

## AN OPERATIONAL PROCEDURE FOR HANKEL TYPE INTEGRALS

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**ABSTRACT.** In this paper, an operational procedure is established to evaluate Hankel type integrals. First, an operator  $L(\theta)$ ,  $\theta \equiv -x \frac{d}{dx}$  is constructed, which defines the integral. Then making use of some basic properties of this operator, an elementary procedure is developed for evaluating integrals for a special class of analytic functions. A few examples are given to illustrate the technique.

**KEY WORDS AND PHRASES.** Differential operator of infinite order, Hankel transform, Mellin transform, Bessel functions, Hypergeometric functions.

**SUBJECT CLASSIFICATION:** 44A15

### 1. INTRODUCTION.

We consider the singular integral of the type

$$\int_0^{\infty} f(xt) J_{\nu}(2t) dt, \quad (1.1)$$

where  $J_{\nu}$  is the usual Bessel function of the first kind of order  $\nu$ ,  $\nu \geq -\frac{1}{2}$ , and  $f(x)$  is a suitable function. This integral can be viewed as defining the Hankel transform of the function  $f$ . In this note, our main aim is to construct an operator  $L(\theta)$ ,  $\theta \equiv -x \frac{d}{dx}$ , so that  $L(\theta)[f(x)]$  defines the integral (1.1), [cf. 1, §9.5; 2]. We then establish properties of the operator  $L(\theta)$ , and with these help of the properties, we shall obtain an operational procedure to evaluate the integral (1.1).

### 2. THE OPERATOR.

It is an easy matter to see that the differential operator  $\theta^n$ , where  $\theta \equiv -x \frac{d}{dx}$  and  $n$  a positive integer, is such that

$$\theta^n [x^{\alpha}] = (-\alpha)^n x^{\alpha},$$

for some constant  $\alpha$ . Then  $p_n(\theta)$ , a polynomial of  $n^{\text{th}}$  degree in  $\theta$ , gives

$$p_n(\theta)[x^{\alpha}] = p_n(-\alpha)x^{\alpha}.$$

Consequently,

$$\begin{aligned} \rho(\theta)[x^\alpha] &= \lim_{n \rightarrow \infty} p_n(\rho)[x^\alpha] \\ &= \lim_{n \rightarrow \infty} p_n(-\alpha)x^\alpha \\ &= p(-\alpha)x^\alpha, \end{aligned}$$

where  $p(s) = \lim_{n \rightarrow \infty} p_n(s)$ , limit of a polynomial. Thus the operator  $p(\theta)$ , is a differential operator of infinite order and has the property that when applied to a power function, simply replaces it with a multiplier. With this understanding, we may write, symbolically,

$$n^{-\theta} = e^{-\theta \ln} = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{(-\ln n)^k}{k!} \theta^k = \lim_{N \rightarrow \infty} p_N(\theta), \text{ say.}$$

Then,

$$\begin{aligned} n^{-\theta} [x^{-s}] &= \lim_{N \rightarrow \infty} p_N(\theta) [x^{-s}] = \lim_{N \rightarrow \infty} p_N(s) x^{-s} \\ &= n^{-s} x^{-s} \end{aligned} \tag{2.1}$$

for some  $s = \sigma + i\tau$ ,  $-\infty < \tau < \infty$ .

If further

$$\begin{aligned} p_n(\theta) &= \prod_{k=1}^n (\nu - 1 + 2k + \theta), \text{ and } s = \sigma + i\tau, \text{ then} \\ p_n(\rho) [x^{-s}] &= \prod_{k=1}^n (\nu - 1 + 2k + \theta) [x^{-s}] \\ &= \prod_{k=1}^n (\nu - 1 + 2k + s) x^{-s}. \end{aligned} \tag{2.2}$$

Next we write

$$\begin{aligned} x^{-s} &= \frac{a - \theta}{a - \theta} [x^{-s}] \\ &= \frac{1}{a - \theta} (a - \theta) [x^{-s}] = \frac{1}{a - \theta} (a - s) x^{-s} \end{aligned}$$

or,

$$\frac{1}{a - \theta} [x^{-s}] = \left(\frac{1}{a - s}\right) x^{-s}, \tag{2.3}$$

for some constants  $a$  and  $s$ . This defines an operator of the type  $\frac{1}{a - \theta}$ , in the sense that when applied to a power function, simply reproduces it with a multiplier. This property parallels that of the operator which is a polynomial or limit of a polynomial in  $\theta$ . By a repeated application of an operator of the type (2.3), we have

$$\prod_{k=1}^n \left(\frac{1}{\nu - 1 + 2k - \theta}\right) [x^{-s}] = \prod_{k=1}^n \left(\frac{1}{\nu - 1 + 2k - s}\right) x^{-s}, \tag{2.4}$$

for some  $s$ , except where  $s = \nu - 1 + 2k$ ,  $k = 1, 2, \dots, n$ .

