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Presentations of Iwahori-Hecke algebras of p-adic GL(2)

de Waal, André

UNIVERSITY OF CALGARY

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by

André de Waal

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Abstract

In this thesis we define the Iwahori-Hecke algebra for the algebraic group GL_2 over a p-adic field. Besides the definition there are two common presentations used to describe the Iwahori-Hecke algebras in terms of generators and relations. We introduce two more presentations to provide some insight into the structure of the algebra, as well as proving that each of these presentations is isomorphic to the others, mostly through explicit isomorphisms.

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Table of Contents

Chapter 1

Introduction

The Iwahori-Hecke algebra of an algebraic group is useful object of study in representation theory. In [2], Chriss and Khuri-Makdisi review that the category of smooth representations of a split connected reductive algebraic group over a p-adic field can be decomposed into a disjoint union of what are known as Bernstein blocks. They go on to show that the block corresponding to the unramified principal series representation of that algebraic group is isomorphic to the cateogry of modules over the Iwahori-hecke algebra of that same group.

The Iwahori-Hecke algebra for an algebraic group is defined as a convolution algebra over $\mathbb{C}$. Since we care about its module category, it is very helpful to have a description of this algebra in terms of generators and relations. One common presentation of this algebra is given by a list of generators in the form of characteristic functions and relations between them. For an infinite group, the lists of generators and relations in this presentation are generally infinite. Another common presentation of the Iwahori-Hecke algebra for an algebraic group is actually a specialisation of what is usually referred to as the Bernstein-Lusztig presentation of the affine Hecke algebra. In this thesis we will focus on a particular algebraic group, GL_2 over a p-adic field F, and show what this definition and the two previously described presentations look like in this case. Limiting our view to this example, we can be more detailed in how these three presentations interact with each other and give explicit isomorphisms. Furthermore, this allows us to introduce two new presentations as well. The first new presentation introduced is given by finite subsets of generators and relations seen in one of the common presentations, which demonstrates that in the case of GL_2, the Iwahori-Hecke algebra is finitely presented. The other new presentation introduced here is also a finite presentation of the Iwahori-Hecke algebra, but one which allows us to more easily witness the links between the more well-known ones. The hope of this thesis is to build upon already studied results and drawing more intuitively available and precise conclusions for a particular example.

Chapter 2

Algebraic Group GL_2 and its Measure

2.1 The Algebraic Group GL_2 and its (co)root data

In this section we will describe the root datum for the algebraic group GL_2 of two by two invertible matrices over a p-adic field F. The root datum of an algebraic group consists of a character lattice, root system, cocharacter lattice, and coroot system. A more in-depth treatment of these constructions is given by Murnaghan in Section 5 of [8] describing abstract root data and the root datum for an algebraic group.

We begin with a p-adic field F, and for the sake of simplicity, denote by GL_2 the algebraic group GL_2/F over F. The Iwahori-Hecke algebra of GL_2 is tied to its Weyl group, so we will need to discuss roots of GL_2. In order to do this, we first begin by selecting a maximal torus of this algebraic group. Take T to be the algebraic group defined so that for any extension E/F,

$$T(E) = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mid t_1, t_2 \in E^* \right\}.$$

We note that T is in fact a split maximal torus for GL_2, since T is isomorphic to $\mathbb{G}_m^2$ as an algebraic group. As in Murnaghan we can define the character lattice $X^*(T) = \mathrm{AlgGrp}_{/\bar{F}}(T, \mathrm{GL}_1)$. We define two characters $e_1, e_2 \in X^*(T)$ by

$$\begin{aligned} e_i : T(E) &\to \mathrm{GL}_1(E) \\ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} &\mapsto t_i \end{aligned}$$

for any extension E/F and for each of $i = 1, 2$. Since $X^*(T(E))$ consists of only group homomorphisms, we get that it is generated by e_1 and e_2, so $X^*(T) \cong \mathbb{Z}\, e_1 \oplus \mathbb{Z}\, e_2$. Now that we have a character lattice, we can define the corresponding root system.

Definition 2.1.1. [8] Let G be an algebraic group over a field F and let T be a maximal torus of G. A character $\alpha \in X^*(T)$ is a *root* of G if and only if for any extension E/F there exists some nonzero $X \in \mathrm{Lie}\, G(E)$ such that

$$\mathrm{Ad}(t)(X) = \alpha(t)(X)$$

in $\mathrm{Lie}\, G$ for all $t \in T(E)$.

We want to find roots of our algebraic group GL_2. Note $\mathrm{Lie}\, \mathrm{GL}_2 = \mathfrak{gl}_2$. Let E be a field extension of F and suppose $\alpha \in X^*(T)$. Suppose

$$t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

is in $T(E)$. Next we suppose that α is a root of GL_2. Then for some nonzero element of $\mathfrak{gl}_2$

$$X = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

we have

$$\mathrm{Ad}(t)(X) = \alpha(t)(X).$$

Then we have

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} \frac{1}{t_1} & 0 \\ 0 & \frac{1}{t_2} \end{pmatrix} = \begin{pmatrix} \alpha(t)w & \alpha(t)x \\ \alpha(t)y & \alpha(t)z \end{pmatrix}.$$

Computing the multiplication, this is true if and only if

$$\begin{pmatrix} w & \frac{t_1}{t_2}x \\ \frac{t_2}{t_1}y & z \end{pmatrix} = \begin{pmatrix} \alpha(t)w & \alpha(t)x \\ \alpha(t)y & \alpha(t)z \end{pmatrix}.$$

Notice that if $w \neq 0$ or $z \neq 0$, then we must have $\alpha(t) = 1$. However, $\alpha(t) = t_1^{a_1} t_2^{a_2}$ for some integers $a_1, a_2 \in \mathbb{Z}$ since α is a group homomorphism. So if $t_1 \neq t_2$, $\alpha(t) \neq 1$. This contradicts that the above equality holds for all $t \in T$, therefore we must have $w = z = 0$. Next, since we required that X is nonzero, we have $\alpha(t) = \frac{t_1}{t_2}$ or $\alpha(t) = \frac{t_2}{t_1}$. That is, $\alpha = \pm(e_1 - e_2)$. We call $\alpha_1 = e_1 - e_2$ a simple root, and we get the root system $\Phi = \{\alpha_1, -\alpha_1\}$.

This naturally leads us into one description of the Weyl group W for $\mathrm{GL}_2(F)$, the construction of which is again defined by Murnaghan. Through the isomorphism $X^*(T(F)) \cong \mathbb{Z}\, e_1 \oplus \mathbb{Z}\, e_2$, we can define a reflection s_{α_1} by $s_{\alpha_1}(e_1) = e_2$ and $s_{\alpha_1}(e_2) = e_1$. We call this reflection s_{α_1} because it is the group homomorphism with the property $s_{\alpha_1}(\alpha_1) = -\alpha_1$. We can therefore restrict this reflection to the root system Φ. Taking S to be the set of all reflections that we are able to define on Φ, in this case we have $S = \{s_{\alpha_1}\}$. Now we can define the Weyl group as the group generated by reflections of the root system, and therefore the group generated by S, so we get $W = \{\mathrm{id}, s_{\alpha_1}\}$.

Murnaghan also details a dual notion of characters and roots, leading to the cocharacter and coroot lattice, which we will explore here in the context of GL_2. The cocharacter lattice is defined to be $X_*(T) = \mathrm{AlgGrp}_{/\bar{F}}(\mathrm{GL}_1, T)$. We find two cocharacters $f_1, f_2 \in X_*(T)$ given by

$$f_1(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$
$$f_2(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

for all $z \in \mathrm{GL}_1(E)$. Similarly to the character lattice described above, since $X^*(T(E))$ consists of only group homomorphisms, it is generated by f_1 and

f_2, so $X_*(T) \cong \mathbb{Z}\, f_1 \oplus \mathbb{Z}\, f_2$. Then a perfect pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ is determined by

$$\langle e_i, f_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

From this coroots can be defined. For each root $\alpha \in \Phi$, there exists a unique $\alpha^\vee \in X_*(T)$ such that $\langle \beta, \alpha^\vee \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ for all $\beta \in \Phi$ where $(\cdot, \cdot)$ is standard Euclidean inner product on $\langle \Phi \rangle \otimes_{\mathbb{Z}} \mathbb{R}$. In particular for GL_2, for $\alpha_1 = e_1 - e_2$ we can write $\alpha_1^\vee = f_1 - f_2$. Then we get the coroot system $\Phi^\vee = \{\alpha_1^\vee, -\alpha_1^\vee\}$.

2.2 Haar Measure on $\mathrm{GL}_2(F)$

From this point forward we will denote by G the group $GL_2(F)$. Next we want to define a measure on G .

First viewing F as a group under addition, we define the Haar measure on F. We begin by choosing a neighbourhood basis of 0 in F. Denote by q the size of the residue field of F, $q = |\,\mathcal{O}_F\,/\,\mathfrak{p}\,|$, and let ϖ be a uniformizer of the unique maximal ideal $\mathfrak{p}$ of $\mathcal{O}_F$. Since F is a p-adic field, an open ball of radius $r \in \mathbb{R}^{\geq 0}$ is an open ball of radius q^n for some $n \in \mathbb{N}$ without loss of generality. In particular, an open ball of radius q^n about 0 is given by

$$\begin{aligned} B_{q^n}(0) &= \{x \in F \mid \|x - 0\| < q^n\} \\ &= \{x \in F \mid \|x\| \leq q^{n-1}\} \\ &= \{x \in F \mid q^{1-n}\|x\| \leq 1\} \\ &= \{x \in F \mid \|\varpi^{n-1}\|\|x\| \leq 1\} \\ &= \{x \in F \mid \|\varpi^{n-1}x\| \leq 1\} \\ &= \{x \in F \mid \varpi^{n-1}x \in \mathcal{O}_F\} \\ &= \mathfrak{p}^{n-1}\,. \end{aligned}$$

This means that for any neighbourhood U of 0, there exists some n small enough so that $\mathfrak{p}^n \subseteq U$. In other words, the set

$$\{\mathfrak{p}^n \mid n \in \mathbb{N}\}$$

is a neighbourhood basis for 0 in F. Using this information, we can define a Haar measure μ_F on F. It is enough to define $\mu_F(\mathfrak{p}^n)$ for each element $\mathfrak{p}^n$ of the neighbourhood basis, and then extend this measure to all of F via translation since we are requiring that μ_F be a Haar measure. With this we define $\mu_F(\mathfrak{p}^n) = \mu_F(a + \mathfrak{p}^n) = q^{-n}$ for each $n \in \mathbb{N}$ and any $a \in F$. Note that since F is an abelian group, it is unimodular, and so μ_F is both a left and right Haar measure. We can extend this to the product measure μ_{F^4} on F^4.

Now that we have defined a measure μ_{F^4} on F^4, we modify it to create a measure μ on G using μ_{F^4}. First we consider the bijection

$$M_{2\times 2}(F) \longrightarrow F^4$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

where $M_{2\times 2}(F)$ is the set of all 2×2 matrices with entries in F. With this identification in mind, we can define the measure on $M_{2\times 2}(F)$ such that this bijection is measure-preserving. Then G is a sub-measure space of $M_{2\times 2}(F)$, and we denote the measure on G by μ.

Lemma 2.2.1. *The measure μ described above is a left Haar measure on G.*

Proof. To prove that μ is a left Haar measure, we must show that it is invariant under left translation by G. That is, for any $A \subseteq G$ and $h \in G$, we want to show that $\mu(A) = \mu(hA)$. Consider an arbitrary element of G given by

$$g = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

Then for any other element of G,

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have left multiplication of g by h given by the bijection

$$\begin{aligned} T : G &\longrightarrow G \\ \begin{pmatrix} w & x \\ y & z \end{pmatrix} &\longmapsto \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}. \end{aligned}$$

Viewing T as the linear map

$$\begin{aligned} T : \quad F^4 &\longrightarrow F^4 \\ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} &\longmapsto \begin{pmatrix} aw + by \\ ax + bz \\ cw + dy \\ cx + dz \end{pmatrix} \end{aligned}$$

in F^4 we can compute Jacobian matrix to be

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

So computing the absolute determinant of the Jacobian gives us $|ad - bc|^2 = \det(h)^2$, where the absolute value function $|\cdot|$ is the valuation on F. Now let $A \subseteq G$. In order to show that μ is a left Haar measure, it is enough to show that $\mu(hA) = \mu(A)$ for the arbitrary element $h \in G$ from above. Using the same notation for entries of $h \in G$ as above, we have

$$\begin{aligned}
\mu(hA) &= \int_{hA} \frac{d\mu_{F^4}(hg)}{|\det(hg)|^2} \\
&= \int_A \frac{d\mu_{F^4}(g)}{|\det(hg)|^2} |ad - bc|^2 \\
&= \int_A \frac{d\mu_{F^4}(g)}{|\det(h)|^2 |\det(g)|^2} \det(h)^2 \\
&= \int_A \frac{d\mu_{F^4}(g)}{|\det(g)|^2} \\
&= \mu(A).
\end{aligned}$$

This shows us that μ is invariant under left translation, and is therefore a left Haar measure on G. □

Lemma 2.2.2. *The measure μ described above is a right Haar measure on G.*

Proof. To prove that μ is a right Haar measure, we must show that it is invariant under right translation by G. That is, for any $A \subseteq G$ and $h \in G$, we want to show that $\mu(A) = \mu(Ah)$. Consider an arbitrary element of G given by

$$g = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

Then for any other element of G,

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have right multiplication of g by h given by the bijection

$$\begin{aligned}
T : G &\longrightarrow G \\
\begin{pmatrix} w & x \\ y & z \end{pmatrix} &\longmapsto \begin{pmatrix} aw + cx & bw + dx \\ ay + cz & by + dz \end{pmatrix}.
\end{aligned}$$

We can compute the Jacobian by viewing T as the linear map

$$T: \quad F^4 \longrightarrow F^4$$
$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} aw+cx \\ bw+dx \\ ay+cz \\ by+dz \end{pmatrix}$$

in F^4 to get the Jacobian matrix

$$\begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}.$$

So computing the absolute determinant of the Jacobian gives us $|ad-bc|^2 = \det(h)^2$. Now let $A \subseteq G$. In order to show that μ is a right Haar measure, it is enough to show that $\mu(Ah) = \mu(A)$ for the arbitrary element $h \in G$ from above. Using the same notation for entries of $h \in G$ as above, we have

$$\begin{aligned} \mu(Ah) &= \int_{Ah} \frac{d\mu_{F^4}(gh)}{|\det(gh)|^2} \\ &= \int_A \frac{d\mu_{F^4}(g)}{|\det(gh)|^2} |ad-bc|^2 \\ &= \int_A \frac{d\mu_{F^4}(g)}{|\det(g)|^2 |\det(h)|^2} \det(h)^2 \\ &= \int_A \frac{d\mu_{F^4}(g)}{|\det(g)|^2} \\ &= \mu(A). \end{aligned}$$

This shows us that μ is invariant under right translation, and is therefore a left Haar measure on G. □

The previous two lemmas show us that μ is both a left and a right Haar measure for G, which is therefore a unimodular group.

We define the Iwahori subgroup of G to be

$$I = \left\{ \begin{pmatrix} a & b \\ c\varpi & d \end{pmatrix} \mid a, d \in \mathcal{O}_F^*, \quad b, c \in \mathcal{O}_F \right\}$$

and normalise the measure μ such that $\mu(I) = 1$. This μ on G will implicitly be the measure we use going forward.

Chapter 3

Decomposition of Triple Cosets

In order to define and describe the Iwahori-Hecke algebras, we need to introduce some more definitions and results.

We denote by $X^\vee$ the matrix group

$$\left\{ \begin{pmatrix} \varpi^{a_2} & 0 \\ 0 & \varpi^{a_1} \end{pmatrix} \mid a_1, a_2 \in \mathbb{Z} \right\}$$

with the operation of matrix multiplication. This is equivalent to working with the F-points of the cocharacter lattice of GL_2, $X_*(F)$, via the isomorphism

$$\begin{aligned} X_*(F) &\longrightarrow X^\vee \\ a_1 f_1 + a_2 f_2 &\longmapsto \begin{pmatrix} \varpi^{a_2} & 0 \\ 0 & \varpi^{a_1} \end{pmatrix} \end{aligned}$$

for any $a_1, a_2 \in \mathbb{Z}$.

The Weyl group for G is generated by the reflections corresponding to each root as described in the background. This can be described as the group

$W = \{\mathrm{id}, \sigma\}$ where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so W can be viewed as acting on the cocharacter lattice $X^\vee$ as expressed above via conjugation. In this way we see that σ is exactly the reflection s_{α_1} acting by $\sigma \cdot \alpha_1^\vee = -\alpha_1^\vee$ and $\sigma \cdot -\alpha_1^\vee = \alpha_1^\vee$ for the coroot $\alpha_1^\vee$.

Next we introduce the affine Weyl group W_{af}. First define the matrices

$$\rho = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix},$$

which normalizes I, and

$$\begin{aligned} \tau &= \rho\sigma\rho^{-1} \\ &= \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{pmatrix}. \end{aligned}$$

Then W_{af} can be defined as the group generated by $\tilde{S} = \{\sigma, \tau\}$. We can refer to elements of this group as words with letters σ and τ.

With W_{af} defined by these generators, we can introduce a notion of length on W_{af} as well. In order to define length of a word $w \in W_{\mathrm{af}}$, we begin by writing w as a reduced word in letters from $\tilde{S}$. That is, we write w as the product of elements σ and τ using the fewest letters possible. Since $\sigma^2 = \tau^2 = \mathrm{id}$, the reduced expression of any element of W_{af} is an alternating product of σ and τ. Then we say the length of w is the total number of letters in the reduced word expression.

Next we define a group which we call the *extended affine Weyl group* $\tilde{W}$ by

$$\tilde{W} \cong W_{\mathrm{af}} \rtimes \Omega$$

where $\Omega = \langle \rho \rangle$, and elements of Ω act on W_{af} via conjugation. A typical element of $\tilde{W}$ can be written as $w\rho^k$ where $w \in W_{\mathrm{af}}$ and $k \in \mathbb{Z}$, and the product of two arbitrary elements is given by

$$(w_1\rho^{k_1})(w_2\rho^{k_2}) = (w_1(\rho^{k_1}w_2\rho^{-k_1}))\rho^{k_1}\rho^{k_2} = w_1(\rho^{k_1}w_2\rho^{-k_1})\rho^{k_1+k_2}.$$

In order to work with this group in terms of matrix multiplication, we define the matrix group

$$\tilde{\mathrm{W}}_{\mathrm{Mat}} = \left\{ \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \mid a, b \in \mathbb{Z} \right\} \cup \left\{ \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$$

and show that this is isomorphic to the extended affine Weyl group.

Proposition 3.0.1. *For $\tilde{W}$ and $\tilde{\mathrm{W}}_{Mat}$ as defined above, we have $\tilde{W} \cong \tilde{\mathrm{W}}_{Mat}$.*

Proof. Recall our definition of the extended affine Weyl group, $\tilde{W} = W_{\mathrm{af}} \rtimes \Omega$. In other words, $\tilde{W} \cong \langle \sigma, \tau \rangle \rtimes \langle \rho \rangle$. By our definitions of σ, τ, and ρ as matrices, we can define inclusions

$$\begin{aligned} W_{\mathrm{af}} &\hookrightarrow \tilde{\mathrm{W}}_{\mathrm{Mat}} \\ \sigma &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \tau &\longmapsto \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \Omega &\hookrightarrow \tilde{\mathrm{W}}_{\mathrm{Mat}} \\ \rho &\longmapsto \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}. \end{aligned}$$

This gives us an inclusion

$$\iota : W_{\mathrm{af}} \rtimes \Omega \hookrightarrow \tilde{\mathrm{W}}_{\mathrm{Mat}}\,.$$

We show that this inclusion respects the multiplication on $W_{\mathrm{af}} \rtimes \Omega$ we defined above. Let $w_1\rho^{k_1}, w_2\rho^{k_2} \in W_{\mathrm{af}} \rtimes \Omega$. So we have that in $\tilde{\mathrm{W}}_{\mathrm{Mat}}$,

$$\begin{aligned} \iota(w_1\rho^{k_1})\iota(w_2\rho^{k_2}) &= \iota(w_1\rho^{k_1}w_2\rho^{k_2}) \\ &= \iota(w_1(\rho^{k_1}w_2\rho^{-k_1})\rho^{k_1}\rho^{k_2}) \\ &= \iota(w_1(\rho^{k_1}w_2\rho^{-k_1})\rho^{k_1+k_2}) \\ &= \iota((w_1\rho^{k_1})(w_2\rho^{k_2})). \end{aligned}$$

It remains to show that the inclusion ι is a surjection, and therefore a bijection. We let $M \in \tilde{\mathrm{W}}_{\mathrm{Mat}}$ and show that there exists $w \in \tilde{W}$ such that $\iota(w) = M$. There are four cases to consider.

(i) Let $M = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \in \tilde{\mathrm{W}}_{\mathrm{Mat}}$ such that $a+b$ is even. Now choose $w = (\sigma\tau)^{\frac{a-b}{2}} \rho^{a+b} \in \tilde{W}$. Then

$$\begin{aligned} \iota(w) &= \iota((\sigma\tau)^{\frac{a-b}{2}} \rho^{a+b}) \\ &= \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{pmatrix} \right)^{\frac{a-b}{2}} \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}^{a+b} \\ &= \begin{pmatrix} \varpi^{\frac{a-b}{2}} & 0 \\ 0 & \varpi^{\frac{b-a}{2}} \end{pmatrix} \begin{pmatrix} \varpi^{\frac{a+b}{2}} & 0 \\ 0 & \frac{a+b}{2} \end{pmatrix} \\ &= \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \end{aligned}$$

so there exists $w \in \tilde{W}$ such that $\iota(w) = M$.

(ii) Let $M = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \in \tilde{\mathrm{W}}_{\mathrm{Mat}}$ such that $a+b$ is odd. Now choose $w = (\sigma\tau)^{\frac{a-b-1}{2}} \sigma\rho^{a+b} \in \tilde{W}$. Then

$$\begin{aligned} \iota(w) &= \iota((\sigma\tau)^{\frac{a-b-1}{2}} \sigma\rho^{a+b}) \\ &= \begin{pmatrix} \varpi^{\frac{a-b-1}{2}} & 0 \\ 0 & \varpi^{\frac{b-a+1}{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \varpi^{\frac{a+b-1}{2}} \\ \varpi^{\frac{a+b-1}{2}+1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varpi^{\frac{a-b-1}{2}} & 0 \\ 0 & \varpi^{\frac{b-a+1}{2}} \end{pmatrix} \begin{pmatrix} \varpi^{\frac{a+b-1}{2}+1} & 0 \\ 0 & \varpi^{\frac{a+b-1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \end{aligned}$$

so there exists $w \in \tilde{W}$ such that $\iota(w) = M$.

