UNIVERSITY OF CALGARY

Effect of Defects, Inclusions and Inhomogeneities in Elastic Solids

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN MECHANICAL AND MANUFACTURING ENGINEERING

CALGARY, ALBERTA

DECEMBER, 2019

Abstract

This thesis focusses on the theory of materials with defects introduced by John D. Eshelby in the 50s and the 60s, which today we call *Configurational Mechanics* or, in his honour, *Eshelbian Mechanics*. The thesis consists of four interconnected parts. The first part is dedicated to the relation between two of Eshelby's developments: the energy momentum tensor (or *Eshelby stress tensor*), describing the net force on a defect, and the *Eshelby fourth-order tensor*, which relates the strain in an inclusion in an otherwise homogeneous and isotropic matrix to the virtual strain (transformation strain) defining the geometrical misfit between inclusion and matrix, within the theory of small deformations. The second part of the research was prompted by the fact that, although the relation between Eshelby's inclusion problem (Eshelby, 1951, 1975) and Noether's theorem has been mentioned in literature, no explicit relation has ever been given, to the best of our knowledge. In a framework based on modern differential geometry, it is shown that the application of Noether's theorem allows for straightforwardly obtaining the classical results by (Eshelby, 1951, 1975). The third part of the thesis aims at investigating the work of Eshelby (1951, 1975) on configurational forces and of Noll (1967) on material uniformity within a general framework including thermo-elasticity, volumetric growth inertial effects, in which the divergence of the Eshelby stress is called the *Eshelby force*. A differential identity is obtained for the modified Eshelby stress, which includes, as a particular case, the identity found by Epstein and Maugin (1990). Moreover, a differential identity is obtained for what is called the *modified Eshelby power*, representing the time counterpart of the Eshelby force. Then, a relation between the modified Eshelby force and the modified Eshelby power is derived in the dynamical case. Finally, based on the results obtained in the previous parts of the research, a large-deformation counterpart is proposed of the imagined procedure that Eshelby (1957) used to investigate the theory of inclusions in the case of infinitesimal deformations. A mixed multiplicative decomposition of the deformation gradient, in terms of the Bilby-Kröner-Lee and the Noll-Epstein-Maugin decompositions allows for obtaining the large-deformation fourth-order Eshelby tensor, a novel result.

Acknowledgements

First of all, I would like to take this opportunity to thank my great supervisor, Dr. Salvatore Federico, for his comprehensive and unimaginable assistance and support. Salvatore was always accessible either by email or by meeting even on weekends or holidays and offered all kinds of support possible. Moreover, I would like to express my gratitude to Dr. Marcelo Epstein for his endless help and guidance whenever I needed any my research and to Dr. Alfio Grillo (Politecnico di Torino, Italy) for his great contribution to the work on the theory of Eshelby's inclusion in the light of Noether's theorem. I also acknowledge the Libyan Ministry of Higher Education and Omar Al-Mukhtar University for providing financial support during my academic journey.

I would like to extend my sincere thanks and appreciation to Dr. Leping Li for being part of my supervisor committee, to Dr. Richard Wan for serving in my candidacy and thesis defence examination committee and to Dr. Roberto Martinuzzi for serving in my candidacy committee. I am also very grateful to Dr. Chong-Qing Ru (University of Alberta) for kindly serving as the external examiner for my thesis defence.

Furthermore, I am very thankful to all members of my group, especially Amir Hamedzadeh and Kotaybah Hashlamoun, to Salvatore Di Stefano from Dr. Grillo's group, and to Saleh Bawazeer from Dr. Abdulmajeed Mohamad's group, for their help and valuable comments in the past years.

Last, but not least, I must express my deep gratitude to my wife Bodor, for her continued support and precious love while always standing by my side during my difficult times during my study and to my children, Fatihia, Fawzi, Mohamed, Ali and Alyaman, who were my main motivation. I feel exceptionally indebted to my parents and mother-in-law for their spiritual support: without their encouragement throughout my entire life, I would have never succeeded. I would like to dedicate this work to the spirit of my father-in-law Fawzi Bohadula (1957-2019) who so largely contributed to forming my thoughts from the humanistic point of view.

