THE UNIVERSITY OF CALGARY

An Electric Circuit Analogue of a Nonholonomically Constrained Hamiltonian System

by

Charles L. Cuell

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Abstract

An electric circuit analogue of a mechanical system is a circuit for which the equations of motion are the same as for the mechanical system. It may be desirable to study the circuit instead of the mechanical system since circuits are generally simple to build. This thesis describes the process of developing an electric circuit analogue of a particle in $\Re^3$ under the influence of a potential and the nonholonomic constraint $\dot{z} = y\dot{x}$.

Acknowledgements

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I would also like to thank the other graduate students in the department that have made my time here very enjoyable. Finally, my most heartfelt appreciation goes to Sally for supporting my goals and to Samuel for just being there.

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Introduction

The basic purpose of a mechanical system analogue is to develop an equivalent system that is either easier to study, easier to build, or both. This thesis concentrates on building an electric circuit analogue of the *nonholonomic particle*, an example studied in detail in Bates and Śniatycki [2]. It is a free particle in $\Re^3$ with the constraint $\dot{z} = y\dot{x}$. This constraint is the simplest example of a linear nonholonomic constraint. The potential, $U = \frac{(x+y+z)^2}{2C}$ (C a constant), is added to the system to keep the solution within the circuit's operating parameters longer.

We use an electric circuit as the analogous system since implementing constraints is simply a matter of multiplying voltages. There is no need to design and build complicated mechanical devices to ensure that the constraint is being satisfied.

The purpose of this thesis is to examine the feasibility of building the circuit and running experiments. We also need to know how accurate we can make the output. Since the equations of motion for the constrained mechanical system can be derived, we will be able to make a direct comparison between the output of the circuit and the actual motion of the system.

Chapter 1

Some Circuit Theory

1.1 Kirchhoff's Laws

For the purposes of this thesis, a circuit is a collection of electronic components connected by wires and may be modeled using networks or graphs. A network is drawn to show how each component is connected to the others. See Figure 1.1.

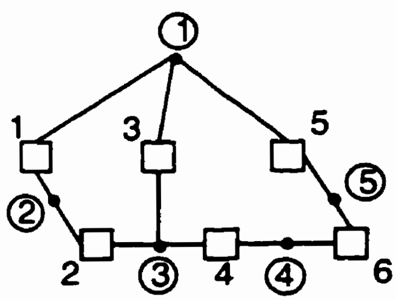


Figure 1.1: **Circuit Diagram**

The boxes are the circuit components, the dots are nodes where the components are connected and the lines are branches (wires) that connect the nodes. Note that a branch goes from node to node and so includes a component. If it is required that each branch contain a component, then the circuit can be drawn as a graph. See Figure 1.2.

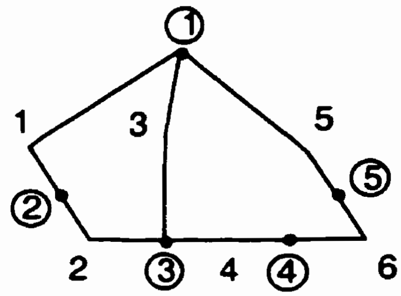


Figure 1.2: **Circuit Graph**

An arbitrary reference direction can be assigned to each branch which gives the circuit the abstract structure of a directed graph. See Figure 1.3. If the current is flowing in the direction of the arrow, then the current has *positive* direction. If it flows in the direction opposite the arrow, it has *negative* direction.

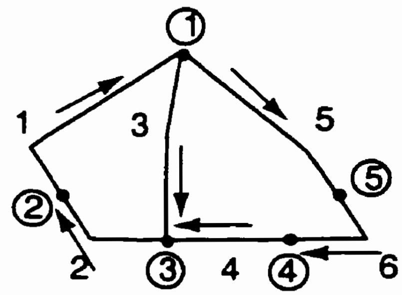


Figure 1.3: **Directed Circuit Graph**

Each node has an electric potential with respect to a *ground* or *reference* node. The ground node is specified ahead of time and the potential at each node is measured with respect to the ground node. Each component induces an electric potential difference between its bounding nodes. This is called the *voltage across the component* or the *branch voltage.*

From this point on, the word circuit will refer to the directed graph representation of a physical electrical network.

Kirchhoff's current law states that the sum of all currents into a node, taking the reference directions into account, is zero. Physically, this is a statement of conservation of charge since a node cannot be a source or a sink for current.

Kirchhoff's voltage law states that the voltage across a branch is equal to the difference in the potentials of its bounding nodes. This, in combination with Kirchhoff's current law, is a statement of conservation of energy. A charge moving from ground to a node will experience a change in energy due to the electric potentials induced by the components. By Kirchhoff's voltage law, the energy change of a charge traversing a single branch must be equal to the difference in the energies required to move the charge from ground to each of the bounding nodes of the branch.

This form of Kirchhoff's voltage law is equivalent to the more familiar form that states that the sum of the branch voltages around a closed loop is zero. Consider a sequence of nodes, with node potentials κ_j, where j denotes the node. The first and last elements are the same, since this is a closed loop. The sequence v_k is the set of branch voltages associated to the closed loop, where nodes m_{j+1} and m_j bound branch b_k. Then

$$\sum_k v_k = \sum_j (\kappa_{j+1} - \kappa_j).$$

This sum is zero, since the end node of the k'th branch is the same as the initial node of the $k+1$ branch, and the initial and final nodes are the same.

Given these two laws, one can write algebraic conditions on the branch voltages and currents in a circuit simply by inspection. To facilitate a more geometric approach, however, consider the following formal definitions of vector spaces of the circuit nodes and branches.

Let a basis for the nodes be denoted by

$$M = \{m_j\}, \qquad j = 1 \ldots n$$

where n is the number of nodes in the circuit. The formal linear combinations over $\Re$ of the m_j's form the vector space

$$C_0 = \{h^j m_j : j = 1 \ldots n\},$$

where identical upper and lower indices in a product indicate summation over the index. This is the arbitrary assignment of a number to each node along with the usual vector space operations. The vector $I \in C_0$ represents a state of the circuit in which there is a net amount of current h^j going into node m_j.

Now let a basis for the branches be denoted by

$$B = \{b_j\}, \qquad j = 1 \ldots p$$

where p is the number of branches in the circuit. Form the set of linear combinations over $\Re$ of the b_j's by

$$C_1 = \{i^j b_j : j = 1 \ldots p\}.$$

An element of C_1 is a vector in which the component i^j is the current in branch b_j.

The vector space duals of C_0 and C_1 are the spaces

$$C^0 = \{\kappa_j \eta^j : j = 1 \ldots n\},$$

and

$$C^1 = \{v_j \beta^j : j = 1 \ldots p\},$$

where η^j is dual to m_j and β^j is dual to b_j. The component κ^j of a vector in C^0 is the potential at node m_j measured with respect to ground. The component v_j of a vector in C^1 is the branch voltage for branch b_j.

C_0 and C^0 have the structure of $\Re^n$ and C_1 and C^1 have the structure of $\Re^p$.

Refer to Figure 1.4 and define the linear map $\partial : C_1 \rightarrow C_0$ by

$$\partial : b_j \rightarrow m_k - m_q. \tag{1.1}$$

Equation 1.1 uses the reference direction for a branch to assign the vector $m_k - m_q$ to the branch b_j. Extend this linearly to the whole vector space. Denote the dual of ∂ by $\partial^* : C^0 \rightarrow C^1$. Then let $\partial^* \eta^m = \epsilon_j^m \beta^j$, for some matrix ϵ_j^m. On the basis of C_1,

$$< \partial^* \eta^m, b_j > = < \eta^m, \partial b_j >,$$

Figure 1.4: **Boundary Map**

$$
\begin{aligned}
&= \; < \eta^m, m_k - m_q >, \\
&= \; \delta^m_k - \delta^m_q .
\end{aligned}
$$

Also,

$$
\begin{aligned}
< \partial^* \eta^m, b_j > &= < \epsilon^m_k \beta^k, b_j >, \\
&= \epsilon^m_k \delta^k_j, \\
&= \epsilon^m_j .
\end{aligned}
$$

So,

$$\epsilon^m_j = \delta^m_k - \delta^m_q, \tag{1.2}$$

where m_k and m_q are the bounding nodes of branch b_j. The (m, j) entry in ϵ^m_j is 1 if branch j is incident on node m, -1 if branch j leaves node m and 0 if branch j doesn't touch node m.

Let $i \in C_1$, $i = i^j b_j$. Then

$$
\begin{aligned}
\partial i &= i^j \partial b_j, & (1.3) \\
&= h^\ell m_\ell, & (1.4)
\end{aligned}
$$

for $h = h^\ell m_\ell \in C_0$. So,

$$\begin{aligned} h^k &= <\eta^k, h^\ell m_\ell>, && (1.5)\\ &= <\eta^k, i^j \partial b_j>, && (1.6)\\ &= i^j <\partial^* \eta^k, b_j>, && (1.7)\\ &= i^j \epsilon_j^k. && (1.8) \end{aligned}$$

Also, let $\Lambda \in C^0$, $\Lambda = \kappa_j \eta^j$. Then, for some $v = v_\ell \beta^\ell$ in C^1,

$$\begin{aligned} \partial^* \Lambda &= \kappa_j \partial^* \eta^j, && (1.9)\\ &= v_\ell \beta^\ell. && (1.10) \end{aligned}$$

So,

$$\begin{aligned} v_k &= <v_\ell \beta^\ell, b_k>, && (1.11)\\ &= <\kappa_j \partial^* \eta^j, b_k>, && (1.12)\\ &= \kappa_j \epsilon_k^j. && (1.13) \end{aligned}$$

The above formalism now provides a way to express Kirchhoff's laws in algebraic form. Consider Equations 1.3 and 1.4. h^ℓ is the net current into node m_ℓ, so a vector $i \in C_1$ satisfies Kirchhoff's current law if $i \in \ker\partial$. That is, if $h^\ell = 0, \ell = 1 \ldots n$. The kernel of ∂ can be computed from Equations 1.8, where the requirements on the i^j's are $i^j \epsilon_j^k = 0$.

Now consider Equations 1.11, 1.12, and 1.13. The image of ∂^* can be computed from these equations. Remember that the (j, k) entry of ϵ_k^j is 1 if branch k is incident on node j, -1 if branch k leaves node j and 0 otherwise. So v_k is the difference in the bounding node potentials of branch k. Thus, Kirchhoff's voltage law can be stated by requiring $v \in \mathrm{im}\partial^*$. The requirements on the components of v are given by Equation 1.13.

Kirchhoff's laws are expressed algebraically in terms of the operators ∂ and ∂^*, which depend only on the topology of the graph. Hence the restrictions that the laws place on the possible dynamics of the circuit are completely topological in origin. Tellegen's Theorem is a classical theorem that states that the net power delivered to the circuit by the circuit components is zero. Since the theorem only depends on Kirchhoff's laws, it is a topological theorem.

The power for a circuit element is given by $v_j i^j$ (single component, so no sum on j), where v_j is the voltage across the j'th component and i^j is the current through it. For a circuit, the total power is the sum of the power in each branch. Given a state vector, (i, v) with $i \in C_1$ and $v \in C^1$, the total power is $i^j v_j$ (sum over $j = 1 \ldots p$) or, in terms of the canonical pairing, the total power is $< v, i >$.