Combining the results (2.1), (2.2) and (2.4), we construct the operator

$$k_n^* (1 - \theta) = \frac{1}{2} n^{-\theta} \prod_{k=1}^n \left(\frac{\nu - 1 + 2k + \theta}{\nu - 1 + 2k - \theta}\right) \tag{2.5}$$

such that

$$k_n^* (1 - \theta) [x^{-s}] = k_n^* (1 - s) x^{-s}, \tag{2.6}$$

where

$$k_n^* (1 - s) = \frac{1}{2} n^{-s} \prod_{k=1}^n \left(\frac{\nu - s + 2k + s}{\nu - 1 + 2k - s}\right), \quad s = \sigma + i\tau, \quad -\infty < \tau < \infty.$$

Next, we shall establish properties of the function  $k_n^*(1-s)$ .

LEMMA 1.  $|k_n^*(1-\sigma-i\tau)| = n^{-\sigma} O(1)$ , as  $|\tau| \rightarrow \infty$ .

This is quite obvious.

LEMMA 2.  $\lim_{n \rightarrow \infty} k_n^*(1-s)$  exists and is uniform on every compact set of the  $s$ -axis.

PROOF. From above,

$$\begin{aligned} k_n^*(1-s) &= \frac{1}{2} e^{-s} \ln n \prod_{k=1}^n \left( \frac{\nu-1+2k+s}{\nu-1+2k-s} \right) \\ &= \frac{1}{2} e^{s(1+\frac{1}{2}+\dots+\frac{1}{n}) \ln n} \prod_{k=1}^n \left( 1 + \frac{2s}{\nu-1+2k-s} \right) e^{-s/k}. \end{aligned}$$

Consider the product

$$\begin{aligned} \prod_{k=1}^n \left( 1 + \frac{2s}{\nu-1+2k-s} \right) e^{-s/k} &= \prod_{k=1}^n \left[ 1 + \left( 1 + \frac{2s}{\nu-1+2k-s} \right) e^{-s/k} - 1 \right] \\ &= \prod_{k=1}^n [1 + a_k(s)]. \end{aligned}$$

Now,

$$\begin{aligned} \lim_{k \rightarrow \infty} k^2 a_k(s) &= \lim_{k \rightarrow \infty} k^2 \left[ \left( 1 + \frac{2s}{\nu-1+2k-s} \right) e^{-s/k} - 1 \right] \\ &= \lim_{k \rightarrow \infty} k^2 \left[ \left( 1 + \frac{2s}{\nu-1+2k-s} \right) \left( 1 - \frac{s}{k} + \frac{1}{2!} \frac{s^2}{k^2} - \dots \right) - 1 \right] \\ &= \lim_{k \rightarrow \infty} k^2 \left[ s \left( \frac{2}{\nu-1+2k-s} - \frac{1}{k} \right) + s^2 \left( \frac{1}{2k^2} - \frac{2}{k(\nu-1+2k-s)} \right) + O\left(\frac{1}{k^3}\right) \right] \\ &= \frac{1}{2} s(1-\nu). \end{aligned}$$

Or,  $a_k(s) = \frac{1}{2} s(1-\nu) o(1)$ , as  $k \rightarrow \infty$  for all finite  $S$ , hence the infinite product convergence uniformly on every compact set of the  $s$ -axis. Also

$$e^{s(1+\frac{1}{2}+\dots+\frac{1}{n}) \ln n} \rightarrow e^{\gamma s}, \text{ as } n \rightarrow \infty,$$

$\gamma$  being the Euler's constant; hence the result.

