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A Modern Introduction to Algebraic Goodwillie Calculus

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A Modern Introduction to Algebraic Goodwillie Calculus

by

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Abstract

In this thesis we will investigate the traditional algebraic Goodwillie calculus of Johnson and McCarthy (see [JM03]) from a more modern perspective. In particular, in the recent paper [BJM11], a new perspective on Goodwillie calculus in categories without a basepoint is given using modern machinery such as model categories, and homotopy limits and colimits. This thesis demonstrates that with relatively few modifications this language may be applied in the traditional setting to recover the same constructions found in [JM03]. As a final example, we investigate a result in [KM02] which demonstrates how this language may be applied to recover André-Quillen homology as the first derivative of a particular functor.

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Chapter 1

Introduction

Goodwillie calculus, also known as the Calculus of Functors, was originally developed by Thomas Goodwillie in a series of three paper during the 1990's and early 2000's (see [Goo90], [Goo92], and [Goo03]). In these papers, Goodwillie develops a construction dubbed the Taylor tower of a functor which in some sense approximates functors between nice homotopical categories such as topological spaces or spectra. While interesting in their own right, these papers are not the subject of this thesis.

The two papers of primary interest to us are [JM03] and [BJM11]. The former paper, authored by Johnson and McCarthy, develops a similar calculus of functors but in a much more algebraic setting. In particular, they concern themselves with functors between categories of chain complexes, and give a tower construction for such functors, analogous to the one given by Goodwillie. The way the tower is defined is markedly different from how the tower in the classical case is given. In the latter paper by Bauer, Johnson, and McCarthy, a third construction is given which is capable of recovering the classical calculus of Goodwillie in special cases, but looks more like a modernized version of algebraic calculus. In this thesis we show that this is exactly the case.

The primary contribution of this thesis is that we clearly articulate the relationship between based and unbased algebraic Goodwillie calculus. In developing the unbased algebraic Goodwillie calculus in this thesis (Chpater 4), we make use of the ideas from [BJM11] so as to make the relationship between the two settings evident. By taking this approach, we obtain a dictionary which may be applied to explicitly translate statements and theorems in the based setting into the language of the unbased setting. The utility of algebraic Goodwillie calculus is that it is accessible to algebraists with little knowledge of homotopy theoretic techniques.

This thesis is useful because it provides techniques to apply the unbased calculus in the algebraic setting.

In terms of required background, we assume the reader is familiar with basic category theory (limits, colimits, functors, natural transformations, etc.), homological algebra (chain complexes, homology, resolutions), and basic algebraic topology (homotopy, singular homology, homotopy groups). Additional topics will be developed in the first two chapters.

In Chapter 2 we will give a summary of the most important results concerning model categories and simplicial model categories. The topics covered are for the most part directly relevant to the topics of this thesis. For the reader who is familiar with these topics, this chapter may be skipped and referred to on an as-needed basis only. For the reader who seeks more depth, there are many existing references which cover these topics. The author is particularly fond of [DS95] for an introduction to model categories, and [Hir03] for a more comprehensive overview.

Chapter 3 provides more background information, though now more technical and more particular to this thesis. Four topics are covered: homotopy (co)limits, cubical diagrams, comonads, and André-Quillen homology. These topics are again explained in just enough detail as to make the remaining chapters clear. Additional references for each of the topics are given in the appropriate chapters.

Algebraic Goodwillie Calculus is developed in Chapter 4 and it is here where the original contributions of this thesis are found. In this chapter we will develop cross effects and the algebraic tower using a model based on ideas from [BJM11]. The major result of this section is Theorem 4.3.4, in which we demonstrate that this new model is in fact equivalent to the original model in [JM03]. Supporting this result, we also prove Proposition 4.2.6.

Finally, in Chapter 5 we demonstrate as an example how André-Quillen homology may be recovered as the first derivative of a certain functor. This section is based on the ideas from [KM02]. We take the time to expand on some details which were glossed over in the original paper, as well as prove some results whose proofs were omitted (for example, Lemma 5.2.2).

One final note: in this thesis, we adopt the convention of omitting punctuation after diagrams and display mode mathematics. The presence of punctuation in such situations is debated in the mathematical community, and we have opted to follow the reasoning of Allen Hatcher who writes, "I think periods and commas in display mode are so ugly that they should never be used. Display mode is something removed from text mode, in another dimension as it were, so vestiges of text mode like punctuation should never appear in display mode" [Hat].

And so we begin!

Chapter 2

Simplicial Model Categories

The bulk of this thesis will require a familiarity with model categories and simplicial categories. This chapter aims to provide a quick summary of the important results in the area. The reader who is already acquainted with these topics may skip this chapter and refer to it on an as-needed basis only. All of the material in this chapter is well-established, and citations may be found throughout for the reader who is interested in a more thorough treatment.

2.1 Model Categories

Model categories, also known as closed model categories or Quillen closed model categories, were developed by Dan Quillen in the late 1960s ([Qui67]). A model structure on a category $\mathbf{C}$ is nothing more than a specification of three distinguished classes of morphisms. However, the axioms which define these classes allow one to carry over many of the ideas about homotopy in spaces to $\mathbf{C}$. Is essence, a model structure allows one to 'do homotopy theory' in a category.

More recently, the theory of ∞-categories or quasicategories may be seen as a generalization of model categories. In the same way, in an ∞-category one has notions of homotopy and many of the ideas of classical homotopy theory carry over. However, this construction, while far more powerful, is also suitably more complex. We will not have need for this machinery, and so we shall content ourselves with the more mundane notion of a model category. A substantial treatment of ∞-categories may be found in [Lur12].

Definition 2.1.1. Let $\mathbf{C}$ be a category. A *model structure* on $\mathbf{C}$ consists of three distinguished classes of morphisms: fibrations, cofibrations, and weak equivalences. Each of these classes is required to contain all identity maps and be closed under composition. A map which is both a weak equivalence and a (co)fibration is known as an *acyclic* or *trivial* (co)fibration. These classes constitute a model structure if the following five axioms hold:

- MC1 The category $\mathbf{C}$ has all finite limits and colimits.
- MC2 (2-out-of-3 axiom) If $f\colon X \to Y$ and $g\colon Y \to Z$ are such that any two of f, g, and $g \circ f$ are weak equivalences, then so is the third.
- MC3 (Retract axiom) If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then f is the same. Note that $f\colon X \to Y$ is a retract of $g\colon A \to B$ by definition if there are morphisms i_1, i_2, r_1, and r_2 making the following diagram commute:

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_X}{\frown} & & \\ X & \xrightarrow{i_1} & A & \xrightarrow{r_1} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i_2} & B & \xrightarrow{r_2} & Y \\ & & \underset{\mathrm{id}_Y}{\smile} & & \end{array} \tag{2.1.1}$$

- MC4 (Lifting axiom) Suppose we have a commuting square:

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \tag{2.1.2}$$

  Then a *lift* $f\colon B \to X$ making the diagram of five arrows commute exists whenever one of the following conditions holds:

  (a) The map i is an acyclic cofibration and p is a fibration,

  (b) The map i is a cofibration and p is an acyclic fibration.

- MC5 (Factorization axiom) Any map $f\colon X \to Y$ in $\mathbf{C}$ may be factored in two ways:

  (a) $f = p \circ i$ where i is an acyclic cofibration and p is a fibration,

(b) $f = p \circ i$ where i is a cofibration and p is an acyclic fibration.

A category equipped with a model structure is known as a *model category*.

Although many of these axioms appear at first sight to be unhelpful, they turn out to provide enough structure to reprove many classic theorems from topology, such as the homotopy lifting or homotopy extension theorems (see [Hat01]). We will see in Example 2.1.7, topological fibrations, cofibrations, and weak equivalences satisfy these axioms.

Note 2.1.2. The axiom MC5 says that all morphisms may be factored in two different ways, but it does *not* say that the factorizations are functorial. Hence these factorizations need not be compatible in any way. However, in almost every interesting example of a model category the factorizations are functorial, and we will make the assumption that all model categories have functorial factorization. Many sources (for example [Hir03]) include functoriality in the statement of MC5 because of this.

Note 2.1.3. If $\mathbf{C}$ is a model category, then by MC1 it has an initial object (which we will denote $\emptyset$) and a terminal object (which we will denote $*$). An object X of $\mathbf{C}$ is said to be *fibrant* if the map $X \to *$ is a fibration, and X is said to be *cofibrant* if $\emptyset \to X$ is a cofibration.

Definition 2.1.4. Let $\mathbf{C}$ be a model category and X be an object of $\mathbf{C}$. A *fibrant replacement* of X is an object X' such that there is an acyclic cofibration $X \to X'$ and X' is fibrant. A *cofibrant replacement* of X is an object X'' such that there is an acyclic fibration $X'' \to X$ and X'' is cofibrant.

By MC5, fibrant and cofibrant replacements always exist, and MC4 together with MC2 implies that any two fibrant (resp. cofibrant) replacements are weakly equivalent. As we are making the assumption that MC5 includes functorial factorization, we thus obtain a cofibrant replacement functor $Q\colon \mathbf{C} \to \mathbf{C}$ and a fibrant replacement functor $R\colon \mathbf{C} \to \mathbf{C}$. These functors are unique up to natural isomorphism.

Note 2.1.5. In an attempt to not draw diagrams for MC4, we will use the following (standard) terminology. If a lift exists in (2.1.2), then we will say that i has the left lifting property (LLP) with respect to p, and that p has the right lifting property (RLP) with respect to i. In this terminology, MC4 states that cofibrations have the LLP with respect to acyclic fibrations, and fibrations have the RLP with respect to acyclic cofibrations.

Note 2.1.6. It is an easy exercise (or see [DS95, Proposition 3.13]) to prove that the lifting properties of MC4 exactly characterize the four classes fibrations, cofibrations, acyclic fibrations, and acyclic cofibrations. That is (for example) a map is a cofibration iff it has the LLP with respect to acyclic fibrations. A corollary to this is that when giving a model structure, it is enough to specify the weak equivalences and the cofibrations, or the weak equivalences and the fibrations, and the remaining class is uniquely determined by lifting properties.

Some examples of model categories are of course in order.

Example 2.1.7. The standard example (and motivating example) of a model category is the category **Top** of topological spaces. The standard model structure says that a map $f\colon X \to Y$ is a...

1. ...weak equivalence if it is a weak homotopy equivalence for every choice of basepoint. That is, for all $x_0 \in X$, $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is a group isomorphism for $n > 0$ and a bijection of sets for $n = 0$.
2. ...fibration if it is a Serre fibration. That is, f has the RLP with respect to inclusions $D^n \times \{0\} \hookrightarrow D^n \times [0, 1]$, where D^n is the unit n-disk.
3. ...cofibration if it is a cofibration of spaces. A cofibration of spaces may be defined to be the maps with the LLP with respect to acyclic Serre fibrations. For example, if we are working with CW complexes, then the inclusion of any subcomplex is a cofibration.

Under this model structure, every space is fibrant, and a space is cofibrant iff it is a retract of a CW complex. A detailed proof that these classes satisfy the axioms of a model category may be found in [Hov00].

Example 2.1.8. The category of chain complexes is another category which has an existing notion of homotopy (chain homotopy). So, as one may expect, the category $\mathbf{Ch}_{\geq 0}(\mathbf{A})$ of bounded below chain complexes of objects in the abelian category $\mathbf{A}$ are a model category. In fact, there are two canonical model structures on this category though they are Quillen equivalent (see Definition 2.1.12). The *projective model structure* says that a morphism $f\colon C \to D$ is a...

1. ...weak equivalence if it is a quasi-isomorphism. That is, $f_*\colon H_*(C) \to H_*(D)$ is an isomorphism.
2. ...fibration if $f_n\colon C_n \to D_n$ is an epimorphism for $n > 0$.
3. ...cofibration if $f_n\colon C_n \to D_n$ is a monomorphism such that coker f_n is projective for all $n \geq 0$.

The other common model structure is the *injective model structure*, but we will not use it in this thesis. When talking about $\mathbf{Ch}_{\geq 0}(\mathbf{A})$ as a model category, we will always be referring to the projective model structure unless otherwise specified. That this constitutes a model structure is proven in [DS95].

In the category of spaces, we may form the homotopy category $\mathrm{Ho}(\mathbf{Top})$ by formally inverting all weak equivalences. The same may be done for any model category and we will generally write $\gamma\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ for the localization functor. The procedure by which the homotopy category of a general model category is constructed is rather technical, but the important result is that the homotopy category $\mathrm{Ho}(\mathbf{C})$ satisfies the property that a map f in $\mathbf{C}$ is a weak equivalence iff $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathbf{C})$. Moreover, it is indeed a localization with respect to the class of weak equivalences, meaning that any functor

$F\colon \mathbf{C} \to \mathbf{D}$ which takes weak equivalences to isomorphisms factors through the homotopy category. For an explicit construction of the homotopy category of a model category, we refer the reader to [DS95, §5].

Definition 2.1.9. Let $\mathbf{C}$ be a model category, and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor ($\mathbf{D}$ need not be a model category). The *left derived functor* of F is the right Kan extension (see [Lan98] for relevant definitions) of F along the localization $\gamma\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$, and is denoted $\mathbf{L}F$. The *right derived functor* of F is the left Kan extension of F along γ and is denoted $\mathbf{R}F$. If $\mathbf{D}$ is a model category as well with localization functor $\tilde{\gamma}\colon \mathbf{D} \to \mathrm{Ho}(\mathbf{D})$, then the *total left (resp. right) derived functor* of F is the left (resp. right) derived functor of $\tilde{\gamma} \circ F$.

Note the reversal of handedness of the definition. While unfortunate, it is standard and it is required for the terminology to be compatible with the familiar notion of a derived functor. If both $\mathbf{C}$ and $\mathbf{D}$ are model categories, $\mathbf{L}F$ and $\mathbf{R}F$ will refer to the total derived functors unless otherwise noted.

The following proposition gives us a sufficient condition for left/right derived functors to exist, and an explicit construction in this case.

Proposition 2.1.10. *Let $\mathbf{C}$ be a model category and $\mathbf{D}$ be any category, and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. Then:*

1. *If F is such that whenever f is a weak equivalence between cofibrant objects in $\mathbf{C}$ then $F(f)$ is an isomorphism in $\mathbf{D}$, then $\mathbf{L}F$ exists and may be given by:*

$$\mathbf{L}F(X) := F(QX)$$

   where Q denotes the cofibrant replacement functor.

2. *If F is such that whenever f is a weak equivalence between fibrant objects in*

$\mathbf{C}$ then $F(f)$ is an isomorphism in $\mathbf{D}$, then $\mathbf{R}F$ exists and may be given by:

$$\mathbf{R}F(X) := F(RX)$$

where R denotes the fibrant replacement functor.

Proof. [DS95, Proposition 9.3] □

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor between model categories, and suppose that G is right adjoint to F (we will indicate this relation by writing $F \dashv G$). A natural question is under what conditions this adjunction descends to the homotopy categories. This is answered in the following theorem (and proven in [DS95]):

Theorem 2.1.11. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor between model categories, and suppose $F \dashv G$, and let $\mathbf{L}F$ and $\mathbf{R}G$ denote their total derived functors. Then:*

1. *If F preserves cofibrations and G preserves fibrations, then we obtain an adjunction $\mathbf{L}F \dashv \mathbf{R}G$ on the homotopy categories.*
2. *If we further have that for every cofibrant object X of $\mathbf{C}$ and every fibrant object Y of $\mathbf{D}$, a map $X \to G(Y)$ is a weak equivalence in $\mathbf{C}$ iff its adjoint map $F(X) \to Y$ is a weak equivalence in $\mathbf{D}$, then the adjunction between $\mathbf{L}F$ and $\mathbf{R}G$ is an equivalence of categories.*

Definition 2.1.12. A pair of functors satisfying Condition 1 of Theorem 2.1.11 is said to be a *Quillen pair* and they form a *Quillen adjunction.* If the functors in addition satisfy Condition 2, then F and G are said to be a *Quillen equivalence* and the categories $\mathbf{C}$ and $\mathbf{D}$ are said to be *Quillen equivalent.*

In Example 2.1.8 we remarked that the injective and projective model structures on $\mathbf{Ch}_{\geq 0}(\mathbf{A})$ are Quillen equivalent. In Subsection 2.3 we will an example of two distinct categories which are Quillen equivalent (in contrast to the above example where it is the same category with two model structures which are Quillen equivalent).

2.2 Nice Model Categories

While the model category axioms on their own allow one to prove many very nice properties, they are insufficient for many purposes. In this section, we will introduce additional properties we often impose on our model categories which make them behave nicer in many situations. These conditions are very technical, but are also very useful for proving that a potential model category is indeed a model category.

Definition 2.2.1. Consider the following pullback diagram in some category $\mathbf{C}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{k} & W \end{array}$$

Let A and B be classes of maps in $\mathbf{C}$. Then A is *stable under base change* if whenever $k \in A$, it follows that $f \in A$ as well. The class A is said to be *stable under pullbacks along* B if whenever $k \in A$ and $h \in B$, then $f \in A$ as well. The terms *stable under cobase change* and *stable under pushouts along* B are defined dually.