(iii) Let $M = \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix} \in \tilde{\mathrm{W}}_{\mathrm{Mat}}$ such that $a+b$ is even. Now choose $w = (\sigma\tau)^{\frac{a-b}{2}}\sigma\rho^{a+b} \in \tilde{W}$. Then

$$\begin{aligned}
\iota(w) &= \iota((\sigma\tau)^{\frac{a-b}{2}}\sigma\rho^{a+b}) \\
&= \begin{pmatrix} \varpi^{\frac{a-b}{2}} & 0 \\ 0 & \varpi^{\frac{b-a}{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^{\frac{a+b}{2}} & 0 \\ 0 & \varpi^{\frac{a+b}{2}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \varpi^{\frac{a-b}{2}} \\ \varpi^{\frac{b-a}{2}} & 0 \end{pmatrix} \begin{pmatrix} \varpi^{\frac{a+b}{2}} & 0 \\ 0 & \varpi^{\frac{a+b}{2}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix}
\end{aligned}$$

so there exists $w \in \tilde{W}$ such that $\iota(w) = M$.

(iv) Finally, let $M = \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix} \in \tilde{\mathrm{W}}_{\mathrm{Mat}}$ such that $a+b$ is odd. Now choose $w = (\sigma\tau)^{\frac{a-b+1}{2}}\rho^{a+b} \in \tilde{W}$. Then

$$\begin{aligned}
\iota(w) &= \iota((\sigma\tau)^{\frac{a-b+1}{2}}\rho^{a+b}) \\
&= \begin{pmatrix} \varpi^{\frac{a-b+1}{2}} & 0 \\ 0 & \varpi^{\frac{b-a-1}{2}} \end{pmatrix} \begin{pmatrix} 0 & \varpi^{\frac{a+b-1}{2}} \\ \varpi^{\frac{a+b-1}{2}+1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix}
\end{aligned}$$

so there exists $w \in \tilde{W}$ such that $\iota(w) = M$.

With all four cases covered, we conclude that ι is a bijection which preserves multiplication, and therefore an isomorphism. This concludes the proof that $\tilde{W} \cong \tilde{\mathrm{W}}_{\mathrm{Mat}}$. □

So every element $w \in \tilde{W}$ can be written as $w = w_0\rho^k$ where w_0 is a word in W_{af} and k is an integer. Through this isomorphism, the length on $\tilde{W}$ is given by $\ell(w_0\rho^k) = \ell(w_0)$ where the right-hand side refers to the length function on W_{af}.

We note two final lemmas which will help us decompose 2×2 matrices in a way that will be helpful to us to write in terms of σ, τ, ρ, and elements of I.

Lemma 3.0.2. *Consider an arbitrary matrix* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(F)$ *with determinant written as* det. *We can decompose this matrix in the following ways.*

I. Suppose α is invertible. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\gamma}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \frac{\det}{\alpha} \end{pmatrix}.$$

II. Suppose β is invertible. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ \delta & -\frac{\det}{\beta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\alpha}{\beta} & 1 \end{pmatrix}.$$

III. Suppose γ is invertible. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \frac{\alpha}{\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ 0 & -\frac{\det}{\gamma} \end{pmatrix}.$$

IV. Suppose δ is invertible. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{\det}{\delta} & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\gamma}{\delta} & 1 \end{pmatrix}.$$

Lemma 3.0.3. *For G, I, σ, τ, and ρ as defined above, we have the following identities:*

$$\rho\sigma\rho^{-1} = \tau, \qquad \rho\tau\rho^{-1} = \sigma, \qquad I\rho = \rho I.$$

These lemmas can be verified through straightforward matrix multiplication.

We now state a case the $\mathrm{GL}_2(F)$ case of a more general proposition given by Robert Lansky in [5] section 3 relating triple I-cosets to double I-cosets using the infrastructure we have set up above.

Proposition 3.0.4. *Let w, w' be arbitrary elements of the extended affine Weyl group $\tilde{W}$ of G. Then*

(i) For all $s \in \tilde{S}$

a) $IsIwI = IswI$ if $\ell(sw) > \ell(w)$,

b) $IsIwI = IswI \cup IwI$ if $\ell(sw) < \ell(w)$.

(ii) If $\ell(ww') = \ell(w) + \ell(w')$ then

$$IwIw'I = Iww'I.$$

In particular, if $s_1, \cdots, s_d \in \tilde{S}$, $\omega \in \Omega$, and $w = s_1 \cdots s_d \, \omega$ is a reduced expression, then

$$Is_1Is_2I \cdots Is_dI\omega I = IwI.$$

Proof. Let $w, w' \in \tilde{W}$. Let

$$g = \begin{pmatrix} a & b \\ c\varpi & d \end{pmatrix}$$

be an arbitrary element of I. That is $a, d \in \mathcal{O}_F^*$, and $b, c \in \mathcal{O}$. We split this into cases, and heavily rely on lemma 3.0.2 for decomposing matrices in our computations.

(i) Let $s \in \tilde{S}$.

a) Suppose $\ell(sw) > \ell(w)$. We know that $IswI \subseteq IsIwI$ because I contains the identity. In order to show that $IsIwI = IswI$, it is sufficient to show that $sIw \subseteq IswI$. We have 4 cases to consider.

(1) Let $s = \sigma$ and $w = (\tau\sigma)^k$ where $k \in \mathbb{N}$. Then for our arbitrary element $g \in I$,

$$\begin{aligned}
&sgw \\
&= \sigma g (\tau\sigma)^k \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c\varpi & d \end{pmatrix} \begin{pmatrix} \varpi^{-k} & 0 \\ 0 & \varpi^k \end{pmatrix} \\
&= \begin{pmatrix} c\varpi^{-k+1} & d\varpi^k \\ a\varpi^{-k} & b\varpi^k \end{pmatrix} \\
&= \begin{pmatrix} 1 & a^{-1}c\varpi \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} a\varpi^{-k} & b\varpi^k \\ 0 & (d - a^{-1}bc\varpi)\varpi^k \end{pmatrix} \quad 3.0.2(III) \\
&= \begin{pmatrix} 1 & a^{-1}c\varpi \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} \varpi^{-k} & 0 \\ 0 & \varpi^k \end{pmatrix} \begin{pmatrix} a & b\varpi^{2k} \\ 0 & d - a^{-1}bc\varpi \end{pmatrix} \\
&= \begin{pmatrix} 1 & a^{-1}c\varpi \\ 0 & 1 \end{pmatrix} \sigma(\tau\sigma)^k \begin{pmatrix} a & b\varpi^{2k} \\ 0 & d - a^{-1}bc\varpi \end{pmatrix} \\
&\in I\sigma(\tau\sigma)^k I = IswI.
\end{aligned}$$

Therefore since g was arbitrary, we have

$$sIw \subseteq IswI.$$

We conclude that

$$IsIwI = IswI.$$

(2) Let $s = \sigma$ and $w = (\tau\sigma)^k\tau$ where $k \in \mathbb{N}$. Then for our arbitrary

$g \in I$,

$$\begin{aligned}
sgw &= \sigma g(\tau\sigma)^k\tau \\
&= \begin{pmatrix} d\varpi^{k+1} & c\varpi^{-k} \\ b\varpi^{k+1} & a\varpi^{-k-1} \end{pmatrix} \\
&= \begin{pmatrix} (d-a^{-1}bc\varpi)\varpi^{k+1} & c\varpi^{-k} \\ 0 & a\varpi^{-k-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{-1}b\varpi^{2k+2} & 1 \end{pmatrix} \quad 3.0.2(IV) \\
&= \begin{pmatrix} d-a^{-1}bc\varpi & c\varpi \\ 0 & a \end{pmatrix} \begin{pmatrix} \varpi^{k+1} & 0 \\ 0 & \varpi^{-k-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{-1}b\varpi^{2k+2} & 1 \end{pmatrix} \\
&= \begin{pmatrix} d-a^{-1}bc\varpi & c\varpi \\ 0 & a \end{pmatrix} (\sigma\tau)^{k+1} \begin{pmatrix} 1 & 0 \\ a^{-1}b\varpi^{2k+2} & 1 \end{pmatrix} \\
&\in I\sigma(\tau\sigma)^k\tau I = IswI.
\end{aligned}$$

Since g was arbitrary, we have

$$sIw = IswI.$$

Therefore

$$IsIwI = IswI.$$

(3) Let $s = \tau$ and $w = (\sigma\tau)^k$ where $k \in \mathbb{N}$. Using the above result of part (1) as well as lemma 3.0.3, we have

$$\begin{aligned}
IsIwI &= I\tau I(\sigma\tau)^k I \\
&= I\rho\sigma\rho^{-1}I(\rho\tau\rho^{-1}\rho\sigma\rho^{-1})^k I \\
&= \rho I\sigma\rho^{-1}I\rho(\tau\sigma)^k\rho^{-1}I \\
&= \rho I\sigma I(\tau\sigma)^k I\rho^{-1} \\
&= \rho I\sigma(\tau\sigma)^k I\rho^{-1} && (1) \\
&= I\rho\sigma\rho^{-1}(\rho\tau\rho^{-1}\rho\sigma\rho^{-1})^k I \\
&= I\tau(\sigma\tau)^k I \\
&= IswI
\end{aligned}$$

as required.

(4) Let $s = \tau$ and $w = (\sigma\tau)^k\sigma$ where $k \in \mathbb{N}$. Using the above result of part (2) as well as lemma 3.0.3, we have

$$\begin{aligned} IsIwI &= I\tau I(\sigma\tau)^k\sigma I \\ &= I\rho\sigma\rho^{-1}I(\rho\tau\rho^{-1}\rho\sigma\rho^{-1})^k\rho\tau\rho^{-1}I \\ &= \rho I\sigma\rho^{-1}I\rho(\tau\sigma)^k\tau I\rho^{-1} \\ &= \rho I\sigma I(\tau\sigma)^k\tau I\rho^{-1} \\ &= \rho I\sigma(\tau\sigma)^k\tau I\rho^{-1} && (2) \\ &= I\rho\sigma\rho^{-1}(\rho\tau\rho^{-1}\rho\sigma\rho^{-1})^k\rho\tau\rho^{-1}I \\ &= I\tau(\sigma\tau)^k\sigma I \\ &= IswI \end{aligned}$$

as required.

b) Suppose $\ell(sw) < \ell(w)$. We know that $IswI \subseteq IsIwI$, again because I contains the identity. In order to show that $IsIwI = IswI \cup IwI$, we must show that $IwI \subseteq IsIwI$ and $IsIwI \subseteq IswI \cup IwI$. It is sufficient then to show $w \in IsIwI$ and $sIw \subseteq IswI \cup IwI$. We have 4 cases to consider.

(5) Let $s = \sigma$ and $w = (\sigma\tau)^k\sigma$ where $k \in \mathbb{N}$. First we show that $w \in IsIwI$, so we write

$$\begin{aligned} w &= (\sigma\tau)^k\sigma \\ &= \begin{pmatrix} 0 & \varpi^k \\ \varpi^{-k} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (\sigma\tau)^k\sigma \begin{pmatrix} 1 & \varpi^{2k} \\ 0 & -1 \end{pmatrix} \\ &\in I\sigma I(\sigma\tau)^k\sigma I = IsIwI, \end{aligned}$$

so $IwI \subseteq IsIwI$. It remains to show that $sIw \subseteq IswI \cup IwI$. Taking our arbitrary element $g \in I$ again,

$$sgw = \sigma g(\sigma\tau)^k\sigma = \begin{pmatrix} d\varpi^{-k} & c\varpi^{k+1} \\ b\varpi^{-k} & a\varpi^k \end{pmatrix}.$$

First suppose $b \in \mathcal{O}_F^*$. Then

$$\begin{aligned}
sgw &= \sigma g (\sigma\tau)^k \sigma \\
&= \begin{pmatrix} d\varpi^{-k} & c\varpi^{k+1} \\ b\varpi^{-k} & a\varpi^k \end{pmatrix} \\
&= \begin{pmatrix} 1 & b^{-1}d \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} b\varpi^{-k} & a\varpi^k \\ 0 & (c\varpi - ab^{-1}d)\varpi^k \end{pmatrix} \qquad 3.0.2(III) \\
&= \begin{pmatrix} 1 & b^{-1}d \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} \varpi^{-k} & 0 \\ 0 & \varpi^k \end{pmatrix} \begin{pmatrix} b & a\varpi^{2k} \\ 0 & c\varpi - ab^{-1}d \end{pmatrix} \\
&= \begin{pmatrix} 1 & b^{-1}d \\ 0 & 1 \end{pmatrix} (\sigma\tau)^k \sigma \begin{pmatrix} b & a\varpi^{2k} \\ 0 & c\varpi - ab^{-1}d \end{pmatrix} \\
&\in I(\sigma\tau)^k \sigma I = IwI
\end{aligned}$$

Conversely, suppose $b \in \mathfrak{p}$. Then

$$\begin{aligned}
sgw &= \sigma g (\sigma\tau)^k \sigma \\
&= \begin{pmatrix} d\varpi^{-k} & c\varpi^{k+1} \\ b\varpi^{-k} & a\varpi^k \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ bd^{-1} & 1 \end{pmatrix} \begin{pmatrix} d\varpi^{-k} & c\varpi^{k+1} \\ 0 & (a - bcd^{-1}\varpi)\varpi^k \end{pmatrix} \qquad 3.0.2(I) \\
&= \begin{pmatrix} 1 & 0 \\ bd^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varpi^{-k} & 0 \\ 0 & \varpi^k \end{pmatrix} \begin{pmatrix} d & c\varpi^{2k+1} \\ 0 & a - bcd^{-1}\varpi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ bd^{-1} & 1 \end{pmatrix} (\tau\sigma)^k \begin{pmatrix} d & c\varpi^{2k+1} \\ 0 & a - bcd^{-1}\varpi \end{pmatrix} \\
&\in I(\tau\sigma)^k I = IswI.
\end{aligned}$$

This accounts for all possible values of b, therefore $sgw \in IswI \cup IwI$. Since g was arbitrary we have $sIw \subseteq IswI \cup IwI$. Then $IsIwI \subseteq IswI \cup IwI$.

Putting this information together, we conclude that $IsIwI = IswI \cup IwI$.

(6) Let $s = \tau$ and $w = (\tau\sigma)^k$ where $k \in \mathbb{N}$. First we show that $w \in IsIwI$, so we write

$$\begin{aligned}
w &= (\tau\sigma)^k \\
&= \begin{pmatrix} \varpi^{-k} & 0 \\ 0 & \varpi^k \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -\varpi & 1 \end{pmatrix} \tau \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix} (\tau\sigma)^k \begin{pmatrix} 1 & \varpi^{2k-1} \\ 0 & -1 \end{pmatrix} \\
&\in I\tau I(\tau\sigma)^k I = IsIwI,
\end{aligned}$$

so $IwI \subseteq IsIwI$. It remains to show that $IsIwI \subseteq IswI \cup IwI$. Taking our arbitrary element $g \in I$ again,

$$\tau g(\tau\sigma)^k = \begin{pmatrix} c\varpi^{-k} & d\varpi^{k-1} \\ a\varpi^{-k+1} & b\varpi^{k+1} \end{pmatrix}.$$

First suppose that $c \in \mathcal{O}_F^*$. Then

$$\begin{aligned}
sgw &= \tau g(\tau\sigma)^k \\
&= \begin{pmatrix} c\varpi^{-k} & d\varpi^{k-1} \\ a\varpi^{-k+1} & b\varpi^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ ac^{-1}\varpi & 1 \end{pmatrix} \begin{pmatrix} c\varpi^{-k} & d\varpi^{k-1} \\ 0 & (b\varpi - ac^{-1}d)\varpi^k \end{pmatrix} && 3.0.2(I) \\
&= \begin{pmatrix} 1 & 0 \\ ac^{-1}\varpi & 1 \end{pmatrix} \begin{pmatrix} \varpi^{-k} & 0 \\ 0 & \varpi^k \end{pmatrix} \begin{pmatrix} c & d\varpi^{2k-1} \\ 0 & b\varpi - ac^{-1}d \end{pmatrix} \\
&\in I(\tau\sigma)^k I = IwI.
\end{aligned}$$

Conversely, suppose $c \in \mathfrak{p}$. Then

$$\begin{aligned}
sgw &= \tau g(\tau\sigma)^k \\
&= \begin{pmatrix} c\varpi^{-k} & d\varpi^{k-1} \\ a\varpi^{-k+1} & b\varpi^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & a^{-1}c\varpi^{-1} \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} a\varpi^{-k+1} & b\varpi^{k+1} \\ 0 & (d - a^{-1}bc\varpi)\varpi^{k-1} \end{pmatrix} \quad 3.0.2(III) \\
&= \begin{pmatrix} 1 & a^{-1}c\varpi^{-1} \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} \varpi^{-k+1} & 0 \\ 0 & \varpi^{k-1} \end{pmatrix} \begin{pmatrix} a & b\varpi^{2k} \\ 0 & (d - a^{-1}bc\varpi)\varpi^{k-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & a^{-1}c\varpi^{-1} \\ 0 & 1 \end{pmatrix} \tau(\tau\sigma)^k \begin{pmatrix} a & b\varpi^{2k} \\ 0 & (d - a^{-1}bc\varpi)\varpi^{k-1} \end{pmatrix} \\
&\in I\tau(\tau\sigma)^k I = IswI.
\end{aligned}$$

Putting this information together, we conclude that $IsIwI = IswI \cup IwI$.

(7) Let $s = \sigma$ and $w = (\sigma\tau)^k$ where $k \in \mathbb{N}$. Using the above result of part (6) as well as lemma 3.0.3, we have

$$\begin{aligned}
IsIwI &= I\sigma I(\sigma\tau)^k I \\
&= I\rho\tau\rho^{-1} I(\rho\tau\rho^{-1}\rho\sigma\rho^{-1})^k I \\
&= \rho I\tau\rho^{-1} I\rho(\tau\sigma)^k \rho^{-1} I \\
&= \rho I\tau I(\tau\sigma)^k I\rho^{-1} \\
&= \rho I\tau(\tau\sigma)^k I\rho^{-1} \cup \rho I(\tau\sigma)^k I\rho^{-1} \qquad (6) \\
&= I\rho\tau\rho^{-1}(\rho\tau\rho^{-1}\rho\sigma\rho^{-1})^k I \cup I(\rho\tau\rho^{-1}\rho\sigma\rho^{-1})^k I \\
&= I\sigma(\sigma\tau)^k I \cup I(\sigma\tau)^k I \\
&= IswI \cup IwI
\end{aligned}$$

as required.

(8) Let $s = \tau$ and $w = (\tau\sigma)^k\tau$ where $k \in \mathbb{N}$. Using the above result

of part (5) as well as remark 3.0.3, we have

$$\begin{aligned}
IsIwI &= I\tau I(\tau\sigma)^k\tau I \\
&= I\rho\sigma\rho^{-1}I(\rho\sigma\rho^{-1}\rho\tau\rho^{-1})^k\rho\sigma\rho^{-1}I \\
&= \rho I\sigma\rho^{-1}I\rho(\sigma\tau)^k\sigma I\rho^{-1} \\
&= \rho I\sigma I(\sigma\tau)^k\sigma I\rho^{-1} \\
&= \rho I\sigma(\sigma\tau)^k\sigma I\rho^{-1} \cup \rho I(\sigma\tau)^k\sigma I\rho^{-1} \qquad (5) \\
&= I\rho\sigma\rho^{-1}(\rho\sigma\rho^{-1}\rho\tau\rho^{-1})^k\rho\sigma\rho^{-1}I \cup I(\rho\sigma\rho^{-1}\rho\tau\rho^{-1})^k\rho\sigma\rho^{-1}I \\
&= I\tau(\tau\sigma)^k\tau I \cup I(\tau\sigma)^k\tau I \\
&= IswI \cup IwI
\end{aligned}$$

as required.

With all 8 of these cases covered, we can conclude that whenever $\ell(sw) > \ell(w)$, $IsIwI = IswI$ and when $\ell(sw) < \ell(w)$, $IsIwI = IswI \cup IwI$.

(ii) Suppose $\ell(ww') = \ell(w) + \ell(w')$. Since $w, w' \in \tilde{W}$, we can write reduced forms $w = s_1 s_2 \cdots s_r \rho^m$ and $w' = s'_{r+1} s'_{r+2} \cdots s'_d \rho^n$ for $s_i, s'_i \in W_{\text{af}}$ for all i, and with $r, d \in \mathbb{N}$ and $m, n \in \mathbb{Z}$. Let $\rho^m s'_j \rho^{-m} = s_j$ for each $r + 1 \leq j \leq d$. In particular this tells us

$$\rho^m s'_{r+1} s'_{r+2} \cdots s'_d = s_{r+1} s_{r+2} \cdots s_d \rho^m.$$

Note that this conjugation by ρ^m acts as the identity here when m is even. Then by repeated application of part (i) of this proposition,

$$\begin{aligned}
IwIw'I &= Is_1s_2\cdots s_r\rho^m Is'_{r+1}s'_{r+2}\cdots s'_d\rho^n I \\
&= Is_1s_2\cdots s_r Is_{r+1}s_{r+2}\cdots s_d I\rho^{m+n} \\
&= Is_1Is_2\cdots s_r Is_{r+1}s_{r+2}\cdots s_d I\rho^{m+n} \\
&\qquad\vdots \qquad\qquad \text{repeating above step} \\
&= Is_1Is_2I\cdots Is_r Is_{r+1}s_{r+2}\cdots s_d I\rho^{m+n} \\
&= Is_1Is_2I\cdots Is_r s_{r+1}s_{r+2}\cdots s_d I\rho^{m+n} \\
&\qquad\vdots \qquad\qquad \text{repeating above step} \\
&= Is_1s_2\cdots s_d I\rho^{m+n} \\
&= Is_1s_2\cdots s_r\rho^m s'_{r+1}s'_{r+2}\cdots s'_d\rho^n I \\
&= Iww'I.
\end{aligned}$$

In particular, if $m = 0$, then

$$Is_1I\cdots Is_dI\rho^nI = Is_1s_2\cdots s_d\rho^nI.$$

□

This proves the proposition, and in fact we get the following corollary as well.

