{T}}$ is any vector in which the coefficients sum to unity, then $(\mathbf{P}^{\mathbf{T}})^n\mathbf{V}^{\mathbf{T}}$ will converge to the required solution as n increases.

Another method is the use of the Monte Carlo method, in which a sequence of values $X_1, X_2, \ldots, X_n$ are chosen as a random sample from a population with probability density function $f(x)$. Such a sequence can be generated using software or a table of random numbers and a table of the cumulative distribution of X_t. Alternatively, $\{X_t; t \in [0, n]\}$ may be actual observed inflows at a dam site.The distribution of Z_t is then estimated by finding the observed set of values of Z_t which would occur if the sequence $\{X_t; t \in [0, n]\}$ was the actual input into the dam. This method enables us to estimate how the probability distribution of Z_t changes with a relatively small change in the capacity of the dam K or the release rule. This method is considered in Chapter 6 of this dissertation.

It is always possible to obtain the stationary probability distribution of Z_t numerically, by solving (4.3) for known values of the $\{p_i\}$, and Moran(1954, 1959) has done so for discrete approximations to various Gamma-type input distributions, and dams of different capacities K. Here we consider a solution for the geometric input with discrete release.

4.2 Modified Equations for the Finite Dam

If we assume that X_t, Z_t and U_t are discrete, then the probability distribution $R_i(i = 0, 1, \ldots)$ of U_t which can be written as the convolution

$$R_i = \sum_{r=0}^{K-M} p_{i-r} V_r \quad (p_s = 0 \text{ if } s < 0)$$

satisfies the infinite matrix equation

$$\mathbf{R} = \mathbf{R}\mathbf{Q} \tag{4.4}$$

together with $\sum_{i=0}^{\infty} R_i = 1$, where $\mathbf{R}$ is the row vector of stationary probabilities of U_t, and

$$\mathbf{Q} = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{K-M-1} & p_{K-M} & p_{K-M+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_0 & p_1 & p_2 & \cdots & p_{K-M-1} & p_{K-M} & p_{K-M+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & p_0 & p_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{4.5}$$

(4.4) can be written as an infinite set of equations

$$\begin{aligned} R_0 &= p_0 R_0 + p_0 R_1 + \cdots + p_0 R_m, \\ R_1 &= p_1 R_0 + p_1 R_1 + \cdots + p_1 R_m + p_0 R_{m+1}, \\ &\vdots \\ R_{K-M-1} &= p_{K-M-1} R_0 + \cdots + p_{K-M-1} R_m + p_{K-M-2} R_{M+1} + \cdots + p_0 R_{K-1}, \\ R_{K-M} &= p_{K-M} R_0 + \cdots + p_{K-M} R_M + p_{K-M-1} R_{M+1} + \cdots + p_0 \left(R_K + R_{K+1} + \cdots \right), \\ &\vdots \\ R_{K-M+s} &= p_{K-M+s} R_0 + \cdots + p_{K-M+s} R_M + p_{K-M+s-1} R_{M+1} + \cdots + p_s \left(R_K + R_{K+1} + \cdots \right), \\ &\vdots \end{aligned} \tag{4.6}$$

(4.6) is an infinite set of equations for the stationary distributions of an infinite Markov chain. (4.6) also represents an analogue of (3.7). The stationary probabilities V_i of the dam content Z_t are readily seen to be

$$V_0 = (R_0 + \cdots + R_M), \quad V_i = R_{M+i} \text{ for } (0 < i < K - M), \quad \text{and}$$
$$V_{K-M} = (R_K + \cdots)$$

4.3 Geometric Input with Discrete Release

There are difficulties involved in dealing with the continuous model described in Chapter 3. Early attempts to find stationary distributions were restricted to the discrete dam. Continuous solutions were obtained from the discrete analogues by direct limiting processes. The first explicit solution for the finite dam was obtained by Moran (1959) for the case of geometric input

$$p_i = (1 - B)\, B^i \quad (0 < B < 1) \tag{4.7}$$

Rather than directly solving (4.2) for the sequence of stationary probabilities P_i by substituting the given p_i in it, a clever reformulation of the problem dependent on the specific form (4.7) of the input distribution allows the equation to be significantly simplified. The problem is transformed into one in which unit amounts of water flow into the dam. At time intervals X_j $(j = 0, 1, \ldots)$, the release is assumed to be such that M units or the total amount in the dam is discharged, whichever is lesser.

Let $A = (1 - B)$ represent the probability of success and B, the probability of failure. Then (4.7) gives the probability that i trials at unit "time" intervals had failed, before success had been achieved. This is equivalent to the probability that a "time" interval X_j is of length i before a release M occurs.

In the scenario presented above, let

$$\pi = (\pi_0, \ldots, \pi_K)$$

be the row vector of stationary probabilities of the dam content U_t, at times just before a trial which may result in a unit input or a release M. The stationary distribution of dam content Z_t after a release is

$$V_0 = (\pi_0 + \ldots + \pi_M) \quad \text{and}$$
$$V_i = \pi_{i+M} \quad \text{for} \quad (1 \leq i \leq K - M).$$

π clearly satisfies the matrix equation

$$\pi = \pi \mathbf{G},$$

where $\sum_{i=0}^{K} \pi_i = 1$ and $\mathbf{G}$ is the stochastic matrix determined by following the following steps:

- insert the geometric distribution with

  $$p_i = AB^i \, (i = 0, 1, \ldots)$$

  in (4.6) to obtain

$$\begin{aligned}
R_0 &= A\,(R_0 + \ldots + R_M)\,, \\
R_1 &= AB\,(R_0 + \ldots + R_M) + AR_{M+1}, \\
R_2 &= AB^2\,(R_0 + \ldots + R_M) + ABR_{M+1} + AR_{M+2}, \\
&\vdots \\
R_{K-M-1} &= AB^{K-M-1}\,(R_0 + \ldots + R_M) + AB^{K-M-2}R_{M+1} + \ldots + AR_{K-1}, \\
R_{K-M} &= AB^{K-M}\,(R_0 + \ldots + R_M) + AB^{K-M-1}R_{M+1} + \ldots + A\,(R_K + R_{K+1} + \ldots)\,, \\
R_{K-M+s} &= BR_{K-M+s-1},
\end{aligned} \tag{4.8}$$

- multiply each equation in (4.8) by B and subtract from the next. Also write

  $$R'_K = R_K + R_{K+1} + \ldots.$$

One obtains a finite set of equations

$$
\begin{aligned}
R_0 &= A\,(R_0 + \ldots + R_M)\,, \\
R_1 - BR_0 &= AR_{M+1}, \\
R_2 - BR_1 &= AR_{M+2}, \\
&\vdots \\
R_{K-M} - BR_{K-M-1} &= AR'_K, \\
R_{K-M+s} &= BR_{K-M+s-1}, \\
R'_K &= BR_{K-1} + BR'_K. \qquad (4.9)
\end{aligned}
$$

where $s = 1, 2, \ldots, M-1$.