Contents

Chapter 1

Introduction

The masterly work by Eshelby (e.g., 1951, 1957, 1961, 1975) in the mechanics of materials with defects has turned out to be of fundamental importance since at least the 1960s. In his classical paper, Eshelby (1951) applied the methods of Classical Field Theory (e.g., Landau and Lifshitz, 1939) to Elasticity and introduced an energy-momentum stress tensor that was completely *material*, as opposed to the Cauchy stress, which is completely spatial, and to the first Piola-Kirchhoff stress, which has a spatial "leg" and a material "leg". Eshelby's energy-momentum stress tensor has been subsequently extensively studied in the context of a completely material mechanics, which is now called *Configurational Mechanics* or *Eshelbian Mechanics*, and has been called *Eshelby stress* by Maugin and Trimarco (1992). In our notation, the Eshelby stress reads

$$\boldsymbol{\mathfrak{E}} = W\,\boldsymbol{I}^T - \boldsymbol{F}^T\boldsymbol{P}, \qquad \mathfrak{E}_A{}^B = W\,\delta_A{}^B - F^a{}_A\,P_a{}^B, \tag{1.1}$$

where W is the elastic energy density per unit reference volume, $\boldsymbol{I}$ is the (material) identity tensor, $\boldsymbol{F}$ is the deformation gradient and $\boldsymbol{P}$ is the first Piola-Kirchhoff stress.

Eshelby (e.g., 1951, 1975) showed that the divergence of the energy-momentum stress tensor represents a material force density, which is directly related to the presence of inhomogeneities in the body. Indeed, the integral of the divergence of the energy-momentum tensor over a region containing a defect is the *net material force* acting on the defect. This force is called a *configurational* *force* and can be thought of as the "price to be paid" to create a change in a certain configuration, e.g., by "inserting" a defect in an otherwise homogeneous material, or to create a different configuration in which the defect has been displaced with respect to an existing configuration. In this sense, the Eshelby stress is seen as the object that captures inhomogeneities and singularities (e.g., Gurtin, 1995; Epstein and Maugin, 2000; Epstein and Elżanowski, 2007; Verron et al., 2009; Maugin, 2011). The close relationship between the configurational force and Eshelby stress has been employed and differently interpreted by several authors.

Gurtin (1995) followed the work by Eshelby (1951) on lattice defects, which identifies configurational forces as those forces able to explain the internal structure of a material, and mentioned that Eshelby's ideas were hinted at by Gibbs (1878) in his work on the equilibrium of heterogeneous substances. Furthermore, Gurtin (1995) clarified the difference between standard and configurational forces: the former describe the response of the body to deformation of the material points, and the latter the response to changes in the internal structure. In the picture provided by Gurtin (1995), whereas standard forces are compatible with the balance laws of linear and angular momentum, the configurational forces obey to an independent balance law. Moreover, Podio-Guidugli (2002) remarked how standard forces are not adequate to describe the body's response to the evolution of the material structure, and illustrated how the configurational force has to be taken into consideration in order to get a full picture of the body's response.

A point of view that is some respects is very different has been taken by Maugin, in a series of works starting from that by Maugin and Trimarco (1992) (see, e.g., the book by Maugin, 2011, for a comprehensive list of the works by Maugin and collaborators in this area). Maugin and Trimarco (1992) showed how, for the elastic case, the configurational force and Eshelby stress can be retrieved by a pull-back of the (covector) equation of the balance of linear momentum. This point in particular has been long debated (Maugin, 2006), as it is in contrast with the view proposed by Gurtin (1995), according to which of configurational forces are totally independent from the balance of standard forces.

Among the other streams of research starting from Eshelby's pioneering works, the second one on which we will concentrate is that of the problem of the inclusion in a homogeneous matrix. Eshelby (1957) studied the elastic field (intended as the collection of the displacement, strain and stress) on an ellipsoidal inclusion in an isotropic elastic matrix. Three cases can be devised (e.g., Balluffi, 2012): *i*) inclusion with geometrical misfit and same material properties of the matrix (or "homogeneous inclusion", as it is often called in the literature), *ii*) inclusion with no geometrical misfit but different elastic properties than the matrix (or "inhomogeneous inclusion"), and *iii*) inclusion with both geometrical misfit and different elastic properties than the matrix (or "general inclusion").