Corollary 3.0.5. *Suppose the assumptions from proposition 3.0.4 hold. Then For all $w \in \tilde{W}$ and $s \in \tilde{S}$, if $\ell(ws) < \ell(w)$ then*

$$IwIsI = IwsI \cup IwI.$$

Proof. Suppose $s \in \tilde{S}$ and $w \in \tilde{W}$ with $\ell(ws) < \ell(w)$. We can write $W = \rho^n s_1s_2\cdots s_{\ell(w)}$ where $s_i \in \tilde{S}$ for each i and $n \in \mathbb{Z}$. Then in order for $\ell(ws) <$

$\ell(w)$, we need that $s_{\ell(w)} = s$. Then using proposition 3.0.4 part (ii), we have

$$\begin{aligned}
IwIsI &= I\rho^n s_1 s_2 \cdots s_{\ell(w)} IsI \\
&= \rho^n I s_1 I s_2 I \cdots I s_{\ell(w)-1} I s_{\ell(w)} IsI \\
&= \rho^n I s_1 I s_2 I \cdots I s_{\ell(w)-1} (I \cup I s_{\ell(w)} I) \\
&= I\rho^n s_1 s_2 \cdots s_{\ell(w)-1} I \cup I\rho^n s_1 s_2 \cdots s_{\ell(w)-1} I s_{\ell(w)} I \\
&= IwsI \cup IwI
\end{aligned}$$

as required. $\square$

Chapter 4

Presentations of the Iwahori-Hecke Algebra

4.1 The Iwahori-Hecke Algebra and its First Presentation

In order to define the Iwahori-Hecke algebra of G and give some alternate presentations of it we will first define its usual Hecke algebra.

Definition 4.1.1. Let G be a locally compact group. Then the *Hecke algebra* of G is the convolution algebra $H(G) = (C_c^\infty(G), *)$ of $\mathbb{C}$-valued functions which are smooth and compactly supported.

Now let K be a compact open subgroup of G and denote by e_K the characteristic function

$$e_K(g) = \begin{cases} \mu(K)^{-1} & g \in K \\ 0 & g \notin K. \end{cases}$$

We can show that e_K is an involution under the operation of convolution.

$$\begin{aligned}(e_K * e_K)(h) &= \int_G e_K(g)e_K(g^{-1}h)dg \\ &= \int_K \mu(K)^{-1}e_K(g^{-1}h)dg \\ &= \mu(K)^{-1}\int_K e_K(h)dg \\ &= \mu(K)^{-1}\mu(K)e_K(h) \\ &= e_K(h)\end{aligned}$$

since $g^{-1}h \in K$ if and only if $h \in gK = K$ when $g \in K$.

The Hecke algebra $H(G)$ has no identity, but using e_K as described above, we can define a related unital algebra with respect to a compact open subgroup K of G. This leads to an algebra denoted $H(G, K)$ as given by Bushnell and Henniart in [1].

Definition 4.1.2. Let K be a compact open subgroup of G. We define $H(G, K)$ for the pair (G, K) to be the $\mathbb{C}$-algebra

$$\begin{aligned}H(G, K) &= e_K * H(G) * e_K \\ &= \{f \in H(G) \mid f(k_1 g k_2) = f(g) \text{ for all } k_1, k_2 \in K, \quad g \in G\}.\end{aligned}$$

In other words, $H(G, K)$ is the ring of compactly supported $(K \times K)$-invariant functions on G under the operation of convolution with identity e_K. In this formulation the action of $(K \times K)$ is given by $(k_1, k_2)f(g) = f(k_1 g k_2^{-1})$.

In the special case when $K = I$ the Iwahori subgroup of G, we call $H(G, I)$ the *Iwahori-Hecke algebra* of G.

For our first presentation of the Iwahori-Hecke algebra, we write a definition of a $\mathbb{C}$-algebra and then prove that it is isomorphic to the $H(G, I)$ defined above.

Definition 4.1.3. For $\tilde{W}$ as defined above, we define a $\mathbb{C}$-algebra $\mathcal{H}_{\tilde{W}} := \mathbb{C}\langle S_w \mid w \in \tilde{W}\rangle / \sim$ with generators $\{S_w \mid w \in \tilde{W}\}$ and relations

(1) For all $w, w' \in \tilde{W}$, if $\ell(ww') = \ell(w) + \ell(w')$, then $S_w S_{w'} = S_{ww'}$, and

(2) If $\ell(w) = 1$, then $S_w^2 = q + (q-1)S_w$.

Theorem 4.1.4. *As algebras over $\mathbb{C}$, $H(G,I) \cong \mathcal{H}_{\tilde{W}}$.*

In order to prove this theorem we need to study the elements of $H(G,I)$ and how convolution looks in this group. Since $H(G,I)$ is a group of compactly supported functions, it is generated by characteristic functions. In particular, since we are only looking at $(I \times I)$-invariant functions, each characteristic function f that we consider has the property that $f(g) = f(a_1 g a_2)$ for all $g \in G$ and $a_1, a_2 \in I$. So we can instead just consider characteristic functions of $I \backslash G / I$.

We will be working with double cosets of the form IwI where $w \in \tilde{W}$ so it will be useful for us to write these as unions of single cosets for some calculations later on. We note the following lemma.

Lemma 4.1.5. *Let $w = s_0 s_1 \cdots s_{\ell(w)-1} \rho^k$ be a reduced word in $\tilde{W}$ via the isomorphism $W_{af} \rtimes \Omega$ with $s_i \in \tilde{S}$ for all i and $k \in \mathbb{Z}$, and let C_w be a set of coset representatives for $\mathcal{O}_F \,/\, \mathfrak{p}^{\ell(w)}$. Then*

$$IwI = \bigsqcup_{n \in N_w} n' w I$$

where:

(i)

$$N_w = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in C_w \right\}$$

if $\ell(w) > 0$ and $s_0 = \sigma$,

(ii)

$$N_w = \left\{ \begin{pmatrix} 1 & 0 \\ x\varpi & 1 \end{pmatrix} \mid x \in C_w \right\}$$

if $\ell(w) > 0$ and $s_0 = \tau$, and

(iii)

$$N_w = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

if $\ell(w) = 0$.

Proof. First we will prove this for $w \in W_{\mathrm{af}}$ using induction on the length of w. We begin with two base cases, $I\sigma I$ and $I\tau I$. Note first that

$$I\sigma I = \bigcup_{h \in I} h\sigma I$$

by the definition of cosets. Now we show that for each $h \in I$, $h\sigma I = n\sigma I$ for some $n \in N_\sigma$. Let

$$h = \begin{pmatrix} a & b \\ c\varpi & d \end{pmatrix}$$

be an arbitrary element of I. We can write

$$b = \sum_{i=0}^{\infty} b_i \varpi^i \qquad \text{and} \qquad d = \sum_{j=0}^{\infty} d_j \varpi^j$$

where b_i, d_j are coset representatives of $\mathcal{O}_F \,/\, \mathfrak{p}$ for all i, j and $d_0 \neq 0$. Then choose

$$n = \begin{pmatrix} 1 & \frac{b_0}{d_0} \\ 0 & 1 \end{pmatrix}.$$

Through straightfoward algebra, $h\sigma I = n\sigma I$ if and only if $\sigma n^{-1} h \sigma I = I$, which is true if and only if $\sigma n^{-1} h \sigma \in I$. We compute

$$\begin{aligned} \sigma n^{-1} h \sigma &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{b_0}{d_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c\varpi & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} d & c\varpi \\ b - \frac{b_0}{d_0} d & a - \frac{b_0}{d_0} c\varpi \end{pmatrix}. \end{aligned}$$

Notice that d and $a - \frac{b_0}{d_0} c\varpi$ are in $\mathcal{O}_F^*$ because a and d are, and $c\varpi \in \mathcal{O}_F$. All that is left to check to show that this matrix is in I is that $b - \frac{b_0 d}{d_0} \in \mathfrak{p}$. We can write

$$\frac{b_0 d}{d_0} = \sum_{j=0}^{\infty} \frac{b_0 d_j}{d_0} \varpi^j.$$

In particular, this tells us that

$$\begin{aligned} b - \frac{b_0 d}{d_0} &= b_0 + \left(\sum_{i=1}^{\infty} b_i \varpi^i \right) - b_0 - \left(\sum_{j=1}^{\infty} \frac{b_0 d_j}{d_0} \varpi^j \right) \\ &= \left(\sum_{i=1}^{\infty} (b_i - \frac{b_0 d_i}{d_0}) \varpi^{i-1} \right) \varpi \end{aligned}$$

which is necessarily in $\mathfrak{p}$, as required. Therefore for all $h \in I$, $h\sigma I = n\sigma I$ for some $n \in N_\sigma$.

Now we show that the $n\sigma I$ are all disjoint. Suppose $n, m \in N_\sigma$. Again, we have $n\sigma I = m\sigma I$ if and only if $\sigma m^{-1} n \sigma \in I$. But here for

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and

$$m = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} \sigma m^{-1} n \sigma &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ x - y & 1 \end{pmatrix} \end{aligned}$$

which is contained in I if and only if $x - y \in \mathfrak{p}$. But x, y are in the set of coset representatives of $\mathcal{O}_F \,/\, \mathfrak{p}$, so $x - y \in \mathfrak{p}$ if and only if $x = y$. Therefore for any $n \neq m$ in N_σ, $n\sigma I \neq m\sigma I$.

Putting these results together, we get

$$I\sigma I = \bigsqcup_{n \in N_\sigma} n\sigma I$$

as required.

Next we prove the second base case. We want to show $I\tau I = \bigsqcup_{n\in N_\tau} n\tau I$. However, we can use facts about ρ to simplify this proof greatly. Using the result from above, we get

$$\begin{aligned} I\tau I &= I\rho\sigma\rho^{-1}I \\ &= \rho I\sigma I\rho^{-1} \\ &= \rho\left(\bigsqcup_{n\in N_\sigma} n\sigma I\right)\rho^{-1} \\ &= \bigsqcup_{n\in N_\sigma} \rho n\sigma I\rho^{-1}. \end{aligned}$$

We notice that for all $n \in N_\sigma$, we can write

$$\rho n = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x\varpi & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} = n'\rho$$

for some $n' \in N_\tau$. In fact, $\rho N_\sigma = N_\tau\rho$. Using this, we finish the equality above by

$$\begin{aligned} I\tau I &= \bigsqcup_{n\in N_\sigma} \rho n\sigma I\rho^{-1} \\ &= \bigsqcup_{n'\in N_\tau} n'\rho\sigma\rho^{-1}I \\ &= \bigsqcup_{n'\in N_\tau} n'\tau I \end{aligned}$$

as required. Now we can use induction to extend this result to words $w \in W_{\mathrm{af}}$ with length $\ell(w) > 1$.

Suppose $IwI = \bigsqcup_{n\in N_w} nwI$ for all w with length $1 \le \ell(w) \le L$ for some integer L. Let w be such a word. Let $s \in \tilde{S}$. If $\ell(ws) < \ell(w)$, then we know $IwsI = \bigsqcup_{n\in N_{ws}}$ by our inductive hypothesis. So suppose $\ell(ws) > \ell(w)$. Then using proposition 3.0.4, we can write

$$\begin{aligned} IwsI &= IwIsI \\ &= \bigsqcup_{n\in N_w} nwIsI && \text{by inductive hypothesis} \\ &= \bigsqcup_{n\in N_w}\bigsqcup_{m\in N_s} nwmsI && \text{by inductive hypothesis.} \end{aligned}$$

Here we need to split into the two cases of the lemma.

(i) Suppose $w = s_0 s_1 \cdots s_{L-1}$ with $s_0 = \sigma$. So we can write for $n \in N_w$, $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. We split this up further so we can choose how to write m as well.

(a) Suppose for some $k > 0$, $w = (\sigma\tau)^k = \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{-k} \end{pmatrix}$, so $\ell(w) = 2k$. Then $s = \sigma$ and we can write $m = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} nwms &= n\begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{-k} \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} s \\ &= n\begin{pmatrix} 1 & y\varpi^{2k} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{-k} \end{pmatrix} s \\ &= nm'ws \end{aligned}$$

where $m' = \begin{pmatrix} 1 & y\varpi^{2k} \\ 0 & 1 \end{pmatrix}$. So

$$\begin{aligned} nm' &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y\varpi^{2k} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x + y\varpi^{2k} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Indexing over all $n \in N_w$ and $m \in N_s$, $x + y\varpi^{2k}$ ranges over all elements of N_{ws} as $x + y\varpi^{2k} = \sum_{i=0}^{2k} x_i \varpi^i$ where each x_i is a coset representative for $\mathcal{O}_F \,/\, \mathfrak{p}$, and $x_{2k} = y$. Therefore

$$\begin{aligned} IwsI &= \bigsqcup_{n \in N_w} \bigsqcup_{m \in N_s} nwmsI \\ &= \bigsqcup_{n \in N_w} \bigsqcup_{m' \in N_s} nm'wsI \\ &= \bigsqcup_{n' \in N_{ws}} n'wsI \end{aligned}$$

as required.

(b) Suppose now that for some $k \geq 0$, $w = (\sigma\tau)^k\sigma = \begin{pmatrix} 0 & \varpi^k \\ \varpi^{-k} & 0 \end{pmatrix}$, so $\ell(w) = 2k{+}1$ and $s = \tau$, which means we can write $m = \begin{pmatrix} 1 & 0 \\ y\varpi & 1 \end{pmatrix}$. Then

$$\begin{aligned} nwms &= n \begin{pmatrix} 0 & \varpi^k \\ \varpi^{-k} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y\varpi & 1 \end{pmatrix} s \\ &= n \begin{pmatrix} 1 & y\varpi^{2k+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \varpi^k \\ \varpi^{-k} & 0 \end{pmatrix} s \\ &= nm'ws \end{aligned}$$

where $m' = \begin{pmatrix} 1 & y\varpi^{2k+1} \\ 0 & 1 \end{pmatrix}$. So

$$\begin{aligned} nm' &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y\varpi^{2k+1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x + y\varpi^{2k+1} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

When we index over all $n \in N_w$ and $m \in N_s$, $x + y\varpi^{2k+1}$ ranges over all elements of N_{ws} as $x + y\varpi^{2k} = \sum_{i=0}^{\infty} x_i\varpi^i$ where each x_i is a coset representative for $\mathcal{O}_F \,/\, \mathfrak{p}$ and $x_{2k+1} = y$. Therefore

$$\begin{aligned} IwsI &= \bigsqcup_{n \in N_w} \bigsqcup_{m \in N_s} nwmsI \\ &= \bigsqcup_{n \in N_w} \bigsqcup_{m' \in N_s} nm'wsI \\ &= \bigsqcup_{n' \in N_{ws}} n'wsI \end{aligned}$$

as required.

Putting these together, we have that $IwI = \bigsqcup_{n \in N_w} nwI$ for all $w \in W_{\mathrm{af}}$ with $\ell(w) > 0$ and $s_0 = \sigma$.

(ii) Now suppose that $w = s_0 s_1 \cdots s_{L-1}$ with $s_0 = \tau$. For each word s_i of length 1 in W_{af}, we denote by $\bar{s}_i$ the length 1 word $\rho s_i \rho^{-1}$ in W_{af}. By extension we can define then for the word w, $\bar{w} = \bar{s_0}\bar{s_1} \cdots \bar{s_{L-1}}$. It is worth noting that $\ell(w) = \ell(\bar{w})$, since $w = \rho \bar{w} \rho^{-1}$. With this in mind, we can write

$$\begin{aligned} IwI =& \rho I \bar{w} I \rho^{-1} \\ &= \bigsqcup_{n \in N_{\bar{w}}} \rho n \bar{w} I \rho^{-1} && \text{from part } (i) \\ &= \bigsqcup_{n \in N_{\bar{w}}} (\rho n \rho^{-1}) \rho \bar{w} \rho^{-1} I \\ &= \bigsqcup_{m \in \rho N_{\bar{w}} \rho^{-1}} mwI \\ &= \bigsqcup_{m \in N_w} mwI && N_w = \rho N_{\bar{w}} \rho^{-1} \end{aligned}$$

as required.

(iii) Finally, for the empty word $w = 1$ of length 0 we have that $IwI = I$.

This concludes the proof that

$$IwI = \bigsqcup_{n \in N_w} nwI$$

for any word $w \in W_{\text{af}}$, so it remains to show that this is true for words $w \in \tilde{W}$. Let $w = w_0\rho^k$ in $\tilde{W}$ where $w_0 \in W_{\text{af}}$ and $k \in \mathbb{Z}$. Then using the above part of the proof, we can write

$$\begin{aligned} IwI &= Iw_0\rho^k I \\ &= Iw_0 I\rho^k \\ &= \bigsqcup_{n \in N_{w_0}} nw_0 I\rho^k \\ &= \bigsqcup_{n \in N_w} nwI \end{aligned}$$

since $\ell(w_0) = \ell(w)$, so $N_{w_0} = N_w$. This concludes this proof for all $w \in \tilde{W}$. $\square$

As a result of the above lemma, we can also see that for any $w \in \tilde{W}$,

$$\begin{aligned} \mu(IwI) &= \mu\left(\bigsqcup_{n \in N_w} nwI\right) \\ &= \sum_{n \in N_w} \mu(nwI) \\ &= |N_w| \\ &= q^{\ell(w)}. \end{aligned}$$

Since we want to work with characteristic functions of $I\backslash G/I$, we introduce an isomorphism between this space and $\tilde{W}$ so that we can use the properties we already know about $\tilde{W}$.

Proposition 4.1.6. *Let G, I, and $\tilde{W}$ be defined as above. Then $I\backslash G/I \cong \tilde{W}$.*

Proof. Let $K \cong \mathrm{GL}_2(\mathcal{O}_F)$. First we see from lemma 3.0.2 we get for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin I$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} c & d \\ 0 & -\frac{ad-bc}{c} \end{pmatrix} \in I\sigma I,$$

so $K \cong I \cup I\sigma I$. Next we show that

$$G \cong \bigcup_{a,b\in\mathbb{Z},\ a\leq b} K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K.$$

Let

$$g = \begin{pmatrix} a\varpi^m & b\varpi^n \\ c\varpi^r & d\varpi^s \end{pmatrix} \in G$$

where $a, b, c, d \in \mathcal{O}_F$ and $m, n, r, s \in \mathbb{Z}$. Since g is invertible, we have $ad\varpi^{m+s} - bc\varpi^{n+r} \neq 0$ so $ad \neq 0$ or $bc \neq 0$. First suppose $ad \neq 0$. Then we consider the following four cases.

(i) Suppose $m - n \geq 0$ and $s - r \geq 0$. Then

$$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^r \end{pmatrix} \begin{pmatrix} \varpi^{m-n} & a^{-1}b \\ cd^{-1} & \varpi^{s-r} \end{pmatrix} \in K \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^r \end{pmatrix} K.$$

(ii) Suppose $m - n \geq 0$ and $s - r < 0$. Then

$$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^s \end{pmatrix} \begin{pmatrix} \varpi^{m-n} & a^{-1}b \\ cd^{-1}\varpi^{r-s} & 1 \end{pmatrix} \in K \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^s \end{pmatrix} K.$$

(iii) Suppose $m - n < 0$ and $s - r \geq 0$. Then

$$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^r \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b\varpi^{n-m} \\ cd^{-1} & \varpi^{s-r} \end{pmatrix} \in K \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^r \end{pmatrix} K.$$

(iv) Suppose $m - n < 0$ and $s - r < 0$. Then

$$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^s \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b\varpi^{n-m} \\ cd^{-1}\varpi^{r-s} & 1 \end{pmatrix} \in K \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^s \end{pmatrix} K.$$

Next suppose that $bc \neq 0$. Then we have four more cases to consider.

