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The Distinguishing Chromatic Number of Wreath Products

by

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Abstract

We investigate the distinguishing number and distinguishing chromatic number of a graph. Extending the results of Collins and Trenk, we find the distinguishing chromatic number of the wreath product of specific graphs. In particular, we find the exact values of $\chi_D(C[G])$ and $\chi_D(P[G])$, where C is a cycle, P is a path, and G is a connected graph. We also obtain an upper bound on $\chi_D(T[G])$, for any tree T. The distinguishing chromatic numbers of a wreath product found in this thesis are with respect to the number of distinct distinguishing r-colourings of G, $\xi_r(G)$. We then provide algorithms to find exact values of $\xi_r(C)$ and $\xi_r(T)$.

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Chapter 1

Introduction

1.1 Overview

The concept of the distinguishing number of a graph was introduced by Albertson and Collins [3] in connection with the following problem posed by Frank Rubin [18].

> Suppose a blind man has a circular ring of keys, each seemingly identical. However, a variety of handles are available for each key, distinguishable by touch. If each key and handle is symmetric, then how many different handles are required for the man to always be able to select the correct key first?

In essence, the question is concerned with how one can organize objects around a circle so as to eliminate all possible symmetries of the arrangement, and in particular, determining the fewest number of different objects required to do so.

The *distinguishing number* of a graph G, $D(G)$, is the fewest number of labels needed to label the vertices of G so as to destroy its symmetries. In this context, the solution to Rubin's problem is to find the distinguishing number of a particular type of graph, namely a cycle.

Instead of circular key rings, Albertson and Collins [3] propose a generalization of Rubin's problem to key "rings" of varying shapes. This notion leads to the distinguishing number of an arbitrary graph, and efforts to find the value of $D(G)$ for an arbitrary graph G.

The distinguishing number of a graph is one example from the vast collection of graph labeling problems. The most well known graph labeling problem is determining the chromatic number of a graph. The *chromatic number* of a graph G, $\chi(G)$, is the fewest number of labels required to label all the vertices of a graph so that any two vertices connected by an edge receive different labels. When this occurs, each vertex label is referred to as the *colour* of the vertex. The assignment of colours to the vertices of a graph is a *graph colouring.*

Graph colouring has a long history which stems from the "map colouring" problem.

> Given a map divided into countries, what is the fewest number of colours required to colour the countries of the map so that countries that share a border receive different colours?

This problem can be modeled with what is known as a planar graph. The notion of the dual of a planar graph can then be used to restate the map colouring problem. Given a planar graph G (corresponding to some map), the dual of G has vertices corresponding to the countries of the map, and hence the map colouring problem is equivalent to colouring the vertices of the dual of G. In 1976, Appel and Haken [6] proved one of the most famous results in graph theory, the Four Colour Theorem. The Four Colour Theorem states that if G is a planar graph, then $\chi(G) \leq 4$. This is a significant result on its own, but also because, in general, determining the chromatic number of an arbitrary graph is a difficult problem.

Collins and Trenk [12], in 2006, marry the concept of the distinguishing number and chromatic number to create the *distinguishing chromatic number* of a graph. The distinguishing chromatic number of a graph G, $\chi_D(G)$, is the fewest number of colours required to colour the vertices of a graph so as to eliminate all symmetry (the use of the term colour rather than label implies that any two vertices joined by an edge receive different colours). Collins and Trenk proceed to determine values of $\chi_D(G)$ for certain graphs G, and determine bounds for arbitrary graphs.

The results of Albertson and Collins [3], and Collins and Trenk [12], have led to a flood of interest in the distinguishing number and the distinguishing chromatic number. A survey of results about distinguishing number, distinguishing chromatic number, and the complexity of determining the distinguishing number and the distinguishing chromatic number are contained in Chapter 2. The remainder of the thesis focuses on determining the distinguishing chromatic number of the wreath product of graphs. In Chapter 3 we examine how one can determine the distinguishing chromatic number of two general classes of wreath products: the wreath product of a cycle with an arbitrary connected graph, and the wreath product of a tree with an connected graph. We obtain an equality for the distinguishing chromatic number of the wreath product of cycles with connected graphs and upper bounds on the distinguishing chromatic number of wreath products of trees with connected graphs. An important concept in determining the distinguishing chromatic number of a wreath product of graphs is the number of distinct distinguishing colourings of a graph. We introduce this notion in Chapter 3, but then expand on this topic in the following chapter. Chapter 4 begins with a discussion of distinct colourings of a graph using k colours. We then find the number of distinct distinguishing colourings of trees and cycles. The final chapter, Chapter 5, provides the concluding findings of the thesis and some open problems to be considered in future work.

1.2 Notation and Terminology

A *graph* G consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge is an unordered pair of vertices. For $u, v \in V(G)$, the edge $e = \{u, v\}$ is written $e = uv$, and we say u and v are *adjacent* vertices, and that vertices u and v are *incident* with edge e. For our purposes, graph means *simple graph*, that is, any edge consists of a pair of distinct vertices, and any pair of vertices determines at most one edge. Note that this ensures that an edge is uniquely determined by its two incident vertices. A graph is said to be *planar* if it is possible to embed its vertices and edges in the plane so that no two edges cross.

Let G be a graph and $u \in V(G)$. The *neighbourhood* of u, denoted $N_G(u)$ is the set of all vertices adjacent to u. The *degree* of u, denoted $d_G(u)$, is the number of elements in $N_G(u)$. If there is no chance of confusion, we simply write $N_G(u)$ as $N(u)$ and $d_G(u)$ as $d(u)$. In general, for any finite set X, we write $|X|$ to denote the number of elements in X, so $d(u) = |N(u)|$. We denote by $\Delta(G)$ the maximum degree of the vertices of G, i.e., if $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ then $\Delta(G) = \max\{d(v_i) \mid 0 \leq i \leq n-1\}$. Again, if there is no chance of confusion, we simply write Δ for $\Delta(G)$.

A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$, and for any $u, v \in V(H)$, if $uv \in E(H)$ then $uv \in E(G)$. For $S \subseteq V(G)$, the *subgraph of G induced by S*, denoted $G[S]$, has as its vertex set S, and for $u, v \in S$, $uv \in E(G[S])$ if and only if $uv \in E(G)$.

A *path* P is a graph in which the vertices can be named so that only consecutive vertices are adjacent. More specifically, it is possible to name the vertices of P as $V(P) = \{v_0, v_1, \ldots, v_{k-1}\}$ for some $k \geq 1$, so that $E(P) = \{v_i v_{i+1} \mid 0 \leq i \leq k-2\}$, and we write $P = v_0 v_1 \cdots v_{k-1}$. Vertices v_0 and v_{k-1} are the *endpoints* of the path P, and we say that P is a path from v_0 to v_{k-1}. We often write P_k to denote a path with k vertices; the length of a path is defined to be the number of edges in the path, so the length of P_k is $k-1$.

A *cycle* C is a graph with at least three vertices, whose vertices can be named so that consecutive vertices are adjacent, along with the extra provision that the first and last vertices are also adjacent, and no others. More specifically, it is possible to name the vertices of C so that $V(C) = \{v_0, v_1, \ldots, v_{k-1}\}$ for some $k \geq 3$, and so that $E(C) = \{v_i v_{i+1} \mid 0 \leq i \leq k-1\}$, where subscripts are taken modulo k. We write $C_k = v_0 v_1 \cdots v_{k-1} v_0$ to denote a cycle with k vertices; the length of a cycle is the number of edges in the cycle (which is equal to the number of vertices in the cycle), so the length of C_k is k. A graph is said to be *acyclic* if it contains no cycles.

A graph G is *connected* if for any pair of vertices $u, v \in V(G)$ there is a path in G from u to v. The *distance* between two vertices u and v is the length of a shortest path from u to v, and is denoted $d_G(u, v)$.

A *tree* T is a graph that is both connected and acyclic. A tree is a *rooted tree* if one of its vertices has been designated the root (the root may be an arbitrary designation). If T is a tree and $u, v \in V(T)$, then there is a unique path in T joining u and v. Let T be a tree with root r. For any vertex u of T the *level* of u is the distance in T between u and r. Thus, the level of r is zero, and the vertices in $N(r)$ are at level one. Any non-root vertex in T with degree one is referred to as a *leaf.* A vertex in T that is neither the root nor a leaf is an *internal* vertex. For a non-leaf vertex $v \in V(T)$, a vertex $x \in N(v)$ such that the level of x is one greater than the level of v is referred to as a *child* of v, and v is the *parent* of x. Any two children, x and y, of v are called *siblings.*

A center of a tree T is a vertex z with the property that $\max\{d_T(z, u) \mid u \in V(T)\}$ is minimized. It is not difficult to prove [6, Exercise 4.1.8] that a tree T has either a unique center or two adjacent centers.

Let G be a graph. The complement of G, denoted $\overline{G}$, is a graph such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid uv \notin E(G), \text{ for } u, v \in V(G)\}$. Thus, each pair of non-adjacent vertices in G is adjacent in $\overline{G}$, and each pair of adjacent vertices in G is non-adjacent in $\overline{G}$.

A *complete graph* K_n is a graph on n vertices in which any two vertices are adjacent. If G is a graph, a subset $I_m \in V(G)$, with $|I_m| = m$, is called an *independent set* if and only if , for any $u, v \in I_m$, $uv \notin E(G)$, i.e., no two vertices in I_m are adjacent. Note that if G is a complete graph on n vertices, then $\overline{G}$ is an independent set of size n.

A *bipartite graph* is a graph, in which the vertex set can be partitioned into two disjoint sets, X and Y, so that any edge connects a vertex in X to one in Y, i.e., X and Y are independent sets. A *complete bipartite graph*, $K_{m,n}$, is a bipartite graph in which

$|X| = m$, $|Y| = n$, and each vertex in X is adjacent to all vertices in Y.

Two graphs G and H are *isomorphic* if and only if there is a bijection $\theta : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\theta(u)\theta(v) \in E(H)$. The bijection θ is called an *isomorphism* between G and H. An *automorphism* of G is simply an isomorphism of G to itself. Note that since the graphs we are considering are simple, an automorphism of G is just a permutation of $V(G)$ that preserves adjacency. We denote by $Aut(G)$ the set of all automorphisms of G. The set $Aut(G)$ under the operation of composition is a group, called the *automorphism group* of G. We say that the graph G realizes the group Q if the group $Aut(G)$ and Q are isomorphic groups, and the statement G realizes Q is denoted by $Aut(G) \cong Q$. For example $Aut(C_n) \cong D_n$, where D_n is the dihedral group of order $2n$, and $Aut(K_n) \cong S_n$, where S_n is the symmetric group of order $n!$.

A vertex labeling of a graph G is a mapping c from $V(G)$ to a set S. If $|S| = k$, then c is called a k-labeling of G, and we usually assume (without loss of generality) that $S = \{0, 1 \ldots, k-1\}$. We say a k-labeling c is a *k-colouring* if no two adjacent vertices are assigned the same label. The chromatic number of a graph G, $\chi(G)$, is the fewest number of colours required for a k-colouring of G. A labeling c of the graph G is *distinguishing* if the only automorphism of G that preserves the vertex labels is the trivial automorphism. If c is a k-labeling of G that is distinguishing, then c is said to be *k-distinguishing.* The fewest number of labels that can be used to give G a distinguishing labeling is called the *distinguishing number of* G, denoted by $D(G)$. The distinguishing chromatic number of G, denoted $\chi_D(G)$, is the fewest number of colours required to give G a distinguishing colouring.

Let G be a graph, $u \in V(G)$ and $c : V(G) \to S$ a labeling of G. We say that c *fixes* u, or that u is *fixed* by c, if and only if any label preserving automorphism $g \in Aut(G)$ has the property that $g(u) = u$.

For any terms and notation not defined here, the reader is directed to [6] or [14].

Chapter 2

A Survey of the Distinguishing Number and the Distinguishing Chromatic Number

2.1 The Distinguishing Number

The Frank Rubin problem, introduced at the start of the Chapter 1 is as follows.

> Suppose a blind man has a circular ring of keys, each seemingly identical. However, a variety of handles are available for each key, distinguishable by touch. If each key and handle is symmetric, then how many different handles are required for the man to always select the correct key first?

Shortly after the question was posed came the somewhat surprising answer. If the key ring has at least six keys, then two handle shapes suffice to identify the keys. If there are three, four, of five keys, then two handle shapes are not sufficient but three are. Albertson and Collins [3] restate Rubin's problem in the context of graphs by defining the distinguishing number of a cycle and, more generally, the distinguishing number of a graph. Using their notation, the answer to Rubin's problem is

$$D(C_n) = \begin{cases} 2 \text{ if } n \geq 6, \\ 3 \text{ if } n = 3, 4, 5, \end{cases}$$

where C_n is the cycle of length n.

Albertson and Collins [3] then restate the problem as a graph labeling problem, formulating the notion of the distinguishing number and providing preliminary results. Recall that the distinguishing number of an arbitrary graph G is the fewest number of labels required to label the vertices of G so as to eliminate all symmetries of G.

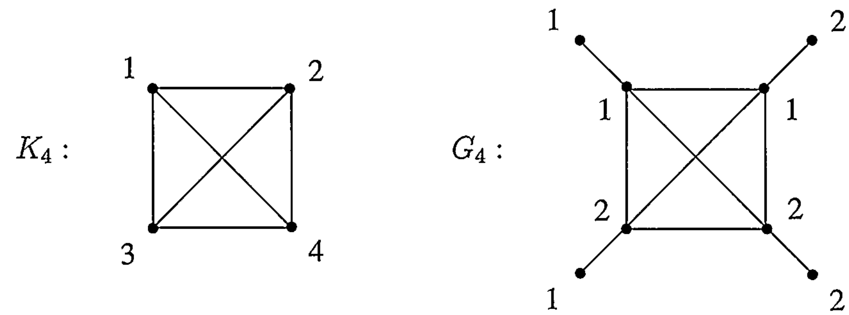

Figure 2.1: $Aut(K_4) \cong Aut(G_4)$, but $D(K_4) \neq D(G_4)$.

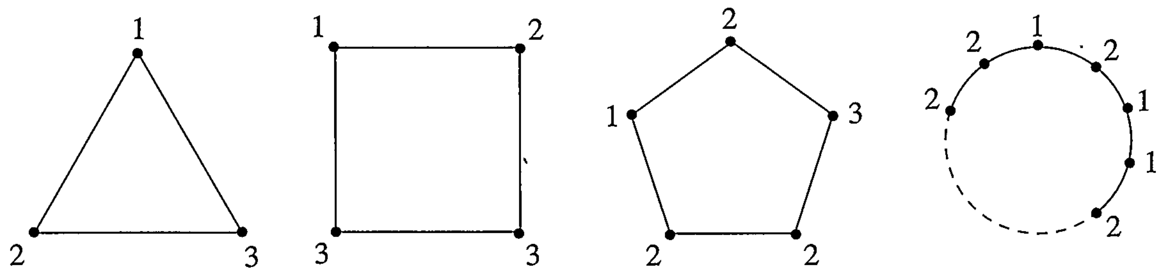

Figure 2.2: Distinguishing labelings of cycles

Albertson and Collins first observe that the distinguishing number of G is equal to the distinguishing number of $\overline{G}$. They also relate the distinguishing number of G to its automorphism group, but note that the distinguishing number of G is not entirely dependent on $Aut(G)$. An example makes this fact clear. Consider the complete graph K_n and the graph G_n, a graph with $2n$ vertices obtained by attaching a single vertex to every vertex of K_n. Figure 2.1 illustrates the case when $n = 4$. Despite the two graphs sharing the same automorphism group, i.e., $Aut(K_n) \cong Aut(G_n) \cong S_n$, $D(K_n) \neq D(G_n)$. Figure 2.1 shows $D(K_4) \leq 4$, and it is clear that fewer labels are not sufficient, so $D(K_4) = 4$. On the other hand, Figure 2.1 also illustrates a distinguishing labeling of G_4 with two labels, implying that $D(G_4) = 2$.

Although for an arbitrary graph G there appears to be no direct correlation between $Aut(G)$ and $D(G)$, Albertson and Collins do present relationships that exist under certain circumstances. In particular, if G is a group for which $Aut(G) \cong D_n$, then $D(G) \leq 3$. This is most relevantly exemplified by C_n, since it pertains to the primary question regarding key rings. Figure 2.2 displays distinguishing labelings of all cycles: C_3, C_4, C_5 and C_n, $n \geq 6$.

Albertson and Collins also show that for a graph G and a nontrivial normal group Γ, if $Aut(G) \cong \Gamma$, then $D(G) = 2$. As a corollary, it follows that for a graph G and a nontrivial abelian group Γ, if $Aut(G) \cong \Gamma$, then $D(G) = 2$. Focusing on graphs with distinguishing number two, it is shown that for any group Γ, there is a graph G such that $Aut(G) \cong \Gamma$ and $D(G) = 2$. The proof of this first requires the existence of a graph G for which $Aut(G) \cong \Gamma$. The existence of such a graph is guaranteed in [13]. Attach a path of length $\log_2 n$ to each of the n vertices in G to obtain a graph G', where $Aut(G') \cong \Gamma$. Since there exists $2^{\log_2 n} = n$ ways to 2-label a path of length $\log_2 n$, we can label each path differently, resulting in a distinguishing label of G'. This is illustrated in Figure 2.1 with $G \cong K_4$, $G' \cong G_4$, and $Aut(G') \cong S_4$.

Albertson and Collins also prove that if G is a graph with $Aut(G) \cong S_4$, then $D(G) = 2$ or $D(G) = 4$. This result is proved by examining the orbits of S_4 on a case by case basis. Examples of both situations are illustrated in Figure 2.1.

Bogstad and Cowen [5], rather than characterizing graphs based on automorphism groups, consider the distinguishing number of two particular families of graphs, hypercubes and squares of hypercubes. The d-dimensional hypercube, Q_d, has 2^d vertices consisting of all binary strings of length d. Vertices in Q_d are adjacent if their strings differ in exactly one entry. The square of a hypercube, Q_d^2, is obtained from Q_d by adding new edges between every pair of vertices that are distance two from each other in Q_d.

Bogstad and Cowen [5] prove that

$$D(Q_d) = \begin{cases} 3 \text{ for } d = 2 \text{ or } d = 3, \\ \\ 2 \text{ otherwise.} \end{cases}$$

A distinguishing labeling of Q_d for $d \geq 4$ is as follows. Let $v_i \in V(Q_d)$, $0 \leq i \leq d$, be a vertex whose first i digits of its binary string are 0 and the remaining $d-i$ digits are 1. In addition let $w \in V(Q_d)$ be the vertex $100\cdots001$. Notice that the vertices $\{v_0, v_1, \ldots, v_d,\}$ form a path of length $d-1$, and of these vertices w is adjacent only to v_{d-1}. Now label the vertices in the set $\{v_0, v_1, \ldots, v_d, w\}$ with label 1, and all other vertices with label 2. This produces a distinguishing labeling of Q_d.

To see this, let G be the subgraph of Q_d induced by the vertices $\{v_0, v_1, \ldots v_d, w\}$. Suppose $\sigma \in Aut(Q_d)$ preserves the labels of Q_d. Then σ, restricted to G, is an automorphism of G. Hence $\sigma(v_{d-1}) = v_{d-1}$, since v_{d-1} is the only vertex on G with degree three. Furthermore, σ must preserve distance between vertices in Q_d. Because of the way Q_d is defined, the distance between two vertices in Q_d is simply the number of digits in which they differ. In Q_d, $d_{Q_d}(v_0, w) = d-1$ while $d_{Q_d}(v_0, v_d) = d$, and thus $\sigma(w) = w$ and $\sigma(v_d) = v_d$. This implies that all vertices of G are fixed by σ.

Now suppose that $x, y \in V(Q_d) \backslash V(G)$ are distinct vertices such that $\sigma(x) = y$. Since distance between vertices in Q_d must be preserved by σ and since $\sigma(v_d) = v_d$, $d(x, v_d) = d(\sigma(x), \sigma(v_d)) = d(y, v_d)$. But $v_d = 00\ldots0$, so this implies that x and y have the same number of digits equal to 0 and the same number of digits equal to 1. Let $x = x_1x_2\ldots x_d$ and $y = y_1y_2\ldots y_d$, and suppose that i is the first digit in which x and y differ. Then $x_j = y_j$ for $j = 1, 2, \ldots, i-1$ and $x_i \neq y_i$. Suppose that a of the digits $x_1, x_2, \ldots x_{i-1}$ are equal to 1, then a of the digits $y_1, y_2, \ldots y_{i-1}$ are also equal to 1. Since x and y differ in the i^{th} digit, $x_i = 1$ and $y_i = 0$, or $x_i = 0$ and $y_i = 1$. Without loss of generality assume that $x_i = 1$ and $y_i = 0$. Denote by b the number of digits among

$x_{i+1}, x_{i+2}, \ldots x_d$ equal to 1. Then x has $a+b+1$ digits equal to 1. Since y also has $a+b+1$ digits equal to 1, and $y_i = 0$, there are $b+1$ digits among $y_{i+1}, y_{i+2}, \ldots y_d$ that are equal to 1. Since $v_i = 00\ldots011\ldots1$ with the first i digits equal to 0,

$$\begin{aligned} d(x, v_i) &= a + 1 + (d-i) - b \\ &= d + a - b + 1 - i \end{aligned}$$

and

$$\begin{aligned} d(y, v_i) &= a + (d-i) - (b+1) \\ &= d + a - b - 1 - i \end{aligned}$$

We conclude that x and y are not the same distance from v_i, and thus must be fixed by σ. Since all the vertices in Q_d with label 1 are fixed, and all the vertices in Q_d with label 2 are fixed, there does not exist a nontrivial automorphism σ that preserves labels.