Definition 2.2.2. A model category $\mathbf{C}$ is said to be *right proper* if the class of weak equivalences is stable under pullbacks along fibrations, and it is said to be *left proper* if the class of weak equivalences is stable under pushouts along cofibrations. A model category which is both left and right proper is said to be *proper*.

It is worth mentioning that in any model category, one may prove that the classes of fibrations and acyclic fibrations are stable under base change, and the classes of cofibrations and acyclic cofibrations are stable under cobase change (this is a relatively straightforward exercise using the model category axioms). We will see in Proposition 3.1.8 an example of when this property is useful. The categories $\mathbf{Top}$, $\mathbf{Ch}_{\geq 0}(\mathbf{A})$, and $\mathbf{sSet}$ (see Section 2.3) are all proper model categories.

The remaining properties we often desire are unfortunately more technical. We give a very brief summary of these properties, but the reader is encouraged to consult a reference such has [Hir03] or [Hov00] for a more complete treatment. Proofs will be omitted, but may be found in the aforementioned references.

The properties all hinge on our category and its objects being 'small' enough. In particular, small objects give rise to the *small object argument* which is a crucial tool for proving that many model categories satisfy axiom MC5, and its hypotheses occur in the statement of one of our properties.

Definition 2.2.3. A category $\mathbf{C}$ is said to be *locally small* if the collection of morphisms between any two objects is a proper set[1]. The category $\mathbf{C}$ is *small* if the collection of objects in the category form a proper set. The category $\mathbf{C}$ is (co)complete if it has all small (co)limits, where a small (co)limit is a (co)limit over a small category.

We next give the definition of a small object for completeness, though we will not need this definition for our purposes.

Definition 2.2.4. Let $\mathbf{C}$ be a (locally small) category with filtered colimits. An object X of $\mathbf{C}$ is *small* if there is a regular cardinal κ such that the functor $\mathrm{Hom}_{\mathbf{C}}(X, -)$ commutes with κ-filtered colimits.

Definition 2.2.5. A category $\mathbf{C}$ is said to be *locally presentable* if:

1. $\mathbf{C}$ is locally small and cocomplete,
2. There is a *set* of objects S such that every object of $\mathbf{C}$ is the colimit of objects in S (that is $\mathbf{C}$ is *generated* by S),
3. Every object of S is small.

[1]It is common in literature to refer to a locally small category simply as a category, as most categories people care about are locally small.

As mentioned earlier, the small object argument is of crucial importance. The lemma is again quite technical in nature and is explained more thoroughly in a reference such as [Hov00]. It is given as follows:

Theorem 2.2.6 (Small Object Argument)**.** *Let* $\mathbf{C}$ *be a cocomplete category, and let* I *be a set of morphisms in* $\mathbf{C}$ *such that the domains of the morphisms in* I *are small relative to transfinite compositions of pushouts of maps of* I *(the collection of such maps is commonly denoted* $\mathrm{cell}(I)$*). Then every morphism in* $\mathbf{C}$ *factors functorially as a map in* $\mathrm{cell}(I)$*, followed by a map which has the right lifting property with respect to* I*.*

If I is a set of morphisms which satisfies the hypotheses of the small object argument, then we say that I *admits the small object argument.* Note that every locally presentable category satisfies the hypotheses of the small object argument for any set I, as every object is small by hypothesis.

Definition 2.2.7. A model category $\mathbf{C}$ is said to be *cofibrantly generated* if there are sets I and J of maps in $\mathbf{C}$ such that:

1. both I and J admit the small object argument,
2. a morphism has the RLP with respect to J iff it is a fibration,
3. a morphism has the RLP with respect to I iff it is an acyclic fibration.

If $\mathbf{C}$ is cofibrantly generated and locally presentable, then it is said to be *combinatorial.*

By MC5, it follows that every map in I is a cofibration, and every map in J is an acyclic cofibration, and hence we call I the set of generating cofibrations, and J the set of generating acyclic cofibrations. Using the recognition theorem for cofibrantly generated model categories (see [Hov00, Theorem 2.1.19]) is generally one of the most straightforward ways to prove that a category is a model category (though this is still challenging). We conclude this section with some examples of cofibrantly generated model categories.

Example 2.2.8. The category **Top** is cofibrantly generated (but not locally presentable). The generating cofibrations and acyclic cofibrations are:

$$I = \{S^{n-1} \hookrightarrow D^n : n \geq 0\}$$
$$J = \{D^n \times \{0\} \hookrightarrow D^n \times [0,1] : n \geq 0\}$$

The category **sSet** of simplicial sets (see Section 2.3 for the definition of this category, along with an explanation of the notation) is also cofibrantly generated:

$$I = \{\partial\Delta[n] \hookrightarrow \Delta[n] : n \geq 0\}$$
$$J = \{\Lambda^r[n] \hookrightarrow \Delta[n] : n > 0, 0 \leq r \leq n\}$$

However, it is also locally presentable, as it is generated by the set $S = \{\Delta[n] : n \geq 0\}$ and hence **sSet** is combinatorial.

Example 2.2.9. If **A** is an abelian category, then $\mathbf{Ch}_{\geq 0}(\mathbf{A})$ is cofibrantly generated. Let $K(R,n)$ denote the chain complex which is 0 except in dimension n where it is R (so it is an "Eilenberg-Maclane chain complex"). Let $D_n(R)$ denote the chain complex which is 0 except in dimensions $(n-1)$ and n where it is R, and the map between them is the identity map. Then the generating cofibrations and acyclic cofibrations are:

$$I = \{K(R, n-1) \to D_n(R) : n \geq 0\}$$
$$J = \{0 \to K(R,n) : n \geq 1\}$$

See [DS95, §7] for a full proof of this fact.

2.3 Simplicial Sets

Simplicial constructions are abundant in modern algebraic topology for many reasons. For our purposes, simplicial objects will appear as they provide a method to impose a notion of homotopy on categories which otherwise don't have a notion of homotopy (and so there is no natural definition of a weak equivalence). A very good general reference for simplicial sets and simplicial homotopy theory is [GJ99].

Definition 2.3.1. Let $\mathbf{\Delta}$ be the category whose objects are sets $[n] := \{0, 1, \ldots, n\}$ for each finite n, and whose morphisms are order preserving maps. That is, $k \leq \ell$ implies that $f(k) \leq f(\ell)$. This category is known as the *cosimplicial indexing category*, and its opposite $\mathbf{\Delta}^{\mathrm{op}}$ is known as the *simplicial indexing category.*

It may be helpful to note that $\mathbf{\Delta}$ may be more succinctly described as (a relabeling of) the category of finite non-zero von Neumann ordinals.

Definition 2.3.2. Let $\mathbf{C}$ be a category. The *category of simplicial objects in* $\mathbf{C}$ is the functor category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C})$ and is denoted $\mathbf{sC}$. In other words, a simplicial object of $\mathbf{C}$ is a contravariant functor $\mathbf{\Delta} \to \mathbf{C}$.

If $\mathbf{C}$ is (co)complete, then so is $\mathbf{sC}$, and limits and colimits are done level-wise. That is, if $F \colon \mathbf{I} \to \mathbf{sC}$, then

$$(\operatorname*{colim}_{i \in \mathbf{I}} F(i))_n = \operatorname*{colim}_{i \in \mathbf{I}} F(i)_n, \qquad (\lim_{i \in \mathbf{I}} F(i))_n = \lim_{i \in \mathbf{I}} F(i)_n$$

In particular, products and coproducts are formed level-wise.

The classical example of a simplicial category is when $\mathbf{C} = \mathbf{Set}$ in which case we recover simplicial sets. Simplicial sets will play an important role as this category has a particularly nice geometric interpretation. As a preview to their geometric nature, we offer our first (and perhaps most important) example of a simplicial set.

Definition 2.3.3. The *standard n-simplex* is the representable functor $\Delta[n] := \mathrm{Hom}_{\mathbf{\Delta}}(-,[n])$.

Note that $\Delta[n]_m = \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$, and by Yoneda's lemma, this is isomorphic to the set of natural transformations (i.e. morphisms of simplicial sets) $\Delta[m] \to \Delta[n]$. This in fact holds more generally. If X is any simplicial set, then there is a bijection between the n-simplices of X and the set of morphisms of simplicial sets $\Delta[n] \to X$. Justification for referring to the objects $\Delta[n]$ as simplices will be given shortly.

Definition 2.3.4. Let $d^i\colon [n-1] \to [n]$ and $s^i\colon [n+1] \to [n]$ for $0 \le i \le n$ be defined by:

$$d^i(k) := \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \ge i \end{cases} \qquad s^i(k) := \begin{cases} k & \text{if } k \le i \\ k-1 & \text{if } k > i \end{cases}$$

Then, if X is a simplicial object of a category $\mathbf{C}$, we define the *face maps* to be $d_i := X(d^i)\colon X_n \to X_{n-1}$ and the *degeneracy maps* to be $s_i := X(s^i)\colon X_n \to X_{n+1}$.

Note that there will be $n+1$ face maps $d_i\colon X_n \to X_{n-1}$, and $n+1$ degeneracy maps $s_i\colon X_n \to X_{n+1}$, with $0 \le i \le n$ as in Definition 2.3.4.

Proposition 2.3.5. *If X is a simplicial object of $\mathbf{C}$, then the face and degeneracy maps satisfy the following identities:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, \quad \text{if } i < j \\ s_i s_j &= s_{j+1} s_i, \quad \text{if } i \le j \\ d_i s_j &= s_{j-1} d_i, \quad \text{if } i < j \\ d_i s_i &= d_{i+1} s_i = 1 \\ d_i s_j &= s_j d_{i-1}, \quad \text{if } i > j+1 \end{aligned}$$

Proof. These identities may all be expressed in terms of the d^i's and s^i's, and the respective identities have straight-forward verifications. □

Historically, a simplicial set was defined to be a collection of sets X_n together with maps d_i and s_i as defined above which satisfy the identities in Proposition 2.3.5. However, with the language of category theory, we are able to avoid working with such a point-set description.

Note 2.3.6. It will be important later to note that the category of simplicial sets is a closed symmetric monoidal category[2]. The monoidal product is given by the categorical product, and the internal hom is given level-wise by

$$\mathrm{Map}(X,Y)_n := \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta[n], Y)$$

with face and degeneracy maps acting on the $\Delta[n]$ component. See [Hov00, Proposition 4.2.8] for a proof.

The crucial construction for bridging the gap between simplicial objects and something geometric is the geometric realization functor. We give its construction now for simplicial categories, but we shall see in Subsection 2.4 how it generalizes to other simplicial categories.

First we recall a construction from category theory.

Definition 2.3.7. Let $\mathbf{C}$ be a small category and let $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ be a functor. The *end* of F is denoted $\int_{c\in\mathbf{C}} F(c,c)$ and is the equalizer in the diagram:

$$\int_{c\in\mathbf{C}} F(c,c) \longrightarrow \prod_{c\in\mathbf{C}} F(c,c) \rightrightarrows \prod_{c\to c'} F(c,c')$$

The maps occurring in the equalizer are as follows: if s_c denotes the map

$$\prod_{f_i\colon c\to c_i} F(c,f_i)\colon F(c,c) \to \prod_{f_i} F(c,c_i)$$

[2]Intuitively, a *monoidal category* is a category equipped with a bifunctor $-\otimes-\colon \mathbf{C}\times\mathbf{C}\to\mathbf{C}$ which gives $\mathbf{C}$ the structure of a monoid. It is *closed* if it has an internal hom. See [Lan98]

then one map is $\prod_{c\in\mathbf{C}} s_c$. The other map is defined similarly. If t_c is the map

$$\prod_{f_i\colon c_i\to c} F(f_i, c)\colon F(c,c) \to \prod_{f_i} F(c_i, c)$$

then the other map is $\prod_{c\in\mathbf{C}} t_c$.

The *coend* of F is denoted $\int^{c\in\mathbf{C}} F(c,c)$ and is the coequalizer in the diagram:

$$\coprod_{c\to c'} F(c', c) \rightrightarrows \coprod_{c\in\mathbf{C}} F(c,c) \longrightarrow \int^{c\in\mathbf{C}} F(c,c)$$

The maps occurring in the coequalizer are as follows: if $f\colon c \to c'$ and $p_f\colon F(c', c) \to \coprod_{c\in\mathbf{C}} F(c,c)$ is the composition

$$F(c', c) \xrightarrow{F(c', f)} F(c', c') \longrightarrow \textstyle\coprod_{c\in\mathbf{C}} F(c,c)$$

then one map is $\coprod_{f\in\mathbf{Mor(C)}} p_f$. The other map is defined similarly. If $f\colon c \to c'$ and $q_f\colon F(c', c) \to \coprod_{c\in\mathbf{C}} F(c,c)$ is the composition

$$F(c', c) \xrightarrow{F(f, c')} F(c,c) \longrightarrow \textstyle\coprod_{c\in\mathbf{C}} F(c,c)$$

then the other map is $\coprod_{f\in\mathbf{Mor(C)}} q_f$.

A basic property of ends and coends is that any limit or colimit may be writen as the end or coend over some diagram. For more a proof of this fact, as well as more details on ends and coends, the reader is encouraged to consult a reference such as [Lan98].

As an example of an end, we offer the following. If $F, G\colon \mathbf{C} \to \mathbf{D}$ are functors, then one has:

$$\int_{c\in\mathbf{C}} \mathrm{Hom}_{\mathbf{D}}(F(c), G(c)) \cong \mathrm{Nat}(F, G)$$

where $\mathrm{Nat}(F, G)$ denotes the set of natural transformations from F to G. This example occurs is more detail in [Lan98, Section IX.5]. For coends, we have the following definition/example:

Definition 2.3.8. Let X be a simplicial set. The *geometric realization* of X is denoted $|X|$ and is the topological space given by the coend

$$|X| := \int^{n \in \mathbf{\Delta}} X_n \times \Delta^n$$

where Δ^n is the standard n-simplex of topology, and X_n is given the discrete topology.

The standard construction of a coequalizer implies that we may explicitly compte the geometric realization of a simplicial set using the formula:

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the relation $\sim$ identifies:

1. For $x \in X_{n+1}$ and $p \in \Delta^n$, we have $(d_i(x), p) \sim (x, D_i(p))$.
2. For $x \in X_{n-1}$ and $p \in \Delta^n$, we have $(s_i(x), p) \sim (x, S_i(p))$.

Here, the maps d_i and s_i are the standard face and degeneracy maps, and the maps D_i and S_i are the standard face inclusion maps and face collapsing maps in topology. This definition is a bit unwieldy, but it does allow one to perform explicit computations.

Example 2.3.9. We have a number of well-known properties of the geometric realization. We refer the reader to a standard reference such as [GJ99] for proofs.

1. $|\Delta[n]| \cong \Delta^n$.
2. Geometric realization preserves colimits. That is, if $F\colon \mathbf{I} \to \mathbf{sSet}$ then

$$|\operatorname*{colim}_{i \in \mathbf{I}} F(i)| \cong \operatorname*{colim}_{i \in \mathbf{I}} |F(i)|$$

3. For any simplicial set X, $|X|$ is compactly generated. Furthermore, if **CGH** denotes the category of compactly generated Hausdorff spaces, and $- \times_{CG} -$ denotes the product in this category (sometimes known as the Kelley space product), then $|X \times Y| \cong |X| \times_{CG} |Y|$.

4. Let $\tilde{X} \colon \Delta \downarrow X \to \mathbf{sSet}$ be the forgetful functor taking a map $\Delta[n] \to X$ (which specifies an n-simplex of X) to $\Delta[n]$. Then we have an alternative characterization of the geometric realization:

$$X \cong \operatorname*{colim}_{\Delta \downarrow X} \tilde{X} \cong \operatorname*{colim}_{\Delta \downarrow X} \Delta[n], \qquad |X| \cong \operatorname*{colim}_{\Delta \downarrow X} |\tilde{X}| \cong \operatorname*{colim}_{\Delta \downarrow X} \Delta^n$$

   See [GJ99, Lemma 2.1] for a proof.

Item 2 of Example 2.3.9 suggests that perhaps geometric realization is a left adjoint. This is indeed the case.

Proposition 2.3.10. *The functor* $|-| \colon \mathbf{sSet} \to \mathbf{Top}$ *has a right adjoint* $\mathrm{Sing} \colon \mathbf{Top} \to \mathbf{sSet}$ *given level-wise by:*

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$$

Proof. The proof is most straightforward when we use the definition of the geometric realization from Point 4 in Example 2.3.9. With this fact, we have:

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{Top}}(\operatorname*{colim}_{\Delta \downarrow X} \Delta^n, Y) \cong \lim_{\Delta \downarrow X} \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y) = \lim_{\Delta \downarrow X} \mathrm{Sing}(Y)_n$$

Now, we note that by Yoneda's lemma, $\mathrm{Sing}(Y)_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n], \mathrm{Sing}(Y))$. Thus, we may continue our string of isomorphisms as

$$\cdots \cong \lim_{\Delta \downarrow X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n], \mathrm{Sing}(Y)) \cong \mathrm{Hom}_{\mathbf{sSet}}(\operatorname*{colim}_{\Delta \downarrow X} \Delta[n], \mathrm{Sing}(Y)) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Sing}(Y))$$

This establishes $|-|$ as the left adjoint of Sing. ∎

We thus have an adjunction between the model category **Top** and the category of simplicial sets. One would hope that some of the homotopy theory from spaces could be transported to simplicial sets. This is indeed the case, and the adjunction turns out to behave very nicely. Before giving the model structure, we need some terminology in order to define the fibrations.