(v) Suppose $m - r \geq 0$ and $s - n \geq 0$. Then

$$g = \begin{pmatrix} ac^{-1}\varpi^{m-r} & 1 \\ 1 & b^{-1}d\varpi^{s-n} \end{pmatrix} \begin{pmatrix} \varpi^r & 0 \\ 0 & \varpi^n \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \in K \begin{pmatrix} \varpi^r & 0 \\ 0 & \varpi^n \end{pmatrix} K.$$

(vi) Suppose $m-r \geq 0$ and $s-n<0$. Then

$$g = \begin{pmatrix} ac^{-1}\varpi^{m-r} & \varpi^{n-s} \\ 1 & b^{-1}d \end{pmatrix} \begin{pmatrix} \varpi^{r} & 0 \\ 0 & \varpi^{n} \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \in K \begin{pmatrix} \varpi^{r} & 0 \\ 0 & \varpi^{n} \end{pmatrix} K.$$

(vii) Suppose $m-r<0$ and $s-n \geq 0$. Then

$$g = \begin{pmatrix} ac^{-1} & 1 \\ \varpi^{r-m} & b^{-1}d\varpi^{s-n} \end{pmatrix} \begin{pmatrix} \varpi^{m} & 0 \\ 0 & \varpi^{n} \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \in K \begin{pmatrix} \varpi^{m} & 0 \\ 0 & \varpi^{n} \end{pmatrix} K.$$

(viii) Suppose $m-r<0$ and $s-n<0$. Then

$$g = \begin{pmatrix} ac^{-1} & \varpi^{n-s} \\ \varpi^{r-m} & b^{-1}d \end{pmatrix} \begin{pmatrix} \varpi^{m} & 0 \\ 0 & \varpi^{s} \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \in K \begin{pmatrix} \varpi^{m} & 0 \\ 0 & \varpi^{s} \end{pmatrix} K.$$

So we see that any $g \in G$ can be written as $K \begin{pmatrix} \varpi^{a} & 0 \\ 0 & \varpi^{b} \end{pmatrix} K$ for some integers a and b. However, we observe

$$\begin{aligned} K \begin{pmatrix} \varpi^{a} & 0 \\ 0 & \varpi^{b} \end{pmatrix} K &= K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^{b} & 0 \\ 0 & \varpi^{a} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K \\ &= K \begin{pmatrix} \varpi^{b} & 0 \\ 0 & \varpi^{a} \end{pmatrix} K \end{aligned}$$

so we can write

$$G \cong \bigcup_{a,b\in\mathbb{Z},\ a\leq b} K \begin{pmatrix} \varpi^{a} & 0 \\ 0 & \varpi^{b} \end{pmatrix} K.$$

For any $a, b \in \mathbb{Z}$ such that $a \leq b$, there is an integer $k \geq \mathbb{Z}$ such that $b = a + k$. So we can write

$$\begin{pmatrix} \varpi^{a} & 0 \\ 0 & \varpi^{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^{k} \end{pmatrix} \begin{pmatrix} \varpi^{a} & 0 \\ 0 & \varpi^{a} \end{pmatrix} = (\tau\rho)^{k}\rho^{2a}.$$

Now using this and the fact that $K \cong I \cup I\sigma I$, we get for all $a \le b$,

$$\begin{aligned}
&K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K \\
&= K(\tau\rho)^k \rho^{2a} K \\
&= (I \cup I\sigma I)(\tau\rho)^k \rho^{2a}(I \cup I\sigma I) \\
&= I(\tau\rho)^k\rho^{2a}I \cup I(\tau\rho)^k\rho^{2a}I\sigma I \cup I\sigma I(\tau\rho)^k\rho^{2a}I \cup I\sigma I(\tau\rho)^k\rho^{2a}I\sigma I \\
&= I(\tau\rho)^k\rho^{2a}I \cup (I(\tau\rho)^{k-1}\rho^{2a+1}I \cup I(\tau\rho)^k\rho^{2a}I) \cup I\sigma(\tau\rho)^k\rho^{2a}I \\
&\quad \cup (I\sigma(\tau\rho)^{k-1}\rho^{2a}I \cup I\sigma(\tau\rho)^k\rho^{2a}I) \\
&= I(\tau\rho)^k\rho^{2a}I \cup I(\tau\rho)^{k-1}\rho^{2a+1}I \cup I\sigma(\tau\rho)^k\rho^{2a}I \cup I\sigma(\tau\rho)^{k-1}\rho^{2a+1}I \\
&= I \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} I \cup I \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix} I \cup I \begin{pmatrix} 0 & \varpi^b \\ \varpi^a & 0 \end{pmatrix} I \cup I \begin{pmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{pmatrix} I.
\end{aligned}$$

Finally we see that this is a disjoint union since for each of these matrices w, we have $\mu(IwI) = q^{\ell(w)}$ from lemma 4.1.5. So cosets IwI in the above decomposition have measures

$$\begin{aligned}
\mu(I(\tau\rho)^k\rho^{2a}I) &= q^{\ell((\tau\rho)^k\rho^{2a})} &&= q^k \\
\mu(I(\tau\rho)^{k-1}\rho^{2a+1}I) &= q^{\ell((\tau\rho)^{k-1}\rho^{2a+1})} &&= q^{k-1} \\
\mu(I\sigma(\tau\rho)^k\rho^{2a}I) &= q^{\ell(\sigma(\tau\rho)^k\rho^{2a})} &&= q^{k+1} \\
\mu(I\sigma(\tau\rho)^{k-1}\rho^{2a+1}) &= q^{\ell(\sigma(\tau\rho)^{k-1}\rho^{2a+1})} &&= q^k,
\end{aligned}$$

so the only two which may be equal are the first and last. When $a = b$ we can see this just becomes the disjoint union

$$I \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^a \end{pmatrix} I \sqcup I \begin{pmatrix} 0 & \varpi^a \\ \varpi^a & 0 \end{pmatrix} I$$

since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin I$. When $a \neq b$, we observe that

$$I \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} I = I \begin{pmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{pmatrix} I$$

if and only if there exist matrices $\begin{pmatrix} x & y \\ t\varpi & z \end{pmatrix}, \begin{pmatrix} x' & y' \\ t'\varpi & z' \end{pmatrix} \in I$ such that

$$\begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \begin{pmatrix} x & y \\ t\varpi & z \end{pmatrix} = \begin{pmatrix} x\varpi^a & y\varpi^a \\ t\varpi^{b+1} & z\varpi^b \end{pmatrix}$$
$$= \begin{pmatrix} x'\varpi^b & y'\varpi^a \\ t'\varpi^{b+1} & z'\varpi^a \end{pmatrix}$$
$$= \begin{pmatrix} x' & y' \\ t'\varpi & z' \end{pmatrix} \begin{pmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{pmatrix}.$$

But we can see from the first entries alone we can see this is not true since $x, x' \in \mathcal{O}_F^*$ and $a \neq b$, so $x\varpi^a \neq x'\varpi^b$. Therefore we get the disjoint union

$$K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K$$
$$\cong I \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} I \sqcup I \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix} I \sqcup I \begin{pmatrix} 0 & \varpi^b \\ \varpi^a & 0 \end{pmatrix} I \sqcup I \begin{pmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{pmatrix} I.$$

Since the identity matrix multiplied by any nonzero power of ϖ is not contained in K, we have that each choice of a, b such that $a \leq b$ makes

$$K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K$$

unique. Therefore we get that

$$G \cong \bigcup_{a \leq b} K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K \cong \bigsqcup_{w \in \tilde{W}} IwI.$$

In particular this tells us that $\tilde{W}$ gives us exactly a list of coset representatives for $I\backslash G/I$. In other words, this gives us $\tilde{W} \cong I\backslash G/I$ as required.

□

Now keeping in mind the correspondence $\tilde{W} \cong I\backslash G/I$, we will treat $\tilde{W}$ as a set of coset representatives of $I\backslash G/I$ and define a characteristic function on $I\backslash G/I$ in the following way. For a word $w \in \tilde{W}$, define the characteristic function

$$T_w(h) = \begin{cases} 1 & \text{if } h \in IwI \\ 0 & \text{else.} \end{cases}$$

Lemma 4.1.7. *For words $w, w' \in \tilde{W}$, the support of the convolution $T_w * T_{w'}$ is supp$(T_w * T_{w'}) = IwIw'I$.*

Proof. Note that for arbitrary $h \in G$,

$$\begin{aligned}(T_w * T_{w'})(h) &= \int_G T_w(g) T_{w'}(g^{-1}h)dg \\ &= \int_{IwI} T_{w'}(g^{-1}h)dg.\end{aligned}$$

We need to show that $h \in \text{supp}(T_w * T_{w'})$ if and only if $h \in IwIw'I$. From above, we see that $h \in \text{supp}(T_w * T_{w'})$ if and only if

$$\int_{IwI} T_{w'}(g^{-1}h)dg \neq 0.$$

Since $T_{w'}$ is a non-negative function, this is true if and only if there exists $g \in IwI$ such that $T_{w'}(g^{-1}h) \neq 0$, or equivalently $g^{-1}h \in Iw'I$. Multiplying on the left by g, we see that this is true if and only if $h \in gIw'I \subseteq IwIw'I$. □

Using lemma 4.1.5, we can prove another lemma that will be useful for our main result.

Lemma 4.1.8. *Let $w, w' \in W_{af}$ such that $\ell(ww') = \ell(w) + \ell(w')$. Then for $n \in N_w$, $ww' \in nwIw'I$ if and only if n is the identity matrix.*

Proof. First we observe that when n is the identity matrix,

$$ww' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} w' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in nwIw'I$$

so it only remains to prove that when n is not the identity matrix, $ww' \notin nwIw'I$. From here on, we will assume that $\ell(w) > 0$ and n is not the identity matrix.

Notice that $ww' \in nwIw'I$ if and only if $w^{-1}n^{-1}ww' \in Iw'I$. Applying lemma 4.1.5, this is true if and only if for some $m \in N_{w'}$, $w^{-1}n^{-1}ww' \in mw'I$, or equivalently, $(w')^{-1}m^{-1}w^{-1}n^{-1}ww' \in I$. We break this into cases depending on the structures of the reduced forms of w and w', given by $s_0 s_1 \cdots s_{\ell(w)-1}$ and $s'_0 s'_1 \cdots s'_{\ell(w')-1}$ respectively.

(i) Suppose $s_0 = \sigma$. Now let

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_w.$$

Note that since n is not the identity matrix, $x\varpi^{-\ell(w)} \notin \mathcal{O}_F$.

(1) First consider $w = (\sigma\tau)^k$ for some positive integer k. Note here $\ell(w) = 2k$. Notice also that this means in order for $\ell(ww') = \ell(w) + \ell(w')$, $\ell(w') = 0$ or $s'_0 = \sigma$. In both cases, by lemma 4.1.5, an arbitrary element of $N_{w'}$ is of the form

$$m = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in N_{w'}.$$

(a) Suppose $w' = (\sigma\tau)^t$ for some non-negative integer t. Using some straightforward conjugation calculations, we can see that for any $m \in N_{w'}$,

$$\begin{aligned}
&(w')^{-1}m^{-1}w^{-1}n^{-1}ww' \\
&\quad = (\tau\sigma)^t \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} (\tau\sigma)^k \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} (\sigma\tau)^k (\sigma\tau)^t \\
&\quad = (\tau\sigma)^t \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x\varpi^{-2k} \\ 0 & 1 \end{pmatrix} (\sigma\tau)^t \\
&\quad = (\tau\sigma)^t \begin{pmatrix} 1 & -x\varpi^{-2k} - y \\ 0 & 1 \end{pmatrix} (\sigma\tau)^t \\
&\quad = \begin{pmatrix} 1 & (-x\varpi^{-2k} - y)\varpi^{-2t} \\ 0 & 1 \end{pmatrix} \notin I
\end{aligned}$$

since $(-x\varpi^{-2k} - y)\varpi^{-2t} = (-x\varpi^{-\ell(w)} - y)\varpi^{-\ell(w')} \notin \mathcal{O}_F$.

(b) Suppose $w' = (\sigma\tau)^t\sigma$ for some non-negative integer t. Using the same methods as above, we can see that for any $m \in N_{w'}$,

$$\begin{aligned}
&(w')^{-1}m^{-1}w^{-1}n^{-1}ww' \\
&\qquad = \sigma(\tau\sigma)^t \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} (\tau\sigma)^k \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} (\sigma\tau)^k (\sigma\tau)^t \sigma \\
&\qquad = \sigma(\tau\sigma)^t \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x\varpi^{-2k} \\ 0 & 1 \end{pmatrix} (\sigma\tau)^t \sigma \\
&\qquad = \sigma(\tau\sigma)^t \begin{pmatrix} 1 & -x\varpi^{-2k} - y \\ 0 & 1 \end{pmatrix} (\sigma\tau)^t \sigma \\
&\qquad = \sigma \begin{pmatrix} 1 & (-x\varpi^{-2k} - y)\varpi^{-2t} \\ 0 & 1 \end{pmatrix} \sigma \\
&\qquad = \begin{pmatrix} 1 & 0 \\ (-x\varpi^{-2k} - y)\varpi^{-2t} & 1 \end{pmatrix} \notin I
\end{aligned}$$

since $(-x\varpi^{-2k} - y)\varpi^{-2t} = (-x\varpi^{-\ell(w)} - y)\varpi^{-\ell(w')-1} \notin \mathfrak{p}$.

(2) Next consider $w = (\sigma\tau)^k\sigma$ for some non-negative integer k. Note here $\ell(w) = 2k + 1$. This time we note that here we have that $\ell(w') = 0$ or $s_0' = \tau$, therefore in either case lemma 4.1.5 tells us an arbitrary element of $N_{w'}$ is of the form

$$m = \begin{pmatrix} 1 & 0 \\ y\varpi & 1 \end{pmatrix}.$$

(c) Suppose $w' = (\tau\sigma)^t$ for some non-negative integer t. We can see that

$$
\begin{aligned}
&(w')^{-1}m^{-1}w^{-1}n^{-1}ww' \\
&\qquad = (\sigma\tau)^t \begin{pmatrix} 1 & 0 \\ -y\varpi & 1 \end{pmatrix} \sigma(\tau\sigma)^k \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} (\sigma\tau)^k \sigma(\tau\sigma)^t \\
&\qquad = (\sigma\tau)^t \begin{pmatrix} 1 & 0 \\ -y\varpi & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & -x\varpi^{-2k} \\ 0 & 1 \end{pmatrix} \sigma(\tau\sigma)^t \\
&\qquad = (\sigma\tau)^t \begin{pmatrix} 1 & 0 \\ -y\varpi & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x\varpi^{-2k} & 1 \end{pmatrix} (\tau\sigma)^t \\
&\qquad = (\sigma\tau)^t \begin{pmatrix} 1 & 0 \\ -y\varpi - x\varpi^{-2k} & 1 \end{pmatrix} (\tau\sigma)^t \\
&\qquad = \begin{pmatrix} 1 & 0 \\ (-y\varpi - x\varpi^{-2k})\varpi^{-2t} & 1 \end{pmatrix} \notin I
\end{aligned}
$$

since $(-y\varpi - x\varpi^{-2k})\varpi^{-2t} = (-y - x\varpi^{\ell(w)})\varpi^{-\ell(w')}\varpi \notin \mathfrak{p}$.

(d) Suppose $w' = (\tau\sigma)^t\tau$ for some non-negative integer t. We can see that

$$
\begin{aligned}
&(w')^{-1}m^{-1}w^{-1}n^{-1}ww' \\
&\qquad = \tau(\sigma\tau)^t \begin{pmatrix} 1 & 0 \\ -y\varpi & 1 \end{pmatrix} \sigma(\tau\sigma)^k \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} (\sigma\tau)^k \sigma(\tau\sigma)^t \tau \\
&\qquad = \tau(\sigma\tau)^t \begin{pmatrix} 1 & 0 \\ -y\varpi & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x\varpi^{-2k} & 1 \end{pmatrix} (\tau\sigma)^t \tau \\
&\qquad = \tau(\sigma\tau)^t \begin{pmatrix} 1 & 0 \\ -y\varpi - x\varpi^{-2k} & 1 \end{pmatrix} (\tau\sigma)^t \tau \\
&\qquad = (\tau\sigma)^t \begin{pmatrix} 1 & (-y\varpi - x\varpi^{-2k})\varpi^{-2} \\ 0 & 1 \end{pmatrix} (\sigma\tau)^t \\
&\qquad = \begin{pmatrix} 1 & (-y\varpi - x\varpi^{-2k})\varpi^{-2t-2} \\ 0 & 1 \end{pmatrix} \notin I
\end{aligned}
$$

since $(-y\varpi - x\varpi^{-2k})\varpi^{-2t-2} = (-y - x\varpi^{-\ell(w)})\varpi^{-\ell(w')} \notin \mathcal{O}_F$.

(ii) Now suppose that $s_0 = \tau$. Let

$$n = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in N_w.$$

Notice since $\rho^{-1} I \rho = I$, we can say

$$(w')^{-1} m^{-1} w^{-1} n^{-1} w w' \in I$$

if and only if

$$\rho(w')^{-1} m^{-1} w^{-1} n^{-1} w w' \rho^{-1} \in I.$$

We still need to show that $ww' = nwIw'I$. Conjugating by ρ gives us the equivalent statement $\rho(ww')\rho^{-1} \in \rho(nwIw'I)\rho^{-1}$ where we can write

$$\begin{aligned} \rho(nwIw'I)\rho^{-1} &= (\rho n \rho^{-1})(\rho w \rho^{-1})(\rho I \rho^{-1})(\rho w' \rho^{-1})(\rho I \rho^{-1}) \\ &= (\rho n \rho^{-1})\bar{w} I \bar{w'} I \end{aligned}$$

We can see that $\rho N_w \rho^{-1} = N_{\bar{w}}$, so in particular $\rho n \rho^{-1} \in N_{\bar{w}}$. Recognising that $\bar{s_0} = \sigma$, this statement falls under one of the cases mentioned above, and is therefore already proven. This tells us that the equivalent statement is also true, so $ww' \in nwIw'I$ if and only if n is the identity as required.

□

Lemma 4.1.9. *Let $w \in \tilde{W}$ and $k \in \mathbb{Z}$. Then $T_w * T_{\rho^k} = T_{w\rho^k}$ and $T_{\rho^k} * T_w = T_{\rho^k w}$.*

Proof. Let $w \in \tilde{W}$ and $h \in G$. First we show that $T_w * T_\rho = T_{w\rho}$. Using lemma 3.0.3 we observe

$$\begin{aligned}
(T_w * T_\rho)(h) &= \int_G T_w(g)T_\rho(g^{-1}h)dg \\
&= \int_G T_w(hk^{-1})T_\rho(k)dk & k = g^{-1}h \\
&= \int_{I\rho I} T_w(hk^{-1})dk \\
&= \int_{I\rho} T_w(h\rho^{-1}n^{-1})d(n\rho) & 3.0.3 \\
&= \int_I T_w(h\rho^{-1})dn \\
&= \mu(I)T_w(h\rho^{-1}) \\
&= T_{w\rho}(h)
\end{aligned}$$

so $T_w * T_\rho = T_{w\rho}$. Repeating this process inductively shows us that $T_w * T_{\rho^k} = T_{w\rho^k}$ for all $k \geq 0$. By some of the same techniques,

$$\begin{aligned}
(T_\rho * T_w)(h) &= \int_G T_\rho(g)T_w(g^{-1}h)dg \\
&= \int_{I\rho I} T_w(g^{-1}h)dg \\
&= \int_{\rho I} T_w(k^{-1}\rho^{-1}h)d(\rho k) & 3.0.3 \\
&= \int_I T_w(k^{-1}\rho^{-1}h)dk \\
&= \int_I T_w(\rho^{-1}h)dk \\
&= \mu(I)T_w(\rho^{-1}h) \\
&= T_{\rho w}(h)
\end{aligned}$$

so $T_\rho * T_w = T_{\rho w}$. Again this tells us by induction that $T_{\rho^k} * T_w = T_{\rho^k w}$ for all $k \geq 0$.

We can use nearly identical arguments to show that $T_w * T_{\rho^{-1}} = T_{w\rho^{-1}}$ and $T_{\rho^{-1}} * T_w = T_{\rho^{-1}w}$, so by induction $T_w * T_{\rho^k} = T_{w\rho^k}$ and $T_{\rho^k} * T_w = T_{\rho^k w}$ for all integers $k < 0$ as well. Gathering these results together, we have proved the lemma for all integers k. $\square$

Proposition 4.1.10. *Let $w, w' \in W_{af}$. If $\ell(ww') = \ell(w) + \ell(w')$, then $T_w * T_{w'} = T_{ww'}$.*

Proof. Suppose $w, w' \in W_{\text{af}}$ with $\ell(ww') = \ell(w) + \ell(w')$. Then by proposition 3.0.4, we know that $\text{supp}(T_w * T_{w'}) = IwIw'I = Iww'I$. Since the convolution of characteristic functions is a characteristic function, it is sufficient to show that $(T_w * T_{w'})(ww') = 1$ to show that $T_w * T_{w'} = T_{ww'}$. Observe using lemma 4.1.8

$$
\begin{aligned}
(T_w * T_{w'})(ww') &= \int_G T_w(g) T_{w'}(g^{-1}ww')dg \\
&= \int_{IwI} T_{w'}(g^{-1}ww')dg \\
&= \sum_{m \in N_w} \int_{mwI} T_{w'}(k^{-1}w^{-1}m^{-1}ww')dmdk \\
&= \sum_{m \in N_w} \int_I T_{w'}(w^{-1}m^{-1}ww')dk \\
&= \sum_{m \in N_w} T_{w'}(w^{-1}m^{-1}ww') \\
&= T_{w'}(w^{-1}ww') \qquad \text{4.1.8} \\
&= T_{w'}(w') \\
&= 1.
\end{aligned}
$$

$\square$

Corollary 4.1.11. *Let $w, w' \in \tilde{W}$. If $\ell(ww') = \ell(w) + \ell(w')$, then $T_w * T_{w'} = T_{ww'}$.*

Proof. Suppose $w, w' \in \tilde{W}$ with $\ell(ww') = \ell(w) + \ell(w')$. Then $w = w_0\rho^a$ and $w' = w'_0\rho^b$ for some $w_0, w'_0 \in W_{\text{af}}$ and $a, b \in \mathbb{Z}$. We can express $\rho^a w'_0 = v'_0\rho^a$ for some $v'_0 \in W_{\text{af}}$ which depends on the parity of a. This gives us $ww' =$

$w_0\rho^a w_0'\rho^b = w_0 v_0' \rho^{a+b}$, where $\ell(w_0 v_0') = \ell(w_0) + \ell(v_0')$. Then by lemma 4.1.9,

$$\begin{aligned}
T_w * T_{w'} &= T_{w_0\rho^a} * T_{w_0'\rho^b} \\
&= T_{w_0} * T_{\rho^a} * T_{w_0'} * T_{\rho^b} \\
&= T_{w_0} * T_{\rho^a w_0'} * T_{\rho^b} \\
&= T_{w_0} * T_{v_0'\rho^a} * T_{\rho^b} \\
&= T_{w_0} * T_{v_0'} * T_{\rho^a} * T_{\rho^b} \\
&= T_{w_0 v_0'} * T_{\rho^a \rho^b} \\
&= T_{w_0 v_0' \rho^a \rho^b} \\
&= T_{w_0 \rho^a w_0' \rho^b} \\
&= T_{ww'}
\end{aligned}$$

□

Now we have a concrete computation of convolution $T_w * T_{w'}$ when $\ell(ww') = \ell(w) + \ell(w')$. In order to complete writing $H(G, I)$ in terms of generators and relations, we still need to show what this convolution looks like for $\ell(ww') \neq \ell(w) + \ell(w')$. First we can compute some of the simpler cases. To start, we compute the convolution $T_\sigma * T_\sigma$.