In a similar way Bogstad and Cowen also prove that

$$D(Q_d^2) = \begin{cases} 4 \text{ for } d = 2 \text{ or } d = 3, \\ 2 \text{ otherwise.} \end{cases}$$

Albertson [1] generalizes the work of Bogstad and Cowen to Cartesian powers of graphs. The Cartesian product of two graphs G and H, denoted $G \square H$, has vertex set $V(G) \times V(H)$ and $(u_1, v_1)(u_2, v_2) \in E(G \square H)$ if $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. This can be thought of as a product for which any edge in G and any edge in H produces four edges (a square) in the graph $G \square H$. Occasionally we take repeated Cartesian products of a graph with itself, $G \square G \square \cdots \square G$ (k times). This is referred to as a Cartesian power of G, and is denoted $G^{[k]}$. Note that $Q_d = K_2^{[d]}$.

A graph is *prime* if it can not be written as a Cartesian product of two non-trivial graphs. If the graphs H and G can be written as Cartesian products of prime graphs, say,

$H = H_1 \square H_2 \square \cdots \square H_k$ and $G = G_1 \square G_2 \square \cdots \square G_l$, then H and G are relatively prime if $H_i \neq G_j$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$.

Albertson [1] conjectures that for a connected graph G there exists a value R so that for all $r \geq R$, $D(G^{[r]}) = 2$. The conjecture appears more likely to be true when he shows that for a connected prime graph G, and $n \geq 4$, $D(G^{[n]}) = 2$. Moreover, if $|V(G)| \geq 5$, then $D(G^{[n]}) = 2$, for all $n \geq 3$.

Imrich and Klavžar [15] verify Albertson's conjecture [1] by proving that if G is a connected graph (not necessarily prime) such that $|V(G)| \geq 3$, then for all $n \geq 3$, $D(G^{[n]}) = 2$. They then exhaust the work on distinguishing labelings of Cartesian powers by stating that for $G \neq K_2, K_3$, and $n \geq 2$, $D(G^{[n]}) = 2$. In summary, for any connected graph G, and $n \geq 2$,

$$D(G^{[n]}) = \begin{cases} 3 \text{ for } G^{[n]} \in \{K_2^{[2]}, K_2^{[3]}, K_3^{[2]}\}, \\ 2 \text{ otherwise.} \end{cases}$$

Imrich and Klavžar also show that for relatively prime graphs G and H, $D(G \square H) = 2$ when $|V(H)| \leq |V(G)| < 2^{|V(H)|} - |V(H)|$.

Albertson and Boutin [2] present results concerning the distinguishing number of the Kneser graph using determining sets, an approach that had not been previously used. For $n \geq 2k + 1$, the Kneser graph, denoted $K_{n:k}$, has as its vertex set the set of all $\binom{n}{k}$, k-element subsets of $\{1, 2, \ldots, n\}$. For $u, v \in V(K_{n:k})$, $uv \in E(K_{n:k})$ if the subsets corresponding to u and v are disjoint. Figure 2.3 shows $K_{5:2}$ and $K_{6:3}$. Consider the graph G and let $S \subseteq V(G)$. The set S is a *determining set* of G if any automorphism of G can be uniquely described by an automorphism of the subgraph of G induced by the vertices of S. An example of this is depicted in Figure 2.1, where the vertices of K_4 form a determining set of G_4

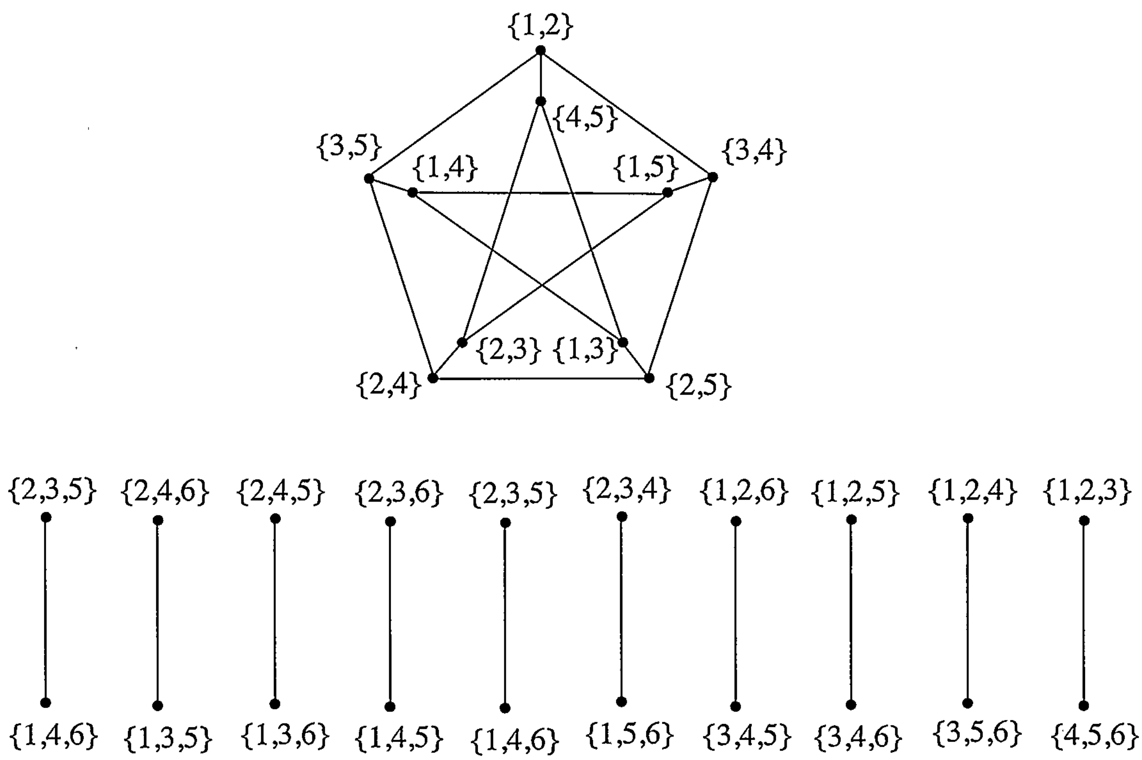

Figure 2.3: $K_{5:2}$ and $K_{6:3}$

Albertson and Boutin [2] prove that for $n \geq 6$ and $k \geq 2$, $D(K_{n:k}) = 2$. To prove this, they find a determining set $S \subseteq V(K_{n:k})$ with the property that the only automorphism of $K_{n:k}[S]$ is the trivial automorphism, i.e., $Aut(K_{n:k}[S]) \cong e$. By assigning the label 1 to all the vertices in S and the label 2 to all other vertices in $V(K_{n:k})$ they obtain a distinguishing labeling of $K_{n:k}$. This is illustrated in the following example.

Consider the graph $K_{10:3}$. To find a distinguishing labeling of $K_{10:3}$, it suffices by a result of [3] to find a distinguishing labeling of its complement, $\overline{K_{10:3}}$. Consider the

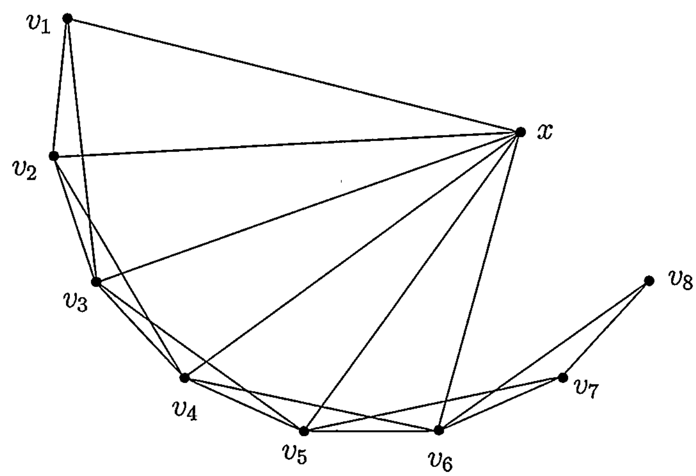

Figure 2.4: The determining set of $\overline{K_{10:3}}$

vertices that correspond to the following subsets of $\{1, 2, \ldots, 10\}$.

$$v_1 = \{1, 2, 3\}$$
$$v_2 = \{2, 3, 4\}$$
$$v_3 = \{3, 4, 5\}$$
$$v_4 = \{4, 5, 6\}$$
$$v_5 = \{5, 6, 7\}$$
$$v_6 = \{6, 7, 8\}$$
$$v_7 = \{7, 8, 9\}$$
$$v_8 = \{8, 9, 10\}$$
$$x = \{2, 4, 6\}$$

Let $V = \{v_1, v_2, \ldots, v_8\}$, $S = V \cup \{x\}$, and let $G = \overline{K_{10:3}}[S]$, the subgraph of $\overline{K_{10:3}}$ induced by the vertices of S (see Figure 2.4). Boutin shows in [7] that S is indeed a determining set of $\overline{K_{10:3}}$. One can verify that $Aut(G) \cong e$ by first noting that both v_8 and x are fixed under any automorphism of G, since their degrees are unique in G. Also,

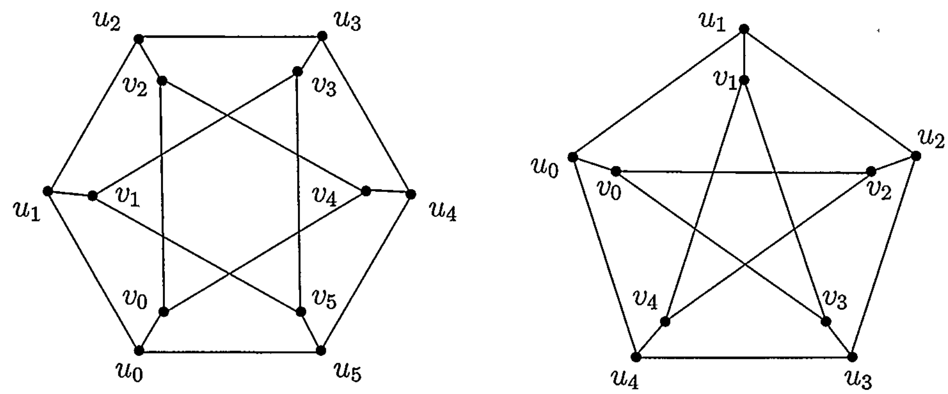

Figure 2.5: $P(6,2)$ and $P(5,2)$

note that the only automorphism of G that preserves adjacency among the vertices in V are those that simultaneously interchange v_1 and v_8, v_2 and v_7, v_3 and v_6, and v_4 and v_5. Since v_8 is fixed, all $v_i \in V(G)$ are fixed, and thus $Aut(G) \cong e$. Thus, any label preserving automorphism of $\overline{K_{10:3}}$ fixes the determining set S. Since all automorphisms of $\overline{K_{10:3}}$ are uniquely described by the automorphisms of S, $V(\overline{K_{10:3}})$ is also fixed, and thus the labeling is distinguishing. It now follows that the labeling is distinguishing for $K_{10:3}$.

Collins and Trenk [12] find exact values of $D(G)$ for certain families of graphs. Their results are summarized in Table 2.1. In addition, they provide a Brooks' Theorem type upper bound on $D(G)$. Brooks' Theorem states that for any graph G, $\chi(G) \leq \Delta(G)+1$ with equality if and only if G is a complete graph or a cycle of odd length. Their analogue to Brooks' Theorem states that for any connected graph G, $D(G) \leq \Delta(G)+1$ with equality if and only if G is $K_{\Delta,\Delta}$, $K_{\Delta+1}$ or C_5. For trees, Collins and Trenk have a stronger result. They prove that for any tree T, $D(T) \leq \Delta(T)$. Furthermore, equality is achieved only if $G = T_{\Delta,h}$, where $T_{\Delta,h}$ is a tree in which any vertex in $V(T_{\Delta,h})$ that is not a leaf has degree Δ, and all leaves are distance h from the root.

Graph G	$D(G)$
K_n	n
Complement of K_n	n
$K_{a_1^{j_1}, a_2^{j_2}, \ldots, a_r^{j_r}}$	min $\{p \mid \binom{p}{a_i} \geq j_i$ for all $i\}$
P_{2n}	2
P_{2n+1}	2
C_4	3
C_5	3
C_6	2
$C_n, n \geq 7$	2
Petersen graph, $P(5,2)$	3
$T_{\Delta,h}, \Delta \geq 2$	Δ
Tree T, $T \neq T_{\Delta,h}, K_1, K_2$	$\leq \Delta - 1$
G connected	$\leq \Delta + 1$

Table 2.1: Collins and Trenk distinguishing number results.

Weigand and Jacobson [21] study the distinguishing number of generalized Petersen graphs. The generalized Petersen graph $P(n,r)$, with $n \geq 3, r \geq 1$, and $2r < n$, has the following structure: $V(P(n,r)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ where $u_0, u_1, \ldots, u_{n-1}$ induce an n-cycle. In addition $u_i v_i \in E(P(n,r))$ and $v_i v_{i+r} \in E(P(n,r))$, where indices are taken modulo n. Figure 2.5 shows $P(6,2)$ and $P(5,2)$. Weigand and Jacobson prove that

$$D(P(n,r)) = \begin{cases} 3 \text{ for } P(n,r) \in \{P(4,1), P(5,2)\}, \\ 2 \text{ otherwise.} \end{cases}$$

2.2 The Distinguishing Chromatic Number

Collins and Trenk [12] define the distinguishing chromatic number, $\chi_D(G)$ of a graph G, as the fewest number of labels required to give G a distinguishing labeling that is also a colouring. Collins and Trenk find exact values of $\chi_D(G)$ for certain families of graphs and upper bounds on $\chi_D(G)$ for arbitrary connected graphs and trees. Their results are

Graph G	$\chi_D(G)$
K_n	n
Complement of K_n	n
$K_{a_1^{j_1},a_2^{j_2},\ldots,a_r^{j_r}}$	$\Sigma j_i a_i$
P_{2n}	2
P_{2n+1}	3
C_4	4
C_5	3
C_6	4
$C_n, n \geq 7$	3
Petersen graph	4
$T_{\Delta,h}, \Delta \geq 2$	$\Delta + 1$
Tree T, $T \neq T_{\Delta,h}, K_1, K_2$	$\leq \Delta$
G connected	$\leq 2\Delta$

Table 2.2: Collins and Trenk distinguishing chromatic number results.

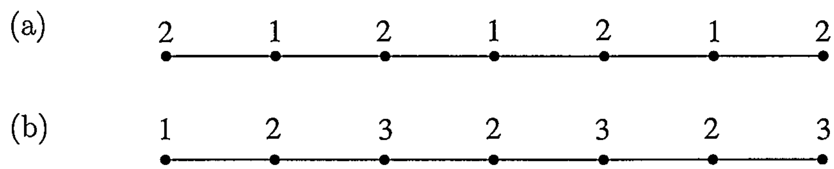

Figure 2.6: A colouring of P_7 with two colours and a distinguishing 3-colouring of P_7.

summarized in Table 2.2, and we will elaborate on some of their findings.

Consider the statement $\chi_D(P_{2n+1}) = 3$. This result follows from the fact that the only non-trivial automorphism of P_{2n+1} is a reflection in the line through the middle vertex of P_{2n+1}. If $P_{2n+1} = v_0 v_1 \cdots v_{2n}$, then the only nontrivial automorphism of P_{2n+1} maps $v_i \rightarrow v_{2n-i}$, $0 \leq i \leq n-1$. In 2-colouring P_{2n+1} we are forced to colour the vertices in an alternating sequence which in turn is not distinguishing (see Figure 2.6 (a)). However, using three colours allows us to colour vertex v_0 with one colour, and alternately colour the remaining $2n$ vertices with the two unused colours (see Figure 2.6 (b)). Since the only colour preserving automorphism that map v_0 to v_{2n} is the trivial automorphism, this colouring is distinguishing.

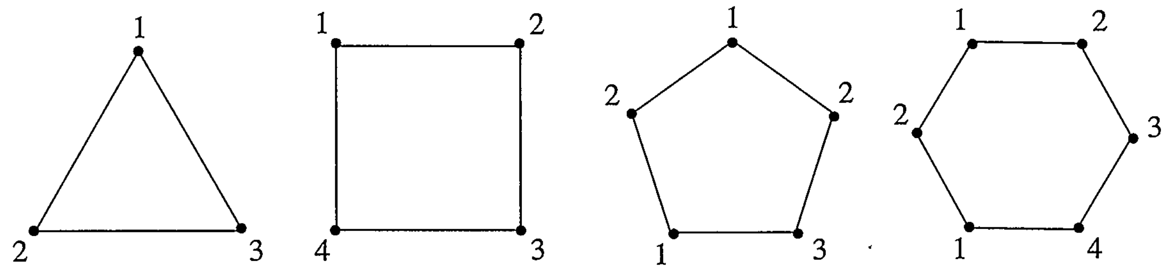

Figure 2.7: Distinguishing colourings of C_3, C_4, C_5, C_6.

In addition to providing distinguishing colouring of C_3, C_4, C_5, and C_6 (Figure 2.7), we also demonstrate a distinguishing colouring for C_n, $n \geq 7$. First note that there is no distinguishing 2-colouring of C_n for any n. As with paths, a 2-colouring of a cycle is forced to have colours alternating. If the colours of a cycle $C = v_0v_1 \cdots v_{n-1}v_0$ alternate so that vertices with even index receive one colour and vertices of odd index receive a different colour, then cycles of odd length are not coloured, since v_0 and v_{n-1} are assigned the same colour. A cycle of even length, $C_{2n} = v_0v_1 \cdots v_{2n-1}v_0$, will have colours preserved under reflection in the line that passes through the vertices v_0 and v_n, and thus is not distinguishing.

We refer the reader to Figure 2.8, for examples of distinguishing 3-colouring of cycles of length at least seven. To see that these colourings are indeed distinguishing we first consider the case of cycles of odd length.

Let $C_{2n+1} = v_0v_1 \cdots v_{2n}v_0$, and define $c : V(C_{2n+1}) \to \{1, 2, 3\}$ as follows. Set $c(v_0) = 1$, and then alternate colours 2 and 3 the rest of the way around the cycle. Any nontrivial automorphism of C_{2n+1} that preserves colours must fix v_0. Since the cycle has an odd number of vertices, the only such automorphism must reflect C_{2n+1} through the line passing through v_0 and edge $v_{n-1}v_n$. But $c(v_n) \neq c(v_{n-1})$, so there is no nontrivial automorphism of C_{2n+1} that preserves the colouring c.

For $C_{2n} = v_0v_1 \cdots v_{2n-1}v_0$, define $c : V(C_{2n}) \to \{1, 2, 3\}$ as follows. Set $c(v_0) =$

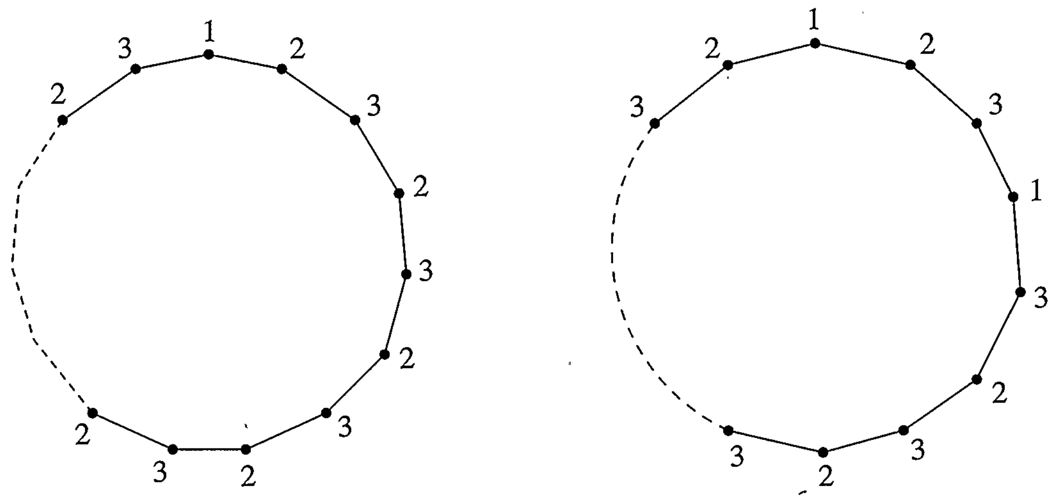

Figure 2.8: A distinguishing 3-colouring of C_n where n is odd, and n is even.

$c(v_3) = 1$, $c(v_i) = 2$ for odd values of $i \neq 3$, and $c(v_i) = 3$ for even values of $i \neq 0$. There are exactly two vertices with colour 1, namely v_0 and v_3, so any nontrivial automorphism of C_{2n}, say g, must either fix v_0 and v_3 or interchange v_0 and v_3. However, only the trivial automorphism fixes v_0 and v_2. Therefore, g must interchange v_0 and v_3. This would mean that $g(v_1) = v_2$ and $g(v_2) = v_1$, but $c(v_1) \neq c(v_2)$, so there is no nontrivial automorphism of C_{2n} that preserves the colouring c.

Given an arbitrary connected graph G, Collins and Trenk [12] provide a second analogue to Brooks' Theorem, this time for the distinguishing chromatic number. This analogue to Brooks' Theorem states that for any connected graph G, $\chi_D(G) \leq 2\Delta(G)$ with equality if and only if G is $K_{\Delta,\Delta}$ or C_6. For trees, Collins and Trenk have a stronger result. They prove that for any tree T, $\chi_D(T) \leq \Delta(T) + 1$. Furthermore, equality is only achieved if $G = T_{\Delta,h}$.