In fact,

$$\begin{aligned} \lim_{n \rightarrow \infty} k_n^*(1-s) &= \lim_{n \rightarrow \infty} \frac{1}{2} n^{-s} \prod_{k=1}^n \left( \frac{\nu-1+2k+s}{\nu-1+2k-s} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} n^{-s} \frac{1}{2^{\nu-1}} \frac{1}{2^s} \prod_{k=1}^n \left( \frac{2k}{\nu-1+2k-s} \right) \cdot \lim_{n \rightarrow \infty} n^{\frac{1}{2}\nu-\frac{1}{2}s} \prod_{k=1}^n \left( \frac{\nu-1+2k+s}{2k} \right) \\ &= \frac{1}{2} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)} = k^*(1-s), \text{ say,} \end{aligned} \tag{2.7}$$

using Euler's product for  $\Gamma$ -functions, [3, p.11].

Note that the function

$$\begin{aligned} k^*(s) &= \frac{1}{2} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}s)} \\ &= \mathcal{M}[J_\nu(2x), s], \end{aligned} \tag{2.8}$$

the Mellin transform of  $J_\nu(2x)$ ,  $s = \sigma + i\tau$ ,  $-\infty < \tau < \infty$ , and  $0 < \sigma < \nu+1$ , [4, p.326].

Now, we define the operator

$$k^*(1-\theta) = \frac{1}{2} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\theta)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\theta)}, \quad \theta \equiv -x \frac{d}{dx} \tag{2.9}$$

LEMMA 3.  $K^*(1-\theta)[x^{-s}] = k^*(1-s)x^{-s}$ , where  $k^*(1-s)$  is defined above.

PROOF.  $k^*(1-\theta)[x^{-s}] = \lim_{n \rightarrow \infty} k_n^*(1-\theta)[x^{-s}]$ , where by using Euler's product for

$\Gamma$ -functions,  $k^*(1-\theta) = \lim_{n \rightarrow \infty} k_n^*(1-\rho) = \frac{1}{2} \lim_{n \rightarrow \infty} n^{-\theta} \prod_{k=1}^n \left( \frac{\nu-1+2k+\theta}{\nu-1+2k-\theta} \right)$ , as in (2.5) above.

Now, using the results (2.6) and (2.7) we have

$$\begin{aligned} k^*(1-\theta)[x^{-s}] &= \lim_{n \rightarrow \infty} k_n^*(1-\theta)[x^{-s}] \\ &= \lim_{n \rightarrow \infty} k_n^*(1-s)x^{-s} \\ &= k^*(1-s)x^{-s}, \end{aligned}$$

as desired.

3. THE INTEGRAL.

THEOREM 1. Let  $f(x)$  be such that  $f^*(s) = \mathcal{M}[f(x):s] \in L(\sigma-i\infty, \sigma+i\infty)$  and  $k(x) = J_\nu(2x)$ ,  $\nu \geq -\frac{1}{2}$  and  $0 < \sigma < \nu + 1$ . Then

$$k^*(1-\theta)[f(x)] = \int_0^\infty f(xt)J_\nu(2t)dt, \tag{3.1}$$

where  $k^*(1-\theta)$  is the operator defined by (2.8) above.

PROOF. Since  $f^*(s)$  defines the Mellin transform of  $f(x)$ , we may write

$$\begin{aligned} k^*(1-\theta)[f(x)] &= k^*(1-\theta) \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)x^{-s}ds \\ &= \lim_{n \rightarrow \infty} k_n^*(1-\theta) \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)x^{-s}ds \\ &= \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)k_n^*(1-\theta)[x^{-s}]ds \\ &= \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)k_n^*(1-s)x^{-s}ds, \end{aligned}$$

due to Lemma 3. To justify bringing the operator  $k_n^*(1-\theta)$  inside the integral sign, we have simply to show that the resulting integral

$$\int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)k_n^*(1-s)x^{-s}ds \tag{3.2}$$

is uniformly convergent for all finite  $x$ . This is so, since

$$\begin{aligned} \left| \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)k_n^*(1-s)x^{-s}ds \right| &= \left| \int_{-\infty}^{\infty} f^*(\sigma+ir)k_n^*(1-\sigma-ir)x^{-\sigma-ir}idr \right| \\ &= O(n^{-\sigma})x^{-\sigma} \int_{-\infty}^{\infty} |f^*(\sigma+ir)|dr < \infty, \end{aligned}$$

for  $x > 0$ ,  $\sigma > 0$ , since  $f^*(s) \in L(\sigma-i\infty, \sigma+i\infty)$  and by using the results of Lemma 1. In fact the integral (3.2) converges absolutely, as well. Together with the results of Lemma 2 and 3, we can then apply Lebesgue's limit theorem, to obtain

$$\begin{aligned}
 k^*(1-\theta)[f(x)] &= \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s) k_n^*(1-s) x^{-s} ds \\
 &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s) \left[ \lim_{n \rightarrow \infty} k_n^*(1-s) x^{-s} \right] ds \\
 &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s) k^*(1-s) x^{-s} ds \\
 &= \int_0^\infty f(xt) J_\nu(2t) dt,
 \end{aligned}$$

due to the Parseval theorem for Mellin transforms [5, chapt. II], and since  $k^*(s) = \mathcal{M}[J_\nu(2t); s]$ . Hence the theorem.