From lemma 4.1.7, we can see that $\operatorname{supp}(T_\sigma * T_\sigma) = I\sigma I\sigma I$. Further, proposition 3.0.4 gives us that $I\sigma I\sigma I = I \cup I\sigma I$. Since this is the convolution of $(I \times I)$-invariant functions, the result is also invariant under left or right translation by I. Therefore it is enough to compute $(T_\sigma * T_\sigma)(1)$ and $(T_\sigma * T_\sigma)(\sigma)$ to completely determine the function. First we compute $(T_\sigma * T_\sigma)(1)$ using lemma 4.1.5,

$$
\begin{aligned}
(T_\sigma * T_\sigma)(1) &= \int_G T_\sigma(g) T_\sigma(g^{-1}) dg \\
&= \int_{I\sigma I} T_\sigma(g^{-1}) dg \\
&= \sum_{n \in N_\sigma} \int_{n\sigma I} T_\sigma(g^{-1}) dg && \text{by 4.1.5} \\
&= \sum_{n \in N_\sigma} \int_I T_\sigma(k^{-1}\sigma n^{-1}) dk && \text{setting } g = n\sigma k \\
&= \sum_{n \in N_\sigma} \int_I T_\sigma(\sigma) dk \\
&= |N_\sigma| \mu(I) \\
&= q.
\end{aligned}
$$

Next, we can compute

$$
\begin{aligned}
(T_\sigma * T_\sigma)(\sigma) &= \int_G T_\sigma(g) T_\sigma(g^{-1}\sigma) dg \\
&= \int_{I\sigma I} T_\sigma(g^{-1}\sigma) dg \\
&= \sum_{n \in N_\sigma} \int_{n\sigma I} T_\sigma(g^{-1}\sigma)) dg && \text{by 4.1.5} \\
&= \sum_{n \in N_\sigma} \int_I T_\sigma(k^{-1}\sigma n^{-1}\sigma) dk && \text{setting } g = n\sigma k \\
&= \sum_{n \in N_\sigma} \int_I T_\sigma(\sigma n^{-1}\sigma) dk \\
&= \sum_{n \in N_\sigma} T_\sigma(\sigma n^{-1}\sigma) \mu(I) \\
&= \sum_{n \in N_\sigma} T_\sigma(\sigma n^{-1}\sigma).
\end{aligned}
$$

Notice that for an arbitrary element $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_\sigma$, we have

$$\begin{aligned} \sigma n^{-1}\sigma &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \end{aligned}$$

By the definition of N_σ, we know that here x is a representative of $\mathcal{O}_F \,/\, \mathfrak{p}$. In particular, $x = 0$ or $x \in \mathcal{O}_F^*$. When $x \neq 0$, we have $\sigma n^{-1}\sigma \notin I$, so $\sigma n^{-1}\sigma \in I\sigma I$. Conversely if $x = 0$, then $\sigma n^{-1}\sigma \in I$, so $\sigma n^{-1}\sigma \notin I\sigma I$. Therefore

$$\begin{aligned} (T_\sigma * T_\sigma)(\sigma) &= \sum_{n \in N_\sigma} T_\sigma(\sigma n^{-1}\sigma)\mu(I) \\ &= |N_\sigma| - 1 \\ &= q - 1. \end{aligned}$$

Putting this information together, and using that this convolution is invariant under left and right translation by I, we can write

$$T_\sigma^2 = T_\sigma * T_\sigma = qe_I + (q-1)T_\sigma.$$

As a second base case, we must compute the convolution $T_\tau * T_\tau$. However, we can use previous results to show

$$\begin{aligned} T_\tau * T_\tau &= T_{\rho\sigma\rho^{-1}} * T_{\rho\sigma\rho^{-1}} \\ &= T_\rho * T_\sigma * T_{\rho^{-1}\rho} * T_\sigma * T_{\rho^{-1}} \\ &= T_\rho * T_\sigma^2 * T_{\rho^{-1}} \\ &= T_\rho * (qe_I + (q-1)T_\sigma) * T_{\rho^{-1}} \\ &= qT_{\rho\rho^{-1}} + (q-1)T_{\rho\sigma\rho^{-1}} \\ &= qe_I + (q-1)T_\tau \end{aligned}$$

We have now computed $T_w * T_{w'}$ in the cases that $\ell(w) + \ell(w') = \ell(ww')$, $w = w' = \sigma$, and $w = w' = \tau$. This completely determines the multiplication in this algebra, but we first more generally prove the case when $\ell(ww') \neq \ell(w) + \ell(w')$ using this result, then prove an isomorphism between $H(G, I)$ and a ring of generators and relations.

Proposition 4.1.12. *Let $w, w' \in W_{af}$ such that $\ell(ww') \neq \ell(w) + \ell(w')$. We write $w = s_1 s_2 \cdots s_{\ell(w)}$ and $w' = s'_1 s'_2 \cdots s'_{\ell(w')}$. Then*

$$T_w T_{w'} = q^m T_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_i w'}$$

where $r_i = s'_1 s'_2 \cdots s'_i s'_{i+1} s'_i \cdots s'_1$ and $m = \min(\ell(w), \ell(w'))$.

Proof. We prove this lemma by induction on the length of $\ell(w')$. When $\ell(w') = 0$ then $\ell(ww') = \ell(w)$ for any $w \in W_{\mathrm{af}}$, so the first case we consider is when $\ell(w') = 1$. Here we can write $w' = s'_1$. In order for $\ell(ww') \neq \ell(w) + \ell(w')$ to be true, we must have $s_{\ell(w)} = s'_1$. So we get

$$\begin{aligned} T_w T_{w'} &= T_{s_1 s_2 \cdots s_{\ell(w)-1}} T^2_{s_{\ell(w)}} \\ &= T_{s_1 s_2 \cdots s_{\ell(w)-1}} (q e_I + (q-1) T_{s_{\ell(w)}}) \\ &= q T_{s_1 s_2 \cdots s_{\ell(w)-1}} + (q-1) T_w \\ &= q T_{ww'} + (q-1) T_{ws_1 w'} \end{aligned}$$

as required.

Now suppose that for some positive integer k, if $\ell(w') = k$ then

$$T_w T_{w'} = q^m T_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_i w'}.$$

For our inductive step we want to show that for $s'_{k+1} \in W_{\mathrm{af}}$ such that $\ell(w' s'_{k+1}) = k + 1$,

$$T_w T_{w' s'_{k+1}} = q^{\min(\ell(w), k+1)} T_{ww' s'_{k+1}} + \sum_{i=0}^{\min(\ell(w), k+1)-1} q^i (q-1) T_{wr_i w'}$$

In order to prove this, we use that

$$\begin{aligned}
T_w T_{w's'_{k+1}} &= (T_w T_{w'})T_{s'_{k+1}} \\
&= \left[q^m T_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_iw'} \right] T_{s'_{k+1}} \\
&= q^m T_{ww'} T_{s'_{k+1}} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_iw'} T_{s'_{k+1}}
\end{aligned}$$

so we need to be able to compute $T_{ww'}T_{s'_{k+1}}$ and $T_{wr_iw'}T_{s'_{k+1}}$ by comparing $\ell(ww's'_{k+1})$ with $\ell(ww')+\ell(s'_{k+1})$ and $\ell(wr_iw's'_{k+1})$ with $\ell(wr_iw')+\ell(wr_iw's'_{k+1})$. For all i such that $0 \leq i \leq m-1$,

$$\begin{aligned}
\ell(wr_iw') &= \ell((s_1 s_2 \cdots s_{\ell(w)})(s'_1 s'_2 \cdots s'_i s'_{i+1} s'_i \cdots s'_1)(s'_1 s'_2 \cdots s'_k)) \\
&= \ell(s_1 s_2 \cdots s_{\ell(w)-(i+1)} s'_{i+1} s'_{i+2} \cdots s'_k) \\
&= (\ell(w) - (i+1)) + (k - i) \\
&= \ell(w) + k - 2i - 1
\end{aligned}$$

and

$$\begin{aligned}
\ell(wr_iw's'_{k+1}) &= \ell((s_1 s_2 \cdots s_{\ell(w)})(s'_1 s'_2 \cdots s'_i s'_{i+1} s'_i \cdots s'_1)(s'_1 s'_2 \cdots s'_{k+1})) \\
&= \ell(s_1 s_2 \cdots s_{\ell(w)-(i+1)} s'_{i+1} s'_{i+2} \cdots s'_k s'_{k+1}) \\
&= (\ell(w) - (i+1)) + ((k+1) - i) \\
&= \ell(w) + k - 2i \\
&= \ell(wr_iw') + \ell(s'_{k+1}).
\end{aligned}$$

This tells us that for all $0 \leq i \leq m-1$, we have $T_{wr_iw'}T_{s'_{k+1}} = T_{wr_iw's'_{k+1}}$. There are two cases to consider here which affect the convolution $T_w T_{w's'_{k+1}}$.

(1) Suppose $\ell(w) \leq \ell(w')$. This means we have

$$\ell(ww') = k - \ell(w).$$

Then $m = \min(\ell(w), \ell(w's_{k+1})) = \ell(w)$. Then

$$\begin{aligned}
\ell(ww's'_{k+1}) &= \ell(s_1 s_2 \cdots s_{\ell(w)} s'_1 s'_2 \cdots s'_k s'_{k+1}) \\
&= \ell(s'_{\ell(w)+1} s'_{\ell(w)+2} \cdots s'_{k+1}) \\
&= \ell(ww') + \ell(s'_{k+1})
\end{aligned}$$

and for all i such that $0 \leq i \leq \ell(w) - 1$,

$$\begin{aligned}
\ell(wr_iw') &= \ell((s_1s_2\cdots s_{\ell(w)})(s'_1s'_2\cdots s'_is'_{i+1}s'_i\cdots s'_1)(s'_1s'_2\cdots s'_k)) \\
&= \ell(s_1s_2\cdots s_{\ell(w)-(i+1)}s'_{i+1}s'_{i+2}\cdots s'_k) \\
&= (\ell(w)-(i+1)) + (k-i) \\
&= \ell(w) + k - 2i - 1
\end{aligned}$$

and

$$\begin{aligned}
\ell(wr_iw's'_{k+1}) &= \ell((s_1s_2\cdots s_{\ell(w)})(s'_1s'_2\cdots s'_is'_{i+1}s'_i\cdots s'_1)(s'_1s'_2\cdots s'_{k+1})) \\
&= \ell(s_1s_2\cdots s_{\ell(w)-(i+1)}s'_{i+1}s'_{i+2}\cdots s'_ks'_{k+1}) \\
&= (\ell(w)-(i+1)) + ((k+1)-i) \\
&= \ell(w) + k - 2i \\
&= \ell(wr_iw') + \ell(s_{k+1}).
\end{aligned}$$

Therefore we can compute

$$\begin{aligned}
T_wT_{w's'_{k+1}} &= T_wT_{w'}T_{s'_{k+1}} \\
&= \left[q^{\ell(w)}T_{ww'} + \sum_{i=0}^{\ell(w)-1} q^i(q-1)T_{wr_iw'}\right]T_{s'_{k+1}} \\
&= q^{\ell(w)}T_{ww's'_{k+1}} + \sum_{i=0}^{\ell(w)-1} q^i(q-1)T_{wr_iw's_{k+1}}
\end{aligned}$$

where $\ell(w) = \min\{\ell(w), k+1\}$, as required.

(2) Now suppose $\ell(w) > \ell(w')$. Then $m = \min\{\ell(w), \ell(w')\} = \ell(w') = k$, and $\min\{\ell(w), \ell(w's'_{k+1})\} = \ell(w's'_{k+1}) = k+1$. This means that

$$\ell(ww') = \ell(w) - k.$$

Then

$$\begin{aligned}
\ell(ww's'_{k+1}) &= \ell(s_1s_2\cdots s_{\ell(w)}s'_1s'_2\cdots s'_{k+1}) \\
&= \ell(s_1s_2\cdots s_{\ell(w)-(k+1)}) \\
&= \ell(ww') - 1 \\
&\neq \ell(ww') + \ell(s'_{k+1}).
\end{aligned}$$

So we can then compute

$$\begin{aligned}
T_w T_{w's'_{k+1}} &= q^m T_{ww'} T_{s'_{k+1}} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_iw'} T_{s'_{k+1}} \\
&= q^k (q T_{ww's'_{k+1}} + (q-1) T_{ww'}) + \sum_{i=0}^{k-1} q^i (q-1) T_{wr_iw's'_{k+1}} \\
&= q^{k+1} T_{ww's'_{k+1}} + \sum_{i=0}^{k} q^i (q-1) T_{wr_iw's'_{k+1}}
\end{aligned}$$

where the last equality is true because

$$\begin{aligned}
wr_k w' s'_{k+1} &= w(s'_1 s'_2 \cdots s'_k s'_{k+1} s'_k \cdots s'_2 s'_1)(s'_1 s'_2 \cdots s'_k s'_{k+1}) \\
&= w(s'_1 s'_2 \cdots s'_k) \\
&= ww'.
\end{aligned}$$

Therefore we have proved the equality we desired, since $\min \ell(w), k+1 = k+1$.

Putting the base case as well as the above two inductive cases together, we have proved that for $w, w' \in W_{\text{af}}$, if $\ell(ww') \neq \ell(w) + \ell(w')$, then

$$T_w T_{w'} = q^m T_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_iw'}$$

and the lemma is proved. □

We would like to extend the above lemma to include all elements $w, w' \in \tilde{W}$ as a corollary.

Corollary 4.1.13. *Let $w, w' \in \tilde{W}$ such that $\ell(ww') \neq \ell(w) + \ell(w')$ and write their reduced forms as $w = s_1 s_2 \cdots s_{\ell(w)} \rho^k$ and $w' = s'_1 s'_2 \cdots s'_{\ell(w')} \rho^{k'}$. Then*

$$T_w T_{w'} = q^m T_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_iw'}$$

where r_i and m are defined as in proposition 4.1.12.

Proof. First we write elements $w_0 = s_1 s_2 \cdots s_{\ell(w)}$ and $w_0' = s_1' s_2' \cdots s_{\ell(w')}'$ of the affine Weyl group. Define $v_0 = \rho^{-k} w_0 \rho^k \in W_{\mathrm{af}}$ so that $w = \rho^k v_0$. Then since $v_0, w_0' \in W_{\mathrm{af}}$, we can use proposition 4.1.12 to compute

$$\begin{aligned} T_w T_{w'} &= T_{\rho^k} T_{v_0} T_{w_0'} T_{\rho^{k'}} \\ &= T_{\rho^k} \left(q^m T_{v_0 w_0'} + \sum_{i=0}^{m-1} q^i (q-1) T_{v_0 r_i w_0'} \right) T_{\rho^{k'}} \\ &= q^m T_{\rho^k v_0 w_0' \rho^{k'}} + \sum_{i=0}^{m-1} q^i (q-1) T_{\rho^k v_0 r_i w_0' \rho^{k'}} \\ &= q^m T_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) T_{w r_i w'} \end{aligned}$$

as required. □

Now we have all the tools we need in order to express $H(G, I)$ in terms of generators and relations by describing an isomorphism with a free algebra modulo some relations. With this we are now able to prove theorem 4.1.4 stating that $H(G, I) \cong \mathcal{H}_{\tilde{W}}$.

Proof. In order to show these are isomorphic, we need to define an isomorphism. We define a map

$$\begin{aligned} \varphi : H(G, I) &\longrightarrow \mathcal{H}_{\tilde{W}} \\ T_w &\longmapsto S_w. \end{aligned}$$

This completely determines a $\mathbb{C}$-vector space homomorphism from $H(G, I)$ since every function with compact support can be described as a linear combination of characteristic functions of the form T_w. It remains to show that this map preserves multiplication. We know that for two generators $T_w, T_{w'}$ where $w, w' \in \tilde{W}$ the convolution $T_w T_{w'}$ depends on the value of $\ell(ww')$. If

$\ell(ww') = \ell(w) + \ell(w')$ then we have

$$\begin{aligned}\varphi(T_w T_{w'}) &= \varphi(T_{ww'}) \\ &= S_{ww'} \\ &= S_w S_{w'} \\ &= \varphi(T_w)\varphi(T_{w'}).\end{aligned}$$

Conversely suppose $\ell(ww') \neq \ell(w) + \ell(w')$. Then the same arguments from proposition 4.1.12 and corollary 4.1.13 apply here, replacing T_w with S_w in the proof, so

$$S_w S_{w'} = q^m S_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) S_{wr_iw'}.$$

With this, we can compute

$$\begin{aligned}\varphi(T_w T_{w'}) &= \varphi\left(q^m T_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) T_{wr_iw'}\right) \\ &= q^m \varphi(T_{ww'}) + \sum_{i=0}^{m-1} q^i (q-1) \varphi(T_{wr_iw'}) \\ &= q^m S_{ww'} + \sum_{i=0}^{m-1} q^i (q-1) S_{wr_iw'} \\ &= S_w S_{w'} \\ &= \varphi(T_w)\varphi(T_w).\end{aligned}$$

Putting these two cases together, we see that φ preserves multiplication, and is therefore a homomorphism of $\mathbb{C}$-algebras. It remains to show that it is invertible. We can consider the map

$$\begin{aligned}\psi : \mathcal{H}_{\tilde{W}} &\longrightarrow H(G, I) \\ S_w &\longmapsto T_w.\end{aligned}$$

Using almost identical arguments to those above, this is a homomorphism of $\mathbb{C}$-algebras, and we can see that $\varphi \circ \psi = \mathrm{id}_{H(G,I)}$ and $\psi \circ \varphi = \mathrm{id}_{\mathcal{H}_{\tilde{W}}}$. Therefore φ and ψ are inverses of each other, and $H(G, I) \cong \mathcal{H}_{\tilde{W}}$. $\square$

4.2 Finite Presentation

Theorem 4.1.4 gives a way to express $H(G, I)$ in terms of an infinite list of generators and relations with the presentation $\mathcal{H}_{\tilde{W}}$. In this chapter we will further reduce these to finite lists of generators corresponding to generators of $\tilde{W}$, and a finite list of relations.

Definition 4.2.1. For $\tilde{W}$ as defined above, we define a $\mathbb{C}$-algebra $\mathcal{H}_{\tilde{W}}^{\text{fin}}$ by generators $\{S_\sigma, S_\rho, S_{\rho^{-1}}\}$ and relations

(1) $S_\rho S_{\rho^{-1}} = 1$

(2) $S_{\rho^{-1}} S_\rho = 1$

(3) $S_\sigma^2 = (q-1)S_\sigma + q$

(4) $S_\rho S_\sigma S_{\rho^{-1}} = S_{\rho^{-1}} S_\sigma S_\rho$.

Theorem 4.2.2. *Let $\mathcal{H}_{\tilde{W}}$ and $\mathcal{H}_{\tilde{W}}^{fin}$ be defined as above. Then as $\mathbb{C}$-algebras, $\mathcal{H}_{\tilde{W}} \cong \mathcal{H}_{\tilde{W}}^{fin}$.*

Proof. First we show that there is an embedding of $\mathbb{C}$-algebras. There is an embedding of $\mathbb{C}$-modules given by

$$\iota : \mathcal{H}_{\tilde{W}}^{\text{fin}} \hookrightarrow \mathcal{H}_{\tilde{W}},$$

since the set of generators of $\mathcal{H}_{\tilde{W}}^{\text{fin}}$ is a subset of the generators of $\mathcal{H}_{\tilde{W}}$. We just need to show that this preserves multiplication by preserving these relations. That is, we need to show that $\iota(S_\rho)\iota(S_{\rho^{-1}}) = 1 = \iota(S_{\rho^{-1}})\iota(S_\rho)$ and $\iota(S_\sigma^2) = (q-1)\iota(S_\sigma) + q$. For the first two relations, recognising that $\ell(\rho) + \ell(\rho^{-1}) = 0 = \ell(\rho\rho^{-1}) = \ell(\rho^{-1}\rho)$, we have

$$\iota(S_\rho)\iota(S_{\rho^{-1}}) = S_\rho S_{\rho^{-1}} = 1 = \iota(1) = \iota(S_\rho S_{\rho^{-1}})$$

and

$$\iota(S_{\rho^{-1}} \iota S_\rho) = S_{\rho^{-1}} S_\rho = 1 = \iota(1) = \iota(S_{\rho^{-1}} S_\rho).$$

Moving onto the third relation, since ι is an embedding of modules we see that

$$\begin{aligned}\iota(S_\sigma)^2 &= S_\sigma^2 \\ &= (q-1)S_\sigma + q \\ &= \iota(S_\sigma^2).\end{aligned}$$

Finally, the fourth relation can be seen using the fact that $\rho\sigma\rho^{-1} = \tau = \rho^{-1}\sigma\rho$ in the following way:

$$\begin{aligned}\iota(S_\rho S_\sigma S_{\rho^{-1}}) &= S_\rho S_\sigma S_{\rho^{-1}} \\ &= S_{\rho\sigma\rho^{-1}} \\ &= S_{\rho^{-1}\sigma\rho} \\ &= S_{\rho^{-1}} S_\sigma S_\rho \\ &= \iota(S_{\rho^{-1}} S_\sigma S_\rho).\end{aligned}$$

Therefore we can conclude that ι is an inclusion of $\mathbb{C}$-algebras.

Now we just want to show that this map is a surjection as well. So we start by writing elements S_w in terms of S_σ, S_ρ, and $S_{\rho^{-1}}$. Using the length relation in $\mathcal{H}_{\tilde{W}}$, we have for any $w = s_1 s_2 \cdots s_{\ell(w)}\rho^k$ where each $s_i \in \{\sigma, \tau\}$ and $k \in \mathbb{Z}$,

$$\begin{aligned}S_w &= \begin{cases} S_{s_1} S_{s_2} \cdots S_{s_{\ell(w)}} (S_\rho)^k & \text{if } k \geq 0 \\ S_{s_1} S_{s_2} \cdots S_{s_{\ell(w)}} (S_{\rho^{-1}})^{-k} & \text{if } k < 0 \end{cases} \\ &= S_{s_1} S_{s_2} \cdots S_{s_{\ell(w)}} (S_{\rho^{\varepsilon(k)}})^{|k|}\end{aligned}$$

where ε is the function which represents the sign of an integer,

$$\varepsilon(k) := \begin{cases} 1 & k \geq 0 \\ -1 & k < 0 \end{cases}.$$

Further we use that

$$\begin{aligned}S_\tau &= S_{\rho\sigma\rho^{-1}} \\ &= S_\rho S_\sigma S_{\rho^{-1}}.\end{aligned}$$

Then for each of the S_{s_i} above, we have $S_{s_i} = S_\sigma$ or $S_{s_i} = S_\tau = S_\rho S_\sigma S_{\rho^{-1}}$. Therefore any S_w can be written as a product of S_σ, S_ρ, and $S_{\rho^{-1}}$, and so ι is surjective. This defines an isomorphism of $\mathbb{C}$-algebra modules $\mathcal{H}_{\tilde{W}}^{\text{fin}} \cong \mathcal{H}_{\tilde{W}}$ as required. □

This theorem demonstrates that the Iwahori-Hecke algebra $H(G, I)$ itself is finitely generated by three characteristic functions and four relations.

4.3 Modified Bernstein-Lusztig presentation

Next we once again find an isomorphism to provide another presentation of the Iwahori-Hecke algebra $H(G, I)$.

Definition 4.3.1. We define the $\mathbb{C}$-algebra $\mathcal{H}_t$ by generators $\{S, t, t^{-1}\}$ and relations

(1) $tt^{-1} = 1$

(2) $t^{-1}t = 1$

(3) $S^2 = (q-1)S + q$

(4) $StSt = tStS$.

This time we prove that a specific map is an isomorphism between these two presentations, since this makes $\mathcal{H}_t$ a very useful connecting step between the previous presentations and the Berstein-Lusztig presentation that will be defined in the next chapter.

Theorem 4.3.2. *For $\mathcal{H}_{\tilde{W}}^{fin}$ and $\mathcal{H}_t$ as defined above, the map*

$$\begin{aligned} \varphi : \mathcal{H}_t &\longrightarrow \mathcal{H}_{\tilde{W}}^{fin} \\ S &\longmapsto S_\sigma \\ t &\longmapsto q^{1/2}(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho) \\ t^{-1} &\longmapsto q^{-1/2} S_{\rho^{-1}} S_\sigma \end{aligned}$$

defines an isomorphism of $\mathbb{C}$-algebras.

