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VIX-linked GMMB under affine GARCH models and its Diffusion Limits

Chen, Yuyu

UNIVERSITY OF CALGARY

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by

Yuyu Chen

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Abstract

In variable annuity (VA) industry, to compensate for the liability coming from embedded riders in VA, insurer usually charge a fixed percentage of investment fund as the riders fee. However, the traditional fixed-fee structure would misalign insurer's income and liability and in consequence cause risk management challenges for insurer. In 2013, the Chicago Board of Options Exchange (CBOE) suggests linking riders fee in variable annuity with VIX index in a white paper and shows that VIX-linked fee structure can help to re-align insurer's income and liability using non-parametric models.

Affine GARCH models are used in this work to analyze VIX-linked fee structure for VA with guarantee minimum maturity benefit (GMMB). A closed-form solution to GMMB has been derived and is used to determine a fair fee structure. Comparison between fixed-fee structure and VIX-linked fee structure has been been shown by numerical examples.

Keywords: Variable annuity, VIX-linked fee, GMMB, affine GARCH models, closed-form solution, diffusion limits.

Preface

This thesis is an original work by the author. No part of this thesis has been previously published.

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Chapter 1

Introduction

Variable annuity is an insurance contract which allows the insured to participate in the investment activity and pays a series of installments to the policyholder. Those payments to the insured will fluctuate based on the performance of the investment fund. In general variable annuities can be purchased by a single payment or a series of installments, and for simplicity we only consider single payment contracts in this thesis. Since the insured may suffer from financial losses when the market is volatile, the insurance company provides some guarantee options or riders embedded in a variable annuity, such as guaranteed minimum maturity benefits (GMMBs), guaranteed minimum income benefits (GMIBs) and guaranteed minimum withdrawal benefits (GMWBs).

In this thesis, we only consider GMMBs which act as a protection for the policyholder against poor performance of the market. At the maturity of the contract, a GMMB guarantees the insured to receive the greater of the investment fund value or the guaranteed amount if the insured is still alive. In this case the insurer will assume downside market risk, and the GMMB acts as a put option to the policyholder.

GMMBs are funded by a series of insurance fees directly charged from the investment fund, and the fee is usually a fixed percentage of the investment fund value. Though the fixed fee rate structure has been widely used in the industry, it can cause discrepancy between the insurer's income and its market liability. Indeed, due to leverage effect, equity returns are negatively correlated with market volatility, thus if the market is going down, the liability will increase as the volatility increases while less fee is charged as equity return decreases. In other words, the insurer's income becomes less when more fees are actually needed to compensate for the insurer's increased liability.

Moreover, as the variable annuity is generally a long-term contract, it is very expensive to hedge using long-term options as these options have high prices due to the lack of suppliers. Hedging program using short-term options is also a great cost, as the option price will be influenced by the fluctuation of the market. The mismatch of income and liability will also cause other issues for insurers. On one hand, it will encourage the policyholder to surrender the contract (lapsation risk) when the market is stable as more fees are charged than needed. On the other hand, if the market is in financial crisis, the fee charged is less than needed and this will encourage the policyholders to stay in the contract and reduce their incentive to surrender (adverse selection).

In this case, the Chicago Board of Option Exchange(CBOE) suggested that the rider fee should be related to the volatility index (VIX) in 2013. CBOE has shown that a VIX-linked fee structure can realign insurer's income and liability by using non-parametric analysis. A toy model has also been used by Bernard et al. [2016] to draw the same conclusion. However, the lack of a strong parametric model and sounded quantitative analysis tools may expose the variable annuity industry to significant systematic risk.

A strong quantitative investigation on the effects of a VIX-linked fee structure has first been done by Cui et al. [2017] who use the Heston's model (Heston [1993]) to analyze GMMBs. With the analytic property of Heston's model, a closed-form solution to pricing GMMB has been derived and is used to show that the VIX-linked fee structure can reduce the sensitivity of insurer's net liability to market liability. The literature was extended by analyzing a more complex guarantee GMWB in Kouritzin and MacKay [2017] where jumps are added to the market model. Analyzing GMWB is more complex than GMMB for its path-dependent feature and an analytic form of the associated liability may not exist. They derived a weak solution to the investment fund value and the numerical analysis from Monte Carlo simulations presents a consistent result for VIX-linked fee structures as in Cui et al. [2017].

Although Cui et al. [2017] and Kouritzin and MacKay [2017] have investigated the effects of VIX-linked fee structure under continuous-time stochastic volatility models, this work has never been done in a discrete-time setting. A popular alternative to continuous-time stochastic volatility models is the discrete-time GARCH models. We intend to do the analysis under GARCH models, not only because of the ease of estimation but also for their remarkable performance to capture the stylized features of financial series. More specifically, we assume the equity dynamics follow affine GARCH models, since they lead to semi-closed form solutions for European style options.

Under affine GARCH mdoels, conditional moment generating function the logarithm of spot asset can be derived and semi-closed form solution to option price can be obtained using the conditional moment generating function. Since a GMMB is a put option to the policyholder, by similar techniques we are able to derive semi-closed form solution to insurer's liability associated with GMMB under affine GARCH models. The HN-GARCH model (Heston and Nandi [2000]) is the first affine GARCH model which leads to semi-closed form solutions to option pricing under a Gaussian assumption on the driving noise process. Since Monte Carlo simulations are needed to price European options under non-affine GARCH models, affine GARCH models can save time and reduce computation burdens. However, under the HN-GARCH model, the equity dynamics is assumed to follow conditional Gaussian distribution which can not capture short term behavior of equity option smiles. This work was extended in Christoffersen et al. [2006] by including negative skewness and heavy kurtosis using the Inverse Gaussian (IG) distribution for the GARCH innovations. Since the IG-GARCH model converges to the HN-GARCH model by taking some specific parametrization (see Christoffersen et al. [2006]), we do not expect a great improvement on option pricing from the HN-GARCH model to the IG-GARCH model. In fact, Christoffersen et al. [2006] showed that the IG-GARCH model outperforms the HN-GARCH model in-sample and up to ten weeks out-of-sample and it also has a better fit for deep in-the-money call options. However, it performs worse than the HN-GARCH model for longer out-of-sample periods. Both the HN-GARCH model and the IG-GARCH model have the Heston's model (Heston [1993]) as their continuous-time limits, depending on the constraints imposed on the underlying parameters.

To estimate the model parameters, we use maximum likelihood estimation (MLE) which is one of the most important estimation methods for GARCH models. The performance of two popular MLEs, return-only MLE and joint MLE with both returns and VIX have been compared in the recent literature. The joint estimation with VIX is first proposed by Hao and Zhang [2013] who showed such estimation methodology can improve GARCH models' s ability to fit market VIX, assuming VIX errors are independent. The performance of the joint estimation on option pricing was assessed by Kanniainen et al. [2014] who added autoregressive disturbances to the VIX and showed that GARCH models have better performance on option pricing under joint MLE. Joint MLE with VIX under the HN-GARCH model and the IG-GARCH model has been investigated by Wang et al. [2017] and Chorro et al. [2016] respectively.

The main work of this thesis is threefold. First discrete-time affine GARCH models with Gaussian and non-Gaussian innovation are used to assess the effects of the VIX-linked fee structure. Second, semi-closed form solutions to the insurer's liability are derived and a fee charging scheme at discrete time is considered so that so that a fair fee structure can be obtained numerically. Thirdly, the diffusion limits of fair total fee rate have been investigated and convergence results of fair total fee rate are shown numerically.

The rest of the context is organized as follows. Chapter 2 introduces the two affine GARCH models, the HN-GARCH model and the IG-GARCH model, and present the analytic expressions for VIX. In the next chapter, fund dynamics under affine GARCH models and the corresponding conditional moment generating functions of log fund value are discussed. Next, semi-closed form solutions for GMMB and risk-neutral rider fees are derived, so that a fair fee scheme can be obtained numerically. In chapter 4, the diffusion limit of GMMB under the HN-GARCH model is discussed, and we investigate the convergence of the GARCH-based fair fee rate to the continuous-time counterpart. In the last chapter, we use MLE to estimate model parameters and effect of VIX-linked fee structure is assessed by numerical examples. The last chapter concludes this thesis.

Chapter 2

Market Model and Fee Structure

The affine GARCH models are famous for their semi-closed form solution to pricing an European style option, with the use of inverse Fourier transformation on the conditional moment generating function. In this chapter, we introduce two affine GARCH models, the HN-GARCH model and the IG-GARCH model as our market models. After that we discuss about how to charge the fees and how to link rider fees with the VIX index, thus the fee structure is defined. This chapter serves as the preparation for the derivation of the fund dynamics.

2.1 Market Model

Consider an economy with filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}^t)$, where $\mathcal{F}^t$ represents the information generated by the dynamics of underlying asset over time t, $t \in \mathcal{T}=\{t|t = 0, 1, 2 \ldots, T\}$, T is the maturity of the contract.

In this thesis, we consider to use discrete-time affine GARCH models which can lead to several advantages. First, compared with continuous-time stochastic volatility models, discrete-time GARCH models can make better use of observable daily data to estimate the model parameters and the stylized facts exhibited by time series can also be captured by GARCH models. Second, compared with the non-affine GARCH models, affine GARCH models provide semi-closed form solutions to option pricing issues which can help to reduce the calculation burdens since Monte-Carlo simulation is needed on option pricing under non-affine GARCH models.

2.1.1 Heston-Nandi Model

As the first affine GARCH model, the HN-GARCH model (Heston and Nandi [2000]) has been widely used in the literature for its analytic property which leads to semi-closed form solutions for the conditional moment generating function of the logarithm asset price process. Under the HN-GARCH model, the underlying asset has the following dynamics under P (physical) measure:

$$\log \frac{S_{t+1}}{S_t} = r + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1}, \tag{2.1}$$

$$h_{t+1} = \omega + \beta h_t + \alpha (z_t - \gamma \sqrt{h_t})^2, \tag{2.2}$$

$$z_t \sim N(0,1). \tag{2.3}$$

Here S_t is the equity price, r is the continuously compounded interest rate for each time interval. h_t represents the conditional variance of the log-return processes. γ measures the leverage effect between log return process and the conditional variance. z_t follows a standard normal distribution. The unconditional expectation of h_t is called the long-term variance, and the long-term variance under physical measure is given:

$$\overline{h} = E^P[h_t] = \frac{\omega + \alpha}{1 - \beta - \alpha\gamma^2}.$$

Using local risk-neutral valuation relationship (LRNVR) proposed by Duan [1995], which can be conveniently used to derive the risk-neutral measure of a GARCH model when the driving noise follows a standard Gaussian distribution, we can derive the risk-neutral measure for the Heston-Nandi model:

Definition 2.1.1. A pricing measure Q is said to satisfy the locally risk-neutral valuation relationship (LRNVR) if the measure Q is mutually absolutely continuous with respect to the measure P, and satisfies the following conditions:

- $\log \frac{S_{t+1}}{S_t}|\mathcal{F}_t$ is normally distributed under Q;
- $E_Q[\frac{S_{t+1}}{S_t}|\mathcal{F}_t] = e^r$;
- $Var^Q[\frac{S_{t+1}}{S_t}|\mathcal{F}_t] = Var^P[\frac{S_{t+1}}{S_t}|\mathcal{F}_t]$ almost surely with respect to measure P.

Proposition 2.1.1. *Suppose the asset dynamics follow the HN-GARCH model from (2.1)-(2.3) under physical measure, the risk-neutral dynamics based on the LRNVR are given by:*

$$\log \frac{S_{t+1}}{S_t} = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z^*_{t+1}, \tag{2.4}$$

$$h_{t+1} = \omega + \beta h_t + \alpha(z^*_t - \gamma^*\sqrt{h_t})^2, \tag{2.5}$$

$$z^*_t \sim N(0, 1). \tag{2.6}$$

Here $z^*_{t+1} = z_{t+1} + (\lambda + \frac{1}{2})\sqrt{h_{t+1}}$,and $\gamma^* = \gamma + \lambda + \frac{1}{2}$

Proof See Section A.1.1 in the Appendix

Here z^*_t follows a standard normal distribution under risk-neutral measure. The log-return process follows a conditionally Gaussian distribution. We also need to note that the conditional variance remains unchanged after the change of measure based on LRNVR. The long-term variance under Q measure is:

$$\overline{h}^* = E^Q[h_t] = \frac{\omega + \alpha}{1 - \beta - \alpha\gamma^{*2}}.$$

2.1.2 IG-GARCH Model

Since the affine GARCH model with Gaussian innovation can not capture the negative skewness and high kurtosis frequently exhibited by financial time series, the IG-CARCH model introduced by Christoffersen et al. [2006] uses inverse Gaussian distribution as the

GARCH innovation to capture the heavy tails of financial series. Following is the dynamic of underlying asset if it follows the IG-GARCH model under P (physical) measure :

$$\log \frac{S_{t+1}}{S_t} = r + \nu h_{t+1} + \eta y_{t+1}, \tag{2.7}$$

$$h_{t+1} = \omega + b h_t + c y_t + a h_t^2 / y_t. \tag{2.8}$$

Here r is the continuously compounded interest in each time interval. h_t represents the conditional variance of the log return. y_t follows a inverse Gaussian distribution with degree of freedom $\delta_t = h_t/\eta^2$. The density function of inverse Gaussian distribution with support $(y > 0)$ is given by:

$$\begin{aligned} P(y;\delta) &= \int_0^y \frac{\delta}{\sqrt{2\pi z^3}} e^{-\frac{1}{2}(\sqrt{z} - \frac{\delta}{\sqrt{z}})^2} dz, \\ &= N(\frac{-\sigma}{\sqrt{y}} + \sqrt{y}) + e^{2\delta} N(\frac{-\delta}{\sqrt{y}} - \sqrt{y}) \end{aligned}$$

With integration on the above density function we can derive a generalization of the moment function:

$$E\left[\exp(\phi y + \frac{\theta}{y})\right] = \frac{\delta}{\sqrt{\delta^2 - 2\theta}} \exp(\delta - \sqrt{(\delta^2 - 2\theta)(1 - 2\phi)})$$

and the moments:

$$\begin{aligned} &E[y] = \delta,\ Var[y] = \delta, \\ &Skew[y] = \frac{3}{\sqrt{\delta}},\ Kurt[y] = \frac{15}{\delta} \end{aligned}$$

The unconditional expectation of h_t is called the long-term variance. The long-term variance under physical measure is:

$$\overline{h} = E^P[h_t] = (\omega + \eta^4 a)/(1 - a\eta^2 - b - c/\eta^2).$$

To obtain the Q (risk-neutral) measure under IG-GARCH model, we consider to use conditional Esscher transformation which can be applicable when driving noise follows a non-normal distribution.

Definition 2.1.2. Let θ_t be an $\mathcal{F}_t$ predictable process. The probability measure Q^{ess} defined by:

$$\frac{dQ^{ess}}{dP}|_{\mathcal{F}_T} = \prod_{t=1}^{T} \frac{\exp(\theta_t s_t)}{M^P_{s_t|\mathcal{F}_{t-1}}(\theta_t)}$$

is called the conditional Esscher transformed measure of P, generated by the return processes $s_t = S_t/S_{t-1}$ and the family θ of Esscher parameters, with respect to the filtration $\mathcal{F}$. The Esscher parameter θ_t is the solution of the following equality:

$$M^P_{s_t|\mathcal{F}_{t-1}}(1+\theta_t) = e^{r_t} M^P_{s_t|\mathcal{F}_{t-1}}(\theta_t)$$

The above equality can be verified by using the martingale constraint that $E^Q[e^{s_t}] = e^r$.