Thus the equation (3.1) defines  $k^*(1-\theta)$  as an integral operator, having the property that

$$k^*(1-\theta)[x^\alpha] = k^*(1+\alpha)x^\alpha,$$

for some  $\alpha$ , due to Lemma 3. In light of the operational calculus generated by the operator  $k^*(1-\theta)$ , one can now evaluate the integrals of the Hankel type, using the operational procedures. For instance if  $f(x)$  is analytic and expressed in a power series

$$f(x) = \sum_{n=0}^\infty C_n x^{n+\alpha}, \quad |x| < r,$$

for some  $r$  and  $\alpha$ , then

$$\begin{aligned}
 k^*(1-\theta)[f(x)] &= k^*(1-\theta) \sum_{n=0}^\infty C_n x^{n+\alpha} \\
 &= \sum_{n=0}^\infty C_n k^*(1-\theta)[x^{n+\alpha}] \\
 &= \sum_{n=0}^\infty C_n k^*(1+n+\alpha) x^{n+\alpha}, \quad |x| < R,
 \end{aligned}$$

for some  $R$ , where

$$k^*(1+n+\alpha) = \frac{1}{2} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2}\alpha)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}n - \frac{1}{2}\alpha)}.$$

The above analysis is justified provided

$$\lim_{n \rightarrow \infty} \left| \frac{n C_{n+1}}{C_n} \right| = 0(1).$$

Hence we now have an operational procedure for evaluating the given integrals, in fact,

$$\int_0^\infty f(xt) J_\nu(2t) dt = \sum_{n=0}^\infty C_n k^*(1+n+\alpha) x^{n+\alpha}, \tag{3.3}$$

where  $f(x) = \sum_{n=0}^\infty C_n x^{n+\alpha}$  and  $k^*(s) = \mathcal{M}[J_\nu(t); s] = \frac{1}{2} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}s)}.$

4. SPECIAL CASES AND EXAMPLES.

Consider the case when  $\nu = \frac{1}{2}$ . Then from (3.1), we have

$$\frac{1}{\sqrt{\pi}} \int_0^\infty f(xt) \frac{\sin(2t)}{\sqrt{t}} dt = k^*(1-\theta)[f(x)] \tag{4.1}$$

where  $k^*(1-\theta) = \frac{1}{2} \frac{\Gamma(\frac{3}{4} - \frac{1}{2}\theta)}{\Gamma(\frac{3}{4} + \frac{1}{2}\theta)}$ . If for instance, we let  $f(x) = x^\mu$ ,  $|\mu| < \frac{1}{2}$ , then

$$\begin{aligned} \frac{x^\mu}{\sqrt{\pi}} \int_0^\infty t^{\mu-\frac{1}{2}} \sin(2t) dt &= \frac{1}{2} \frac{\Gamma(\frac{3}{4} - \frac{1}{2}\theta)}{\Gamma(\frac{3}{4} + \frac{1}{2}\theta)} [x^\mu] \\ &= \frac{1}{2} \frac{\Gamma(\frac{3}{4} - \frac{1}{2}\mu)}{\Gamma(\frac{3}{4} + \frac{1}{2}\mu)} x^\mu, \end{aligned}$$

or,

$$\int_0^\infty t^{\mu-\frac{1}{2}} \sin(2t) dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{3}{4} - \frac{1}{2}\mu)}{\Gamma(\frac{3}{4} + \frac{1}{2}\mu)}.$$

Putting  $\mu = -\frac{1}{2}$ , gives us the classical result,

$$\int_0^\infty \frac{\sin(2t)}{t} dt = \frac{\pi}{2}.$$

Similarly, by letting  $\nu = -\frac{1}{2}$  in (3.1), we have

$$\frac{1}{\sqrt{\pi}} \int_0^\infty f(xt) \frac{\cos(2t)}{\sqrt{t}} dt = k^*(1-\theta)[f(x)], \tag{4.2}$$

where

$$k^*(1-\theta) = \frac{1}{2} \frac{\Gamma(\frac{1}{4} - \frac{1}{2}\theta)}{\Gamma(\frac{1}{4} + \frac{1}{2}\theta)}.$$