Let T be a tree and let $v \in V(T)$ be a center of T. Recall that T has either one center or two adjacent centers. We first consider the case where v is the unique center of T, and designate v as the root of T. Colour v with colour 1, and colour the children of v differently from each other using the colours from the set $\{2, 3, \ldots, \Delta + 1\}$. At this point, colour 1 is retired and not used again. Now inductively colour the remaining vertices of T. Consider

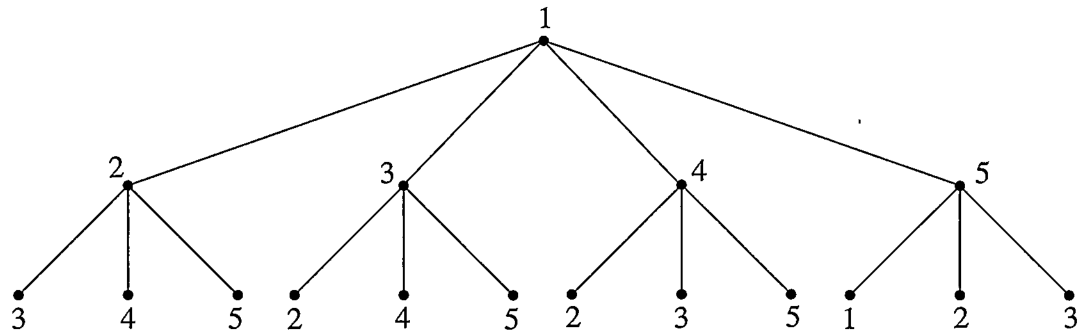

Figure 2.9: A distinguishing 5-colouring of a $T_{4,3}$

an internal vertex $u \in V(T)$ with arbitrary colour $i \in \{2, 3, \ldots, \Delta + 1\}$. Let the children of u be the set of vertices S, and note that $|S| \leq \Delta(T) - 1$. By colouring the vertices in S differently from each other, using the set of colours $\{2, 3, \ldots \Delta + 1\} \setminus \{i\}$, we obtain a distinguishing colouring of T (see Figure 2.9). To see that the colouring is distinguishing we first note that any automorphisms of T (not necessarily colour preserving) must fix v, in order to preserve the distance from v to any other vertex of the graph. Since the children of any vertex are coloured differently from each other, the only automorphism of T that can preserve colours is the one that permutes parent vertices, not children. However, the only vertex in T that is not a child of another vertex is v, the root, which we have established as fixed. This implies that the entire graph is fixed under colour preserving automorphisms, so the colouring is distinguishing.

If T has two adjacent centers, u and v, begin by colouring u with colour 1 and v with colour 2. Then colour the vertices in $N(u)\backslash\{v\}$ and $N(v)\backslash\{u\}$ using the same inductive construction from the previous case. Then both $N(u)\backslash\{v\}$ and $N(v)\backslash\{u\}$ have a distinguishing colouring, and u and v can not be interchanged (they receive different colours). Therefore, this produces a distinguishing colouring of T.

Weigand and Jacobson [21] study the distinguishing chromatic number of generalized Petersen graphs. Weigand and Jacobson prove that

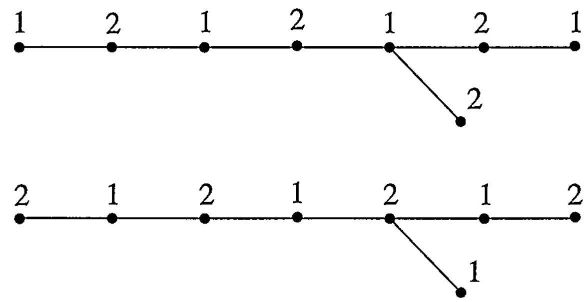

Figure 2.10: The two 2-colourings of G

$$\chi_D(P(n,r)) = \begin{cases} 4 \text{ for } P(n,r) \in \{P(4,1), P(5,2)\}, \\ 3 \text{ otherwise.} \end{cases}$$

Choi, Hartke, and Kaul [10] consider Cartesian products and show that for any sufficiently large value of r, the distinguishing chromatic number of the Cartesian product of r copies of G will be at most the $\chi(G)+1$. More formally, given a connected graph G with at least two vertices, there exists an integer R such that for all $r \geq R$,

$$\chi(G) \leq \chi_D(G^{[r]}) \leq \chi(G)+1.$$

Choi, Hartke, and Kaul also consider the graph obtained by Cartesian products of complete graphs, $K_{t_1} \Box K_{t_2} \cdots \Box K_{t_r}$. They prove for $r \geq 5$ and $t_i \geq 2$, $i = 1, \ldots, r$,

$$\max\{t_i \mid 1 \leq i \leq r\} \leq \chi_D(K_{t_1} \Box K_{t_2} \cdots \Box K_{t_r}) \leq \max\{t_i \mid 1 \leq i \leq r\} + 1.$$

As a corollary, they obtain the following result for d-dimensional hypercubes: for all $d \geq 5$, $\chi_D(Q_d) = 3$.

Collins, Trenk, and Hovey [11] find connections between the distinguishing chromatic number of a graph and the automorphism group of the graph. They first prove that if G is a graph with $\chi_D(G) = 2$, then $Aut(G)$ is either the trivial group or the group on two elements $\mathbf{Z}_2$. The key to the proof of this theorem lies in the fact that the only non-trivial colour preserving automorphism of a 2-coloured graph is the one that interchanges the two colours. The graph depicted in Figure 2.10 is an example of a graph G with $\chi_D(G) = 2$ and $Aut(G) \cong \mathbf{Z}_2$. They extend their result to a graph with non-isomorphic components $G_1, G_2, \dots G_q$ and p_i isomorphic copies of G_i, $1 \leq i \leq q$. It is shown that for $k = \sum_{i=1}^{q} \binom{p_i}{2} + q$, if $\chi_D(G) = 2$, then

$$Aut(G) = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 = \mathbf{Z}_2^k.$$

Collins, Trenk, and Hovey [11] also show that for any finite group Γ and for any $r \geq 3$, there exists a graph G so that $Aut(G) = \Gamma$ and $\chi_D(G) = r$. They also relate the chromatic number of a graph to the distinguishing chromatic number. For a graph G, a prime p, and an integer n, if $Aut(G) = \mathbf{Z}_{p^n}$ then $\chi_D(G) \leq \chi(G) + 1$. Once more they extend their theorem to a more complicated situation. Let $\{p_1, p_2, \dots, p_k\}$ be a set of distinct primes and $n = p_1^{i_1} p_2^{i_2} \dots p_k^{i_k}$ for $i_j \in \mathbb{N}$. If G is a graph in which $|Aut(G)| = n$, then

$$\chi_D(G) \leq \chi(G) + i_1 + i_2 + \cdots + i_k.$$

Furthermore if $Aut(G)$ is an abelian group and $Aut(G) = \mathbf{Z}_{p_1}^{n_1} \times \cdots \times \mathbf{Z}_{p_k}^{n_k}$ for some k where $p_1, \dots, p_k$ are primes, (not necessarily distinct) then

$$\chi_D(G) \leq \chi(G) + k.$$

This inequality is the best possible since there exists a graph G for which $Aut(G) = \mathbf{Z}_{p_1}^{n_1} \times \cdots \times \mathbf{Z}_{p_k}^{n_k}$ and $\chi_D(G) = \chi(G) + k$.

Laflamme and Seyffarth [17] improve, for bipartite graphs, the result of Collins and Trenk [12], that $\chi_D(G) \leq 2\Delta$ with equality if and only if $G \in \{K_{\Delta,\Delta}, C_6\}$. They prove that if G is a connected bipartite graph with $\Delta(G) \geq 3$, then $\chi_D(G) \leq 2\Delta - 2$ unless $G \in \{K_{\Delta,\Delta-1}, K_{\Delta,\Delta}\}$. This result is best possible in the sense that there exists an infinite family of bipartite graphs with maximum degree $\Delta \geq 3$ and distinguishing chromatic number $2\Delta - 2$ for any $\Delta \geq 3$. The result of Laflamme and Seyffarth also disproves

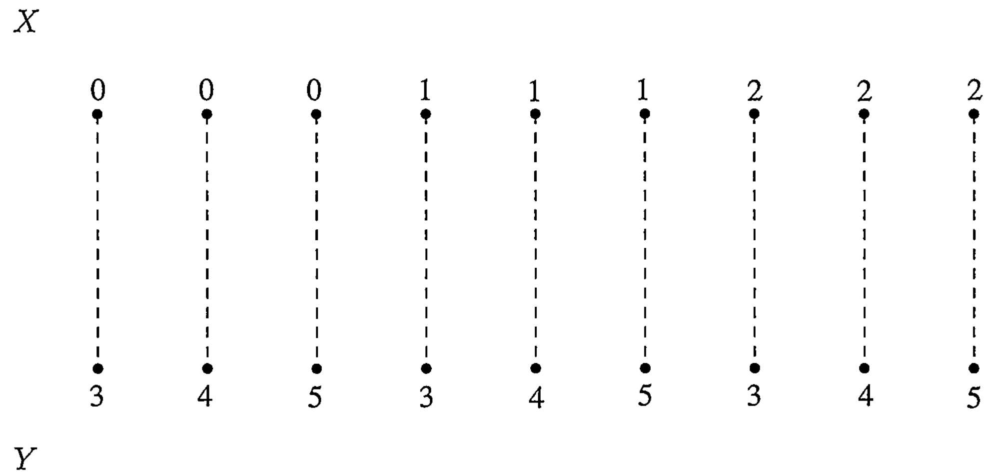

Figure 2.11: A distinguishing colouring of $K_{9,9} - M$

a conjecture of Collins and Trenk [12] that there is no connected graph G such that $\chi_D(G) = 2\Delta - 1$.

Consider the complete bipartite graph $K_{n,n}$, where $V(K_{n,n}) = X \cup Y$, with $X = \{x_0, x_1, \ldots, x_{n-1}\}$, and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. We define the *complete bipartite graph minus the perfect matching*, $K_{n,n} - M$, as a graph for which $V(K_{n,n} - M) = X \cup Y$ and $E(K_{n,n} - M) = E(K_{n,n}) - \{x_i y_i \mid 0 \leq i \leq n-1\}$ where $0 \leq i \leq n-1$. We display $(K_{9,9} - M)$ with a distinguishing colouring in Figure 2.11, where the dotted lines represent edges of the perfect matching that are missing from the complete bipartite graph, and it is understood that the remaining edge between X and Y all exist.

Laflamme and Seyffarth [17] show that $\chi_D(K_{n,n} - M) = \lceil 2\sqrt{n} \rceil$, a somewhat surprising result when compared to the distinguishing chromatic number of the complete bipartite graph, a graph with only n additional edges but with the significantly larger distinguishing chromatic number, namely $2n$.

To see that $\chi_D(K_{n,n} - M) \leq \lceil 2\sqrt{n} \rceil$, we demonstrate a distinguishing colouring of $(K_{n,n} - M)$ with $\lceil 2\sqrt{n} \rceil$ colours when n is a perfect square (the argument when n is not a perfect square is similar, but requires requires a more complex analysis). Let $n = q^2$,

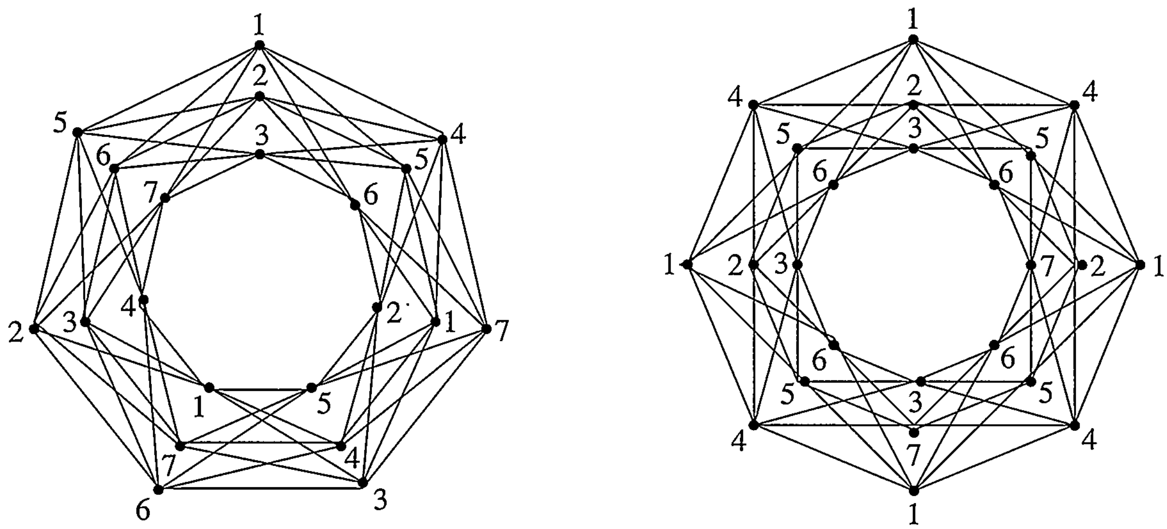

Figure 2.12: Distinguishing colouring of $C_7[I_3]$, and $C_8[I_3]$.

$q \in \mathbb{N}$, and consider $K_{n,n} - M$, where $V(K_{n,n} - M) = (X, Y)$ for $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. Consider the following colouring of $K_{n,n} - M$ with $2\sqrt{n} = 2q$ colours. Colour vertices $x_{i \cdot q+0}, x_{i \cdot q+1} \cdots x_{i \cdot q+q-1}$ with colour i, for $0 \leq i \leq q-1$. For each j, $0 \leq j \leq q-1$, colour vertices in the set $\{y_{j+i \cdot q} \mid 0 \leq i \leq q-1\}$, with colour $q+j$. This colouring is shown in Figure 2.11 in the case $n = 3^2$. Any two adjacent vertices in $K_{n,n} - M$ have different labels, so this is indeed a colouring, not just a labeling. The colouring has been defined so that if $c(x_i) = c(x_j)$, $0 \leq i \neq j \leq n-1$, then $c(y_i) \neq c(y_j)$. Since x_i is adjacent to all vertices in $Y - \{y_i\}$, there is no colour preserving automorphism of G that maps x_i to x_j. Analogously, there is no colour preserving automorphism of G that maps y_i to y_j, and hence $\chi_D(K_{n,n} - M) \leq \lceil 2\sqrt{n} \rceil$.

Tang [20] determines the distinguishing chromatic number of the wreath product of a cycle of length n with an independent set of size m. The wreath product of a graph H with a graph G, denoted $H[G]$, has $V(H[G]) = V(H) \times V(G)$, with vertices (u_1, v_1) and (u_2, v_2) adjacent if $u_1 = u_2$ and $v_1v_2 \in E(G)$, or if $u_1u_2 \in E(H)$ (see Figure 2.12 for examples of $C_7[I_3]$ and $C_8[I_3]$). This can be thought of as replacing each vertex in H with a copy of G and each edge in H with a complete bipartite graph between copies of

G that correspond to adjacent vertices in H.

Tang shows that for all $n \geq 3$ and $m \geq 1$,

$$\chi_D(C_n[I_m]) = \begin{cases} 3m & n = 3, \\ 2m+2 & n = 4, \\ \lceil m/\binom{n-1}{2} \rceil + 2m & n \geq 5 \text{ and odd}, \\ 2m+1 & n \geq 6 \text{ and even}. \end{cases}$$

The examples in Figure 2.12 show distinguishing colours of $C_7[I_3]$ and $C_8[I_3]$, each with seven colours, showing that the bound in Tang's result can be achieved.

2.3 Complexity Results

When Albertson and Collins [3] formalize the distinguishing number of a graph, they also discuss the complexity of determining $D(G)$. Even the complexity of determining whether or not $D(G) = 1$ for a given graph G has not been resolved. Deciding if $D(G) = 1$ is equivalent to deciding if $Aut(G) \cong e$, and is a candidate for a problem that lies somewhere between being in P and being NP-complete [16]. As a result, determining if $D(G) = 1$ may be a difficult problem.

Russell and Sundaram [19] show that determining if $D(G) > k$ is an AM class problem, the set of languages for which there are Arthur-Merlin games. They are then able to show that the problem of determining if $D(G) > k$ is not in co-NP.

Cheng [8] focuses her attention on the complexity of finding the distinguishing number of trees and forests. Using a recursive formula, she finds the number of "inequivalent" distinguishing k-labelings of a tree or forest on n vertices. This algorithm runs in $O(n \log n)$ time. The notion of "inequivalent" distinguishing k-colourings will be described in detail in later chapters, where we use the identical notion that we define independently of Cheng [9][8].

Arvind, Cheng, and Devanur [4]. show that the distinguishing number of a planar graph can be found in polynomial time. Given a planar graph G with n vertices all with degree three, $D(G)$ can be found in $O(n^3 \log^3 n)$ time. Also, for any planar graph G on n vertices (with no restriction on degree), $D(G)$ can be found in $O(n^5 \log^3 n)$ time.

Cheng [9] extends her work to determine the complexity of computing the distinguishing chromatic numbers of arbitrary graphs, rooted trees, and planar graphs. She also generalizes her work in [8] to find the complexity of determining the distinguishing number of an arbitrary interval graph, and the complexity of determining the distinguishing chromatic number of an interval graph (an *interval graph* is a graph whose vertices can be represented by intervals in the real line such that two vertices are adjacent if and only if their corresponding intervals have a non-empty intersection).

Cheng [9] show that for $k \geq 3$, determining if a graph G has a distinguishing k-colouring is NP-hard. Also, if G is planar and has $\Delta(G) \leq 5$, then to determine if G has a distinguishing colouring with k colours is NP-hard. If T is a rooted tree, then finding $\chi_D(T)$ can be done in $O(n^3 \log^3 \Delta(T))$ time. Finally Cheng shows that the distinguishing number and the distinguishing chromatic number an interval graph can be determined in $O(n^3 \log^3 n)$ time. The final result comes from an algorithm that counts the number of different labelings (colourings) using k labels (colours). This algorithm can be run to determine the smallest value of k that produces at least one labeling (colouring) using k labels (colours).

Chapter 3

Distinguishing Colourings of the Wreath Product

3.1 Preliminaries

In this chapter, we examine the distinguishing chromatic number of the wreath product of graphs, extending earlier work of Tang [20]. Recall that the *wreath product* of graph H with G, $H[G]$, has vertex set $V(H) \times V(G)$; two vertices (u, v) and (u', v') are adjacent in $H[G]$ if and only if either $uu' \in E(H)$, or $u = u'$ and $vv' \in E(G)$. In essence, $u \in V(H)$ corresponds to a copy G_u of G, and an edge $uv \in E(H)$ corresponds to a complete bipartite graph between $V(G_u)$ and $V(G_v)$.

Let G be a graph and c a vertex colouring of G. We write (G, c) for the graph G along with the colouring c, and say that (G, c) is a colouring of G. If F is a subgraph of G, then (F, c) is the subgraph of F along with the colouring c restricted to F.

Lemma 3.1.1. *If c is a distinguishing colouring of $H[G]$, and $u \in V(H)$, then (G_u, c) is a distinguishing colouring of G_u.*

Proof. This proof proceeds by contradiction. Let $u \in V(H)$ and let $g \in Aut(G)$ be a non-trivial colour preserving automorphism of G, i.e., $(gG_u, c) \cong (G_u, c)$. Define $g^* \in Aut(H[G])$ as the automorphism of $H[G]$ that maintains the automorphism g on $V(G_u)$, and acts as the identity on all other vertices of $H[G]$. Then $(g^*H[G], c)) \cong (H[G], c))$, so g^* is a nontrivial automorphism of $H[G]$ that preserves colours, contradicting the fact that c is a distinguishing colouring of $H[G]$. □

This lemma is used to obtain the following lower bound on the distinguishing chromatic number of $H[G]$.

Lemma 3.1.2. *For any graph H with at least one edge and for any graph G,*

$$\chi_D(H[G]) \geq 2\chi_D(G).$$

Proof. Take a distinguishing colouring of the graph $H[G]$ and let $uv \in E(H)$. Since uv corresponds to a complete bipartite graph between the copies G_u and G_v of G in $H[G]$, G_u and G_v must be coloured by disjoint sets of colours. By Lemma 3.1.1, each of G_u and G_v are coloured with at least $\chi_D(G)$ colours, and thus $\chi_D(H[G]) \geq 2\chi_D(G)$. □

Remark 3.1.1. From the proof of Lemma 3.1.2, we see that in a distinguishing colouring of $H[G]$, if $uv \in E(H)$, the colour sets of G_u and G_v must be disjoint.

Tang [20] evaluates $\chi_D(C_n[I_m])$, where C_n is a cycle of length n and I_m is an independent set of m vertices. This is an attractive problem due to the symmetry in the graph. Tang's main result determines the exact value of $\chi_D(C_n[I_m])$.

Theorem 3.1.1. *For all $n \geq 3$ and $m \geq 1$,*

$$\chi_D(C_n[I_m]) = \begin{cases} 3m & n = 3, \\ 2m+2 & n = 4, \\ \lceil m/\binom{n-1}{2} \rceil + 2m & n \geq 5 \text{ and odd}, \\ 2m+1 & n \geq 6 \text{ and even}. \end{cases}$$

We extend Tang's work to the more general case of determining the exact value of $\chi_D(C_n[G])$ for an arbitrary graph G. However, to do so we must first establish the notion of *distinct distinguishing colourings* of a graph G.

3.2 Distinct Distinguishing Colourings

In this section we introduce the notion of distinct distinguishing colourings of a graph. Despite this topic's thorough examination in Chapter 4, we require a basic understanding of this concept in order to present our results about distinguishing colourings of wreath products of graphs.

One can show that a graph has a distinguishing colouring using k colours simply by providing such a colouring. However, this colouring may not be the only k-colouring of the graph that is distinguishing. There may be many different colourings that produce distinguishing colourings, and it is important for us to carefully formulate a definition of what it means for two distinguishing colourings of be different. This concept motivates the following definition and example.

Definition 3.2.1. Two colourings of a graph G, c_1 and c_2, are *similar colourings* if for some $g \in Aut(G), c_1(u) = c_2(g(u))$ for each $u \in V(G)$. Two colourings that are not similar are *distinct colourings.* We denote the number of distinct distinguishing r-colourings of G as $\xi_r(G)$.