Proposition 2.1.2. *Suppose the asset dynamics follows the IG-GARCH model from (2.5)-(2.6) under physical measure, the risk-neutral dynamics based on conditional Esscher transformation are given by:*

$$\log \frac{S_{t+1}}{S_t} = r + \nu^* h^*_{t+1} + \eta^* y^*_{t+1}, \tag{2.9}$$

$$h^*_{t+1} = \omega^* + b h^*_t + c^* y^*_t + a^* h^{*2}_t / y^*_t. \tag{2.10}$$

where

$$\nu^* = \nu(\frac{\eta^*}{\eta})^{-\frac{3}{2}}, y^*_{t+1} = y_{t+1}(\frac{\eta^*}{\eta})^{-1},$$
$$\omega^* = \omega(\frac{\eta^*}{\eta})^{\frac{3}{2}}, c^* = c(\frac{\eta^*}{\eta})^{\frac{5}{2}}, a^* = a(\frac{\eta^*}{\eta})^{-\frac{5}{2}}.$$

Proof See Section A.1.2 in the Appendix. (For more details see Christoffersen et al. [2006] Appendix B)

Unlike the HN-GARCH model, after the transformation from physical measure to risk-neutral measure, the risk-neutral conditional variance can be expressed by $h^*_{t+1} = (\eta^*/\eta)^{\frac{3}{2}} h_{t+1}$. As indicated by Christoffersen et al. [2006], the IG-GARCH model can capture the stylized fact that the risk-neutral variance is in general greater than historical variance. The long-term variance under Q measure is:

$$\overline{h}^* = E^Q[h^*_t] = (\omega^* + \eta^{*4} a)/(1 - a^*\eta^{*2} - b - c^*/\eta^{*2}).$$

2.2 Fee Structure

We assume that the fee rate paid directly from the VA account is composed of two parts. The first part is called investment management fee rate, denoted by c^{inv}, which is used to compensate the managers' service of managing the underlying investment funds. The second part is called rider fee rate or rider charge rate, denoted by c_t, and it goes to the insurer in order to cover the cost of financial guarantees provided by the insurer.

We assume the investment management fee rate to be constant and non-negative at the inception of the contract. We denote the total fee rate to be c^{tot}_t, and it equals to the sum of the investment management fee rate and the rider fee rate:

$$c^{tot}_t = c^{inv} + c_t.$$

Here c^{inv} and c_t are the continuously-compounded fee rates for each period, but in practice we charge those fees at discrete time. Although both the investment management fee and the rider fee are charged at the same time, we assume that the investment management fee is charged right before the rider fee.

2.2.1 VIX Index

The VIX index is used to measure the future expectation of daily volatility implied from at-the-money SP 500 index option prices. According to Hao and Zhang [2013], the VIX Index can be calculated as the arithmetic average of the expected annualized variance over the next month with n trading days under risk-neutral measure. We define a proxy $V_t(n)$ for VIX, as the arithmetic average of the expected daily variance over the next month, and $V_t(n)$ can be regarded as a daily version of the VIX. We assume there are 22 trading days in a month and 252 trading days in one year.

$$V_t(n) = \frac{1}{n}\sum_{k+1}^{n} E_t^Q[h_{t+k}],$$

$$VIX_t = 100\sqrt{252V_t(22)}.$$

Here n is the number of trading days in one month thus $n = 22$, h_{t+k} is the conditional variance, Q is risk-neutral measure. Analytic calculation formulas for $V_t(n)$ under the HN-GARCH model and the IG-GARCH model have been derived using iterative conditional expectation in Wang et al. [2017] and Chorro et al. [2016] respectively. Both Wang et al. [2017] and Chorro et al. [2016] showed that $V_t(n)$ is a weighted average of current conditional variance of next period and the long-term variance.

Proposition 2.2.1. *If equity dynamic follows the HN-GARCH model, then the VIX proxy V_t is a weighted average of current conditional variance of next period and the long-term variance under risk-neutral measure*

$$V_t(n) = (1 - \Gamma(n))\overline{h} + \Gamma(n)h_{t+1}.$$

where $\overline{h} = \frac{\omega+\alpha}{1-\beta-\alpha\gamma^{*2}}$,$\Gamma(n) = \frac{1-\tilde{\beta}^n}{n(1-\tilde{\beta})}$,$\tilde{\omega} = \omega + \alpha$,$\tilde{\beta} = \beta + \alpha\gamma^{*2}$.

Proposition 2.2.2. *If equity dynamic follows the IG-GARCH model, then the VIX proxy V_t is a weighted average of current conditional variance of next period and the long-term* *variance under risk-neutral measure:*

$$V_t(n) = (1 - \Gamma^*(n))\overline{h^*} + \Gamma^*(n)h^*_{t+1}.$$

where $\psi = a^*(\eta^*)^2 + b + c^*/(\eta^*)^2$, $\overline{h}^* = \frac{(\omega^* + (\eta^*)^4 a)}{(1-\psi)}$, $\Gamma^*(n) = \frac{1-\psi^n}{n(1-\psi)}$.

2.2.2 Fee Structure linked with VIX

The purpose of a VIX-linked fee structure is to realign the insurer's liability and its income. The value of the guarantee provided by the insurer will increases as the market volatility increases, in this case more fee income need to be charged for compensation. Following Cui et al. [2017], We assume that the rider fee has a positive relationship with the arithmetic average of the expected daily variance over next month $V_t(22)$:

$$c_t = \overline{c} + \frac{m}{252}(\frac{VIX_{t-1}}{100})^2 = \overline{c} + mV_{t-1}(22).$$

Here $\overline{c} > 0$ is the based fee rate and $m > 0$ is the multiplier of $V_t(n)$. Since $c^{tot}_t = c^{inv} + c_t$, we can get:

Under Heston-Nandi:

$$c^{tot}_t = M + N(h_t - \overline{h}).$$

$$\text{Here } M = c^{inv} + \overline{c} + m\overline{h} > 0,\ N = m\Gamma(22).$$

Under IG-GARCH:

$$c^{tot}_t = M^{IG} + N^{IG}(h^*_t - \overline{h}^*).$$

$$\text{Here } M^{IG} = c^{inv} + \overline{c} + m\overline{h}^* > 0,\ N^{IG} = m\Gamma^{IG}(22).$$

The dynamic of c^{tot}_t is very similar with that under continuous-time stochastic volatility model (Cui et al. [2017]). Here M (M^{IG}) can be interpreted as the "long-term fee rate", and since $\overline{h}$ ($\overline{h}^*$) is the long-term variance, N (N^{IG}) can be regarded as the "volatility risk premium rate". The positiveness of N (N^{IG}) means insurer will receive more fee income when volatility goes up, and this is exactly the purpose of the VIX-linked fee structure which intends to realign insurer' income and liability.

Chapter 3

GMMB Pricing and Fair Fee Structure

In this chapter we provide the theoretical foundations of the analysis on a VIX-linked fee structure. With the market models and fee structure specified in the last chapter, we first derive the fund dynamics under both the HN-GARCH model and the IG-GARCH model. Next, using the analytic property of the affine GARCH models, we derive the conditional moment generating function of log fund value, which can be used to price the GMMB. After that, more details of the fee structure are discussed and an analytic calculation formula of expected risk-neutral ride fees is derived. Thus, a fair structure can be defined by letting the price of GMMB equal to the expected risk-neutral rider fees.

3.1 Fund Dynamics and Moment Generating Function

The change of investment fund value is composed of two parts, the first part is from the dynamics of the underlying asset and the second part is the charing of fees. Since the total fee rate defined in the last chapter is continuously-compounded for each time interval, we have the following fund dynamics from time t to t+1 :

$$\frac{F_{t+1}}{F_t} = \frac{S_{t+1}}{S_t} e^{-c_{t+1}^{tot}}.$$

By using the above equity dynamics and fee structure defined in Chapter 2, we are able to derive the risk-neutral fund dynamics under the HN-GARCH model and the IG-GARCH model.

Proposition 3.1.1. *If equity dynamics follow the HN-GARCH model, then the insured's investment account value has the following dynamics under risk-neutral measure:*

$$\log \frac{F_{t+1}}{F_t} = (r - M + N\overline{h}) - (\frac{1}{2} + N)h_{t+1} + \sqrt{h_{t+1}} z_{t+1}^*, \tag{3.1}$$

$$h_{t+1} = \omega + \beta h_t + \alpha(z_t - \gamma^* \sqrt{h_t})^2. \tag{3.2}$$

Proposition 3.1.2. *If equity dynamics follow the IG-GARCH model, then the insured's investment account value has the following dynamics under risk-neutral measure:*

$$\log \frac{F_{t+1}}{F_t} = (r - M^{IG} + N^{IG}\overline{h^*}) + (\nu^* - N^{IG})h_{t+1}^* + \eta^* y_{t+1}^*, \tag{3.3}$$

$$h_{t+1}^* = \omega^* + bh_t^* + c^* y_t^* + a^* h_t^{*2}/y_t^*. \tag{3.4}$$

Expressions for the parameters or distributions with superscripts (*) are under risk-neutral measure can be found in the previous chapter.

3.1.1 Moment Generating Function

Following the proof of Heston and Nandi [2000] and Christoffersen et al. [2006], we can derive semi-closed form solutions to the conditional moment generating function of the logarithm of investment fund value under the HN-GARCH model and the IG-GARCH model respectively. The coefficients in each expression can be obtained numerically by some terminal conditions. The following conditional moment generating functions are derived under risk-neutral measure.

Proposition 3.1.3. *If asset dynamics follow the HN-GARCH model, the conditional moment generating function of log(F_T) under Q measure is given by*

$$f_t(\phi) = E_t^Q[F_T^\phi] = \exp[\phi \log F_t + A(t) + B(t)h_{t+1}].$$

where

$$A(t) = A(t+1) + \phi(r - M + N\overline{h}) + B(t+1)\omega - \frac{1}{2}\log(1 - 2B(t+1)\alpha); \tag{3.5}$$

$$B(t) = \phi(\gamma^* - \frac{1}{2} - N) + \beta B(t+1) - \frac{1}{2}\gamma^{*2} + \frac{(\phi - \gamma^*)^2}{2(1 - 2B(t+1)\alpha)}. \tag{3.6}$$

These coefficients can be calculated recursively from the terminal conditions $A(T) = 0$ and $B(T) = 0$.

Proposition 3.1.4. *If asset dynamics follow the IG-GARCH model, the conditional moment generating function of log(F_T) under Q measure is given by*

$$f_t(\phi) = E_t^Q[F_T^\phi] = \exp[\phi \log F_t + A^*(t) + B^*(t)h_{t+1}^*].$$

where

$$A^*(t) = A^*(t+1) + \phi(r - M^* + N^*\overline{h}^*) + B^*(t+1)\omega^* - \frac{1}{2}\log(1 - 2a(\eta^*)^4 B^*(t+1)); \tag{3.7}$$

$$B^*(t) = \phi(\nu^* - N^*) + bB^*(t+1) + (\eta^*)^{-2} - (\eta^*)^{-2}\sqrt{(1 - 2a^*(\eta^*)^4 B^*(t+1))(1 - 2c^* B^*(t+1) - 2\eta^*\phi)}. \tag{3.8}$$

These coefficients can be calculated recursively from the terminal conditions $A^*(T) = 0$ and $B^*(T) = 0$.

3.2 GMMB Pricing and Fair Fee Structure

In this section, a semi-closed form solution to the price of GMMB is derived using Fourier inversion on the conditional moment generating function of log fund value. Next, we construct a fee charging scheme which allows us to obtain a analytic calculation formula on the expected risk-neutral rider fees. By letting the price of a GMMB equal to the expected risk-neutral rider fees, we are able to obtain a fair fee structure numerically.

3.2.1 GMMB Pricing

The GMMBs are guarantees or riders embedded in variable annuities, which will compensate the policyholder the greater of the fund value F_T and the guaranteed amount G at the maturity of the contract, on the condition that the policyholder is still alive. Regarding the investment fund value as the underlying asset, a GMMB is actually a put option embedded in a variable annuity. Common types of guaranteed amount G include:

1. A full refund of the initial fund value: $G = F_0$.
2. A refund of initial fund value with a rolling up rate π: $G = F_0^{\pi}$.

In this thesis we adopt the second type guarantee amount. From the insurer' perspective we can express the cost of GMMB at maturity if the policyholder is still alive.

$$(G - F_T)_+ = \max(G - F_T, 0).$$

We use τ_t to represent the future life time of a policyholder aged x at time t, assuming τ_t is independent of the dynamics of the underlying fund. $\Pi(t)$ is use to represent the no-arbitrage value of the guarantee's cost at time t, it is given by:

$$\Pi(t) = E_t^Q[e^{-r(T-t)}(G - F_T)_+ 1_{(\tau_t > T-t)}].$$

We assume $1_{(\tau_t > T-t)}$ equal to 1. This restriction can be easily relaxed and our result will only be affected by multiplying a constant. We can simplify the above expression as:

$$\Pi(t) = E_t^Q[e^{-r(T-t)}(G - F_T)_+].$$

Since the affine GARCH models have semi-closed form solutions on option pricing issues, semi-closed form solutions to a GMMB also can be derived under affine GARCH models. To calculate $\Pi(t)$, we need to know the probability distribution of F_T, and here we focus on $\log(F_T)$. Previously we have derived the conditional moment generating function $f_t(\phi)$ of $\log(F_t)$, since $f_t(iu)$ is the characteristic function of $\log(F_t)$, we can calculate the risk-neutral probability of $\log(F_t)$ by inverting the characteristic function under risk-neutral measure using inverse Fourier transform.

Proposition 3.2.1. *At time t, the expected risk-neutral value of a GMMB with guarantee amount G is worth:*

$$\Pi(t) = e^{-r(T-t)} \left[G \times Q^{(1)} \left(\log F_T < \log G \right) - E_t^Q[F_T] \times Q^{(2)} \left(\log F_T < \log G \right) \right].$$

where,

$Q^{(1)}(\cdot)$ *is the risk-neutral probability and* $Q^{(2)}(\cdot)$ *is a new probability measure*

$$Q^{(1)} \left(\log F_T < \log G \right) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Re[\frac{G^{-iu} f_t(iu)}{iu} du], \tag{3.9}$$

$$Q^{(2)} \left(\log F_T < \log G \right) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Re[\frac{G^{-iu} f_t(iu+1)}{iu f_t(1)} du]. \tag{3.10}$$

3.2.2 Fair Fee Structure

In Hardy [2003], it has been assumed that the fees for guarantees or riders are charged at discrete time and the fee rate is constant. We follow part of the assumption to charge the investment management fees and the rider fees at discrete time but with a dynamic rider fee rate consisted of a base fee rate and a VIX-linked fee rate, while the investment management fee rate is still a constant.

Although we are using continuously-compounded fee rate in the fund dynamics to calculate the expected risk-neutral value of a GMMB, discrete-time fee rate is used here to calculate expected risk-neutral rider fees for simplicity on the expression. Since only the rider fees are used to compensate for liability, in order to separate rider fees with the investment management fees, we assume that the investment management fee is charged right before the rider fee. Now we can express fund dynamic in from time t to $t+1$ by:

$$\begin{aligned}\frac{F_{t+1}}{F_t} &= \frac{S_{t+1}}{S_t}\exp(-c_{t+1}^{tot}),\\ &= \frac{S_{t+1}}{S_t}\exp(-c^{inv})\exp(-c_{t+1}),\\ &= \frac{S_{t+1}}{S_t}(1-dc^{inv})(1-dc_{t+1}).\end{aligned}$$

Here dc^{inv} and dc_{t+1} are discrete-time fee rate corresponding to the continuously-compounded fee rate c^{inv} and c_{t+1}, and we have the relationship:

$$\exp(-c^{inv}) = (1-dc^{inv}),$$

$$\exp(-c_{t+1}) = (1-dc_{t+1}).$$

Now we can decompose the change of fund value between time t and t+1 to three parts A_{t+1}, B_{t+1} and C_{t+1}:

$$F_{t+1} = \underbrace{F_t\frac{S_{t+1}}{S_t}}_{A_{t+1}} - \underbrace{F_t\frac{S_{t+1}}{S_t} * dc^{inv}}_{B_{t+1}} - \underbrace{F_t\frac{S_{t+1}}{S_t} * (1-dc^{inv}) * dc_{t+1}}_{C_{t+1}}.$$

A_{t+1} represents the fund value at time t+1 if no fee is charged, B_{t+1}represents the investment management fee charged at time t+1 and C_{t+1} represents the rider fee charged at time t+1, then the rider fee charged at time t+1 can be expressed by

$$C_{t+1} = F_{t+1} - A_{t+1} - B_{t+1}. \tag{3.11}$$

We use $\Lambda(t)$ to denote the expected future risk-neutral value of the rider fees conditional on $\mathcal{F}_t$:

Definition 3.2.1. The expected risk-neutral value of the rider fees conditional on $\mathcal{F}_t$ is the summation of all the future expected risk-neutral rider fees C_i $(t+1 \leq i \leq T)$:

$$\Lambda(t) = \sum_{i=t}^{T-1} e^{-r(i-t+1)} E_t^Q [F_i \frac{S_{i+1}}{S_i} * (1 - dc^{inv}) * dc_{i+1}].$$

With the relationship (3.11), we are able to derive a analytic calculation formula for the expected risk-neutral rider fees.