Definition 2.3.11. The *boundary* of the simplicial set $\Delta[n]$, denoted $\partial\Delta[n]$, is the simplicial set obtained by removing the unique non-degenerate n-simplex, which corresponds to the map $\mathrm{id}_{[n]}$. That is, $\partial\Delta[n]_k = \Delta[n]_k$ for $0 \leq k < n$, but $\partial\Delta[n]_n = \Delta[n]_n \backslash \{\mathrm{id}_{[n]}\}$.

For example, the only non-degenerate simplices of $\partial\Delta[1]$ are the 0-simplices, and as usual, $\partial\Delta[1]_0 = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[0], \Delta[1])$. By Yoneda's lemma, this is equal to $\mathrm{Hom}_{\mathbf{\Delta}}([0],[1]) = \{0 \mapsto 0, 0 \mapsto 1\}$. Thus, $\partial\Delta[1] \cong \Delta[0] \coprod \Delta[0]$, mimicking the case in the topological setting where $\partial\Delta^1 = \Delta^0 \coprod \Delta^0$ (or perhaps less cryptically, $\partial I = * \coprod *$).

Definition 2.3.12. The (n,k)-*horn* for $0 \leq k \leq n$ is the simplicial set $\Lambda^k[n]$ given levelwise by

$$(\Lambda^k[n])_m = \{s \in \Delta[n]_m \cong \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]) : [n]\backslash\{k\} \not\subseteq \mathrm{im}\, s\}.$$

Geometrically, we may view $\Lambda^k[n]$ as the union of all but one of the faces of $\Delta[n]$.

We have natural inclusion maps $\Lambda^k[n] \hookrightarrow \partial\Delta[n] \hookrightarrow \Delta[n]$.

Definition 2.3.13. A map of $f\colon X \to Y$ of simplicial sets is a *Kan fibration* if it has the RLP with respect to the inclusion of horns. Diagramatically, f is a Kan fibration if whenever the solid line diagram commutes, there is a map (represented by the dashed line) which makes the whole diagram commute:

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \overset{x}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \Delta[n] & \xrightarrow[y]{} & Y \end{array} .$$

A simplicial set X is said to be a *Kan complex* if the map $X \to *$ is a Kan fibration. In this case, we say X satisfies the *Kan condition*.

Because an n-simplex of Y is determined by a map $y\colon \Delta[n] \to Y$, we may think of a Kan fibration as being a map of simplicial sets for which we can lift simplices of Y to simplices of X along f.

Importantly, we have the following major result.

Theorem 2.3.14. *The category* **sSet** *of simplicial sets is a model category. A map* $f\colon X \to Y$ *is a...*

1. *...weak equivalence if* $|f|\colon |X| \to |Y|$ *is a weak homotopy equivalence of spaces.*
2. *...fibration if it is a Kan fibration.*
3. *...cofibration if* $f_n\colon X_n \to Y_n$ *is injective for all* $n \geq 0$*.*

Moreover, if we equip **Top** *with the model structure of Example 2.1.7 then the adjunction* $|-| \dashv \mathrm{Sing}$ *is a Quillen equivalence. This result is proven in [Hov00].*

The other major result we will need is the Dold-Kan correspondence. In doing so, we will turn our attention from simplicial sets to simplicial objects in an abelian category **A**. We will not provide the proof, though we will sketch one of the two functors involved.

Theorem 2.3.15 (Dold-Kan)**.** *Let* **A** *be an abelian category. Then the categories* $\mathbf{Ch}_{\geq 0}(\mathbf{A})$ *and* **sA** *are equivalent (in the usual sense of equivalence of categories). Moreover, under under this equivalence, chain homotopy corresponds to simplicial homotopy.*

A proof of this theorem may be found in [Wei94], along with a definition of *simplicial homotopy.*

The *normalized chain complex* of a simplicial set X is the chain complex NX given degree-wise by

$$(NX)_k := \bigcap_{i=0}^{n-1} \ker d_i$$

and the differential is given as $\partial = (-1)^n d_n$, where d_i for $0 \leq i \leq n$ are the face maps for the simplicial set X. This defines a functor $N\colon \mathbf{sA} \to \mathbf{Ch}_{\geq 0}(\mathbf{A})$ and is an equivalence of categories. As N is an equivalence of categories, we have an inverse functor $K\colon \mathbf{Ch}_{\geq 0}(\mathbf{A}) \to \mathbf{sA}$. This functor has no standard name in the literature, and its construction is unfortunately rather unwieldy. As we will not have a use for the explicit construction, we omit it here. There is a third, perhaps more obvious, chain complex one may construct from a simplicial object. Namely, for an object X of $\mathbf{sA}$, we may define a chain complex CX by setting $(CX)_k = X_k$ and defining the differential as the alternating sum of the face maps. That is, $\partial = \sum_{i=0}^{n}(-1)^i d_i$. This is the so-called *(unnormalized) chain complex* associated to X. It is clear that NX is a chain subcomplex of CX, and one may in fact show that it occurs as a direct summand of CX. Hence, we have a natural projection map $CX \to NX$ which one may show is in fact a quasi-isomorphism.

2.4 Simplicial Model Categories

Simplicial model categories are a particularly nice class of model categories whose model structure plays nicely with the model structure on simplicial sets. The work in this section may be originally attributed to Quillen in his paper [Qui67], though there are now many wonderful modern references, such as [GJ99].

Definition 2.4.1. A *simplicial category* is a category $\mathbf{C}$ which is enriched in simplicial sets (we will use $\mathbf{Hom}(X,Y)$ to denote the enriched Hom-set), and such that we have isomorphisms $\mathbf{Hom}(X,Y)_0 \cong \mathrm{Hom}_{\mathbf{C}}(X,Y)$ which commute with the enriched composition. The simplicial set $\mathbf{Hom}(X,Y)$ is often known as a *homotopy function complex.*

A simplicial model category will be a model category that is also a simplicial category which is tensored and cotensored over simplicial sets, and satisfies the *homotopy lifting extension theorem.* These are defined as follows:

Definition 2.4.2. A simplicial model category is a simplicial category $\mathbf{C}$ which is also a model category such that the following two additional axioms are satisfied:

1. SMC6 (Tensoring and cotensoring) The category **C** is tensored and cotensored over **sSet**. Explicitly, **C** is *tensored* (or *copowered*) over **sSet** if for each object X of **C** and each simplicial set K, we have an object $X \otimes K$ of **C** such that there are isomorphisms

$$\mathbf{Hom}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathbf{Hom}(X, Y))$$

which are natural in X, Y, and K. The notation Map denotes the internal hom for simplicial sets (see Note 2.3.6). The category **C** is *cotensored* (or *powered*) over **sSet** if for each object X of **C** and each simplicial set K, we have an object X^K of **C** such that there are isomorphisms

$$\mathbf{Hom}(X, Y^K) \cong \mathrm{Map}(K, \mathbf{Hom}(X, Y))$$

which are natural in X, Y, and K.

2. SMC7 (Homotopy lifting) For every cofibration $i\colon A \to B$ and fibration $p\colon X \to Y$ in **C**, the map of simplicial sets

$$\mathbf{Hom}(B, X) \xrightarrow{i^* \times p_*} \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B, Y)$$

is a fibration which is also a weak equivalence if i or p is.

One may show (for example, [GS06, Proposition 3.3]) that SMC7 implies MC4, and so SMC7 may be seen as a generalization of the lifting properties to respect the homotopical nature of the function complexes.

Most of the categories we are interested in are simplicial model categories. The category of simplicial sets is the trivial example, with the tensor being given by the categorical product, and the cotensor being the internal hom. Less trivially, we have the category **CGH** of compactly generated Hausdorff spaces (see [GJ99] for a proof). In this case, the tensor and cotensor are given by first taking the geometric realization of the simplicial set. That is:

$$X \otimes K := X \times |K|^3, \qquad X^K := X^{|K|}$$

The space $X^{|K|}$ is the space of continuous functions $|K| \to X$ with the compact open topology. It is important that we work with the category of compactly generated Hausdorff spaces instead of the full category of all spaces, as we do not have the natural isomorphisms required by the tensor in this larger category.

An important source of simplicial model categories for us comes from the fact that if $\mathbf{C}$ is a simplicial model category, then $\mathbf{sC}$ is a again a simplicial model category. The model structure we impose is the so called *Reedy model structure.* The weak equivalences are maps which are level-wise weak equivalences. For more information on the model structure, the reader should consult a reference such as [Ree74]. Importantly, we have have an adjunction similar to that in Proposition 2.3.10.

Proposition 2.4.3. *If* $\mathbf{C}$ *is a simplicial model category, then the* geometric realization *functor* $| - |\colon \mathbf{sC} \to \mathbf{C}$ *defined by:*

$$|X| := \int^{n \in \mathbf{\Delta}} X_n \otimes \Delta[n]$$

and the singular set *functor* $\mathrm{Sing}\colon \mathbf{C} \to \mathbf{sC}$ *defined level-wise by:*

$$\mathrm{Sing}(X)_n := X^{\Delta[n]}$$

are such that $|-|$ *is left adjoint to* Sing *and the adjunction is a Quillen equivalence.*

This construction may be used to obtain a model structure on bisimplicial sets. Or,

[3] The product here is the Kelley product, which is in general different from the usual product of spaces.

via the Dold-Kan correspondence, this also produces a model structure on the category of bicomplexes of objects in an abelian category. We will make use of this in Section 4.3.

Chapter 3

A Bit of Homotopy and Algebra

Having outlined the model category setting we will be using extensively in this thesis, we now turn to some more specific constructions in this setting. The purpose of this chapter is to provide brief introductions to some more homotopical properties (homotopy limits and colimits, and cubical diagrams) as well as some more algebraic concepts (comonads and André-Quillen homology).

3.1 A Crash-Course in Homotopy (Co)limits

Homotopy (co)limits are a surprisingly complex solution to a natural and easy to pose problem. To motivate the construction, consider the following diagrams:

$$\begin{array}{ccc} S^1 & \xrightarrow{i} & D^2 \\ \downarrow{\scriptstyle i} & & \\ D^2 & & \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} \tag{3.1.1}$$

where $i\colon S^1 \to D^2$ is the inclusion of the boundary of S^1 as the boundary of the unit disk D^2. These two pushout diagrams are an objectwise homotopy equivalence. As algebraic topologists (or more generally, people working in a model category) tend to only work up to weak equivalence, it would be desirable for the pushouts to be at least weak homotopy equivalent. However, the pushout of the first is two-sphere S^2, and the second has pushout equal to $*$.

Homotopy (co)limits rectify this by defining a construction which is similar enough to a regular (co)limit to deserve the name, but has the additional property that it is invariant under weak equivalence. We will outline here two perspectives on the topic. The first is the Bousfield-Kan construction ([BK72]) which gives us an explicit formula we may use to compute homotopy (co)limits in specific instances. The second is more abstract and treats the homotopy (co)limit as a derived functor in the model category sense. This is the approach take in [DHKS05].

In this section, $\mathbf{C}$ refers to a pointed simplicial model category. In this setting we have a notion of homotopy, and so we may attempt to define homotopy limits and colimits. Before giving models for homotopy (co)limits, we present a 'bucket list' of properties we would like (and in fact have).

1. If $F, G\colon \mathbf{I} \to \mathbf{C}$ are two diagrams and $\eta\colon F \to G$ is a natural transformation whose components are weak equivalences then $\operatorname{holim}_{\mathbf{I}} F \simeq \operatorname{holim}_{\mathbf{I}} G$ and $\operatorname{hocolim}_{\mathbf{I}} F \simeq \operatorname{hocolim}_{\mathbf{I}} G$.

2. We would like natural maps between the homotopy (co)limit and ordinary (co)limit. In particular, we will see that we get maps $\operatorname{hocolim}_{\mathbf{I}} F \to \operatorname{colim}_{\mathbf{I}} F$ and $\lim_{\mathbf{I}} F \to \operatorname{holim}_{\mathbf{I}} F$.

3. If $F\colon \mathbf{I} \times \mathbf{J} \to \mathbf{C}$, then we would like a 'Fubini's theorem' analogous to what we have for ordinary limits and colmits. That is, we would like:

$$\operatorname{holim}_{\mathbf{I}\times\mathbf{J}} \simeq \operatorname{holim}_{i\in\mathbf{I}} \operatorname{holim}_{j\in\mathbf{J}} F(i,j) \simeq \operatorname{holim}_{i\in\mathbf{J}} \operatorname{holim}_{j\in\mathbf{I}} F(i,j)$$

$$\operatorname{hocolim}_{\mathbf{I}\times\mathbf{J}} \simeq \operatorname{hocolim}_{i\in\mathbf{I}} \operatorname{hocolim}_{j\in\mathbf{J}} F(i,j) \simeq \operatorname{hocolim}_{i\in\mathbf{J}} \operatorname{hocolim}_{j\in\mathbf{I}} F(i,j)$$

We now offer two perspectives on how we may obtain the properties above.

3.1.1 A Concrete Construction

In many cases, it is most useful if we have an explicit construction of homotopy (co)limits. The construction is very technical, so we will direct the reader who is interested in proofs to an existing resource such as [Hir03, Chapter 18]. Nevertheless, we will summarize one possible model for homotopy (co)limits. This construction is originally due to Bousfield and Kan (see [BK72]).

Definition 3.1.1. If $\mathbf{D}$ is any small category, then we denote by $N(\mathbf{D})$ the *nerve* of $\mathbf{D}$. This is the simplicial set whose n-simplicies are strings of n composable morphisms (the 0-simplices are the objects of $\mathbf{D}$), and the face and degeneracy maps are given by deleting the i-th morphism or inserting the identity in the i-th spot respectively.

Definition 3.1.2 ([BK72])**.** Let $\mathbf{I}$ be a small category, and let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor (i.e. an $\mathbf{I}$-diagram). If F is objectwise fibrant, then we define the homotopy limit as the end:

$$\mathrm{holim}_{\mathbf{I}}\, F := \int_{i\in\mathbf{I}} F(i)^{N(\mathbf{I}\downarrow i)}$$

If F is objectwise cofibrant, we define the homotopy colimit as the coend:

$$\mathrm{hocolim}_{\mathbf{I}}\, F := \int^{i\in\mathbf{I}} F(i) \otimes N(i\downarrow \mathbf{I})^{\mathrm{op}}$$

A word about the notation is in order. If $i \in \mathbf{I}$, we denote by $\mathbf{I}\downarrow i$ the category of *objects over* i. That is, the objects are maps $j \to i$, and morphisms commuting triangles. Similarly $i \downarrow \mathbf{I}$ is the category of *objects under* i. Hence, the objects are maps $i \to j$ and morphisms are again commuting triangles.

3.1.2 The Derived Functor Perspective

A more abstract approach is often useful. We focus our attention on homotopy colimits as everything may be dualized and we have analogous statements for homotopy limits.

Definition 3.1.3. Let $\mathbf{I}$ be a small category, and let $\mathbf{C}$ be a cofibrantly generated model category. The *projective model structure* on $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ has weak equivalences and fibrations given object-wise. The cofibrations are thus determined.

The colimit may be viewed as a functor $\operatorname{colim}\colon \operatorname{Fun}(\mathbf{I},\mathbf{C}) \to \mathbf{C}$ which has a right adjoint $\Delta\colon \mathbf{C} \to \operatorname{Fun}(\mathbf{I},\mathbf{C})$ given by the constant diagram functor. Moreover, if $\operatorname{Fun}(\mathbf{I},\mathbf{C})$ is given the projective model structure, then this is in fact a Quillen adjunction. Hence, we may consider the total left derived functor

$$\mathbf{L}\operatorname{colim}\colon \operatorname{Ho}(\operatorname{Fun}(\mathbf{I},\mathbf{C})) \to \operatorname{Ho}(\mathbf{C})$$

Definition 3.1.4. The homotopy colimit is the total left derived functor of colim. That is,

$$\operatorname{hocolim} F := \mathbf{L}\operatorname{colim} F$$

In any model category, the left derived functor is obtained by applying the functor to a cofibrant replacement. Hence, if $F\colon \mathbf{I} \to \mathbf{C}$ is a diagram and QF denotes its cofibrant replacement, then

$$\operatorname{hocolim} F = [\operatorname{colim} QF]$$

where the square brackets denote the image of $\operatorname{colim} QF$ under the localization $\mathbf{C} \to \operatorname{Ho}(\mathbf{C})$.

In the case of homotopy limits, we use the obvious dual version of the projective model structure, known as the *injective model structure*. One must be careful, as the injective model structure isn't a model structure unless $\mathbf{C}$ is a combinatorial model category (a stronger condition than being cofibrantly generated). We then define the homotopy limit to be the total right derived functor of the limit functor, which is given as

$$\operatorname{holim} F := \mathbf{R}\lim F = [\lim RF]$$

where RF denotes a fibrant replacement of F (in the injective model structure).