Remark 3.2.1. Let c_1 and c_2 be colourings of a graph G, and suppose that there exists a colour f and a vertex $u \in V(G)$ so that $c_1(u) = f$ and $c_2(x) \neq f$ for any $x \in V(G)$. Then c_1 and c_2 are distinct colourings of G.

In [9], Cheng defines the notion of inequivalent colourings. Her definition is analogous to our definition of distinct. Cheng also develops a recursive algorithm to find the number of inequivalent distinguishing colourings of an interval graph. We prove a result that is used to determine the number of distinct distinguishing colourings of a tree. Since trees are a subset of the family of interval graphs, this result is the same as Cheng's, but was proved independently. We also prove results that enable us to determine the number of distinct distinguishing r-colourings of a cycle.

Example 3.2.1. Consider the three colourings of the graph G in Figure 3.1, c_1, c_2, and c_3. First notice that c_1 and c_2 are similar. By calling upon the automorphism that corresponding to the reflection of G in the line passing through v_0 and v_3, we see that

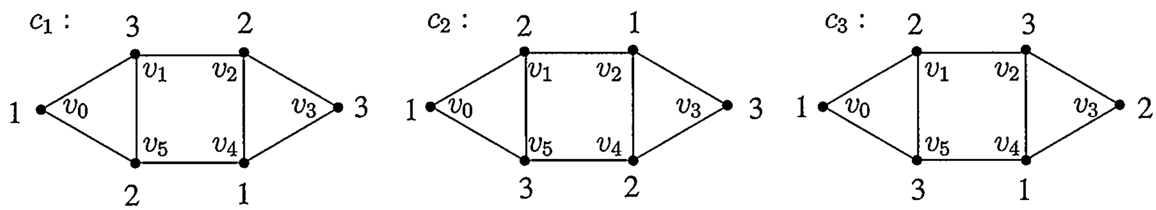

Figure 3.1: Three colourings of G

c_1 and c_2 are indeed similar. On the other hand, c_2 and c_3 are distinct colourings of G. There can not exist any $g \in Aut(G)$ so that for each $u \in V(G)$, $c_2(u) = c_3(g(u))$. This is clear since c_3 assigns colour 2 to a vertex of degree two in G, and c_2 does not. For the same reasoning, c_1 and c_3 are also distinct colourings of G. Since the colourings c_1, c_2, and c_3 are distinguishing, we may conclude that $\xi_3(G) \geq 2$.

The importance of this concept is in the fact that, despite using the same colour set, two distinctly coloured copies of G can not be mapped to each other under any colour preserving automorphism. This is a useful tool when working with the wreath product of two graphs, say $H[G]$, since the vertices of H corresponds to copies of G.

3.3 Wreath Products with Cycles

As was mentioned earlier, we extend the work of Tang [20], who studied the wreath product of a cycle with an independent set of vertices. In this section we find the distinguishing chromatic number of the wreath product of a cycle with an arbitrary connected graph. Before doing so, we first establish an important lemma.

Lemma 3.3.1. *Let G be a connected graph and let $C_n = v_0v_1 \cdots v_{n-1}v_0$ where $n \geq 4$. If $g \in Aut(C_n[G])$, then for any i, $0 \leq i \leq n-1$, g maps G_{v_i} to G_{v_j} for some j, $0 \leq j \leq n-1$.*

Proof. Suppose to the contrary, that for some $u \in V(C_n)$, and $x, y \in V(G_u)$ with $xy \in E(G_u)$, that there is a $g \in Aut(C_n[G])$ such that $g(x) \in V(G_a)$ and $g(y) \in V(G_b)$ for some $a, b \in V(C_n), a \neq b$. Without loss of generality, we may assume that $u = v_0$ (so that $x, y \in V(G_{v_0})$). Since $g(x) \in V(G_a)$, and the neighbours of vertices in G_a are in G_{a-1}, G_a, or G_{a+1} (all subscripts are taken modulo n), it follows that $b \in \{a-1, a, a+1\}$. Since we have assumed $b \neq a$, $b = a-1$ or $b = a+1$, and we may assume without loss of generality that $b = a+1$ and thus $g(y) \in V(G_{a+1})$. Now, all vertices in $V(G_{v_1})$ and $V(G_{v_{n-1}})$ are adjacent to both x and y, and thus, under g, must be mapped to $V(G_a)$ and $V(G_{a+1})$. However, this is impossible since the $2|V(G)|+2$ vertices of $V(G_{v_1}) \cup V(G_{v_{n-1}}) \cup \{x, y\}$ can not be mapped injectively to the $2|V(G)|$ vertices of $V(G_a) \cup V(G_{a+1})$. Thus, any pair of adjacent vertices in G_u must be mapped to the same copy of G in $C_n[G]$ under g. Since G is connected, it follows that all vertices of G_u must be mapped to the same copy of G in $C_n[G]$ under g. □

Lemma 3.3.1 implies that any automorphism of $C_n[G]$ corresponds to an automorphism of C_n. The next result deals with the situation when n is even.

Theorem 3.3.1. *For all n even, $n \geq 4$, and any connected graph G,*

$$\chi_D(C_n[G]) = \begin{cases} 2\chi_D(G) & \text{if } \xi_{\chi_D(G)}(G) \geq 2, \\ 2\chi_D(G)+1 & \text{if } \xi_{\chi_D(G)}(G) = 1 \text{ and } n \geq 6, \\ 2\chi_D(G)+2 & \text{if } \xi_{\chi_D(G)}(G) = 1 \text{ and } n = 4. \end{cases}$$

Proof. Let n be even, $n \geq 4$, and suppose $C_n = v_0 v_1 \cdots v_{n-1} v_0$. Let S_0 and S_1 be disjoint sets (of colours), each containing $\chi_D(G)$ elements. First assume $\xi_{\chi_D(G)}(G) \geq 2$. Then there exist distinct $\chi_D(G)$-colourings c_0 and c_1 of G that use colour set S_0. There also exist distinct $\chi_D(G)$-colourings d_0 and d_1 of G using colour set S_1. We obtain a distinguishing colouring c of $C_n(G)$ by defining the colouring on the copies of G in $C_n[G]$

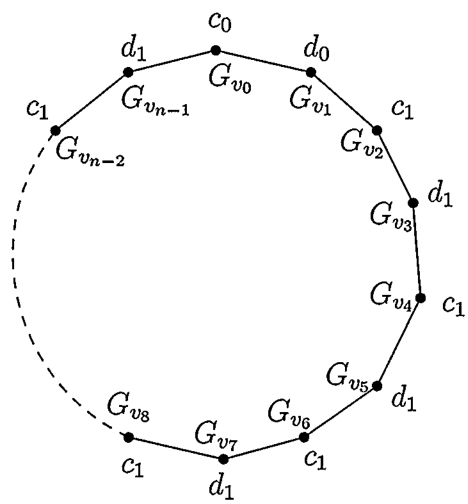

Figure 3.2: A distinguishing colouring of $C_n[G]$, for an even $n \geq 6$, where G is connected and $\xi_{\chi_D(G)}(G) \geq 2$.

as follows. Set

$$(G_{v_0}, c) = (G, c_0),$$

$$(G_{v_1}, c) = (G, d_0),$$

$$(G_{v_i}, c) = (G, c_1), \text{ for even } i,\ 2 \leq i \leq n-2,$$

$$(G_{v_i}, c) = (G, d_1), \text{ for odd } i,\ 3 \leq i \leq n-1.$$

Since the colour set used to colour $G_{v_1}, G_{v_3}, \ldots, G_{v_{n-1}}$ is disjoint from the set used to colour $G_{v_0}, G_{v_2} \ldots G_{v_{n-2}}$, the resulting labeling is a colouring. Furthermore, the colouring c is distinguishing because only G_{v_0} and G_{v_1} have the colourings c_0 and d_0 respectively, so any colour preserving $g \in Aut(C_n[G])$ must fix both G_{v_0} and G_{v_1}. However the only $g \in Aut(C_n[G])$ that permutes the copies of G and fixes G_{v_0} and G_{v_1} is one that fixes all copies of G, i.e., the identity. This proves that $\chi_D(C_n[G], c) \leq 2\chi_D(G)$. By Lemma 3.1.2, it follows that $\chi_D(C_n[G], c) = 2\chi_D(G)$.

For the second part of the theorem, suppose that $\xi_{\chi_D(G)}(G) = 1$ and $n \geq 6$. To obtain a distinguishing colouring of $C_n[G]$ using $2\chi_D(G) + 1$ colours, we use four colourings of

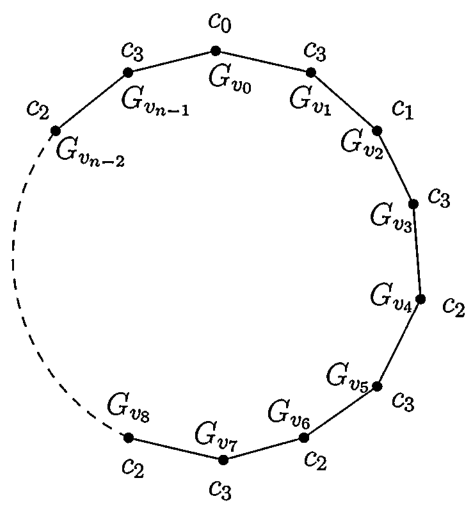

Figure 3.3: A distinguishing colouring of $\chi_D(C_n[G])$, for an even $n \geq 6$, where G is connected and $\xi_{\chi_D(G)}(G) = 1$.

G, defined on four different colour sets. Let

$$S_0 = \{0, 1, \ldots, \chi_D(G) - 1\},$$

$$S_1 = \{1, 2, \ldots, \chi_D(G)\},$$

$$S_2 = \{0, 1, \ldots, \chi_D(G) - 2, \chi_D(G)\},$$

$$S_3 = \{\chi_D(G) + 1, \chi_D(G) + 2, \ldots, 2\chi_D(G) - 1, 2\chi_D(G)\}.$$

Consider the colourings of G, $(G, c_0), (G, c_1), (G, c_2), (G, c_3)$ with colour sets S_0, S_1, S_2, S_3, respectively. We define a distinguishing colouring c of $C_n[G]$, as follows.

$$(G_{v_0}, c) = (G, c_0),$$

$$(G_{v_1}, c) = (G, c_3),$$

$$(G_{v_2}, c) = (G, c_1),$$

$$(G_{v_i}, c) = (G, c_3) \text{ for } i \text{ odd, } 3 \leq i \leq n-1,$$

$$(G_{v_i}, c) = (G, c_2) \text{ for } i \text{ even, } 4 \leq i \leq n-2.$$

Since G_{v_0} and G_{v_2} are the only copies of G to have the colourings c_0 and c_1 respectively, any colour preserving $g \in Aut(C_n[G])$ that permutes the copies of G and preserves colours must fix both G_{v_0} and G_{v_2}. However the only $g \in Aut(C_n(G))$ that fixes G_{v_0} and G_{v_2} is the one that fixes all copies of G, i.e., the identity. Thus c is a distinguishing colouring of $C_n[G]$ with $2\chi_D(G)+1$ colours. By Lemma 3.1.2, $\chi_D(C_n[G]) = 2\chi_D(G)$ or $2\chi_D(G)+1$.

To prove that $2\chi_D(G)$ colours are not sufficient for a distinguishing colouring of $C_n[G]$, suppose $C_n[G]$ has a distinguishing colouring with $2\chi_D(G)$ colours. By Remark 3.1.1, we may assume that G_{v_0} is coloured by a set S_0 of $\chi_D(G)$ colours, that G_{v_1} is coloured with a set S_1 of $\chi_D(G)$ colours, and that $S_0 \cap S_1 = \emptyset$. For $i \in \{1,2\}$, let c_i be a distinguishing colourings of G_{v_i}, with colour set S_i. Since $\xi_{\chi_D}(G) = 1$, c_0 and c_1 are uniquely determined. It follows from Remark 3.1.1 that there is a unique colouring, c, of $C_n[G]$ with the $2\chi_D(G)$ colours of $S_0 \cup S_1$, and that c satisfies

$$(G_{v_i}, c) = (G, c_0) \text{ for } i \text{ even,}$$

$$(G_{v_i}, c) = (G, c_1) \text{ for } i \text{ odd.}$$

However this colouring is not distinguishing. Consider the automorphism $g \in Aut(C_n[G])$ defined by

$$g(G_i) = g(G_{i+2}),$$

where the subscripts are taken modulo n. Then g is a non-trivial colour preserving automorphism of G, and thus $\chi_D(C_n[G]) \geq 2\chi_D(G)+1$. Therefore, $\chi_D(C_n[G]) = 2\chi_D(G)+1$.

For the final part of the theorem, suppose $\xi_{\chi_D(G)}(G) = 1$ and, $n = 4$. Define four colour sets as follows.

$$\begin{aligned}
S_0 =& \{0, 1, \ldots, \chi_D(G)-1\},\\
S_1 =& \{\chi_D(G), \chi_D(G)+1, \ldots, 2\chi_D(G)-1\},\\
S_2 =& \{0, 1, \ldots, \chi_D(G)-2, 2\chi_D(G)\},\\
S_3 =& \{\chi_D(G), \chi_D(G)+1, \ldots, 2\chi_D(G)-2, 2\chi_D(G)+1\}.
\end{aligned}$$

Since $|S_i| = \chi_D(G)$ for each i, $0 \le i \le 3$, G has a distinguishing colouring, c_i, using the colours of S_i. Furthermore, $(G, c_0), (G, c_1), (G, c_2), (G, c_3)$ are distinct because $S_i \neq S_j$ for all i, j where $i \neq j$ and $i, j \in \{0, 1, 2, 3\}$.

Let $C_4 = v_0v_1v_2v_3v_0$. Define a colouring of $C_4[G]$ by colouring the copies of G in $C_4[G]$ as follows.

$$(G_{v_0}, c) = (G, c_0),$$
$$(G_{v_1}, c) = (G, c_1),$$
$$(G_{v_2}, c) = (G, c_2),$$
$$(G_{v_3}, c) = (G, c_3).$$

Since the four colourings on the copies of G are distinct, there is no colour preserving $g \in Aut(C_4[G])$ that permutes copies of G. As a result c is a distinguishing colouring of $C_4[G]$. This shows that $\chi_D(C_4[G]) \le 2\chi_D(G) + 2$. By Lemma 3.1.2, $\chi_D(C_4[G])$ is equal to $2\chi_D(G)$, $2\chi_D(G) + 1$, or $2\chi_D(G) + 2$. Now it remains to show that $C_4[G]$ does not have a distinguishing colouring with $2\chi_D(G) + 1$ colours.

Consider $C_4[G]$ where $C_4 = v_0v_1v_2v_3v_0$, and let c be a distinguishing colouring of $C_4[G]$. Then (G_{v_0}, c) must be distinct from (G_{v_2}, c), and (G_{v_1}, c) must be distinct from (G_{v_3}, c). If not, then there is a colour preserving automorphism of $C_4[G]$ that interchanges G_{v_1} and G_{v_3}, and fixes G_{v_0} and G_{v_2}, or there is a colour preserving automorphism that interchanges G_{v_0} and G_{v_2}, and fixes G_{v_1} and G_{v_3}, respectively.

Suppose $(C_4[G], c)$ has a distinguishing colouring with the $2\chi_D(G) + 1$ colours, say $\{0, 1, \ldots, 2\chi_D(G)\}$. Let S_0, S_1, S_2, S_3 denote the subsets of colours used on $G_{v_0}, G_{v_1}, G_{v_2}, G_{v_3}$, respectively. Then for each i, $0 \le i \le 3$, (subscripts taken modulo 4)

$$|S_i| \ge \chi_D(G), \tag{3.1}$$

$$S_i \cap S_{i+1} = \emptyset, \tag{3.2}$$

$$S_i \neq S_{i+2}. \tag{3.3}$$

We may conclude from this that $|S_i| = \chi_D(G)$ for all i, for if $|S_i| = \chi_D(G) + 1$ then there would be only one possible colour set disjoint from S_i. This would then imply that $S_{i+1} = S_{i-1}$, which violates 3.3. Without loss of generality, we may assume that

$$S_0 = \{0, 1, \ldots, \chi_D(G) - 1\},$$
$$\text{and } S_1 = \{\chi_D(G), \chi_D(G) + 1, \ldots, 2\chi_D(G) - 1\}.$$

Since $S_1 \cap S_2 = \emptyset$, $S_2 \subset \{0, 1, \ldots, \chi_D(G) - 1, 2\chi_D(G)\}$. Also since $S_2 \neq S_0$, $2\chi_D(G) \in S_2$. Similarly, $S_3 \cap S_0 = \emptyset$, implies that $S_3 \subset \{\chi_D(G), \chi_D(G)+1, \ldots, 2\chi_D(G)-1, 2\chi_D(G)\}$. But since $S_3 \neq S_1$, $2\chi_D(G) \in S_3$. Thus $2\chi_D(G) \in S_2 \cap S_3$, a contradiction. So $(C_4[G], c)$ does not have a distinguishing colouring with $2\chi_D(G) + 1$ colours, and thus $\chi_D(C_4[G]) = 2\chi_D(G) + 2$.

□

The next theorem extends the result of Theorem 3.3.1 to cycles with odd length. Although the colouring described in the proof of Theorem 3.3.2 appears complicated, Example 3.3.1 helps to illustrate the specifics.

Theorem 3.3.2. *For all odd $n \geq 3$ and any connected graph G,*

$$\chi_D(C_n[G]) = 2\chi_D(G) + \left\lceil \chi_D(G) / \left(\frac{n-1}{2}\right) \right\rceil.$$

Proof. Let $C_n = v_0 v_1 \cdots v_{n-1} v_0$, and let $q = \lceil \chi_D(G) / \left(\frac{n-1}{2}\right) \rceil$. Define the following $2\left\lceil \frac{\chi_D(G)}{q} \right\rceil + 1$ sets of colours, whose entries are taken modulo $(2\chi_D(G) + q)$. Let

$$S_k = \{k\chi_D(G), k\chi_D(G) + 1, \ldots, (k+1)\chi_D(G) - 1\},$$

for $0 \leq k \leq 2\left\lceil \frac{\chi_D(G)}{q} \right\rceil$.

Before proceeding, we should note that S_i is disjoint from S_{i+1}, for $0 \leq i \leq n-1$. This is true since S_i and S_{i+1} constitute $2\chi_D(G)$ consecutive colours of the set of colours with cardinality greater than $2\chi_D(G)$.

A second important fact is that, for $0 \leq i \neq j \leq 2\left\lceil \frac{\chi_D(G)}{q} \right\rceil$, $S_i \neq S_j$. To show this, it suffices to show that $i\chi_D(G) \neq j\chi_D(G)$ modulo $(2\chi_D(G)+q)$. These terms form the sequence,

$$0, \chi_D(G) - 0q, 2\chi_D(G) - 0q, \chi_D(G) - q, 2\chi_D(G) - q, \ldots, \chi_D(G) - rq,$$
$$2\chi_D(G) - rq, \ldots, \chi_D(G) - \left(\left\lceil \frac{\chi_D(G)}{q} \right\rceil - 1\right) q, 2\chi_D(G) - \left(\left\lceil \frac{\chi_D(G)}{q} \right\rceil - 1\right) q,$$

where $0 \leq r \leq \left\lceil \frac{\chi_D(G)}{q} \right\rceil - 1$. Since

$$\left\lceil \frac{\chi_D(G)}{q} \right\rceil < \frac{\chi_D(G)}{q} + 1,$$
$$\left\lceil \frac{\chi_D(G)}{q} \right\rceil - 1 < \frac{\chi_D(G)}{q},$$
$$\left(\left\lceil \frac{\chi_D(G)}{q} \right\rceil - 1\right) q < \chi_D(G).$$

From this, we deduce that for every r, $0 \leq rq < \chi_D(G)$. Therefore,

$$0 < \chi_D(G) - rq \leq \chi_D(G)$$

and

$$\chi_D(G) < 2\chi_D(G) - rq \leq 2\chi_D(G).$$

Since $\chi_D(G) - rq$ and $2\chi_D(G) - rq$ are both strictly decreasing as r increases, we may conclude that $i\chi_D(G) \neq j\chi_D(G)$ modulo $(2\chi_D(G)+q)$.

Finally, we claim that S_0 is disjoint from $S_{2\left\lceil \frac{\chi_D(G)}{q} \right\rceil}$. We have

$$\begin{aligned} S_0 &= \{0, 1, \ldots, \chi_D(G) - 1\}, \\ S_{2\left\lceil \frac{\chi_D(G)}{q} \right\rceil} &= \left\{2\chi_D(G) - q\left\lceil \frac{\chi_D(G)}{q} \right\rceil + q, 2\chi_D(G) - q\left\lceil \frac{\chi_D(G)}{q} \right\rceil + q + 1, \ldots, \right. \\ &\quad \left. 2\chi_D(G) - q\left\lceil \frac{\chi_D(G)}{q} \right\rceil + q + \chi_D(G) - 1\right\}. \end{aligned}$$

Since we have already shown $\chi_D(G) < 2\chi_D(G) - rq$, we know that there is no element in $S_{2\lceil\frac{\chi_D(G)}{q}\rceil}$ so small that it is also in S_0. However we must also show that there is no element in $S_{2\lceil\frac{\chi_D(G)}{q}\rceil}$ so large that it is contained in S_0 when taken modulo $2\chi_D(G) + q$.

Since $\frac{\chi_D(G)}{q} \leq \left\lceil \frac{\chi_D(G)}{q} \right\rceil$, the largest possible values that can be contained in $S_{2\lceil\frac{\chi_D(G)}{q}\rceil}$ are

$$\begin{aligned}&\{2\chi_D(G) - \chi_D(G) + q, 2\chi_D(G) - \chi_D(G) + q + 1, \ldots,\\ &2\chi_D(G) - \chi_D(G) + q + \chi_D(G) - 1\}\\ =&\{\chi_D(G), \chi_D(G) + 1, \ldots, 2\chi_D(G) - 1\},\end{aligned}$$

and this set is also disjoint from S_0 when the values are taken modulo $2\chi_{D(G)}(G) + q$. We may now say that the proposed colouring is indeed a colouring and not just a labeling.