Proposition 3.2.2. *The expected future risk-neutral value of the rider fees conditional on $\mathcal{F}_t$ can be expressed by:*

$$\Lambda(t) = F_t - e^{-r(T-t)} E_t^Q [F_T] - dc^{inv} \sum_{i=t}^{T-1} e^{-r(i-t)} E_t^Q [F_i].$$

Proof See Section A.3 in the Appendix

The above results are obtained by iterative conditional expectation. Since the rider fees are used to compensate the cost of a GMMB, under a fair fee structure, it should be set that at the inception of the contract $t = 0$, the expected risk-neutral value of rider fees equal to the expected risk-neutral value of the GMMB.

Definition 3.2.2. A fair fee structure is a pair $(\bar{c}^*$, $m^*)$ that satisfies

$$\Lambda(0) = \Pi(0).$$

The fair fee structure is not unique, any pair of $\bar{c}$ and m which can satisfy the above condition is a fair fee structure. To get a fair fee structure, one need to fix one of $\bar{c}$ or m, and use the equality above to solve for the other.

3.2.3 Net liability

In the previous sections, we have derived the expressions for the expected risk-neutral gross liability $\Pi(t)$ for a GMMB and the expected risk-neutral fees $\Lambda(t)$ used to compensate the GMMB. For the purpose of risk management, insurer should hedge net liability, which is defined as the difference between $\Pi(t)$ and $\Lambda(t)$:

Definition 3.2.3. The net liability assumed by insurer at time t $t \in \mathcal{T}=\{t|t=0,1,2\ldots,T\}$ is the difference between liability and income:

$$\Pi^{Net}(t) = \Pi(t) - \Lambda(t).$$

The reason we need to use net liability instead of gross liability to hedge the risk is because financial risk will influence both gross liability and fee income, if we only use gross liability we may overlook the risk while net liability incorporate the gross liability and fee income and thus combine the effects of financial risk from them.

Chapter 4

Diffusion Limits of Fund Dynamic under the HN-GARCH Model

In chapter 2 and chapter 3, we introduce the way to analyze the effects of a VIX-linked fee structure under affine GARCH models. In this chapter, we intend to generalize the analysis to the case when financial events happen at a higher frequency and also investigate the theoretic results as time interval shrinks. First, we derive the fund dynamics under the discretized HN-GARCH model, and similarly as the previous chapters, the conditional moment generating function and the analytic calculation formula for the expected risk-neutral rider fees are also derived. With the standard approach proposed by Nelson [1990] to derive weak convergence results of GARCH models, we obtain the weak diffusion limits of fund dynamics under the HN-GARCH model and thus bring a connection to the analysis of a VIX linked fee structure from discrete-time setting to continuous-time setting.

4.1 Discretized Dynamics of Investment Fund

Previously we have analyzed the VIX-linked fee structure under a market model at a daily pace. In this section we will do the analysis at the situation when financial events happen at a higher frequency under the HN-GARCH model. We first derive the discretized fund dynamics and try to generalize the results which has been investigated in the previous chapters. These generalizations can be used to connect our work with the analysis in Cui et al. [2017] under continuous-time stochastic volatility model.

4.1.1 Discretized Fund dynamics under P

To derive the diffusion limit of fund dynamics under the HN-GARCH model, we need to first discretize the market model. Consider a n-indexed discrete-time financial market with filtered probability space $(\Omega^n, \mathcal{F}^{(n)}, \mathcal{F}_t^{(n)})$, $t \in \mathcal{T}=\{t|t = 0, \frac{1}{n}, \frac{2}{n} \dots, \frac{nT}{n}\}$, T is the maturity of the contract, and the equally spaced time interval between two consecutive trading times is $\Delta = \frac{1}{n}$, n is the number of discretized intervals for each each day. We present the discretized HN-GARCH model under physical measure as following:

$$\log \frac{S_{k\Delta}}{S_{(k-1)\Delta}} = (r + \lambda h_{k\Delta})\Delta + \sqrt{\Delta}\sqrt{h_{k\Delta}} z_{k\Delta}, \tag{4.1}$$

$$h_{(k+1)\Delta} = \omega(\Delta) + \beta(\Delta)h_{k\Delta} + \alpha(\Delta)(z_{k\Delta} - \gamma(\Delta)\sqrt{h_{k\Delta}})^2, \tag{4.2}$$

$$z_{k\Delta} \sim N(0, 1). \tag{4.3}$$

Here, $S_{k\Delta}$ represent the asset price at time $t = k\Delta$, $h_{(k+1)\Delta}$ is the conditional variance of log return and $z_{k\Delta}$ is the driving noise and it follows a standard Gaussian distribution. We let the parameters $\omega(\Delta)$, $\beta(\Delta)$, $\alpha(\Delta)$ and $\gamma(\Delta)$ depend on the length of time interval Δ. $\gamma(\Delta)$ is used to describe the lever effect, that is the negative correlation between asset return and the conditional variance. The persistence has the constraint that $d(\Delta) = \beta(\Delta)+\alpha(\Delta)\gamma^2(\Delta) < 1$.

The change of discretized fund is composed of two parts, the first part is from the dynamics of underlying asset which is described by (4.1)-(4.3) and the second part is the charging of fees at continuously-compounded fee rate $c_{k\Delta}^{tot}$. This process can be expressed as following:

$$\frac{F_{k\Delta}}{F_{(k-1)\Delta}} = \frac{S_{k\Delta}}{S_{(k-1)\Delta}} \exp(-c_{k\Delta}^{tot} \times \Delta). \tag{4.4}$$

In (4.4) $c_{k\Delta}^{tot}$ is the total fee rate at time $k\Delta$. In section 2, we have proven that the daily version of VIX index V_t is a weight average of the current conditional variance in the next period and the long-term variance. Letting $V_t(n;\Delta)$ denote the daily version of VIX under the discretized HN-GARCH model, we can derive a similar expression for $V_t(n;\Delta)$.

Proposition 4.1.1. *If equity dynamic follows the discretized HN-GARCH model, then VIX proxy $V_{k\Delta}(n;\Delta)$ is a weighted average of current conditional variance of next period and the long-term variance under physical measure*

$$V_{k\Delta}(n;\Delta) = (1 - \Gamma(n;\Delta))\overline{h}(\Delta) + \Gamma(n;\Delta)h_{(k+1)\Delta}.$$

where

$$\overline{h}(\Delta) = \frac{\omega(\Delta) + \alpha(\Delta)}{1 - \beta(\Delta) - \alpha(\Delta)\gamma^2(\Delta)}, \ \Gamma(n;\Delta) = \frac{\Delta(1 - \tilde{\beta}(\Delta)^{\frac{n}{\Delta}})}{n(1 - \tilde{\beta}(\Delta))},$$
$$\tilde{\omega}(\Delta) = \omega(\Delta) + \alpha(\Delta), \quad \tilde{\beta}(\Delta) = \beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta).$$

Proof See Section A.4 in the Appendix

Here, $\overline{h}(\Delta)$ is the long-term variance under the discretized HN-GARCH model. Note that the expression of VIX under non-discretized HN-GARCH model has been derived in Wang et al. [2017], and proposition.4.1.1 can be regarded as a generalization of Wang et al. [2017] 's work. Next, we link $V_{k\Delta}(n,;\Delta)$ with the rider fee rate. We assume that the rider fee has a positive relationship with the arithmetic average of the expected discretized daily variance over next month with 22 trading days:

$$c_{k\Delta} = \overline{c} + \frac{m}{252}(\frac{VIX_{(k-1)\Delta}}{100})^2 = \overline{c} + mV_{(k-1)\Delta}(22;\Delta).$$

Since $c_{k\Delta}^{tot} = c^{inv} + c_{k\Delta}$, we can get:

$$c_{k\Delta}^{tot} = M(\Delta) + N(\Delta)(h_{k\Delta} - \overline{h}(\Delta)).$$

$$\text{where } M(\Delta) = c^{inv} + \overline{c} + m\overline{h}(\Delta) > 0,\ N(\Delta) = m\Gamma(22;\Delta).$$

We use $f_{k\Delta}$ to denote the log return of investment fund $f_{k\Delta} = \log \frac{F_{k\Delta}}{F_{(k-1)\Delta}}$, $(k = 1, 2, \cdots, n)$. Now, with (4.1)-(4.4) we can derive discretize fund dynamics under the HN-GARCH model as following:

$$f_{k\Delta} = \left[\left(r - M(\Delta) + N(\Delta)\overline{h}\right) + (\lambda - N(\Delta))\, h_{k\Delta}\right]\Delta + \sqrt{\Delta}\sqrt{h_{k\Delta}}z_{k\Delta}, \tag{4.5}$$

$$h_{(k+1)\Delta} = \omega(\Delta) + \beta(\Delta)h_{k\Delta} + \alpha(\Delta)(z_{k\Delta} - \gamma(\Delta)\sqrt{h_{k\Delta}})^2. \tag{4.6}$$

Note that the discretized fund dynamics from (4.5)-(4.6) under physical measure can be verified by discretizing the fund dynamics in (3.1)-(3.2) directly.

4.1.2 Discretized Fund dynamics under Q

Similarly, the discretized fund dynamics under risk-neutral measure can be obtained. First the dynamics of underlying asset under risk-neutral measure is derived through LRNVR which is applicable when the driving noise follows a standard Gaussian distribution.

Proposition 4.1.2. *If the underlying asset follows the discretized dynamics from (4.1)-(4.3) under physical measure, its discretized dynamics under risk-neutral measure based on LRNVR are:*

$$\log \frac{S_{k\Delta}}{S_{(k-1)\Delta}} = (r - \frac{1}{2}h_{k\Delta})\Delta + \sqrt{\Delta}\sqrt{h_{k\Delta}}z_{k\Delta}^*, \tag{4.7}$$

$$h_{(k+1)\Delta} = \omega(\Delta) + \beta(\Delta)h_{k\Delta} + \alpha(\Delta)(z_{k\Delta}^* - \gamma^*(\Delta)\sqrt{h_{k\Delta}})^2, \tag{4.8}$$

$$z_{k\Delta}^* \sim N(0,1). \tag{4.9}$$

where $z^*_{k\Delta} = z_{k\Delta} + (\lambda + \frac{1}{2})\sqrt{h_{k\Delta}}\sqrt{\Delta}$ *and* $\gamma^*(\Delta) = \gamma(\Delta) + \sqrt{\Delta}(\lambda + 1/2)$.

Note that Badescu et al. [2017]) has shown the above LRNVR transformation is a special case of the risk-measure transformation under variance dependent pricing kernel by letting the market price of variance risk equal to zero. Next, we derive the VIX proxy $V^*_{k\Delta}(n;\Delta)$ under risk-neutral measure.

Proposition 4.1.3. *If equity dynamic follows discretized HN-GARCH model, then the VIX proxy* $V^*_{k\Delta}(n;\Delta)$ *is a weighted average of current conditional variance of next period and the long-term variance under risk-neutral measure*

$$V^*_{k\Delta}(n;\Delta) = (1 - \Gamma^*(n;\Delta))\overline{h}^*(\Delta) + \Gamma^*(n;\Delta)h_{(k+1)\Delta}.$$

where

$$\overline{h}^*(\Delta) = \frac{\omega(\Delta) + \alpha(\Delta)}{1 - \beta(\Delta) - \alpha(\Delta)\gamma^{*2}(\Delta)},\ \Gamma^*(n;\Delta) = \frac{\Delta(1 - \tilde{\beta}^*(\Delta)^{\frac{n}{\Delta}})}{n(1 - \tilde{\beta}^*(\Delta))},$$
$$\tilde{\omega}(\Delta) = \omega(\Delta) + \alpha(\Delta), \quad \tilde{\beta}^*(\Delta) = \beta(\Delta) + \alpha(\Delta)\gamma^{*2}(\Delta).$$

Next, we link the $V^*_{k\Delta}(n;\Delta)$, with the rider fee rate. We assume that the rider fee has a positive relationship with the arithmetic average of the expected discretized daily variance over next month with 22 trading days.

$$c^{tot}_{k\Delta} = M^*(\Delta) + N^*(\Delta)(h_{k\Delta} - \overline{h}^*(\Delta)).$$

$$\text{where } M^*(\Delta) = c^{inv} + \overline{c} + m\overline{h}^*(\Delta) > 0,\ N^*(\Delta) = m\Gamma^*(22;\Delta).$$

Following (4.4), the fees are charged from the investment fund so that we can obtain the discretized fund dynamics under risk-neutral measure:

Proposition 4.1.4. *If the discretized fund dynamics under physical measure follow (4.7)-*

(4.9), the discretized fund dynamics under risk-neutral measure are given:

$$f_{k\Delta} = [(r - M^*(\Delta) + N^*(\Delta)\overline{h}) - (\frac{1}{2} + N^*(\Delta))h_{k\Delta}]\Delta + \sqrt{\Delta}\sqrt{h_{k\Delta}}z_{k\Delta}, \tag{4.10}$$

$$h_{(k+1)\Delta} = \omega(\Delta) + \beta(\Delta)h_{k\Delta} + \alpha(\Delta)(z_{k\Delta} - \gamma^*(\Delta)\sqrt{h_{k\Delta}})^2. \tag{4.11}$$

where $\gamma^*(\Delta) = \gamma(\Delta) + \sqrt{\Delta}(\lambda + 1/2)$.

As in the case when financial events happen at a daily pace, we can also derive the conditional moment generating function of the log fund value at discretized time interval with iterative expectation.