And if  $f(x) = x^\mu$ ,  $|\mu| < \frac{1}{2}$ , we have from above

$$\int_0^\infty t^{\mu-\frac{1}{2}} \cos(2t) dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1}{4} - \frac{1}{2}\mu)}{\Gamma(\frac{1}{4} + \frac{1}{2}\mu)}.$$

Putting  $\mu = 0$ , give us the well-known result,

$$\int_0^\infty \frac{\cos(2t)}{\sqrt{t}} dt = \frac{\sqrt{\pi}}{2}.$$

We now consider a few examples to illustrate the procedure given in formula (3.3).

1. Let  $f(x) = x^\mu e^{-x}$ . Then from (3.3), we have

$$\int_0^\infty (xt)^\mu e^{-xt} J_\nu(2t) dt = \frac{1}{2} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}n)}{n! \Gamma(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2}n)} x^{n+\mu}. \tag{4.3}$$

The series on the right-hand side converges absolutely for  $|x| < 2$ , where  $|\nu| < 1 + \mu$  and  $|\nu| \geq -\frac{1}{2}$ . Using the functional equation  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$ , and splitting the series

into odd and even terms. We obtain from the right hand side of (4.3),

$$\begin{aligned} \frac{1}{2\sqrt{\pi}} \cos \frac{\pi}{2}(\mu-\nu) \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu + n) \Gamma(\frac{1}{2} - \frac{1}{2}\nu + \frac{1}{2}\mu + n)}{n! \Gamma(n + \frac{1}{2})} \left(\frac{x}{2}\right)^{2n} \\ + \frac{1}{2\sqrt{\pi}} \sin \frac{\pi}{2}(\mu-\nu) \sum_{n=0}^\infty \frac{(-1)^n \Gamma(1 + \frac{1}{2}\nu + \frac{1}{2}\mu + n) \Gamma(1 - \frac{1}{2}\nu + \frac{1}{2}\mu + n)}{n! \Gamma(n + \frac{3}{2})} \left(\frac{x}{2}\right)^{2n+1} \end{aligned}$$

Hence,

$$\int_0^\infty t^\mu e^{-xt} J_\nu(2t) dt = \frac{1}{2\pi} \cos \frac{\pi}{2}(\mu-\nu) \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \mu\right) \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \mu\right) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \mu, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \mu; \frac{1}{2}; -\frac{x^2}{4}\right) + \frac{1}{2\pi} \sin \frac{\pi}{2}(\mu-\nu) \Gamma\left(1+\frac{1}{2}, \frac{1}{2}, \mu\right) \Gamma\left(1-\frac{1}{2}, \frac{1}{2}, \mu\right) x {}_2F_1\left(1+\frac{1}{2}, \frac{1}{2}, \mu, 1-\frac{1}{2}, \frac{1}{2}, \mu; \frac{3}{2}; -\frac{x^2}{4}\right) \tag{4.4}$$

due to the result (3.3). Some special cases of this result such as when  $\mu = \nu$  and  $\mu = 0$  can easily be derived. The range of the result (4.4) can be extended to  $x > 0$ , by analytic continuation.

2. Let  $f(x) = x^{-\lambda} J_\mu(n)$ . Then from (3.3), we have

$$\int_0^\infty (xt)^{-\lambda} J_\nu(xt) J_\nu(2t) dt = \sum_{n=0}^\infty \frac{(-1)^n \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda+n\right) x^{2n+\mu-\lambda}}{n! 2^{1+2n+\mu} \Gamma(\mu+n+1) \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda-n\right)} \tag{4.5}$$

Now,

$$\Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda-n\right) \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda+n\right) = \frac{\pi}{(-1)^n \sin \frac{\pi}{2}(1-\nu+\mu-\lambda)} = (-1)^n \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda\right) \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda\right)$$

Therefore the right hand side of (4.5), then gives

$$= \frac{2^{-\mu-1} x^{\mu-\lambda}}{\Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda\right) \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda\right)} \cdot \sum_{n=0}^\infty \frac{\Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda+n\right) \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda+n\right)}{n! \Gamma(1+\mu+n)} \left(\frac{x^2}{4}\right)^n = \frac{2^{-\mu-1} \Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda\right) x^{\mu-\lambda}}{\Gamma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda\right) \Gamma(1+\mu)} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \lambda; 1+\mu; \frac{x^2}{4}\right),$$

where  $|x| < 2, \operatorname{Re}(\mu+\nu) + 1 > \operatorname{Re} \lambda > -1$ , [6, p.48].