The set of $(K+1)$ equations for the quantities $R_0, R_1, \ldots, R'_K$ can be solved directly if we start at the top of the dam. In other words R_{K-1} can be obtained in terms of R'_K, and R_{K-s} in terms of R_{K-s+j} $(j \geq 1)$ for $s > 1$. Moran (1959) has obtained an explicit formula for solving this set of equations. For $r \geq 1$, the general solution of equations (4.9) has been obtained as

$$
\begin{aligned}
\frac{R_{K-r}}{R'_K} = \frac{\pi_{K-r}}{\pi_K} &= S(r,1)A - S(r-M,2)A^2 + S(r-2M,3)A^3 - \ldots \\
&= \sum_{q=1}^{\infty} S\left(r-(q-1)\,M, q\right) A^q \,(-1)^{q-1} \qquad (4.10)
\end{aligned}
$$

where

$$S(n,k) = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} B^{-n} - \begin{pmatrix} n-M-1 \\ k-1 \end{pmatrix} B^{M-n} \tag{4.11}$$

The alternating series (4.10) is finite, for as a result of the above relations, all terms from some point onwards are zero. For instance if $M = 1, K = 5$ and $r = 3$, then

$$\begin{aligned} \frac{\pi_2}{\pi_5} &= S(3,1)\,A - S(2,2)\,A^2 + S(1,3)\,A^3 - \dots \\ &= \frac{A}{B^2}\left(B^{-1} - 1 - A\right) \end{aligned}$$

which is also the result obtained if one were to solve (4.9) directly. Thus from (4.10), the required stationary distributions can be obtained. A continuous analogue is presented in Chapter 5.

4.4 Expectation of Dam Content : Another Approach

Consider the diagram of a simple dam in operation given in figure (4.1). X_t is now the input at time $t+1$ while Z_t remains the dam content at time t. M is the amount of water released in the interval $(t, t+1)$, and K is assumed to be large enough that the probability of overflow is negligible.

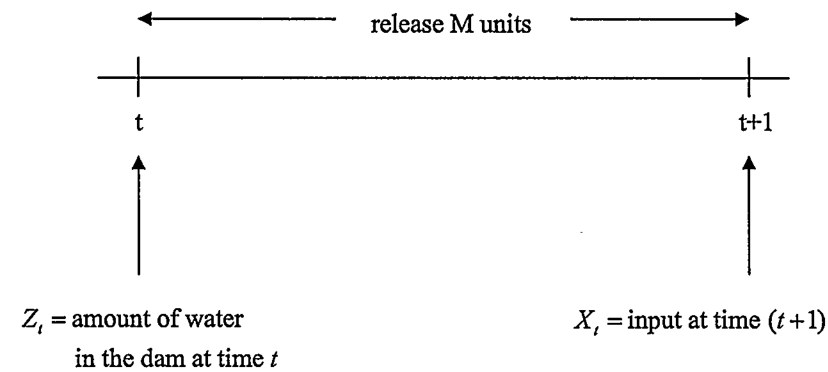


Figure 4.1: A Simple Dam in Operation (b)

Then

$$Z_{t+1} = \begin{cases} X_t & \text{if } Z_t < M \\ Z_t + X_t - M & \text{if } Z_t \geq M \end{cases}$$

Let

$$P(Z_t = n) = p_n, \quad n = 0, 1, 2, \ldots$$

$$P(X_t = m) = q_m, \quad m = 0, 1, 2, \ldots$$

The expression for Z_{t+1} can be written in the form

$$Z_{t+1} = Z_t + X_t - M + \epsilon_t \tag{4.12}$$

where

$$\epsilon_t = M - Z_t \textbf{ when } Z_t < M$$

$$= 0 \textbf{ when } Z_t \geq M$$

Next we take the expected value of (4.12):

$$
\begin{aligned}
E\left(Z_{t+1}\right) &= E\left(Z_t - M + X_t + \epsilon_t\right) \\
\lim_{t\to\infty} E\left(Z_{t+1}\right) &= \lim_{t\to\infty} E\left(Z_t - M + X_t + \epsilon_t\right) \\
0 &= -M + E\left(X\right) + E\left(\epsilon\right) \qquad (4.13)
\end{aligned}
$$

The expression in (4.13) being due to the fact that $E\left(Z_{t+1}\right) = E\left(Z_t\right)$ i.e stationarity. Note also that $\epsilon \geq 0 \Rightarrow M - E\left(X\right) > 0$ or $E\left(X\right) < M$.

4.4.1 Probability Generating Functions

Let

$$E\left(s^X\right) = F\left(s\right) = \sum_{m=0}^{\infty} s^m q_m$$

and

$$E\left(s^Z\right) = H\left(s\right) = \sum_{n=0}^{\infty} s^n p_n$$

be the probability generating functions of the random variables X_t and Z_t respectively.

Hence,

$$E\left(s^{Z_{t+1}}\right) = E\left(s^{Z_t + X_t - M + \epsilon_t}\right)$$

Taking limits on both sides, we have

$$\lim_{t\to\infty} E\left(s^{Z_{t+1}}\right) = \lim_{t\to\infty} E\left(s^{Z_t+X_t-M+\epsilon_t}\right)$$

Thus

$$H(s) = \sum_{k=0}^{M-1} E\left(S^{Z-M+X+\epsilon}\backslash Z = k\right) P\left(Z = k\right)$$
$$+E\left(S^{Z-M+X+\epsilon}\backslash Z \geq M\right) P\left(Z \geq M\right)$$

The first term in the above expression represents a situation where there is insufficient water to release M units, while the second term represents a situation where sufficient water is available to release M.