Proposition 4.1.5. *If asset dynamics follows discretized HN-GARCH from (4.7)-(4.9), the conditional moment generating function of* $log(F_T)$ *under risk-neutral measure is given by*

$$f_{k\Delta}(\phi) = E^Q_{k\Delta}(F_T^{\phi}) = \exp[\phi \log F_{k\Delta} + A(k\Delta) + B(k\Delta)h_{(k+1)\Delta}].$$

where

$$\begin{aligned} A(k\Delta;\Delta) =& A((k+1)\Delta;\Delta) + \phi(r - M^*(\Delta) + N^*(\Delta)\overline{h})\Delta + B((k+1)\Delta;\Delta)\omega(\Delta) \\ & - \frac{1}{2}\log(1 - 2B((k+1)\Delta;\Delta)\alpha(\Delta)); \\ B(k\Delta;\Delta) =& \phi(-\frac{1}{2} - N^*(\Delta))\Delta + \left(\alpha(\Delta)\gamma^{*2}(\Delta) + \beta(\Delta)\right) B((k+1)\Delta;\Delta) \\ & + \frac{(\phi\sqrt{\Delta} - 2\alpha(\Delta)\gamma^*(\Delta)B((k+1)\Delta;\Delta))^2}{2(1 - 2B((k+1)\Delta;\Delta)\alpha(\Delta))}. \end{aligned}$$

With the above conditional moment generating function of risk-neutral discretized fund dynamics, we are able to derive a semi-closed form solution to the price of GMMB using inverse Fourier transformation on the conditional moment generating function. For a fair fee scheme, we need to let the liability of GMMB equal to the expected risk-neutral rider fees. To find a numerical solution to expected risk-neutral rider fees, we represent the process

(4.4) as:

$$\frac{F_{k\Delta}}{F_{(k-1)\Delta}} = \frac{S_{k\Delta}}{S_{(k-1)\Delta}} \exp(-c^{tot}_{k\Delta} \times \Delta), \quad (4.12)$$

$$= \frac{S_{k\Delta}}{S_{(k-1)\Delta}} \exp(-c^{inv} \times \Delta) \exp(-c_{k\Delta} \times \Delta), \quad (4.13)$$

$$= \frac{S_{k\Delta}}{S_{(k-1)\Delta}} (1 - dc^{inv(\Delta)})(1 - dc_{k\Delta}). \quad (4.14)$$

where

$$(1 - dc^{inv(\Delta)}) = \exp(-c^{inv} \times \Delta), \ (1 - dc_{k\Delta}) = \exp(-c_{k\Delta} \times \Delta).$$

Here $c^{inv*(\Delta)}$ and $c^{*}_{k\Delta}$ are fee rates charged at each step and they are equivalent to the continuously-compounded fee rate c^{inv} and $c_{k\Delta}$. From (4.12)-(4.14), we can see for each time interval with length Δ, a continuously compounded fee rate $c^{tot}_{k\Delta}$ is charged. Similarly with the process of (3.11), we are able to derive a analytic calculation formula for the expected future risk-neutral value of rider fees at discretized time with (4.14).

Proposition 4.1.6. *Suppose the fund value follows the dynamics from (4.10)-(4.11) under risk-neutral measure, the expected future risk-neutral value of the rider fees conditional on* $\mathcal{F}_t$*, where* $t = k\Delta$ $(k = 0,\ 1,\ \ldots\ ,\ T/\Delta)$ *can be expressed by:*

$$\Lambda^{(\Delta)}(t) = F_t - e^{-r(T-t)} E^Q_t[F_T] - dc^{inv(\Delta)} \textstyle\sum_{i=t}^{T-\Delta} e^{-r(i-t)} E^Q_t[F_i].$$

where

$$dc^{inv(\Delta)} = 1 - \exp(-c^{inv} \times \Delta).$$

4.2 Diffusion Limit of HN-GARCH model

The HN-GARCH model has a diffusion limit to the continuous-time stochastic volatility model Heston's model (See Heston and Nandi [2000] and Badescu et al. [2017]) which has been used to investigated the effects of VIX-linked fee structure in Cui et al. [2017]. In this section we use the standard approach introduced in Nelson [1990] to derive the diffusion limits of fund dynamics under both physical measure and risk-neutral measure. The characteristic function log fund value under risk-neutral measure and the analytic calculation formula for the expected risk-neutral rider fees are presented.

4.2.1 Diffusion Limits of Fund Dynamics under P

To connect the analysis of a VIX-linked fee structure under the discrete-time GARCH model with a continuous-time stochastic volatility model, we first derive a weak diffusion limit of the HN-GARCH model and deduct fees from the investment fund to get the diffusion limit of fund dynamics. In Nelson [1990], a standard approach is proposed to derive a weak diffusion limit of Markov process with some sets of conditions, and this approach is applicable to derive diffusion limits of arch-type models. Following Francq and Zakoian [2011], we present the following weak convergence theorem and use it to derive the diffusion limits of fund dynamics under physical measure.

Proposition 4.2.1. *If there exists a continuous mapping μ and σ from $\mathbb{R}^d$ to $\mathbb{R}^d$ and $\mathcal{M}_{p\times d}$ respectively, such that for all $r > 0$ and for some $\sigma > 0$,*

$$\lim_{\Delta\to 0}\sup_{||f||<r}\left|\frac{1}{\Delta}E[f_{k\Delta} - f_{(k-1)\Delta}|f_{(k-1)\Delta} = f] - \mu(f)\right| = 0,$$

$$\lim_{\Delta\to 0}\sup_{||f||<r}\left|\frac{1}{\Delta}\mathrm{Var}[f_{k\Delta} - f_{(k-1)\Delta}|f_{(k-1)\Delta} = f] - \sigma(f)\sigma(f)^{\mathrm{T}}\right| = 0,$$

$$\lim_{\Delta\to 0}\sup_{||f||<r}(\frac{1}{\Delta})^{(2+\sigma)/2}\mathrm{E}[\|f_{k\Delta} - f_{(k-1)\Delta}\|^{2+\sigma}|f_{(k-1)\Delta} = f] < \infty.$$

Then if the equation

$$df_t = u(f_t)dt + \sigma(f_t)dW_t, f_0 = a.$$

admits a solution.(f_t) which is unique in law and if f_0 converges in law to a, then $f_{k\Delta}$

converges in law to f_t

For the convenience to calculate moments, conditional joint cumulant generating function (CGF) is useful on deriving to the diffusion limits of fund dynamics. The CGF of log return and its conditional variance under the HN-GARCH model has been investigated in Badescu et al. [2017]. A one-step joint CGF of $s_{k\Delta} = \log(S_{k\Delta}/S_{(k-1)\Delta})$ and $h_{(k+1)\Delta}$ conditional on $\mathcal{F}^{(n)}_{(k-1)\Delta}$ under physical measure has been derived.

Proposition 4.2.2. *If the underlying asset following discretized HN-GARCH model under physical measure, one-step joint CGF of* $s_{k\Delta}$ *and* $h_{(k+1)\Delta}$ *conditional on* $\mathcal{F}^{(n)}_{(k-1)\Delta}$ *is*

$$\begin{aligned} C_{(s_{k\Delta},h_{(k+1)\Delta})}(\phi s_{k\Delta} + \psi h_{(k+1)\Delta}) &= logE^Q[exp(\phi s_{k\Delta} + \psi h_{(k+1)\Delta})|\mathcal{F}^{(n)}_{(k-1)\Delta}], \\ &= A(\phi, \psi; (k-1)\Delta, k\Delta) + B(\phi, \psi; (k-1)\Delta, k\Delta)h_{(k+1)\Delta}. \end{aligned}$$

where

$$\begin{aligned} A(\phi, \psi; (k-1)\Delta, k\Delta) &= \phi r\Delta + \psi\omega(\Delta) - \frac{1}{2}\log(1 - 2\phi\alpha(\Delta)); \\ B(\phi, \psi; (k-1)\Delta, k\Delta) &= \phi\lambda\Delta + \psi(\alpha(\Delta)\gamma^2(\Delta) + \beta(\Delta)) + \frac{(\phi\sqrt{\Delta} - 2\psi\alpha(\Delta)\gamma(\Delta))^2}{2(1 - 2\psi\alpha(\Delta))}. \end{aligned}$$

This joint CGF of $(f_{k\Delta}, h_{(k+1)\Delta})$ will be used to calculate the moments of log return and its conditional variance. By make following assumption on model parameters, we are able to use the weak convergence theorem to derive the diffusion limit of the HN-GARCH model.

$$\begin{aligned} &\omega(\Delta) = \omega\Delta, \ \alpha(\Delta) = \alpha\Delta, \\ &\gamma(\Delta) = \frac{\gamma}{\sqrt{\Delta}}, \ \kappa = 1 - \beta - \alpha\gamma^2, \\ &\beta(\Delta) = 1 - \kappa\Delta - \alpha(\Delta)\gamma^2(\Delta). \end{aligned}$$

Proposition 4.2.3. *If the underlying asset following HN-GARCH model under physical* *measure, the weak convergence diffusion limit of underlying asset under physical measure is*

$$d \log S_t = (r + \lambda h_t)dt + \sqrt{h_t} dW_{1t}, \tag{4.15}$$

$$dh_t = \kappa(\theta - h_t)dt + \sigma\sqrt{h_t} dW_{2t}, \tag{4.16}$$

where

$$\kappa = 1 - \beta - \alpha\gamma^2, \ \theta = (\omega + \alpha)/\kappa, \ \sigma = 2\alpha\gamma.$$

Proof See Section A.5 in the Appendix

With Ito's lemma, (4.15)-(4.16) is equivalent to the following dynamics.

$$\frac{dS_t}{S_t} = \left(r + (\lambda + \frac{1}{2})h_t \right) dt + \sqrt{h_t} dW_{1t}, \tag{4.17}$$

$$dh_t = \kappa(\theta - h_t)dt + \sigma\sqrt{h_t} dW_{2t}, \tag{4.18}$$

From section 4.1 we know that total fee rate $c_{k\Delta}^{tot} = M(\Delta) + N(\Delta)(h_{k\Delta} - \overline{h}(\Delta))$ with $M(\Delta) = c^{inv} + \overline{c} + m\overline{h}(\Delta) > 0$, $N(\Delta) = m\Gamma^{(\Delta)}(22)$. Using the parameters specification in this section we are able to show that:

Corollary 4.2.1. *If the underlying asset follows discretized HN-GARCH dynamics under P, the coefficients of total fee rate $M(\Delta)$ and $N(\Delta)$ on GMMB have following result as time interval goes to 0*

$$\lim_{\Delta \to 0} M(\Delta) = c^{inv} + \overline{c} + m\theta, \ \lim_{\Delta \to 0} N(\Delta) = \frac{m(1 - e^{-\kappa n})}{n\kappa}.$$

Here θ is the long-term variance and it can be calculated with the constraints we put on the parameters of the discretized HN-GARCH model. With the above corollary and the weak convergence of conditional variance to the instantaneous variance, the total fee rate

$c_{k\Delta}^{tot}$ has a weak diffusion limit to a instantaneous fee rate.

$$c_t^{tot} = M(0) + N(0)(h_t - \theta).$$

Here, $M(0) = c^{inv} + \bar{c} + m\theta$ and $N(0) = \frac{m(1-e^{-\kappa n})}{n\kappa}$. Now we can deduct the instantaneous fee from the investment fund with following dynamics to get the diffusion limits of the investment fund.

$$\frac{dF_t}{F_t} = \frac{dS_t}{S_t} - c_t^{tot} dt.$$

Proposition 4.2.4. *If the underlying asset following HN-GARCH model under physical measure, the weak convergence diffusion limit of fund dynamics under P is*

$$\frac{dF_t}{F_t} = \left(r - M(0) + N(0)\theta + \left(\lambda + \frac{1}{2} - N(0) \right) h_t \right) dt + \sqrt{h_t} dW_{1t},$$
$$dh_t = \kappa(\theta - h_t)dt + \sigma\sqrt{h_t} dW_{2t},$$

where

$$\kappa = 1 - \beta - \alpha\gamma^2, \ \theta = (\omega + \alpha)/\kappa, \ \sigma = 2\alpha\gamma.$$

Here the Brownian motions W_{1t} and W_{2t} are correlated with $E[dW_{1t}W_{2t}] = \rho dt$, where $\rho = -1$, and we can also write the above dynamics with only one Brownian motion.

4.2.2 Diffusion Limits of Fund Dynamics under Q

Similarly we can derive the risk-neutral diffusion limits of the HN-GARCH model and deduct instantaneous fee rate from investment fund to get risk-neutral diffusion limits of investment fund. The risk-neutral diffusion limit of HN-GARCH is derived:

Proposition 4.2.5. *If the underlying asset following HN-GARCH model , the weak conver-*

gence diffusion limit of underlying asset under Q is

$$\frac{dS_t}{S_t} = (r + h_t)\, dt + \sqrt{h_t} dW_{1t}, \tag{4.19}$$

$$dh_t = \kappa^*(\theta^* - h_t)dt + \sigma\sqrt{h_t} dW_{2t}, \tag{4.20}$$

where

$$\kappa^* = \kappa - \sigma(\lambda + \frac{1}{2}),\ \ \theta^* = (\omega + \alpha)/\kappa^*,\ \ \sigma = 2\alpha\gamma.$$

Here θ^* is the risk-neutral long-term variance. Under risk-neutral measure we have,

Corollary 4.2.2. *If the underlying asset follows discretized HN-GARCH dynamics under Q, the coefficients of total fee rate $M^*(\Delta)$ and $N^*(\Delta)$ on GMMB have following result as time interval goes to 0*

$$\lim_{\Delta\to 0} M^*(\Delta) = c^{inv} + \overline{c} + m\theta^*,\ \ \lim_{\Delta\to 0} N^*(\Delta) = \frac{m(1 - e^{-\kappa^* n})}{n\kappa^*}.$$

After deducting the instantaneous fee rate we get:

Proposition 4.2.6. *If the underlying asset following HN-GARCH model under risk-neutral measure, the weak convergence diffusion limit of fund dynamics under Q is*

$$\frac{dF_t}{F_t} = [r - M^*(0) - N^*(0)(h_t - \overline{h})]dt + \sqrt{h_t} dW_{1t}, \tag{4.21}$$

$$dh_t = \kappa^*(\theta^* - h_t)dt + \sigma\sqrt{h_t} dW_{2t} \tag{4.22}$$

where

$$\kappa^* = \kappa - \sigma(\lambda + \frac{1}{2}),\ \ \theta^* = (\omega + \alpha)/\kappa^*,\ \ \sigma = 2\alpha\gamma.$$

Comparing the above result with the stochastic differential dynamics of fund value in Cui et al. [2017], they are very similar except that the correlation coefficient of Brownian motions may not be -1 of in Cui et al. [2017]. This similarity is not surprising as the HN-GARCH model has a weak convergence diffusion limit to Hston's model and the fee scheme used in this thesis is similar with Cui et al. [2017].

The closed-fom solution to GMMB and risk neutral values of rider fees under diffusion limit have been derived in Cui et al. [2017]. This results can be used to calculate a fair fee scheme as time interval shrinks and we expect to see the fair fee rate converges. We let $X_t = \log \frac{F_T}{F_t}$ conditional on $\mathcal{F}_t$, and the conditional characteristic function of X_t is present here.

Proposition 4.2.7. *Suppose fund dynamics follow (4.6)-(4.7), the conditional characteristic function of X_t conditional on $\mathcal{F}_t$ is:*

$$\begin{aligned}\gamma(\mu, t) = &\exp[i\mu X_t + i\mu(r - M(0) + N(0)\theta^*)(T-t) + \kappa\theta^*(T-t)\frac{q}{\sigma^2} \\ &+ \frac{2\kappa^*\theta^*}{\sigma^2}\log\frac{1-g}{1-ge^{-d(T-t)}} + \frac{h_t q}{\sigma^2}\frac{1-e^{-d(T-t)}}{1-ge^{-d(T-t)}}].\end{aligned}$$

where $d = \sqrt{(\kappa + i\sigma\mu)^2 + \sigma^2((2N(0)+1)i\mu + \mu^2)}$, $q = \kappa - d + i\sigma\mu$, $g = q/(q+2d)$.

Note that the above proposition is a special case of the result (17) on page 11 of Cui et al. [2017] as the correlation of Brownian motions ρ is -1 in this thesis. The expected future risk-neutral fees under diffusion limit has also been derived by Cui et al. [2017].

Proposition 4.2.8. *Suppose fund dynamics follow (4.6)-(4.7), the risk-neutral fees conditional on $\mathcal{F}_t$ can be derived as:*

$$E\left[\int_t^T e^{-r(u-t)} c_\mu F_\mu d_\mu\right] = F_t - e^{-r(T-t)}E_t[F_T] - c^{inv}\int_t^T e^{-r(\mu-t)}E_t[F_i]d\mu.$$

The above result is very similar with Proposition 3.2.2, the expected risk-neutral value of rider fees measured at discrete-time setting, and it can be regarded as the continuous-time version of Proposition 3.2.2. The expected fund value in this solution can be solved by the characteristic function of F_t.