Briefly, Eshelby imagines to have an ellipsoidal cavity in the body. The shape of this cavity is mapped into a new shape by means of a *transformation strain*: this determines the geometrical misfit (or the geometrical misfit equivalent to a misfit in material properties) of the inclusion. Then the new shape is assigned material properties (either the same of the matrix or different ones), and subjected to surface tractions, so that it is deformed until it exactly matches the original shape of the cavity in the body. Once the inclusion is put back and welded into the cavity, the tractions are released and the inclusion and the matrix are allowed to relax elastically, thereby attaining a *cancelling strain* (which is discontinuous at the inclusion-matrix interface). For an ellipsoidal inclusion, this procedure can be studied in terms of a fourth-order tensor $\mathbb{S}$, now commonly called *Eshelby tensor* (e.g. Weng, 1990), which maps the transformation strain $\boldsymbol{\epsilon}^*$ into the cancelling strain $\boldsymbol{\epsilon}^C$:

$$\boldsymbol{\epsilon}^C = \mathbb{S} : \boldsymbol{\epsilon}^*, \qquad \epsilon^C_{ij} = \mathrm{S}_{ijkl}\, \epsilon^*_{kl}. \tag{1.2}$$

We emphasise here that Eshelby's fourth-order tensor $\mathbb{S}$ can be readily defined for an *isotropic* matrix. Since the ellipsoidal inclusion can describe several cases, ranging from cracks (extremely oblate ellipsoid) to fibres (extremely prolate ellipsoid), this approach has been used widely in the literature on linear composite material with inclusions (e.g., Walpole, 1966a,b, 1969; Tandon and Weng, 1984; Weng, 1990; Federico et al., 2004), in order to find the overall properties starting from the properties of the individual constituents.

This thesis was originally motivated by:

1. The lack of an explicit relation between Eshelby stress and Eshelby fourth-order tensor in the case of small deformations, even in Eshelby's original works (e.g., Eshelby, 1951, 1957);

2. The lack of a large-deformation counterpart of Eshelby's treatment of the ellipsoidal inclusion in terms of the fourth-order tensor; the only attempt we are aware of was that by Nemat-Nasser (1999), but we believe that this work may be affected by some technical issues.

Thus, the intention was to work on a *discrete inclusion*, which would eventually be set to be ellipsoidal.

The first goal was in fact achieved in the paper.

> Alhasadi, M.F., Federico, S., 2017. Relation Between Eshelby's Tensor and Stress within an Ellipsoidal Inclusion, *Acta Mechanica*, 228, 1045-1069

in which we found the relation between the two Eshelby tensors in the *interior* of an ellipsoidal inclusion, in all three cases of "homogeneous", "inhomogeneous" and "general" inclusion.

However, the path to the second goal, i.e., extending the theory of the ellipsoidal inclusion to large deformations, was harder that initially expected. In order to overcome the various difficulties that we encountered, we first reinterpreted Eshelby's variational treatment of the defect problem (Eshelby, 1975) in the light of Noether's theorem. This resulted in the paper

> Federico, S., Alhasadi, M.F., Grillo, A., in press. Eshelby's Inclusion Theory in the Light of Noether's Theorem, *Mathematics and Mechanics of Complex Systems*, manuscript 190206-Federico, submitted 2019-02-06, accepted with revisions 2019-04-12, resubmitted 2019-06-15, accepted in final form 2019-08-18

in which we showed how a straightforward application of Noether's theorem yields all of Eshelby's results on the configurational force, the inhomogeneity of the elastic energy and the imagined procedure of “repositioning” of the defect. Then, it was necessary to enrich our treatment by incorporating the theory of *continuously distributed defects*, stemming from the work by Noll (1967), whose relation with the original work by Eshelby (1951) was shown by Epstein and Maugin (1990). Noll’s theory expresses the inhomogeneity of a material body as a “remapping” of the neighbourhoods of each point, which creates residual stresses. This is in fact equivalent to Eshelby’s material force, which arises whenever the material is inhomogeneous. We extended the work by Epstein and Maugin (1990) to the dynamical thermomechanical case in the paper

> Alhasadi, M.F., Epstein, M., Federico, S., 2019. Eshelby Force and Power for Uniform Bodies, *Acta Mechanica*, 230, 1663-1684

With the tools employed in these two works (i.e., a precise explanation of Eshelby’s thought experiment and the correct multiplicative decomposition that describes it, in Federico et al., 2019; Alhasadi et al., 2019, respectively), it was then possible to return to the case of the discrete inclusion and to successfully extend Eshelby’s work on the ellipsoidal inclusion (Eshelby, 1957) to the case of large deformations. This resulted in a manuscript that has recently been submitted:

> Alhasadi, M.F., Federico, S., Eshelby’s Inclusion Problem in Large Deformations, *Proceedings of the Royal Society Series A*, submitted 2019-11-03

The four papers on which this thesis is based reported in Chapters 2-5:

- Chapter 2, based on Alhasadi and Federico (2017);
- Chapter 3, based on based on Federico et al. (2019);
- Chapter 4, based on Alhasadi et al. (2019);
- Chapter 5, based on Alhasadi and Federico (2019).