It is worth mentioning that the category **Top** is *not* combinatorial, as it isn't locally presentable. However, it is possible to find nice subcategories of **Top** which are. For example, [FR08, Theorem 3.9] gives a coreflective subcategory of **Top** which has a model structure with the same cofibrations and weak equivalences, but which is combinatorial.

3.1.3 Examples and Additional Properties

Applying our construction in Subsection 3.1.1, we may offer some explicit constructions of homotopy colimits over small diagrams. We give the constructions in **Top** to facilitate geometric intuition, though these constructions of course work more generally.

Example 3.1.5. If $f\colon A \to B$ is a map between CW complexes (so that they are cofibrant), then $\operatorname{hocolim} f \simeq M_f$, the mapping cylinder of f. Note that if f is a cofibration, then M_f deformation retracts onto B, and in this case, the map $\operatorname{hocolim} F \to \operatorname{colim} F$ is a weak equivalence.

If f is a map of chain complexes, then $\operatorname{hocolim} F$ is quasi-isomorphic to the algebraic mapping cylinder.

Example 3.1.6. Let F be the pushout diagram below:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\big\downarrow & & \\ Y & & \end{array}$$

where A, X, and Y are all CW complexes. The Bousfield-Kan construction of the homotopy colimit (Definition 3.1.2) gives us that the homotopy pushout is the quotient of the coproduct (writing I for the unit interval/1-simplex $[0,1]$, and where $A_f = A_g = A$ is used for notational clarity):

$$\operatorname{hocolim} F \simeq \left(X \coprod (A_f \times I) \coprod A \coprod (A_g \times I) \coprod Y \right) / \sim$$

The equivalence relation $\sim$ makes the identifications:

1. $(a,0) \in A_f \times I$ is identified with $f(a) \in X$
2. $(a,0) \in A_g \times I$ is identified with $g(a) \in Y$.

3. $(a, 1) \in A_f \times I$ or $(a, 1) \in A_g \times I$ is identified with $a \in A$.

Returning to the example at the beginning of this section, using the Bousfield-Kan construction we see that the homotopy pushout of both diagrams in (3.1.1) is homotopy equivalent to S^2, illustrating that the homotopy pushout, unlike the ordinary pushout, is weak homotopy invariant.

We give one example of a homotopy limit. In general, homotopy limits are not gluing constructions, but rather function spaces and are thus more difficult to visualize and intuit.

Example 3.1.7. Let $f\colon A \to B$ be a map of spaces. As every space is fibrant, we need no hypotheses on the spaces X, Y, and A to apply the Bousfield-Kan construction. Doing so yields that $\operatorname{holim} f$ is the subspace of $A \times B^I$ consisting of pairs (a, γ) such that $\gamma(0) = a$. In other words, $\operatorname{holim} f$ is the classical pathspace P_f of f.

The following property will be used in the Proposition 3.2.3.

Proposition 3.1.8. *If* $\mathbf{C}$ *is a right proper model category, then the map from the ordinary pullback along a fibration to the homotopy pullback of the same diagram is a weak equivalence. If* $\mathbf{C}$ *is left proper, then the map from the homotopy pushout along a cofibration to the ordinary pushout of the same diagram is a weak equivalence.*

Proof. See [Hir03, Corollary 13.3.8]. □

3.1.4 Caveats

Before wrapping up our introduction, we present some cautions to be wary of when working with homotopy (co)limits.

1. In light of Subsection 3.1.2, we may only ever work in the homotopy category when discussing homotopy (co)limits. This construction implies that they are only well-defined up to weak equivalence, so one should in general avoid

discussing properties of homotopy (co)limits which are not invariant under weak equivalence.

2. Homotopy (co)limits do not satisfy any sort of universal property. Moreover, we in general don't have a canonical map $F(i) \to \operatorname{hocolim}_{\mathbf{I}} F$ or $\operatorname{holim}_{\mathbf{I}} F \to F(i)$. Such maps exist, but they are only well-defined up to weak equivalence. Again, this construction lives in the homotopy category!

3. Having accepted that homotopy (co)limits are objects in the homotopy category, it's tempting to think that perhaps a homotopy (co)limit as just a (co)limit in the homotopy category. This is again false! The homotopy category of a model category is generally neither complete nor cocomplete, even if the original category was.

The above can make working with homotopy (co)limits a bit of an art.

3.2 A Summary of the 'Yoga of Cubical Diagrams'

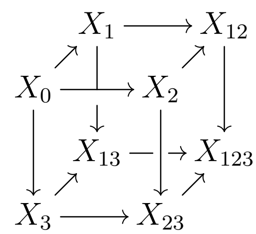
We begin with a brief survey of the theory of cubical diagrams. A standard reference for this section is [Goo92, Chapter 1].

In this section, $\mathbf{C}$ refers to a pointed simplicial model category (for example $\mathbf{Top}_*$ or $\mathbf{Ch}_{\geq 0}(R)$). By a *pointed category*, we mean a category with an object $*$ which is both initial and terminal. We will present this section homotopically, but we will have cause later to use a strict version. All results carry over by replacing homotopy limits and colimits with regular limits and colimits, unless otherwise specified.

Definition 3.2.1. We let $\underline{n} = \{1, \ldots, n\}$. An *n-cube* in a category $\mathbf{C}$ is a functor $\mathscr{X} : \mathcal{P}(\underline{n}) \to \mathbf{C}$ where $\mathcal{P}(\underline{n})$ is the power set of $\underline{n}$ viewed as a category with morphisms given by inclusion. We will set $\mathcal{P}_0(\underline{n}) := \mathcal{P}(\underline{n}) \backslash \{\emptyset\}$ and $\mathcal{P}_1(\underline{n}) := \mathcal{P}(\underline{n}) \backslash \{\underline{n}\}$. We will write $\mathscr{X}_S$ instead of $\mathscr{X}(S)$.

For example, below we have (from left to right) a 1-cube, a 2-cube, and 3-cube (where X_{13} is shorthand for $X_{\{1,3\}}$):

$$X_0 \longrightarrow X_1 \qquad \begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$



Given a cube, we have a number of important constructions.

Definition 3.2.2. Let $\mathscr{X}$ be an n-cube.

1. The map $\alpha(\mathscr{X})$ is defined to be the canonical composition

$$\mathscr{X}_\emptyset \to \lim \mathscr{X}|_{\mathcal{P}_0(\underline{n})} \to \text{holim}\, \mathscr{X}|_{\mathcal{P}_0(\underline{n})}$$

2. The map $\beta(\mathscr{X})$ is defined to be the canonical composition

$$\mathscr{X}|_{\mathcal{P}_1(\underline{n})} \to \text{colim}\, \mathscr{X}|_{\mathcal{P}_1(\underline{n})} \to X_{\underline{n}}$$

3. The *total fibre* of $\mathscr{X}$ is defined as $\text{tfib}(\mathscr{X}) := \text{hofib}\, \alpha(\mathscr{X})$.

4. The *total cofibre* of $\mathscr{X}$ is defined as $\text{tcofib}(\mathscr{X}) := \text{hocofib}\, \beta(\mathscr{X})$.

Recall that the homotopy fibre of a map $f\colon X \to Y$ is defined to be the homotopy pullback of f along $* \to Y$. The homotopy cofibre of f is the homotopy pushout of f along $X \to *$. When working in a non-homotopical setting, we will use the same notation for the strict total fibre and cofibre.

As one may suspect, a map of n-cubes $\mathscr{X} \to \mathscr{Y}$ is a natural transformation. Note that by using the components of the natural transformation, we may build an $(n+1)$-cube $\mathscr{Z}$ such that $\mathscr{Z}_S = \mathscr{X}_S$ if $n+1 \notin S$ and $\mathscr{Z}_S = \mathscr{Y}_{S\setminus\{n+1\}}$ if $n+1 \in S$. Clearly, the converse holds as well and an n-cube may be seen as a map of $(n-1)$-cubes. This interpretation proves very useful, and leads to the following lemma:

Lemma 3.2.3. *Let* $\mathbf{C}$ *be a right proper simplicial model category. If* $\mathscr{Z}\colon \mathscr{X} \to \mathscr{Y}$ *is a map of* n*-cubes in* $\mathbf{C}$*, then:*

$$\mathrm{tfib}(\mathscr{Z}) \simeq \mathrm{hofib}(\mathrm{tfib}(\mathscr{X}) \to \mathrm{tfib}(\mathscr{Y}))$$

Proof. First, let $\mathscr{W}$ be the 2-cube below:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array} \tag{3.2.1}$$

Using the fact that we can write iterated homotopy limits as a homotopy limit over a product category (see the 'bucket list'in Section 3.1), we obtain that:

$$\mathrm{tfib}(\mathscr{W}) \simeq \mathrm{holim}\left(\begin{array}{ccccc} * & \longrightarrow & * & \longleftarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longleftarrow & C \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & C & \longleftarrow & C \end{array}\right) \tag{3.2.2}$$

The homotopy limit of the bottom row is weak homotopy equivalent to A by the assumption that $\mathbf{C}$ is right proper (see Proposition 3.1.8)[1]. Hence, taking homotopy limits horizontally and then vertically yields the traditional definition of the total fibre. However, taking homotopy limits vertically and then horizontally gives us that $\mathrm{tfib}(\mathscr{W}) \simeq \mathrm{hofib}(\mathrm{hofib}(f) \to \mathrm{hofib}(g))$. This proves the lemma in the case when $\mathscr{Z}$ is a map of 1-cubes. For $\mathscr{Z}\colon \mathscr{X} \to \mathscr{Y}$

[1]If we are working non-homotopically, then the limit of this row is isomorphic to A so the result still holds.

a map of n-cubes, we consider the 2-cube:

$$\begin{array}{ccc} \mathscr{X}_\emptyset & \xrightarrow{\alpha(\mathscr{X})} & \operatorname{holim} \mathscr{X}|_{\mathcal{P}_0(\underline{n})} \\ \downarrow & & \downarrow \\ \mathscr{Y}_\emptyset & \xrightarrow{\alpha(\mathscr{Y})} & \operatorname{holim} \mathscr{Y}|_{\mathcal{P}_0(\underline{n})} \end{array} \tag{3.2.3}$$

By above, the total fibre of this square is $\operatorname{hofib}(\operatorname{tfib}(\mathscr{X}) \to \operatorname{tfib}(\mathscr{Y}))$, and so it suffices to show that this is in fact computing $\operatorname{tfib}(\mathscr{Z})$. However, this follows from again writing iterated homotopy limits as a limit over a product. Namely, we may use this to show that:

$$\operatorname{holim}(\mathscr{Y}_\emptyset \to \operatorname{holim} \mathscr{Y}|_{\mathcal{P}_0(\underline{n})} \leftarrow \operatorname{holim} \mathscr{X}|_{\mathcal{P}_0(\underline{n})}) \simeq \operatorname{holim} \mathscr{Z}|_{\mathcal{P}_0(\underline{n+1})} \tag{3.2.4}$$

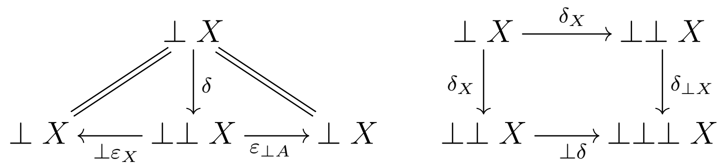
This fact finishes the proof. □

In Goodwillie's traditional homotopy calculus, notions of homotopy Cartesian and co-Cartesian are needed in order to define *n-excisive* – the analogue of degree n. In the algebraic setting, this construction has been replaced by cross effects functors, so we will have no need to develop this material.

3.3 Comonads

Comonads will play a prominent role in many of the constructions used. We provide a quick introduction, and refer the reader unfamiliar with the subject to a standard reference such as [Wei94]. It is worth noting that all of this section dualizes to obtain a theory of monads, cosimplicial objects, and cohomology theories, though we will not have a use for this construction and so omit it.

Definition 3.3.1. A *comonad* (or *cotriple*) in a category $\mathbf{C}$ is a comonoid in the category of endofunctors of $\mathbf{C}$. Explicitly, this means we have an endofunctor $\perp\colon \mathbf{C} \to \mathbf{C}$ (the comonoid) with natural transformations $\varepsilon\colon \perp\to \mathrm{id}_{\mathbf{C}}$ (the *counit*) and $\delta\colon \perp\to\perp\perp$ (the *comultiplication*) such that for each object X of $\mathbf{C}$:



Counit identity Coassociativity identity

We recall the following facts (see [Wei94]), which we will record in the following lemma:

Lemma 3.3.2. *The following facts about comonads hold:*

1. *If $F \dashv G$, then $\perp = FG$ is a comonad with counit ε and comultiplication given component-wise by $\delta_X = F(\eta_{G(X)})$, where η and ε are the unit and counit of the adjunction respectively.*

2. *If $\perp$ is a comonad in $\mathbf{C}$ and X is an object in $\mathbf{C}$, then we may form a simplicial object $\perp^{*+1} X$ (living in $\mathbf{sC}$) by defining the n-simplices to be $(\perp^{*+1} X)_n :=\perp^{n+1} X$. The i-th face and degeneracy maps are given by applying the counit and comulitplication maps respectively in the i-th spot. Moreover, the counit serves as an augmentation $(\perp^{*+1} X)_0 =\perp X \to X$ and hence we have a simplicial map $\perp^{*+1} X \to X$ where we are viewing X as a constant simplicial object.*

Comonads give rise to homology theories in a natural way. In particular, in Section 3.4 we will see that a particular comonad gives rise to André-Quillen homology.

Definition 3.3.3. Let $(\perp, \varepsilon, \delta)$ be a comonad on a category $\mathbf{C}$. Let $\mathbf{A}$ be an abelian category, and let $E\colon \mathbf{A} \to \mathbf{C}$ be a functor. The *comonad homology* of X in $\mathbf{A}$ with coefficients in $\mathbf{C}$ (relative to $\perp$) is defined by $H_n(X; E) := \pi_n(E(\perp^{*+1} X))$.

This general construction turns out to specialize to many well-known constructions in homological algebra.

Example 3.3.4. Fix a commutative ring R and consider the adjunction $U \dashv F$ where $U\colon \mathbf{Mod}_R \to \mathbf{Set}$ is the forgetful functor and $F\colon \mathbf{Set} \to \mathbf{Mod}_R$ is the free R-module functor. This gives rise to a comonad which we will denote by $\perp$. If we define $E(M) = M \otimes_R N$ for a given R-module N, then the comonad homology recovers the Tor functors:

$$H_n(M; E) = \mathrm{Tor}_n^R(M, N)$$

This example is explored in more detail in [Wei94].

André-Quillen homology arises from a comonad in a slightly different way. However, André-Quillen cohomology does arise as a cohomology theory from a comonad as in Definition 3.3.3 by taking E to be contravariant so that we get cohomology. As our interests lie in André-Quillen homology, this construction will not be needed directly.

3.4 André-Quillen Homology

In Chapter 5 we will show that André-Quillen homology may be recovered as the derivative of a particular functor. We take this opportunity to provide a brief introduction to André-Quillen homology, though the material will not be needed until Chapter 5.

André-Quillen homology is a homology theory for algebras which may be viewed as the derived functor of the cotangent complex. It was originally developed in parallel by Quillen in [Qui70] and André in [And74] and is widely considered to be the 'correct' homology theory for algebras. One may view André-Quillen homology as a special case of Quillen homology, which is in general the derived functor of an abelianization functor. While interesting, we will not need this approach, and we refer the reader who is interested in this perspective to a reference such as [GS06]. We will provide a purely algebraic construction, as it is this construction which will be most useful to us. We will use [Wei94] and [Iye07] as our primary references for the material in this section. All the results in this section are well-known and are either proven or left as exercises in the aforementioned references.

Throughout this section, k denotes a commutative unital ring, A will be a commutative k-algebra, and M will be a left A-module. Note that because A is commutative, any left A-module is a bimodule with the right action being given by $m \cdot a = am$, $a \in A, m \in M$. The action will generally be denoted by am unless we wish to emphasize the fact that we are dealing with an action and not multiplication, in which case the action will be denoted $a \cdot m$. Our first goal is to define the cotangent complex of an algebra.

Definition 3.4.1. An *k-linear derivation of A with coefficients in M* is a k-linear map $\delta\colon A \to M$ satisfying the Leibnitz rule: $\delta(ab) = \delta(a)b + a\delta(b)$.

The space of all such derivations is denoted by $\mathrm{Der}_k(A, M)$. One may define an A-module structure on $\mathrm{Der}_k(A, M)$ by setting $(a\delta)(b) = a\delta(b)$. We observe that $M \mapsto \mathrm{Der}_k(A, M)$ defines an endofunctor on the category of A-modules. As we shall see in Proposition 3.4.2, the functor $\mathrm{Der}_k(A, -)$ is actually representable in the category of A-modules, and we denote the module representing it by $\Omega_{A/k}$. This A-module is known as the module of *Kähler differentials* for A over k.