The remainder of this proof will be dedicated to showing that the colouring is distinguishing. Since $|S_k| = \chi_D(G)$ for each k, $0 \leq k \leq 2\left\lceil \frac{\chi_D(G)}{q} \right\rceil$, G has a distinguishing colouring using the colours of S_k. Let (G, c_k) denote a distinguishing colouring of G using colour set S_k. Then, by Remark 3.2.1,

$$(G, c_0), (G, c_1), \ldots, (G, c_{2\lceil\chi_D(G)/q\rceil})$$

are distinct distinguishing colourings of G. We obtain a distinguishing colouring c of $C_n[G]$ by defining the colouring of the copies of G in $C_n[G]$. Set

$$(G_{v_k}, c) = (G, c_k)$$

for $0 \leq k \leq 2\left\lceil \frac{\chi_D(G)}{q} \right\rceil$, and for $\left\lceil \frac{\chi_D(G)}{q} \right\rceil \leq j \leq \frac{n-3}{2}$, let

$$\begin{aligned}(G_{v_{2j+1}}, c) &= (G, c_{2\lceil\chi_D(G)/q\rceil-1})\\ (G_{v_{2j+2}}, c) &= (G, c_{2\lceil\chi_D(G)/q\rceil}).\end{aligned}$$

Since G_{v_0} and G_{v_1} are the only copies of G to have the colourings c_0 and c_1 respectively, any colour preserving $g \in Aut(C_n[G])$ that permutes the copies of G and preserves colour must fix both G_{v_0} and G_{v_1}. However the only $g \in Aut(C_n[G])$ that fixes G_{v_0} and G_{v_1} is the one that fixes all copies of G, i.e., the identity. Thus c is a distinguishing colouring of $C_n[G]$.

To prove that $C_n[G]$ has no distinguishing colourings with fewer than

$$2\chi_D(G) + \left\lceil \chi_D(G) / \binom{n-1}{2} \right\rceil$$

colours, suppose the contrary, that

$$\chi_D(C_n[G]) < 2\chi_D(G) + \left\lceil \chi_D(G) / \left(\frac{n-1}{2}\right) \right\rceil .$$

Since $\chi_D(C_n[G])$ and $\lceil \chi_D(G) / (\frac{n-1}{2}) \rceil$ are both integers, this implies that

$$\begin{aligned} \chi_D(C_n[G]) &< 2\chi_D(G) + \chi_D(G) / \left(\frac{n-1}{2}\right), \\ \frac{n-1}{2} &< \frac{(n-1) \cdot \chi_D(G)}{\chi_D(C_n[G])} + \frac{\chi_D(G)}{\chi_D(C_n[G])}, \\ \frac{n-1}{2} &< \frac{n \cdot \chi_D(G)}{\chi_D(C_n[G])}. \end{aligned}$$

Since n is odd, $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$, so

$$\frac{n \cdot \chi_D(G)}{\chi_D(C_n[G])} > \left\lfloor \frac{n}{2} \right\rfloor .$$

Each of the n copies of G must be coloured with at least $\chi_D(G)$ colours from a set of $\chi_D(C_n[G])$ colours, so by the Pigeonhole Principle, there is some colour that appears on at least $\frac{n \cdot \chi_D(G)}{\chi_D(C_n[G])}$ copies of G. But $\frac{n \cdot \chi_D(G)}{\chi_D(C_n[G])} > \lfloor \frac{n}{2} \rfloor$ implies that such a colour appears on more than half of the copies of G. Thus, there exists some i such that G_{v_i} and $G_{v_{i+1}}$ contain a common colour, contradicting Remark 3.1.1. □

Example 3.3.1. We provide a distinguishing colouring of $C_n[G]$ where $n = 11$ and G is an arbitrary connected graph with $\chi_D(G) = 6$. In this case,

$$q = \left\lceil \chi_D(G) / \left(\frac{n-1}{2}\right) \right\rceil = \left\lceil \frac{6}{5} \right\rceil = 2$$

This means we require $2\chi_D(G) + q = 2(6) + 2 = 14$ colours. According to the proof of Theorem 3.3.2, we define colour set $S_i, 0 \leq i \leq 6$, as follows. For $0 \leq i \leq 2\lceil\frac{\chi_D(G)}{q}\rceil = 2\lceil\frac{6}{2}\rceil = 6$ and

$$S_i = \{i\chi_D(G), i\chi_D(G)+1, \ldots, (i+1)\chi_D(G)-1\}, \text{ so}$$

$$S_0 = \{0, 1, 2, \ldots \chi_D(G) - 1\} = \{0, 1, 2, 3, 4, 5\}$$

$$S_1 = \{6, 7, 8, 9, 10, 11\}$$

$$S_2 = \{12, 13, 0, 1, 2, 3\}$$

$$S_3 = \{4, 5, 6, 7, 8, 9\}$$

$$S_4 = \{10, 11, 12, 13, 0, 1\}$$

$$S_5 = \{2, 3, 4, 5, 6, 7\}$$

$$S_6 = \{8, 9, 10, 11, 12, 13\}$$

Let (G, c_k) denote a distinguishing colouring of G using colour set S_k. We obtain a distinguishing colouring c of $C_{11}[G]$ by defining the colourings of the copies of G in $C_{11}[G]$. Set

$$(G_{v_k}, c) = (G, c_k)$$

for $1 \leq k \leq 2\left\lceil\frac{\chi_D(G)}{q}\right\rceil = 6$, and for $2 = \left\lceil\frac{\chi_D(G)}{q}\right\rceil \leq j \leq \frac{n-3}{2} = 4$, let

$$(G_{v_{2j+1}}, c) = (G_{v_7}, c) = (G_{v_9}, c) = (G, c_5)$$

$$(G_{v_{2j+2}}, c) = (G_{v_8}, c) = (G_{v_{10}}, c) = (G, c_6).$$

3.4 Wreath Products with Paths

To prove our result for wreath products with paths, we require an analogue to Lemma 3.3.1 for paths rather then cycles. In fact, we have a slightly stronger result that holds for trees, and will be used again is Section 3.5.

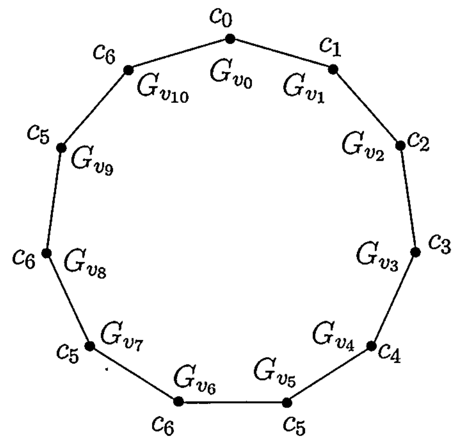

Figure 3.4: A proper distinguishing colouring $C_{11}[G]$, where $\chi_D(G) = 6$.

Lemma 3.4.1. *Let G be a connected graph and let T be a tree with at least three vertices. If $g \in Aut(T[G])$, then for any $u \in V(T)$, g maps G_u to G_w for some $w \in V(T)$.*

Proof. The proof of this result requires us to consider two cases. Suppose T contains a vertex v that is not the endpoint of a path of length two in T. It follows that T is a tree that is isomorphic to the complete bipartite graph $K_{1,m}$ for some $m \geq 2$, and that v is the vertex in the part of size one. In $T[G]$, any vertex $u \in V(G_v)$ has degree greater than $2|V(G)|$, whereas a vertex $u \in V(G_x), x \neq v$, has degree at most $|V(G)| + |V(G)| - 1$. This implies that any automorphism of $T[G]$ must map G_v to G_v.

Next, assume that u is an endpoint of a path of length two in T, and let $x, y \in V(G_u)$ be adjacent in $T[G]$, i.e., $xy \in E(G_u)$. Suppose that there exists a $g \in Aut(T[G])$ such that $g(x) \in V(G_a)$ and $g(y) \in V(G_b)$ for some $a, b \in V(T)$, $a \neq b$. Since $xy \in E(T[G])$, $ab \in E(T)$. Let $u' \in V(T)$ be a vertex that is distance two from u in T, i.e., there is a path of length two in T between u and u'. Choose $z \in V(T[G])$ with $z \in V(G_{u'})$. In $T[G]$, z is distance two from both x and y. Since g is an automorphism of $T[G]$, $g(z)$ must be distance two from both $g(x)$ and $g(y)$. If $g(z) \in V(G_a)$, then it is distance one from $g(y)$, a contradiction. Similarly, if $g(z) \in V(G_b)$, then it is distance one from $g(x)$, also a contradiction. Since T is acyclic, there is no vertex in T that is distance two from both a and b, hence there is no copy G_p of G so that $z \in V(G_p)$ is distance two from both x and y, and therefore no such automorphism g exists. Since G is connected, it follows that all vertices of G_u must be mapped to the same copy of G in $C_n[G]$ under g, and hence G_u is mapped to G_w for some $w \in V(T)$.

□

As a consequence of this Lemma, any automorphism of $T[G]$ corresponds to an automorphism of T.

Theorem 3.4.1. *For any graph G and any $n \geq 2$*

$$\chi_D(P_n[G]) = \begin{cases} 2\chi_D(G) & \text{if } n \text{ is even },\\ 2\chi_D(G) & \text{if } \xi_{\chi_D(G)}(G) \geq 2 \text{ and } n \text{ is odd },\\ 2\chi_D(G)+1 & \text{if } \xi_{\chi_D(G)}(G) = 1 \text{ and } n \text{ is odd }. \end{cases}$$

Proof. We begin by noting that if an acyclic graph H has a single center vertex v, then for all $g \in Aut(H)$, $g(v) = v$. If H has two adjacent centers , u and v, then for each $g \in Aut(H)$, either $g(v) = v$ and $g(u) = u$, or $g(v) = u$ and $g(u) = v$.

Let $P_{2n} = v_0v_1 \cdots v_{2n-1}$ be a path with an even number of vertices. By Lemma 3.1.2, it is sufficient to give a distinguishing colouring of $P_{2n}[G]$ with $2\chi_D(G)$ colours. Suppose (G, c_0) and (G, c_1) are distinct distinguishing colourings of G on disjoint colour sets $S_0 = \{0, 1, \cdots, \chi_D(G)-1\}$ and $S_1 = \{\chi_D(G), \chi_D(G)+1, \ldots, 2\chi_D(G)-1\}$, respectively. Define the colouring c on $P_{2n}[G]$ by colouring the copies of G in $P_{2n}[G]$ as follows, $0 \leq i \leq n-1$,

$$(G_{v_{2i}}, c) = (G, c_0),$$
$$(G_{v_{2i+1}}, c) = (G, c_1).$$

Then for all i, G_{v_i} and $G_{v_{i+1}}$ are coloured with disjoint colour sets. Suppose $g \in Aut(P_{2n}[G])$ is a nontrivial colour preserving automorphism of $P_{2n}[G]$. Since the only non-trivial automorphism of P_{2n} is the one that interchanges v_i with v_{2n-i-1}, it follows from Lemma 3.4.1 that g interchanges G_{v_i} with $G_{v_{2n-i-1}}$. In particular, when $i = n-1$, g interchanges $G_{v_{n-1}}$ with G_{v_n}. However $G_{v_{n-1}}$ and G_{v_n} are coloured with different colour sets, so g cannot interchange them, a contradiction. Therefore, c is a distinguishing colouring of $P_{2n}[G]$ with $2\chi_D(G)$ colours, showing that $\chi_D(P_{2n}[G]) \leq 2\chi_D(G)$. Using Lemma 3.1.2, we get $\chi_D(P_{2n}[G]) = 2\chi_D(G)$.

Let $P_{2n-1} = v_0v_1 \cdots v_{2n-2}$ be a path with an odd number of vertices and assume $\xi_{\chi_D(G)}(G) \geq 2$. Let (G, c_0) and (G, c_1) be distinguishing colours of G using the colour set $S_0 = \{0, 1, \ldots, \chi_D(G) - 1\}$, and let (G, d_0) be a distinguishing colouring of G that uses colour set $S_1 = \{\chi_D(G), \chi_D(G) + 1, \ldots, 2\chi_D(G) - 1\}$. Define the colouring c on $P_{2n-1}[G]$ by colouring the copies of G in $P_{2n-1}[G]$ as follows.

$$(G_{v_0}, c) = (G, c_1),$$

$$(G_{v_i}, c) = (G, d_0) \text{ for } i \text{ odd},$$

$$(G_{v_i}, c) = (G, c_0) \text{ for } i \text{ even}, i \neq 0.$$

Then for all i, G_{v_i} and $G_{v_{i+1}}$ are coloured with disjoint colour sets. Suppose $g \in Aut(P_{2n-1}[G])$ is a nontrivial colour preserving automorphism of $P_{2n-1}[G]$. Since the only non-trivial automorphism of P_{2n-1} is the one that interchanges v_i with v_{2n-i-2}, it follows from Lemma 3.4.1 that g interchanges G_{v_i} with $G_{v_{2n-i-2}}$. In particular, when $i = 0$, g interchanges G_{v_0} with $G_{v_{2n-2}}$. However (G_{v_0}, c) and $(G_{v_{2n-2}}, c)$ are distinct colourings of G, so g cannot interchange them, a contradiction. Therefore, c is a distinguishing colouring of $P_{2n-1}[G]$, showing that $\chi_D(P_{2n-1}[G]) \leq 2\chi_D(G)$. By Lemma 3.1.2, $\chi_D(P_{2n-1}[G]) = 2\chi_D(G)$.

Finally, suppose $P_{2n-1} = v_0v_1 \cdots v_{2n-2}$ is a path, with an odd number of vertices, and assume $\xi_{\chi_D(G)}(G) = 1$. Define three different colour sets as follows.

$$S_0 = \{0, 1, \ldots, \chi_D(G) - 1\},$$

$$S_1 = \{\chi_D(G), \chi_D(g) + 1, \ldots, 2\chi_D(G) - 1\},$$

$$S_2 = \{0, 1, \ldots, \chi_D(G) - 2, 2\chi_D(G)\}.$$

Let $(G, c_0), (G, c_1)$ and (G, c_2) be distinct distinguishing colourings of G with colour sets S_0, S_1 and S_2, respectively. Define the colouring c on $P_n[G]$ by colouring the copies of G in $P_n[G]$ as follows. Let

$$(G_{v_0}, c) = (G, c_2),$$

$$(G_{v_{2i-1}}, c) = (G, c_1), 1 \leq i \leq n - 1,$$

$$(G_{v_{2i}}, c) = (G, c_0), 1 \leq i \leq n - 1.$$

Then for all i, G_{v_i} and $G_{v_{i+1}}$ are coloured with disjoint colour sets. Suppose $g \in Aut(P_{2n-1}[G])$ is a nontrivial colour preserving automorphism of $P_{2n-1}[G]$. Since the only non-trivial automorphism of P_{2n-1} is the one that interchanges v_i with v_{2n-i-2}, it follows from Lemma 3.4.1 that g interchanges G_{v_0} and $G_{v_{2n-2}}$. However, this mapping is not colour preserving because G_{v_0} and $G_{v_{2n-2}}$ are distinctly coloured. Therefore c is a distinguishing colouring of $P_{2n-1}[G]$ and hence $\chi_D(P_{2n-1}[G]) \leq 2\chi_D(G) + 1$. By Lemma 3.1.2, it follows that $\chi_D(P_{2n-1}[G])$ equals $2\chi_D(G)$ or $2\chi_D(G) + 1$.

Suppose that $\chi_D(P_n[G]) = 2\chi_D(G)$. By Remark 3.1.1, G_0 and G_1 must be coloured with disjoint sets of colours, S_0 and S_1, respectively. However, $|S_0|, |S_1| \geq \chi_D(G)$, so $|S_0 \cup S_1| \geq 2\chi_D(G)$. Since the number of colours allowed is exactly $2\chi_D(G)$, it follows that $|S_0| = |S_1| = \chi_D(G)$. For $i \in \{0, 1\}$, define c_i to be a distinguishing colouring of G using colour set S_i. Since $\xi_{\chi_D(G)}(G) = 1$, c_i is unique, and by Remark 3.1.1, it follows that the only colourings of $P_{2n-1}[G]$ with the $2\chi_D(G)$ are

$$(G_{v_i}, c) = (G, c_0) \text{ for } i \text{ even,}$$

$$(G_{v_i}, c) = (G, c_1) \text{ for } i \text{ odd.}$$

or

$$(G_{v_i}, c) = (G, c_0) \text{ for } i \text{ odd,}$$

$$(G_{v_i}, c) = (G, c_1) \text{ for } i \text{ even.}$$

In either case, there is a colour preserving automorphism $g \in Aut(P_{n-1}[G])$ that interchanges G_{v_i} and $G_{v_{2n-i-2}}$, for $0 \leq i \leq n-1$. Therefore, c is not a distinguishing colouring of $P_{2n-1}[G]$, and thus $\chi_D(P_{2n-1}[G]) = 2\chi_d(G) + 1$. □

3.5 Wreath Products with Trees

The next result generalizes the results in the previous section from paths to trees. In what follows, assume that a rooted tree T is drawn in the plane in such a fashion that the children of each vertex have a natural ordering from left to right.

Theorem 3.5.1. *Let T be a tree with root v_0, with bipartition $X = \{v_0, v_1, \ldots, v_{n-1}\}$, and $Y = \{u_1, u_2, \ldots, u_{m-1}\}$. Then $\chi_D(T[G]) \leq r + p$, where r is the smallest integer such that $\xi_r(G) \geq \max\{d(v_0), d(v_1) - 1, d(v_2) - 1, \ldots,, d(v_{n-1}) - 1\}$, and p the smallest integer such that $\xi_p(G) \geq \max\{d(u_i) \mid 1 \leq i \leq m-1\}$.*

Proof. Choose the smallest r and p such that

$$\xi_r(G) \geq \max\{\ d(v_0), d(v_1) - 1, d(v_2) - 1, \ldots, d(v_{n-1}) - 1\},$$

and $\xi_p(G) \geq \max\{d(u_i) \mid 1 \leq i \leq m-1\}$. Let $(G, c_0), (G, c_1), \ldots, (G, c_{\xi_r(G)-1})$ and $(G, d_0), (G, d_1), \ldots, (G, d_{\xi_p(G)-1})$ be distinct distinguishing colourings of G, using the colour sets $S_0 = \{0, 1, \ldots, r-1\}$ and $S_1 = \{r, r+1, \ldots, r+p-1\}$, respectively.

In what follows, copies of G in $T[G]$ that correspond to vertices of X are coloured using (G, d_i), $0 \leq i \leq \xi_p(G) - 1$, and those corresponding to vertices of Y are coloured with (G, c_j), $0 \leq j \leq \xi_r(G) - 1$. To obtain a distinguishing colouring c of $T[G]$, begin by setting $(G_{v_0}, c) = (G, d_{\xi_p(G)-1})$. We then discard the colouring $(G, d_{\xi_p(G)-1})$, ensuring that any colour preserving $g \in Aut(T[G])$ fixes G_{v_0}. We complete the distinguishing colouring of $T[G]$ by colouring the remaining copies of G in $T[G]$ as follows.

If for some $u \in Y$, and some j, $0 \leq j \leq \xi_p(G) - 1, (G_u, c) = (G, c_j)$ then each of the copies of G corresponding to the children of u are coloured distinctly from left to right with colouring $(G, d_i), i = 0, 1, \ldots, d(u)-1$. The choice of r insures that $\xi_r(G) \geq d(u)-1$.

If for some $v \in X$ and some k, $(G_v, c) = (G, d_k), 0 \leq k \leq \xi_r(G) - 1$, then each of the copies of G that correspond to the children of v are coloured distinctly from left to right with colouring $(G, c_j), j = 0, 1, \ldots, d(v)-1$. The choice of p insures that $\xi_p(G) \geq d(v)-1$.

This colouring of $T[G]$ has the following properties:

1. If $xy \in E(T)$ and x is the parent of y, then G_x is coloured before G_y.
2. If $y_1, y_2, \ldots, y_t$ are the children of x in T, then $(G_{y_1}, c), (G_{y_2}, c), \ldots, (G_{y_t}, c)$ are distinct distinguishing colourings of G. The fact that $S_0 \cap S_1 = \emptyset$ and because copies of G corresponding to vertices in X are coloured using S_0, while vertices in Y are coloured using S_1, the result is a colouring of $T[G]$ (not just a labeling).

As a consequence of these properties, if $y_1, y_2, \ldots, y_t$ are the children of $x \in T$, and if a colour preserving automorphism $g \in Aut(T[G])$ fixes G_x, then g also fixes $(G_{y_1}, c), (G_{y_2}, c), \ldots, (G_{y_t}, c)$. It now follows by induction on levels that once G_{v_0} has been coloured so that any colour preserving $g \in Aut(T[G])$ fixes G_{v_0}, the resulting colouring c of $T[G]$ is distinguishing.

□

Chapter 4

Distinct Distinguishing Colourings

4.1 Preliminaries

In Chapter 3 we introduced the notion of the number of distinct distinguishing colourings of a graph G using r colours, denoted $\xi_r(G)$. In this chapter we explore this notion further. Unless otherwise stated, we assume that the colours used in an r-colouring of a graph are $\{0, 1, \ldots, r-1\}$. Recall that two colourings of a graph G, c_1 and c_2, are *similar* colourings if for some $g \in Aut(G), c_1(u) = c_2(g(u))$ for each $u \in V(G)$. Two colourings that are not similar are distinct colourings. Since we often strive for the fewest number of colours to colour a graph, it is of particular interest to find $\xi_{\chi_D(G)}(G)$ for a graph G. The next two examples illustrate these ideas.