Recall that $\epsilon = M - Z$ whenever $Z < M$ and $\epsilon = 0$ otherwise. Thus

$$\begin{aligned} H(s) &= \sum_{k=0}^{M-1} E\left(S^{Z-M+X+M-Z}\backslash Z = k\right) P\left(Z = k\right) + E\left(S^{Z-M+X}\backslash Z \geq M\right) P\left(Z \geq M\right) \\ &= \sum_{k=0}^{M-1} E\left(s^{X}\right) P\left(Z = k\right) + E\left(s^{Z-M+X}\backslash Z \geq M\right) P\left(Z \geq M\right) \\ &= F\left(s\right) P\left(Z \leq M-1\right) + \sum_{n=M}^{\infty} p_n s^{n-M} F\left(s\right) \end{aligned} \tag{4.14}$$

Multiply and divide the second term in (4.14) by s^m to get

$$\begin{aligned} H(s) &= F(s)\,P(Z<M) + \frac{F(s)}{s^m}\sum_{n=M}^{\infty} p_n s^n \\ &= F(s)\,P(Z<M) + \frac{F(s)}{s^m}\left\{\sum_{n=0}^{\infty} p_n s^n - \sum_{n=0}^{M-1} p_n s^n\right\} \\ &= F(s)\,P(Z<M) + \frac{F(s)}{s^m}\left\{H(s) - \sum_{n=0}^{M-1} p_n s^n\right\} \qquad (4.15) \end{aligned}$$

Recall

$$\begin{aligned} H(s) &= \sum_{n=0}^{\infty} s^n p_n \\ &= \sum_{n=0}^{M-1} s^n p_n + \sum_{n=M}^{\infty} s^n p_n = H_1(s) + H_2(s) \end{aligned}$$

where

$$H_1(s) = \sum_{n=0}^{M-1} s^n p_n \text{ and } H_2(s) = \sum_{n=M}^{\infty} s^n p_n$$

$H_1(1)$ gives the probability that there is insufficient water to satisfy a release of M units, while $H_2(1)$ gives the probability that the demand M is satisfied. From (4.15),

$$H(s) = F(s)\left[P(Z<M) + \tfrac{\{H(s)-H_1(s)\}}{s^m}\right]$$

which after some manipulations give

$$H(s) = \frac{F(s)\{s^m P(Z<M) - H_1(s)\}}{s^m - F(s)} \qquad (4.16)$$

Recall that

$$H(s) = \sum_{n=0}^{\infty} s^n p_n$$

Hence,

$$H(1) = \sum_{n=0}^{\infty} p_n = 1 \text{ and } H\prime(1) = \sum_{n=0}^{\infty} np_n = E(Z)$$

Similarly,

$$F(1) = \sum_{m=0}^{\infty} q_m = 1 \text{ and } F\prime(1) = \sum_{m=0}^{\infty} mq_m = E(X)$$

From (4.16),

$$H_1(s) = s^M P(Z < M) - \frac{H(s)s^M}{F(s)} + H(s) \tag{4.17}$$

where $H_1(1) = P(Z < M)$ is the probability that there is insufficient water to satisfy the demand M.

Differentiating (4.17) and substituting $s = 1$, we get

$$H_1\prime(s) = Ms^{M-1} \cdot P(Z < M) - \frac{s^M \cdot H\prime(s)}{F(s)} - Ms^{M-1} + \frac{F\prime(s) \cdot s^M \cdot H(s)}{[F(s)]^2} + H\prime(s) \tag{4.18}$$

and

$$H_1\prime(1) = E(X) - MP\,(Z \geq M) \tag{4.19}$$

respectively. Using L'Hopital's rule to solve for $E(Z)$ in (4.16), one obtains

$$\begin{aligned} E(Z) &= \frac{Var(X) - (E(X))^2 + 2M \cdot E(X) - M^2 \cdot P\,(Z \geq M) - H_1\prime(1) - H_1\prime\prime(1)}{2\,(M - E(X))} \\ &= \frac{Var(X) - (E(X))^2 + 2M \cdot E(X) - M^2 \cdot P\,(Z \geq M) - \sum_{n=0}^{M-1} n^2 p_n}{2\,(M - E(X))} \end{aligned} \tag{4.20}$$

which is an expression for the expected value of dam content in the long run. Note that (4.20) is only satisfied for $E\,(X) < M$.

4.4.2 Solution for M=1

We now obtain a solution for the discrete case in which $M = 1$. Here,

$$Z_{t+1} = Z_t - 1 + X_t + \epsilon_t \tag{4.21}$$

where

$$\epsilon_t = 1 - Z_t \textbf{ when } Z_t < 1$$

$$= 0 \textbf{ when } Z_t \geq 1$$

Taking expectations in (4.21), we obtain

$$E(\epsilon) = 1 - E(X) > 0$$

Note also that $E(X) < 1$ implies that $E(X) < E(Z)$ for the stationary condition to be satisfied.

Using (4.20), we obtain after substitution

$$\begin{aligned} E(Z) &= \frac{Var(X) - (E(X))^2 + 2E(X) - P(Z \geq 1)}{2(1 - E(X))} \\ &= E(X) + \frac{E(X(X-1))}{2(1 - E(X))} \\ &= E(X) + \frac{E(X(X-1))}{2p_0} \end{aligned} \tag{4.22}$$

since $p_0 = 1 - E(X)$.

(4.22) satisfies the condition $E(Z) > E(X)$ since

$$E(X(X-1)) = 2q_2 + 6q_3 + \ldots > 0.$$

(4.22) is verified for $M = 1$ through simulation in Chapter 6.

Chapter 5

Continuous Input

5.1 Continuous Analogue of the Geometric Distribution

A continuous analogue of the discrete solution derived in Chapter 4 is presented in this chapter. We rely on the fact that the geometric distribution converges in distribution to the exponential distribution by a simple limiting process. This approach was considered by Moran(1959) and Gani(1957).