In Chapter 6, we summarise each of the four main chapters of this thesis and discuss the possible future work.

Chapter 2

Relation Between Eshelby's Tensor and Stress within an Ellipsoidal Inclusion

This chapter is based on Alhasadi and Federico (2017)

The work by Eshelby (1951, 1957, 1961, 1975) in the mechanics of materials with defects has turned out to be of fundamental importance since at least the 70s. We are particularly interested in two developments of Eshelby's work: the "material" mechanics that arises from the study of what Eshelby called the energy-momentum tensor, and the problem of the ellipsoidal inclusion.

Eshelby (1951) studied the theory of the *configurational* force on a singularity and showed that this force can be expressed in terms of the surface integral of the tractions caused by the *energy-momentum tensor* as shown in Eq. (2.21). This configurational force is the "driving force" for the repositioning of a defect or, in other words, the "price to pay" to reposition a defect within an otherwise homogeneous matrix. Eshelby's energy-momentum tensor has been subsequently called *Eshelby stress* by Maugin and Trimarco (1992), and this is the terminology we use here. The Eshelby stress (or the closely related Mandel stress) is seen as the object that captures inhomogeneities and singularities (e.g., (Gurtin, 1995; Epstein and Maugin, 2000; Epstein and Elżanowski, 2007; Verron et al., 2009; Weng and Wong, 2009; Maugin, 2011)), or the "driving force" of phenomena of material evolution such as plasticity and growth-remodelling (e.g., (Maugin and Epstein, 1998; Epstein and

Maugin, 2000; Cermelli et al., 2001; Epstein, 2002; Imatani and Maugin, 2002; Grillo et al., 2003, 2005; Epstein, 2009, 2015; Grillo et al., 2016, 2015)), or phase transitions, or evolution of the interfaces among phases (e.g., (Gurtin, 1986, 1993; Gurtin and Podio-Guidugli, 1996; Fried and Gurtin, 1994, 2004)).

Among the other streams of research starting from Eshelby's pioneering works, the second one on which we shall concentrate is that of the problem of an inclusion in a homogeneous matrix. Eshelby (1957) described an inclusion with *geometrical misfit* with the matrix by introducing a *transformation strain* (sometimes called *eigenstrain*). Moreover, he showed that, for an *ellipsoidal* inclusion for which the geometrical misfit is given by a *uniform* transformation strain, the stress and strain fields inside the inclusion are uniform. Eshelby (1957, 1961) solved both the case of an ellipsoidal inclusion with misfitting geometry and same material properties as the matrix (homogeneous inclusion), and of an inclusion with misfitting material properties (inhomogeneous inclusion), by means of a method based on a fourth-order tensor that relates the *cancelling* strain in the inclusion (i.e., the strain that allows the inclusion to elastically "relax" within the matrix) to the transformation strain defining the geometrical misfit (an equivalent method for isotropic inclusions has been proposed by Knops (1964)). This fourth-order tensor is commonly referred to as the *Eshelby (fourth-order) tensor* or *Eshelby's* $\mathbb{S}$*-tensor*. Since the ellipsoidal inclusion can describe several cases, ranging from cracks (extremely oblate ellipsoids with null elasticity tensor) to fibres (extremely prolate ellipsoids), this approach has been used widely in literature on linear composite material with inclusions (e.g., (Walpole, 1966a,b, 1969; Tandon and Weng, 1984; Weng, 1984, 1990; Ru et al., 2001; Federico et al., 2004; Kim et al., 2008)), in order to find the overall properties starting from the properties of the individual constituents (equivalent methods for this problem have been proposed, e.g., by Hill (1965)).