Theorem 1 (Tellegen) *Given Kirchhoff's laws for a circuit, the net power is zero. Algebraically, if* $i \in \ker\partial$ *and* $v \in \mathrm{im}\partial^*$*, then* $< v, i >= 0$.

Proof

$$
\begin{aligned}
< v, i > &= < \partial^* u, i >, \qquad \text{some } u \in C^0 \\
&= < u, \partial i >, \\
&= 0. \quad \square
\end{aligned}
$$

Tellegen's theorem is very general in the sense that it does not require the vectors v and i to be measured from the same circuit or even at the same instant in time. From the vector space point of view, $\ker\partial$ and $\mathrm{im}\partial^*$, as subspaces of $\Re^{2p}$, are orthogonal with respect to the standard inner product.

As an example of the preceding theory, consider the circuit given by the graph in Figure 1.3. From Equation 1.2,

$$
\left[\epsilon_j^m\right] = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{1.14}
$$

Equation 1.14 gives

$$[\partial^{*}]\,\Lambda = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \kappa_5 \\ \kappa_6 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix},$$

where $\Lambda = \kappa_j \eta^j \in C^0$. This gives

$$\begin{aligned} v_1 &= \kappa_1 - \kappa_2, \\ v_2 &= \kappa_2 - \kappa_3, \\ v_3 &= -\kappa_1 + \kappa_3, \\ v_4 &= \kappa_3 - \kappa_4, \\ v_5 &= -\kappa_1 + \kappa_5, \\ v_6 &= \kappa_4 - \kappa_5. \end{aligned}$$

In turn, this gives the conditions

$$\begin{aligned} v_1 + v_2 + v_3 &= 0, \\ -v_3 + v_4 + v_5 + v_6 &= 0, \end{aligned}$$

which is just Kirchhoff's voltage law in loop form. With two conditions on a 6 dimensional vector space, $\dim(\mathrm{im}\partial^{*}) = 4$.

$\ker\partial$ is given by

$$
\begin{aligned}
[\partial]\, i &= \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i^1 \\ i^2 \\ i^3 \\ i^4 \\ i^5 \\ i^6 \end{bmatrix}, \\
&= 0,
\end{aligned}
$$

where $i = i^j b_j \in C_1$. This gives the conditions

$$
\begin{aligned}
i^1 &= i^3 + i^4, \\
i^2 &= i^3 + i^4, \\
i^5 &= i^4, \\
i^6 &= i^4.
\end{aligned}
$$

This gives $\dim(\ker\partial) = 2$.

1.2 Constitutive Equations

The circuit that we are concerned with contains resistors, capacitors, inductors and operational amplifiers (op-amps). Table 1.1 shows the schematic for these circuit elements.

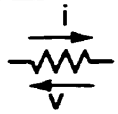
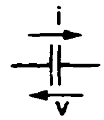
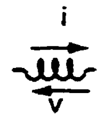
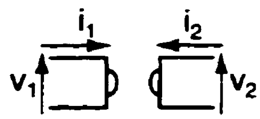
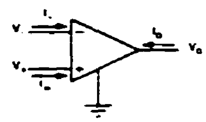
Resistor	
Capacitor	
Inductor	
Gyrator	
Op–Amp	

Table 1.1: Schematics of Basic Circuit Components

A resistor is a component that relates current to voltage

$$v = f(i),$$

for some function f, not necessarily invertible. Most commonly, resistors in a circuit are linear with

$$v = iR,$$

where R is the *resistance* of the resistor. It is a characteristic of the material the resistor is made of.

Capacitors relate voltage to charge

$$v = f(q).$$

Capacitors are also usually linear

$$v = \frac{q}{C},$$

where the constant, C, is the capacitance of the capacitor. It is a characteristic of the dielectric separating capacitor plates.

Inductors relate magnetic flux to current

$$\phi = f(i). \tag{1.15}$$

Magnetic flux is not very conveniently measured, so differentiate 1.15 with respect to time

$$\frac{d\phi}{dt} = \frac{df}{di}\frac{di}{dt}. \tag{1.16}$$

A changing magnetic flux produces a voltage, and we define $L = \frac{df}{di}$. Equation 1.16 can be written

$$v = L\frac{di}{dt}.$$

L is usually a constant.

A gyrator is a two port element defined by

$$i^1 = Gv_2, \tag{1.17}$$

$$i^2 = -Gv_1, \tag{1.18}$$

where G is the *conductance* (reciprocal of resistance). If port 2 is terminated by a capacitor with capacitance C, equations 1.17 and 1.18 become

$$i^1 = G(v_2), \tag{1.19}$$

$$= -G\frac{q^2}{C}, \tag{1.20}$$

where q^2 is the charge on the capacitor. Differentiate 1.20 with respect to time

$$\frac{di^1}{dt} = -G\frac{dq^2}{dt} \tag{1.21}$$

$$= -Gi^2. \tag{1.22}$$

Using 1.18 and solving for v_1, Equation 1.22 is

$$v_1 = \frac{C}{G^2}\frac{di^1}{dt}. \tag{1.23}$$

Equation 1.23 is the constitutive equation for an inductor. Thus, when port 2 is terminated by a capacitor, a gyrator behaves as an inductor with inductance $\frac{C}{G^2}$. This is important, since the *quality* of a gyrator simulated inductance can be considerably higher that that of a traditional wire–wrapped coil inductor. This means less power dissipation and more accurate output. See Bruton [5] for a discussion on the quality of an inductor and simulated inductor. The circuit for a simulated inductor is shown in Figure 3.5, and is used in the circuit described in Chapter 4.

An operational amplifier (op–amp) is defined by the relations

$$i_- = 0, \tag{1.24}$$

$$i_+ = 0, \tag{1.25}$$

$$v_o = A(v_+ - v_-), \tag{1.26}$$

where A is some large constant on the order of 10^4 to 10^6. It is assumed that $-E < v_o < E$, where E is the power supply voltage. An equivalent circuit is given by figure 1.5.

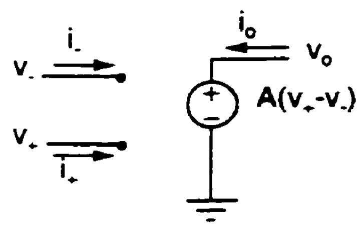


Figure 1.5: **Equivalent Operational Amplifier Circuit**

Circuit 1.5 facilitates the use of Kirchhoff's laws when an op–amp appears in a circuit.

Since A in equation 1.26 is large, it is often useful to assume $A = \infty$ and $v_+ - v_- = 0$. Then the op–amp is described by

$$i_- = 0 \tag{1.27}$$

$$i_+ = 0 \tag{1.28}$$

$$v_+ - v_- = 0. \tag{1.29}$$

An assumption implicit in equations 1.27, 1.28, and 1.29 is that v_o will be whatever is necessary to maintain $v_+ - v_- = 0$. These equations describe an *ideal* op–amp.

1.3 Power and Energy

The instantaneous power delivered to a circuit component is defined by

$$P(t) = v(t)i(t),$$

for a branch voltage v and branch current i. From a mechanical viewpoint, power is the rate at which work is being done on a component by the rest of the circuit. This is written as

$$P(t) = \left.\frac{dW}{dt}\right|_t,$$

where W is the work. For a circuit with resistors, inductors and capacitors, the work being done by the circuit on the components in changing the state from $(i(t_0), v(t_0))$ to $(i(t), v(t))$ along a path Γ is

$$\begin{aligned} W(t) &= \int_{t_0}^{t} \sum_{\lambda} v_\lambda i^\lambda dt + \int_{t_0}^{t} \sum_{\gamma} v_\gamma i^\gamma dt + \int_{t_0}^{t} \sum_{\rho} v_\rho i^\rho dt, \\ &= \int_{t_0}^{t} \sum_{\lambda} L_\lambda i^\lambda \frac{di^\lambda}{dt} dt + \int_{t_0}^{t} \sum_{\gamma} C_\gamma v_\gamma \frac{dv_\gamma}{dt} dt + \int_{t_0}^{t} \sum_{\rho} R(i^\rho)^2 dt, \\ &= \int_{\Gamma} \sum_{\lambda} L_\lambda i^\lambda di^\lambda + \int_{\Gamma} \sum_{\gamma} C_\gamma v_\gamma dv_\gamma + \int_{t_0}^{t} \sum_{\rho} R(i^\rho)^2 dt, \\ &= \sum_{\lambda} L_\lambda \left.\frac{(i^\lambda)^2}{2}\right|_{i^\lambda(t_0)}^{i^\lambda(t)} + \sum_{\gamma} C_\gamma \left.\frac{v_\gamma^2}{2}\right|_{v_\gamma(t_0)}^{v_\gamma(t)} + \int_{t_0}^{t} \sum_{\rho} R(i^\rho)^2 dt. \end{aligned} \tag{1.30}$$

The first sum in Equation 1.30 gives the energy change in the inductors and the second sum gives the energy change in the capacitors. If the path Γ is closed, then the first two terms of Equation 1.30 are zero and

$$W(t) = \int_{t_0}^{t} \sum_{\rho} R(i^\rho)^2 dt. \tag{1.31}$$

So any energy absorbed by the inductors or capacitors is returned to the circuit. The resistor branches, however, always absorb energy. This is seen by noting that the integrand in Equation 1.31 is always positive. Hence the integral is always positive. This means that the circuit always does positive work on the resistors. This is called *dissipation*. The energy absorbed by the resistors is converted to heat. This is a non-reversible process, so the energy is not recoverable by the circuit.

The fact that inductors and capacitors return the energy they absorb leads us to define the energy of a circuit by

$$E(t) = \sum_{\lambda} L_{\lambda} \frac{(i^{\lambda})^2}{2} + \sum_{\gamma} C_{\gamma} \frac{v_{\gamma}^2}{2}. \tag{1.32}$$

By Tellegen's theorem, $P = \frac{dW}{dt} = 0$ for the circuit as a whole. From Equations 1.30 and 1.32

$$\begin{aligned} \frac{dW}{dt} &= \frac{dE}{dt} + \sum_{\rho} R(i^{\rho})^2, \\ &= 0. \end{aligned}$$

So that

$$\frac{dE}{dt} = -\sum_{\rho} R(i^{\rho})^2.$$

This shows that the energy dissipates in a circuit through the resistors.

1.4 The Brayton–Moser Equations for a Circuit

1.4.1 Resistor, Inductor, Capacitor Circuits

The purpose of this section is to derive a set of ordinary differential equations that describe how the state of the circuit changes with time. These equations were first derived by Brayton and Moser in [4]. The following geometric formulation of the theory was done by Smale in [14].