Suppose

$$B = e^{-\mu\delta} \quad \text{and} \quad X = r\delta$$

where μ remains fixed, $\delta \to 0$ and $r \to \infty$ in such a way that X is kept finite. Then

$$\begin{aligned} P\left(x \leq \frac{X}{\delta} = r \leq x + dx\right) & \\ &= P\left(x \leq X = r\delta \leq x + dx\right) \\ &= \lim_{\delta \to 0} \left(1 - e^{-\mu\delta}\right) e^{-\mu r\delta} \delta^{-1} dx \\ &= \mu e^{-\mu x} dx \end{aligned}$$

Also write

$$K\delta = (MN + U)\delta = k, \text{ and } M\delta = m$$

where N is an integer, $0 \leq U < M$, and $K, M \to \infty$ while k, m are kept fixed. Moran (1959) considered the limit of

$$\delta^{-1}\frac{R_{K-r}}{R_{K'}} = \delta^{-1}\left\{S(r,1)A - S(r-M,2)A^2 + S(r-2M,3)A^3 - \ldots\right\}.$$

In general for $q > 0$,

$$\lim \delta^{-1} S\left\{r-(q-1)M, q\right\} A^q = \lim \delta^{-1}\left(1-e^{-\mu\delta}\right)^q$$
$$\left\{\binom{r-(q-1)M-1}{q-1} e^{\{[r-(q-1)M]\delta\mu\}} - \binom{r-qM-1}{q-1} e^{(r-qM)\delta\mu}\right\} \tag{5.1}$$

(5.1) was obtained using the expression contained in (4.10). Note that

$$\begin{aligned}\binom{r-lM-1}{q-1} &= \frac{(r-lM-1)!}{(r-lM-1-(q-1)!\,(q-1)!)} \\ &= \frac{(r-lM-1)!}{(r-lM-q)!\,(q-1)!} \\ &= \frac{(r-lM-1)\,(r-lM-2)\cdots(r-lM-q)!}{(r-lM-q)!\,(q-1)!} \\ &< \frac{(r-lM)^{q-1}}{(q-1)!} \\ &= \frac{(X\delta^{-1}-lm\delta^{-1})^{q-1}}{(q-1)!} \\ &= \frac{(X-lm)^{q-1}\,\delta^{1-q}}{(q-1)!}\end{aligned} \tag{5.2}$$

asymptotically for M, r large and l fixed. Since $\left(1-e^{-\mu\delta}\right)^q$ is asymptotically equal to $(\mu\delta)^q$,

$$\lim \delta^{-1} S\{r-(q-1)M, q\} A^q$$

is asymptotically equal to

$$\delta^{-1}(\mu\delta)^q \left\{ \frac{[X-(q-1)m]^{q-1}\delta^{1-q}}{(q-1)!} e^{\{[X\delta^{-1}-(q-1)m\delta^{-1}]\delta\mu\}} \right\}$$

$$-\delta^{-1}(\mu\delta)^q \left\{ \frac{(X-qm)^{q-1}\delta^{1-q}}{(q-1)!} e^{[(X\delta^{-1}-qm\delta^{-1})\delta\mu]} \right\}$$

$$= (\mu)^q \left\{ \frac{[X-(q-1)m]^{q-1}}{(q-1)!} e^{\{[X-(q-1)m]\mu\}} - \frac{(X-qm)^{q-1}}{(q-1)!} e^{[(X-qm)\mu]} \right\} \quad (5.3)$$

for small values of $(\mu\delta)$, as long as $X - qm \geq 0$.

If $(q-1)m < X < qm$ the limit becomes

$$\mu^q \left\{ \frac{[X-(q-1)m]^{q-1}}{(q-1)!} e^{\{[X-(q-1)m]\mu\}} \right\}$$

and if $X < (q-1)m$,

$$S\{r-(q-1)M, q\} A^q = 0$$

Moran(1959) has obtained the formula for $R_{K'}$ given by

$$R_{K'} = (1+J)^{-1}$$

where

$$J = \sum_{q=1}^{q=N+1} \int_0^K \frac{\mu^q}{(q-1)!} (-1)^{q-1} \left\{ [X - (q-1)\,m]^{q-1} \exp\,[X - (q-1)\,m]\,\mu - (X - qm)^{q-1} \exp\,[(X - qm)\,\mu] \right\} dX$$

and

$$p\,(k - X) = (1 + J)^{-1} \sum_{q=1}^{q=N+1} \frac{(-1)^{q-1}\,\mu^q}{(q-1)!} \left\{ [X - (q-1)\,m]^{q-1} \exp\,\{[X - (q-1)\,m]\,\mu\} - (X - qm)^{q-1} \exp\,[(X - qm)\,\mu] \right\}. \tag{5.4}$$

(5.4) is the corresponding continuous stationary distribution of Z_t on the new "time" scale in which $\exp x = 0$ if $x < 0$. In contrast to (3.7), equation (5.4) can be solved analytically for various values of K desired. Thus a solution can be obtained for the stationary distribution of Z_t by taking advantage of the special relationship between the geometric distribution and the exponential distribution. This method has also been applied to situations in which input is Poisson, with discrete or continuous releases [8].

Chapter 6

Simulation Study

6.1 Use of Monte Carlo Methods in Dam Theory

The application of Monte Carlo methods to dam problems can be traced back to Barnes (1954), who simulated an artificial sequence of stream flows into a dam over a period longer than was actually recorded[19]. Gani (1955) has considered a Monte Carlo solution for the matrix equation (4.2) of the discrete dam. The authors considered an artificial realization of n trials such that the dam content takes the values $0, 1, \ldots, K-M$ with frequencies $n_0, n_1, \ldots, n_{K-M}$ respectively. They used the values

$$\hat{P}_i = \frac{n_i}{n} \tag{6.1}$$

where $i = 0, 1, \ldots, K - M$ as the estimators of the stationary probabilities $\{P_i\}$ of the dam content. These are consistent, and asymptotically normally distributed (Bartlett, 1951); they are also unbiased, but only asymptotically, so that to get rid of the bias, the process should be allowed to run for a little while before starting on the sequence on which the estimates are based (Gani 1957). For large n, $n \cdot var\left(\hat{P}_i\right)$ tends asymptotically to a quantity independent of n; the variances of the estimators can therefore be obtained by splitting the realization into subsections, and estimating the variance of the whole from the variance of the observed values in the subsections. These subsequences are not independent, but serious bias can be avoided by taking a gap between them (Gani 1957).

In this dissertation, we consider

1. the direct simulation of a sequence of continuous stream flow $\{X_t\}$ into a dam of finite capacity K at discrete time periods $t = 0, 1, \ldots, n$. We then determine the effect of dam capacity K on the probabilities $\{P_i\}$ of dam content.

2. the simulation of a sequence of discrete stream flow $\{X_t\}$ into a dam of infinite capacity at discrete time periods $t = 0, 1, \ldots, n$. Dam content at each time period is obtained using the formulation discussed in section (4.4). The theoretical results obtained for the expectation of dam content in section (4.4) is then compared to the approximations from the simulation procedure.