Example 4.1.1. Consider the graph G shown in Figure 4.1. We define the elements, R, F_h, and F_v in the automorphism group of G as follows. Let R represent a rotation of G by 180°, i.e., one where

$$R(v_i) = v_{i+3},$$

(subscripts are taken modulo 6). Let F_h represent a reflection (or flip) in the horizontal line passing through v_0 and v_3, i.e., F_h fixes v_0 and v_3, and interchanges vertices v_1 and

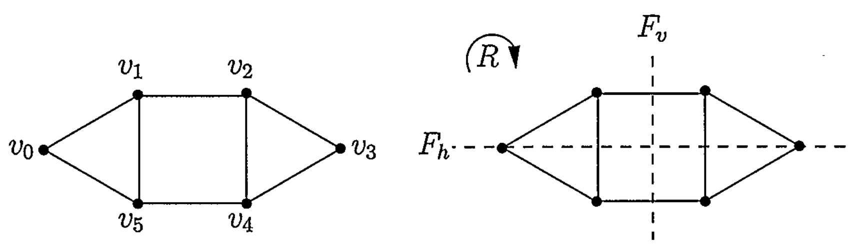

Figure 4.1: The graph G and its automorphisms.

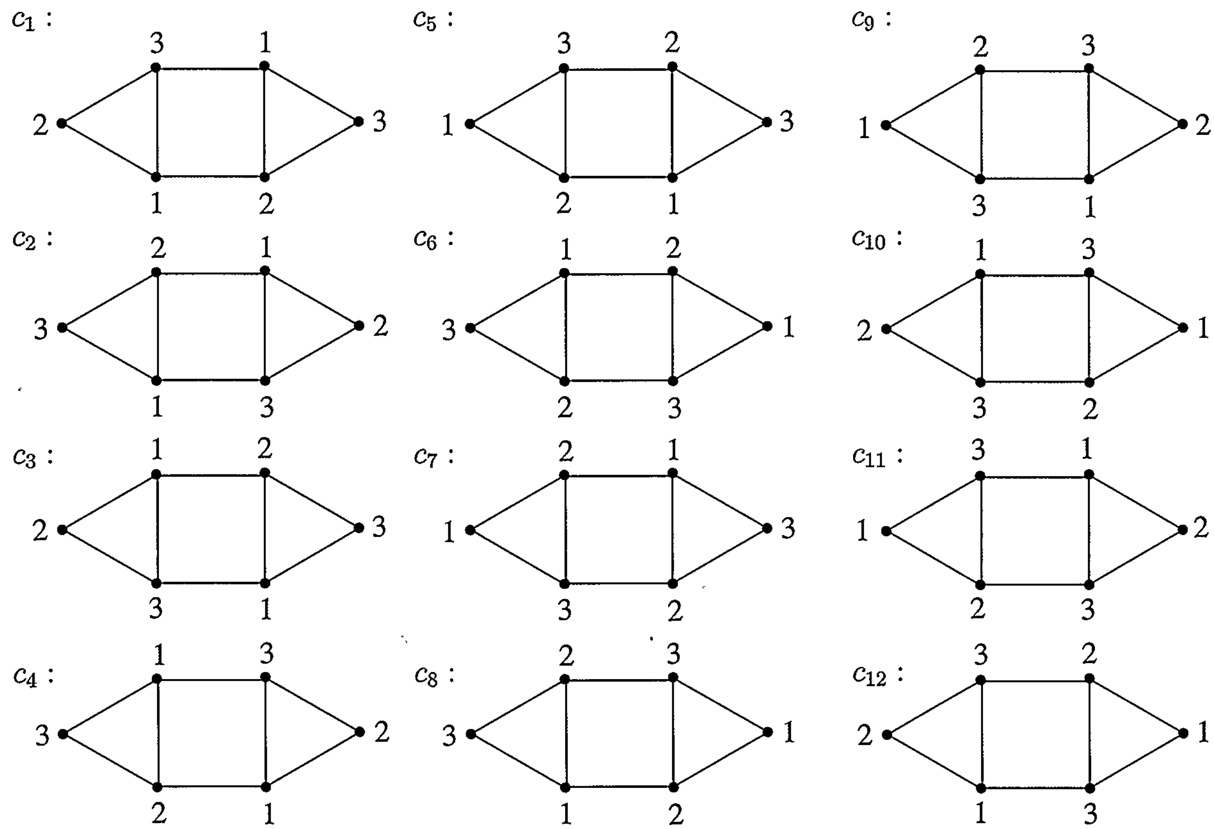

Figure 4.2: All the distinguishing 3-colourings of G.

v_5, and v_2 and v_4. Let F_v represent a reflection in the vertical line passing through edges v_1v_2 and v_5v_4, i.e., F_v interchanges vertices v_0 and v_3, v_1 and v_2, and v_5 and v_4. In Figure 4.2 we provide all possible distinguishing 3-colourings of the graph G. In order to keep the figures from becoming too cluttered, the vertex names (v_0 through v_5) have been omitted; the name of a vertex can be deduced from its position in the figure, and corresponds to its name in Figure 4.1. Note that after performing R, F_h, and F_v on (G, c_1) we obtain (G, c_2), (G, c_3), and (G, c_4), respectively. We then conclude that all the colourings of G in the first column are similar. In fact any two colourings in the same column are similar. However, any two colourings of G from different columns are distinct. In this particular case $\xi_3(G) = 3$. This is because three colourings from different columns will all be distinct, but any selection of four colourings of G will contain at least a pair of similar colourings.

Example 4.1.2. Consider the graph G shown in Figure 4.3. We define the elements, R and F, in the automorphism group of G as follows. Let R represent a clockwise rotation of G by 120°, i.e., one where

$$G(v_i) = v_{i+2},$$

(subscripts are taken modulo 6). Let F represent a reflection in the line passing through v_1 and v_4. In Figure 4.4 we provide all the possible 3-colourings of the graph G. In order to keep the figures from becoming too cluttered, the vertex names (v_0 through v_5) have been omitted; the name of a vertex can be deduced from its position in the figure, and corresponds to its name in Figure 4.3. All the colourings of G are similar; by performing some combination of R and F on (G, c_1) we can obtain any of the other colourings. As a result $\xi_3(G) = 1$. Often such graphs require additional consideration due to this trait.

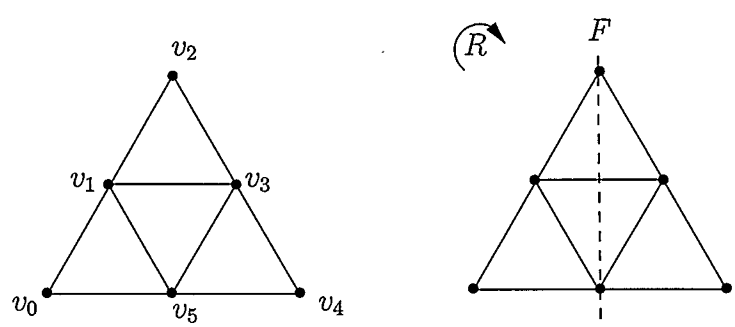

Figure 4.3: The graph G and its automorphisms.

We also provide some additional examples in Table 4.1 to further solidify the concept. Note we put restrictions on the number of colours used. If these restrictions are not met, then there are no distinct distinguishing r-colourings of G.

Despite the ease of evaluating $\xi_r(G)$ in the examples provided thus far, this certainly is not always the case. Consider the path of odd length, P_{2n}. Note that $\chi_D(P_{2n}) = 2$ and $\xi_2(P_{2n}) = 1$, but even determining $\xi_3(P_{2n})$ is not trivial. In addition, the path

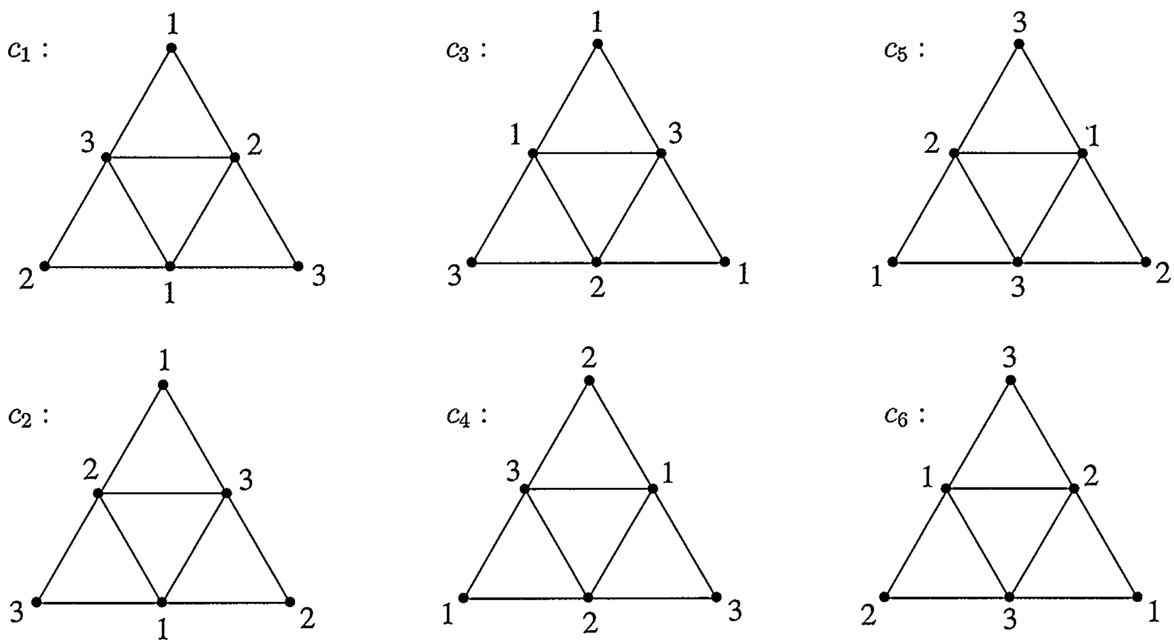

Figure 4.4: All the distinguishing colourings of G.

Graph G	$\chi_D(G)$	$\xi_{\chi_D(G)}(G)$	$\xi_r(G)$
K_n	n	1	$\binom{r}{n}$ for $r \geq n$
I_n	n	1	$\binom{r}{n}$ for $r \geq n$
$K_{m,n}$	$m+n$	$\binom{m+n}{n}$	$\binom{r}{m} + \binom{r-m}{n}$ for $r \geq m+n$

Table 4.1: Preliminary examples

of even length, P_{2n+1}, has $\chi_D(P_{2n+1}) = 3$, but it is not trivial to determine $\xi_3(P_{2n+1})$. The procedure to calculate both $\xi_3(P_{2n})$ and $\xi_3(P_{2n+1})$ follows from a result later in this chapter.

4.2 The Number of Distinct Distinguishing Colourings of a Tree

We begin our investigation of trees, but first we must introduce some additional notation.

Let T be a tree and $w \in V(T)$. We denote by $T(w)$ the tree T rooted at vertex w, and for $u \in N(w)$, $T_w(u)$ denotes the component of $T(w) - w$ rooted at u. Now

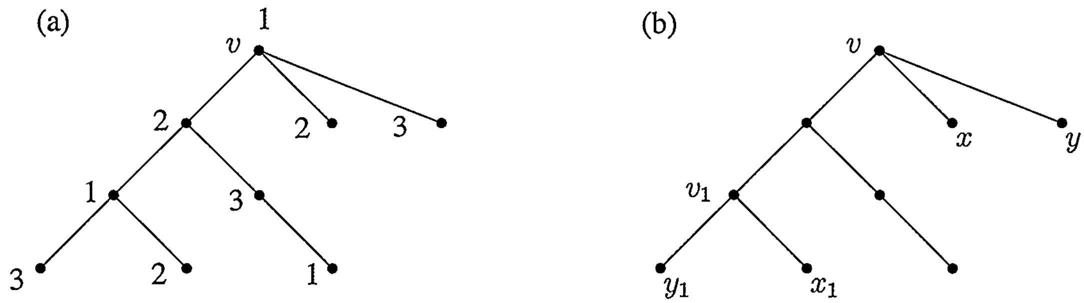

Figure 4.5: Distinguishing colouring of a rooted tree.

suppose $a, b \in N(w)$. We say that $T_w(a)$ and $T_w(b)$ are *isomorphic rooted trees* if there is an isomorphism $f : T_w(a) \to T_w(b)$ such that $f(a) = b$. Otherwise, $T_w(a)$ and $T_w(b)$ are non-isomorphic rooted trees. This isomorphism induces an equivalence relation on $\{T_w(x) \mid x \in N(w)\}$, the rooted components of $T(w) - w$. We define $n[T(w)]$ to be the number of equivalence classes, and write $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{n[T(w)]}\}$ for the set of equivalence classes. If $T_a(w), T_b(w) \in \mathcal{T}_i$ for some i, then $\xi_r(T_a(w)) = \xi_r(T_b(w))$. To simplify notation we write $\xi_r(\mathcal{T}_i)$ for the number of distinct distinguishing r-colourings of any tree in $\mathcal{T}_i$.

Let T be a tree and $v \in V(T)$. The automorphism group of the rooted tree $T(v)$ is defined as

$$Aut(T(v)) = \{g \in Aut(T) \mid g(v) = v\}.$$

We define a colouring c of a rooted tree $T(v)$ to be distinguishing if and only if the only colour preserving automorphism $g \in Aut(T(v))$ is the identity. Figure 4.5 (a) gives an example of a rooted tree, $T(v)$ with a distinguishing colouring. Note that because v is fixed there is no non-trivial colour preserving automorphism of $T(v)$. However, if the tree were not rooted, and we simply name it T, then there is a $g \in Aut(T)$ which is colour preserving, namely the automorphism of T that interchanges v and v_1, x and x_1, and y and y_1, while fixing the remaining three vertices, see Figure 4.5 (b).

Lemma 4.2.1. *Let $r \geq \chi_D(T(w))$, and let $\mathcal{D}_i$ denote the set of r-distinguishing colourings of $T(w)$ in which w receives colour i. Then $|\mathcal{D}_i| = |\mathcal{D}_j|$ for all $0 \leq i, j \leq r - 1$.*

Proof. Let $c \in \mathcal{D}_i$, and define $f : \mathcal{D}_i \to \mathcal{D}_j$ so that f interchanges colours i and j in c. Then f is a bijection between $\mathcal{D}_i$ and $\mathcal{D}_j$, and hence $|D_i| = |D_j|$. □

Remark 4.2.1. Fix some colour a, $0 \leq a \leq r-1$. As a consequence of Lemma 4.2.1 there are $\frac{\xi_r(T(w))}{r}$ distinct r-distinguishing colourings of $T(w)$ in which w receives colour a, and $\xi_r(T(w)) \cdot \frac{r-1}{r}$ distinct r-distinguishing colourings of $T(w)$ in which w receives some colour other than a.

Theorem 4.2.1. *Let T be a tree, $w \in V(T)$ and $r \in \mathbb{N}$. Then the number of distinct distinguishing r-colourings of the rooted tree $T(w)$ is given by*

$$\xi_r(T(w)) = r \prod_{i=1}^{n[T(w)]} \binom{\xi_r(\mathcal{T}_i)(r-1))/r}{|\mathcal{T}_i|}.$$

Proof. Consider $T(w)$ and colour w with some colour a, $0 \leq a \leq r-1$. Choose $q \in N(w)$ such that $T_w(q) \in \mathcal{T}_i$, for some $1 \leq i \leq n[T(w)]$. By Remark 4.2.1 there are $\xi_r(T_w(q)) \cdot \frac{r-1}{r}$ ways to colour $T_w(q)$ so that q does not receive colour a. Thus there are

$$\binom{\xi_r(\mathcal{T}_i)(r-1))/r}{|\mathcal{T}_i|}$$

ways to colour the elements of $\mathcal{T}_i$ so that no two elements have similar colourings. Since there is no automorphism of $T(w)$ that maps $T' \in \mathcal{T}_i$ to $T'' \in \mathcal{T}_j$, $i \neq j$ and $1 \leq i, j \leq n[T(w)]$, it suffices to assign distinct distinguishing colourings to the elements of $\mathcal{T}_i$, $1 \leq i \leq n[T(w)]$. There are r choices for colouring w, so it follows that

$$\xi_r(T(w)) = r \prod_{i=1}^{n[T(w)]} \binom{\xi_r(\mathcal{T}_i)(r-1))/r}{|\mathcal{T}_i|}.$$

□

Theorem 4.2.1 can be used recursively to find $\xi_r(T(w))$ for any rooted tree $T(w)$ since the formula uses the number of distinct distinguishing colourings of rooted subtrees of $T(w)$. Before illustrating this with an example, we have the following corollary.

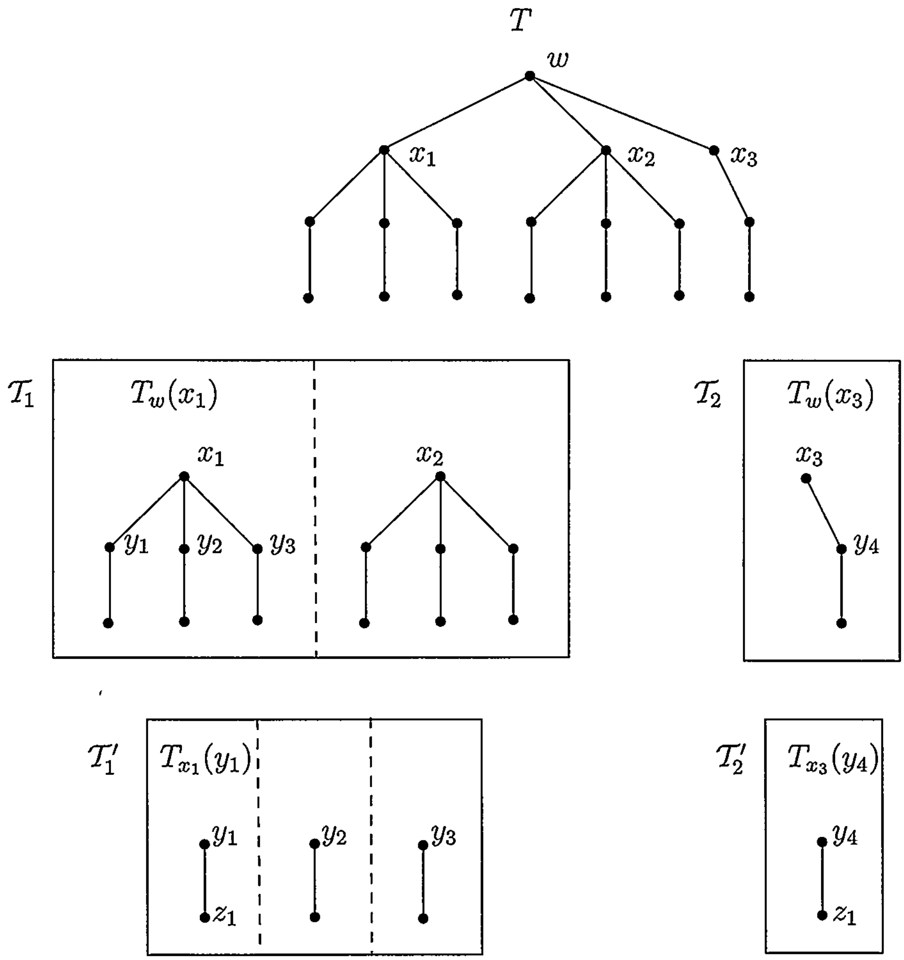

Figure 4.6: The decomposition of tree T

Corollary 4.2.1. *Let T be a tree with a single vertex w as its center. Then for any $r \in \mathbb{N}$,*

$$\xi_r(T) = r \prod_{i=1}^{n[T(w)]} \binom{\xi_r(\mathcal{T}_i)(r-1))/r}{|\mathcal{T}_i|}.$$

Proof. Since T has center w, any automorphism of T must fix w, so $Aut(T) = Aut(T(w))$. □

Example 4.2.1. We determine the number of distinct distinguishing 4-colourings of the tree T in Figure 4.6, which we root at its unique center vertex w. This is done by recursively decomposing each subtree formed by removing its root and considering all other resulting trees. At each stage of this decomposition, the trees that are isomorphic as rooted subtrees are grouped into equivalence classes.

Note that $T_{x_1}(y_1)$ is a complete graph on two vertices, and thus

$$\xi_4(T_{x_1}(y_1)) = 4 \cdot 3 = 12.$$

In addition $T_{x_3}(y_4)$ is a complete graph on two vertices so

$$\xi_4(T_{x_3}(y_4)) = 12.$$

Using Theorem 4.2.1 we get

$$\xi_4(T_w(x_1)) = 4\begin{pmatrix} \xi_4(\mathcal{T}'_1) \times 3/4 \\ 3 \end{pmatrix} = 4\binom{(12 \cdot 3)/4}{3} = 4\binom{9}{3} = 336.$$

and

$$\xi_4(T_w(x_3)) = 4\begin{pmatrix} \xi_4(\mathcal{T}'_{\in}) \times 3/4 \\ 1 \end{pmatrix} = 4\binom{(12 \cdot 3)/4}{1} = 4\binom{9}{1} = 36.$$

Applying Theorem 4.2.1 again gives us

$$\xi_4(T) = 4\begin{pmatrix} \xi_4(\mathcal{T}_1) \times 3/4 \\ 2 \end{pmatrix} \times \begin{pmatrix} \xi_4(\mathcal{T}_2) \times 3/4 \\ 1 \end{pmatrix} = 4\binom{(336 \cdot 3)/4}{2} \times \binom{(36 \cdot 3)/4}{1} = 3415608$$

We next use Theorem 4.2.1 to obtain an expression for $\xi_r(T)$ in the case where, instead of a unique center, the tree T has two adjacent centers.