Denote the subspace of C_1 of currents in resistor branches by $\mathcal{R}$, in inductor branches by $\mathcal{L}$, and in capacitor branches by $\mathcal{C}$. Denote the subspace of C^1 of voltages across resistor branches by $\mathcal{R}^*$, across inductor branches by $\mathcal{L}^*$ and across capacitor branches by $\mathcal{C}^*$. The voltage and current subspaces are dual for their respective components and

$$C_1 \times C^1 = (\mathcal{R} \times \mathcal{L} \times \mathcal{C}) \times (\mathcal{R}^* \times \mathcal{L}^* \times \mathcal{C}^*) .$$

Let $K = \ker\partial \times \mathrm{im}\partial^*$. The resistor characteristic for each resistor branch is a one dimensional submanifold of $\mathcal{R} \times \mathcal{R}^*$ defined by

$$\Delta_\rho = \{(i^\rho, v_\rho) \in \mathcal{R}_\rho \times \mathcal{R}^*_\rho : v_\rho = f(i^\rho)\}, \tag{1.33}$$

where ρ refers to the branch the resistor is in. The product Δ of the Δ_ρ's is a closed submanifold of $\mathcal{R} \times \mathcal{R}^*$. Let $\pi' : K \to \mathcal{R} \times \mathcal{R}^*$ be the natural projection. Then $\Sigma = \pi'^{-1}(\Delta)$ is a submanifold of K. This is the space of physical states of the circuit.

Let $\pi : \Sigma \to \mathcal{L} \times \mathcal{C}^*$ be the natural projection. For every point $x \in \Sigma$ such that $D\pi(x) : T_x\Sigma \to T_{\pi(x)}(\mathcal{L} \times \mathcal{C}^*)$ is an isomorphism, π is a diffeomorphism for a neighbourhood of x. For the purposes of Brayton–Moser theory, it is assumed that π is a global diffeomorphism. In other words, Σ is covered by a single coordinate chart given by π.

The previous statement relies on the truth of the following proposition.

Proposition 1

$$dim(\Sigma) \;=\; dim(\mathcal{L} \times \mathcal{C}^*).$$

Proof

First, we show that $\dim(K) = p$, where p is the number of branches.

$$\begin{aligned} \dim(C_1) &= \dim(\ker\partial) + \dim(\mathrm{im}\partial), \\ &= \dim(\ker\partial) + \dim(\mathrm{im}\partial^*), \end{aligned} \tag{1.34}$$

and

$$\dim(C^0) = \dim(\ker\partial^*) + \dim(\mathrm{im}\partial^*). \tag{1.35}$$

$\ker(\partial^*)$ has dimension 1, since the only way that all voltages can be zero is when all node potentials are the same. $\dim(C_1) = p$ and

$\dim(C^0) = n$, so Equations 1.34 and 1.35 give

$$\dim(\ker\partial) = p - (n-1). \tag{1.36}$$

Using the fact that $\dim(K) = \dim(\ker\partial) + \dim(\mathrm{im}\partial^*)$ and Equation 1.35, we get

$$\begin{aligned} \dim(K) &= p - (n-1) + (n-1), \\ &= p. \end{aligned}$$

Then

$$\begin{aligned} \dim(\Sigma) &= \dim(K) - \dim(\mathcal{R}), \\ &= \dim(\mathcal{L} \times \mathcal{C}^*). \quad \square \end{aligned}$$

Define the metric J on $\mathcal{L} \times \mathcal{C}^*$ by

$$J = -\sum_\lambda L_\lambda (di^\lambda)^2 + \sum_\gamma C_\gamma (dv_\gamma)^2,$$

where the sum over λ is over the inductor branches and the sum over γ is over the capacitor branches. L_λ is the inductance in branch λ and C_γ is the capacitance in branch γ. Now define $I = \pi^* J$, the pullback of J to Σ by the natural projection. At the points for which $D\pi$ is an isomorphism, I is nondegenerate.

Define the *potential* one form on Σ by

$$dP = d\sum_\gamma v_\gamma i^\gamma + \sum_\rho v_\rho di^\rho, \tag{1.37}$$

where the sum over ρ is over the resistor branches.

By Tellegen's theorem, $\sum_j v_j di^j = 0$ since it's a one form on K and so vanishes on vectors tangent to Σ. Rewrite this as

$$\sum_\rho v_\rho di^\rho + \sum_\lambda v_\lambda di^\lambda + \sum_\gamma v_\gamma di^\gamma = 0. \tag{1.38}$$

Now,

$$d\sum_\gamma v_\gamma i^\gamma = \sum_\gamma i^\gamma dv_\gamma + \sum_\gamma v_\gamma di^\gamma. \tag{1.39}$$

Using Equation 1.39, Equation 1.38 can be rewritten as

$$\sum_\rho v_\rho di^\rho + \sum_\lambda v_\lambda di^\lambda + d\sum_\gamma v_\gamma i^\gamma - \sum_\gamma i^\gamma dv_\gamma = 0. \tag{1.40}$$

Or, using the definition of dP in Equation 1.37, we get

$$dP = -\sum_\lambda v_\lambda di^\lambda + \sum_\gamma i^\gamma dv_\gamma. \tag{1.41}$$

Now, $v_\lambda = L_\lambda \frac{di^\lambda}{dt}$ and $i^\gamma = \frac{dv_\gamma}{dt}$ so that Equation 1.41 becomes

$$dP \;=\; -\sum_\lambda L_\lambda \frac{di^\lambda}{dt} di^\lambda + \sum_\gamma C_\gamma \frac{dv_\gamma}{dt} dv_\gamma. \tag{1.42}$$

Or,

$$dP = I^\flat X_P, \tag{1.43}$$

where $X_P = (\frac{di^\lambda}{dt}, \frac{dv_\gamma}{dt})$. The vector field X_P is the tangent vector field to the trajectories of motion of the circuit so that Equation 1.43 is the equation of motion for the circuit. Written out explicitly, they are a set of ordinary differential equations on the manifold Σ. These are the Brayton–Moser equations.

Classic Example

Consider the circuit in Figure 1.6 and its graph in Figure 1.7.

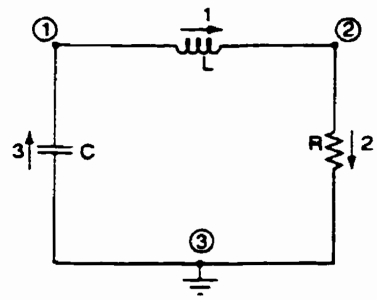


Figure 1.6: **Damped Harmonic Oscillator Circuit**

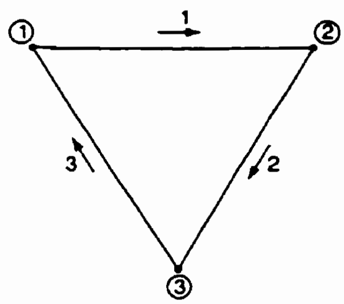


Figure 1.7: **Graph for Damped Harmonic Oscillator Circuit**

A state vector for this circuit is $x = (i^1, i^2, i^3, v_1, v_2, v_3) \in \Re^6$. Kirchhoff's laws are given by the matrix ϵ^j_k from Equation 1.2.

$$[\partial^{\bullet}] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \tag{1.44}$$

and

$$[\partial] = [\partial^{*}]^{T} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}. \tag{1.45}$$

So $i \in \ker\partial$ if

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i^{1} \\ i^{2} \\ i^{3} \end{bmatrix} = 0. \tag{1.46}$$

And $v \in \mathrm{im}\partial^{*}$ if

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{1.47}$$

So,

$$\begin{aligned} K &= \ker\partial \times \mathrm{im}\partial^{*}, \\ &= \mathrm{span}\{\partial_{i^1} + \partial_{i^2} + \partial_{i^3}, \partial_{v_2} - \partial_{v_1}, \partial_{v_3} - \partial_{v_1}\}. \end{aligned}$$

The resistor characteristic is given by $v_2 = i^2 R$, so that

$$\Delta = \{(i^2, i^2 R)\}.$$

And

$$\begin{aligned} \Sigma &= \pi'^{-1}(\Delta), \\ &= \mathrm{span}\{\partial_{i^1} + \partial_{i^2} + \partial_{i^3} - R\partial_{v_1} + R\partial_{v_2}, \partial_{v_3} - \partial_{v_1}\}. \end{aligned}$$

Σ is a graph over $\mathcal{L} \times \mathcal{C}^{*}$ by

$$\Sigma = \{(i^1, i^1, i^1, -i^1 R - v_3, i^1 R, v_3)\}.$$

Hence, the natural projection $\pi : \Sigma \to \mathcal{L} \times \mathcal{C}^*$ is a diffeomorphism. Brayton–Moser theory then applies.

On Σ,

$$\begin{aligned} dP &= d\sum_{\gamma} v_\gamma i^\gamma + \sum_{\rho} v_\rho di^\rho, \\ &= i^1 dv_3 + (v_3 + i^1 R) di^1. \end{aligned}$$

This gives the Brayton–Moser equations as

$$\begin{aligned} \begin{bmatrix} \frac{di^1}{dt} \\ \frac{dv_3}{dt} \end{bmatrix} &= X_P, \\ &= \Gamma^* dP, \\ &= \begin{bmatrix} -\frac{1}{L} & 0 \\ 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} v_3 + i^1 R \\ i^1 \end{bmatrix}, \\ &= \begin{bmatrix} -\frac{1}{L}(v_3 + i^1 R) \\ \frac{i^1}{C} \end{bmatrix}. \end{aligned}$$

1.4.2 Resistorless Circuits

In the case where a circuit has no resistors, the power dissipated is zero. That is, energy in the circuit is conserved. In terms of mechanical systems the motion of the system can be shown to be due to the influence of a potential that is analogous to the electric field energy stored in the capacitors.

With no resistors,

$$\Sigma = K,$$

and

$$\begin{aligned} dP &= d\sum_{\gamma} v_\gamma i^\gamma, \\ &= \sum_{\gamma} v_\gamma di^\gamma + \sum_{\gamma} i^\gamma dv_\gamma. \end{aligned}$$

If $\pi : \Sigma \to \mathcal{L} \times \mathcal{C}^*$ is a diffeomorphism, then the i^γ's are linear functions of the i^λ's. The equations of motion are then given by the vector field

$$\begin{aligned} X_P &= I^\sharp dP, \\ &= -\sum_{\lambda} \frac{1}{L_\lambda}\left(\sum_{\gamma} v_\gamma \frac{\partial i^\gamma}{\partial i^\lambda}\right)\partial_{i^\lambda} + \sum_{\gamma} \frac{1}{C_\gamma} i^\gamma \partial_{v_\gamma}. \end{aligned}$$

Consider the vector space $\mathcal{Q} \times C_1 \times C^1$, where $\mathcal{Q}$ is the vector space of the net amount of charge to flow through the inductors. The components of $\mathcal{Q}$ are q^λ, which is the net charge to flow through the λ'th inductor so that $\dot{q}^\lambda = i^\lambda$. Let

$$\bar{\Sigma} = \{(q^\lambda, i^\lambda, i^\gamma, v_\lambda, v_\gamma) : (q^\lambda) \in \mathcal{Q}, (i^\lambda, i^\gamma) \in \ker\partial, (v_\lambda, v_\gamma) \in \operatorname{im}\partial^*\}.$$

Then,

$$\bar{\pi}\left(\bar{\Sigma}\right) = \Sigma,$$

where

$$\bar{\pi} : (q^\lambda, i^\lambda, i^\gamma, v_\lambda, v_\gamma) \mapsto (i^\lambda, i^\gamma, v_\lambda, v_\gamma).$$

The vector field for the circuit equations of motion is then given on $\bar{\Sigma}$ by

$$\bar{X}_P = -\sum_{\lambda} \frac{1}{L_\lambda}\left(\sum_{\gamma} v_\gamma \frac{\partial i^\gamma}{\partial i^\lambda}\right)\partial_{i^\lambda} + \sum_{\gamma} \frac{1}{C_\gamma} i^\gamma \partial_{v_\gamma} + \sum_{\lambda} i^\lambda \partial_{q^\lambda}. \tag{1.48}$$

Note that $\pi_* \tilde{X}_P = X_P$. The last two sums in Equation 1.48 give

$$\frac{dv_\gamma}{dt} = \frac{i^\gamma}{C_\gamma},$$
$$\frac{dq^\lambda}{dt} = i^\lambda.$$

These integrate to give

$$v_\gamma = \frac{q^\gamma}{C_\gamma}, \tag{1.49}$$

which are the constitutive equations for the capacitors in the circuit. Restricted to the surface defined by Equation 1.49,

$$\tilde{X}_P = -\sum_\lambda \frac{1}{L_\lambda}\left(\sum_\gamma \frac{q^\gamma}{C_\gamma}\frac{\partial q^\gamma}{\partial q^\lambda}\right)\partial_{i^\lambda}, \tag{1.50}$$

where we have made the substitution

$$\frac{\partial i^\gamma}{\partial i^\lambda} = \frac{\partial q^\gamma}{\partial q^\lambda},$$

remembering that i^γ and q^γ are linear functions of the i^λ's and the q^λ's respectively.