Proposition 3.4.2. *We may identify $\Omega_{A/k}$ with the module I/I^2 where $I = \ker(\mu)$ for the map $\mu\colon A \otimes_k A \to A$ given by $\mu(a \otimes b) = ab$.*

Proof. We will show that the module I/I^2 does represent $\mathrm{Der}_k(A, -)$, and hence serves as a model for $\Omega_{A/k}$. First note that I is itself an A-module with A acting on the left-hand factor of the tensor product. That is $a \cdot (x \otimes y) = (x \otimes y) \cdot a = ax \otimes y$. We seek a bijection:

$$\mathrm{Hom}_{\mathbf{Mod}_A}(I/I^2, M) \cong \mathrm{Der}_k(A, M)$$

Define $\varphi\colon \mathrm{Hom}_{\mathbf{Mod}_A}(I/I^2, M) \to \mathrm{Der}_k(A, M)$ by $\varphi(f) = f \circ d$ where $d\colon A \to I/I^2$ is given by $d(a) = [1 \otimes a - a \otimes 1]$. Clearly $\mu d(a) = 0$, so d is well-defined, and it is clearly k-linear.

We note that this map is in fact a derivation:

$$\begin{aligned}
d(ab) - d(a)b - ad(b) &= (1 \otimes ab - ab \otimes 1) - (1 \otimes a - a \otimes 1) \cdot b - a \cdot (1 \otimes b - b \otimes 1) \\
&= 1 \otimes ab - ab \otimes 1 - b \otimes a + ab \otimes 1 - a \otimes b + ab \otimes 1 \\
&= 1 \otimes ab - b \otimes a - a \otimes b + ab \otimes 1 \\
&= d(a)d(b) \in I^2
\end{aligned}$$

Hence, $d(ab) = d(a)b + ad(b)$ in I/I^2, so d is a derivation. It is easy to show that that $\varphi(f)$ is a derivation as well and that φ is an A-module homomorphism, and hence φ is well-defined.

Conversely, define a map $\psi\colon \operatorname{Der}_k(A, M) \to \operatorname{Hom}_{\mathbf{Mod}_k}(I/I^2, M)$ by $\psi(\delta) = f_\delta$ where $f_\delta(a \otimes b) = a\delta(b)$. Then f_δ is well-defined, for if $(a \otimes b)(c \otimes d) = ac \otimes bd \in I^2$, then

$$f_\delta(ac \otimes bd) = ac\delta(bd) = ac(\delta(b)d + b\delta(d)) = acd\delta(b) + abc\delta(d) = 0$$

because $ab = cd = 0$ and using the commutativity of A. Both ψ and f_δ may be readily checked to be A-module homomorphisms. Next, note that if d is the derivation from the preceding paragraph, then $f_\delta \circ d = \delta$. Indeed,

$$f_\delta(d(a)) = f_\delta(1 \otimes a - a \otimes 1) = \delta(a) - a\delta(1) = \delta(a)$$

because $\delta(1) = 0$. This shows that $\varphi\psi = \mathrm{id}$. Further,

$$f_{g \circ d}(a \otimes b) = ag(d(b) = ag(1 \otimes b - b \otimes 1) = g(a \otimes b - ab \otimes 1) = g(a \otimes b)$$

because $ab = 0$ since $a \otimes b \in \ker \mu$. This demonstrates that $\psi\varphi = \mathrm{id}$, establishing the desired bijection. $\square$

Corollary 3.4.3. *The A-module $\Omega_{A/k}$ of Kähler differentials satisfies the following universal* *property: given any derivation $\delta\colon A \to M$, there is a unique A-linear map $\Omega_{A/k} \to M$ such that*

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega_{A/k} \\ & \searrow^{\delta} & \downarrow{\scriptstyle \exists!} \\ & & M \end{array}$$

We will need the following lemma later.

Lemma 3.4.4. *For A an augmented k-algebra, then given a map $q\colon P \to A$ of k-algebras with P free, we have an isomorphism:*

$$\Omega_{P/k} \otimes_P A \cong J/J^2$$

where J is the kernel of the map $P \otimes_k A \to A$ given by $p \otimes a \mapsto q(p)a$.

Proof. Write I for the kernel of the multiplication map $P \otimes_k P \to P$ so that we have a short exact sequence of k-modules :

$$0 \longrightarrow I \longrightarrow P \otimes_k P \longrightarrow P \longrightarrow 0$$

Applying the functor $- \otimes_P A$ to the sequence yields another short exact sequence of the form:

$$0 \longrightarrow I \otimes_P A \longrightarrow P \otimes_k A \longrightarrow A \longrightarrow 0$$

Although A is not flat as a P-module, the functor preserves the exactness of this particular sequence because P is a projective k-module and thus the sequence splits. Hence, we obtain an isomorphism $I \otimes_P A \cong J$. Now, we may identify $\Omega_{P/k}$ as the cokernel in the following short exact sequence

$$0 \longrightarrow I^2 \longrightarrow I \longrightarrow \Omega_{P/k} \longrightarrow 0$$

As P is a polynomial algebra, $\Omega_{P/k}$ is free (see [Iye07]), and so the same argument as above implies that the sequence remains exact after applying $- \otimes_P A$. Finally, using the isomorphism $I \otimes_P A \cong J$ from above yields that $\Omega_{P/k} \otimes_P A \cong J/J^2$. □

We now turn to defining the cotangent complex of an algebra. Before doing so, note that the forgetful functor $U\colon \mathbf{CommAlg}_k \to \mathbf{Set}$ has a right adjoint $k[-]\colon \mathbf{Set} \to \mathbf{CommAlg}_k$ given by taking the polynomial algebra on a set. This gives rise to a comonad $\perp$ on $\mathbf{CommAlg}_k$.

Definition 3.4.5. For a commutative k-algebra A, we define the *cotangent complex* $\mathbb{L}_{A/k}$ to be the simplicial A-module given by

$$\mathbb{L}_{A/k} := A \otimes_{\perp^{*+1} A} \Omega_{(\perp^{*+1} A)/k}$$

where the tensor product is to be taken level-wise (both in the subscript and the right operand).

Note that we have the following isomorphism given by the restriction/extension of scalars adjunction:

$$\mathrm{Der}_k(\perp^{*+1} A, M) \cong \mathrm{Hom}_{\mathbf{sMod}_{\perp^{*+1} A}}(\Omega_{(\perp^{*+1} A)/k}, M) \cong \mathrm{Hom}_{\mathbf{sMod}_A}(\mathbb{L}_{A/k}, M)$$

This mirrors the result in Proposition 3.4.2, and so one should think of the cotangent complex as a simplicial version of the module of Kähler differentials.

Another characterization of the cotangent complex will be useful:

Proposition 3.4.6. *Let $Q\colon \perp^{*+1} A \to A$ be the augmentation map, and consider $J := \ker(\perp^{*+1} A \otimes_k A \xrightarrow{\varepsilon} A)$, where ε is given by $\varepsilon(x \otimes a) = Q(x)a$, then there is an isomorphism of simplicial A-modules:*

$$\mathbb{L}_{A/k} \cong J/J^2$$

Proof. Observe that because $\perp^{n+1} A$ is a free k module for all n, Lemma 3.4.4 implies that we have a levelwise isomorphism

$$(J/J^2)_n \cong A \otimes_{\perp^{n+1} A} \Omega_{\perp^{n+1} A/k}$$

Moreover, these isomorphisms commute with the face and boundary maps because the face and boundary maps for both are obtained by holding the A factor in the tensor product fixed and applying the face and boundary maps of the simplicial object $\perp^{*+1} A$. □

This characterization will be key in Section 5.3.

André-Quillen homology is now defined in a manner very similar to how Tor is defined as an example of comonad homology, except that it is not actually an instance of comonad homology, as the simplicial object is not a comonad itself, but is rather just assembled from one.

Definition 3.4.7. The André-Quillen homology of A with coefficients in the module M is defined as

$$AQ_n(A/k, M) := \pi_n(M \otimes_A \mathbb{L}_{A/k})$$

Clearly, $AQ_n(A/k, A) \cong \pi_n(\mathbb{L}_{A/k})$ and $AQ_n(k/k, M) \cong 0$ for all $n \geq 0$. We will write $AQ_n(A/k)$ in place of $AQ_n(A/k, A)$.

Slightly less trivially, we have the following:

Proposition 3.4.8. *There is an isomorphism:*

$$AQ_0(A/k, M) \cong M \otimes_A \Omega_{A/k}$$

Proof. This is purely formal. The augmented simplicial set $U(\perp^{*+1} A) \to UA$ is aspherical, meaning that $\perp^{*+1} A \to A$ is a simplicial resolution/cofibrant replacement of A (meaning $\pi_0(\perp^{*+1} A) \cong A$ and the higher homotopy groups vanish). It is now a general fact from homological algebra that if we apply $M \otimes_A (A \otimes_- \Omega_{-/k})$ level-wise to $\perp^{*+1} A$, the resulting chain complex (under the Dold-Kan correspondence) will have the 0-th homology group equal to $M \otimes_A (A \otimes_A \Omega_{A/k}) \cong M \otimes_A \Omega_{A/k}$. Hence, $\pi_0(M \otimes_A \mathbb{L}_{A/k}) = AQ_0(A/k, M) \cong M \otimes_A \Omega_{A/k}$. $\square$

There is of course significantly more could say about André-Quillen homology. However, for the purposes of this paper we will need little more than the definition. For the reader who is interested, [Iye07] provides a very readable introduction to the subject with a different, more algebraic approach. Alternatively, [GS06] provides an introduction from a model category perspective, treating André-Quillen homology as a derived abelianization functor.

Chapter 4

Algebraic Goodwillie Calculus

Algebraic (also known as discrete) Goodwillie calculus is another flavour of Goodwillie's homotopy calculus. Algebraic calculus was first developed by Johnson and McCarthy in their paper *Deriving calculus with cotriples* (see [JM03]), and phrased in a more general setting by Bauer, Johnson, and McCarthy in their paper *Cross effects and calculus in an unbased setting* (see [BJM11]). In this section, we will develop both simultaneously and contrast the constructions.

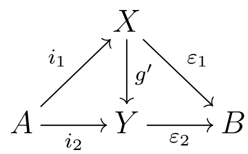
4.1 The Setting

In [JM03] where the based setting is developed, functors from a pointed category $\mathbf{C}$ (meaning there is an object $*$ which is both initial and terminal) with finite coproducts to an abelian category $\mathbf{A}$ are studied. Throughout the rest of this chapter, when working in the based setting, $\mathbf{C}$ and $\mathbf{A}$ will refer to categories with these hypotheses. We will borrow the notation for the coproduct in $\mathbf{Top}_*$ and write $X \vee Y$ for the coproduct of objects X and Y in $\mathbf{C}$ to emphasize the fact that $\mathbf{C}$ is pointed.

The setting is more general in [BJM11], and we take some time now to introduce the notation.

Definition 4.1.1. If $\mathbf{C}$ is a category and $f\colon A \to B$ is a morphism in $\mathbf{C}$, we define the category $\mathbf{C}_f$ to be the category whose objects are factorizations $A \stackrel{i}{\to} X \stackrel{\varepsilon}{\to} B$ of f (which we will write as (X, i, ε), or simply X if the structure maps are clear, and where a morphism

$g\colon (X, i_1, \varepsilon_1) \to (Y, i_2, \varepsilon_2)$ is a map $g'\colon X \to Y$ in $\mathbf{C}$ such that we have a commuting diagram:



The unbased setting studies functors $\mathbf{C}_f \to \mathbf{D}$ where $\mathbf{C}$ and $\mathbf{D}$ are simplicial model categories and $\mathbf{D}$ is stable, pointed, and right proper. This notation will be used without comment when working in the unbased setting for the remainder of this chapter. Note that for any object X of $\mathbf{C}$, we have $\mathbf{C}_{\mathrm{id}_X}$ is a pointed category with basepoint X. Also, note that $\mathbf{C}_f$ inherits a model structure from $\mathbf{C}$ by defining a map to be a weak equivalence/cofibration/fibration if its underlying map in $\mathbf{C}$ is.

If we have a functor $F\colon \mathbf{C} \to \mathbf{A}$ as in the based setting where $\mathbf{C}$ is a (not necessarily simplicial) model category, we get a functor $F\colon \mathbf{sC}_f \to \mathbf{sA}$ in the unbased setting with f the identity map on the basepoint in $\mathbf{C}$. Often times it is even easier as $\mathbf{C}$ and $\mathbf{D}$ will already be simplicial model categories in which case we need only take f to be the identity map on the basepoint to pass to the 'unbased' setting.

4.2 Cross effects

Cross effects are the functors we will use to recognize polynomial functors. The idea is borrowed from the classical setting of real-valued functions, where the $(n+1)$st cross effect of a function $f\colon \mathbb{R} \to \mathbb{R}$ is a function $\mathrm{cr}_{n+1}\colon \mathbb{R}^{n+1} \to \mathbb{R}$ which vanishes iff f is polynomial of degree n. A categorification of this notion is how we will characterize degree n functors, and later, we will use this to define our polynomial approximations.

4.2.1 The Based Setting

The definition of the cross effects presented here will differ from the definition given in [JM03]. However, as we shall see in Proposition 4.2.6 our definition is equivalent. This choice was made because our definition of the cross effects will translate very easily to the definition used in the unbased setting.

First, we wish to restrict our attention to reduced functors.

Definition 4.2.1. A functor $F\colon \mathbf{C} \to \mathbf{A}$ is said to be *reduced* if $F(*) = 0$.

With this hypothesis, we will define the cross effect functors.

Note that because $\mathbf{C}$ is pointed, the compositions $* \to X \xrightarrow{\text{id}} X$ and $* \to Y \xrightarrow{0} X$ are equal where the map 0 is the composition $Y \to * \to X$. Hence, the universal property of the pushout gives us a map $X \vee Y \to X$ (and similarly for Y). Thus, we have a commuting square:

$$\begin{CD} X \vee Y @>>> X \\ @VVV @VVV \\ Y @>>> * \end{CD}$$

If we apply F to this square, the universal property of pullbacks gives us a map $\alpha\colon F(X \vee Y) \to F(X) \oplus F(Y)$ (recall that our target category is abelian and so has finite biproducts[1]). Moreover, we also have a commuting square

$$\begin{CD} * @>>> X \\ @VVV @VVV \\ Y @>>> X \vee Y \end{CD}$$

and if we apply F to it, the universal property for pushouts yields a map $\iota\colon F(X) \oplus F(Y) \to$

[1] Recall that a biproduct of X and Y is an object Z which satisfies the universal property of both the product and coproduct of X and Y.

$F(X \vee Y)$. Putting this together, we get a commuting diagram:

$$
\begin{array}{ccccc}
0 & \longrightarrow & F(X) & & \\
\downarrow & & \downarrow & & \\
F(Y) & \longrightarrow & F(X) \oplus F(Y) & & \\
 & & \downarrow{\scriptstyle \iota} & & \\
 & & F(X \vee Y) & \xrightarrow{\alpha} & F(X) \oplus F(Y)
\end{array}
$$

Further, the two possible compositions from 0 in the top left to $F(X) \oplus F(Y)$ in the bottom right are trivially equal, and so by the uniqueness of the universal map in a pushout, the composition $\alpha\iota$ is the identity. Hence, we get a split exact sequence as follows:

$$0 \to \ker \alpha \to F(X \vee Y) \stackrel{\alpha}{\to} F(X) \oplus F(Y) \to 0$$

The kernel of α above is defined to be the second cross effect of F at (X, Y) (we will see the definition of cross effects in general in Definition 4.2.3) and is denoted $\operatorname{cr}_2 F(X, Y)$. Note that because the sequence is split, we get the identity:

$$\operatorname{cr}_2 F(X, Y) = \frac{F(X \vee Y)}{F(X) \oplus F(Y)}$$

This construction has a very nice alternative formulation:

Lemma 4.2.2. *If F is reduced, then we may equivalently define $\operatorname{cr}_2 F(X, Y)$ to be the total fibre:*

$$\operatorname{cr}_2 F(X, Y) = \operatorname{tfib} \left(\begin{array}{ccc} F(X \vee Y) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(*) \end{array} \right)$$

Proof. By definition, the total fibre of the given square is the kernel (that is, fibre) of the map

$$F(X \vee Y) \to \lim(F(Y) \to F(*) \leftarrow F(X)) = F(X) \oplus F(Y)$$

This map is just α giving us the result. $\square$

We may now define $\operatorname{cr}_n F$ for arbitrary n inductively. For ease of notation, we will begin by making some definitions. Note that we have a map

$$X_1 \vee \cdots \vee X_n \to X_1 \vee \cdots \vee \hat{X}_i \vee \cdots \vee X_n$$

for any $1 \leq i \leq n$ given as the identity except on the i-th component where it is the unique map to the basepoint. Then given any tuple $\vec{X} = (X_1, \ldots, X_n)$, we may define an n-cube $\bigvee \vec{X} \colon \mathcal{P}(\underline{n}) \to \mathbf{C}$ given by

$$\bigvee \vec{X}(S) = \bigvee_{i \in \underline{n} \setminus S} X_i$$

where the maps are defined as above, and we take the coproduct to be $*$ if the indexing set is empty.