As mentioned above, a key point of Eshelby's method for the ellipsoidal inclusion (Eshelby, 1957) is that, if the transformation strain (which defines the geometrical misfit) is *uniform*, the resulting stress and strain field inside the inclusion are uniform. In two dimensions, this naturally occurs for an elliptical inclusion. In addition, Eshelby (1961) conjectured that this condition holds, under *any* remote loadings, *exclusively* for *ellipsoidal* inclusions. Rodin (1996) and Markenscoff (1997) showed that it is indeed impossible to obtain a uniform stress field within a polygonal (in the plane) or a polyhedral (in space) inclusion. By means of complex variable approaches, Sendeckyj (1970) and Ru and Schiavone (1996) proved a stronger version of Eshelby's conjecture for the cases of plane elasticity and anti-plane shear elasticity, respectively. The situation is summarised by Kang and Milton (2008): the *weak* Eshelby's conjecture holds for *any* remote loadings, and the *strong* one for *a single* remote loading.Kang and Milton (2008) and Liu (2008) proved a weak version of the conjecture, whereas Ammari et al. Ammari et al. (2010) proved a version of the conjecture in between the weak and the strong ones. Finally, we note that, in the non-linear elastic case, under some specific conditions, the first Piola-Kirchhoff stress is uniform for an inclusion of arbitrary shape (Ru et al., 2005; Kim et al., 2008).

It is somewhat surprising that there is virtually no intersection between these two fields of research, although both arise from Eshebly's work. On the one hand, we have a group of researchers who work on inhomogeneity theory utilising the Eshelby stress as a prime tool and, on the other hand, we have another group of researchers who work on the similar problem of ellipsoidal inclusions in a homogenous matrix utilising the Eshelby fourth-order tensor. Even in Eshelby's original works (e.g., Eshelby, 1951, 1957)), there seems to be no obvious relationship between the two. Therefore, the objective of this paper is to study the relationship between the Eshelby stress and the Eshelby fourth-order tensor within an ellipsoidal inclusion, for the case of small deformations and linear elasticity.

The motivation for this work is twofold. First, if the Eshelby stress in an inclusion can be found directly from the geometrical characteristics of the system, the elastic properties and the external loads, then one can immediately find the net (configurational) force on the inclusion. For the case of the ellipsoidal inclusion, the components of the Eshelby fourth-order tensor are well known (see, e.g., (Eshelby, 1957; Qiu and Weng, 1990)) and provide all details about the geometry (up to the transformation strain). This could be particularly useful for composite materials reinforced by ellipsoidal or fibre-like inclusions, which have traditionally been studied with Eshelby's method.

Second, we believe that it is important to explicitly show the relation between these two important streams of Eshelby's work, from the epistemological point of view. In passing, we shall attempt to present the relevant theory that is already known in an organic manner, hoping that it can be found to be useful in a didactic setting.

2.1 Theoretical Background

In this section, we elucidate the notation that we employ and report some fundamental results relevant to this work. Although we shall generally use index-free notation throughout the following sections, sometimes it will be useful to work in index notation. Therefore, here we present most expressions in both notations. In index notation, the customary summation convention for repeated indices is enforced throughout and a subscript preceded by a comma, as in $f_{,i}$, denotes partial differentiation with respect to the i-th variable or, more rigorously, the directional derivative with respect to the i-th basis vector $\boldsymbol{e}_i$, i.e., $f_{,i} = \partial_{\boldsymbol{e}_i} f = \partial f / \partial x_i$. We work in a Cartesian setting, so we do not distinguish between partial and covariant derivatives, and between contravariant and covariant indices.

2.1.1 General Relations

We work in the small-displacement setting and therefore we *do not* distinguish the reference and current configurations of a body $\mathcal{B}$. Given a displacement field $\boldsymbol{u}$ on the body $\mathcal{B}$, the displacement gradient is defined as

$$\boldsymbol{h} = \operatorname{grad} \boldsymbol{u}, \qquad h_{ij} = u_{i,j}, \tag{2.1}$$

and can be decomposed into its symmetric and skew-symmetric parts as

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\boldsymbol{h} + \boldsymbol{h}^T\right), \qquad \epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2.2a}$$

$$\boldsymbol{\varphi} = \frac{1}{2}\left(\boldsymbol{h} - \boldsymbol{h}^T\right), \qquad \varphi_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right), \tag{2.2b}$$

where the superscript T denotes transposition, $\boldsymbol{\epsilon}$ is the infinitesimal strain tensor and $\boldsymbol{\varphi}$ is the infinitesimal rotation tensor.