Theorem 4.2.2. *Let T be a tree with adjacent centers u and v, and let $T(u)$ and $T(v)$ denote rooted subtrees of $T\backslash\{uv\}$ rooted at u and v, respectively. Then for any $r \in \mathbb{N}$,*

$$\xi_r(T) = \begin{cases} \frac{r-1}{r}\xi_r(T(u))\xi_r(T(v)) & \text{if } T(u) \text{ and } T(v) \text{ are not isomorphic rooted trees,} \\ \frac{r-1}{2r}\xi_r(T(u))\xi_r(T(v)) & \text{if } T(u) \text{ and } T(v) \text{ are isomorphic rooted trees.} \end{cases}$$

Proof. Let T be a tree with two adjacent center u and v, and let $g \in Aut(T)$. Then either $g(u) = u$ and $g(v) = v$, or $g(u) = v$ and $g(v) = u$. The latter occurs if and only if $T(u)$ and $T(v)$ are isomorphic as rooted trees, so we consider the two cases: *(i)* $T(u)$ and $T(v)$ are not isomorphic as rooted trees; *(ii)* $T(u)$ and $T(v)$ are isomorphic as rooted trees.

Suppose that $T(u)$ and $T(v)$ are not isomorphic as rooted trees. Then if $g \in Aut(T)$, $g(u) = u$ and $g(v) = v$. This implies that for each of the $\xi_r(T(u))$ distinct distinguishing r-colourings of $T(u)$, there are $\frac{r-1}{r}\xi_r(T(v))$ distinct distinguishing r-colourings of $T(v)$ with v coloured differently from u. Therefore, there are $\frac{r-1}{r}\xi_r(T(u))\xi_r(T(v))$ distinct distinguishing r-colourings of T.

Now suppose that $T(u)$ and $T(v)$ are isomorphic as rooted trees, and let c be a distinguishing colouring of T. Then $c(u) \neq c(v)$, and the colouring d obtained from c by interchanging the colours on vertices coloured $c(u)$ and $c(v)$ is also a distinguishing colouring of T. Since $T(u)$ and $T(v)$ are isomorphic as rooted trees, it follows that $(T(u), d)$ and $(T(v), c)$ are isomorphic as rooted, coloured trees, as are $(T(v), d)$ and $(T(u), c)$. This gives us a way of pairing up the $\frac{r-1}{r}\xi_r(T(u))\xi_r(T(v))$ distinguishing r-colourings of T obtained by taking one of the $\xi_r(T(u))$ distinct distinguishing r-colouring of $T(u)$ and one of the $\frac{r-1}{r}\xi_r(T(v))$ distinct distinguishing r-colourings of $T(v)$. The colourings in each pair are not distinct, while colouring from different pairs are distinct. Therefore the number of distinct distinguishing r-colourings of T is

$$\xi_r(T) = \frac{1}{2} \cdot \frac{r-1}{r}\xi_r(T_u) \cdot \xi_r(T_v) = \frac{r-1}{2r}\xi_r(T_u) \cdot \xi_r(T_v).$$

□

Example 4.2.2. We determine the number of distinct distinguishing 4-colourings of the tree T in Figure 4.7, which has two adjacent centers. This is done by considering, separately, the trees $T(u)$ and $T(v)$ from the decomposition of T. Since $T(u)$ and $T(v)$

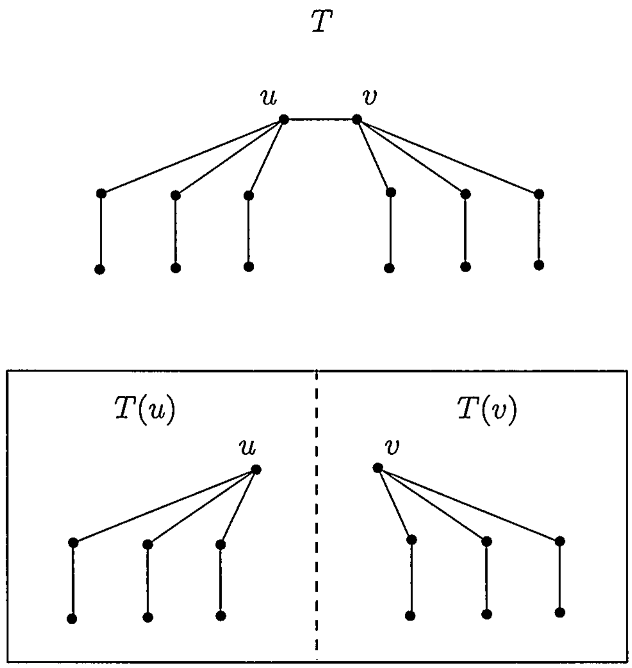

Figure 4.7: The decomposition of T, a tree with two adjacent centers.

are isomorphic as rooted trees, we consider the second case of Theorem 4.2.2. From Example 4.2.1 we have that

$$\xi_4(T(u)) = \xi_4(T(v)) = 336.$$

Therefore, it follows that

$$\xi_4(T) = \frac{3}{2 \cdot 4}\xi_4(T(u)) \cdot \xi_4(T(v)) = \frac{3}{8} \cdot 336 \cdot 336 = 42336.$$

4.3 The Chromatic Polynomial

Let G be a graph. The chromatic polynomial for G, denoted $P(G, x)$ is the number of different colourings of G using x colours. The chromatic polynomial was first defined by Birkoff [6] as a possible approach to solving the Four Colour Theorem. It has been proven that recursively using edge deletion and edge contraction allows the value to be computed for a fixed number of colours. Let G be a graph and $uv \in E(G)$. We denote by $G\backslash\{uv\}$ the graph obtained from G by deleting the edge uv (but not the vertices u and v). The graph $G/\{uv\}$ is the graph obtained from G by contracting the edge uv, and is obtained from G by deleting vertices u and v, and adding a new vertex, say w, so that the neighbourhood of w in $G/\{uv\}$ is equal to $N_G(u) \cup N_G(v)$.

Theorem 4.3.1. *[6] Let G be a graph and $x \in \mathbb{N}$. Then for any $uv \in E(G)$,*

$$P(G,x) = P(G \setminus \{uv\}, x) - P(G/\{uv\}, x).$$

The following two lemmas provide some common known results about the chromatic polynomial of paths and cycles.

Lemma 4.3.1. *If P_n denotes a path with n vertices, then $P(P_n, r) = r(r-1)^{n-1}$.*

Proof. Let $P_n = v_0 v_1 \cdots v_{n-1}$. If we colour the vertices of P_n in numerical order starting from v_0, then there are r colours available to colour v_0. Since adjacent vertices are coloured differently, there are exactly $r-1$ colours available to colour v_1. Similarly, there are $r-1$ colours available to colour v_2. Continuing in this fashion, we get $P(P_n, r) = r(r-1)^{n-1}$. □

Lemma 4.3.2. *If C_n denotes the cycle on n vertices, then*

$$P(C_n, r) = (r-1)^n + (-1)^n(r-1).$$

Proof. The proof follows by induction on n. For $n = 3$,

$$\begin{aligned} P(C_3, r) &= r(r-1)(r-2) \\ &= r^3 - 3r^2 + 2r \end{aligned}$$

and

$$\begin{aligned} (r-1)^3 + (-1)^3(r-1) &= r^3 - 3r^2 + 3r - 1 - r + 1 \\ &= r^3 - 3r^2 + 2r. \end{aligned}$$

Thus

$$P(C_3, r) = (r-1)^3 + (-1)^3(r-1),$$

and establishes the basis for the induction. Suppose $n \geq 4$ and that $uv \in E(C_n)$. Then, by Theorem 4.3.1, $P(C_n, r) = P(C_n \backslash \{uv\}, r) - P(C_n/\{uv\}, r)$. Notice that $C_n \backslash \{uv\} \cong P_n$ and $C_n/\{uv\} \cong C_{n-1}$. Thus

$$\begin{aligned} P(C_n, r) &= P(P_n, r) - P(C_{n-1}, r) \\ &= r(r-1)^{n-1} - (r-1)^{n-1} - (-1)^{n-1}(r-1) \\ &= (r-1)^{n-1}(r-1) + (-1)^n(r-1) \\ &= (r-1)^n + (-1)^n(r-1). \end{aligned}$$

□

4.4 Distinct Distinguishing Colouring of Cycles

In this section, we derive an expression for the number of distinct distinguishing colourings of C_n, a cycle on n vertices. The precise expression depends on the parity of the cycle, and uses the chromatic polynomial of a cycle. The basic technique for expressing $\xi_r(C_n)$ is to count the number of r-colourings of C_n that are not distinguishing, and subtracting this from $P(C_n, r)$. Suppose $C_k = v_0 v_1 \cdots v_{k-1} v_0$, $k \geq 3$. If a colouring of C_k is not distinguishing, then C_k has a nontrivial colour preserving automorphism, and such an automorphism of C_k is either a reflection or a rotation.

First suppose that a colouring c of C_k is preserved by a nontrivial rotation. Then there exits an integer p, $1 < p < k$, such that $p|k$ and such that

$$c(v_i) = c(v_{i+\frac{k}{p}})$$

for all i, $0 \leq i \leq k-1$ (here and in what follows, subscripts are taken modulo k). By induction, it follows that for each i, $0 \leq i \leq k-1$

$$c(v_i) = c(v_{i+j\frac{k}{p}})$$

for all integers j. Thus, such a colouring c is completely defined by the colours of the vertices of the path $v_0v_1\cdots v_{\frac{k}{p}-1}$, which we refer to as the initial segment of C_k with respect to the colouring c.

Definition 4.4.1. Let $C_k = v_0v_1\cdots v_{k-1}v_0$, and suppose p is an integer such that $p|k$ and $1 < p < k$. We define a set of r-colourings of C_n as follows: $c \in \mathcal{R}^p_{k,r}$ if and only if $c(v_i) = c(v_{i+k/p})$ for all i, $0 \leq i \leq p-1$. Then $\mathcal{R}^p_{k,r}$ consists of all r-colourings of C_k that are preserved by a rotation through $\frac{k}{p}$.

The following technical lemma is used to simplify the expression for $\xi_r(C_n)$.

Lemma 4.4.1. *Let $\mathcal{R}^p_{k,r}$ denote the set of all r-colourings of C_k that are preserved by rotation through $\frac{k}{p}$ for some $1 < p < k$. If $1 < p' < p$ and $p'|p$, then $\mathcal{R}^p_{k,r} \subseteq \mathcal{R}^{p'}_{k,r}$.*

Proof. Let $p = qp'$ where $q > 1$, and let $c \in \mathcal{R}^p_{k,r}$. Then

$$c(v_i) = c(v_{i+\frac{k}{p}})$$

for $i \in \mathbb{Z}$ (throughout subscripts are taken modulo k), and hence

$$c(v_i) = c(v_{i+j\frac{k}{p}})$$

for $j \in \mathbb{Z}$. In particular, for $j = q$, we have

$$c(v_i) = c(v_{i+q\frac{k}{p}}) = c(v_{i+\frac{k}{p'}}),$$

since $p = qp'$. Therefore, $c \in \mathcal{R}^{p'}_{k,r}$. □

Corollary 4.4.1. *Let $\mathcal{R}^p_{k,r}$ denote the set of all r-colourings of C_k that are preserved by rotation through $\frac{k}{p}$ for some $1 < p < k$. If $p = p_1p_2$ for $1 < p_i < p$, $i \in \{1,2\}$ then $\mathcal{R}^{p_1}_{k,r} \cap \mathcal{R}^{p_2}_{k,r} = \mathcal{R}^p_{k,r}$.*

Proof. By Lemma 4.4.1, $\mathcal{R}^{p}_{k,r} \subseteq \mathcal{R}^{p_1}_{k,r}$. Similarly $\mathcal{R}^{p}_{k,r} \subseteq \mathcal{R}^{p_2}_{k,r}$. Therefore

$$\mathcal{R}^{p}_{k,r} \subseteq \mathcal{R}^{p_1}_{k,r} \cap \mathcal{R}^{p_2}_{k,r}$$

.

Now suppose $c \in \mathcal{R}^{p_1}_{k,r} \cap \mathcal{R}^{p_2}_{k,r}$. Then for $i, j \in \mathbb{Z}$ (throughout subscripts are taken modulo k)

$$c(v_i) = c(v_{i+j\frac{k}{p_1}}) = c(v_{i+m\frac{k}{p_2}}).$$

This implies

$$\begin{aligned} c(v_i) &= c(v_{i+j\frac{k}{p_1}+m\frac{k}{p_2}}) \\ &= c(v_{i+\frac{mk+jk}{p_1 p_2}}) \\ &= c(v_{i+(m+j)\frac{k}{p_1 p_2}}). \end{aligned}$$

Since $(m+j) \in \mathbb{Z}$, $c \in \mathcal{R}^{p}_{k,r}$, thus $\mathcal{R}^{p_1}_{k,r} \cap \mathcal{R}^{p_2}_{k,r} \subseteq \mathcal{R}^{p}_{k,r}$. We now conclude that $\mathcal{R}^{p_1}_{k,r} \cap \mathcal{R}^{p_2}_{k,r} = \mathcal{R}^{p}_{k,r}$ □

Remark 4.4.1. Notice that $|\mathcal{R}^{p}_{k,r}| = P(C_{k/p}, r)$. A colouring $c \in \mathcal{R}^{p}_{k,r}$ is determined by $c(v_i)$, $0 \leq i \leq k/p - 1$. In addition, $c(v_{\frac{k}{p}-1}) = c(v_{k-1}) \neq c(v_0)$, so there is a one-to-one correspondence between elements of $\mathcal{R}^{p}_{k,r}$ and r-colourings of $C_{k/p}$.

Example 4.4.1. A colouring in $\mathcal{R}^{p}_{k,r}$ is determined by the colours on the vertices of the initial segment, $v_0 v_1 \cdots v_{\frac{k}{p}-1}$. Here we consider a particular case to illustrate Corollary 4.4.1. For $k = 3^2 \cdot 5 = 45$ and $r = 4$ and let $c_1 \in \mathcal{R}^{15}_{45,4}$, $c_2 \in \mathcal{R}^{5}_{45,4}$, and $c_3 \in \mathcal{R}^{3}_{45,4}$. The inital segments of c_1, c_2, and c_3, are shown in Figure 4.8.

Despite the initial segments for c_1, c_2, and c_3 having different lengths, each c_i, $1 \leq i \leq 3$, produces the same colouring of C_{45}. In the case of c_1 notice that $c_1(v_0) \neq c_1(v_{\frac{k}{p}-1}) = c_1(v_2)$, thus ensuring that $c_1(v_{\frac{k}{p}}) = c_1(v_3) \neq c_1(v_2) = c_1(v_{\frac{k}{p}-1})$.

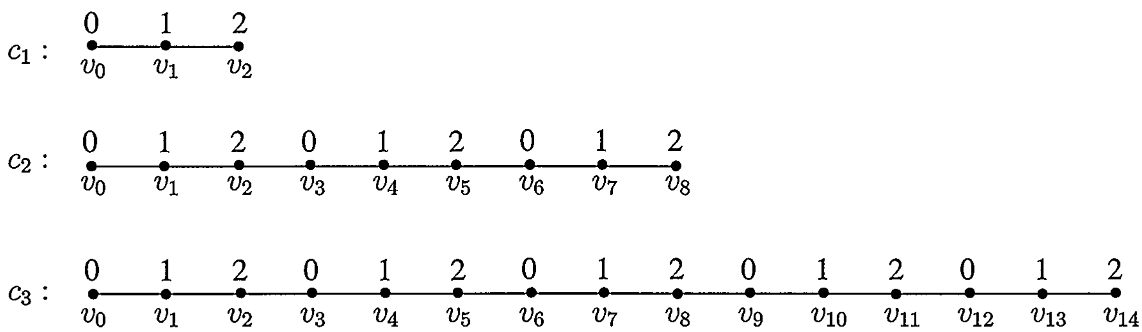

Figure 4.8: Colourings of initial segments

Suppose that $k \geq 3$ is odd and that c is a colouring of $C_k = v_0 v_1 \cdots v_{k-1} v_0$ that is not distinguishing. Then any nontrivial colour preserving automorphism $g \in Aut(C_k)$ must be a rotation, since a reflection would require a pair of adjacent vertices $u, v \in V(C_k)$ to have $g(u) = v$ and $g(v) = u$ which is impossible since $c(u) \neq c(v)$ and g preserves colours.

Theorem 4.4.1. *Let k be odd, $k = p_1^{j_1} p_2^{j_2} \dots p_q^{j_q}$, for distinct primes $p_i > 2$ and $j_i \in \mathbb{N}$. Then*

$$\xi_r(C_k) = \frac{P(C_k, r) - \left|\bigcup_{i=1}^{q} \mathcal{R}_{k,r}^{p_i}\right|}{2k}.$$

Proof. Let $C_k = v_0 v_1 \cdots v_{k-1} v_0$, and suppose c is a non-distinguishing r-colouring of C_k. Then $c \in \mathcal{R}_{k,r}^{p}$ for some p such that $p|k$ and $1 < p < k$. Thus the set of non-distinguishing r-colourings of C_k can be described by

$$\bigcup_{\substack{p|k \\ 1<p<k}} \mathcal{R}_{k,r}^{p}.$$

Suppose $p = p_1^{t_1} \times p_2^{t_2} \times \cdots \times p_q^{t_q}$, $0 \leq t_i \leq j_i$ and $1 \leq i \leq q$. Then, as a consequence of Corollary 4.4.1

$$\bigcup_{\substack{p|k \\ 1<p<k}} \mathcal{R}_{k,r}^{p} = \bigcup_{i=1}^{q} \mathcal{R}_{k,r}^{p_i}.$$

Since the total number of r-colourings of C_k is $P(C_k, r)$, the number of distinguishing r-colourings of C_k is

$$P(C_k, r) - \left|\bigcup_{i=1}^{q} \mathcal{R}_{k,r}^{p_i}\right|. \tag{4.1}$$

To find the number of distinct distinguishing colourings of C_k we divide by $2k$, the order of the dihedral group D_k. Therefore

$$\xi_r(C_k) = \frac{P(C_k, r) - \left|\bigcup_{i=1}^{j} \mathcal{R}_{k,r}^{p_i}\right|}{2k}.$$

□

By using $|\mathcal{R}_{k,r}^{p}| = P(C_{k/p}, r)$, Lemma 4.3.2, and the Principle of Inclusion-Exclusion, one can expand (4.1) and obtain an expression for $\xi_r(C_k)$ in terms of the chromatic polynomial of cycles. This can be evaluated to give the exact value of $\xi_r(C_k)$.

Example 4.4.2. In this example we count the number of distinct distinguishing r-colourings of C_k, where $k = 3^2 \cdot 5 = 45$.

To find $\xi_r(C_{45})$ we use the result from Theorem 4.4.1, and obtain

$$\begin{aligned}
\xi_r(C_{45}) &= \frac{P(C_{45}, 4) - \left|\mathcal{R}_{45,4}^{3} \bigcup \mathcal{R}_{45,4}^{5}\right|}{2k} \\
&= \frac{P(C_{45}, 4) - \left(\left|\mathcal{R}_{45,4}^{3}\right| + \left|\mathcal{R}_{45,4}^{5}\right| - \left|\mathcal{R}_{45,4}^{3} \bigcap \mathcal{R}_{45,4}^{5}\right|\right)}{2(45)} \\
&= \frac{P(C_{45}, 4) - \left(\left|\mathcal{R}_{45,4}^{3}\right| + \left|\mathcal{R}_{45,4}^{5}\right| - \left|\mathcal{R}_{45,4}^{15}\right|\right)}{2(45)} \\
&= \frac{P(C_{45}, r) - (P(C_{15}, r) + P(C_9, r) - P(C_3, r))}{90}
\end{aligned}$$

Before finding an expression for the number of distinct distinguishing colours for a cycle of even length, it is necessary to first characterize non-distinguishing r-colourings of even length cycles that are preserved by reflection through a line containing antipodal vertices of the cycle, in addition to being preserved by a nontrivial rotation. Note that for such an r-colouring, we may assume, without loss of generality, that the vertices of C_{2k} are labelled so that there is a line of reflection through v_0 and v_k.

Let $C_{2k} = v_0 v_1 \cdots v_{2k-1} v_0$ and suppose p is an integer such that $1 < p \leq k$ and such that $p|k$. We define $\mathcal{F}^p_{2k,r}$ as the set of all r-colourings of C_{2k} for which

$$c(v_i) = c(v_{-i})$$
$$c(v_i) = c(v_{i+2k/p}),$$

(throughout, subscripts are taken modulo $2k$). Then $\mathcal{F}^p_{2k,r}$ consists of all r-colourings of C_{2k} that are preserved by rotation through $\frac{2k}{p}$ and by reflection in the line containing v_0 and v_k.

A colouring c in $\mathcal{F}^p_{2k,r}$ is determined by the colours of the vertices on the path $v_0 v_1 \ldots v_{\frac{k}{p}}$. The reasons for this are twofold. First suppose $v_0, v_1, \ldots v_{\frac{k}{p}}$ have been coloured (i.e., labeled so that adjacent vertices have different colours). Since there is a reflection in the line containing v_0 and v_k it follows that

$$\begin{aligned} c(v_{2k-1}) &= c(v_1) \\ c(v_{2k-2}) &= c(v_2) \\ \vdots \quad &= \quad \vdots \\ c(v_{(2k-\frac{k}{p})+2}) &= c(v_{\frac{k}{p}-2}) \\ c(v_{(2k-\frac{k}{p})+1}) &= c(v_{\frac{k}{p}-1}) \\ c(v_{(2k-\frac{k}{p})}) &= c(v_{\frac{k}{p}}) \end{aligned}$$

Thus the colours of $v_{2k-\frac{k}{p}}, v_{2k-\frac{k}{p}+1}, \ldots, v_{2k-1}$ are forced by the reflection. Furthermore, since c is preserved by rotation through $\frac{2k}{p}$, it follows that.

$$\begin{aligned}
c(v_{\frac{k}{p}+1}) &= c(v_{(2k-\frac{k}{p})+1}) = c(v_{\frac{k}{p}-1}) \\
c(v_{\frac{k}{p}+2}) &= c(v_{(2k-\frac{k}{p})+2}) = c(v_{\frac{k}{p}-2}) \\
c(v_{\frac{k}{p}+3}) &= c(v_{(2k-\frac{k}{p})+3}) = c(v_{\frac{k}{p}-3}) \\
\vdots \quad &= \quad \vdots \quad = \quad \vdots \\
c(v_{\frac{2k}{p}-1}) &= \quad c(v_{2k-1}) \quad = c(v_1).
\end{aligned}$$

Now we have coloured $v_0, v_1, \ldots, v_{\frac{2k}{p}-1}$. Since the colouring is preserved by rotation through $\frac{2k}{p}$, it follows by induction that for each i, $0 \leq i \leq 2k-1$,

$$c(v_i) = c(v_{i+j\frac{2k}{p}}),$$

for all $j \in \mathbb{Z}$, where subscripts are taken modulo $2k$. Following our earlier definition, the intial segment of C_{2k}, with respect to c is the path $v_0 v_1 \cdots v_{\frac{2k}{p}-1}$.