As a special case if  $\lambda = \nu-\mu-1$ , then

$$\int_0^\infty (xt)^{-\nu+\mu+1} J_\mu(xt) J_\nu(2t) dt = \frac{2^{-\mu-1} x^{2\mu-\nu+1}}{\Gamma(\nu-\mu)} {}_2F_1\left(1-\nu+\mu, 1+\mu; 1+\mu; \frac{x^2}{4}\right) = \frac{2^{-\mu-1} x^{2\mu-\nu+1}}{\Gamma(\nu-\mu)} \left(1 - \frac{x^2}{4}\right)^{\nu-\mu-1}, \quad |x| < 2.$$

3. Let  $f(x) = x^{\nu-M-1} \prod_{i=1}^k J_{\mu_i}(a_i x)$ ,  $M = \sum_{i=1}^k \mu_i$ ,  $a_i > 0$  and  $0 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{k}{2} + \frac{3}{2}$ .

Or,

$$f(x) = x^{\nu-1} \prod_{i=1}^k x^{-\mu_i} J_{\mu_i}(a_i x) = \prod_{i=1}^k \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{a_i}{2}\right)^{2n+\mu_i}}{n! \Gamma(n+\mu_i+1)} x^{2n+\nu-1}$$

Then

$$\int_0^\infty (xt)^{\nu-M-1} \prod_{i=1}^k J_{\mu_i}(a_i xt) J_\nu(2t) dt = k^* (1-0) \left[ \prod_{i=1}^k \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{a_i}{2}\right)^{2n+\mu_i}}{n! \Gamma(n+\mu_i+1)} \cdot x^{2n+\nu-1} \right] = \prod_{i=1}^k \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{a_i}{2}\right)^{2n+\mu_i}}{n! \Gamma(n+\mu_i+1)} k^* (2n+\nu) x^{2n+\nu-1}$$

$$\begin{aligned}
 &= \frac{1}{2} \prod_{i=1}^k \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{a_i}{2}\right)^{2n+\mu_i} \Gamma(\nu+n)}{n! \Gamma(n+\mu_i+1) \Gamma(1-n)} x^{2n+\nu-1} \\
 &= \frac{1}{2} \prod_{i=1}^k x^{\nu-1} \left[ \frac{\left(\frac{a_i}{2}\right)^{\mu_i} \Gamma(\nu)}{\Gamma(\mu_i+1)} + \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{a_i}{2}\right)^{2n+\mu_i} \Gamma(\nu+n)}{n! \Gamma(n+\mu_i+1) \Gamma(1-n)} x^{2n+\nu-1} \right] \\
 &= \frac{\Gamma(\nu)}{2} x^{\nu-1} \prod_{i=1}^k \frac{\left(\frac{a_i}{2}\right)^{\mu_i}}{\Gamma(\mu_i+1)}, \quad [6, \text{p.54}].
 \end{aligned}$$

4. Finally we will derive a general result formally.

Let

$$f(x) = x^{2\lambda} G_{p,q}^{m,n} \left[ cx^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right], \quad p + q < 2(m+n).$$

Then

$$\begin{aligned}
 &\int_0^{\infty} f(xt) J_{\nu}(2t) dt = k^*(1-\theta) [f(x)] \\
 &= k^*(1-\theta) \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=n+1}^p \Gamma(a_j-s)} x^{2s+2\lambda} c^s ds \\
 &= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=n+1}^p \Gamma(a_j-s)} c^s k^*(1+2s+2\lambda) x^{2s+2\lambda} ds \\
 &= \frac{1}{4\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s) \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + s + \lambda\right)}{\prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=n+1}^p \Gamma(a_j-s) \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - s - \lambda\right)} c^s x^{2s+2\lambda} ds \\
 &= \frac{1}{2} x^{2\lambda} G_{p+2,q}^{m,n+1} \left[ cx^2 \left| \begin{matrix} \frac{1}{2} + \frac{1}{2}\nu - \lambda, a_1, \dots, a_p, \frac{1}{2} + \frac{1}{2}\nu - \lambda \\ b_1, \dots, b_q \end{matrix} \right. \right],
 \end{aligned}$$

with the usual conditions on parameters, [6, p.91].

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