Definition 4.2.3. For a functor $F \colon \mathbf{C} \to \mathbf{A}$, we define the *n-th cross effect* of F at a tuple $\vec{X} = (X_1, \ldots, X_n)$ as

$$\operatorname{cr}_n F(\vec{X}) := \operatorname{tfib}\left(F\left(\bigvee \vec{X}\right)\right)$$

We write $\perp_n F(X)$ as a shorthand for $\operatorname{cr}_n F(X, \ldots, X)$ (the reason for this notation will become clear later).

Note that the definition was made for *all* functors, and not just reduced ones. In particular, we see $\operatorname{cr}_1 F(X) = \ker(F(X) \to F(*))$. Moreover, this map splits and so $\operatorname{cr}_1 F(X) \cong F(X)/F(*)$, the reduction of F. If F is reduced, then the $n = 2$ construction in this definition is precisely the result of Lemma 4.2.2. Having defined the cross effects, we begin with two preliminary observations:

Lemma 4.2.4 ([JM03], Proposition 1.2)**.** *For $F \colon \mathbf{C} \to \mathbf{A}$ we have:*

1. *If $\sigma \in \Sigma_n$, then $\operatorname{cr}_n F(X_1, \ldots, X_n) \cong \operatorname{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$. That is, $\operatorname{cr}_n F$ is symmetric.*

2. If $X_i = *$ for any $1 \le i \le n$, then $\operatorname{cr}_n F(X_1, \dots, X_n) \cong *$

Proof. The first one is clear since coproducts commute. The second follows by repeatedly applying the fact that the total fibre of an n-cube is is the fibre of the total fibres of its $(n-1)$-cube faces obtained by writing it as a map of $(n-1)$-cubes. □

For the next lemma, observe that by fixing the first $(n-1)$ variables we may view $\operatorname{cr}_n F$ as a function of one variable, and then consider the second cross effects of it. In particular, we have the following:

Lemma 4.2.5 ([BJM11], Lemma 3.7)**.** *For $F\colon \mathbf{C} \to \mathbf{A}$, there is an isomorphism:*

$$\operatorname{cr}_n F(X_1, \dots, X_n) \cong \operatorname{cr}_2(\operatorname{cr}_{n-1} F(X_1, \dots, X_{n-2}, -))(X_{n-1}, X_n)$$

Proof. We begin with the right hand side of the isomorphism. By definition, this is the total fibre of the square:

$$\begin{array}{ccc} \operatorname{cr}_{n-1} F(X_1, \dots, X_{n-2}, X_{n-1} \vee X_n) & \longrightarrow & \operatorname{cr}_{n-1} F(X_1, \dots, X_{n-2}, X_{n-1}) \\ \downarrow & & \downarrow \\ \operatorname{cr}_{n-1} F(X_1, \dots, X_{n-2}, X_n) & \longrightarrow & \operatorname{cr}_{n-1} F(X_1, \dots, X_{n-2}, *) = * \end{array} \tag{4.2.1}$$

By definition, each corner is the total fibre of a certain $(n-1)$-cube, and hence we can write (4.2.1) as a 2-cube of $(n-1)$-cubes using the non-homotopical analog of Lemma 3.2.3. In particular, it is the total fibre of the $(n+1)$-cube:

$$\begin{array}{ccc} F\left(\bigvee \vec{X}_0\right) & \longrightarrow & F\left(\bigvee \vec{X}_1\right) \\ \downarrow & & \downarrow \\ F\left(\bigvee \vec{X}_2\right) & \longrightarrow & F\left(\bigvee \vec{X}_{12}\right) \end{array} \tag{4.2.2}$$

where

$$\vec{X}_0 = (X_1, \dots, X_{n-2}, X_{n-1} \vee X_n)$$
$$\vec{X}_1 = (X_1, \dots, X_{n-2}, X_{n-1})$$
$$\vec{X}_2 = (X_1, \dots, X_{n-2}, X_n)$$
$$\vec{X}_{12} = (X_1, \dots, X_{n-2}, *)$$

Each of the four $(n-1)$-cubes in (4.2.2) may be written as a map of $(n-2)$-cubes as follows: let $\mathscr{Z}_T$ be the $(n-1)$-cube $F(\bigvee X_T)$ where $T \subseteq \{1,2\}$ and set $\mathscr{X}_T(S) = \mathscr{Z}_T(S)$ and $\mathscr{Y}_T(S) = \mathscr{Z}_T(S \cup \{n-1\})$ for $S \subseteq \{1, \dots, n-2\}$. Applying Lemma 3.2.3 again, it follows we may write (4.2.2) as the fibre of the map of 2-cubes constructed out of either $\mathscr{X}_T$'s or $\mathscr{Y}_T$'s. However, the one constructed out of $\mathscr{Y}_T$'s has a vanishing total fibre, as all four $\mathscr{Y}_T$'s are the same. Hence, it follows that the total fibre of (4.2.2) is the total fibre of the n-cube:

$$\begin{array}{ccc} \mathscr{X}_0 & \longrightarrow & \mathscr{X}_1 \\ \downarrow & & \downarrow \\ \mathscr{X}_2 & \longrightarrow & \mathscr{X}_{12} \end{array} \tag{4.2.3}$$

However, this is exactly the definition of $\mathrm{cr}_n F(X_1, \dots, X_n)$ in Definition 4.2.3, finishing the proof. □

Proposition 4.2.6. *There is a recursive formula for the cross effects given by:*

$$\mathrm{cr}_n F(X_1, \dots, X_n) \oplus \mathrm{cr}_{n-1} F(X_1, \dots, X_{n-1}) \oplus \mathrm{cr}_{n-1} F(X_1, \dots, X_{n-2}, X_n)$$
$$\cong \mathrm{cr}_{n-1} F(X_1, \dots, X_{n-2}, X_{n-1} \vee X_n)$$

Proof. This is just Lemma 4.2.2 together with Lemma 4.2.5. □

The result in Proposition 4.2.6 is given as the definition of the cross effects in [JM03].

Note 4.2.7. We may recursively apply Proposition 4.2.6 to $F(X_1 \vee \cdots \vee X_n)$ to obtain:

$$\begin{aligned} F(X_1 \vee \cdots \vee X_n) &\cong \operatorname{cr}_1 F(X_1 \vee \cdots \vee X_n) \oplus F(*) \\ &\cong \operatorname{cr}_2 F(X_1, X_2 \vee \cdots \vee X_n) \oplus \operatorname{cr}_1 F(X_1) \oplus \operatorname{cr}_1 F(X_2 \vee \cdots \vee X_n) \oplus F(*) \\ &\cong \cdots \\ &\cong F(*) \oplus \left(\bigoplus_{k=1}^{n} \left(\bigoplus_{1 \leq j_1 \leq \cdots j_k \leq n} \operatorname{cr}_k(X_{j_1}, \ldots, X_{j_k}) \right) \right) \end{aligned}$$

Note in particular that we have a unique projection map $\pi \colon F(X_1 \vee \cdots \vee X_n) \to \operatorname{cr}_n F(X_1, \ldots, X_n)$ and a unique inclusion $\iota \colon \operatorname{cr}_n F(X_1, \ldots, X_n) \to F(X_1 \vee \cdots \vee X_n)$.

Having defined the cross effects in general, we may now define what it means for a functor to be polynomial.

Definition 4.2.8. A functor $F \colon \mathbf{C} \to \mathbf{A}$ is said to have *degree* n if $\operatorname{cr}_{n+1} F \cong *$. It is linear if it is both degree 1 and reduced. If $\mathbf{A}$ is the category of chain complexes in some abelian category, then we will weaken this notion to only requiring that $\operatorname{cr}_{n+1} F$ be quasi-isomorphic to $*$.

Note 4.2.9. In fact, when we write the category $\mathbf{Ch}_{\geq 0}(\mathbf{A})$ we are implicitly talking about the category of chain complexes with the localizing subcategory of the class of quasi-isomorphisms (or using the language of 2.1 it is the homotopy category of $\mathbf{Ch}_{\geq 0}(\mathbf{A})$). When discussing functors $F \colon \mathbf{Ch}_{\geq 0}(\mathbf{A}_1) \to \mathbf{Ch}_{\geq 0}(\mathbf{A}_2)$ we will also take the convention that these preserve the localization, meaning they preserve quasi-isomorphisms. Hence it follows that every such functor is in fact reduced.

Note 4.2.10. In the classical setting, we define the cross effects in a way analogous to the definition in Proposition 4.2.6. If $f\colon \mathbb{R} \to \mathbb{R}$, then we set:

$$\begin{aligned}
\operatorname{cr}_0 f(x) &= f(0) \\
\operatorname{cr}_1 f(x) &= f(x) - f(0) \\
&\vdots \\
\operatorname{cr}_n f(x_1, \ldots, x_n) &= \operatorname{cr}_{n-1} f(x_1, \ldots, x_{n-2}, x_{n-1} + x_n) \\
&\quad - (\operatorname{cr}_{n-1} f(x_1, \ldots, x_{n-1}) + \operatorname{cr}_{n-1} f(x_1, \ldots, x_{n-2}, x_n)
\end{aligned}$$

One can show by induction that if f is a polynomial of degree n, then $\operatorname{cr}_{n+1} f$ is identically 0.

We present a handful of trivial examples of degree n functors below. More involved examples will be given later.

Example 4.2.11. If $\mathbf{A}$ is an abelian category, then the identity functor $\operatorname{id}\colon \mathbf{A} \to \mathbf{A}$ is linear. Indeed, it is reduced, and

$$\operatorname{cr}_2 \operatorname{id}(A, B) = \frac{\operatorname{id}(A \oplus B)}{\operatorname{id}(A) \oplus \operatorname{id}(B)} \cong 0$$

This is a stark contrast to the traditional Goodwillie tower of [Goo03]. In this setting, the identity functor $\mathbf{Top}_* \to \mathbf{Top}_*$ is non-linear, and in fact does not have finite degree.

Slightly less trivially, we have the following:

Example 4.2.12. Let E be any reduced homology theory on the category $\mathbf{CW}_*$ of pointed CW complexes which satisfies the dimension axiom[2]. Then for $n \in \mathbb{Z}$, $E_n\colon \mathbf{CW}_* \to \mathbf{Ab}$ is linear because of the wedge axiom, implying that $E_n(X \vee Y) \cong E_n(X) \oplus E_n(Y)$. Note that if we drop the assumption that E satisfies the dimension axiom, then it is no longer true that E is necessarily linear, as in this case $\operatorname{cr}_2 E_n(X, Y) \cong E_n(*)$ which is potentially non-zero.

[2] By a reduced homology theory, we mean in the sense of the Eilenberg-Steenrod axioms. See [Hat01, Section 2.3] for an explicit statement of these axioms.

An explicit example of a non-linear functor is as follows:

Example 4.2.13. Fix k a commutative ring, and let $T^n\colon \mathbf{Mod}_k \to \mathbf{Mod}_k$ be defined by $T^n(A) = A^{\otimes n}$. Then T^n is degree n (see [JM03]).

Definition 4.2.14. A functor $G\colon \mathbf{C}^n \to \mathbf{A}$ is said to be *n-multireduced* if $G(X_1, \ldots, X_n) \cong *$ whenever $X_i = *$ for any $1 \leq i \leq n$. We will denote by $\mathrm{Fun}_n(\mathbf{C}^n, \mathbf{A})$ the category of n-reduced functors. We will write $\mathrm{Fun}_*(\mathbf{C}, \mathbf{A})$ instead of $\mathrm{Fun}_1(\mathbf{C}, \mathbf{A})$, as this is the category of reduced functors.

Note 4.2.15. It is worth mentioning that the category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ will in general be a *large* category unless $\mathbf{C}$ is small. This means that the hom sets in $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ are not proper sets.

By Lemma 4.2.4, cr_n is n-reduced. Moreover, it is clear that the assignment $F \mapsto \mathrm{cr}_n F$ is functorial, and thus cr_n is a functor $\mathrm{Fun}(\mathbf{C}, \mathbf{A}) \to \mathrm{Fun}_n(\mathbf{C}^n, \mathbf{A})$. The key observation in the construction of the algebraic tower is that this functor has an adjoint.

Proposition 4.2.16 ([JM03], Example 1.8)**.** *The functor* cr_n *admits a left adjoint* $\Delta_n\colon \mathrm{Fun}_n(\mathbf{C}^n, \mathbf{A}) \to \mathrm{Fun}(\mathbf{C}, \mathbf{A})$ *defined by*

$$\Delta_n(F)(X) = F(X, \ldots, X)$$

Sketch of Proof. The proof in [JM03] proceeds by constructing natural inverse maps

$$\Phi\colon \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C},\mathbf{A})}(\Delta_n F, G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^n,\mathbf{A})}(F, \mathrm{cr}_n G)$$

$$\Psi\colon \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^n,\mathbf{A})}(F, \mathrm{cr}_n G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C},\mathbf{A})}(\Delta_n F, G)$$

using the maps π and ι from Note 4.2.7. ∎

Note that $\perp_n F = \Delta_n \mathrm{cr}_n(F)$, where $\perp_n$ is defined in Definition 4.2.3. Hence, $\perp_n$ forms a comonad.

The following fact will be used in Chapter 5. While stated in [JM03], the details were left to the reader. We present them here in full.

Proposition 4.2.17 ([JM03], Example 1.7)**.** *Let* $F\colon \mathbf{C} \to \mathbf{A}$*. Then* $p_nF := \operatorname{coker}(\perp_{n+1} F \xrightarrow{\varepsilon} F)$ *has degree* n*, where the map* ε *is the counit in adjunction* $\Delta_n \dashv \operatorname{cr}_n$*.*

Proof. First, note that by recursively expanding out the definition of the cross effect in Proposition 4.2.6 we find that:

$$\operatorname{cr}_n F(X_1, \dots, X_n) \cong \operatorname{coker}\left(\bigoplus_{k=1}^{n} F(X_1 \vee \cdots \vee \hat{X}_k \vee \cdots \vee X_n) \to F\left(\bigvee_{k=1}^{n} X_k\right)\right)$$

In our case, p_nF being a colimit (the cokernel is the pushout along the map to the terminal object) implies that it commutes with direct sums and other cokernels. Hence, we have that p_nF commutes with cross effects, and so:

$$\operatorname{cr}_{n+1}(p_nF) = \operatorname{cr}_{n+1} \operatorname{coker}(\perp_{n+1} F \to F) \cong \operatorname{coker}(\operatorname{cr}_{n+1}(\perp_{n+1} F) \to \operatorname{cr}_{n+1} F)$$

Now, the map $\operatorname{cr}_{n+1}(\varepsilon)\colon \operatorname{cr}_{n+1}(\perp_{n+1} F) \to \operatorname{cr}_{n+1} F$ has a section $\eta_{\operatorname{cr}_{n+1} F}$ by the unit-counit formula, and hence the map $\operatorname{cr}_{n+1}(\varepsilon)$ is a (split) epimorphism. Since epimorphisms are stable under cobase change, it follows that the map $* \to \operatorname{cr}_{n+1}(p_nF)$ is an epimorphism, and hence $\operatorname{cr}_{n+1}(p_nF) \cong *$. □

4.2.2 The Unbased Setting

With our alternate formulation of the cross effects functor, the definition carries over relatively directly to the unbased case (and it was in fact the unbased case which inspired the perspective take in our treatment of the based case). We only must modify our definition of $\bigvee \vec{X}$ (see Definition 4.2.3 for an explanation of this notation) in light of the fact that we don't have a basepoint. Recall that in the unbased setting, our domain category consists of factorizations of a fixed map $f\colon A \to B$. This means that B is a terminal object in this setting. Hence, we have maps:

$$X_1 \amalg_A \cdots \amalg_A X_i \amalg_A \cdots \amalg_A X_n \to X_1 \amalg_A \cdots \amalg_A B \amalg_A \cdots \amalg_A X_n$$

for any $1 \leq i \leq n$, where the B on the right hand side has replaced X_i. Then given any tuple $\vec{X} = (X_1, \ldots, X_n)$, we may define an n-cube $\coprod \vec{X} \colon \mathcal{P}(\underline{n}) \to \mathbf{C}$ given by

$$\coprod \vec{X}(S) = \coprod_{i \in \underline{n}} Z_i$$

where $Z_i = X_i$ if $i \notin S$ and $Z_i = B$ if $i \in S$. With this, we now make the obvious definition:

Definition 4.2.18. For a functor $F \colon \mathbf{C}_f \to \mathbf{D}$, we define the *$n$-th cross effect* of F at a tuple $\vec{X} = (X_1, \ldots, X_n)$ as

$$\mathrm{cr}_n F(\vec{X}) := \mathrm{tfib}\left(F\left(\coprod \vec{X}\right)\right)$$

It is worth pointing out that the results in Lemma 4.2.5 and Lemma 4.2.4 hold, and the proofs are almost identical. However, the result of Proposition 4.2.6 doesn't carry over as we do not have an explicit formula for the second cross effect.