Equation 1.50 is a vector field defined in terms of the inductor branches only. Since $\dot{q}^\lambda = i^\lambda$, the space $\mathcal{Q} \times \mathcal{L}$ is the same as the tangent bundle, $T\mathcal{Q}$, of $\mathcal{Q}$.

Let $F = \sum_\lambda L_\lambda \ddot{q}^\lambda \partial_{q^\lambda}$. In terms of mechanical systems, this is the force field on $\mathcal{Q}$ that produces the motion given by the vector field $\tilde{X}_P$. Then

$$F = -\sum_\lambda \left(\sum_\gamma \frac{q^\gamma}{C_\gamma}\frac{\partial q^\gamma}{\partial q^\lambda}\right)\partial_{q^\lambda}.$$

Proposition 2 *$F = -\nabla U$, where $U = \sum_\gamma \frac{(q^\gamma)^2}{2C_\gamma}$ is a potential function on Q.*

Proof

$$\begin{aligned}
\nabla U &= \sum_\lambda \frac{\partial}{\partial q^\lambda} U \partial_{q^\lambda}, \\
&= \sum_\lambda \frac{\partial}{\partial q^\lambda} \left(\sum_\gamma \frac{(q^\gamma)^2}{2C_\gamma} \right) \partial_{q^\lambda}, \\
&= \sum_\lambda \left(\sum_\gamma \frac{q^\gamma}{C_\gamma} \frac{\partial q^\gamma}{\partial q^\lambda} \right) \partial_{q^\lambda}. \quad \Box
\end{aligned}$$

Given the potential U, a Lagrangian for the circuit can be written as

$$\ell = \sum_\lambda \frac{L_\lambda}{2} (\dot{q}^\lambda)^2 - \sum_\gamma \frac{(q^\gamma)^2}{2C_\gamma}. \tag{1.51}$$

The first term on the right hand side of Equation 1.51 is the energy in the inductors and the second term is the energy in the capacitors. Thinking in terms of mechanical systems, the energy in the inductors is analogous to the kinetic energy of a mechanical system with a Lagrangian given by Equation 1.51 and the energy in the capacitors is analogous to potential energy.

The equations of motion for a mechanical system given by the Lagrangian in Equation 1.51 are, for the k'th component

$$\frac{\partial \ell}{\partial q^k} - \frac{d}{dt}\left(\frac{\partial \ell}{\partial \dot{q}^k} \right) = 0, \tag{1.52}$$

$$\sum_\gamma \frac{q^\gamma}{C_\gamma} \frac{\partial q^\gamma}{\partial q^k} + L_k \ddot{q}^k = 0. \tag{1.53}$$

The Brayton–Moser equations for a resistorless circuit are given by the vector field $\bar{X}_P$ in Equation 1.50. Rewritten with $\dot{q}^k = i^k$, the equations are

$$\ddot{q}^k = -\frac{1}{L_k}\left(\sum_{\gamma}\frac{q^{\gamma}}{C_{\gamma}}\frac{\partial q^{\gamma}}{\partial q^k}\right).$$

Or,

$$\sum_{\gamma}\frac{q^{\gamma}}{C_{\gamma}}\frac{\partial q^{\gamma}}{\partial q^k} + L_k\ddot{q}^k = 0. \tag{1.54}$$

Equations 1.53 and 1.54 are identical. We say that the circuit is an *electric circuit analogue* of the mechanical system given by the Lagrangian in Equation 1.51.

1.4.3 Op–Amps in a Circuit

To facilitate the application of Brayton–Moser theory for circuits containing op-amps it is useful to introduce two more types of circuit elements: the *nullator* and *norator*. These are defined in Bruton [5].

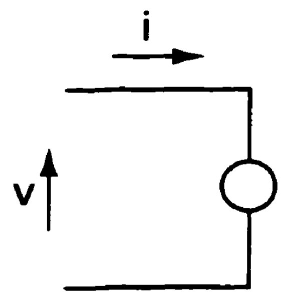


Figure 1.8: **The Nullator**

The schematic for a nullator is shown in Figure 1.8. The constitutive equations are

$$v = 0,$$

$$i = 0.$$

The norator is shown in Figure 1.9.

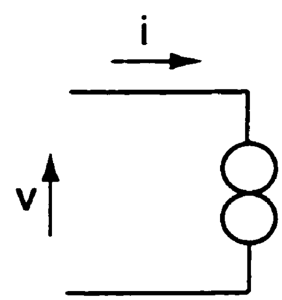


Figure 1.9: **The Norator**

The constitutive equations are

$$v = arbitrary,$$

$$i = arbitrary.$$

Considering the equations for an ideal op–amp given in 1.27, 1.28, and 1.29, we can use a nullator and a norator to represent an ideal op–amp. See Figure 1.10. This device is called a *nullor*.

The nullator represents the inputs of an ideal op–amp, since there is no current into them and no voltage drop across them. The norator represents the output, since

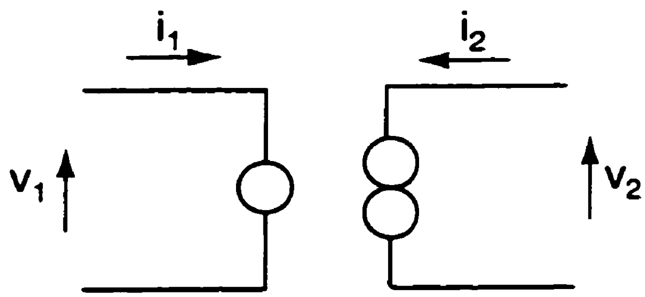


Figure 1.10: **The Nullor Op–Amp**

there is no current constraint, and the output of the ideal op–amp will be whatever is necessary to force a zero voltage drop across the inputs.

For the purposes of writing the Brayton–Moser equations, the nullator and norator can be included as resistive elements. The nullator has a resistor characteristic of $(0, 0)$ and the norator has characteristic (i, v). An op–amp is therefore made up of two resistive elements that together produce a two dimensional characteristic (the whole $i - v$ plane given by the norator). So the set of physical states, Σ, can be constructed in the same manner with its dimension still being equal to the dimension of $\mathcal{L} \times \mathcal{C}^{*}$.

Chapter 2

The Circuit Analogy

It was shown in section 1.4.2 that a circuit with no resistors is the electric analogue of some mechanical system with Lagrangian 1.51. It is obvious that this is a Hamiltonian system with momenta given by the Legendre transformation $p_{q^\lambda} = L_\lambda \dot{q}^\lambda$. The converse problem is whether a Hamiltonian system has an electric circuit analog. The rest of this thesis is devoted to studying the electric circuit analogue of a particle in $\Re^3$ with potential $U = \frac{(x+y+z+Q)^2}{2C}$ and the nonholonomic constraint $\dot{z} = y\dot{x}$.

2.1 Unconstrained System

The basic dynamical system for which an analogy will be discussed is a particle in $\Re^3$ under the influence of the potential $U = \frac{(x+y+z+Q)^2}{2C}$, where Q and C are constants. As shown in Section 2.4.2, this is the potential given analogously by the energy in a capacitor.

The circuit shown in Figure 2.1 is a capacitor in parallel with three inductors (of equal inductance). From left to right the branches will be referred to as as the w, x, y, and z branches. The electric field energy of the capacitor is $\frac{w^2}{2C}$, where w is the charge on the capacitor and C is its capacitance. The magnetic field energy of each inductor is $\frac{L(\dot{x}^i)^2}{2}$, where $\dot{x}^i$ is the current through the i'th inductor.

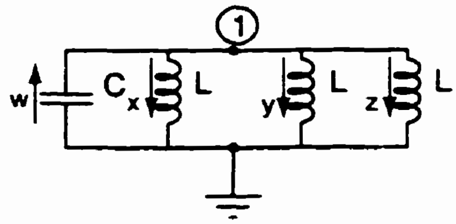


Figure 2.1: **Harmonic Oscillator Circuit**

Kirchhoff's Current Law at node 1 gives the equation

$$\dot{w} - \dot{x} - \dot{y} - \dot{z} = 0.$$

Or, after integrating,

$$w - x - y - z - Q = 0, \tag{2.1}$$

where Q is a constant of integration. The electric field energy of the capacitor is then

$$U = \frac{(x+y+z+Q)^2}{2C}.$$

Using Brayton–Moser theory, or applying Kirchhoff's voltage law, the equations for the circuit are

$$L\ddot{x} + \frac{x+y+z+Q}{C} = 0, \tag{2.2}$$

$$L\ddot{y} + \frac{x+y+z+Q}{C} = 0, \tag{2.3}$$

$$L\ddot{z} + \frac{x+y+z+Q}{C} = 0. \tag{2.4}$$

Now consider the mechanical system consisting of a particle of mass m in $\Re^3$ under the influence of a force due to the potential

$$U = \frac{(x+y+z+Q)^2}{2C}, \tag{2.5}$$

where Q and C are constants. The Lagrangian for the system is

$$\ell = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{(x+y+z+Q)^2}{2C}, \tag{2.6}$$

and the Euler-Lagrange equations are

$$m\ddot{x} + \frac{x+y+z+Q}{C} = 0, \tag{2.7}$$

$$m\ddot{y} + \frac{x+y+z+Q}{C} = 0, \tag{2.8}$$

$$m\ddot{z} + \frac{x+y+z+Q}{C} = 0. \tag{2.9}$$

Since these are identical to the circuit equations, the circuit is an electric analogue of the mechanical system.

Let

$$r = x + y + z + Q.$$

Then

$$\ddot{r} = \ddot{x} + \ddot{y} + \ddot{z}.$$

Then equations 2.2, 2.3, and 2.4 can be added to give

$$\ddot{r} + \frac{3r}{LC} = 0.$$

Or, for the mechanical system, add equations 2.7, 2.8, and 2.9 to get

$$\ddot{r} + \frac{3r}{mC} = 0.$$

This is a simple harmonic oscillator with spring constant $\frac{3}{C}$.