6.2 Simulation Procedure : Continuous Input

Moran (1954) has stated that "The best distribution for the fitting of an observed distribution of X_t (stream flow) is probably the gamma type distribution". In the field of hydrology, scientists have also assumed that stream flow follow gamma type distributions in various studies e.g Insua et al. (1996), Insua & Salewicz (1995) etc. Hence our decision to assume that the sequence of input into the dam follow the exponential distribution and we write

$$f(X_t) = \frac{1}{\theta} e^{-\frac{X_t}{\theta}}$$

where $X_t > 0$ and θ is the mean of the distribution. Exponential samples consisting of 600, 1100, 1600 and 2100 sample points were generated using the SAS© software (Version 9.0). Given particular values of K (dam capacity), M (periodical release) and Z_0 (initial dam content), we calculated the subsequent values of the dam content and the stationary probabilities of the dam going dry. This was done in order to determine the effects of dam size and amount of periodic release on the probability of the dam going dry. An estimate of this probability is given by the proportion of times $Z_t = 0$.

The first 100 values in each realization were dropped in order to minimize the effects of initial conditions on the distribution of dam content.

Assuming the Monte Carlo realization is given by $Z_1, Z_2, \ldots, Z_n$ where n is the number of realizations. Suppose also that the system has been allowed to run for a while before the Monte Carlo realizations were recorded so that the influence of initial conditions can be ignored. Let

$$I_t = \begin{cases} 1 & \text{if } Z_t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence $P(Z = 0)$ can be estimated by

$$P(Z = 0) = \widehat{p} = \frac{\sum_{j=1}^{n} I_j}{n}.$$

From the formulation given above, one can safely assume that each I_t follows a Bernoulli distribution and we write

$$E(I_t) = P\{Z_t = 0\} = p \quad \text{and}$$
$$Var(I_t) = E(I_t^2) - \{E(I_t)^2\} = p(1-p)$$

p, if it exists, is the stationary probability that the dam runs dry. The I_t's are also assumed to be independent and identically distributed. As mentioned in earlier chapters, the assumption of independence for the I_t's is reasonable since the time period being considered is yearly.

6.3 Variance and Confidence Interval

By the Central Limit Theorem

$$\frac{\widehat{p}-p}{\sqrt{Var(\widehat{p})}} \sim N(0,1).$$

Hence, an approximate $1-\alpha$ confidence interval for p is given by the formula

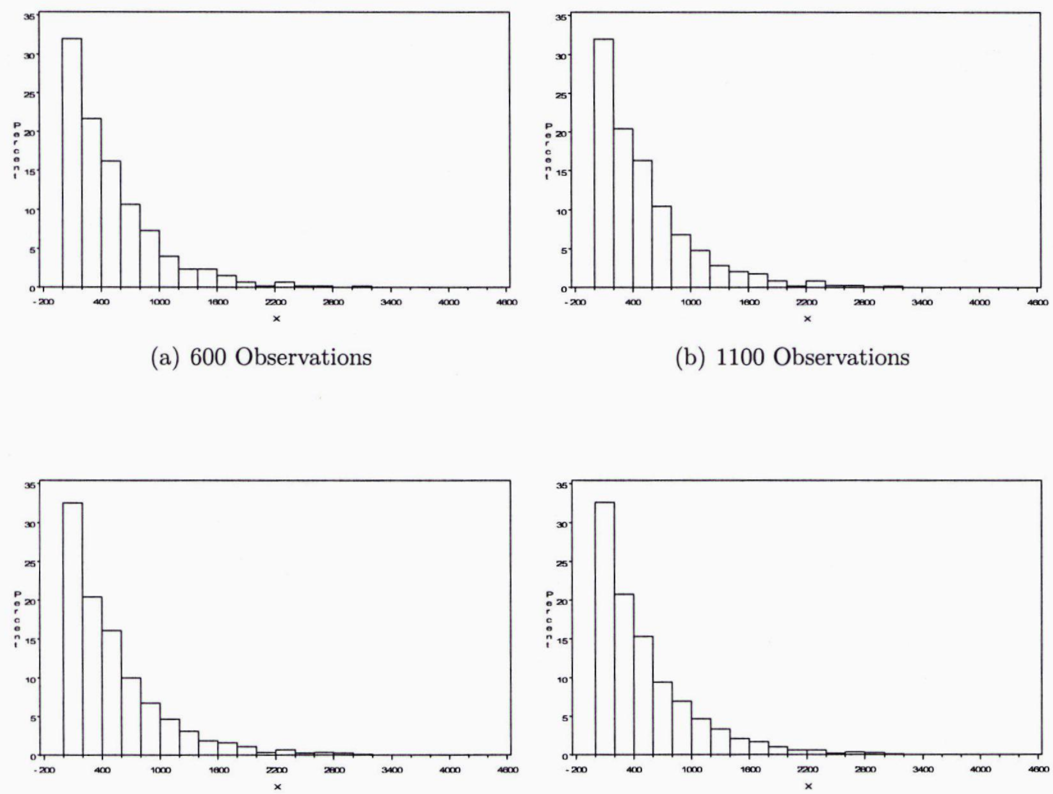
$$\widehat{p} \pm Z_{\alpha/2}\sqrt{Var\left(\widehat{p}\right)}. \tag{6.2}$$

Inverting the scores' test, an approximate confidence interval is also given by

$$\frac{2\widehat{p}+\frac{Z^2_{\alpha/2}}{n} \pm \sqrt{\left(2\widehat{p}+\frac{Z^2_{\alpha/2}}{n}\right)^2 - 4\widehat{p}^2\left(1+\frac{Z^2_{\alpha/2}}{n}\right)}}{2\left(1+\frac{Z^2_{\alpha/2}}{n}\right)} \tag{6.3}$$

6.4 Simulation Results : Continuous Input

First we present the histograms of simulated exponential samples using $\theta = 500$:



(a) 600 Observations

(b) 1100 Observations

(c) 1600 Observations

(d) 2100 Observations

Figure 6.1: Histograms of Exponential Input

The histograms of dam content Z_t for the various sample sizes considered can be found in Appendix B. Here

$$\theta = 500, M = 500, \text{ and } K = 800$$

where θ is the expected mean input, M is the release rate per time period, and K is the capacity of the dam.

6.5 Probability of Emptiness(Overflow) for some K

Table (6.1) gives probabilities of emptiness and overflow(in brackets) for arbitrary dams of capacities $600, 800, 1000$ and 1200, with exponential inputs of mean $\theta = 500$ and $M = 500$.