To simplify this proof, it will be helpful to have the following lemmas related to some of the relations we have defined previously.

Lemma 4.3.3. *Let H be a $\mathbb{C}$-algebra and suppose there exists $S \in H$ such that $S^2 = (q-1)S + q$ where $q \in \mathbb{C}$. Then $S^{-1} = q^{-1}S + (q^{-1} - 1)$.*

Proof. Through straightforward computations we observe that

$$\begin{aligned}(q^{-1}S + (q^{-1}-1))S &= q^{-1}S^2 + (q^{-1}-1)S \\ &= q^{-1}((q-1)S + q) + (q^{-1}-1)S \\ &= (1-q^{-1})S + 1 + (q^{-1}-1)S \\ &= 1\end{aligned}$$

and so $S^{-1} = q^{-1}S + (q^{-1} - 1)$.

□

Lemma 4.3.4. *Suppose H is a group with elements a, a^{-1}, and S such that $aSa^{-1} = a^{-1}Sa$. Then $a^2S = Sa^2$.*

Proof. Through straightforward computation we observe that

$$\begin{aligned}a^2S &= a(aSa^{-1})a \\ &= a(a^{-1}Sa)a \\ &= Sa^2\end{aligned}$$

as required. □

Now we prove theorem 4.3.2, showing that $\mathcal{H}^{\text{fin}}_{\tilde{W}} \cong \mathcal{H}_t$.

Proof. We begin showing that φ is a morphism of $\mathbb{C}$-algebras. We will go through each of the four relations and show they are respected by φ. First using lemma 4.3.3 we observe

$$\begin{aligned}\varphi(t)\varphi(t^{-1}) &= (q^{1/2}(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho))(q^{-1/2}S_{\rho^{-1}}S_\sigma) \\ &= q^{-1}S_\sigma S_\rho S_{\rho^{-1}}S_\sigma + (q^{-1}-1)S_\rho S_{\rho^{-1}}S_\sigma \\ &= q^{-1}S_\sigma^2 + (q^{-1}-1)S_\sigma \\ &= 1 && \text{by 4.3.3} \\ &= \varphi(tt^{-1})\end{aligned}$$

so φ respects the first relation. Reversing the order, we get

$$\begin{aligned}
\varphi(t^{-1})\varphi(t) &= (q^{-1/2}S_{\rho^{-1}}S_\sigma)(q^{1/2}(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho)) \\
&= q^{-1}S_{\rho^{-1}}S_\sigma S_\sigma S_\rho + (q^{-1}-1)S_{\rho^{-1}}S_\sigma S_\rho \\
&= S_{\rho^{-1}}(q^{-1}S_\sigma^2 + (q^{-1}-1)S_\sigma)S_\rho \\
&= S_{\rho^{-1}}(1)S_\rho && \text{by 4.3.3} \\
&= 1 \\
&= \varphi(t^{-1}t)
\end{aligned}$$

so the second relation is respected as well. Next for the quadratic relation, we get

$$\begin{aligned}
\varphi(S)^2 &= S_\sigma^2 \\
&= (q-1)S_\sigma + q \\
&= \varphi((q-1)S + q) \\
&= \varphi(S^2)
\end{aligned}$$

as required. Finally we will show that relation (4) is respected. We will compute both $\varphi(StSt)$ and $\varphi(tStS)$ separately and show that they are equal. Observe

$$\begin{aligned}
\varphi(S)\varphi(t)\varphi(S)\varphi(t) &= (S_\sigma(q^{1/2}(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho)))^2 \\
&= q(S_\sigma(q^{-1}S_\sigma + (q^{-1}-1))S_\rho)^2 \\
&= q(S_\sigma^{-1}S_\sigma S_\rho)^2 && \text{by 4.3.3} \\
&= qS_\rho^2
\end{aligned}$$

and by lemma 4.3.4,

$$\begin{aligned}
&\varphi(t)\varphi(S)\varphi(t)\varphi(S)\\
&\quad = ((q^{1/2}(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho))S_\sigma(q^{1/2}(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho))S_\sigma)\\
&\quad = q(q^{-2}S_\sigma S_\rho S_\sigma^2 S_\rho S_\sigma + q^{-1}(q^{-1}-1)S_\sigma S_\rho S_\sigma S_\rho S_\sigma\\
&\qquad + (q^{-1}-1)q^{-1}S_\rho S_\sigma^2 S_\rho S_\sigma + (q^{-1}-1)^2 S_\rho S_\sigma S_\rho S_\sigma)\\
&\quad = q(q^{-1}(1-q^{-1})S_\sigma S_\rho S_\sigma S_\rho S_\sigma + q^{-1}S_\sigma S_\rho^2 S_\sigma\\
&\qquad + q^{-1}(q^{-1}-1)S_\sigma S_\rho S_\sigma S_\rho S_\sigma - (q^{-1}-1)^2 S_\rho S_\sigma S_\rho S_\sigma\\
&\qquad + (q^{-1}-1)S_\rho^2 S_\sigma + (q^{-1}-1)S_\rho^2 S_\sigma + (q^{-1}-1)^2 S_\rho S_\sigma S_\rho S_\sigma)\\
&\quad = q((1-q^{-1})S_\sigma S_\rho^2 + S_\rho^2 + (q^{-1}-1)S_\sigma S_\rho^2)\\
&\quad = qS_\rho^2 \qquad\qquad \text{by 4.3.4.}
\end{aligned}$$

Together, these computations show us that

$$\varphi(S)\varphi(t)\varphi(S)\varphi(t) = \varphi(StSt) = \varphi(tStS) = \varphi(t)\varphi(S)\varphi(t)\varphi(S).$$

This concludes the justification that φ is a morphism of $\mathbb{C}$-algebras.

Now we define a map in the other direction to show that φ is an isomorphism. So we define

$$\begin{aligned}
\psi : \mathcal{H}_{\tilde{W}}^{\text{fin}} &\longrightarrow \mathcal{H}_t\\
S_\sigma &\longmapsto S\\
S_\rho &\longmapsto q^{-1/2}tS\\
S_{\rho^{-1}} &\longmapsto q^{-1/2}St^{-1} + (q^{-1/2}-q^{1/2})t^{-1}.
\end{aligned}$$

We must again show that this defines a morphism of $\mathbb{C}$-algebras. That is, we must show that the relations

$$\begin{gathered}
\psi(S_\sigma)^2 = \psi((q-1)S_\sigma + q),\\
\psi(S_\rho)\psi(S_{\rho^{-1}}) = 1 = \psi(S_{\rho^{-1}})\psi(S_\rho),\\
\psi(S_\rho)\psi(S_\sigma)\psi(S_{\rho^{-1}}) = \psi(S_{\rho^{-1}})\psi(S_\sigma)\psi(S_\rho)
\end{gathered}$$

are satisfied.

For the first relation, we observe

$$\begin{aligned}\psi(S_\sigma^2) &= S^2\\ &= (q-1)S + q\\ &= (q-1)\psi(S_\sigma) + q\\ &= \psi((q-1)S_\sigma + q)\end{aligned}$$

as required. Now we move onto other relations. Again using lemma 4.3.3, we compute

$$\begin{aligned}\psi(S_\rho^{-1})\psi(S_\rho) &= (q^{-1/2}St^{-1} + (q^{-1/2} - q^{1/2})t^{-1})(q^{-1/2}tS)\\ &= (q^{1/2}(q^{-1}St^{-1} + (q^{-1}-1)t^{-1}))(q^{-1/2}tS)\\ &= q^{-1}St^{-1}tS + (q^{-1}-1)t^{-1}tS\\ &= q^{-1}S^2 + (q^{-1}-1)S\\ &= 1 && \text{by } 4.3.3\\ &= \psi(S_{\rho^{-1}}S_\rho)\end{aligned}$$

and

$$\begin{aligned}\psi(S_\rho)\psi(S_{\rho^{-1}}) &= (q^{-1/2}tS)(q^{-1/2}St^{-1} + (q^{-1/2} - q^{1/2})t^{-1})\\ &= (q^{-1/2}tS)(q^{1/2}(q^{-1}St^{-1} + (q^{-1}-1)t^{-1}))\\ &= t(q^{-1}S^2 + (q^{-1}-1)S)t^{-1}\\ &= t(1)t^{-1} && \text{by } 4.3.3\\ &= 1\end{aligned}$$

as required. Finally, we compute $\psi(S_\rho)\psi(S_\sigma)\psi(S_{\rho^{-1}})$ and $\psi(S_{\rho^{-1}})\psi(S_\sigma)\psi(S_\rho)$ separately, then show that they are both equal. First we observe that by lemma 4.3.3, we can write

$$\psi(S_{\rho^{-1}}) = (q^{-1/2}St^{-1} + (q^{-1/2} - q^{1/2})t^{-1}) = q^{1/2}t^{-1}S^{-1}.$$

This lets us more easily compute the following,

$$\begin{aligned}
\psi(S_\rho)\psi(S_\sigma)\psi(S_{\rho^{-1}}) &= (q^{-1/2}St)(S)(q^{1/2}t^{-1}S^{-1}) \\
&= StSt^{-1}S^{-1} \\
&= t^{-1}(tStS)t^{-1}S^{-1} \\
&= t^{-1}(StSt)t^{-1}S^{-1} \\
&= t^{-1}St.
\end{aligned}$$

Now computing the other multiplication, we observe

$$\begin{aligned}
\psi(S_{\rho^{-1}})\psi(S_\sigma)\psi(S_\rho) &= (q^{1/2}t^{-1}S^{-1})(S)(q^{-1/2}St) \\
&= t^{-1}St.
\end{aligned}$$

So we have

$$\psi(S_\rho)\psi(S_\sigma)\psi(S_{\rho^{-1}}) = t^{-1}St = \psi(S_{\rho^{-1}})\psi(S_\sigma)\psi(S_\rho)$$

as required.

Therefore ψ is a morphism of $\mathbb{C}$-algebras, and it remains to show that φ and ψ are inverses of each other. We will show that the composition of these two functions sends each generator to itself for both algebras.

First we show that $\psi \circ \varphi = \mathrm{id}_{\mathcal{H}_t}$. We compute

$$\psi \circ \varphi(S) = \psi(S_\sigma) = S,$$

and

$$\begin{aligned}
\psi \circ \varphi(t) &= \psi(q^{1/2}(q^{-1}S_\rho S_\sigma + (q^{-1}-1)S_\rho)) \\
&= q^{-1/2}\psi(S_\rho)\psi(S_\sigma) + (q^{-1/2} - q^{1/2})\psi(S_\rho) \\
&= q^{-1/2}(q^{-1/2}tS)(S) + (q^{-1/2} - q^{1/2})(q^{-1/2}tS) \\
&= q^{-1}tS^2 + (q^{-1}-1)tS \\
&= q^{-1}t((q-1)S + q) + (q^{-1}-1)tS \\
&= t && \text{by 4.3.3,}
\end{aligned}$$

and finally

$$
\begin{aligned}
\psi \circ \varphi(t^{-1}) &= \psi(q^{-1/2} S_\sigma S_{\rho^{-1}}) \\
&= q^{-1/2} \psi(S_\sigma) \psi(S_{\rho^{-1}}) \\
&= q^{-1/2}(S)(q^{-1/2} S t^{-1} + (q^{-1/2} - q^{1/2}) t^{-1}) \\
&= q^{-1} S^2 t^{-1} + (q^{-1} - 1) S t^{-1} \\
&= (q^{-1} S^2 + (q^{-1} - 1) S) t^{-1} \\
&= t^{-1} && \text{by 4.3.3.}
\end{aligned}
$$

It remains to show that $\varphi \circ \psi = \mathrm{id}_{\mathcal{H}^{\mathrm{fin}}_{\tilde{W}}}$. We compute

$$
\varphi \circ \psi(S_\sigma) = \varphi(S) = S_\sigma,
$$

and

$$
\begin{aligned}
\varphi \circ \psi(S_\rho) &= \varphi(q^{-1/2} t S) \\
&= q^{-1/2} \varphi(t) \varphi(S) \\
&= q^{-1/2}(q^{1/2}(q^{-1} S_\rho S_\sigma + (q^{-1} - q) S_\rho)) S_\sigma \\
&= q^{-1} S_\rho S_\sigma^2 + (q^{-1} - 1) S_\rho S_\sigma \\
&= S_\rho (q^{-1} S_\sigma^2 + (q^{-1} - 1) S_\sigma) \\
&= S_\rho && \text{by 4.3.3,}
\end{aligned}
$$

and finally

$$
\begin{aligned}
\varphi \circ \psi(S_\rho^{-1}) &= \varphi(q^{-1/2} S t^{-1} + (q^{-1/2} - q^{1/2}) t^{-1}) \\
&= q^{-1/2} \varphi(S) \varphi(t^{-1}) + (q^{-1/2} - q^{1/2}) \varphi(t^{-1}) \\
&= q^{-1/2}(S_\sigma)(q^{-1/2} S_\sigma S_{\rho^{-1}}) + (q^{-1/2} - q^{1/2})(q^{-1/2} S_\sigma S_{\rho^{-1}}) \\
&= (q^{-1} S_\sigma^2 + (q^{-1} - 1) S_\sigma) S_{\rho^{-1}} \\
&= S_{\rho^{-1}} && \text{by 4.3.3}
\end{aligned}
$$

This concludes the proof that φ and ψ are inverses of each other, and therefore φ is an isomorphism.

□

Now we have another finite presentation of the Iwahori-Hecke algebra which is useful for connecting the previous presentations to the following one.

4.4 Bernstein-Lusztig presentation

Finally, we have one more presentation of the Iwahori-Hecke algebra to consider, known as the Bernstein-Lusztig presentation.

The following is a definition of the Bernstein-Lusztig presentation of the affine Hecke algebra for $G = \mathrm{GL}_2(F)$. It is worth noting that the third relation in this definition is made significantly simpler by restricting to the $\mathrm{GL}_2(F)$ case.

Definition 4.4.1. Let $X^\vee = X_*(T)(F)$ be the F-points of the cocharacter lattice of G and let ν be an indeterminate. The *Bernstein-Lusztig presentation of the affine Hecke algebra* is the $\mathbb{C}[\nu]$-algebra with basis
$\{T_s\theta_x \mid s \in W, \quad x \in X^\vee\}$ and relations:

(i) $(T_\sigma + 1)(T_\sigma - \nu^2) = 0$

(ii) $\theta_x\theta_{x'} = \theta_{x+x'}$ if $x, x' \in X^\vee$

(iii) $\theta_x T_\sigma - T_\sigma\theta_{\sigma(x)} = (\nu^2 - 1)\frac{\theta_x - \theta_{\sigma(x)}}{1-\theta_{-\alpha^\vee}}$.

Remark 4.4.2. The Bernstein-Lusztig presentation is especially useful for understanding the structure of the Iwahori-Hecke algebra, and in particular gives us relations which greatly simplify the process of finding the centre of the algebra. Notice that relation *(ii)* shows us that each θ_x commutes with each other. Then relation *(iii)* in the above definition can be used to find the centre of this algebra. Whenever $x = \sigma(x)$, θ_x commutes with every element in the algebra by this relation. Moreover it can be seen that the centre of this algebra is generated by elements $\theta_{f_1+f_2}$ and $\theta_{f_1} + \theta_{f_2}$ since these are the minimal elements which are invariant under the action of W.

In order for the relations in definition 4.4.1 to make sense within the context of the basis in the definition, we must recognise that the term

$$\frac{\theta_x - \theta_{\sigma(x)}}{1 - \theta_{-\alpha^\vee}}$$

found in the last relation in the above definition can be described using basis elements.

Lemma 4.4.3. *For $x \in X^\vee$ and θ_x as described in the definition above, $\frac{\theta_x - \theta_{\sigma(x)}}{1-\theta_{-\alpha^\vee}}$ is a linear combination of elements θ_y for $y \in X^\vee$.*

Proof. Let $x = a_1 f_1 + a_2 f_2 \in X^\vee$. First we suppose $a_1 - a_2 \geq 1$. Then using the equation for the partial sum of a geometric series,

$$\begin{aligned}\frac{\theta_x - \theta_{\sigma(x)}}{1 - \theta_{-\alpha^\vee}} &= \frac{\theta_{f_1}^{a_1}\theta_{f_2}^{a_2} - \theta_{f_1}^{a_2}\theta_{f_2}^{a_1}}{1 - \theta_{-\alpha^\vee}} \\ &= \frac{\theta_{f_1}^{a_1}\theta_{f_2}^{a_2}(1 - \theta_{f_1}^{a_2-a_1}\theta_{f_2}^{a_1-a_2})}{1 - \theta_{-\alpha^\vee}} \\ &= \theta_{f_1}^{a_1}\theta_{f_2}^{a_2}\frac{(1 - \theta_{-\alpha^\vee}^{a_1-a_2})}{1 - \theta_{-\alpha^\vee}} \\ &= \theta_{f_1}^{a_1}\theta_{f_2}^{a_2}\sum_{i=0}^{a_1-a_2-1}\theta_{-\alpha^\vee}^{i} \\ &= \sum_{i=0}^{a_1-a_2-1}\theta_{(a_1-i)f_1+(a_2+i)f_2}\end{aligned}$$

which is exactly a linear combination of elements of the form $T_s\theta_y$ for $y \in X^\vee$ where $s = 1$ and therefore $T_s = 1$. Now we consider the cases $a_1 - a_2 = 0$ and $a_1 - a_2 \leq -1$. If $a_1 - a_2 = 0$, we have

$$(q-1)\frac{\theta_x - \theta_{\sigma(x)}}{1 - \theta_{-\alpha^\vee}} = (q-1)\frac{\theta_x - \theta_x}{1 - \theta_{-\alpha^\vee}} = 0.$$

If $a_1 - a_2 \leq -1$, then $a_2 - a_1 \geq 1$ so we observe

$$\begin{aligned}(q-1)\frac{\theta_x - \theta_{\sigma(x)}}{1 - \theta_{-\alpha^\vee}} &= (1-q)\frac{\theta_{\sigma(x)} - \theta_x}{1 - \theta_{-\alpha^\vee}} \\ &= \sum_{i=0}^{a_2-a_1-1}(1-q)\theta_{(a_1+i)f_1+(a_2-i)f_2}\end{aligned}$$

by the same work as above, so this is a linear combination of elements of the form $T_s\theta_y$ as well. $\square$

This allows us to understand the relations of the affine Hecke algebra in relation to the vector space basis given in the definition. Now we specialise the affine Hecke algebra by sending the indeterminate ν to the complex value $q^{1/2}$ and denote the resulting algebra by $\mathcal{H}_{\mathrm{BL}}$. We prove that this specialisation of the affine Hecke algebra for is another presentation of the Iwahori-Hecke algebra for G.

Theorem 4.4.4. *Consider $\mathcal{H}_t$ as defined in theorem 4.3.2 and $\mathcal{H}_{BL}$ defined as above. Then $\mathcal{H}_t \cong \mathcal{H}_{BL}$.*

In order to prove this we first introduce some more results and definitions. We can view $\mathcal{H}_{\mathrm{BL}}$ as an algebra with generators T_σ, θ_{f_1}, and θ_{f_2} and the same relations given in the definition. It is straightforward to see that we get each basis element from these three generators since if $x = a_1f_1 + a_2f_2$, then $\theta_x = \theta_{f_1}^{a_1}\theta_{f_2}^{a_2}$, so we just multiply by T_σ as needed.

We define a semidirect product $X^\vee \rtimes W$ and show that $\tilde{W} \cong X^\vee \rtimes W$. In order to do this, we must first define an action of W on $X^\vee$. We choose the action determined by $\sigma \cdot (a_1f_1 + a_2f_2) = a_2f_1 + a_1f_2$ for an arbitrary element $a_1f_1 + a_2f_2 \in X^\vee$. Then $X^\vee \rtimes W$ is the set $X^\vee \times W$ with multiplication defined for arbitrary $x_1, x_2 \in X^\vee$ and $w_1, w_2 \in W$ by $(x_1 \rtimes w_1)(x_2 \rtimes w_2) = (x_1 + (w_1 \cdot x_2) \rtimes w_1w_2)$.

Using the isomorphism from proposition 3.0.1, we need only to show that $X^\vee \rtimes W \cong \tilde{\mathrm{W}}_{\mathrm{Mat}}$ in order to show that $X^\vee \rtimes W \cong \tilde{W}$.

Lemma 4.4.5. *For the extended affine Weyl group $\tilde{W}$, the Weyl group W, and X as defined above, we have the isomorphism $\tilde{W} \cong X^\vee \rtimes W$.*

Proof. If we show that $\tilde{\mathrm{W}}_{\mathrm{Mat}} \cong X^\vee \rtimes W$ then using proposition 3.0.1 we are done. We define inclusions

$$\begin{aligned} X^\vee &\hookrightarrow \tilde{\mathrm{W}}_{\mathrm{Mat}} \\ a_1 f_1 + a_2 f_2 &\longmapsto \begin{pmatrix} \varpi^{a_2} & 0 \\ 0 & \varpi^{a_1} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} W &\hookrightarrow \tilde{\mathrm{W}}_{\mathrm{Mat}} \\ \sigma &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Putting these together, we get an inclusion

$$\iota : X^\vee \rtimes W \hookrightarrow \tilde{\mathrm{W}}_{\mathrm{Mat}}$$

and in particular

$$\begin{aligned} \iota(a_1 f_1 + a_2 f_2 \rtimes \sigma) &= \iota(a_1 f_1 + a_2 f_2 \rtimes 1)\iota(0 \rtimes \sigma) \\ &= \begin{pmatrix} \varpi^{a_2} & 0 \\ 0 & \varpi^{a_1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varpi^{a_2} \\ \varpi^{a_1} & 0 \end{pmatrix} \end{aligned}$$

and it remains only to show that ι respects multiplication. Suppose $x = a_1 f_1 + a_2 f_2 \in X^\vee$, $y = b_1 f_1 + b_2 f_2 \in X^\vee$ and $w, w' \in W$. We split this into four cases and show that

$$\iota(x \rtimes w)\iota(y \rtimes w') = \iota((x \rtimes w)(y \rtimes w'))$$

in each case.