Using this example as a guide, charge is considered to be analogous to displacement and current to be analogous to velocity. The electric field energy in the capacitor is analogous to the potential energy of the particle. The magnetic field energy in each inductor is analogous to the kinetic energy due to each component of the particle's velocity.

In Cartesian coordinates, $p_{x^i} = m\dot{x}^i$, so $F = \dot{p}_{x^i} = m\ddot{x}^i$ is the i'th component of the force on the particle. For the circuit in Figure 2.1, $v_i = L\ddot{x}^i$ for the voltage across the i'th inductor. From this, the voltage is analogous to force and inductance is analogous to mass. A voltage drop in the i'th branch is analogous to applying a force in the x^i direction.

Note that an identification is not made between force and voltage in the capacitor branch. The reason for this is because the voltage across a capacitor is proportional to the charge on the capacitor, not proportional to the second time derivative of charge. In addition, the variable w has been eliminated via Kirchhoff's current law so that the equations only involve the x^i's and the inductor branches.

2.2 Constrained System

A constraint is given by specifying a one form, ϕ, on configuration space and requiring that all velocity vectors lie in the kernel of the one form. If ϕ is an exact one form, then the constraint determines a surface in configuration space that the particle must remain within. This is called a *holonomic constraint*. A nonintegrable constraint is called *nonholonomic*.

The constraint imparts a force on the particle to cause its motion to always satisfy the constraint. According to D'Alembert's principle, this constraint force is always perpendicular to the motion of the particle, and hence does no work on the particle. The vector field, $X = g^{\sharp}\phi$, where g is the standard metric on $\Re^3$, is perpendicular to all motions of the system. The force of constraint is given by the field $F = \lambda X$, where λ is an undetermined scalar function on configuration space. D'Alembert's principle determines the direction of F, but it's magnitude can vary from point to point, hence the function λ is necessary. λ is sometimes called a *Lagrange undetermined multiplier*.

In order to build an electric circuit analog of a mechanical system with a constraint given by ϕ, the constraint force, F, must be explicitly included in the circuit. Since a force is analogous to a voltage drop, each inductor branch in the circuit must have a voltage drop of the same magnitude as the corresponding component of the constraint force. The following is based on the unconstrained system described above.

Consider the nonholonomic mechanical constraint

$$\phi \;=\; dz - y\,dx, \tag{2.10}$$

which can also be written as

$$\dot{z} - y\dot{x} \;=\; 0. \tag{2.11}$$

D'Alembert's principle gives the force that maintains this constraint as

$$F = \lambda(-y, 0, 1),$$

where λ is a Lagrange undetermined multiplier.

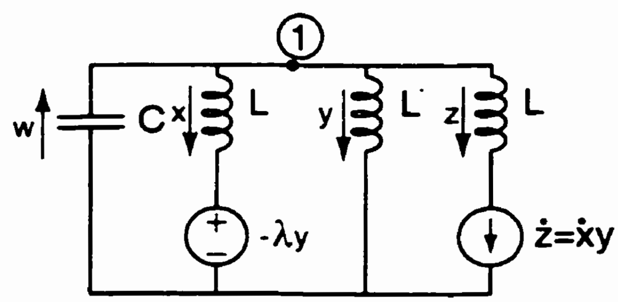


Figure 2.2: **Constrained Harmonic Oscillator Circuit**

To build a circuit that emulates constraint 2.11, a voltage drop of magnitude $-y\lambda$ needs to be added to the x branch and another voltage drop of magnitude λ in the z branch. A circuit that symbolically satisfies the constraint is shown in Figure 2.2. In the z branch a voltage controlled current source has been inserted to force the constraint to be satisfied. Since $v_z = L\ddot{z}$, the voltage across the z inductor is determined by the changing current produced by the current source. In order for Kirchhoff's voltage law to be satisfied there must be a voltage drop across the current source. Let this voltage drop have magnitude λ. Now add a voltage controlled voltage source to the x branch to give a voltage drop of $-y\lambda$. Since the constraint force does not have a y component, no additional voltage drop in the y branch is necessary.

The Lagrangian for the constrained mechanical system (Equation 2.11) is the same as the Lagrangian for the unconstrained system (Equation 2.6). Taking the nonholonomic constraint into account, the equations of motion are modified by the constraint force. Thus, the equations are

$$m\ddot{x} + \frac{x+y+z+Q}{C} - \lambda y = 0, \tag{2.12}$$

$$m\ddot{y} + \frac{x+y+z+Q}{C} = 0, \tag{2.13}$$

$$m\ddot{z} + \frac{x+y+z+Q}{C} + \lambda = 0. \tag{2.14}$$

Using Kirchhoff's voltage law, the equations for the circuit in Figure 2.2 are

$$L\ddot{x} + \frac{x+y+z+Q}{C} - \lambda y = 0, \tag{2.15}$$

$$L\ddot{y} + \frac{x+y+z+Q}{C} = 0, \tag{2.16}$$

$$L\ddot{z} + \frac{x+y+z+Q}{C} + \lambda = 0. \tag{2.17}$$

Identifying m with L, equations 2.12, 2.13, and 2.14 are identical to equations 2.15, 2.16, and 2.17.

Eliminating λ and using $\ddot{z} = y\ddot{x} + \dot{y}\dot{x}$ (which comes from differentiating equation 2.11) the final form of the Euler-Lagrange Equations 2.12, 2.13 and 2.14 are

$$\ddot{x} + \frac{y}{1+y^2}\dot{x}\dot{y} + \left(\frac{1+y}{1+y^2}\right)\frac{x+y+z+Q}{mC} = 0, \tag{2.18}$$

$$\ddot{y} + \frac{x+y+z+Q}{mC} = 0, \tag{2.19}$$

$$\dot{z} - y\dot{x} = 0, \tag{2.20}$$

Hamilton's equations for the circuit can be written by letting $p_{x^i} = L\dot{x}^i$ and $\dot{p}_w = \frac{w}{C}$. Kirchhoff's voltage law then gives

$$\begin{aligned} \dot{p}_w + \dot{p}_x - \lambda y &= 0, \\ \dot{p}_w + \dot{p}_y &= 0, \\ \dot{p}_w + \dot{p}_z + \lambda &= 0. \end{aligned}$$

The constraint $\dot{z} - y\dot{x} = 0$, written as a constraint on phase space, is $p_z - yp_x = 0$. Using Kirchhoff's Current Law, eliminating λ, and using $\dot{p}_z = \dot{y}p_x + y\dot{p}_x$ the equations can be written in Hamiltonian form as

$$\begin{aligned} \dot{x} &= \frac{p_x}{L}, \\ \dot{y} &= \frac{p_y}{L}, \\ \dot{z} &= \frac{yp_x}{L}, \\ \dot{p}_x &= -\frac{y}{1+y^2}\left(\frac{p_x p_y}{L}\right) - \left(\frac{1+y}{1+y^2}\right)\frac{x+y+z+Q}{C}, \end{aligned}$$

$$\dot{p}_y = -\frac{x+y+z+Q}{C}.$$

We now derive Hamilton's equations for the mechanical system using the method of Bates and Śniatycki [2]. The Hamiltonian is

$$h = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{w^2}{2C},$$

with constraints

$$\begin{aligned} w - x - y - z - Q &= 0, \\ dz - ydx &= 0. \end{aligned}$$

The first constraint is holonomic, so it can be substituted directly into the Hamiltonian to obtain

$$h = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{(x+y+z+Q)^2}{2C}.$$

To eliminate the nonholonomic constraint we restrict our attention to the submanifold of phase space that satisfies the constraint. Namely the set

$$M = \{(x, y, z, p_x, p_y, p_z) : p_z - yp_x = 0\}.$$

Since M is the graph of a function, $x, y, z, p_x,$ and, p_y are coordinates.

We also only consider motions of the system that satisfy the constraint form pulled back to phase space by the cotangent bundle projection. Thus, the allowable motions must have tangent vectors in the distribution

$$\begin{aligned} H &= \ker\{\pi^*(dz - ydx)\}\bigcap TM \\ &= \ker\{dz - ydx\}\bigcap TM, \\ &= \operatorname{span}\{y\partial_z + \partial_x, \partial_y, \partial_{p_x}, \partial_{p_y}\}, \end{aligned}$$

where π is the canonical projection from $T^*Q \to Q$ and Q is the configuration space.

The canonical symplectic two form on phase space, $T^*\Re^3$, is

$$\omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z.$$

Hamilton's equations are

$$X \lrcorner \omega = dh + \lambda\pi^*(dz - ydx),$$

where λ is a Lagrange undetermined multiplier. Restricted to M, the canonical symplectic two form is

$$\omega_M = dx \wedge dp_x + dy \wedge dp_y - p_x dy \wedge dz + ydz \wedge dp_x.$$

Since $\omega_M|_H = \omega_H$ is nondegenerate, Hamilton's equations can be written as

$$X \lrcorner \omega_H = dh_H,$$

where dh_H is dh restricted to vectors in H. There is no term involving the Lagrange multiplier since $\pi^*(dz - ydx)(v) = 0$ for all $v \in H$.

The Hamiltonian vector field, X, contracted with ω_M is

$$X \lrcorner \omega_M = -\dot{p}_x dx + (\dot{z}p_x - \dot{p}_y)dy - (\dot{y}p_x + y\dot{p}_x)dz + (\dot{x} + y\dot{z})dp_x + \dot{y}dp_y.$$

The differential of the Hamiltonian is

$$dh = \frac{x+y+z+Q}{C}dx + \left(\frac{yp_x^2}{m} + \frac{x+y+z+Q}{C}\right)dy + \frac{x+y+z+Q}{C}dz + \frac{1+y^2}{m}p_x dp_x + \frac{p_y}{m}dp_y.$$

Evaluating the left and right hand sides of Hamilton's equations on H gives

$$\begin{aligned}
\langle X \lrcorner \omega_M, y\partial_z + \partial_x \rangle &= -y\dot{y}p_x - y^2\dot{p}_x - \dot{p}_x, \\
\langle X \lrcorner \omega_M, \partial_y \rangle &= \dot{z}p_x - \dot{p}_y, \\
\langle X \lrcorner \omega_M, \partial_{p_x} \rangle &= \dot{x} + y\dot{z}, \\
\left\langle X \lrcorner \omega_M, \partial_{p_y} \right\rangle &= \dot{y}.
\end{aligned}$$

And

$$\begin{aligned}
\langle dh, y\partial_z + \partial_x \rangle &= (1+y)\left(\frac{x+y+z+Q}{C}\right), \\
\langle dh, \partial_y \rangle &= \frac{yp_x^2}{m} + \frac{x+y+z+Q}{C}, \\
\langle dh, \partial_{p_x} \rangle &= \frac{1+y^2}{m}p_x, \\
\left\langle dh, \partial_{p_y} \right\rangle &= \frac{p_y}{m}.
\end{aligned}$$

Equating the appropriate left and right hand sides gives

$$-y\dot{y}p_x - y^2\dot{p}_x - \dot{p}_x = (1+y)\left(\frac{x+y+z+Q}{C}\right),$$

$$\begin{aligned}
\dot{z}p_x - \dot{p}_y &= \frac{yp_x^2}{m} + \frac{x+y+z+Q}{C}, \\
\dot{x} + y\dot{z} &= \frac{1+y^2}{m}p_x, \\
\dot{y} &= \frac{p_y}{m}.
\end{aligned}$$

These are four equations in five variables. To get the fifth equation, use the fact that $X \in H$ so that

$$X = a(y\partial_z + \partial_x) + b\partial_y + c\partial_{p_x} + d\partial_{p_y}.$$

Equating coefficients with the Hamiltonian vector field for h

$$X = \dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z + \dot{p}_x\partial_{p_x} + \dot{p}_y\partial_{p_y},$$

gives

$$\dot{z} = y\dot{x}.$$

This gives us five linear equations in the five variables $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{p}_x$, $\dot{p}_y$ which results in Hamilton's equations

$$\begin{aligned}
\dot{x} &= \frac{p_x}{m}, \\
\dot{y} &= \frac{p_y}{m}, \\
\dot{z} &= \frac{yp_x}{m}, \\
\dot{p}_x &= -\frac{y}{1+y^2}\left(\frac{p_xp_y}{m}\right) - \left(\frac{1+y}{1+y^2}\right)\frac{x+y+z+Q}{C}, \\
\dot{p}_y &= -\frac{x+y+z+Q}{C}.
\end{aligned}$$

Identifying m with L shows that the circuit and mechanical system equations are the same.