Definition 4.2.19. A functor $G \colon \mathbf{C}_f^n \to \mathbf{D}$ is said to be *weakly n-reduced* if $G(X_1, \ldots, X_0) \simeq *$ if $X_i \simeq *$ for any $1 \leq i \leq n$. We will denote by $\mathrm{Fun}_n(\mathbf{C}_f^n, \mathbf{D})$ the category of homotopy classes of weakly reduced functors.

As in the based setting, cr_n defines a functor $\mathrm{Fun}(\mathbf{C}_f, \mathbf{D}) \to \mathrm{Fun}_n(\mathbf{C}_f^n, \mathbf{D})$. Moreover it has an adjoint Δ_n up to homotopy given by the same formula. However, the proof of this fact becomes surprisingly more subtle. We refer the reader to [BJM11] for a full proof of this fact.

4.3 The Algebraic Tower

Having established the existence of left adjoints to the cross effects functors, we can now give the construction of the polynomial approximations. In both settings, the definition of the polynomial approximation to a functor F will make use of the adjunction $\Delta_n \dashv \mathrm{cr}_n$ to construct a comonad $\perp_n F := \Delta_n \mathrm{cr}_n(F)$.

4.3.1 The Based Setting

In the based setting, [JM03] gives a definition for functors $\mathbf{C} \to \mathbf{A}$ for an arbitrary abelian category, and a slightly different definition when the abelian category is some form of chain complexes (in some abelian category). We will only give the construction in the case when the target category is chain complexes, as this will be the case of interest to us, and is also the case which translates most directly to the definition in the unbased case.

Definition 4.3.1. For $F\colon \mathbf{C} \to \mathbf{Ch}_{\geq 0}(\mathbf{A})$ a reduced functor, we define n-th polynomial approximation P_nF to be:

$$P_nF := \mathrm{MappingCone}[| \perp_{n+1}^{*+1} F| \to |\mathrm{id}^{*+1}F| \simeq F]$$

Denote by p_nF the induced map $F \to P_nF$.

We are requiring our functors to be reduced, as this is a requirement for the comonad $\perp_{n+1}$ to exist (by Proposition 4.2.16). If F is not reduced, then we may replace F by $\mathrm{cr}_1 F$, which is reduced. Note that the map p_nF of this theorem differs from the object p_nF of Proposition 4.2.17. This notation is unfortunate, though standard.

Again, our definition differs from the definition of P_nF given in [JM03]. We will see that our definition is weakly equivalent to the one given there, and we take our definition as it makes it clear that the construction of the tower given in [BJM11] is indeed a generalization of the original construction in the based setting. Before giving our proof, we recall a number of theorems which we will use for the proof.

Firstly, we will need the Dold-Kan theorem. For a quick review of this theorem and the notation involved, see Theorem 2.3.15. For more details, the reader should consult a reference such as [Wei94].

Next, we will need the Dold-Puppe Theorem, a generalization of the Eilenberg-Zilber theorem. We recall here the statement of this theorem, and refer the reader to [GJ99, Theorem 2.5] for a proof.

Theorem 4.3.2 (Dold-Puppe)**.** *Let A be a bisimplicial abelian group. Let $C(A)$ denote the* Moore bicomplex *of A, which is the bicomplex obtained from A by applying the functor C level-wise. Then $|A|$ and $\operatorname{Tot} C(A)$ are chain homotopy equivalent.*

Finally, we will need the spectral sequences associated to a bicomplex. In particular, the following theorem will be needed:

Theorem 4.3.3. *Let C be a bicomplex. Denote the homology in the horizontal direction by $H'_*(C)$ and in the vertical direction by $H''_*(C)$. Then there are two first quadrant spectral sequences ${}'E^r_{p,q}$ and ${}''E^r_{p,q}$ such that:*

$$ {}'E^2_{p,q} = H'_p(H''_q(C)) \implies H_{p+q}(\mathrm{Tot}(C)), \qquad {}''E^2_{p,q} = H''_q(H'_p(C)) \implies H_{p+q}(\mathrm{Tot}(C)) $$

An introduction to spectral sequences, including a proof of this fact may be found in [Wei94, Section 5.6].

Theorem 4.3.4. *For $F\colon \mathbf{C} \to \mathbf{Ch}_{\geq 0}(\mathbf{A})$ define:*

$$ P^k_n F := \mathrm{MappingCone}[N(\perp^{*+1}_{n+1} F_k) \to N(\mathrm{id}^{*+1} F_k) \simeq F_k] $$

where $F_k(X)$ is the object occurring in the k-th spot of the chain-complex $F(X)$. Define $P'_n F$

to be the total complex of the bicomplex

$$\cdots P_n^3 F \to P_n^2 F \to P_n^1 F \to P_n^0 F$$

Then $P'_n F(X)$ is quasi-isomorphic to $P_n F(X)$ for every object X of $\mathbf{C}$.

Proof. Using [JM03, Definition 2.4], $P_n^k F$ is naturally chain homotopy equivalent to the chain complex $C^{\perp_{n+1}}(F_k)$ where $C_m^{\perp_{n+1}}(F_k) = \perp_{n+1}^m F_k$ (with the convention $\perp_{n+1}^0 F_k = F_k$) with boundary maps given by $\partial_n = \sum_{i=0}^{n-1}(-1)^i d_i$ for $n > 1$ and $\partial_1 = \varepsilon$. Hence, by definition $P'_n F(X)$ is the total complex of the bicomplex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \perp^2 F(X)_2 & \longrightarrow & \perp^2 F(X)_1 & \longrightarrow & \perp^2 F(X)_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \perp F(X)_2 & \longrightarrow & \perp F(X)_1 & \longrightarrow & \perp F(X)_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & F(X)_2 & \longrightarrow & F(X)_1 & \longrightarrow & F(X)_0
\end{array} \tag{4.3.1}$$

Conversely, if we let K be the inverse to N in the Dold-Kan correspondence, then we note that by applying K level-wise to the $\perp_{n+1}^{*+1} F$ we obtain a bisimplicial object of $\mathbf{A}$ which we will denote $K(\perp_{n+1}^{*+1} F)$. By the Dold-Puppe theorem, there is a quasi-isomorphism from $C|K(\perp_{n+1}^{*+1} F)|$ to the totalization of the Moore bicomplex associated to $K(\perp_{n+1}^{*+1} F)$, which we will denote $C(K(\perp_{n+1}^{*+1} F))$. Moreover, as the Moore complex and the normalized chain complex are also quasi-isomorphic, it follows that $C|K(\perp_{n+1}^{*+1} F)| \cong N|K(\perp_{n+1}^{*+1} F)|$, and interpreting the geometric realization as the diagonal, it is clear that the right hand side is weakly equivalent to $|\perp_{n+1}^{*+1} F|$. On the other hand, the spectral sequence of a double complex implies that taking normalization in the same direction as we applied K is a quasi-isomorphism after taking totalization. So summarizing, we have:

$$| \perp_{n+1}^{*+1} F| \cong N|K(\perp_{n+1}^{*+1} F)| \xrightarrow{\simeq} \operatorname{Tot} C(\perp_{n+1}^{*+1} F)$$

Since the mapping cone is invariant (up to quasi-isomorphism) under quasi-isomorphisms, this implies that

$$P_n F(X) \simeq \text{MappingCone}[\operatorname{Tot} C(\perp_{n+1}^{*+1} F(X)) \to F(X)]$$

However, the standard construction of the mapping cone implies that this is in fact equal to the total complex of the bicomplex in (4.3.1), finishing the proof. □

The next lemma indicates that the functors $P_n F$ as defined are indeed 'polynomial approximations' to F.

Lemma 4.3.5 ([JM03], Lemma 2.11)**.** *For $F\colon \mathbf{C} \to \mathbf{Ch}_{\geq 0}(\mathbf{A})$, we have that:*

1. *The functor $P_n F$ is degree n.*
2. *If F is degree n, then $p_n F\colon F \to P_n F$ is a quasi-isomorphism.*
3. *The map $p_n F$ is universal among all maps from F to a degree n functor. Explicitly, this means that if G is degree n, and $\theta\colon F \to G$ is a natural transformation, then there exists a unique natural transformation $\tilde{\theta}$ such that $\theta = \tilde{\theta} \circ p_n F$.*

Note that we have natural transformations $q_n\colon P_n \to P_{n-1}$ given by the composition

$$\operatorname{cr}_n F(X, \ldots, X) \longrightarrow \operatorname{cr}_{n-1} F(X \vee X, X, \ldots, X) \longrightarrow \operatorname{cr}_{n-1} F(X, \ldots, X)$$

where the first map is an inclusion (Proposition 4.2.6), and the second is $\operatorname{cr}_{n-1} F(+, \mathrm{id}, \ldots, \mathrm{id})$ (+ denotes the fold map given by the universal property of the coproduct). Moreover, the maps assemble to form a commuting diagram as follows:

$$\begin{array}{ccccccccc} & & & & F & & & & \\ & & & & \downarrow{\scriptstyle p_nF} & & & & \\ \cdots \longrightarrow & P_{n+1}F & \xrightarrow[q_{n+1}F]{} & & P_nF & \xrightarrow[q_nF]{} & P_{n-1}F \longrightarrow & \cdots \longrightarrow & P_0F \end{array} \tag{4.3.2}$$

4.3.2 The Unbased Setting

The definition of the towers P_nF given in Definition 4.3.1 carries over directly by noticing that the mapping cone is nothing more than a homotopy cofibre in the category of chain complexes. Hence, in the unbased setting we make the following definition.

Definition 4.3.6. For $F\colon \mathbf{C}_f \to \mathbf{D}$, we define the n-th polynomial approximation Γ_nF to be:

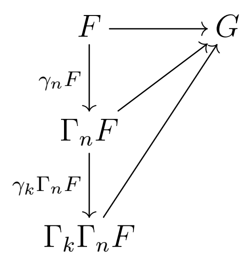
$$\Gamma_nF := \text{hocofib}[| \perp_{n+1}^{*+1} F| \to |\text{id}^{*+1}F| \simeq F|]$$

An analogous statement of Lemma 4.3.5 holds for the functors Γ_n. Moreover, we still have natural transformations $\Gamma_nF \to \Gamma_{n-1}F$ making the diagram analogous to (4.3.2) commute.

We have the following fact in both the based and unbased setting. This fact is stated in [JM03], though we present a different proof here. In particular, we give a proof which applies in the unbased case as well.

Lemma 4.3.7. *If $k < n$, then $\Gamma_k(\Gamma_nF) \simeq \Gamma_kF$. This holds in the based case with P_n in place of Γ_n.*

Proof. Let $\alpha\colon F \to G$ be any natural transformation, where G is degree k. Then since G is also degree n, the map factors through a map $\Gamma_nF \to G$. Since G is degree k, this factors through a map $\Gamma_k\Gamma_n F \to G$. Summarizing, we have a commuting diagram:



Hence, $\Gamma_k\Gamma_n F$ satisfies the same universal property as $\Gamma_k F$ (see Lemma 4.3.5), and hence they must be weakly equivalent. $\square$

4.4 The Layers of the Tower

The layers of the algebraic tower are to serve as an analogue of the individual terms in a traditional Taylor series. In this setting, the n-th term of the Taylor series may be found by taking the difference between the n-th Taylor polynomial and the $(n-1)$-st. In the categorical setting, we make a similar definition.

Definition 4.4.1. If $F\colon \mathbf{C} \to \mathbf{A}$ is a functor in the based setting, then we define the *n-th layer* of F as

$$D_n F := \mathrm{hofib}(P_n F \xrightarrow{q_n F} P_{n-1} F)$$

If $F\colon \mathbf{C}_f \to \mathbf{D}$ is a functor in the unbased setting, then we may define the n-th layer of F in the same way, with $\Gamma_n F$ in place of $P_n F$.

In the setting of traditional Taylor series, these layers are homogeneous polynomials of degree n in the sense that the Taylor approximation of degree k is zero for $k < n$ and an isomorphism for $k \geq n$. We make this definition in our setting as well.

Definition 4.4.2. A functor F in either setting is said to be *homogeneous of degree n* or *n-homogeneous* if $P_k F \simeq *$ for $k < n$ and $P_k F \simeq F$ for $k \geq n$. Of course, we use $\Gamma_n F$ in this definition in the unbased setting.

Note that if $P_{n-1}F \simeq *$, then it follows that F is n-homogeneous by Lemma 4.3.7. As one would hope, the following fact holds:

Proposition 4.4.3 ([JM03], Remark 2.14.7). *The layer D_nF is degree n (in both the based and unbased setting). It is n-homogeneous in the unbased setting if the target category of F is stable (for example the category of spectra), and in the based setting if we take our target category to be* $\mathbf{Ch}(\mathbf{A})$*, the category of* unbounded *chain complexes of objects in a cocomplete abelian category* $\mathbf{A}$.[3]

Proof. As the cross effect is defined as a homotopy limit, it commutes with the homotopy fibre (another homotopy limit). Hence,

$$\mathrm{cr}_{n+1} D_nF = \mathrm{cr}_{n+1} \mathrm{hofib}(\Gamma_n F \to \Gamma_{n-1}F) \simeq \mathrm{hofib}(\mathrm{cr}_{n+1} \Gamma_n F \to \mathrm{cr}_{n+1} \Gamma_{n-1}F) \simeq 0$$

So D_nF is degree n and so $\Gamma_k D_n F \simeq D_nF$ for $k \geq n$.

For the remaining result, the key observation is that in both cases, the target category is stable in the ∞-category sense. For the reader who is interested, [Lur09] would be the standard reference, though we only need the fact (which is part of the definition of being stable) that all homotopy fibre sequences are also homotopy cofibre sequences in a stable ∞-category. Hence, if H denotes the functor $\mathrm{hofib}(F \to G)$, then:

$$\mathrm{hofib}(| \perp_{*+1}^{n+1} F| \to | \perp_{*+1}^{n+1} G)|) \simeq | \perp_{*+1}^{n+1} H|$$

Now, consider the diagram obtained by taking homotopy cofibres horizontally and then

[3] *The polynomial approximations P_nF for a functor whose codomain is $\mathbf{Ch}(\mathbf{A})$ are defined using the obvious extension of the equivalent definition in Theorem 4.3.4.*

homotopy fibres vertically:

$$\begin{array}{ccccc}
|\perp_{*+1}^{n+1} H| & \longrightarrow & H & \longrightarrow & \mathrm{hofib}(\Gamma_n F \to \Gamma_n G) \\
\downarrow & & \downarrow & & \downarrow \\
|\perp_{*+1}^{n+1} F| & \longrightarrow & F & \longrightarrow & \Gamma_n F \\
\downarrow & & \downarrow & & \downarrow \\
|\perp_{*+1}^{n+1} G| & \longrightarrow & G & \longrightarrow & \Gamma_n F
\end{array}$$

Since iterated homotopy fibres commute and the bottom two rows are fibre sequences (as our category is stable), it follows that the top row is a fibre sequence, and hence a cofibre sequence as well. Thus, $\mathrm{hofib}(\Gamma_n F \to \Gamma_n G)$ is weakly equivalent to $\Gamma_n \,\mathrm{hofib}(F \to G)$. Hence if $k < n$, then:

$$\Gamma_k D_n F = \Gamma_k \,\mathrm{hofib}(\Gamma_n F \to \Gamma_{n-1} F) \simeq \mathrm{hofib}(\Gamma_k \Gamma_n F \xrightarrow{\simeq} \Gamma_k \Gamma_{n-1} F) \simeq *$$

□

It is worth mentioning that the proof given above differs from the proof in [JM03], and that this result does not occur in [BJM11].

In the based case, [JM03] give an exact formulation of the layers of the tower. For the remainder of this section, we restrict to the based case, though it is likely that similar results hold in the unbased setting.

Definition 4.4.4. If $F\colon \mathbf{C}^n \to \mathbf{A}$, the *n-multilinearizaton* of F, denote by $D_1^{(n)} F$, is the functor obtained as the composition:

$$D_1^{(n)} F := D_1^1 \circ D_1^2 \circ \cdots \circ D_1^n F$$

where $D_1^i F$ is D_1 applied to the functor obtained by holding all but the i-th variable of F constant.

The following theorem may be found in [JM03, Lemmas 3.8, 3.9].

Theorem 4.4.5. *Let* $F\colon \mathbf{C} \to \mathbf{Ch}_{\geq 0}(\mathbf{A})$*. Then* D_nF *is naturally equivalent to* $D_1^{(n)} \operatorname{cr}_n F_{h\Sigma_n}$*. If* F *is degree* n*, then the multilinearization is unnecessary, and* F *is naturally equivalent to* $\operatorname{cr}_n F_{h\Sigma_n}$

For a group G and a G-object A of $\mathbf{A}$, we let A_{hG} denote the *homotopy orbits* of A under the action of G. Explicitly, if we let $H\colon G \to \mathbf{A}$ be the functor defining the G-action on A (where we treat G as a category with one element), then

$$A_{hG} := \operatorname{hocolim}_G H$$

In the case of Theorem 4.4.5, $A = D_1^{(n)} \operatorname{cr}_n F(X_1, \ldots, X_n)$ has a Σ_n action given by permuting the X_i's. Note that while $\operatorname{cr}_n F$ is symmetric in its arguments, its multilinearization $D_1^{(n)} \operatorname{cr}_n F$ is not.