(i) Suppose $w = w' = 1$. Then

$$\begin{aligned}
\iota(x \rtimes 1)\iota(y \rtimes 1) &= \begin{pmatrix} \varpi^{a_2} & 0 \\ 0 & \varpi^{a_1} \end{pmatrix} \begin{pmatrix} \varpi^{b_2} & 0 \\ 0 & \varpi^{b_1} \end{pmatrix} \\
&= \begin{pmatrix} \varpi^{a_2+b_2} & 0 \\ 0 & \varpi^{a_1+b_1} \end{pmatrix} \\
&= \iota((a_1+b_1)f_1 + (a_2+b_2)f_2 \rtimes 1) \\
&= \iota(x + 1 \cdot y \rtimes 1) \\
&= \iota((x \rtimes 1)(y \rtimes 1))
\end{aligned}$$

as required.

(ii) Suppose $w = \sigma$ and $w' = 1$. Then

$$\begin{aligned}
\iota(x \rtimes \sigma)\iota(y \rtimes 1) &= \begin{pmatrix} 0 & \varpi^{a_2} \\ \varpi^{a_1} & 0 \end{pmatrix} \begin{pmatrix} \varpi^{b_2} & 0 \\ 0 & \varpi^{b_1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \varpi^{a_2+b_1} \\ \varpi^{a_1+b_2} & 0 \end{pmatrix} \\
&= \iota((a_1+b_2)f_1 + (a_2+b_1)f_2 \rtimes \sigma) \\
&= \iota((a_1f_1 + a_2f_2) + (b_2f_1 + b_1f_2) \rtimes \sigma) \\
&= \iota(x + \sigma \cdot y \rtimes \sigma) \\
&= \iota((x \rtimes \sigma)(y \rtimes 1))
\end{aligned}$$

as required.

(iii) Suppose $w = 1$ and $w' = \sigma$.

$$\begin{aligned}
\iota(x \rtimes 1)\iota(y \rtimes \sigma) &= \begin{pmatrix} \varpi^{a_2} & 0 \\ 0 & \varpi^{a_1} \end{pmatrix} \begin{pmatrix} 0 & \varpi^{b_2} \\ \varpi^{b_1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \varpi^{a_2+b_2} \\ \varpi^{a_1+b_1} & 0 \end{pmatrix} \\
&= \iota((a_1+b_1)f_1 + (a_2+b_2)f_2 \rtimes \sigma) \\
&= \iota(x + 1 \cdot y \rtimes \sigma) \\
&= \iota((x \rtimes 1)(y \rtimes \sigma))
\end{aligned}$$

as required.

(iv) Suppose $w = w' = \sigma$. Then

$$\begin{aligned}\iota(x \rtimes \sigma)\iota(y \rtimes \sigma) &= \begin{pmatrix} 0 & \varpi^{a_2} \\ \varpi^{a_1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \varpi^{b_2} \\ \varpi^{b_1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varpi^{a_2+b_1} & 0 \\ 0 & \varpi^{a_1+b_2} \end{pmatrix} \\ &= \iota((a_1+b_2)f_1 + (a_2+b_1)f_2 \rtimes 1) \\ &= \iota((a_1f_1 + a_2f_2) + (b_2f_1 + b_1f_2) \rtimes 1) \\ &= \iota(x + \sigma \cdot y \rtimes 1) \\ &= \iota((x \rtimes \sigma)(y \rtimes \sigma))\end{aligned}$$

as required.

Therefore ι is an injective homomorphism. It remains to show that ι is surjective.

Suppose $\begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \in \tilde{\mathrm{W}}_{\mathrm{Mat}}$. Then we can choose $bf_1 + af_2 \rtimes 1$ so that

$$\iota(bf_1 + af_2 \rtimes 1) = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}.$$

Now suppose $\begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix} \in \tilde{\mathrm{W}}_{\mathrm{Mat}}$. Then we choose $bf_1 + af_2 \rtimes \sigma$ so that

$$\iota(bf_1 + af_2 \rtimes \sigma) = \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix}.$$

Therefore ι is a bijective homomorphism of groups, and thus an isomorphism $X \rtimes W \cong \tilde{\mathrm{W}}_{\mathrm{Mat}}$. □

Next we define a subgroup of $X^\vee$

$$\mathrm{X}_{\mathrm{dom}} = \{x \in X^\vee \mid \langle \alpha_1, x \rangle \geq 0\}.$$

In particular, for $x = a_1 f_1 + a_2 f_2 \in X^\vee$, we get

$$\begin{aligned}\langle \alpha_1, x\rangle &= \langle f_1 - f_2, a_1 f_1 + a_2 f_2\rangle \\ &= a_1 - a_2.\end{aligned}$$

With this in mind, we can write

$$\begin{aligned}\mathrm{X}_{\mathrm{dom}} &= \{a_1 f_1 + a_2 f_2 \in X^\vee \mid a_1 - a_2 \geq 0\} \\ &= \{a_1 f_1 + a_2 f_2 \in X^\vee \mid a_1 \geq a_2\}.\end{aligned}$$

This group is useful to work with because of the following two lemmas.

Lemma 4.4.6. *Suppose $x \in \mathrm{X}_{dom}$. Then $x = (\tau\sigma)^n \rho^{2k}$ or $x = (\tau\sigma)^n \tau \rho^{2k+1}$ for some $n, k \in \mathbb{Z}$ and $n \geq 0$.*

Proof. Let $x \in \mathrm{X}_{\mathrm{dom}}$. That is,

$$x = \begin{pmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{pmatrix}$$

for some $a, b \in \mathbb{Z}$ such that $a \geq b$. Then $a - b \geq 0$ and is either even or odd, so we consider these cases separately.

(i) Suppose $a - b = 2n$ for some $n \in \mathbb{Z}$, $n \geq 0$. We also choose $k = a + b$. Adding and subtracting these equations together gives us $a = k + n$ and $b = k - n$ respectively. In this case,

$$\begin{aligned} w &= \begin{pmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{pmatrix} \\ &= \begin{pmatrix} \varpi^{k-n} & 0 \\ 0 & \varpi^{k+n} \end{pmatrix} \\ &= \begin{pmatrix} \varpi^{-n} & 0 \\ 0 & \varpi^{n} \end{pmatrix} \begin{pmatrix} \varpi^{k} & 0 \\ 0 & \varpi^{k} \end{pmatrix} \\ &= (\tau\sigma)^n \rho^{2k} \end{aligned}$$

with $n \geq 0$ as required.

(ii) Alternatively suppose that $a - b = 2n + 1$ for some $n \in \mathbb{Z}$, $n \geq 0$. Next choose $k = a + b$. Again we take both the sum and difference of these equations to get $a = k + n + 1$ and $b = k - n$. We observe

$$\begin{aligned} w &= \begin{pmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{pmatrix} \\ &= \begin{pmatrix} \varpi^{k-n} & 0 \\ 0 & \varpi^{k+n+1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varpi^{-n-1} \\ \varpi^{n+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \varpi^k \\ \varpi^{k+1} & 0 \end{pmatrix} \\ &= (\sigma\tau)^n \tau \rho^{2k+1} \end{aligned}$$

with $n \geq 0$ as required.

Therefore the lemma is proved for all values of $x \in \mathrm{X}_{\mathrm{dom}}$. □

Lemma 4.4.7. *Suppose $w, w' \in \mathrm{X}_{dom}$. Then $\ell(ww') = \ell(w) + \ell(w')$.*

Proof. Suppose that $w, w' \in \mathrm{X}_{\mathrm{dom}}$. Then by lemma 4.4.6, we have the following four cases to consider.

(i) Suppose $w = (\tau\sigma)^n \rho^{2k}$ and $w' = (\tau\sigma)^{n'} \rho^{2k'}$ for some $n, n', k, k' \in \mathbb{Z}$, $n, n' \geq 0$. Then $\ell(w) = 2n$, $\ell(w') = 2n'$, and

$$\begin{aligned} ww' &= (\tau\sigma)^n \rho^{2k} (\tau\sigma)^{n'} \rho^{2k'} \\ &= (\tau\sigma)^n (\tau\sigma)^{n'} \rho^{2k} \rho^{2k'} \\ &= (\tau\sigma)^{n+n'} \rho^{2(k+k')} \end{aligned}$$

and so $\ell(ww') = 2(n + n')$. Therefore $\ell(ww') = 2n + 2n' = \ell(w) + \ell(w')$.

(ii) Suppose $w = (\tau\sigma)^n \rho^{2k}$ and $w' = (\tau\sigma)^{n'} \tau \rho^{2k'+1}$ for some $n, n', k, k' \in \mathbb{Z}$, $n, n' \geq 0$. Then $\ell(w) = 2n$, $\ell(w') = 2n' + 1$, and

$$\begin{aligned} ww' &= (\tau\sigma)^n \rho^{2k} (\tau\sigma)^{n'} \tau \rho^{2k'+1} \\ &= (\tau\sigma)^n (\tau\sigma)^{n'} \tau \rho^{2k} \rho^{2k'+1} \\ &= (\tau\sigma)^{n+n'} \tau \rho^{2(k+k')+1} \end{aligned}$$

and so $\ell(ww') = 2(n+n')+1$. Therefore $\ell(ww') = 2n+2n'+1 = \ell(w)+\ell(w')$.

(iii) Suppose $w = (\tau\sigma)^n\tau\rho^{2k+1}$ and $w' = (\tau\sigma)^{n'}\rho^{2k'}$ for some $n, n', k, k' \in \mathbb{Z}$, $n, n' \geq 0$. Then $\ell(w) = 2n+1$, $\ell(w') = 2n'$, and

$$\begin{aligned} ww' &= (\tau\sigma)^n\tau\rho^{2k+1}(\tau\sigma)^{n'}\rho^{2k'} \\ &= (\tau\sigma)^n\tau(\sigma\tau)^{n'}\rho^{2k+1}\rho^{2k'} \\ &= (\tau\sigma)^n(\tau\sigma)^{n'}\tau\rho^{2(k+k')+1} \\ &= (\tau\sigma)^{n+n'}\tau\rho^{2(k+k')+1} \end{aligned}$$

and so $\ell(ww') = 2(n+n')+1$. Therefore $\ell(ww') = 2n+1+2n' = \ell(w)+\ell(w')$.

(iv) Suppose $w = (\tau\sigma)^n\tau\rho^{2k+1}$ and $w' = (\tau\sigma)^{n'}\tau\rho^{2k'+1}$ for some $n, n', k, k' \in \mathbb{Z}$, $n, n' \geq 0$. Then $\ell(w) = 2n+1$, $\ell(w') = 2n'+1$, and

$$\begin{aligned} ww' &= (\tau\sigma)^n\tau\rho^{2k+1}(\tau\sigma)^{n'}\tau\rho^{2k'+1} \\ &= (\tau\sigma)^n\tau(\sigma\tau)^{n'}\sigma\rho^{2k+1}\rho^{2k'+1} \\ &= (\tau\sigma)^{n+n'+1}\rho^{2(k+k'+1)} \end{aligned}$$

and so $\ell(ww') = 2(n+n'+1)$. Therefore $\ell(ww') = 2n+1+2n'+1 = \ell(w)+\ell(w')$.

This covers all possibilities of w and w', and therefore proves that $\ell(ww') = \ell(w)+\ell(w')$ for all $w, w' \in \mathrm{X}_{\mathrm{dom}}$. $\square$

Lemma 4.4.8. *Suppose H is a $\mathbb{C}$-algebra with elements S and t such that $S^2 = (q-1)S + q$ and $StSt = tStS$ where $q \in \mathbb{C}$. Then for any integers $n, m \in \mathbb{Z}$,*

(i) $t^n(StS)^m = (StS)^m t^n$

(ii) $(StS)^m t^m = (StSt)^m$

(iii) $(StSt)^m t^n = t^n(StSt)^m$

(iv) $(StSt)^m(StS)^n = (StS)^n(StSt)^m$

(v) $(StSt)^m S = S(StSt)^m$.

Proof. (i) We first prove by induction that $t^n(StS) = (StS)t^n$. In our base case we have $n = 0$, and $StS = StS$ as required. Now if $t^k StS = StSt^k$ for some $k \in \mathbb{Z}$, we show that $t^{k+1}StS = StSt^{k+1}$ and $t^{k-1}StS = StSt^{k-1}$. First we assume the inductive hypothesis for $k \geq 0$ and observe

$$\begin{aligned} t^{k+1}StS &= t(t^k StS) \\ &= t(StSt^k) \qquad \text{by inductive hypothesis} \\ &= (tStS)t^k \\ &= (StSt)t^k \\ &= StSt^k. \end{aligned}$$

Next assume the inductive hypothesis for $k \leq 0$ and we observe

$$\begin{aligned} t^{k-1}StS &= t^{-1}(t^k StS) \\ &= t^{-1}(StSt^k) \\ &= t^{-1}(StSt)t^{-1}t^k \\ &= t^{-1}(tStS)t^{k-1} \\ &= StSt^{k-1}. \end{aligned}$$

By the two parts of the induction above, we have that $t^n StS = StSt^n$ for all $n \in \mathbb{Z}$.

Now we prove that $t^n(StS)^m = (StS)^m t^n$ by induction on m, using the result above. Our base case is when $m = 0$, we have $t^n = t^n$ as required. Now suppose our inductive hypothesis $t^n(StS)^k = (StS)^k t^n$ holds for $k \geq 0$. Then we observe

$$\begin{aligned} t^n(StS)^{k+1} &= t^n(StS)^k(StS) \\ &= (StS)^k t^n (StS) && \text{by inductive hypothesis} \\ &= (StS)^k (StS) t^n && \text{by above result} \\ &= (StS)^{k+1} t^n. \end{aligned}$$

Next assume our inductive hypothesis for $k \leq 0$ and observe

$$\begin{aligned} t^n(StS)^{k-1} &= t^n(StS)^k(StS)^{-1} \\ &= (StS)^k t^n (StS)^{-1} && \text{by inductive hypothesis} \\ &= (StS)^k (StS)^{-1}(StS) t^n (StS)^{-1} \\ &= (StS)^{k-1} t^n (StS)(StS)^{-1} && \text{by above result} \\ &= (StS)^{k-1} t^n. \end{aligned}$$

Therefore by the two inductions and base case, we have that $t^n(StS)^m = (StS)^m t^n$ for all $n, m \in \mathbb{Z}$.

(ii) We prove by induction that $(StS)^m t^m = (StSt)^m$. For the base case we can see immediately that

$$(StS)^0 t^0 = 1 = (StSt)^0.$$

Now suppose that for some integer $k \geq 0$ we have

$$(StS)^k t^k = (StSt)^k.$$

Then we can observe

$$\begin{aligned}(StS)^{k+1}t^{k+1} &= (StS)^k(StSt)t^k \\ &= (StS)^k(tStS)t^k \\ &= (StS)^k t(StS)t^k \\ &= (StS)^k t^{k+1}(StS) \qquad \text{by part } (i) \\ &= (StS)^k t^k (tStS) \\ &= (StSt)^{k+1} \qquad \text{by inductive hypothesis}\end{aligned}$$

as required.

(iii) Now we prove that $(StSt)^m t^n = t^n(StSt)^m$. Using parts (i) and (ii) we see

$$\begin{aligned}(StSt)^m t^n &= (StS)^m t^{n+m} \\ &= t^n(StS)^m t^m \\ &= t^n(StSt)^m\end{aligned}$$

as required.

(iv) Next we show that $(StSt)^m(StS)^n = (StS)^n(StSt)^m$. Using parts (i) and (ii) again, we get

$$\begin{aligned}(StSt)^m(StS)^n &= (StS)^m t^m (StS)^n \\ &= (StS)^{m+n} t^m \\ &= (StS)^n(StSt)^m\end{aligned}$$

as required.

(v) Next we show that $(StSt)^m S = S(StSt)^m$ by induction on m. When $m = 0$, it is straightforward to see

$$(StSt)^0 S = S = S(StSt)^0.$$

Now suppose that for some $k \geq 0$ we have

$$(StSt)^k S = S(StSt)^k.$$

Then we observe

$$\begin{aligned}
(StSt)^{k+1}S &= (StSt)^k(StSt)S \\
&= (StSt)^k S(tStS) \\
&= S(StSt)^k(StSt) \qquad \text{by inductive hypothesis} \\
&= S(StSt)^{k+1}
\end{aligned}$$

as required. Now suppose that the same inductive hypothesis holds for some $k \leq 0$. Then

$$\begin{aligned}
(StSt)^{k-1}S &= (StSt)^{-1}(StSt)^k S \\
&= (StSt)^{-1}S(StSt)^k \qquad \text{by inductive hypothesis} \\
&= t^{-1}S^{-1}t^{-1}S^{-1}S(StSt)^k \\
&= t^{-1}S^{-1}t^{-1}(StSt)^k \\
&= SS^{-1}t^{-1}S^{-1}t^{-1}(StSt)^k \\
&= S(StSt)^{k-1}
\end{aligned}$$

as required. Therefore by induction we have $(StSt)^m S = S(StSt)^m$ for all integers m.

□

Lemma 4.4.9. *Suppose H is a ring with the same assumptions as the previous lemma. Then for any integers m and n we have*

(i) if $m - n \geq 1$ then

$$(q^{-1}StS)^m t^n S - S(q^{-1}StS)^n t^m = (q-1)\sum_{i=0}^{m-n-1}(q^{-1}StS)^{m-i}t^{n+i},$$

(ii) and if $m - n \leq -1$ then

$$(q^{-1}StS)^m t^n S - S(q^{-1}StS)^n t^m = (1-q)\sum_{i=m-n}^{-1}(q^{-1}StS)^{m-i}t^{n+i}.$$

Proof. (i) We begin by assuming that $m-n \geq 1$, so we can write $m = n+c$ for some integer $c \geq 1$, and prove this by induction on c. For our base case, let $c = 1$. Then using lemma 4.4.8 we can see

$$\begin{aligned}
(q^{-1}StS)^{n+1}t^nS - S(q^{-1}StS)^nt^{n+1} \\
&= (q^{-1}StS)(q^{-1}StSt)^nS - S(q^{-1}StSt)^nt \\
&= (q^{-1}StSt)^n[(q^{-1}StS)S - St] \\
&= (q^{-1}StSt)^n[q^{-1}StS^2 - St] \\
&= (q^{-1}StSt)^n[(q-1)q^{-1}StS + St - St] \\
&= (q-1)(q^{-1}StS)^{n+1}t^n \\
&= (q-1)\sum_{i=0}^{0}(q^{-1}StS)^{n+1-i}t^{n+i}
\end{aligned}$$

as required. Now we assume for some $k \geq 1$ that

$$(q^{-1}StS)^{n+k}t^nS - S(q^{-1}StS)^nt^{n+k} = (q-1)\sum_{i=0}^{k-1}(q^{-1}StS)^{n+k-i}t^{n+i}. \qquad (*)$$

Then

$$\begin{aligned}
&(q^{-1}StS)^{n+k+1}t^nS - S(q^{-1}StS)^nt^{n+k+1} \\
&\quad= (q^{-1}StS)^{n+k}(q^{-1}StS)t^nS - S(q^{-1}StS)^nt^{n+k+1} \\
&\quad= (q^{-1}StS)^{n+k}t^nq^{-1}StS^2 - S(q^{-1}StS)^nt^{n+k+1} \\
&\quad= (q-1)(q^{-1}StS)^{n+k+1}t^n + (q^{-1}StS)^{n+k}t^nSt - S(q^{-1}StS)^nt^{n+k+1} \\
&\quad= (q-1)(q^{-1}StS)^{n+k+1}t^n + \left[(q^{-1}StS)^{n+k}t^NS - S(q^{-1}StS)^nt^{n+k}\right]t \\
&\quad= (q-1)(q^{-1}StS)^{n+k+1}t^n + \left[\sum_{i=0}^{k-1}(q^{-1}StS)^{n+k-i}t^{n+i}\right]t \qquad \text{by } (*) \\
&\quad= (q-1)(q^{-1}StS)^{n+k+1}t^n + \sum_{i=1}^{k}(q^{-1}StS)^{n+k+1-i}t^{n+i} \\
&\quad= (q-1)\sum_{i=0}^{k}(q^{-1}StS)^{n+k+1-i}t^{n+i}
\end{aligned}$$

as required. Then by induction, for all integers m and n such that $m - n \geq 1$, we have

$$(q^{-1}StS)^m t^n S - S(q^{-1}StS)^n t^m = (q-1)\sum_{i=0}^{m-n-1}(q^{-1}StS)^{m-i}t^{n-i}.$$

(ii) Now suppose $m - n \leq -1$, so we write $n = m + c$ for some integer $c \geq 1$, and we perform induction on c. For our base case, let $c = 1$. Then using lemma 4.4.8 we have

$$\begin{aligned}(q^{-1}StS)^m t^{m+c} S - &S(q^{-1}StS)^{m+c}t^m \\ &= (q^{-1}StSt)^m[tS - S(q^{-1}StS)] \\ &= (q^{-1}StSt)^m[tS - (q-1)(q^{-1}StS) - tS] \\ &= (1-q)(q^{-1}StS)^{m+1}t^m \\ &= (1-q)\sum_{i=-1}^{-1}(q^{-1}StS)^{m-i}t^{m+1-i}\end{aligned}$$

as required. Now we assume for some $k \geq 1$ that

$$(q^{-1}StS)^m t^{m+k} S - S(q^{-1}StS)^{m+k}t^m = (1-q)\sum_{i=-k}^{-1}(q^{-1}StS)^{m-i}t^{m+c+i}. \qquad (**)$$

Then

$$\begin{aligned}&(q^{-1}StS)^m t^{m+k+1} S - S(q^{-1}StS)^{m+k+1}t^k \\ &\quad= (q^{-1}StS)^m t^{m+k+1} S - (q-1)(q^{-1}StS)^{m+k+1}t^m - tS(q^{-1}StS)^{m+k}t^m \\ &\quad= (1-q)(q^{-1}StS)^m + k + 1t^m + t[(q^{-1}StS)^m t^{m+k} S - S(q^{-1}StS)^{m+k}t^m] \\ &\quad= (1-q)(q^{-1}StS)^{m+k+1}t^m + (1-q)\sum_{i=-k}^{-1}(q^{-1}StS)^{m-i}t^{m+k+i+1} \qquad \text{by } (**) \\ &\quad= (1-q)\sum_{i=-(k+1)}^{-1}(q^{-1}StS)^{m-i}t^{m+k+i+1}\end{aligned}$$

as required. Then by induction, we have that for all integers m and n such that $m - n \leq -1$,

$$(q^{-1}StS)^m t^n S - S(q^{-1}StS)^n t^m = (1-q)\sum_{i=m-n}^{-1}(q^{-1}StS)^{m-i}t^{n+i}.$$

□

Now we have what we need to prove theorem 4.4.4 and show that $\mathcal{H}_t \cong \mathcal{H}_{\mathrm{BL}}$.