Using Fourier inversion on the characteristic function of X_t, we can calculate the price of GMMB under diffusion limit. And by letting the price equal to the expected risk-neutral rider fees, we can calculate the fee rate under diffusion limit and we expect that the fair total fee rate under discretized HN-GARCH will converge to the fair total fee rate in diffusion limit as time interval shrinks.

Chapter 5

Numerical Results

Based the theoretic results obtained in the previous chapters, we provide some numerical examples to analyze the effects of a VIX-linked fee structure. First, we describe the financial series which we use to estimate the model parameters with two different maximum likelihood estimation methods. Next, we discuss about how the market models and the estimation methods will influence the fair fee structure. With the fair fee structure, the analysis of a VIX-linked fee structure is conducted and show that the VIX-linked fee structure can realign the insurer's income and its liability. At last, convergence results of total fee rate are studied numerically.

5.1 Data and Estimation

In this section we first describe the data we are using and present some descriptive statistics of the dataset. After that, return-only maximum likelihood estimation and joint maximum likelihood estimation are introduced. The model parameters of the HN-GARCH model and the IG-GARCH model are estimated using these two estimation methods, and the results are stored in a table.

5.1.1 Data

To implement the parameter estimation, we need to use the information from 2 time series, SP 500 index and VIX. In our thesis, we use SP 500 and VIX from January 5, 1999 to December 31, 2013, both series are collected from Yahoo Finance. Log return of SP 500 is calculated from the closed price. We use 3-month U.S. Treasury bill rate as the risk-free interest rate, 3-month U.S. Treasury bill rate can be collected from Federal Reserve website. We provide some descriptive statistics for SP 500, VIX and log return in Table 3.1.

Table 5.1: Descriptive statistics of SP 500, VIX and log return

	N	$Mean$	Std	Min	Max	$Median$	$Skew$	$Kurt$
SP 500	3772	1242.0	207.09	676.53	1848.4	1253.5	0.0161	2.9024
VIX	3772	21.622	8.8642	9.8900	80.860	20.155	1.9401	9.3870
$Log - Return$	3772	0.0001	0.0130	-0.095	0.1096	0.0006	-0.167	10.453

5.1.2 Parameter Estimation

In this thesis, we are using Maximum Likelihood Estimation(MLE) to estimate the parameters, although this method is very sensitive to initial values when we use computer programs to optimize the likelihood function. To address this problem, we generate 1000 random initial parameters for the Heston-Nandi model, 20000 initial parameters for the IG-GARCH model, and choose those estimates with the largest likelihood as our final results. We use two types of MLE here.

The first method is the straight forward MLE with the log returns of underlying equity only which we call return-only MLE. The density function of log-return which are given below:

$$f^{HN}(R_t) = \frac{1}{\sqrt{2\pi h(t)}} e^{-\frac{1}{2}\frac{(R_t - r - \lambda h_t)^2}{h_t}}$$

$$f^{IG}(R_t) = \frac{h_t}{-\eta\sqrt{2\pi(R_t - r - \nu h_t)^3}} e^{-\frac{1}{2}[\sqrt{\frac{R_t - r - \nu h_t}{\eta}} - \frac{h_t}{\eta^2}\sqrt{\frac{R_t - r - \nu h_t}{\eta}}]^2}$$

Using the density function of log-return, the likelihood function under the HN-GARCH model and the IG-GARCH model, correspondingly denoted by L_r^{HN} and L_r^{IG}, are derived under physical measure. The log-likelihood function under both models is derived by $\log(L_r) = \sum_{t=1}^{n} \log f(R_t)$

Under HN-GARCH:

$$\log(L_r^{HN}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}(\log(h_t) + (R_t - r - \lambda h_t + \frac{1}{2}h_t)^2/h_t)$$

Under IG-GARCH:

$$\log(L_r^{IG}) = -n\log(-\eta\sqrt{2\pi}) - \frac{1}{2}\sum_{t=1}^{n}[\log\frac{(\eta y_t)^3}{h_t^2} + (\sqrt{y_t} - \frac{h_t}{\eta}\sqrt{\frac{1}{y_t}})^2]$$

where $R_t = \frac{S_(t)}{S_(t-1)}$ is the log return,$y_t = \frac{R_t - r - \nu ht}{\eta}$.

The second method is joint MLE using both log-returns of underlying asset and VIX index data. Hao and Zhang [2013] first proposed the joint MLE using both returns and VIX, they assumed the error between market VIX and model VIX follows a *i.i.d* normal distribution with zero mean. While Kanniainen et al. [2014] assumed this error to follow an auto-regressive disturbance. In this thesis, we prefer using the the assumption in Hao and Zhang [2013].

We assume the VIX error follows a *i.i.d* normal distribution with zero mean:

$$u_t = (VIX^{Mkt} - VIX^{Mod})/\sqrt{252} \times 100$$
$$u_t \sim i.i.dN(0, s_v^2)$$

s_v^2 is estimated by the sample variance, and the log-likelihood function of VIX is

$$\log(L_v) = -\tfrac{n}{2}\log(2\pi\bar{s}_v^2) - \tfrac{1}{s_v^2}\textstyle\sum_{t=1}^{n} u_t^2$$

The joint MLE parameters can be obtained by maximizing the sum of return log-likelihood and VIX log-likelihood.

$$\log(L_{vr}) = \log(L_r) + \log(L_v)$$

5.1.3 Estimated Parameters

In this subsection we provide the results from return-only MLE and joint MLE under the HN-GARCH model and the IG-GARCH model, and compare the models' abilities to fit the dynamics of spot prices and VIX index. Results are stored in the table 5.2. In this table, the estimated parameters from return-only MLE and joint MLE are presented with the standard errors.

From this table, we observe that these two models present close long-term variance and persistence using the same estimation method. This coincides with the fact that IG-GARCH converges to Heson-Nandi when η goes to 0. We notice that the IG-GARCH has a better performance than Heston-Nandi regardless of the estimation method which can be checked by the log-likelihood and AIC/BIC value. Especially when using joint estimation, both log-likelihoods (return likelihood and VIX likelihood) for the IG-GARCH are higher than those for the HN-GARCH model. Moreover, under return-only MLE, $M_3 = -0.1449$, $M_4 = 3.0350$, and under joint MLE, $M_3 = -0.3746$, $M_4 = 3.2339$. This result is reasonable since the driving noise in IG-GARCH can capture negative skewness and high kurtosis of financial series and it can have a better fit to the spot prices and the VIX index than a standard Gaussian distribution.

	$HN\text{-}GARCH_R$	$HN\text{-}GARCH_{R,VIX}$		$IG\text{-}GARCH_R$	$IG\text{-}GARCH_{R,VIX}$
λ	0.2832	-0.4999	ν	1585.53	613.59
	(1.226e-04)	(1.82e-03)		(1.03e-21)	(1.13e-19)
ω	5.312e-09	6.6016e-09	ω	0	1e-10
	(7.335e-07)	(1.129e-08)		(7.496e-15)	(3.05e-13)
β	0.7651	0.7731	b	-19.125	-1.78
	(0.0176)	(1.827e-04)		(9.195e-19)	(7.24e-17)
α	3.557e-06	2.534e-06	a	2.146e+07	3.47e+05
	(2.142e-07)	(3.792e-08)		(5e-25)	(1.94e-22)
γ	242.3598	291.5444	c	4.60e-06	4.92e-06
	(2.044e-4)	(9.713e-06)		(3.67e-11)	(8.09e-11)
-	-	-	η	-6.30e-04	-1.63e-03
	-	-		(3.278e-15)	(3.91e-14)
Skew(M_3)	0	0	Skew(M_3)	-0.1449	-0.3746
Kurt(M_4)	3	3	Kurt(M_4)	3.0350	3.2339
h_∞	1.37e-04	2.95e-04	h_∞	1.71e-04	2.94e-04
Pers-P	0.9740	0.9914	Pers-P	0.9823	0.99168
Pers-Q	0.9745	0.9907	Pers-Q	0.9770	0.9917
LL_R	1.186e+04	1.174e+04	LL_R	1.179e+04	1.179e+04
LL_{VIX}	-	1.719e+04	LL_{VIX}	-	1.720e+04
$LL_{R,VIX}$	-	2.893e+04	$LL_{R,VIX}$	-	2.899e+04
AIC	-2.371e+04	-5.785e+04	AIC	-2.378e+04	-5.795e+04
BIC	-2.368e+04	-5.782e+04	BIC	-2.375e+04	-5.792e+04

Table 5.2: Results from return-only-MLE and joint MLE under HN-GARCH and IG-GARCH. Subscript R refers to the results from return-only-MLE and Subscript R, VIX refer to the results from joint estimation. Standard errors are presented in parenthesis below estimated parameter results. Skew and Kurt represent the skewness and kurtosis for the driving noise ,h_∞ denotes the long-term variance,Pers-P and Pers-Q represent the persistence under physical and risk-neutral measure respectively. The likelihood value, Akaike and Bayesian information criteria values are also presented.

5.2 Fair Fee Rate

As shown in section 3.2.2, to calculate a fair fee structure we need to let the expected risk-neutral value of riders fee equal to the price of GMMB. Next, fix one of $\overline{c}$ or m and solve the equation for the other. We use recent data on average expense ratios for mutual funds in the United States to set the investment management fee, c_{inv}. For GMMB's rolling rate π, we usually assume it to be less than risk-free interest rate. Time to maturity is set to be 10 years (2520 days). Besides the parameters for the market models estimated from maximum likelihood estimation, we set other parameters in table 5.3, and c_{inv} and r are stated as continuously-compounded annualized rate. If not specified we will use those parameters.

$Parameters$	F_0	T	c_{inv}	G	r
$Value$	100	10	0.0075	$F_0e^{\pi T}$	0.02

Table 5.3: Variable Annuity Parameters

In following sections, we present the results of fair fee structure under both the HN-GARCH model and the IG-GARCH model, and analyze how market parameters from different estimation methods, return-only-estimation and VIX-estimation influence the fair fee structure. The relationships between $\overline{c}, c_{inv}, \pi$ and m are also investigated.

Figure 5.1 and 5.2 shows the paths of expected present value (EPV) of GMMB and the risk-neutral rider fees as m increases under return-only estimation. Results for joint estimation are given in Figure 5.3 and 5.4. We set $T = 365$ (days) and $\pi = 0$. The EPV of GMMB has a positive relationship with multiplier m, as a higher fee rate will decrease the value of investment fund and thus increase the insurer's liability. We can see both the EPV of GMMB and the risk-neutral rider fees depend on multiplier m, and the riders fee increases faster with m than GMMB. The fair fee multiplier m is the intersection of the two lines.