If inertia is neglected (static or quasi-static case) and in the absence of external volume forces, the balance of linear momentum for a continuum body states the vanishing of the divergence of the (Cauchy) stress tensor, i.e.,

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}, \qquad \sigma_{ij,j} = 0. \tag{2.3}$$

For a linear elastic body, the stress $\boldsymbol{\sigma}$ is related to the infinitesimal strain $\boldsymbol{\epsilon}$ via the linear elasticity tensor $\mathbb{L}$, i.e.,

$$\boldsymbol{\sigma} = \mathbb{L} : \boldsymbol{\epsilon}, \qquad \sigma_{ij} = \mathrm{L}_{ijkl}\, \epsilon_{kl}, \tag{2.4}$$

where the colon ":" denotes double contraction (in this case, of the last two legs of $\mathbb{L}$ and the two legs of $\boldsymbol{\epsilon}$). The elasticity tensor enjoys both major (or diagonal) and minor (or pair) symmetries, i.e., for every second-order tensors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{a} : \mathbb{L} : \boldsymbol{b} = \boldsymbol{b} : \mathbb{L} : \boldsymbol{a}, \qquad \mathrm{L}_{ijkl} = \mathrm{L}_{klij}, \tag{2.5a}$$

$$\boldsymbol{a} : \mathbb{L} : \boldsymbol{b} = \boldsymbol{a}^T : \mathbb{L} : \boldsymbol{b} = \boldsymbol{a} : \mathbb{L} : \boldsymbol{b}^T = \boldsymbol{a}^T : \mathbb{L} : \boldsymbol{b}^T, \qquad \mathrm{L}_{ijkl} = \mathrm{L}_{jikl} = \mathrm{L}_{ijlk} = \mathrm{L}_{jilk}. \tag{2.5b}$$

The elastic energy density per unit volume of linear elasticity is the quadratic form

$$w = \frac{1}{2}\, \boldsymbol{\sigma} : \boldsymbol{\epsilon} = \frac{1}{2}\, \boldsymbol{\epsilon} : \mathbb{L} : \boldsymbol{\epsilon}, \qquad w = \frac{1}{2}\, \sigma_{ij}\, \epsilon_{ij} = \frac{1}{2}\, \epsilon_{ij}\, \mathrm{L}_{ijkl}\, \epsilon_{kl}. \tag{2.6}$$

Note that, because of the symmetry of the (Cauchy) stress $\boldsymbol{\sigma}$ and the minor symmetry of the elasticity tensor $\mathbb{L}$, it is equivalent to express the elastic energy w as a function of the strain $\boldsymbol{\epsilon}$ or of the displacement gradient $\boldsymbol{h}$, whose skew-symmetric part $\boldsymbol{\varphi}$ is "filtered" by the double contraction with $\boldsymbol{\sigma}$ or $\mathbb{L}$, i.e., we can write Eq. (2.4) in the alternative form

$$\boldsymbol{\sigma} = \mathbb{L} : \boldsymbol{h}, \qquad \sigma_{ij} = \mathrm{L}_{ijkl}\, h_{kl}, \tag{2.7}$$

and Eq. (2.6) in the alternative form

$$w = \frac{1}{2}\,\boldsymbol{\sigma} : \boldsymbol{h} = \frac{1}{2}\,\boldsymbol{h} : \mathbb{L} : \boldsymbol{h}, \qquad w = \frac{1}{2}\,\sigma_{ij}\,h_{ij} = \frac{1}{2}\,h_{ij}\,\mathrm{L}_{ijkl}\,h_{kl}. \tag{2.8}$$

Given a certain elasticity tensor $\mathbb{L}$, the terminology *elastic field* X denotes the collection of a given displacement field $\boldsymbol{u}^X$, the corresponding strain $\boldsymbol{\epsilon}^X$ (derived according to Eq. (2.2a)), and the corresponding stress $\boldsymbol{\sigma}^X$ (derived via Eq. (2.4)). Two elastic fields X and Y associated with the same elasticity tensor $\mathbb{L}$ are called *corresponding fields*, and, by virtue of the major symmetry of the elasticity tensor (i.e., $\mathbb{L} = \mathbb{L}^T$, $\mathrm{L}_{ijkl} = \mathrm{L}_{klij}$), are such that

$$\boldsymbol{\sigma}^X : \boldsymbol{\epsilon}^Y = \boldsymbol{\sigma}^Y : \boldsymbol{\epsilon}^X, \qquad \sigma^X_{ij}\epsilon^Y_{ij} = \sigma^Y_{ij}\epsilon^X_{ij}. \tag{2.9}$$