Chapter 5

André-Quillen Homology and the Algebraic Goodwillie Tower

The material in this chapter is based on work in [KM02]. We will focus on their result in Section 7.1 where it is shown that one may recover André-Quillen homology (and its higher analogues) as the layers of a particular based functor. In this section, we will walk through the proof of this fact and outline some of the details which were omitted in [KM02]. At the end of the section, we explore the possibility of generalizing this fact to the unbased case.

5.1 The Setting and Basic Facts

Throughout this chapter, k will denote a simplicial commutative ring containing $\mathbb{Q}$. The category $\mathbf{sCommAlg}_k$ does not have a base point (k is initial while 0 is terminal), and so to move to the based setting, we consider the category $\mathbf{s}(k\backslash\mathbf{CommAlg}/k)$ of *simplicial* augmented commutative k-algebras. One should recognize this category as $(\mathbf{sCommAlg}_k)_{\mathrm{id}_k}$, using the notation of Definition 4.1.1. For $A \in \mathbf{s}(k\backslash\mathbf{CommAlg}/k)$, we write $\varepsilon_A \colon A \to k$ for the augmentation map, and write $I(A)$ for its kernel (the so called *augmentation ideal*). Note that we get a short exact sequence

$$0 \longrightarrow I(A) \longrightarrow A \xrightarrow{\varepsilon} k \longrightarrow 0$$

and the map $k \to A$ giving A the structure of a k-algebra provides a splitting. Hence we have an isomorphism $A \cong k \oplus I(A)$ which is natural in A. This implies that I is in fact a functor $\mathbf{s}(k\backslash\mathbf{CommAlg}/k) \to \mathbf{sMod}_k$, and it is this functor that we will use to recover André-Quillen homology.

Definition 5.1.1. If A is a simplicial commutative algebra, then the *homotopy groups* are given by $\pi_n(A) := H_n(NA)$, where NA is the normalized chain complex of A (see Theorem 2.3.15). We say that a simplicial k-algebra is *n-connected* if $\pi_i(A) = 0$ for $i \leq n$. A map $f\colon A \to B$ of simplicial k-algebras is *n-connected* if it induces an isomorphism on the first $(n-1)$ homotopy groups, and a surjection on π_n.

We will need the following result shortly. A proof is given [KM02, Lemma 2.14].

Lemma 5.1.2. *If k is a commutative ring containing $\mathbb{Q}$ and L is an i-connected augmented simplicial k-algebra which is free in each dimension, then the map $I(L) \to I/I^2(L)$ is $(2i+1)$-connected.*

The category $\mathbf{sCommAlg}_k$ is a model category with model structure given as follows:

Theorem 5.1.3. *The category $\mathbf{sCommAlg}_k$ is a model category where a map $f\colon A \to B$ is a...*

1. *weak equivalence if $f_*\colon \pi_*(A) \to \pi_*(Y)$ is an isomorphism.*
2. *fibration if $N(f)\colon NA \to NB$ is surjective in dimensions greater than 0, where N denotes the normalized chain complex of the Dold-Kan correspondence.*

The cofibrations are thus determined.

Proof. See [GS06, Theorem 4.17]. □

We next wish to determine what a cofibrant replacement in $\mathbf{sCommAlg}_k$ looks like. First, we recall a definition from commutative algebra:

Definition 5.1.4. Let M be a module over a commutative ring R. Then the *tensor algebra* of M is the graded algebra $T(M)$ which in dimension n is $T^n(M) := M^{\otimes n}$ and $T^0(M) = R$. The *symmetric algebra* of M is the quotient $S(M)$ of $T(M)$ by the ideal generated by all differences of products $m \otimes n - n \otimes m$, with $m, n \in M$.

We may extend this definition level-wise to obtain a symmetric algebra functor $\mathbf{sMod}_k \to \mathbf{sCommAlg}_k$.

Now, the same criteria of being a weak equivalence or fibration applied to a a simplicial k-module gives a model structure on $\mathbf{sMod}_k$. Moreover, the maps $S\colon \mathbf{sMod}_k \to \mathbf{sCommAlg}_k$ given by the the symmetric algebra functor, and the forgetful functor $U\colon \mathbf{sCommAlg}_k \to \mathbf{sMod}_k$ are Quillen functors with S being the left Quillen functor. This implies $S(M)$ is cofibrant whenever M is a cofibrant simplicial module. Using the small object argument (see for example [Hir03, Proposition 10.5.16]) to construct a functorial cofibrant replacement in $\mathbf{sMod}_k$, we see that this construction in fact yields a free simplicial module, and so we have functorial free (cofibrant) replacement in $s\mathbf{Mod}_k$ given by $M \mapsto k[X_M]$ where X_M is some set depending functorially on M. Using the adjunction $S \dashv U$ and the fact that it is Quillen and thus preserves weak equivalences between cofibrant objects, it follows that we thus have a natural map $A \mapsto S \circ k[X_{UA}]$ which is a weak equivalence. Hence, this functor defines a functorial cofibrant replacement in $\mathbf{sCommAlg}_k$.

Definition 5.1.5. For $A \in \mathbf{sCommAlg}$ (or $\mathbf{s}(k\backslash\mathbf{CommAlg}/k)$), we let $L_A := S \circ k[X_{UA}]$. The simplicial algebra L_A is known as a *free resolution* of A.

Note that the assignment $A \mapsto L_A$ is functorial.

5.2 The Taylor Tower of I

Our goal of this section is to compute P_1. As $P_1 I$ is defined using the comonad $\perp_{n+1}^{*+1}$, our first goal is to compute $\operatorname{cr}_n I$. For this, we have the following:

Lemma 5.2.1 ([KM02], Lemma 4.3)**.** *There is an isomorphism*

$$\operatorname{cr}_n I(A_1, \ldots, A_n) \cong \bigotimes_{i=1}^{n} I(A_i)$$

Proof. The proof is a straightforward proof by induction using Proposition 4.2.6. As I is reduced, the theorem holds trivially for $n = 1$. For $n = 2$, we first note that

$$A_1 \otimes_k A_2 \cong (k \oplus I(A_1)) \otimes_k (k \oplus I(A_2)) \cong k \oplus I(A_1) \oplus I(A_2) \oplus (I(A_1) \otimes_k I(A_2))$$

and hence,

$$I(A_1 \otimes_k A_2) \cong (A_1 \otimes_k A_2)/k \cong I(A_1) \oplus I(A_2) \oplus (I(A_1) \otimes_k I(A_2))$$

Hence,

$$\operatorname{cr}_2 I(A_1, A_2) \cong \frac{I(A_1 \otimes_k A_2)}{I(A_1) \oplus I(A_2)} \cong I(A_1) \otimes_k I(A_2)$$

To rest of the proof is now just induction on n. □

In particular, this means that $\perp_n I = I^{\otimes n}$, which we will write as I^n.

The following lemma is given in both [JM03] and [KM02], but a proof is given in neither. We present a proof here.

Lemma 5.2.2 ([JM03], Lemma 6.6)**.** , *If* $G \colon \mathbf{C} \to \mathbf{C}'$ *is a reduced coproduct preserving functor and* $F \colon \mathbf{C}' \to \mathbf{Ch}_{\geq 0}(\mathbf{A})$ *then* $P_n(F \circ G) \cong (P_n F) \circ G$.

Proof. If $\vec{X} = (X_1, \ldots, X_n)$, then because G is coproduct preserving, we note that:

$$F \circ G \left(\bigvee \vec{X} \right) \cong F \left(\bigvee G(\vec{X}) \right)$$

where $G(\vec{X}) = (G(X_1), \ldots, G(X_n))$. Hence, $\operatorname{cr}_n(F \circ G)(X_1, \ldots X_n) \cong \operatorname{cr}_n F(G(X_1), \ldots G(X_n))$, and so $\perp_n (F \circ G) \cong (\perp_n F) \circ G$. Looking at the simplicial object associated to our comonad thus gives us that $\perp_n^{*+1} (F \circ G)(X) \cong (\perp_n^{*+1} F)(G(X))$. Finally:

$$
\begin{aligned}
P_n(F \circ G)(X) &= \mathrm{MappingCone}(|\perp_n^{*+1} (F \circ G)(X)| \to F \circ G(X)) \\
&\cong \mathrm{MappingCone}(|\perp_n^{*+1} F(G(X))| \to F \circ G(X)) \\
&= (P_n F)(G(X))
\end{aligned}
$$

□

Before presenting the main theorem of this section, we will need an alternative characterization of the first derivative of a functor $F\colon \mathbf{Ch}_{\geq 0}(\mathbf{Mod}_R) \to \mathbf{Ch}_{\geq 0}(\mathbf{Mod}_S)$ for R and S unital rings. This characterization is by the same authors as [JM03] and is given in their earlier paper [JM98].

Definition 5.2.3. Let $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ be a functor. The *prolongation* of F is a functor $\mathbf{F}\colon \mathbf{Ch}_{\geq 0}(\mathbf{Mod}_R) \to \mathbf{Ch}_{\geq 0}(\mathbf{Mod}_S)$ given by $\mathbf{F} = N \circ F \circ K$, where N and K are the functors occurring in the Dold-Kan correspondence (Theorem 2.3.15) and F is applied to a simplicial module level-wise.

Given a functor F as above which is also reduced, they give a construction $D_1 F$ known as the Dold-Puppe stabilization and is defined by:

$$D_1 F(X) := \operatorname*{colim}_{n \to \infty} \mathbf{F}(X[n])[-n]$$

where, for a chain complex C, $C[k]$ is the chain complex given in dimension n by $C[k]_n = C_{k+n}$. Using the Dold-Kan correspondence, we may obtain a similar formula when our functor is from $\mathbf{s}(k\backslash\mathbf{CommAlg}/k)$ to $\mathbf{sMod}_k$. Using the result of the Appendix in [JM98], we see that if F is the simplicial prolongation of a functor (meaning it is applied level-wise), then D_1F is a functor $\mathbf{s}(k\backslash\mathbf{CommAlg}/k) \to \mathbf{Ch}_{\geq 0}(\mathbf{sMod}_k)$ and is given by:

$$D_1F(X) = \operatorname*{colim}_{n\to\infty} F(d(B^nX))[-n] \tag{5.2.1}$$

where B denotes the bar construction and B^n denotes iterating the bar construction n times. Hence, B^nX is an n-simplicial object, and $d(B^nX)$ denotes the diagonal of this object. It is unimportant for our purposes as to how exactly the bar construction is defined. It will suffice for us to think of iterating the bar construction as a simplicial version of the shift functor $X \mapsto X[n]$ on chain complexes. The details of how this works may be found in the appendix to [JM98].

In this setting, [JM98, Lemma 5.4] states that the map $N \circ F \to D_1(F)$ is a natural quasi-isomorphism iff F is additive (or equivalently, reduced and of degree 1 in the sense of Definition 4.2.8). Hence, if we temporarily denote by $D_1'F$ the functor given in (5.2.1), then given any degree 1 functor G (which is automatically reduced by Note 4.2.9 and a map $F \to G$, we obtain a commuting diagram as follows:

$$\begin{array}{ccc} F & \longrightarrow & D_1'F \\ \downarrow & & \downarrow \\ G & \xrightarrow{\simeq} & D_1'G \end{array}$$

As the bottom map is a quasi-isomorphism, it is invertible up to chain homotopy, and hence $D_1'F$ satisfies the same universal property as D_1F, and hence $D_1F \simeq D_1'F$. It is this characterization of the derivative which we will use in the proof of Theorem 5.2.5.

Note 5.2.4. Before proceeding to the main theorem of this section, it is crucial to note that if X is an i-connected object of $\mathbf{s}(k\backslash\mathbf{CommAlg}/k)$, then BX is $(i+1)$-connected, and if C is an i-connected chain complex, then $C[-1]$ is $(i-1)$-connected. This can be seen by observing that the iterated bar construction is just the shift functor after applying the Dold-Kan theorem (see [JM98]), and the shift functor has this property. Applying this interpretation of the derivative to Lemma 5.1.2 means that for any augmented simplicial k-algebra L which is free in each dimension, we have that the map $D_1I(L) \to D_1I/I^2(L)$ is a weak equivalence (the down shift functor commutes with colimits).

Theorem 5.2.5 ([KM02], Theorem 4.5)**.** *We may identify P_1I as the left derived functor (in the model category sense) of I/I^2.*

Proof. First, we observe that $I/I^2 = p_1F$ by Lemma 5.2.1, and by Proposition 4.2.17 it follows that I/I^2 is linear. Hence, the universal property of P_1I implies there is a natural transformation $P_1I \to I/I^2$ factoring the natural map $I \to I/I^2$. The fact that the isomorphism $A \cong k \oplus I(A)$ is natural in A implies that I preserves weak equivalences, and hence P_1 does as well. Hence, precomposing with the free resolution functor L yields a map

$$P_1I \xrightarrow{\simeq} P_1I \circ L \longrightarrow I/I^2 \circ L$$

It is this map we claim is a weak equivalence. As the first map is already a weak equivalence, it suffices to show that the second map is as well.

The spectral sequence of a bicomplex implies that it is sufficient to show that $P_1I(L_n) \simeq I/I^2(L_n)$ for each $n \geq 0$. As $L_n = S(M)$ for a free simplicial k-module, it thus suffices to show that $P_1(I \circ S) \simeq I/I^2 \circ S$. Since S preserves coproducts (it is left adjoint to the forgetful functor to k-modules), Lemma 5.2.2 implies we have that

$$P_1(I/I^2 \circ S) \simeq P_1(I/I^2) \circ S \simeq I/I^2 \circ S$$

where the last equality is because I/I^2 is linear. Finally, we consider the following diagram:

$$\begin{array}{ccc} D_1(I \circ S) & \xrightarrow{\simeq} & D_1(I/I^2 \circ S) \\ \simeq\downarrow & & \downarrow\simeq \\ P_1(I \circ S) & \longrightarrow & P_1(I/I^2 \circ S) \end{array}$$

As both I and I/I^2 are reduced, the two vertical maps are weak equivalences. The top map is a weak equivalence due to the remark in Note 5.2.4. Hence, the two-out-of-three axiom for model categories implies the bottom map is as well, finishing the proof. □

5.3 Recovering André-Quillen Homology

In this section, we will make precise what exactly is meant by the statement 'André-Quillen homology is the derivative of I/I^2'. Most of the work has in fact already been done, and what remains is assembling the pieces to obtain this statement.

To begin, we first notice that by Proposition 3.4.2, it follows that

$$\Omega_{A/k} = I/I^2(A \otimes_k A)$$

where the augmentation map for $A \otimes_k A$ is the fold map given by the universal property of the coproduct. The crucial observation is that the cotangent complex of a functor is in fact just a cofibrant replacement of the functor $\Omega_{-/k}$.

Proposition 5.3.1. *We have a weak equivalence* $\mathbb{L}_{A/k} \simeq I/I^2(A \otimes_k L_A)$.

Proof. This fact is essentially Proposition 3.4.6. Using the notation of this proposition, all that remains to do is verify that $\perp^{*+1} A$ is indeed a cofibrant replacement of A. However this is immediate as it is free in each dimension (and thus cofibrant) and has homotopy vanishing in dimensions other than zero, where the zeroth homotopy group is A. By general model category theory, we thus have a weak equivalence $\perp^{*+1} A \to L_A$, and so as I/I^2 is a homotopy functor, we have that:

$$\mathbb{L}_{A/k} \cong I/I^2(A \otimes_k \perp^{*+1} A) \xrightarrow{\simeq} I/I^2(A \otimes L_A)$$

□

Hence, we have a composition of functors:

$$A \longrightarrow L_A \longrightarrow A \otimes_k L_A \longrightarrow I/I^2(A \otimes_k L_A) \simeq \mathbb{L}_{A/k}$$

Writing this another way, we have that $\mathbb{L}_{A/k} \simeq D_1 I(A \otimes_k -)(A)$, as $D_1 I$ is computed by taking a cofibrant replacement of A (meaning replace A with L_A) before evaluating. The fact that we must precompose I with $A \otimes_k -$ before looking at D_1 essentially stems from the fact that before doing so, our category has no basepoint. Thus, the functor $A \otimes_k -$ may be seen as adding a basepoint so that we may apply the based construction of the tower.

Chapter 6

Future Work

To conclude, we offer two potential topics for future research.

As was remarked at the end of Chapter 5, the functor $A \otimes_k -$ may be viewed as adding a basepoint. This suggests that perhaps the unbased construction of the tower may be applied directly to the functor I. It seems unlikely to this author that André-Quillen homology would be again recovered, as the n-th cross effect of I will not be the n-th tensor power in general. It is presently an open question as to what exactly is recovered by this construction.

Another possible direction for future work could involve looking into yet another setting for Goodwillie calculus. In his book *Higher Algebra* ([Lur12]), Jacob Lurie has reworked classical Goodwillie calculus from the ground up using the language of ∞-categories. However, it is not known how or whether his work can be applied in the algebraic or unbased cases. It seems likely that this is doable, and that doing so would help bridge the numerous 'flavours' of Goodwillie calculus currently in existence.

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