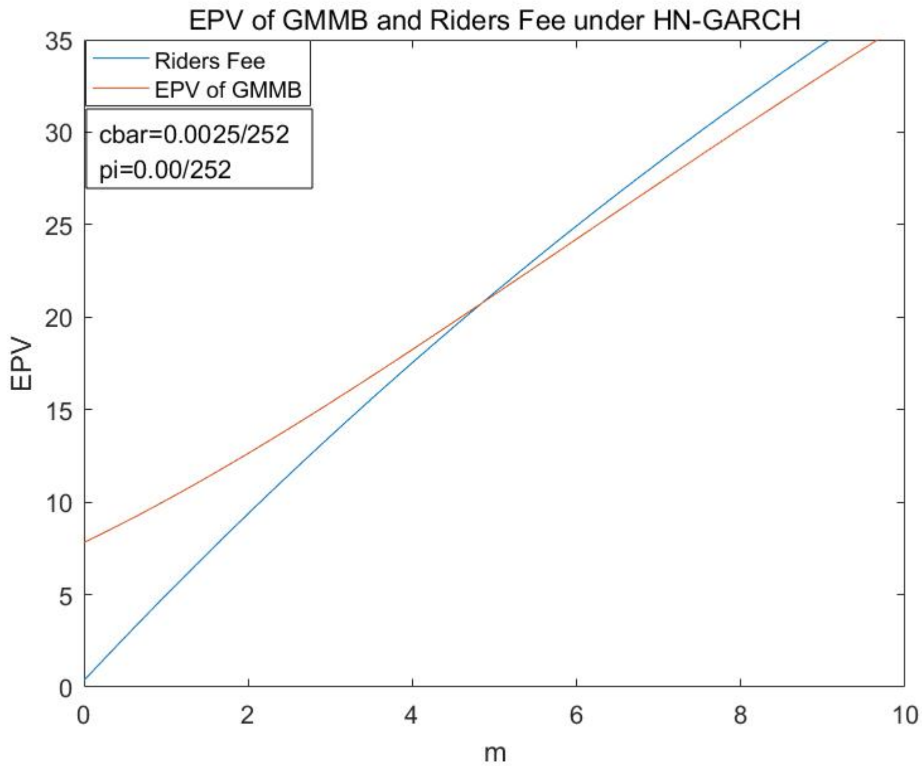

Figure 5.1: HN-GARCH under return-only estimation

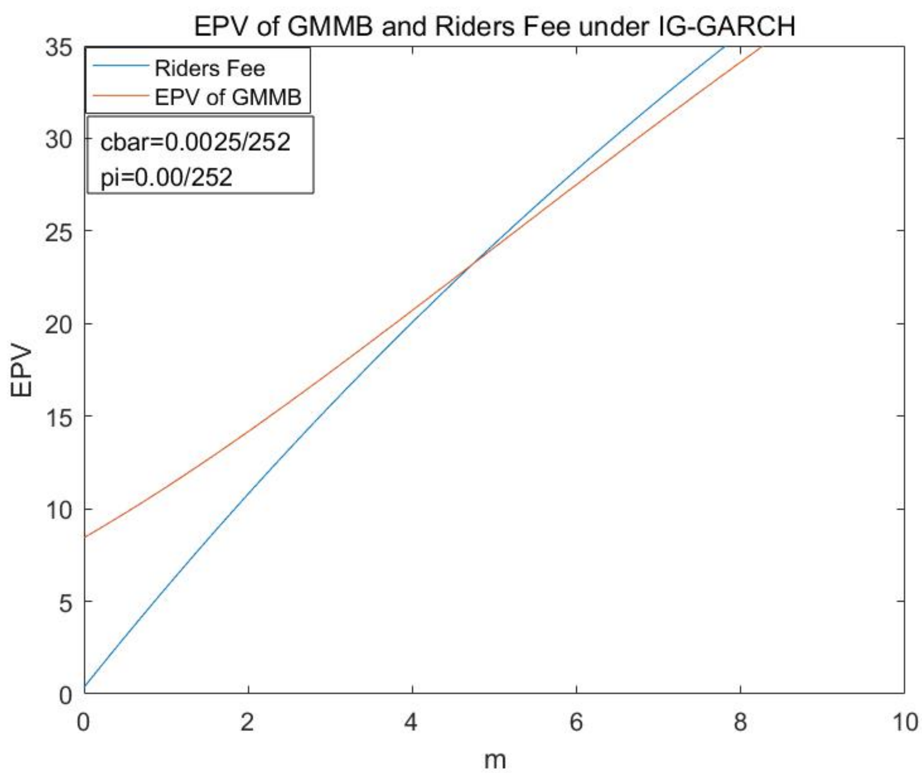

Figure 5.2: IG-GARCH under return-only Estimation

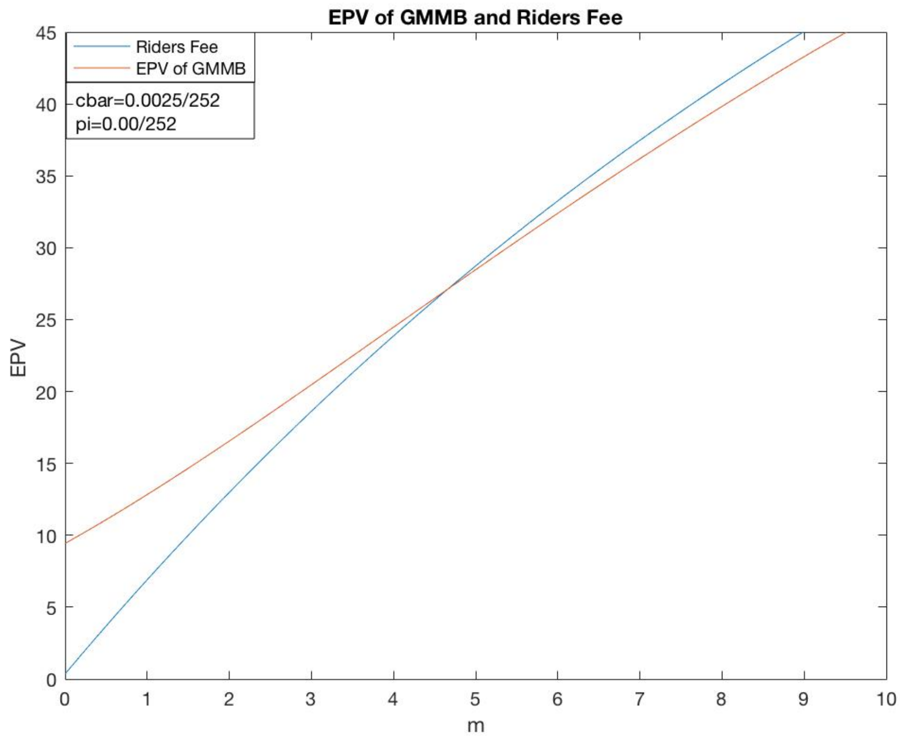

Figure 5.3: HN-GARCH under joint estimation

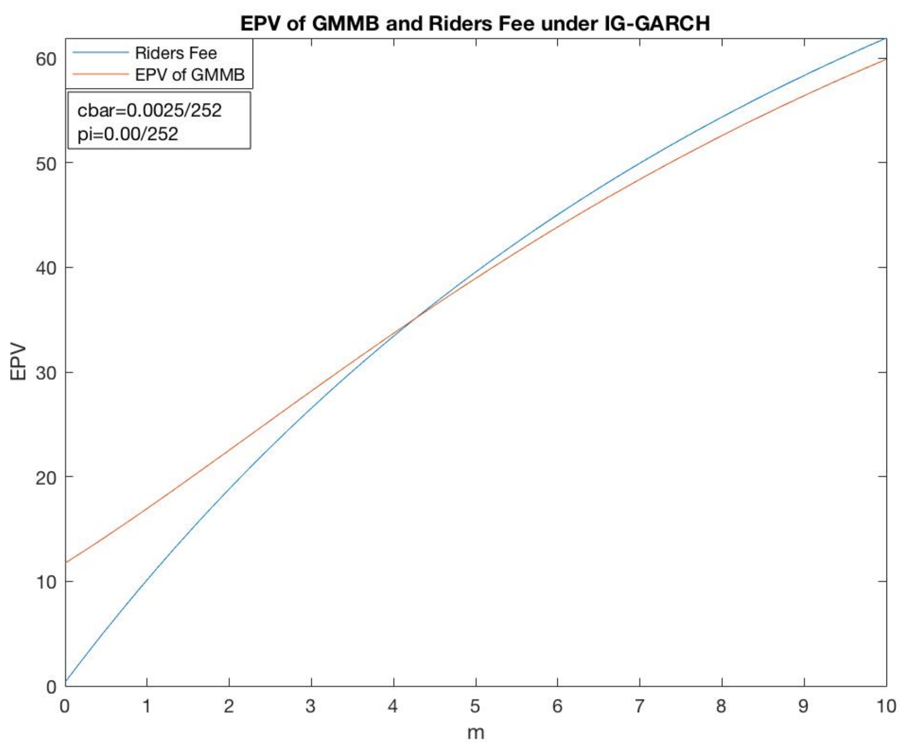

Figure 5.4: IG-GARCH under joint estimation

5.2.1 Fair Multiplier m

Table 5.4 presents the fair multiplier m for different values of c_{inv}, $\overline{c}$ and π under the HN-GARCH model. Regardless of the estimation method, we first analyze how based fee rate $\overline{c}$, investment management fee rate c_{inv} and rolling-up rate π influence m.

The fair m has a positive correlation with the rolling-up rate π. As a higher rolling-up rate will increase the value of guarantee, then a higher fee is needed to compensate for the liability. And clearly the rolling-up fee rate has a significant impact on fair multiplier especially when annualized π increases from 1.00% to 1.50%.

The fair multiplier decreases as based fee rate increases. This is intuitive since the value of guarantee is covered by ride fee which is composed by two parts with one part to be a fixed fee rate $\overline{c}$ and another part to be a flexible fee influenced by the VIX Index. If based fee rate increases the flexible fee does not need to be so high to cover the whole guarantee.

As the investment management fee increases, the fair multiplier m increase. Indeed, a higher c_{inv} will deplete the fund value at a faster speed and in consequence increase the liability of financial guarantee. While c_{inv} is not used to compensate for the insurer's liability, a higher rider fee rate has to be charged to finance the guarantee.

The above result holds under both return-only estimation and VIX-estimation. We observe that the fair multiplier m from VIX-estimation is always less than that under return-only estimation. The difference between fair multipliers under different estimation methods has a significant increase when rolling-up rate increases. While the change of c_{inv} and $\overline{c}$ does not have a obvious impact on these differences.

Fair multiplier m for different values of c_{inv}, $\overline{c}$ and π under the IG-GARCH model is also given in table 5.5. Similar with the HN-GARCH model, m will increase as the rolling-up rate increases, and the increase is obvious when annualized π increases from 1.00% to 1.50%. Also, multiplier m is a decreasing function of base fee rate. The positive relationship between m and c_{inv} still holds under the IG-GARCH model.

	HN GARCH Model			
	$\overline{c} = 0.25\%$	$\overline{c} = 0.45\%$	$\overline{c} = 0.65\%$	$\overline{c} = 2.00\%$
	$c_{inv} = 0.65\%$			
$\pi = 0.00\%$	0.780	0.715	0.650	-0.105
	(0.745)	(0.700)	(0.656)	(0.131)
$\pi = 1.00\%$	1.526	1.460	1.394	0.618
	(1.350)	(1.303)	(1.257)	(0.709)
$\pi = 1.50\%$	2.797	2.730	2.663	1.876
	(2.291)	(2.242)	(2.194)	(1.63)
	$c_{inv} = 0.75\%$			
$\pi = 0.00\%$	0.815	0.751	0.686	-0.071
	(0.772)	(0.727)	(0.683)	(0.157)
$\pi = 1.00\%$	1.625	1.559	1.493	0.716
	(1.421)	(1.374)	(1.328)	(0.779)
$\pi = 1.50\%$	3.197	3.131	3.064	2.287
	(2.527)	(2.479)	(2.431)	(1.871)
	$c_{inv} = 0.85\%$			
$\pi = 0.00\%$	0.852	0.788	0.723	-0.034
	(0.800)	(0.756)	(0.711)	(0.185)
$\pi = 1.00\%$	1.735	1.669	1.603	0.825
	(1.499)	(1.452)	(1.405)	(0.856)
$\pi = 1.50\%$	3.723	3.658	3.593	2.834
	(2.816)	(2.769)	(2.722)	(2.171)

Table 5.4: **Daily fair multiplier** m under **HN-GARCH** from return-only estimation and joint estimation. Results from joint estimation are presented in brackets under those from return-only estimation. $c_{inv}, \overline{c}$ and r are given in annualized term.

	IG GARCH Model			
	$\overline{c} = 0.25\%$	$\overline{c} = 0.45\%$	$\overline{c} = 0.65\%$	$\overline{c} = 2.00\%$
	$c_{inv} = 0.65\%$			
$\pi = 0.00\%$	0.768	0.712	0.657	0.006
	(0.659)	(0.633)	(0.606)	(0.294)
$\pi = 1.00\%$	1.449	1.392	1.334	0.661
	(1.103)	(1.075)	(1.046)	(0.714)
$\pi = 1.50\%$	2.564	2.506	2.447	1.758
	(1.735)	(1.705)	(1.675)	(1.324)
	$c_{inv} = 0.75\%$			
$\pi = 0.00\%$	0.799	0.744	0.688	0.036
	(0.677)	(0.651)	(0.624)	(0.311)
$\pi = 1.00\%$	1.535	1.478	1.420	0.746
	(1.147)	(1.119)	(1.090)	(0.757)
$\pi = 1.50\%$	2.883	2.825	2.767	2.086
	(1.859)	(1.829)	(1.799)	(1.448)
	$c_{inv} = 0.85\%$			
$\pi = 0.00\%$	0.833	0.777	0.721	0.069
	(0.696)	(0.669)	(0.643)	(0.329)
$\pi = 1.00\%$	1.630	1.573	1.515	0.839
	(1.194)	(1.165)	(1.137)	(0.803)
$\pi = 1.50\%$	3.288	3.232	3.175	2.510
	(2.002)	(1.972)	(1.942)	(1.593)

Table 5.5: **Daily fair multiplier** m under **IG-GARCH** from return-only estimation and joint estimation. Results from joint estimation are presented in brackets under those from return-only estimation. $c_{inv}, \overline{c}$ and r are given in annualized term.

We also solved for fair base fee rate $\overline{c}$ for different m and rolling-up rate under the HN-GARCH and the IG-GARCH in table 5.6 and 5.7. $\overline{c}$ has a negative correlation with m since both $\overline{c}$ and m work together to compensate for the liability. And as rolling-up rate increases $\overline{c}$ has to be greater because a higher liability needs to be covered.

	HN GARCH Model			
	$m = 0$	$m = 0.5$	$m = 0.75$	$m = 2$
	$c_{inv} = 0.75\%$			
$\pi = 0.00\%$	2.780%	1.227%	0.452%	-3.419 %
	(3.705%)	(1.465%)	(0.349%)	(-5.195%)
$\pi = 0.05\%$	5.169%	3.654%	2.897%	-0.882 %
	(6.356%)	(4.200%)	(3.126%)	(-2.219%)
$\pi = 1.00\%$	9.962%	8.433%	7.670%	3.869 %
	(10.955%)	(8.812%)	(7.746%)	(2.459%)

Table 5.6: **Annualized fair base fee rate** $\bar{c}$ under **HN-GARCH** from return-only estimation and joint estimation. Results from joint estimation are presented in brackets under those from return-only estimation. c_{inv} and r are given in annualized term.

	IG GARCH Model			
	$m = 0$	$m = 0.5$	$m = 0.75$	$m = 2$
	$c_{inv} = 0.75\%$			
$\pi = 0.00\%$	3.248%	1.446%	0.547%	-4.058%
	(5.490%)	(1.715%)	(-0.151%)	(-9.443%)
$\pi = 0.05\%$	5.928%	4.183%	3.312%	-1.364%
	(8.735%)	(5.151%)	(3.376%)	(-5.682%)
$\pi = 1.00\%$	12.052%	10.221%	9.311%	3.297%
	(14.047%)	(10.514%)	(8.773%)	(-0.690%)

Table 5.7: **Annualized fair base fee rate** $\bar{c}$ under **IG-GARCH** from return-only estimation and joint estimation. Results from joint estimation are presented in brackets under those from return-only estimation. c_{inv} and r are given in annualized term.

5.2.2 Fair Total Fee Rate

To assess the effect of a VIX-linked fee structure on the total fee rate when market is stable or volatile, we set market volatility to be $0.75\overline{h}$, $\overline{h}$, $2\overline{h}$, $3\overline{h}$ and calculate the fair total fee rate. The maturity is set to be 10 years and the rolling-up rate is $\pi = 0$. The total fee rate is given in table 5.8. The total fee rate increases as market volatility increases and in consequence more fees can be used to hedge for the larger higher and reduce insurer's risk. However, when market volatility is high ($h_t = 2\overline{h}$ or $3\overline{h}$), the total fee may become too high and may result in a negative return from the investment fund and can increase surrender risk. To prevent such risk, a potential cap should be given consideration on the fee rate during periods of high volatility.

The total fee rate c_t^{tot} from joint estimation is always greater than that from return-only estimation under both the HN-GARCH model and the IG-GARCH model. Also the IG-GARCH model has a higher c_t^{tot} than the HN-GARCH model regardless of the estimation method, and c_t^{tot} underthe IG-GARCH model is more sensitive to market volatility's increase. We also observe that total fee rate c_t^{tot} under the IG-GARCH model from joint estimation become extremely high when market volatility increase. As a consequence, using the IG-GARCH as market model with parameters estimated from joint estimation provides the most conservative pricing strategy for a fair fee structure.

5.3 Comparison of Fee structures

The goal of a VIX-linked fee structure is to realign insurer's income and liability associated with guarantees. In this section we will present the paths of net liability and fee income under the fixed fee structure and the VIX-linked fee structure. To get a fixed fee structure we just need to set the multiplier to be zero. We simulated the paths of net liability of GMMB and riders fee under the HN-GARCH model with parameters from different MLEs, and the simulation is done under physical measure. The effects of VIX-linked fee under

	HN GARCH Model			*IG GARCH Model*		
	$\overline{c} = 0.25\%$	$\overline{c} = 0.45\%$	$\overline{c} = 0.65\%$	$\overline{c} = 0.25\%$	$\overline{c} = 0.45\%$	$\overline{c} = 0.65\%$
	$c_{inv} = 0.75\%$			$c_{inv} = 0.75\%$		
$h_t = 0.75\overline{h}$	3.276%	3.296%	3.316%	3.683%	3.696%	3.710%
	(4.341%)	(4.347%)	(4.353%)	(6.960%)	(6.925%)	(6.890%)
$h_t = 1.00\overline{h}$	3.818%	3.795%	3.772%	4.369%	4.334%	4.300%
	(5.294%)	(5.244%)	(5.195%)	(8.806%)	(8.697%)	(8.589%)
$h_t = 2.00\overline{h}$	5.985%	5.791%	5.596%	7.111%	6.886%	6.661%
	(9.106%)	(8.835%)	(8.565%)	(16.18%)	(15.79%)	(15.39%)
$h_t = 3.00\overline{h}$	8.152%	7.786%	7.421%	9.854%	9.438%	9.223%
	(12.92%)	(12.43%)	(11.93%)	(23.57%)	(22.87%)	(22.18%)

Table 5.8: **Annualized total fee rate** c_t^{tot} under **HN-GARCH** and **IG-GARCH** from return-only estimation and joint estimation. Results from joint estimation are presented in brackets under those from return-only estimation. $c_{inv}, \overline{c}$ and r are given in annualized term.

different estimation methods will be investigated.

Figure 5.5 and 5.6 present the resulting simulation paths of the net liability and riders fee under fixed-fee structure and VIX-linked fee structure respectively. Simulation results under both models and both estimation methods are presented. Under the fixed-fee structure, the insurer's fee income has a negative correlation with its net liability. As fee income is a fixed percentage of the fund value, it will decreases when net liability increases, and cause a risk management problem for insurance company in consequence. Under the VIX-linked fee structure, the insurer's fee income is positively correlated with the liability, which serves as an evidence for the fact that VIX-linked fee structure can help to re-align the insurer's income and liability.

However, we can see clearly from the graphs that as market become volatile, the fee rates under the VIX-linked fee structure become much higher than the fee rates under fixed-fee structure, and fee rates under the VIX-linked fee structure may drag the investment fund 's ability to make profit. Thus, a potential cap on the rider fees under the VIX-linked fee structure should be considered.