Chapter 3

The Circuit

3.1 The Components

The schematic for a circuit that obeys constraint 2.11 is shown in Figure 3.1. A larger diagram is shown in Appendix A.

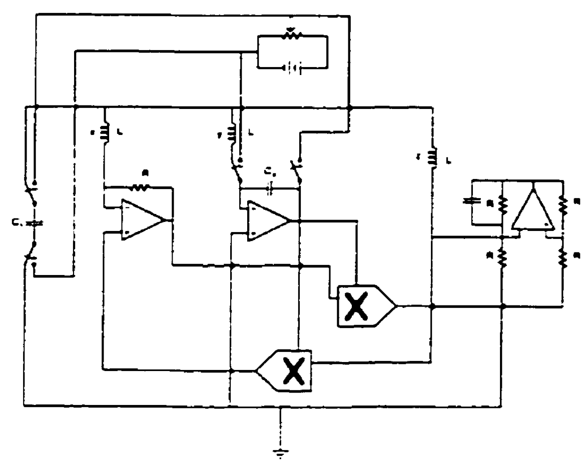


Figure 3.1: **Schematic for Constrained Harmonic Oscillator Circuit**

The constraint to be realized is $\dot{z} - y\dot{x} = 0$. In other words, the current in the z branch needs to be the product of the current in the x branch and the net charge that has gone through the y branch. These quantities are first converted to voltages for easier manipulation, then multiplied by a commercial multiplier. The output of the multiplier is sent to a voltage controlled current source in the z branch.

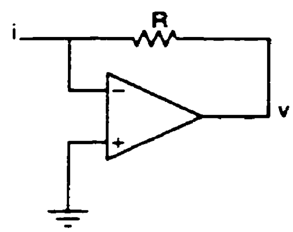


Figure 3.2: **Current to Voltage Converter**

The circuit for converting current to voltage is shown in Figure 3.2. The output is $v = -iR$. In the circuit shown in Figure 3.1, the non-inverting input is not connected to ground, but to the output of a voltage multiplier. The purpose of this will be made clear in the next section.

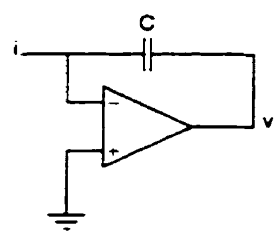


Figure 3.3: **Integrator**

Figure 3.3 shows the circuit used to convert charge to voltage. Current is the time derivative of the net charge that has gone through a branch. Hence, charge is given by the integral of current. The voltage across the capacitor is $v(t) = \frac{q(t)}{C}$, where $q(t)$ is the charge on the capacitor with its time dependence explicitly shown.

The output of the operational amplifier is $v = -\frac{1}{C}\int_0^t i(t')dt'$. This circuit is called an integrator.

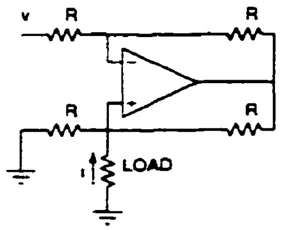


Figure 3.4: **Voltage Controlled Current Source**

The voltage controlled current source is shown in Figure 3.4. This circuit draws the current $i = \frac{v}{R}$ through a grounded load (represented by the resistor labelled 'LOAD'). For our purposes, the grounded load is the entire circuit.

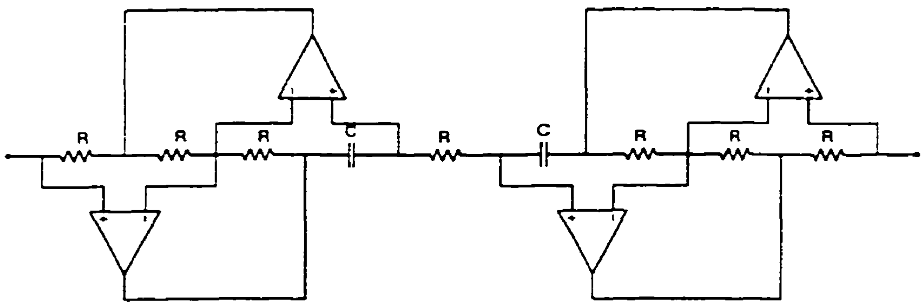


Figure 3.5: **Simulated Inductor**

The inductors in circuit 3.1 are simulated by the circuit shown in Figure 3.5. The inductance is given by $L = R^2C$. See Bruton [5] for a detailed explanation of this circuit.

3.2 The Constraint in the Circuit

We start with the standard harmonic oscillator circuit as described in Chapter 2 and shown in Figure 2.1. The current in the x branch is converted to a voltage using the circuit in Figure 3.2. The output will be denoted by

$$v_x = -i_x R.$$

The current through the y branch is integrated to give

$$v_y = -\frac{y}{C},$$

where $y = \frac{1}{C}\int_0^t i(t')dt'$, the net charge through this branch. The outputs, v_x and v_y, go to the inputs of another voltage multiplier.

The output of the multiplier is scaled by $\frac{1}{10V}$. This serves the double purpose of keeping the voltage to reasonable values and fixing the units. Since the output of the multiplier is the product of two voltages, the scaling factor needs units of $\frac{1}{\text{volts}}$ in order for the output to have units of volts. At this point we have

$$v_{xy} = \frac{v_x v_y}{10V}.$$

This goes to the input of the voltage controlled voltage source, where the current through the z inductor is forced to be

$$i_z = \frac{v_{xy}}{R}$$

$$
\begin{aligned}
&= \frac{v_x v_y}{(10V)R} \\
&= \frac{i_x R \frac{y}{C}}{(10V)R} \\
&= \frac{i_x y}{(10V)C}
\end{aligned}
$$

Or, writing $i_x = \dot{x}$ and $i_z = \dot{z}$

$$\dot{z} = \frac{\dot{x}y}{(10V)C}. \tag{3.1}$$

Kirchhoff's voltage law around a loop including the z branch shows that there is a voltage drop λ across the current source. The constraint, as given by Equation 3.1, gives an analogous constraint force of

$$F = \lambda\left(-\frac{y}{(10V)C}, 0, 1\right). \tag{3.2}$$

Equation 3.2 shows that a voltage drop of $-\lambda\frac{y}{(10V)C}$ is needed in the x branch. The output of the integrator is $-\frac{y}{C}$. This and the voltage across the current source, λ, are used as inputs to another multiplier. The output of the multiplier is then

$$v = -\lambda\frac{y}{(10V)C}. \tag{3.3}$$

The voltage in 3.3 is precisely what is needed. The output of the multiplier is then sent to the noninverting input of the current to voltage converter in the x branch to provide the necessary voltage drop.

The initial conditions that can be set for the circuit are

$$x(0) = z(0) = 0,$$

$$\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0,$$

$$y(0) = D, \tag{3.4}$$

where D is a parameter that we can control (the charge on the capacitor). With these initial conditions, $Q = 0$ in equation 2.1.

To achieve condition 3.4, a voltage, v, is placed across capacitor C_2 (Figure 3.1) to induce the charge vC_2 on the capacitor. This charge corresponds to an initial y displacement in the analogous mechanical system. There is an initial force acting on the particle due to the potential given in Equation 2.5. This force is

$$F(0) = -\frac{1}{C_1}\left(y(0), y(0), y(0)\right).$$

This implies that the voltage applied across C_2 also needs to be applied across C_1 so that each inductor will have a voltage of magnitude $\frac{y(0)}{C_1}$ applied across it. Note that $C_2 = C_1$ so that the initial conditions in terms of the charges on the capacitors are equal.

The initial conditions are valid for $t < 0$. At $t = 0$, the switch is flipped, and the capacitors are brought into the main circuit.

3.3 Building the Circuit

This section is devoted to the little details necessary to make the circuit function.

The circuit was built on a protoboard in an aluminum case for shielding. The simulated inductors were wire wrapped on separate perf boards for modularity.

The op-amps used in the circuit and inductors are LF347 quad op-amps. Individual op-amps could be used, but we found that the quads reduced the signal propagation time enough to eliminate the self oscillations experienced with the single op-amps. Also, the quad op-amps allowed for a more compact and tidy circuit.

In order to filter out power supply noise, 0.1μF capacitors were placed across the op-amps's positive power supply input and ground. The same was done for the negative power supply input.

100μF polarized capacitors were placed from the positive power supply line to ground. Same for the negative power supply line.

A 0.01μF capacitor was placed in parallel with the negative feedback resistor on the current source to reduce self oscillations in that circuit.

The capacitors used for the circuit were measured to be within 1% of each other. The capacitors used for the inductors had values with a maximum difference of 5% of each other. The resistors for the circuit had a maximum difference in value of 0.04%. For the inductors, the difference was 1%.

The switch used is a 4 pole, single throw mechanical switch. No noise was evident when switching. A digital switch should be used for a sensitive experiment.

3.4 The Experiment

The purpose of building this circuit was to test to see if an electric circuit analogue of a mechanical system is accurate enough to claim that the dynamics of the circuit are the same as the dynamics of the mechanical system. Even though the equations for the circuit and the mechanical system are identical, the circuit experiences energy loss due to the inherent resistance of conductors. So we cannot expect the actual dynamics to be identical.

The experiment was run for various initial values of y and with all other initial conditions zero. This restriction was for the purpose of simplicity. Data was collected for $\dot{x}$ and y using a computer oscilloscope. $\dot{z}$ was computed using the circuit constraint from Equation 3.1, rather than measured directly from the circuit. This was also done for simplicity. The error in computing the value rather than measuring it directly is the accuracy that one can measure the resistance for the resistors in the current source. This is 0.005% for 1KΩ resistors.