Table 6.1: Probability of Emptiness/Overflow

No. of realizations	K=600	K=800	K=1000	K=1200
500	0.61(0.30)	0.55(0.23)	0.44(0.18)	0.40(0.14)
1000	0.60(0.33)	0.52(0.26)	0.41(0.21)	0.36(0.19)
1500	0.59(0.33)	0.51(0.26)	0.40(0.22)	0.35(0.19)
2000	0.60(0.33)	0.52(0.26)	0.41(0.21)	0.36(0.19)

Clearly the possibility of the dam going dry or overflowing can be reduced by taking into account the capacity of the reservoir. The greater the reservoir capacity, the smaller the probability of emptiness/overflow. A larger capacity reservoir also ensures that the reservoir is operable. From Table 6.1, we note that for $K = 600$, the dam goes dry approximately 60% of the time while an overflow occurs about 30% of the time. This means there is always either an overflow or emptiness in the dam being considered. As K increases, the situation is less dire and the dam can be operated with less difficulties.

Also considered were the confidence intervals for the various probabilities earlier determined. In particular, we compared the confidence intervals obtained using equations (6.2) and (6.3) labelled in table 6.2 as CI1 and CI2 respectively.

We note that equations (6.2) and (6.3) are approximately equivalent for determining the confidence intervals for the proportions considered. It is more

Table 6.2: 95% Confidence Intervals for Overflow Probability: $K = 800$

No. of realizations	$\widehat{p}$	CI1	CI2
500	0.2340	[0.1969,0.2711]	[0.1990,0.2731]
1000	0.2640	[0.2367,0.2913]	[0.2376,0.2922]
1500	0.2633	[0.2410,0.2856]	[0.2417,0.2862]
2000	0.2605	[0.2413,0.2797]	[0.2417,0.2802]

accurate though, to use CI2 in obtaining the confidence intervals of proportions.

6.6 Simulation Procedure : Expectation of Dam Content

Using the release rule presented in Section (4.4), we have obtained simulated realizations of dam content for 1000 samples, each of size $n = 10$. The discrete distribution considered for the input is the geometric distribution with *pdf*

$$P(X = x) = p\,(1-p)^x; \qquad x = 0, 1, \ldots \tag{6.4}$$

with p and $(1-p)$ as defined in Section (4.3). For this distribution,

$$E(X) = (1-p)/p \quad \text{and} \quad Var(X) = (1-p)/p^2$$

From (4.22), the expectation of Z_t for $M = 1$ was obtained as

$$E(Z) = E(X) + \frac{E(X(X-1))}{2(1-E(X))}$$

where $E(X) < 1$.

For each sequence of realization of dam contents, the following confidence intervals were calculated

$$\overline{Z}_i \pm t_{(\alpha/2),df=n}\sqrt{Var\left(\overline{Z_i}\right)}$$

where $i = 1, 2, \ldots, n+1$. From Table 6.3, we see that the expected success rates for finding $E(Z)$ in the confidence intervals are quite close to the actualizations from the different samples for $p = 0.7$. The implication is that the theoretical value of $E(Z)$ derived in Section (4.4) is quite accurate.

Table 6.3: Simulation Result for Expectation of Dam Content

p	0.7
E(X)	0.4286
Var(X)	0.6122
E(Z)	0.75
Confidence Interval	95%
Observed	92%

Chapter 7

Conclusion and Recommendations

7.1 Conclusion

Dams are constructed mainly for the purpose of creating a permanent reservoir of water for use at a later time. The water trapped in the reservoir can be put into various uses including irrigation, electricity generation, provision of potable water, recreational activities and so on. Dams are very useful, but their benefits must be weighed against the backdrop of costly consequences of events such as dam failures, flooding and other environmental/health hazards. In Alberta, the provincial government owns 200 out of the approximately 1400 dams located in the province (*Source:* Government of Alberta). The regulation of dam and canal safety is the responsibility of Alberta Environment (AENV). The activities of AENV include: setting standards of practice for dam safety, provision of technical review with regards to dam safety, and performing safety inspections for dams.

Dam content is required to be kept at an "optimum level" in such a way that the dam satisfies the purpose for which it was built. A dry dam is undesirable since the demands for water may not be satisfied in a timely manner. A dam which is always full creates a dangerous situation, since there may not be adequate storage space to catch unexpected incoming flood. Based on a rule of operation, the dam's operator must decide how much water to release at each time period in order to keep the dam running at an "optimal level". Since dam content is a direct consequence of reservoir inflow, there is a need to study the relationship between the distributions of these two quantities in order to determine an effective dam control strategy. Reservoir inflow is stochastic; hence the need to formulate dam control problems as stochastic, rather than deterministic problems.

Moran's work constitutes the fundamental framework on which the probability theory of dam control has been built over the years. In this thesis, a review of the probability theory of dam control has been given with a focus on Moran's pioneering work in this area of research. We considered a dam of finite capacity into which flows independent and identically distributed inputs. The first two moments of the stationary distribution of dam content have been derived from an earlier result obtained by Moran(1959). A theory was also developed for the expectation of dam content given that reservoir input is discrete and dam capacity is assumed to be very large.

Simulation studies were conducted to investigate the long term effects of reservoir capacity on dam overflow and emptiness, assuming exponentially distributed inflows. As expected, the higher the capacity of the dam, the lower the probabilities of emptiness and overflow. Simulation studies were also conducted to verify the theoretical expectation of dam content, which was found to be approximately reliable given geometric distributed inflows.

As we have only considered a single dam in our work, it will be interesting to investigate the distribution of dam content for dams in series. For instance, one may consider two dams, 1 and 2, in series, so that the output and overflow from dam 1 flows into dam 2. From the literature review conducted, this is an open research area requiring extensive research work.

Another interesting research possibility would be to consider the case where we have a flood-way and a flood-plain below the dam, such as in Calgary. The flood-way is lower and would flood first when excessive water were released from the dam. The flood-plain would flood next, after which the whole community would be flooded, such as in Bowness in Calgary. When one realized that flooding was inevitable, I believe an optimal strategy would allow the early flooding of the flood-way and the houses therein, in order to try to avoid having to flood the flood-plain and the houses there. Similarly, the flooding of the flood-plain would be done early enough to try to avoid flooding the whole community. This would cause some political turmoil, but would be an interesting stochastic optimization problem.