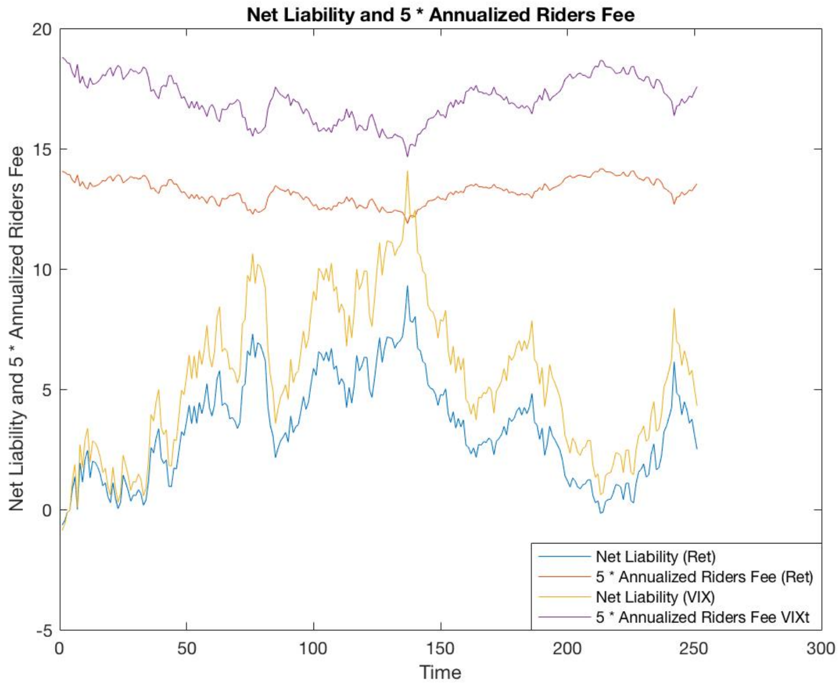

Figure 5.5: Fixed Percentage Fee

Ret-only estimation: $\overline{c} = 0.1282/252,\ m = 0$

Joint estimation: $\overline{c} = 0.1524/252,\ m = 0$

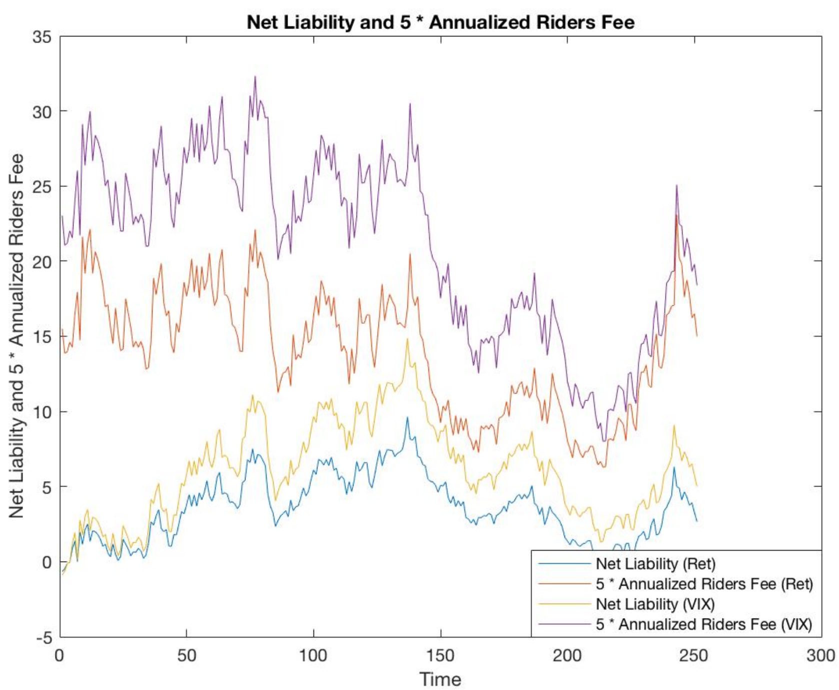

Figure 5.6: VIX-linked Fee

Ret-only estimation:$\overline{c} = 0.0025/252,\ m = 4.8466$

Joint estimation: $\overline{c} = 0.0025/252,\ m = 4.6589$

5.4 Convergence of Fair Fee Rate

In section 4, the discretized fund dynamics under the HN-GARCH model and its diffusion limits have been obtained and analysis of GMMB under those dynamics has been done. By letting the price of GMMB equal to the expected risk-neutral rider fees, we are able to calculate a fair fee structure and also a fair fee rate. Although the convergence result of GMMB's price and the expected risk-neutral rider fees have not been proven theoretically, here we check the convergence results of the fair total fee rate numerically.

Here we test the convergence of total fee rate when the maturity is 0.5, 1, 2 and 3 months with 22 trading in each month, using market parameters estimated from both ret-only estimation and joint estimation. The length of each period is set to be $1/2^n, (n = 1, 2, \dots, 8)$. Here we set the minimum interval length to be 1/256 to reduce the calculation burden as the algorithm for the HN-GARCH model involves recursive relationships on the coefficients, and the minimum length 1/256 seems good enough to show how the convergence goes. The convergence results of total fee rate can be found in table 5.9. All the fair fee rates are calculated under risk-neutral measure.

From the table, we can see that the fair total fee rate converges very quickly to the diffusion limit rates as time interval shrinks under all maturities. However, longer maturity leads to a faster convergence than shorter maturity. With other criteria being the same, total fee rates from joint estimation are greater than those from return-only estimation, and this is consistent with what we find when market model is not discretized, that joint estimation provides a more conservative strategy to find a fair fee structure than return-only estimation

.

Total Fee Rate(%) for GMMB and its Diffusion Limit								
	Maturity in Months							
	Ret-only Estimation				Joint Estimation			
Intraday Periods	0.5	1	2	3	0.5	1	2	3
1	0.7182	0.4820	0.3144	0.2419	1.0081	0.6967	0.4740	0.3727
2	0.7298	0.4867	0.3161	0.2427	1.0244	0.7044	0.4772	0.3745
4	0.7358	0.4891	0.3169	0.2432	1.0329	0.7083	0.4788	0.3754
8	0.7388	0.4920	0.3173	0.2434	1.0372	0.7103	0.4796	0.3758
16	0.7403	0.4908	0.3175	0.2435	1.0394	0.7112	0.4801	0.3760
32	0.7410	0.4911	0.3176	0.2435	1.0405	0.7117	0.4803	0.3762
64	0.7412	0.4913	0.3177	0.2436	1.0410	0.7120	0.4804	0.3762
128	0.7414	0.4913	0.3177	0.2436	1.0413	0.7121	0.4804	0.3762
256	0.7414	0.4913	0.3177	0.2436	1.0415	0.7122	0.4804	0.3762
Diffusion limit rates	0.7415	0.4913	0.3178	0.2436	1.0416	0.7123	0.4805	0.3763

Table 5.9: Convergence of total fee rate in percentage under HN-GARCH .

5.5 Conclusion

In this thesis, we extend the analysis of the VIX-linked fee structure in Cui et al. [2017] and Kouritzin and MacKay [2017], who use continuous-time stochastic volatility model as the market models, to a discrete-time setting. Affine GARCH models are used to derive semi-closed form solutions to GMMB and a fee charging scheme has been constructed to derive a analytic calculation formula of the expected risk-neutral rider fees. Then a fair fee structure can be obtained numerically by letting the price of GMMB equal to the expected risk-neutral rider fees. Numerical examples show that VIX-linked fee structure can help the insurer to re-align its income and liability. However, the fee rate under VIX-linked fee structure seems to be too high when the market is relatively volatile and it will decrease the profit ability of investment fund and a potential cap on the fee rate may be considered in consequence. Next the diffusion limit of investment fund dynamics under the HN-GARCH model has been investigated and we connect the analysis of the VIX-linked fee structure in discrete time with the analysis in continuous time in Cui et al. [2017]. Numerically, as time interval shrinks, the convergence results of fair total fee rate as time interval shrinks is tested. In the future research, more advanced GARCH models can be used as the market model and more complex guarantees such as GMWB can be studied at discrete-time setting. Also a variance dependent kernel can be used to find the risk-neutral measure of the market model and we can see how the total fee rate will be influenced when there is market price of variance risk.

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Appendix A

A.1 Proof of Risk-Neutral Measure under Affine GARCH Models

A.1.1 Proof of Proposition 2.1.1

We use locally risk-neutral valuation relationship (LRNVR) to derive the risk-neutral equity dynamics under the HN-GARCH model.

We assume the underlying asset has the following risk-neutral dynamics under the HN-GARCH model:

$$\log \frac{S_{t+1}}{S_t} = v_{t+1} + \sqrt{h_t} z^*_{t+1}$$

v_t is the conditional mean of the log return and by the assumption of the HN-GARCH model, $z^*(t)$ follows a standard normal distribution. Then we get:

$$\begin{aligned} E^Q[\frac{S_{t+1}}{S_t}|\mathcal{F}_t] &= E^Q(e^{v_{t+1}+\sqrt{h_{t+1}}z^*_{t+1}}|\mathcal{F}_t) \\ &= e^{v_{t+1}+\frac{1}{2}h_{t+1}} \end{aligned}$$

By the martingale constraints of LRNVR, $e^{v_{t+1}+\frac{1}{2}h_{t+1}} = e^r$. Then $v_{t+1} = r - \frac{1}{2}h_{t+1}$. From the physical dynamics under HN-GARCH model we have:

$$log(\frac{S_{t+1}}{S_t}) = r + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1} \tag{A.1}$$

$$= r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} z^*_{t+1} \tag{A.2}$$

where $z^*_{t+1} = z_{t+1} + (\lambda + \frac{1}{2})\sqrt{h_{t+1}}$

Next, due to the invariant conditional variance property of LRNVR we can derive the risk-neutral dynamic of conditional variance:

$$h_{t+1} = \omega + \beta h_t + \alpha(z_t - \gamma\sqrt{h_t})^2$$

$$= \omega + \beta h_t + \alpha(z^*_t - \gamma^*\sqrt{h_t})^2$$

where $\gamma^* = \gamma + \lambda + \frac{1}{2}$

A.1.2 Proof of Proposition 2.1.2

We use $s_t = \log \frac{S_t}{S_{t-1}}$ to denote the log-return of asset price and derive the moment generating function of s_t under physical measure with respect to filtration $\mathcal{F}_{t-1}$:

$$M^P_{s_t|\mathcal{F}_{t-1}}(\mu) = \exp\left[\mu(r + \nu h_{t+1}) + \frac{h_{t+1}}{\eta^2}(1 - \sqrt{1 - 2\mu\eta})\right] \tag{A.3}$$

Conditional Esscher transformed is used in the context to derive the risk-neutral measure of equity dynamics under the IG-GARCH model.

With conditional Esscher transformation, we have following relationship:

$$M^Q_{s_t|\mathcal{F}_{t-1}}(z) = \frac{M^P_{s_t|\mathcal{F}_{t-1}}(z + \theta_t)}{M^P_{s_t|\mathcal{F}_{t-1}}(\theta_t)}$$

Using this relationship with (A.3), we get

$$M^Q_{s_t|\mathcal{F}_{t-1}}(z) = \exp\left[z\left(r + \nu h_{t+1}\right) + \frac{h_{t+1}}{\eta^2}\left(\sqrt{1-2\theta_t\eta} - \sqrt{1-2(z+\theta_t)\eta}\right)\right] \tag{A.4}$$

By martingale constraint $E^Q(e^{s_t}) = M^Q_{s_t|\mathcal{F}_{t-1}}(1) = e^r$, we solve for Esscher parameters θ_t:

$$\theta_t = \theta = \frac{1}{2}\left[\eta^{-1} - \frac{1}{\nu^2\eta^3}(1 + \frac{\nu^2\eta^3}{2})\right] \tag{A.5}$$

Plug in (A.5) back to (A.4), we can get:

$$\begin{aligned} M^Q_{s_t|\mathcal{F}_{t-1}}(z) &= \exp\left[z\left(r + \nu h_{t+1}\right) + \frac{h_{t+1}}{\eta^2}(\frac{1}{\nu\eta} + \frac{\nu\eta^2}{2})\left(1 - \sqrt{1 - 2z\eta/(\frac{1}{\nu\eta} + \frac{\nu\eta^2}{2})^2}\right)\right] \\ &= \exp\left[z(r + \nu^* h^*_{t+1}) + \frac{h^*_{t+1}}{\eta^{*2}}(1 - \sqrt{1 - 2z\eta^*})\right] \end{aligned}$$

where $\nu^* = \nu(\frac{\eta^*}{\eta})^{\frac{-3}{2}}$, $h^*_{t+1} = h^{\frac{3}{2}}_{t+1}$, $\eta^* = \eta/(\frac{1}{\nu\eta} + \frac{\nu\eta^2}{2})^2$.

Then we can get the equity dynamics under risk neutral measure.

$$\log\frac{S_{t+1}}{S_t} = r + \nu^* h^*_{t+1} + \eta^* y^*_{t+1}$$

where y^*_{t+1} follows a inverse normal distribution with degree of freedom h^*_{t+1}/η^{*2}. With the relationship $h^*_{t+1} = h^{\frac{3}{2}}_{t+1}$, we are able to derive the dynamics of conditional volatility and complete the proof.

$$h_{t+1} = \omega^* + bh^*_t + c^* y^*_t + a^* h^{*2}_t/y_t \tag{A.6}$$

where $y^*(t+1) = y(t+1)(\frac{\eta^*}{\eta})^{-1}$,$\omega^* = \omega(\frac{\eta^*}{\eta})^{\frac{3}{2}}$, $c^* = c(\frac{\eta^*}{\eta})^{\frac{5}{2}}$, $a^* = a(\frac{\eta^*}{\eta})^{-\frac{5}{2}}$

A.2 Proof of Moment Generating Function

A.2.1 Proof of Proposition 3.1.3

We guess the moment generating function takes the following form:

$$f_t(\phi) = E_t^Q(F_T^{\phi}) = \exp[\phi \log F_t + A(t) + B(t)h_{t+1}]$$

By iterated expectation, we have

$$\begin{aligned}
E_t^Q(F_T^{\phi}) =& E_t^Q\left[E(F_T^{\phi}|\mathcal{F}_{t+1})\right] \\
=& E_t^Q\left[\exp\left(\phi \log F_{t+1} + A(t+1) + B(t+1)h_{t+2}\right)\right] \\
=& E_t^Q[\exp(\phi(\log F_t + (r - P + Q\overline{h}) - (\frac{1}{2} + Q)h_{t+1} + \sqrt{h_{t+1}}z_{t+1}^*) + A(t+1) \\
& + B(t+1)(\omega + \beta h_{t+1} + \alpha(z_{t+1} - \gamma^*\sqrt{h_{t+1}})^2))] \\
=& \exp\left[\phi\left(\log F_t + (r - P + Q\overline{h}) - (\frac{1}{2} + Q)h_{t+1}\right) + A(t+1) + B(t+1)(\omega + \beta h_{t+1})\right] \\
& \times E_t^Q\left[\exp\left(\phi\sqrt{h_{t+1}}z_{t+1}^* + B(t+1)\alpha(z_{t+1} - \gamma^*\sqrt{h_{t+1}})^2\right)\right]
\end{aligned}$$

Next we deal with the expectation:

$$\begin{aligned}
& E_t^Q\left[\exp\left(\phi\sqrt{h_{t+1}}z_{t+1}^* + B(t+1)\alpha(z_{t+1} - \gamma^*\sqrt{h_{t+1}})^2\right)\right] \\
=& E_t^Q\left[\exp\left(B(t+1)\alpha[z_{t+1}^2 + (\frac{\phi}{B(t+1)\alpha} - 2\gamma^*)\sqrt{h_{t+1}}z_{t+1} + \gamma^{*2}h_{t+1}]\right)\right] \\
=& \exp(B(t+1)\alpha[\gamma^{*2} - \frac{1}{4}(\frac{\phi}{B(t+1)\alpha} - 2\gamma^*)^2]h_{t+1}) \\
& \times E_t^Q\left[\exp\left(B(t+1)\alpha(z_{t+1} + \frac{1}{2}(\frac{\phi}{B(t+1)\alpha} - 2\gamma^*)\sqrt{h_{t+1}})^2\right)\right]
\end{aligned}$$