The data for x, $\dot{y}$, and z were computed numerically from the data gathered from the circuit. The programs for computing x and $\dot{y}$ are listed in Appendix B. The program for computing z is identical to the one for computing x, except for filenames. Following are the figures representing the data. Each experimental result is plotted with the theoretical result, as computed numerically by Maple using Equations 2.18, 2.19, 2.20, and the circuit constraint given in Equation 3.1. See Appendix B for the Maple code. Also included in the figures are the results of computing the total energy of the circuit as a function of time, plotted with the theoretical value of the energy computed from the Maple data.

It is apparent from the figures that the experimental data matches up reasonably well with the theoretical results. A small phase difference is noticeable, likely due to extra capacitance from the proto-board and errors in the simulated inductor values.

The computer oscilloscope sampled data fast enough to often record the same value for several consecutive time steps. This caused the numerically computed graphs for x (Figure 3.6), z (Figure 3.8), and $\dot{y}$ (Figure 3.10) to appear "jagged".

In any Hamiltonian system, the energy remains constant. As can be seen in the energy graph (Figure 3.12), however, the computed curve is definitely not constant, leaving some doubt as to the accuracy of the numerical method used by Maple. The measured energy is even worse. The problem likely lies in the accuracy of the the current to voltage converter, the charge to voltage converter, and the voltage controlled current source.

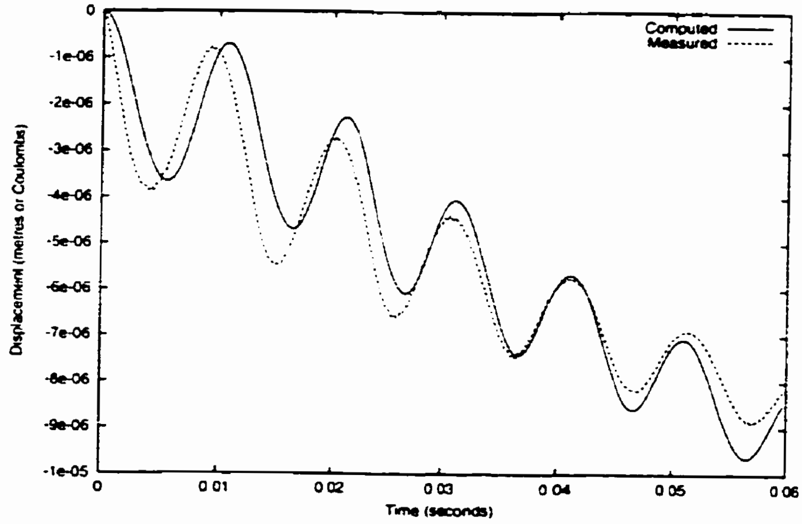


Figure 3.6: **x Displacement**

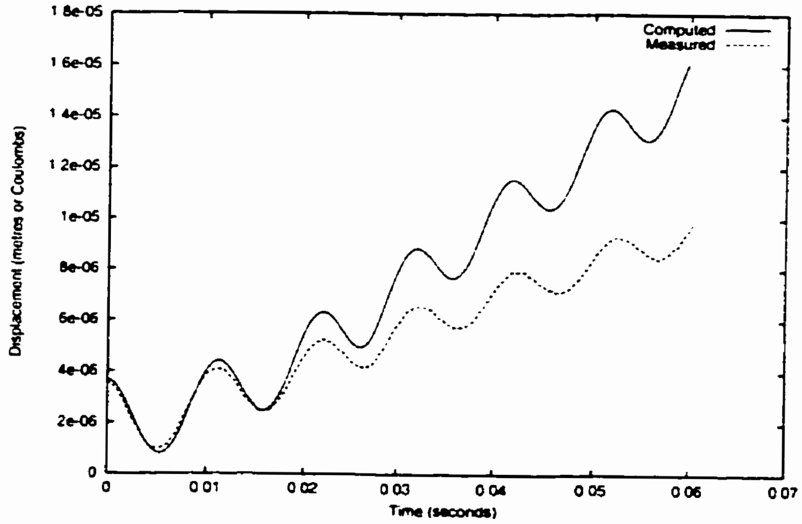


Figure 3.7: **y Displacement**

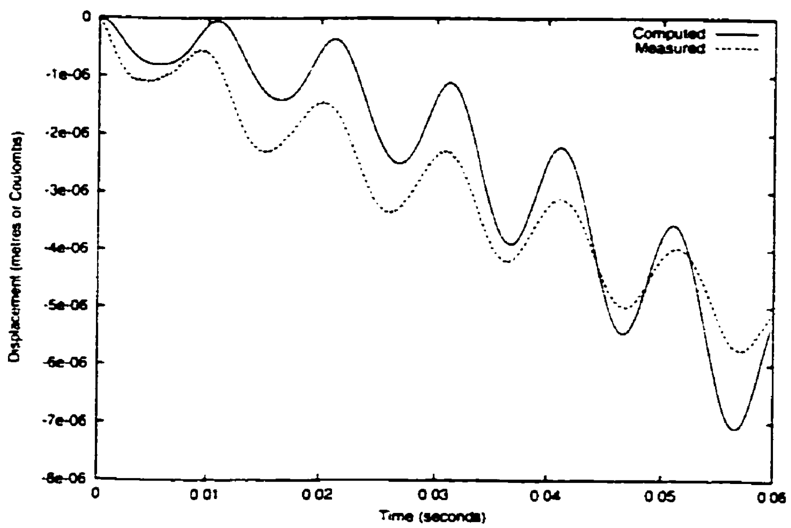


Figure 3.8: **z Displacement**

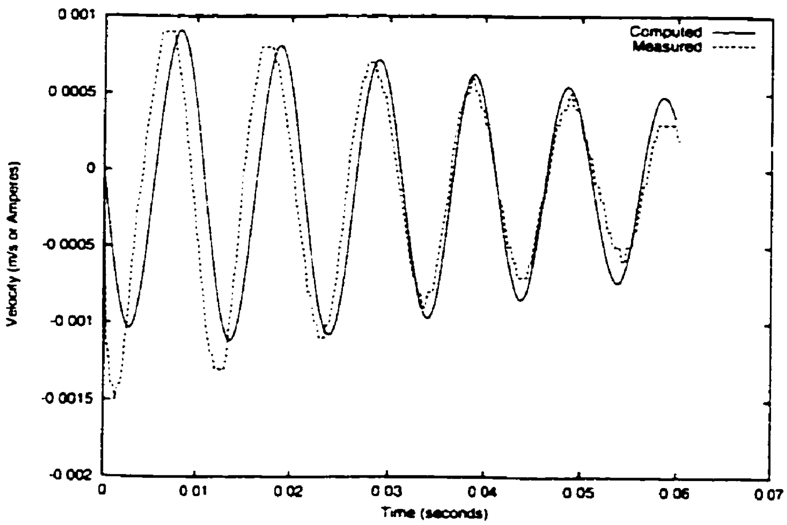


Figure 3.9: **x Velocity**

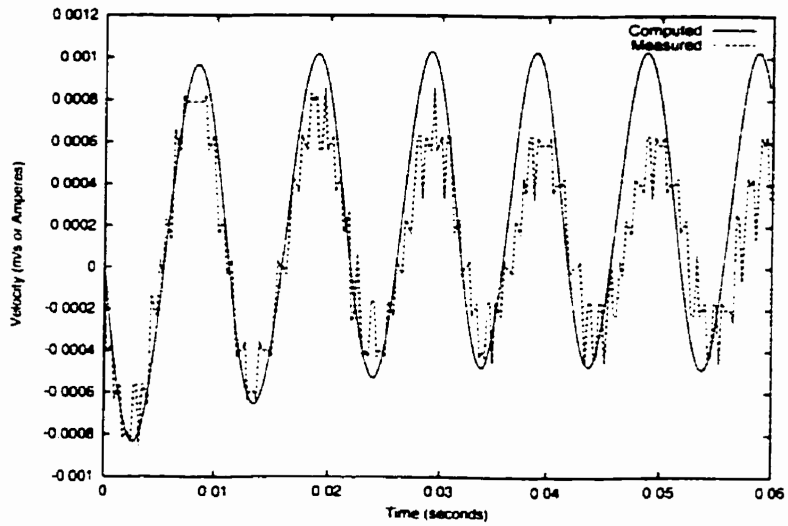


Figure 3.10: **y Velocity**

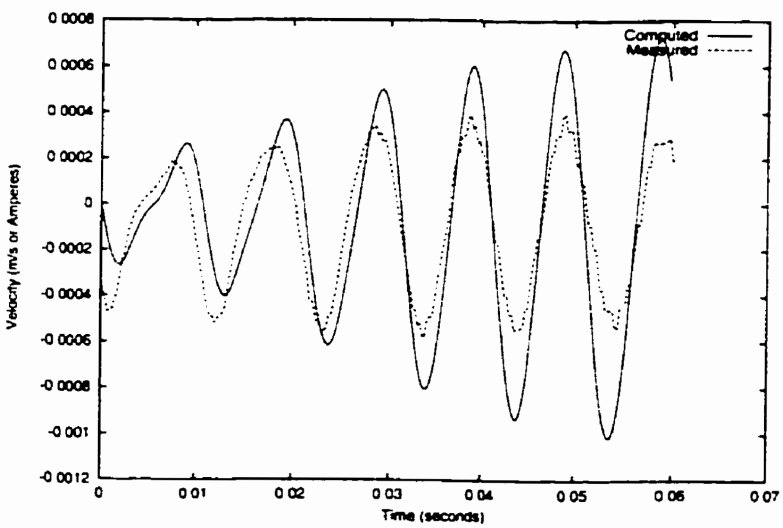


Figure 3.11: **z Velocity**

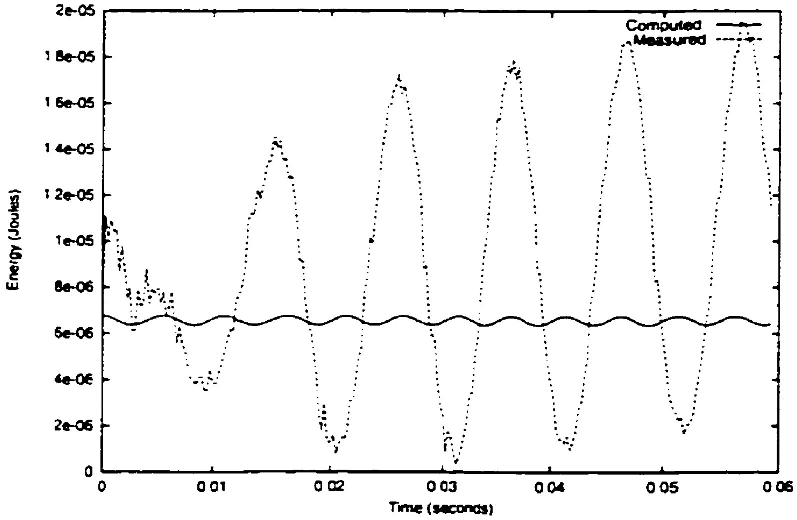


Figure 3.12: **Energy of the Circuit**

Chapter 4

Conclusion and Further Work

The goal of this project was to decide whether it was possible to build an electric circuit analogue of a Hamiltonian system with a nonholonomic constraint and to determine whether such a circuit would be useful in providing information about the dynamics of the system. It is clear from the above results that such a circuit can be built with output accurate enough for running experiments. Higher quality output can be obtained by having the circuit built on a printed circuit board with components of tighter tolerances. This should give a better energy graph as well.