Proof. First we define a map

$$\begin{aligned} \varphi : \mathcal{H}_t &\longrightarrow \mathcal{H}_{\mathrm{BL}} \\ S &\longmapsto T_\sigma \theta_0 = T_\sigma \\ t &\longmapsto T_1 \theta_{f_2} = \theta_{f_2} \\ t^{-1} &\longmapsto T_1 \theta_{-f_2} = \theta_{-f_2}. \end{aligned}$$

We first show that φ respects the relations of $\mathcal{H}_t$. First we see that

$$\begin{aligned} \varphi(S)^2 &= T_\sigma^2 \\ &= (q-1)T_\sigma + q \\ &= \varphi((q-1)S + q) \\ &= \varphi(S^2) \end{aligned}$$

and

$$\begin{aligned} \varphi(t)\varphi(t^{-1}) &= \theta_{f_2}\theta_{-f_2} \\ &= \theta_0 \\ &= 1 \\ &= \varphi(1) \\ &= 1 \\ &= \theta_{-f_2}\theta_{f_2} \\ &= \varphi(t^{-1})\varphi(t) \end{aligned}$$

as required. Finally, we show the relation that $StSt = tStS$. From the relations of $\mathcal{H}_{\mathrm{BL}}$, we can see

$$\theta_{f_1} T_\sigma - T_\sigma \theta_{f_2} = (q-1)\theta_{f_1}$$

which we can rearrange to get

$$T_\sigma \theta_{f_2} = \theta_{f_1}(T_\sigma + 1 - q)$$

and multiplying by T_σ on the right gives us

$$T_\sigma \theta_{f_2} T_\sigma = q\theta_{f_1}$$

by the inverse of T_σ from lemma 4.3.3. Now we can compute

$$\begin{aligned}
\varphi(S)\varphi(t)\varphi(S)\varphi(t) &= T_\sigma \theta_{f_2} T_\sigma \theta_{f_2} \\
&= q\theta_{f_1}\theta_{f_2} \\
&= \theta_{f_2}(q\theta_{f_1}) \\
&= \theta_{f_2} T_\sigma \theta_{f_2} T_\sigma \\
&= \varphi(t)\varphi(S)\varphi(t)\varphi(S)
\end{aligned}$$

as required. Therefore $\varphi : \mathcal{H}_t \to \mathcal{H}_{\mathrm{BL}}$ is a homomorphism of $\mathbb{C}$-algebras. Next we define a map in the other direction. For $x \in X^\vee$, we can write $x = a_1 f_1 + a_2 f_2$ for some $a_1, a_2 \in \mathbb{Z}$. We defined the map ψ by

$$\begin{aligned}
\psi : \mathcal{H}_{\mathrm{BL}} &\longrightarrow \mathcal{H}_t \\
T_1\theta_x = \theta_x &\longmapsto (q^{-1}StS)^{a_1} t^{a_2} \\
T_\sigma \theta_x &\longmapsto S(q^{-1}StS)^{a_1} t^{a_2}
\end{aligned}$$

We now show that ψ respects the relations of $\mathcal{H}_{\mathrm{BL}}$. Recall the list of relations is given by:

(i) $(T_\sigma + 1)(T_\sigma - q) = 0$

(ii) $\theta_x \theta_{x'} = \theta_{x+x'}$ if $x, x' \in X$

(iii) $\theta_x T_\sigma - T_\sigma \theta_{\sigma(x)} = (q-1)\frac{\theta_x - \theta_{\sigma(x)}}{1-\theta_{-\alpha^\vee}}$

We prove that ψ respects each relation independently.

(i) For the first relation, we observe

$$\begin{aligned}\psi(T_\sigma + 1)\psi(T_\sigma - q) &= (S+1)(S-q)\\ &= S^2 + (1-q)S - q\\ &= (q-1)S + q + (1-q)S - q\\ &= 0\\ &= \psi((T_\sigma + 1)(T_\sigma - q)).\end{aligned}$$

(ii) Next let $x, y \in X^\vee$, so we can write $x = a_1 f_1 + a_2 f_2$ and $y = b_1 f_1 + b_2 f_2$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. In particular we have $x + y = (a_1 + b_1) f_1 + (a_2 + b_2) f_2$. Then by lemma 4.4.8,

$$\begin{aligned}\psi(\theta_x \theta_y) &= \psi(\theta_{x+y})\\ &= (q^{-1}StS)^{a_1+b_1} t^{a_2+b_2}\\ &= q^{-(a_1+b_1)}(StS)^{a_1}(StS)^{b_1} t^{a_2} t^{b_2}\\ &= q^{-(a_1+b_1)}(StS)^{a_1} t^{a_2} (StS)^{b_1} t^{b_2} \qquad \text{by } 4.4.8\\ &= ((q^{-1}StS)^{a_1} t^{a_2})((q^{-1}StS)^{b_1} t^{b_2})\\ &= \psi(\theta_x)\psi(\theta_y)\end{aligned}$$

as required.

(iii) Finally we let $x \in X^\vee$ and write $x = a_1 f_1 + a_2 f_2$. We split this into three cases. First suppose that $a_1 - a_2 \geq 1$. We rewrite the third relation using the finite geometric series formula as

$$\begin{aligned}(q-1)\frac{\theta_x-\theta_{\sigma(x)}}{1-\theta_{-\alpha^\vee}} &= (q-1)\frac{\theta_{a_1f_1}\theta_{a_2f_2}-\theta_{a_2f_1}\theta_{a_1f_2}}{1-\theta_{-\alpha^\vee}} \\ &= (q-1)\frac{\theta_{a_1f_1+a_2f_2}(1-\theta_{-f_1}^{a_1-a_2}\theta_{f_2}^{a_1-a_2})}{1-\theta_{-f_1+f_2}} \\ &= (q-1)\frac{\theta_{a_1f_1+a_2f_2}(1-(\theta_{-f_1+f_2})^{a_1-a_2})}{1-\theta_{-f_1+f_2}} \\ &= (q-1)\sum_{i=0}^{a_1-a_2-1}\theta_{a_1f_1+a_2f_2}\theta_{-f_1+f_2}^i \\ &= (q-1)\sum_{i=0}^{a_1-a_2-1}\theta_{(a_1-i)f_1+(a_2+i)f_2}.\end{aligned}$$

Then from lemma 4.4.9, we observe

$$\begin{aligned}\psi(\theta_x)\psi(T_\sigma)-\psi(T_\sigma)\psi(\theta_{\sigma(x)}) &= (q^{-1}StS)^{a_1}t^{a_2}S-S(q^{-1}StS)^{a_2}t^{a_1} \\ &= (q-1)\sum_{i=0}^{a_1-a_2-1}(q^{-1}StS)^{a_1-i}t^{n+i} \\ &= (q-1)\sum_{i=0}^{a_1-a_2-1}\psi(\theta_{(a_1-i)f_1+(a_2+i)f_2})\end{aligned}$$

as required. Now suppose $a_1-a_2\leq -1$. We can once again rewrite the third relation as above, but this time using a modified geometric series formula to get

$$\begin{aligned}(q-1)\frac{\theta_x-\theta_{\sigma(x)}}{1-\theta_{-\alpha^\vee}} &= (q-1)\frac{\theta_{a_1f_1+a_2f_2}(1-(\theta_{-f_1+f_2})^{a_1-a_2})}{1-\theta_{-f_1+f_2}} \\ &= (1-q)\sum_{i=a_1-a_2}^{-1}\theta_{(a_1-i)f_1+(a_2+i)f_2}.\end{aligned}$$

Then by lemma 4.4.9 we have

$$\begin{aligned}\psi(\theta_x)\psi(T_\sigma) - \psi(T_\sigma)\psi(\theta_{\sigma(x)}) &= (q^{-1}StS)^{a_1}t^{a_2}S - S(q^{-1}StS)^{a_2}t^{a_1}\\ &= (1-q)\sum_{i=m-n}^{-1}(q^{-1}StS)^{a_1-i}t^{a_2+i}\\ &= (1-q)\sum_{i=m-n}^{-1}\psi(\theta_{(a_1-i)f_1+(a_2+i)f_2})\end{aligned}$$

as required. Finally, suppose $a_1 = a_2$. Then $\sigma(x) = x$ and so using lemma 4.4.8,

$$\begin{aligned}\psi(\theta_x)\psi(T_\sigma) - \psi(T_\sigma)\psi(\theta_{\sigma(x)}) &= (q^{-1}StS)^{a_1}t^{a_1}S - S(q^{-1}StS)^{a_1}t^{a_1}\\ &= (q^{-1}StSt)^{a_1}S - S(q^{-1}StSt)^{a_1}\\ &= 0\\ &= (q-1)\frac{\psi(\theta_x)-\psi(\theta_x)}{1-\psi(\theta_{-\alpha^\vee})}\\ &= (q-1)\frac{\psi(\theta_x)-\psi(\theta_{\sigma(x)})}{1-\psi(\theta_{-\alpha^\vee})}\end{aligned}$$

as required.

Therefore $\psi : \mathcal{H}_{\mathrm{BL}} \to \mathcal{H}_t$ is a homomorphism of $\mathbb{C}$-algebras. It remains to show that φ and ψ are inverses of each other, and therefore isomorphisms.

First consider $\varphi \circ \psi : \mathcal{H}_{\mathrm{BL}} \to \mathcal{H}_{\mathrm{BL}}$. On basis elements θ_x and $T_\sigma\theta_x$ where $x = a_1f_1 + a_2f_2$, we have

$$\begin{aligned}(\varphi\circ\psi)(\theta_x) &= \varphi((q^{-1}StS)^{a_1}t^{a_2})\\ &= (q^{-1}\varphi(S)\varphi(t)\varphi(S))^{a_1}\varphi(t)^{a_2}\\ &= (q^{-1}T_\sigma\theta_{f_2}T_\sigma)^{a_1}\theta_{f_2}^{a_2}\\ &= \theta_{a_1f_2}\theta_{a_2f_2}\\ &= \theta_x,\end{aligned}$$

and using this same result,

$$\begin{aligned}(\varphi \circ \psi)(T_\sigma \theta_x) &= \varphi(S)(\varphi \circ \psi)(\theta_x) \\ &= T_\sigma \theta_x.\end{aligned}$$

Now we consider $\psi \circ \varphi : \mathcal{H}_t \to \mathcal{H}_t$ and calculate on generators S, t, and t^{-1}. We observe

$$(\psi \circ \varphi)(S) = \psi(T_\sigma) = S,$$

$$\begin{aligned}(\psi \circ \varphi)(t) &= \psi(\theta_{f_2}) \\ &= (q^{-1}StS)^0 t^1 \\ &= t,\end{aligned}$$

and

$$\begin{aligned}(\psi \circ \varphi)(t^{-1}) &= \psi(\theta_{-f_2}) \\ &= (q^{-1}StS)^0 t^{-1} \\ &= t^{-1}.\end{aligned}$$

Therefore φ and ψ are inverse homomorphisms, and thus isomorphisms. This concludes the proof that $\mathcal{H}_t \cong \mathcal{H}_{\mathrm{BL}}$.

□

Chapter 5

Coming full circle (pentagon)

5.1 One Final Isomorphism

We have shown that the five $\mathbb{C}$-algebras are isomorphic,

$$H(G,I) \cong \mathcal{H}_{\tilde{W}} \cong \mathcal{H}_{\tilde{W}}^{\text{fin}} \cong \mathcal{H}_t \cong \mathcal{H}_{\text{BL}},$$

so each can be viewed as a presentation of the Iwahori-Hecke algebra. However each of these presentations offers a different perspective and is useful in various contexts. When working with generators and relations, the Bernstein-Lusztig presentation $\mathcal{H}_{\text{BL}}$ is often used, so it is useful to have a more direct understanding of the relationship between elements in this presentation and the corresponding characteristic function presentation as seen in $\mathcal{H}_{\tilde{W}}^{\text{fin}}$.

Theorem 5.1.1. *Let $\mathcal{H}_{BL}$ and $\mathcal{H}_{\tilde{W}}^{fin}$ be defined as above, with $\mathcal{H}_{BL}$ expressed in terms of a basis of elements θ_x and $T_\sigma\theta_x$ where $x \in X^\vee$. Let $x \in X^\vee$ be defined as $x = a_1 f_1 + a_2 f_2$ and recall that $x \in \mathrm{X}_{dom}$ if and only if $a_1 \geq a_2$.*

Then the maps

$$\varphi : \mathcal{H}_{BL} \longrightarrow \mathcal{H}^{fin}_{\tilde{W}}$$

$$\theta_x \longmapsto \begin{cases} q^{-\ell(x)}(S_\rho S_\sigma)^k S_\rho^{2a_2}, & a_1 = a_2 + k, \quad k \geq 0, \text{ if } X \in \mathrm{X}_{dom} \\ q^{\ell(x)}(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho)^k S_\rho^{2a_1}, & a_2 = a_1 + k, \quad k > 0, \text{ if } X \notin \mathrm{X}_{dom} \end{cases}$$

$$T_\sigma\theta_x \longmapsto \begin{cases} q^{-\ell(x)}S_\sigma(S_\rho S_\sigma)^k S_\rho^{2a_2}, & a_1 = a_2 + k, \quad k \geq 0, \text{ if } X \in \mathrm{X}_{dom} \\ q^{\ell(x)}S_\sigma(q^{-1}S_\sigma S_\rho + (q^{-1}-1)S_\rho)^k S_\rho^{2a_1}, & a_2 = a_1 + k, \quad k > 0, \text{ if } X \notin \mathrm{X}_{dom} \end{cases}$$

and

$$\begin{aligned} \psi : \mathcal{H}^{fin}_{\tilde{W}} &\longrightarrow \mathcal{H}_{BL} \\ S_\sigma &\longmapsto T_\sigma \\ S_\rho &\longmapsto q\theta_{f_2}T_\sigma \\ S_{\rho^{-1}} &\longmapsto q^{1/2}(q^{-1}T_\sigma\theta_{-f_2} + (q-1)\theta_{-f_2}). \end{aligned}$$

are isomorphisms and inverses of each other.

Since these are just compositions of functions already shown to be isomorphisms, we can see that these φ and ψ are isomorphisms as well, and inverses of each other. Finally, for some extra clarity on how elements of $\mathcal{H}_{\mathrm{BL}}$ relate to functions, we can compose φ with the embedding $\iota : \mathcal{H}^{\mathrm{fin}}_{\tilde{W}} \hookrightarrow \mathcal{H}_{\tilde{W}}$ given in the proof of theorem 4.2.2 to get the following isomorphism.

Corollary 5.1.2. *For the inclusion map $\iota : \mathcal{H}^{fin}_{\tilde{W}} \hookrightarrow \mathcal{H}_{\tilde{W}}$ as described in the proof of Theorem 4.2.2 and φ described in Theorem 5.1.1, the composition is* *given by*

$$\iota \circ \varphi : \mathcal{H}_{BL} \longrightarrow \mathcal{H}_{\tilde{W}}$$
$$\theta_x \longmapsto \begin{cases} q^{-\ell(x)/2} S_x & x \in \mathrm{X}_{dom} \\ q^{\ell(x)/2} S_{-x}^{-1} & x \notin \mathrm{X}_{dom} \end{cases}$$

$$T_\sigma \theta_x \longmapsto \begin{cases} q^{-\ell(x)/2} S_\sigma S_x & x \in \mathrm{X}_{dom} \\ q^{\ell(x)/2} S_\sigma S_{-x}^{-1} & x \notin \mathrm{X}_{dom}\,. \end{cases}$$

and this map describes an isomorphism of $\mathbb{C}$*-algebras.*

Since we are working in GL_2, we have the simple case that $x \notin \mathrm{X}_{\mathrm{dom}}$, if and only if $-x \in \mathrm{X}_{\mathrm{dom}}$. More generally for we would want to write a map that sends

$$\theta_x \mapsto q^{(\ell(x_2)-\ell(x_1))/2} S_{x_1} S_{x_2}^{-1}$$

where $x = x_1 - x_2$ for $x_1, x_2 \in \mathrm{X}_{\mathrm{dom}}$. We see in the following lemma that this map is equivalent to the one written above in theorem 5.1.1 in the GL_2 case.

Lemma 5.1.3. *Suppose* $x \in X^\vee$*. We can write* $x = x_1 - x_2$ *for some* $x_1, x_2 \in \mathrm{X}_{dom}$*. Then the product of characteristic functions* $S_{x_1} S_{x_2}^{-1}$ *is independent of choice of* x_1 *and* x_2*.*

Proof. Suppose $x \in X$. If $x \in \mathrm{X}_{\mathrm{dom}}$, we can always write $x = x - 0$, where both $x, 0 \in \mathrm{X}_{\mathrm{dom}}$. If $x \notin \mathrm{X}_{\mathrm{dom}}$, then $x = 0 - (-x)$ where $-x \in \mathrm{X}_{\mathrm{dom}}$. It remains to show that our definition of θ_x is independent of choice of x_1 and x_2.

Now suppose $x = x_1 - x_2 = y_1 - y_2$ where $x_1, x_2, y_1, y_2 \in \mathrm{X}_{\mathrm{dom}}$. We first note that by lemma 4.4.7,

$$S_{x_2} S_{y_2} = S_{x_2+y_2} = S_{y_2} S_{x_2}.$$

Inverting both sides, we get

$$S_{y_2}^{-1} S_{x_2}^{-1} = S_{x_2}^{-1} S_{y_2}^{-1}.$$

Rearranging further, this tells us

$$S_{x_2}^{-1} S_{y_2} = S_{y_2} S_{x_2}^{-1}.$$

We will use this identity in the following calculation. Observe

$$\begin{aligned}
(S_{x_1} S_{x_2}^{-1})(S_{y_1} S_{y_2}^{-1})^{-1} &= S_{x_1} S_{x_2}^{-1} S_{y_2} S_{y_1}^{-1} \\
&= S_{x_1} S_{y_2} S_{x_2}^{-1} S_{y_1}^{-1} && \text{by above calculation} \\
&= S_{x_1+y_2} (S_{y_1+x_2})^{-1} && \text{by lemma 4.4.7} \\
&= S_{x_1+y_2} (S_{x_1+y_2})^{-1} && \text{since } x_1 - x_2 = y_1 - y_2 \\
&= 1.
\end{aligned}$$

Therefore $(S_{x_1} S_{x_2}^{-1})(S_{y_1} S_{y_2}^{-1}) = 1$, and we can conclude that $S_{x_1} S_{x_2}^{-1} = S_{y_1} S_{y_2}^{-1}$ when the conditions of this lemma are satisfied. □

5.2 Conclusion

Now we have shown that each presentation given is isomorphic to $H(G, I)$, and we have maps to take us between these presentations. This is useful to us because each presentation gives us some insight into the Iwahori-Hecke algebra.

The first presentation $\mathcal{H}_{\tilde{W}}$ is often the presentation used to work with this algebra in the literature. It is a common way to view the Iwahori-Hecke algebra and gives us a straightforward way to understand elements as functions as per the original definition of $H(G, I)$. This gives us intuitive understanding of the group without losing sight of the definition, allowing us to write functions in $H(G, I)$ as combinations of characteristic functions whose convolutions we understand. This can streamline the description of modules defined over this algebra.

The next presentation, $\mathcal{H}_{\tilde{W}}^{\text{fin}}$, loses only some of the intuition of the first one, but gives us an explicit finite presentation of the algebra. Following this we have $\mathcal{H}_t$ which loses more of the initial intuition we started with, but does provide us with another finite presentation as well as a stronger connection to the final presentation. It is not generally known that Iwahori-Hecke algebras are finitely presented, so these two presentations do not commonly appear in other works, and are my original contribution to the literature.

The Bernstein-Lusztig presentation $\mathcal{H}_{\mathrm{BL}}$ is usually used as a presentation of an affine Hecke algebra, but has been specialised here to define a presentation of the Iwahori-Hecke algebra. This is a commonly occurring presentation in other works, and allows us to work with a vector space basis, and makes it much easier to describe the centre of the algebra using its last relation, as described in Remark 4.4.2.

Throughout each of these presentations with the isomorphisms provided, we have one generator corresponding to the characterstic function of the double coset $I\sigma I$ of the nontrivial Weyl group element,

$$H(G,I) \longrightarrow \mathcal{H}_{\tilde{W}} \longrightarrow \mathcal{H}_{\tilde{W}}^{\mathrm{fin}} \longrightarrow \mathcal{H}_t \longrightarrow \mathcal{H}_{\mathrm{BL}}$$

$$T_\sigma \longmapsto S_\sigma \longmapsto S_\sigma \longmapsto S \longmapsto T_\sigma,$$

and each follows the relation $S^2 = (q-1)S + q$. As seen in the definition of $\mathcal{H}_{\mathrm{BL}}$, this can also be written as $(S-q)(S+1) = 0$. Therefore for a one-dimensional module of the Iwahori-Hecke algebra, since $H(G,I)$ acts via a homomorphism $H(G,I) \to \mathrm{End}(\mathbb{C}) \cong \mathbb{C}$, the characteristic function T_σ acts by $T_\sigma \mapsto q$ or $T_\sigma \mapsto -1$.

The summary of connections we have made in this thesis are represented by the diagram below, with references pointing to where each isomorphism is described in theorems, proofs, or statements.

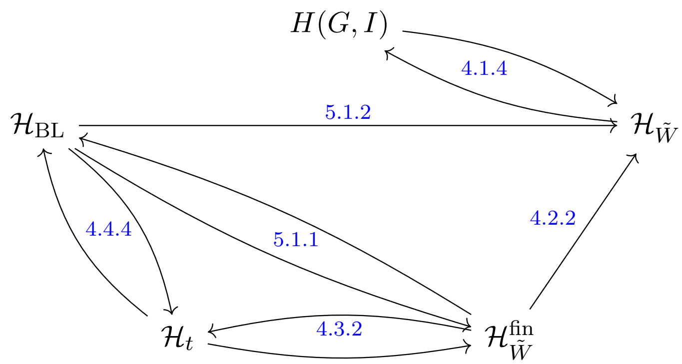

Figure 5.1: Isomorphisms Between Presentations

Three of these five presentations have already been studied well before. However the links between them are often left ambiguous and unexplained in the literature I have found. Restricting to a particular case gives us the ability to work on the level of elements and their interactions with each other.

Some of the arguments in this thesis may be generalisable to other cases of GL_n, although there is work to be done to do so. However staying within this narrower scope means that these results can be tested and expanded upon using modern tools such as proof-checkers, which require more clear-cut data to work with.

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Index of Notation