The form of Stokes' theorem reported below (Eshelby, 1951), applied to an open surface S bounded by the smooth closed curve ∂S, will be useful later (for a proof, see, e.g., (Balluffi, 2012)):

$$\int_S \left(v_{j,j}\,\delta_{ki} - v_{k,i}\right)\,n_k = \int_{\partial S} \varepsilon_{ijk}\,v_j\,\tau_k. \tag{2.10}$$

In coordinate-free notation, this reads

$$\int_S ((\mathrm{div}\,\boldsymbol{v})\,\boldsymbol{i} - \mathrm{grad}\,\boldsymbol{v})^T\,\boldsymbol{n} = \int_{\partial S} \boldsymbol{v} \times \boldsymbol{\tau}. \tag{2.11}$$

In Eqs. (2.10) and (2.11), $\boldsymbol{\tau}$ is the normalised tangent vector to the curve ∂S, $\boldsymbol{v}$ is a vector field, δ_{ki} (Kronecker symbol) are the components of the identity tensor $\boldsymbol{i}$, and ε_{ijk} (permutation symbol) are the components of the Ricci Levi-Civita tensor in Cartesian coordinates. In particular, if the surface S is progressively closed, so that it tends to the boundary $\partial\mathcal{D}$ of some region $\mathcal{D}$ and the boundary ∂S degenerates into a point, the right-hand side tends to zero, and Eq. (2.11) becomes

$$\int_{\partial\mathcal{D}} v_{j,j}\,\delta_{ki}\,n_k = \int_{\partial\mathcal{D}} v_{k,i}\,n_k, \tag{2.12}$$

or

$$\int_{\partial\mathcal{D}} (\operatorname{div} \boldsymbol{v})\, \boldsymbol{n} = \int_{\partial\mathcal{D}} (\operatorname{grad} \boldsymbol{v})^T \boldsymbol{n}. \tag{2.13}$$

Given two corresponding elastic fields X and Y, substitution of $\boldsymbol{v} = \boldsymbol{u}^Y \boldsymbol{\sigma}^X$ and Eq. (2.3) into Eq. (2.13), allows for obtaining the further relation

$$\int_{\partial\mathcal{D}} \sigma_{kl}^X\, u_{k,l}^Y\, \delta_{si}\, n_s = \int_{\partial\mathcal{D}} \left(\sigma_{rs,i}^X\, u_r^Y + \sigma_{rs}^X\, u_{r,i}^Y \right) n_s, \tag{2.14}$$

which, in component-free notation, reads

$$\int_{\partial\mathcal{D}} \left(\boldsymbol{\sigma}^X : \boldsymbol{h}^Y \right) \boldsymbol{n} = \int_{\partial\mathcal{D}} \left(\boldsymbol{u}^Y \operatorname{grad} \boldsymbol{\sigma}^X + (\boldsymbol{h}^Y)^T \boldsymbol{\sigma}^X \right)^T \boldsymbol{n}. \tag{2.15}$$

Finally, the jump of a field f across a given surface S at point $x \in S$ with unit normal $\boldsymbol{n}$ is given by

$$[\![f]\!](x) = \lim_{h \to 0^+} [f(x + h\,\boldsymbol{n}) - f(x - h\,\boldsymbol{n})], \tag{2.16}$$

where, if S is a closed surface, $\boldsymbol{n}$ is understood to be the outward normal.

2.1.2 The Ellipsoidal Inclusion and the Fourth-Order Eshelby Tensor

For the establishment of the problem of an inclusion in an otherwise homogeneous matrix, we essentially follow the procedure outlined by Eshelby (1957) (last paragraph of page 376):

> *We shall solve this problem with the help of a simple set of imaginary cutting, straining and welding operations. Cut round the region which is to transform and remove it from the matrix. Allow the unconstrained transformation to take place. Apply surface tractions chosen so as to restore the region to its original form, put it back in the hole in the matrix and rejoin the material across the cut. The stress is now zero in the matrix and has a known constant value in the inclusion. The applied surface tractions*

> *have become built in as a layer of body force spread over the interface between matrix and inclusion. To complete the solution this unwanted layer is removed by applying an equal and opposite layer of body force; the additional elastic field thus introduced is found by integration from the expression for the elastic field of a point force.*

Perhaps because of our own verbal struggle, we prefer to rethink the part "cut round the region which is to transform and remove it from the matrix; allow the unconstrained transformation to take place" in a slightly different way. Below, we describe Eshelby's procedure in our words and notation.