By using the result of a standard normal distribution can be used to deal the above expectation:

$$E\left[\exp\left(a(z+b)^2\right)\right] = \exp\left(-\frac{1}{2}\log(1-2a) + \frac{ab^2}{1-2a}\right)$$

After some simplification we are able to get the moment generating function in the following result:

$$
\begin{aligned}
f_t(\phi) =& \exp[\phi \log F_t + \left(A(t+1) + \phi(r - P + Q\overline{h}) + B(t+1)\omega - \frac{1}{2}\log(1 - 2B(t+1)\alpha) \right) \\
&+ \left(\phi(\gamma - \frac{1}{2} - Q) + \beta B(t+1) - \frac{1}{2}\gamma^2 + \frac{(\phi-\gamma)^2}{2(1-2B(t+1)\alpha)} \right) h_{t+1}]
\end{aligned}
$$

Then we have the recursive relationships for coefficients:

$$
A(t) = A(t+1) + \phi(r - P + Q\overline{h}) + B(t+1)\omega - \frac{1}{2}\log(1 - 2B(t+1)\alpha) \tag{A.7}
$$

$$
B(t) = \phi(\gamma - \frac{1}{2} - Q) + \beta B(t+1) - \frac{1}{2}\gamma^2 + \frac{(\phi-\gamma)^2}{2(1-2B(t+1)\alpha)} \tag{A.8}
$$

A.2.2 Proof of Proposition 3.1.4

Similarly with the proof of A.2.1, we guess the moment generating function takes the following form:

$$
f_t(\phi) = E_t^Q(F_T^\phi) = \exp[\phi \log F_t + A^*(t) + B^*(t)h_{t+1}^*]
$$

Using iterated expectation:

$$
\begin{aligned}
E_t^Q(F_T^\phi) =& E_t^Q\left[E(F_T^\phi | \mathcal{F}_{t+1}) \right] \\
=& E_t^Q\left[\exp\left(\phi \log F_{t+1} + A^*(t+1) + B^*(t+1)h_{t+2}^* \right) \right] \\
=& E_t^Q[\exp(\phi \left(\log F_t + (r - P^* + Q^*\overline{h^*}) + (\nu^* - Q^*)h_{t+1}^* + \eta^* y_{t+1}^* \right) + A^*(t+1) \\
&+ B^*(t+1)\left(\omega^* + bh_{t+1}^* + c^* y_{t+1}^* + a^* h_{t+1}^{*2}/y_{t+1} \right))] \\
=& \exp\left(\phi \left(\log F_t + (r - P^* + Q^*\overline{h^*}) + (\nu^* - Q^*)h_{t+1}^* \right) + A^*(t+1) + B^*(t+1)(\omega^* + bh_{t+1}^*) \right) \\
&\times E_t^Q[\exp((\phi\eta^* + B^*(t+1)c^*)\, y_{t+1}^* + B^*(t+1)a^* h_{t+1}^{*2}/y_{t+1})]
\end{aligned}
$$

Next, we can use the general moment generating function of inverse Gaussian random variable y to deal with the above expectation:

$$E\left[\exp(\phi y + \frac{\theta}{y})\right] = \frac{\delta}{\sqrt{\delta^2 - 2\theta}} \exp(\delta - \sqrt{(\delta^2 - 2\theta)(1 - 2\phi)})$$

Solve the expectation we get:

$$\begin{aligned} f_t(\phi) = &\exp[\phi \log F_t + \left(A^*(t+1) + \phi(r - P^* + Q^*\overline{h^*}) + B^*(t+1)\omega^* - \frac{1}{2}log(1 - 2a(\eta^*)^4 B^*(t+1))\right) \\ &+ B^*(t)(\phi(\nu^* - Q^*) + bB^*(t+1) + (\eta^*)^{-2} - (\eta^*)^{-2} \\ &\times \sqrt{(1 - 2a^*(\eta^*)^4 B^*(t+1))(1 - 2c^* B^*(t+1) - 2\eta^*\phi)})] \end{aligned}$$

Then,

$$A^*(t) = A^*(t+1) + \phi(r - P^* + Q^*\overline{h^*}) + B^*(t+1)\omega^* - \frac{1}{2}log(1 - 2a(\eta^*)^4 B^*(t+1)) \quad \text{(A.9)}$$

$$B^*(t) = \phi(\nu^* - Q^*) + bB^*(t+1) + (\eta^*)^{-2} - (\eta^*)^{-2}\sqrt{(1 - 2a^*(\eta^*)^4 B^*(t+1))(1 - 2c^* B^*(t+1) - 2\eta^*\phi)} \quad \text{(A.10)}$$

A.3 Proof of Proposition 3.2.2

First, we prove the following equality:

$$\sum_{i=t}^{T-1} e^{-r(i-t+1)} E_t[F_i \frac{S_{i+1}}{S_i} - F_{i+1}] = F_t - E_t[e^{-r(T-t)} F_T] \quad \text{(A.11)}$$

Prove from left handside to the right:

$$\sum_{i=t}^{T-1} e^{-r(i-t+1)} E_t[F_i \frac{S_{i+1}}{S_i} - F_{i+1}]$$
$$= E_t[\sum_{i=t}^{T-1} e^{-r(i-t+1)} * (F_i \frac{S_{i+1}}{S_i} - F_{i+1})]$$
$$= E_t[e^{-r}(F_t \frac{S_{t+1}}{S_t} - F_{t+1}) + e^{-2r}(F_{t+1} \frac{S_{t+2}}{S_{t+1}} - F_{t+2}) + \cdots + e^{-r(T-t)}(F_{T-1} \frac{S_T}{S_{T-1}} - F_T)]$$
$$= E_t[e^{-r} F_t \frac{S_{t+1}}{S_t} - e^{-r} F_{t+1} + \cdots + e^{-r(T-t)} F_{T-1} \frac{S_T}{S_{T-1}} - e^{-r(T-t)} F_T]$$
$$= F_t - E_t[e^{-r(T-t)} F_T] + E_t[\sum_{i=t}^{T-2} F_{i+1} e^{-r(i-t+1)} (\frac{S_{i+2}}{S_{i+1}} e^{-r} - 1)]$$
$$= F_t - E_t[e^{-r(T-t)} F_T]$$

The second term of (A.19) is zero which can be proven by using iterative expectation. Then we have A(15). By using this relationship, we finish the rest of the proof:

$$\Lambda(t) = \sum_{i=t}^{T-1} e^{-r(i-t+1)} E_t[F_i \frac{S_{i+1}}{S_i} * (1 - c^{inv*}) * c^*_{i+1}]$$
$$= \sum_{i=t}^{T-1} e^{-r(i-t+1)} E_t[F_i \frac{S_{i+1}}{S_i} - F_{i+1} - F_i \frac{S_{i+1}}{S_i} * c^{inv*}]$$
$$= \sum_{i=t}^{T-1} e^{-r(i-t+1)} E_t[F_i \frac{S_{i+1}}{S_i} - F_{i+1}] - \sum_{i=t}^{T-1} e^{-r(i-t+1)} E_t[F_i \frac{S_{i+1}}{S_i} * c^{inv*}]$$
$$= F_t - E_t[e^{-r(T-t)} F_T] - \sum_{i=t}^{T-1} c^{inv*} * e^{-r(i-t)} E_t[F_i \frac{S_{i+1}}{S_i}]$$
$$= F_t - e^{-r(T-t)} E_t[F_T] - c^{inv*} * \sum_{i=t}^{T-1} e^{-r(i-t)} E_t[F_i]$$

A.4 Proof of Proposition 4.1.1

With iterative expectation we can easily prove that,

$$E_t^Q[h_{t+k\Delta}] = \tilde{\omega}(\Delta) + \tilde{\beta}(\Delta)E_t^Q[h_{t+(k-1)\Delta}]$$

Then we have

$$\begin{aligned} E_t^Q[h_{t+k\Delta}] &= (1 + \tilde{\beta}(\Delta)^2 + \cdots + \tilde{\beta}(\Delta)^{(k-2))\Delta}\tilde{\omega}(\Delta) + \tilde{\beta}(\Delta)^{(k-1)\Delta}h_{t+\Delta} \\ &= \frac{1 - \tilde{\beta}(\Delta)^{(k-1)\Delta}}{1 - \tilde{\beta}(\Delta)}\tilde{\omega} + \tilde{\beta}(\Delta)^{(k-1)\Delta}h_{t+\Delta} \\ &= (1 - \tilde{\beta}(\Delta)^{(k-1)\Delta})\overline{h} + \tilde{\beta}(\Delta)^{(k-1)\Delta}h_{t+\Delta} \end{aligned}$$

Then, by the definition of $V_{k\Delta}(n;\Delta)$

$$\begin{aligned} V_{k\Delta}(n;\Delta) &= \frac{\Delta}{n}\sum_{k=1}^{n/\Delta} E_t^Q(h_{t+k\Delta}) \\ &= \frac{\Delta}{n}[n/\Delta - (\tilde{\beta}(\Delta)^{(0)\Delta} + \tilde{\beta}(\Delta)^{(1)\Delta} + \cdots + \tilde{\beta}(\Delta)^{(n/\Delta-1)\Delta})\overline{h}(\Delta) + (\tilde{\beta}(\Delta)^{(0)\Delta} + \tilde{\beta}(\Delta)^{(1)\Delta} \\ &\quad + \cdots + \tilde{\beta}(\Delta)^{(n/\Delta-1)\Delta})h_{(k+1)\Delta}], \\ &= (1 - \Gamma(n;\Delta))\overline{h}(\Delta) + \Gamma(n;\Delta)h_{t+\Delta}. \end{aligned}$$

where,

$$\begin{aligned} &\overline{h}(\Delta) = \frac{\omega(\Delta) + \alpha(\Delta)}{1 - \beta(\Delta) - \alpha(\Delta)\gamma^2(\Delta)},\ \Gamma(n;\Delta) = \frac{\Delta(1 - \tilde{\beta}(\Delta)^{\frac{n}{\Delta}})}{n(1 - \tilde{\beta}(\Delta))}, \\ &\tilde{\omega}(\Delta) = \omega(\Delta) + \alpha(\Delta), \quad \tilde{\beta}(\Delta) = \beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta). \end{aligned}$$

A.5 Proof of Proposition 4.2.3

To derive the diffusion limits of fund dynamics under physical measure, we first define the following derivatives:

$$A_{ij} = \frac{\partial A(\phi, \psi; (k-1)\Delta, k\Delta)}{\partial \phi^i \psi^j}|_{\phi=0,\psi=0}, \quad B_{ij} = \frac{\partial B(\phi, \psi; (k-1)\Delta, k\Delta)}{\partial \phi^i \psi^j}|_{\phi=0,\psi=0}$$

Where $A(\cdot)$ and $B(\cdot)$ are the coefficients in the bivariate conditional cumulant generating function. After some calculation we can get the following results. By the famous property of cumulant generating function, we can use these results to evaluate the limiting moments of $F_{k\Delta}$ and $h_{(k+1)\Delta}$.

$$\begin{aligned}
&A_{10} = r\Delta; \qquad A_{01} = \omega(\Delta) + \alpha(\Delta); \\
&A_{20} = 0; \qquad A_{02} = 2\alpha^2(\Delta); \qquad A_{11} = 0; \\
&B_{10} = (\lambda)\Delta; \qquad B_{01} = \alpha(\Delta)\gamma^2(\Delta) + \beta(\Delta); \\
&B_{20} = \Delta; \qquad B_{02} = 4\alpha^2(\Delta)\gamma^2(\Delta); \qquad B_{11} = -2\alpha(\Delta)\gamma(\Delta)\sqrt{\Delta}.
\end{aligned}$$

Before deriving the diffusion limits, we first impose following assumptions on the parameters:

$$\begin{aligned}
&\omega(\Delta) = \omega\Delta, \ \alpha(\Delta) = \alpha\Delta \\
&\gamma(\Delta) = \frac{\gamma}{\sqrt{\Delta}}, \ \beta(\Delta) = 1 - \kappa\Delta - \alpha(\Delta)\gamma^2(\Delta)
\end{aligned}$$

where

$$\kappa = 1 - \beta - \alpha\gamma^2$$

Next, apply the weak convergence theorem and start with the conditional mean:

$$\begin{aligned}
\lim_{\Delta\to 0}\frac{1}{\Delta}E[\Delta F_{k\Delta}|\mathcal{F}^h_{(k-1)\Delta}] &= \lim_{\Delta\to 0}\frac{1}{\Delta}(A_{10}+B_{10}h_{k\Delta}) \\
&= \lim_{\Delta\to 0}\frac{1}{\Delta}(r\Delta+\lambda\Delta h_{k\Delta}) \\
&= r+\lambda h_{k\Delta} \\
\lim_{\Delta\to 0}\frac{1}{\Delta}E[\Delta h_{(k+1)\Delta}|\mathcal{F}^h_{(k-1)\Delta}] &= \lim_{\Delta\to 0}\frac{1}{\Delta}(A_{01}+(B_{01}-1)h_{k\Delta}) \\
&= \lim_{\Delta\to 0}\frac{1}{\Delta}(\omega(\Delta)+\alpha(\Delta)+(\alpha(\Delta)\gamma^2(\Delta)+\beta(\Delta)-1)h_{k\Delta}) \\
&= \omega+\alpha-\kappa h_t \\
&= \kappa(\theta-h_t)
\end{aligned}$$

where $\theta = (\omega+\alpha)/\kappa$

Now we can look at second order moments.

$$\begin{aligned}
\lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{Var}[\Delta F_{k\Delta}|\mathcal{F}^h_{(k-1)\Delta}] &= \lim_{\Delta\to 0}\frac{1}{\Delta}(A_{20}+B_{20}h_{k\Delta}) \\
&= \lim_{\Delta\to 0}\frac{1}{\Delta}(0+\Delta h_{k\Delta}) \\
&= \lim_{\Delta\to 0}(h_{k\Delta}) \\
&= h_t \\
\lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{Var}[\Delta h_{(k+1)\Delta}|\mathcal{F}^h_{(k-1)\Delta}] &= \lim_{\Delta\to 0}\frac{1}{\Delta}(A_{02}+B_{02}h_{k\Delta}) \\
&= \lim_{\Delta\to 0}\frac{1}{\Delta}(2\alpha^2(\Delta)+4\alpha^2(\Delta)\gamma^2(\Delta)h_{k\Delta}) \\
&= \lim_{\Delta\to 0}\frac{1}{\Delta}(2\alpha^2\Delta^2+4\alpha^2\gamma^2\Delta h_{k\Delta}) \\
&= \sigma^2 h_t
\end{aligned}$$

$$
\begin{aligned}
\lim_{\Delta \to 0} \frac{1}{\Delta} \text{Cov}[F_{k\Delta}, \Delta h_{(k+1)\Delta} | \mathcal{F}^h_{(k-1)\Delta}] &= \lim_{\Delta \to 0} \frac{1}{\Delta} (A_{11} + B_{11} h_{k\Delta}) \\
&= \lim_{\Delta \to 0} \frac{1}{\Delta} (-2\alpha(\Delta)\gamma(\Delta)\sqrt{\Delta} h_{k\Delta}) \\
&= -\sigma h_t
\end{aligned}
$$

where $\sigma = 2\alpha\gamma$

So under physical measure, the second moment matrix is:

$$
\Sigma\Sigma^{\mathrm{T}} = \begin{bmatrix} h_t & -\sigma h_t \\ -\sigma h_t & \sigma^2 h_t \end{bmatrix}
$$

By Cholesky decomposition, we get the diffusion coefficient :

$$
\Sigma = \begin{bmatrix} \sqrt{h_t} & 0 \\ -\sigma \sqrt{h_t} & 0 \end{bmatrix}
$$