Other operating rules apart from the ones explored in this thesis will also need to be considered, in order to investigate the versatility of the method proposed in Section (4.4). The operating rules presented in this thesis have been used for illustration purposes.

SAS© software (Version 9.0) was the software of choice in the simulation study. We have used SAS© because of the ease with which the needed random variables can be generated. Matrix manipulation is also a delightful exercise in the **IML** (Interactive Matrix Language) procedure found in SAS©. Codes for the simulation study can be found in the Appendix.

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Appendix A

SAS Codes for the Simulation Study

A.1 Probability of Emptiness

```
   /* z is the amount of water in the reservoir before an input x
   u is the sum of z and x
   w is the amount of overflow, if any
   y is the amount of water released
   uu is the amount of water remaining in the reservoir
   after possible overflow
*/


OPTIONS  LINESIZE = 120 ERRORS = 1  MPRINT ;
/*Defining some macros to be used throughout the simulation*/
%let n = 600 ;
%let k = 800;
%let M = 500;
%let Ave = 500;

/* generating exponential random numbers */
```

```
data rands ;
seed = today() ;
     do i = 1 to &n ;
rgamma  = ranexp(seed)* &Ave;
 output ;
     end ;
run ;
proc print data = rands ;
run;

title "Output from SAS random number generators ... &n observations."  ;
/* the IML procedure for matrices */
proc IML;
use rands;
read all VAR {rgamma} into x;
run;
print x;

/*initializing the various vectors to be used */
z = j(&n, 1, 0);
u = j(&n, 1, 0);
w = j(&n, 1, 0);
y = j(&n, 1, 0);
uu = j(&n, 1, 0);
```

```
ALL = j(&n,5,0);
ZEE = j(&n-100,1,0);

/*initial dam content and release rule*/
z[1] = 0;
do i = 1 to (&n+1);
u[i] = x[i] + z[i];
if u[i] > &k then w[i] = x[i] + z[i] - &k; else w[i] = 0;
if w[i] > 0 then uu[i] = &k; else uu[i] = u[i];
if uu[i] >= &M then y[i] = &M; else y[i] = uu[i];
z[i+1] = uu[i] - y[i];
end;

/*probability of emptiness */
count=0;
do i=102 to (&n+1);
if z[i] = 0 then count = count+1;
end;
print count;
p=count/(&n-100);
print p;

/*Confidence Interval for p */
ZZ=probit(.975);
```

```
q = 1 - p;
LP = p - ZZ*sqrt((p*q)/(&n-100));
UP = p + ZZ*sqrt((p*q)/(&n-100));
print LP UP;

/*Confidence Interval using score's test */
A = (2*p+((ZZ**2)/(&n-100)))**2;
B = 4*p*p*(1+((ZZ**2)/(&n-100)));
BB = A - B;
print A B;
C = sqrt(BB);
D = 2*(1+((ZZ**2)/(&n-100)));
E = C/D;
F = (2*p+((ZZ**2)/(&n-100)))/D;
LPP = F - E;
UPP = F + E;
print LPP UPP;

/*probability of overflow*/
count2=0;
do i=101 to &n;
if u[i] > &K then count2 = count2+1;
end;
print count2;
```

```
p2=count2/(&n-100);
print p2;

/*Confidence Interval for p2(overflow)*/
q2 = 1 - p2;
LP2 = p2 - ZZ*sqrt((p2*q2)/(&n-100));
UP2 = p2 + ZZ*sqrt((p2*q2)/(&n-100));
print LP2 UP2;

/*Confidence Interval for p2 using score's test */
A2 = (2*p2+((ZZ**2)/(&n-100)))**2;
B2 = 4*p2*p2*(1+((ZZ**2)/(&n-100)));
BB2 = A2 - B2;
print A2 B2;
C2 = sqrt(BB2);
D2 = 2*(1+((ZZ**2)/(&n-100)));
E2 = C2/D2;
F2 = (2*p2+((ZZ**2)/(&n-100)))/D2;
LPP2 = F2 - E2;
UPP2 = F2 + E2;
print LPP2 UPP2;

/*putting all realizations in a matrix*/
ALL[ , 1] = x;
```

```
ALL[ , 2] = w;
ALL[ , 3] = y;
ALL[ , 4] = u;
ALL[ , 5] = uu;
print x, z, w, y,u,uu,ALL;

create AllDam from ALL;
append from ALL;
run;
quit;

data AllDam2;
set AllDam;
rename COL1=X COL2=W COL3=Y COL4=U COL5=UU;
run;
quit;

proc print data=AllDam2;
run;

ods select BasicMeasures
BasicIntervals
RobustScale
Quantiles;
```

```
proc univariate data=AllDam2
CIBASIC
CIPCTLNORMAL
ROBUSTSCALE;
var x;
run;

ods select all;

proc univariate data=AllDam2;
histogram x w y u uu;
run;
```