As a part of this process, a theory for writing the differential equations describing a circuit's dynamics was reviewed. Resistorless circuits are shown to be analogous to Hamiltonian systems. The converse problem of deciding whether a Hamiltonian system has an electric circuit analogue is done on a case by case basis, though it should be possible to characterize certain Hamiltonian systems as having electric analogues. Further to this is determining what types of constraints can be realized in an electric circuit analogue. The ultimate goal would be to determine whether or not it is possible to construct an electric analogue of a Hamiltonian system with a nonlinear, nonholonomic constraint.

To further test the implementation of an electric analogue as many invariants as possible should be computed for the mechanical system and measured from the circuit. Invariants for holonomic systems come directly from symmetries in the system (Noether's Theorem). For nonholonomic systems, Noether's Theorem applies when the invariant function has its Hamiltonian vector field in the distribution anhilated by the constraint. Otherwise, the best that can be hoped for is a reduced constraint distribution, $\bar{H}$, that is integrable. See Bates and Śniatycki [2].

In the case studied here, the symmetry is in the direction $\partial_z - \partial_x$ so that the reduced constraint distribution is given by

$$\bar{H} = \text{span}\{(1+y)\partial_s, \partial_y, \partial_{p_x}, \partial_{p_y}\},$$

where (s, y, p_x, p_y) parameterize the reduced phase space, $\bar{M}$. $\bar{H}$ drops a dimension in the hyperplane $y = -1$ so that a direct application of the theory in Bates and Śniatycki [2] is not straightforward. Further analysis could be done using the method in Bates [3].

On the topic of symmetry reduction, an interesting side note is found by examining the Hamiltonian system corresponding to the unconstrained harmonic oscillator circuit in Figure 2.1. The Hamiltonian is

$$h = \frac{1}{2L}(p_x^2 + p_y^2 + p_z^2) + \frac{(x+y+z)^2}{2C},$$

where L and C are constants. This system admits translational symmetries in the directions $\partial_z - \partial_x$ and $\partial_z - \partial_y$. The reduced Hamilton's equations are

$$\begin{aligned} \dot{q} &= \frac{3}{L} p_q, \\ \dot{p_q} &= -\frac{q}{C}, \end{aligned}$$

where q and p_q parameterize the reduced phase space. These are the same equations we get by replacing the three parallel inductors in the harmonic oscillator circuit by a single equivalent inductor. In a sense, the process of replacing parallel inductors with a single equivalent inductor "reduces" the circuit. It would be interesting to see what the reduced circuit would be for the system studied in this thesis, if one exists.

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Appendix A

Parts List

Part	Quantity
LF 347 Quad Op–Amp	1
AD633 Multiplier	2
1KΩ Resistor	5
100μF Capacitor	2
1μF Capacitor	2
0.1μF Capacitor	2
0.01μF Capacitor	1
Potentiometer	1
4 Pole Switch	1
12V Power Supply	1
9V Battery	1

Table A.1: Parts for Circuit

Part	Quantity
LF 347 Quad Op–Amp	1
3.3KΩ Resistor	7
0.68μF Capacitor	2
0.1μF Capacitor	2

Table A.2: Parts for Each Simulated Inductor

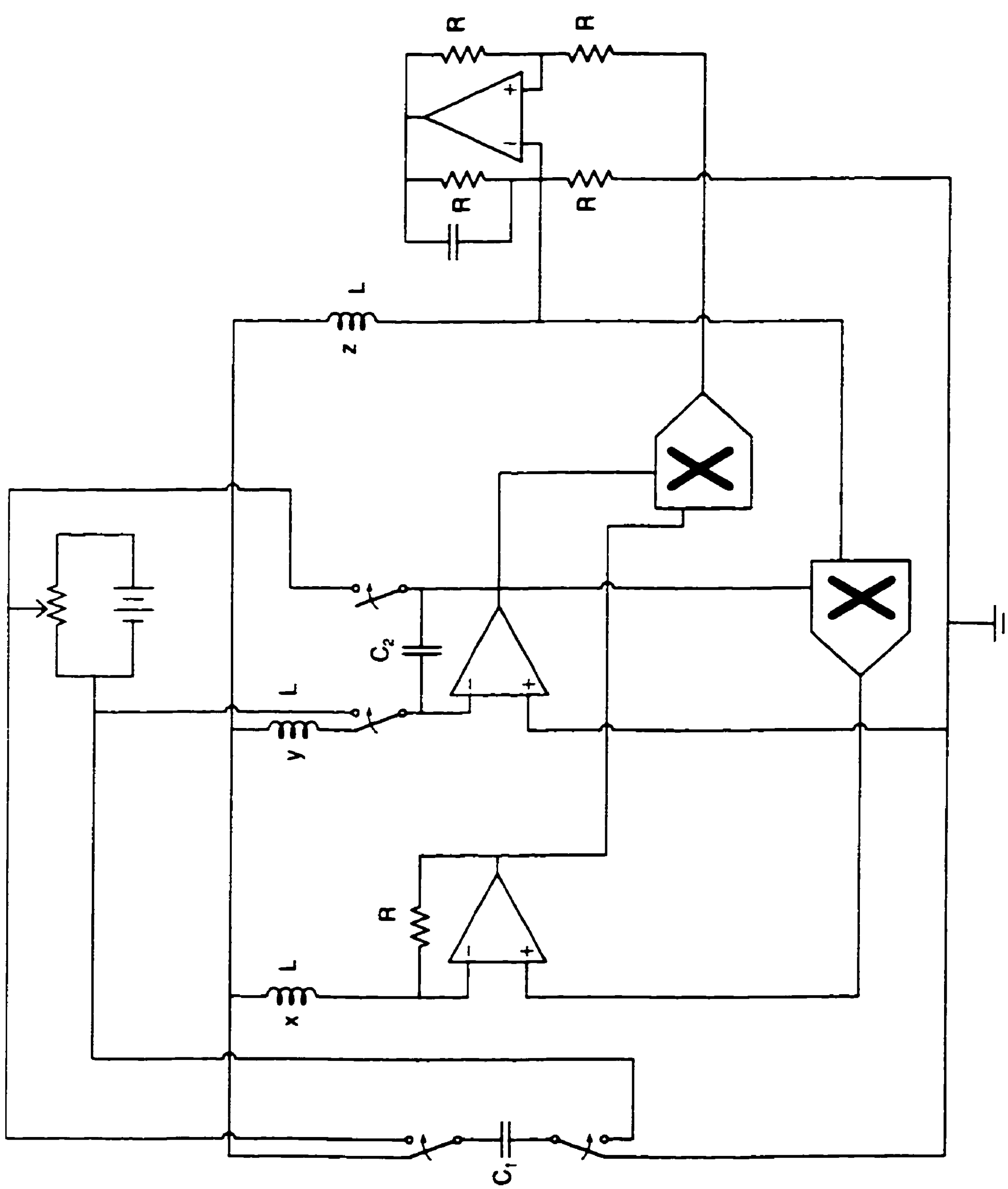

Figure A.1: **Constrained Harmonic Oscillator Circuit**

Appendix B

Computer Programs

```
c     FORTRAN Program to compute x from xdot data
c       using Simpson's rule

      real xdot1,xdot2,xdot3,deltat
      real x1,x2,x3,t1,t2,t3
      integer i

      open(unit=10,file='xdot.dat',status='old')
      open(unit=11,file='x.dat',status='new')

c     time step given by the data file
      deltat=0.000250

      read(10,*) t1,xdot1
      read(10,*) t2,xdot2
      read(10,*) t3,xdot3
```

```
      x1=0.
      x2=.5*deltat*xdot2
      x3=deltat*(xdot1+4.*xdot2+xdot3)/3.
      write(11,100) t1,x1
      write(11,100) t2,x2
      write(11,100) t3,x3

c     Simpson's rule
c     Essentially doing the integration two times, but
c       with different starting points.
      do i=4,240,2
         xdot1=xdot2
         xdot2=xdot3
         t1=t2
         t2=t3
         read(10,*) t3,xdot3
         x2=x2+deltat*(xdot1+4.*xdot2+xdot3)/3.
         write(11,100) t3,x2
```

```
      xdot1=xdot2
      xdot2=xdot3
      t1=t2
      t2=t3
      read(10,*) t3,xdot3
      x3=x3+deltat*(xdot1+4.*xdot2+xdot3)/3.
      write(11,100) t3,x3
    end do

100 format(f8.5,e13.4)

    close(10)
    close(11)

    end
```

```
c     FORTRAN program to compute ydot from y data
c       using numerical differentiation

      real y1,y2,y3,y4,y5,h,ydot,t
      integer i

      open(unit=10,file='y.dat',status='old')
      open(unit=11,file='ydot.dat',status='new')

c     time step given by the data
      deltat=0.000250

      read(10,*) t,y1
      read(10,*) t,y2
      read(10,*) t,y3
      read(10,*) t,y4

      do i=5,242
         read(10,*) t,y5
         ydot=(8.*y4-8.*y2-y5+y1)/(12.*deltat)
```

```
        write(11,100) t-2.*deltat,ydot
        y1=y2
        y2=y3
        y3=y4
        y4=y5
      end do

100   format(f8.5,e13.2)

      close(10)
      close(11)

      end
```

```
#Maple Code for Numerically Solving the Euler-Lagrange
#      Equations
#
#Set some constants
 R1:=1E3;R2:=1.0044E3;L:=7.45;C:=0.99E-6;h:=0.000250;
#
#Define the system as a set of first order ODE's
 sys:={diff(x(t),t)=v(t),
 diff(y(t),t)=w(t),
 diff(v(t),t)+10*R2*(x(t)+y(t)+z(t))*(R2*10*C+R1*y(t))
  /(L*((R2*10*C)^2+(R1*y(t))^2))+
  R1^2*y(t)*v(t)*w(t)/((R2*10*C)^2+(R1*y(t))^2)=0,
 diff(w(t),t)+(x(t)+y(t)+z(t))/(L*C)=0,
 diff(z(t),t)-R1*v(t)*y(t)/(R2*10*C)=0};
#
#Set initial conditions
 inits:={x(0)=0,y(0)=3.7*C,z(0)=0,v(0)=0,w(0)=0};
#
 vars:={x(t),y(t),z(t),v(t),w(t)};
#
```

```
 S:=dsolve(sys union inits,vars,type=numeric,output=listprocedure);
 fx:=subs(S,x(t)):fy:=subs(S,y(t)):fz:=subs(S,z(t)):fv:=subs(S,v(t)):
 fw:=subs(S,w(t)):
#
 A:=array(1..240,1..7):
 for i to 240 do
  a:=fy((i-1)*h):
  b:=fv((i-1)*h):
  A[i,1]:=(i-1)*h:
  A[i,2]:=fx((i-1)*h):
  A[i,3]:=a:
  A[i,4]:=fz((i-1)*h):
  A[i,5]:=b:
  A[i,6]:=fw((i-1)*h):
  A[i,7]:=R1*a*b/(R2*10*C):
 od:
#Write to a data file
 fd:=fopen(linearnumdat,WRITE,TEXT):
 writedata(fd,A,float):
 fclose(linearnumdat):
```