Lemma 4.4.2. *Let $\mathcal{F}^p_{2k,r}$ denote the set of all r-colourings of C_{2k} that are preserved by rotation through $\frac{2k}{p}$ for some $1 < p < k$, and a reflection in the line through v_0 and v_k. If $1 < p' < p$ and $p'|p$, then $\mathcal{F}^p_{2k,r} \subseteq \mathcal{F}^{p'}_{2k,r}$.*

Proof. Let $p = qp'$ where $q > 1$, and let $c \in \mathcal{F}^p_{2k,r}$. Then $c(v_i) = c(v_{i+\frac{2k}{p}})$ for all i, $0 \leq i \leq 2k-1$ and $c(v_i) = c(v_{-i})$ (throughout, subscripts are taken modulo $2k$). This implies that $c(v_i) = c(v_{i+j\frac{2k}{p}})$ for all j, $j \in \mathbb{Z}$. In particular, for $j = q$, we have

$$c(v_0) = c(v_{i+q\frac{2k}{p}}) = c(v_{i+\frac{2k}{p'}})$$

Since $c \in \mathcal{F}^p_{2k,r}$, $c(v_i) = c(v_{-i})$ for all i, $0 \leq i \leq 2k-1$, where subscripts are taken modulo $2k$. Therefore, $c \in \mathcal{F}^{p'}_{2k,r}$.

□

A key point from the proof of Lemma 4.4.2 is that, independent of the value of p, any $c \in \mathcal{F}^p_{2k,r}$ has the property that $c(v_i) = c(v_{-i})$.

Corollary 4.4.2. *Let $\mathcal{F}^p_{2k,r}$ denote the set of all r-colourings of C_{2k} that are preserved by rotation through $\frac{2k}{p}$ for some $1 \leq p \leq k$, and by reflection in the line containing v_0 and v_k. If $p = p_1p_2$ for $1 < p_i < p, i \in \{1,2\}$ then $\mathcal{F}^{p_1}_{2k,r} \cap \mathcal{F}^{p_2}_{2k,r} = \mathcal{F}^p_{2k,r}$.*

Proof. By Lemma 4.4.1, $\mathcal{F}^p_{2k,r} \subseteq \mathcal{F}^{p_1}_{2k,r}$. Similarly $\mathcal{F}^p_{2k,r} \subseteq \mathcal{F}^{p_2}_{2k,r}$. Therefore

$$\mathcal{F}^p_{2k,r} \subseteq \mathcal{F}^{p_1}_{2k,r} \cap \mathcal{F}^{p_2}_{2k,r}$$

.

Now suppose $c \in \mathcal{F}^{p_1}_{2k,r} \cap \mathcal{F}^{p_2}_{2k,r}$. First note that

$$c(v_i) = c(v_{-i}).$$

Secondly,

$$c(v_i) = c(v_{i+j\frac{2k}{p_1}}) = c(v_{i+m\frac{2k}{p_2}}).$$

For any $m, j \in \mathbb{Z}$ (all subscripts are taken modulo $2k$).This then implies that

$$\begin{aligned} c(v_i) &= c(v_{i+j\frac{2k}{p_1}+m\frac{2k}{p_2}}) \\ &= c(v_{i+\frac{m2k+j2k}{p_1p_2}}) \\ &= c(v_{i+(m+j)\frac{2k}{p_1p_2}}) \end{aligned}$$

Since $(m+j) \in \mathbb{Z}$, $c \in \mathcal{F}^p_{2k,r}$, and thus $\mathcal{F}^{p_1}_{2k,r} \cap \mathcal{F}^{p_2}_{2k,r} \subseteq \mathcal{F}^p_{2k,r}$. We conclude that $\mathcal{F}^{p_1}_{2k,r} \cap \mathcal{F}^{p_2}_{2k,r} = \mathcal{F}^p_{2k,r}$ □

Remark 4.4.2. Notice that $|\mathcal{F}^p_{2k,r}| = P(P_{k/p+1}, r)$. A colouring $c \in \mathcal{F}^p_{2k,r}$ is determined by $c(v_i), 0 \leq i \leq (k/p)$. As a result, there is a one-to-one correspondence between elements of $\mathcal{F}^p_{2k,r}$ and r-colourings of $P_{\frac{k}{p}+1}$.

Example 4.4.3. Recall that a colouring in $\mathcal{F}^p_{2k,r}$ is determined by the colours on vertices $v_0, v_1, \ldots, v_{\frac{k}{p}}$. These colours then extend to a colouring of the initial segment

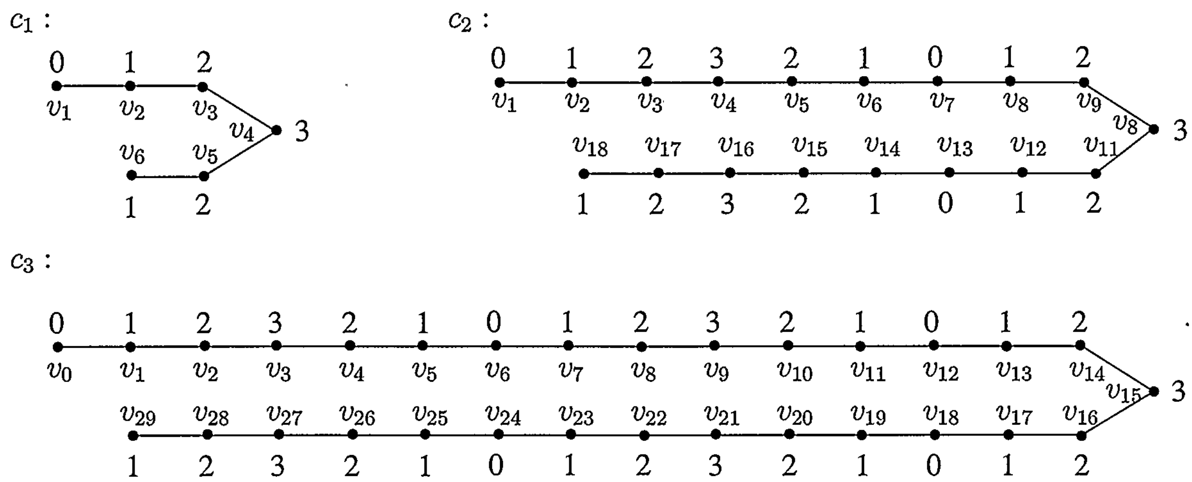

Figure 4.9: Colourings of initial segments

$v_0v_1\cdots v_{\frac{2k}{p}-1}$. Consider $2k = 2\cdot 3^2\cdot 5 = 90$ and $r = 4$. Let $c_1 \in \mathcal{F}^{15}_{90,4}$, $c_2 \in \mathcal{F}^{3}_{90,4}$, and $c_3 \in \mathcal{F}^{2}_{90,4}$. The initial segments of c_1, c_2, and c_3 are shown in Figure 4.9. Here we consider a particular case to illustrate the Corollary 4.4.2.

Despite the initial segments for c_1, c_2 and c_3 having different lengths, each c_i, $1 \le i \le 3$ produces the same colouring of C_{90}.

Theorem 4.4.2. *Let $2k = 2^{j_0}p_1^{j_1}p_2^{j_2}\cdots p_q^{j_q}$, for prime $p_i > 2$ and $j_i \in \mathbb{N}$. Then,*

$$\xi_r(C_{2k}) = \frac{P(C_{2k}, r) - \left|\bigcup_{i=0}^{q} \mathcal{R}^{p_i}_{2k,r}\right| - P(P_{k+1}, r) + \left|\bigcup_{i=1}^{j} \mathcal{F}^{p_i}_{2k,r}\right|}{4k}.$$

Proof. Let $C_{2n} = v_0v_1\ldots v_{2k-1}v_0$ and suppose that c is a non-distinguishing r-colouring of C_{2k}. Then c is preserved by a rotation, a reflection or both. If c is preserved by a rotation, then $c \in \mathcal{R}^{p}_{2k,r}$ for some $p|2k$ and $1 < p < 2k$. If, in addition, c is preserved by reflection, then $c \in \mathcal{F}^{p}_{2k,r}$. If c is preserved by a reflection then, without loss of generality we may assume that there is a line of reflection through v_0 and v_k and that

$$c(v_i) = c(v_{-i})$$

for all i, $1 \le i \le k-1$ and $k+1 \le i \le 2k-1$ (subscripts taken modulo $2k$). This is equivalent to the number of ways to r-colour the path $P_{k+1} = v_0v_1\ldots v_k$ and reflect the colours on the line through v_0 and v_k. The number of r-colourings of P_{k+1} is simply $P(P_{k+1}, r)$. Using inclusion-exclusion, the number of r-colourings of C_{2k} that are not distinguishing is

$$\left|\bigcup_{\substack{p|2k \\ 1<p<2k}} \mathcal{R}^p_{2k,r}\right| + P(P_{k+1}, r) - \left|\bigcup_{\substack{p|2k \\ 1<p<2k}} \mathcal{F}^p_{2k,r}\right|, \tag{4.2}$$

In the first and last terms of this expression, we know that if $p|2k$, then $p = 2^{t_0} p_1^{t_1} p_2^{t_2} \cdots p_q^{t_q}$ for some $0 \leq t_i \leq j_i$ and $0 \leq i \leq q$; to simplify notation, set $p_0 = 2$ so that $p = p_0^{t_0} p_1^{t_1} p_2^{t_2} \cdots p_q^{t_q}$. As a consequence of Lemma 4.4.1 and Lemma 4.4.2, it follows that

$$\left|\bigcup_{\substack{p|2k \\ 1<p<2k}} \mathcal{R}^p_{2k,r}\right| = \left|\bigcup_{i=0}^{q} \mathcal{R}^{p_i}_{2k,r}\right|$$

and

$$\left|\bigcup_{\substack{p|2k \\ 1<p<2k}} \mathcal{F}^p_{2k,r}\right| = \left|\bigcup_{i=0}^{q} \mathcal{F}^{p_i}_{2k,r}\right|$$

To find the number of distinct distinguishing colourings of C_{2k} we divide by $4k$, the order of the dihedral group D_{2k}. It follows that

$$\xi_r(C_{2k}) = \frac{P(C_{2k}, r) - \left|\bigcup_{i=0}^{q} \mathcal{R}^{p_i}_{2k,r}\right| - P(P_{k+1}, r) + \left|\bigcup_{i=0}^{q} \mathcal{F}^{p_i}_{2k,r}\right|}{4k}. \tag{4.3}$$

□

It follows from Remark 4.4.1 and Remark 4.4.2 that

$$|\mathcal{R}^p_{2k,r}| = P(C_{2k/p}, r)$$

and

$$|\mathcal{F}^p_{2k,r}| = P(P_{k/p}, r).$$

Also, Lemma 4.3.2 gives an explicit formula for $P(C_{2k/p}, r)$ in terms of r, k, and p; similarly Lemma 4.3.1 gives an explicit formula for $P(P_{k/p}, r)$ in terms of r, k, and p. Combining these and using the Principle of Inclusion-Exclusion enables us to express $\xi_r(C_{2k})$ in terms of chromatic polynomials of paths and cycles. This expression can then be evaluated to give an exact value of $\xi_r(C_{2k})$.

Example 4.4.4. In this example we count the number of distinct distinguishing r-colourings of C_{2k}, where $2k = 2 \cdot 3^2 \cdot 5 = 90$.

To find $\xi_r(C_{90})$ we use the result of Theorem 4.4.2 and obtain

$$
\begin{aligned}
|D_{90}| \times \xi_r(C_{90}) &= P(C_{90}, r) - \left(\left|\mathcal{R}^2_{90,r} \cup \mathcal{R}^3_{90,r} \cup \mathcal{R}^5_{30,r}\right|\right) - P(P_{46}, r) + \left|\mathcal{F}^3_{90,r} \cup \mathcal{F}^5_{30,r}\right| \\
&= P(C_{90}, r) \\
&\quad - \left(\left|\mathcal{R}^2_{90,r}\right| + \left|\mathcal{R}^3_{90,r}\right| + \left|\mathcal{R}^5_{90,r}\right|\right) \\
&\quad - \left(-\left|\mathcal{R}^2_{90,r} \cap \mathcal{R}^3_{90,r}\right| - \left|\mathcal{R}^2_{90,r} \cap \mathcal{R}^5_{90,r}\right| - \left|\mathcal{R}^3_{90,r} \cap \mathcal{R}^5_{90,r}\right|\right) \\
&\quad - \left(+\left|\mathcal{R}^2_{90,r} \cup \mathcal{R}^3_{90,r} \cup \mathcal{R}^5_{90,r}\right|\right) \\
&\quad - P(P_{46}, r) + \left(\left|\mathcal{F}^3_{90,r}\right| + \left|\mathcal{F}^5_{90,r}\right| - \left|\mathcal{F}^3_{90,r} \cap \mathcal{F}^5_{90,r}\right|\right) \\
&= P(C_{90}, r) \\
&\quad - \left(\left|\mathcal{R}^2_{90,r}\right| + \left|\mathcal{R}^3_{90,r}\right| + \left|\mathcal{R}^5_{90,r}\right| - \left|\mathcal{R}^6_{90,r}\right| - \left|\mathcal{R}^{10}_{90,r}\right| - \left|\mathcal{R}^{15}_{90,r}\right| + \left|\mathcal{R}^{45}_{90,r}\right|\right) \\
&\quad - P(P_{46}, r) + \left(\left|\mathcal{F}^3_{90,r}\right| + \left|\mathcal{F}^5_{90,r}\right| - \left|\mathcal{F}^{15}_{90,r}\right|\right) \\
&= P(C_{90}, r) - P(C_{45}, r) - P(C_{20}, r) - P(C_{18}, r) \\
&\quad + P(C_{15}, r) + P(C_9, r) + P(C_6, r) - P(C_2, r) - P(P_{46}, r) \\
&\quad + P(P_{15}, r) + P(P_9, r) - P(P_3, r)
\end{aligned}
$$

This equation can now be used to find exact values of $\xi_r(c_{90})$ for any r.

Chapter 5

Summary and Future Work

5.1 Summary

The notion of distinguishing colourings of a graph was brought to us by Collins and Trenk [12], who then provide preliminary results about the distinguishing chromatic number of particular families of graphs. They show that, for $n \geq 7$, the distinguishing chromatic number of C_n is

$$\chi_D(C_n) = 3.$$

In addition to this, Collins and Trenk also provide an upper bound for the distinguishing chromatic number of a tree, and prove that, for any tree T,

$$\chi_D(T) \leq \Delta(T) + 1.$$

Extending the result from trees to connected graphs, they also prove that for any connected graph G,

$$\chi_D(G) \leq 2\Delta(G).$$

Laflamme and Seyffarth [17] concentrate their investigation of the distinguishing chromatic number to a particular type of connected graph, connected bipartite graphs. For G a connected bipartite graph with $\Delta(G) \geq 3$, they prove that

$$\chi_D(D) \leq 2\Delta - 2 \text{ unless } G \in \{K_{\Delta,\Delta-1}, K_{\Delta,\Delta}\}.$$

This is indeed the best possible upper bound since there exists infinitely many graphs G for which

$$\chi_D(D) = 2\Delta - 2.$$

Tang [20] provides insights into the distinguishing chromatic number of the wreath product of two graphs. Specifically, Tang determines exactly the distinguishing chromatic number for the wreath product of a cycle with an independent set of vertices. He proves that for all $n \geq 3$ and $m \geq 1$,

$$\chi_D(C_n[I_m]) = \begin{cases} 3m & n = 3, \\ 2m+2 & n = 4, \\ \lceil m/\binom{n-1}{2} \rceil + 2m & n \geq 5 \text{ and odd}, \\ 2m+1 & n \geq 6 \text{ and even}. \end{cases}$$

Throughout this thesis we use and extend the results of our predecessors. We begin by determining the distinguishing chromatic number of the wreath product of a cycle with a connected graph. Using two cases, we show that for all n even, $n \geq 4$, and any connected graph G,

$$\chi_D(C_n[G]) = \begin{cases} 2\chi_D(G) & \text{if } \xi_{\chi_D(G)}(G) \geq 2, \\ 2\chi_D(G)+1 & \text{if } \xi_{\chi_D(G)}(G) = 1 \text{ and } n \geq 6, \\ 2\chi_D(G)+2 & \text{if } \xi_{\chi_D(G)}(G) = 1 \text{ and } n = 4. \end{cases}$$

Secondly we prove that for all odd $n \geq 3$ and any connected graph G,

$$\chi_D(C_n[G]) = 2\chi_D(G) + \left\lceil \chi_D(G)/\binom{n-1}{2} \right\rceil.$$

We then investigate the distinguishing chromatic number of the wreath product of a tree and a connected graph. We prove that

$$\chi_D(T[G]) \leq r + p,$$

where r is the smallest integer such that

$$\xi_r(G) \geq \max\{d(v_0), d(v_1) - 1, d(v_2) - 1, \ldots, , d(v_{n-1}) - 1\},$$

and p is the smallest integer such that $\xi_p(G) \geq \max\{d(u_i) \mid 1 \leq i \leq m-1\}$. Since the results are dependent on $\xi_r(G)$ it is desirable to determine values of $\xi_r(G)$ for particular graphs G.

We obtain results about $\xi_r(G)$ when G is a complete graph, an independent set, a cycle, or a tree. We find a recursive method to count the number of distinct distinguishing colourings of a rooted tree, a result that was proved independently by Cheng [9]. We then prove that for a tree, T, with a single vertex w as its center, and any $r \in \mathbb{N}$,

$$\xi_r(T) = r \prod_{i=1}^{n[T(w)]} \binom{\xi_r(T_i)(r-1))/r}{|T_i|}.$$

On the other hand, if T is a tree with adjacent centers u and v, and any $r \in \mathbb{N}$, then

$$\xi_r(T) = \begin{cases} \frac{r-1}{r}\xi_r(T(u))\xi_r(T(v)) \text{ if } T(u) \text{ and } T(v) \text{ are not isomorphic rooted trees,} \\ \frac{r-1}{2r}\xi_r(T(u))\xi_r(T(v)) \text{ if } T(u) \text{ and } T(v) \text{ are isomorphic rooted trees} \end{cases}$$

(where $T(u)$ and $T(v)$ are the rooted trees obtained from T by deleting edge uv).

Finally, we determine the number of distinct distinguishing colourings of a cycle using the chromatic polynomial. The case where the cycles has even length is more complicated than the case where the cycle have odd length. We prove that if k is odd, $k = p_1^{j_1} p_2^{j_2} \ldots p_q^{j_q}$, for primes $p_i > 2$ and $j_i \in \mathbb{N}$, then

$$\xi_r(C_k) = \frac{P(C_k, r) - \left|\bigcup_{i=1}^{q} \mathcal{R}_{k,r}^{p_i}\right|}{2k}.$$

On the other hand, if the cycle has even length $2k$ and $2k = 2^{j_0} p_1^{j_1} p_2^{j_2} \cdots p_q^{j_q}$, for prime $p_i > 2$ and $j_i \in \mathbb{N}$, then

$$\xi_r(C_{2k}) = \frac{P(C_{2k}, r) - \left|\bigcup_{i=0}^{q} \mathcal{R}_{2k,r}^{p_i}\right| - P(P_{k+1}, r) + \left|\bigcup_{i=1}^{j} \mathcal{F}_{2k,r}^{p_i}\right|}{4k}.$$

Here $\mathcal{R}_{k,r}^{p}$ is the set of all r-colourings of C_k that are preserved by a rotation through $\frac{k}{p}$, and $\mathcal{F}_{2k,r}^{p}$ is the set of all r-colourings of C_{2k} that are preserved by rotation through $\frac{2k}{p}$ and by reflection in the line containing v_0 and v_k.

In the thesis we provide an additional use for number of distinct distinguishing colourings of a graph, applying said number to obtain the distinguishing chromatic number of the wreath product of graphs. We also broaden the work previously done on the distinguishing chromatic number of the wreath product of graphs, further progressing this area of study.

5.2 Future Work

It seems reasonable to believe one can obtain an upper bound of $\chi_D(H[G])$, where H is any bipartite graph. It may be possible to prove that for all $g \in Aut(H[G])$ and all $u \in V(H)$, that $g(G_u) = G_v$ for some $v \in V(H)$. If this can be proven, then, in conjunction with the work of Laflamme and Seyffarth [17] one would be able to provide an upper bound on $\chi_D(H[G])$, using methods similar to those in Chapter 3.

Throughout the thesis, we relied on the fact that $g \in Aut(H[G])$ mapped copies of G to copies of G, which invites the question of when exactly this is true. It is certainly not true in general. For example, consider automorphisms of $K_m[K_n]$. It may be possible to prove that this quality fails (copies of G map to copies of G) only when there exists a $v \in G$ so that $d_G(v) = |V(G)| - 1$. This constraint will allow vertices within a single copy of G to be mapped to different copies of G while maintaining distance within $H[G]$.

It may also be of interest to investigate values of $\chi_D(H[G])$ where G is not a connected graph. Little is known in this case aside from the work of Tang [20], which only covers the most extreme instance, when G is an independent set.

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