A.2 Expectation of Dam Content

```
         /* z(t+1) is the amount of water in the reservoir after
         an input x, M is the release rate
*/


OPTIONS  LINESIZE = 120 ERRORS = 1  MPRINT ;
/*defining various macros to be used in the simulation*/
%let n = 10 ;
```

```
%let M = 1;
%let p = 0.7;
%let nsamples = 1000;

/* For confidence Interval*/
%let pp=0.975;

/*generate random samples of size &n /*
data rands;
seed = today() ;
  do nsample = 1 to &nsamples;
     do i = 1 to &n ;
rgamma  = INT((log(Ranuni(seed)))/(log(1-&p)));
 output ;
     end ;
end;
run ;
quit;

title "Output from SAS random number generators ... &n observations."  ;
proc IML;
use rands;
read all VAR {rgamma} into x;
run;
```

```
/* Defining the vectors needed */
XX = j(&n, &nsamples, 0);
z = j((&n+1), &nsamples, 0);
Z2 = j((&n+1), &nsamples, 0);
sumZ2 = j(1, &nsamples, 0);
sumZ = j(1, &nsamples, 0);
Z_bar = j(1, &nsamples, 0);
sumZSQ = j(1, &nsamples, 0);
sumZSQ2 = j(1, &nsamples, 0);
sumZSQ22 = j(1, &nsamples, 0);
Var_Z = j(1, &nsamples, 0);
SEZ = j(1, &nsamples, 0);
U_Z = j(1, &nsamples, 0);
L_Z = j(1, &nsamples, 0);
count22 = j(1, &nsamples, 0);

/*break down the long sequence of observations x
into various samples in a matrix*/
do i = 1 to &nsamples;
XX[1:&n,i] = x[(&n*i-(&n-1)):&n*i];
end;
run;
```

```
/* Initial content and release rule */
do j = 1 to &nsamples;
z[1,j] = 2;
do i = 1 to &n;
if z[i,j] < &M then z[i+1,j] = XX[i,j]  ;
else z[i+1,j] = XX[i,j] + z[i,j]-&M;
end;
end;

/* calculating the mean and variance of Z(dam content) */
do j = 1 to &nsamples;
sumZ2[j] = 0;
do i = 1 to (&n+1);
Z2[i,j] = z[i,j]*z[i,j];
sumZ2[j] = sumZ2[j] + Z2[i,j];
end;
end;
print sumZ2;

do j = 1 to &nsamples;
sumZ[j] = 0;
do i = 1 to (&n+1);
sumZ[j] = sumZ[j] + z[i,j];
end;
```

```
end;
print sumZ;

do j = 1 to &nsamples;
Z_bar[j] = sumZ[j]/(&n+1);
end;
print Z_bar;

do j = 1 to &nsamples;
sumZSQ[j] = sumz[j]*sumZ[j];
sumZSQ2[j] = sumzSQ[j]/(&n+1);
sumZSQ22[j] = sumZ2[j] - sumZSQ2[j];
end;

do j = 1 to &nsamples;
Var_Z[j] = (sumZSQ22[j])/(&n);
SEZ[j] = SQRT(Var_Z[j]);
end;
print  SEZ Var_Z;

/* Confidence Interval for Z */
t = TINV(&pp,&n,0); /*quantiles from the t-distribution*/
do j = 1 to &nsamples;
U_Z[j] = Z_bar[j]+t*SEZ[j]/SQRT(&n+1);
```

```
L_Z[j] = Z_bar[j]-t*SEZ[j]/SQRT(&n+1);
end;
print L_Z U_Z;

/* Theoretical Expectation and Variance of X */
E_X = (1-&p)/&p;
V_X = (1-&p)/(&p*&p);
print E_X V_X;

/*Calculating the Theorectical Expectation of Z */
E_Z = E_X+((V_X+(E_X*E_X)-E_X)/(2*(1-E_X)));
run;
print E_Z;
run;

/*count the number of samples with E_Z in their confidence intervals*/
count33 = 0;
do j = 1 to &nsamples;
if E_Z >= L_Z[j] & E_Z <= U_Z[j] then count33 = count33+1;
end;
print count33;
run;

create Damcontent from z;
```

```
append from z;

run;

quit;


proc univariate data=Damcontent

CIBASIC

CIPCTLNORMAL

ROBUSTSCALE;

run;
```

Appendix B

Histograms of Dam Content

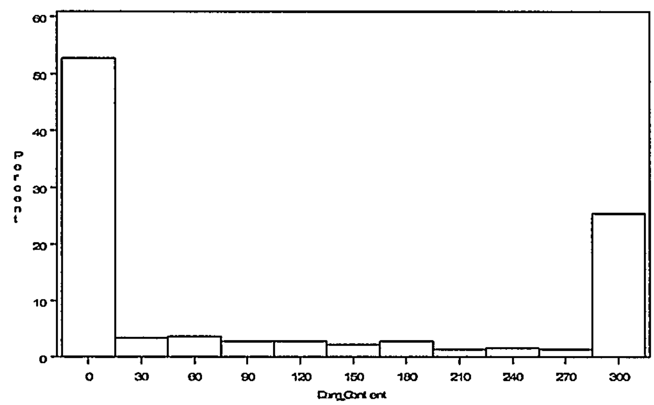

Figure B.1: 500 Observations

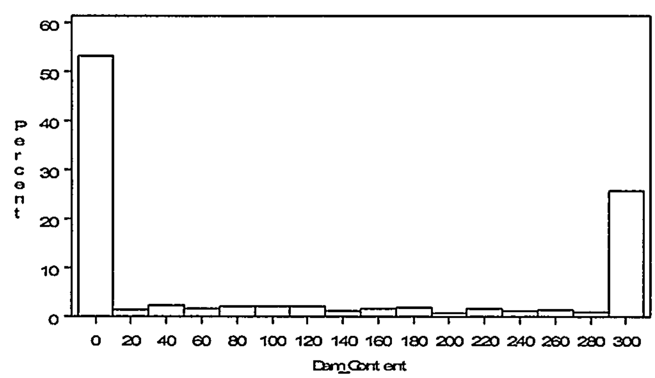

Figure B.2: 1000 Observations

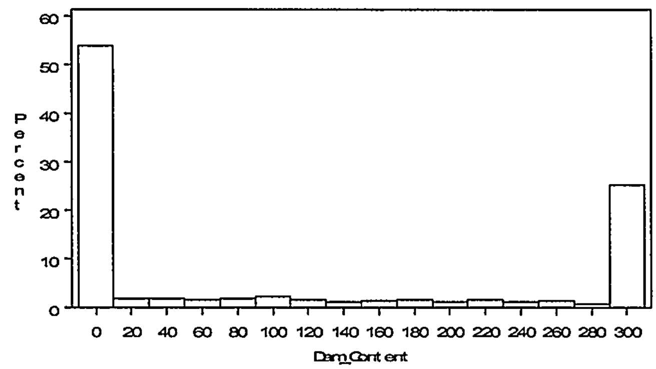

Figure B.3: 1500 Observations

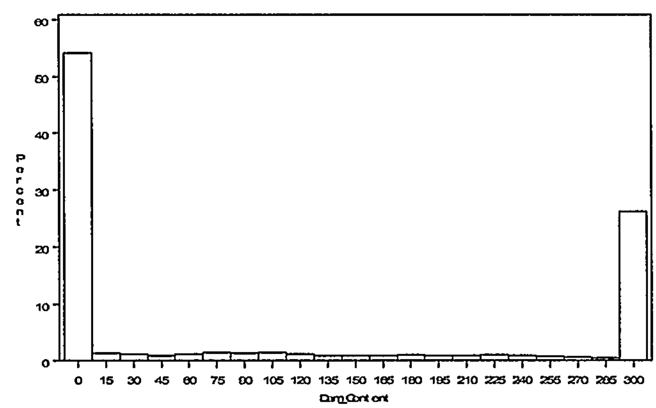

Figure B.4: 2000 Observations