Let $\mathcal{B}$ be a homogeneous elastic body from which a cavity $\mathcal{D}$ is carved. We shall call the region $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$ the *matrix* (where the symbol "\" denotes subtraction of sets). Let $\mathcal{D}^*$ be another homogeneous body, whose shape is obtained by mapping the region $\mathcal{D}$ by means of a *transformation strain* $\boldsymbol{\epsilon}^*$, which determines a *geometrical misfit* between $\mathcal{D}$ and $\mathcal{D}^*$. The material contained in the region $\mathcal{D}^*$ is going to constitute the *inclusion*, and may or may not have the same material properties of the matrix $\mathcal{M}$. Then, in order to fit the shape of the region $\mathcal{D}^*$ into the original shape $\mathcal{D}$, we apply suitable surface tractions on the boundary $\partial\mathcal{D}^*$, and put the inclusion back in the cavity $\mathcal{D}$. Once the inclusion is put back and welded into the cavity, the surface tractions are released and the inclusion and the matrix are allowed to relax elastically, thereby attaining a *cancelling strain* $\boldsymbol{\epsilon}^C$.

We remark that, in the small displacement theory, we are allowed to identify the reference (undeformed) configuration and the current (deformed) configuration, and thus we are going to continue to call the body, the matrix and inclusion with $\mathcal{B}$, $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$ and $\mathcal{D}$, respectively, although at this point they will have changed shape because of the geometrical misfit of matrix and inclusion.

In the absence of applied external tractions, the total strain $\boldsymbol{\epsilon}$ equals the strain $\boldsymbol{\epsilon}^B$ caused by the geometrical misfit *alone*, and it is described, in the inclusion or the matrix, by

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^B = \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*, \tag{2.17a}$$

$$\boldsymbol{\epsilon}^* \neq \mathbf{0}, \quad \text{in } \mathcal{D}, \tag{2.17b}$$

$$\boldsymbol{\epsilon}^* = \mathbf{0}, \quad \text{in } \mathcal{M} = \mathcal{B} \setminus \mathcal{D}, \tag{2.17c}$$

$$[\![\boldsymbol{\epsilon}^C]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}, \tag{2.17d}$$

$$[\![\boldsymbol{\epsilon}]\!] = [\![\boldsymbol{\epsilon}^B]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}. \tag{2.17e}$$

Eq. (2.17d) means that the *cancelling strain* $\boldsymbol{\epsilon}^C$ is discontinuous across the boundary $\partial\mathcal{D}$ of the inclusion, although it is continuous within the inclusion $\mathcal{D}$ and within the matrix $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$. Eq. (2.17e) states that the total strain $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^B$ is generally discontinuous across $\partial\mathcal{D}$ (Hill, 1961; Walpole, 1967). Furthermore, the bonding between matrix and inclusion is assumed to be perfect, i.e., at the matrix-inclusion interface, the displacement field and the traction field are continuous (Eshelby, 1961; Mura, 1987; Balluffi, 2012).

$$[\![\boldsymbol{u}]\!] = [\![\boldsymbol{u}^B]\!] = \mathbf{0}, \quad \text{on } \partial\mathcal{D}, \tag{2.18a}$$

$$[\![\boldsymbol{\sigma}^B]\!]\,\boldsymbol{n} = \mathbf{0}, \quad \text{on } \partial\mathcal{D}. \tag{2.18b}$$

Note that the displacement gradient $\boldsymbol{h} = \operatorname{grad} \boldsymbol{u}$, the symmetric part of which is the strain $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^B$, is discontinuous across $\partial\mathcal{D}$, although the displacement $\boldsymbol{u} = \boldsymbol{u}^B$ is continuous (this has been thoroughly studied by Hill (1961)).

For the case of an ellipsoidal inclusion, to which we shall restrict our analysis throughout this work, the transformation strain $\boldsymbol{\epsilon}^*$ and the cancelling strain $\boldsymbol{\epsilon}^C$ are *uniform* in the inclusion, and are linearly related via the fourth-order *Eshelby tensor* $\mathbb{S}$ (Eshelby, 1957; Balluffi, 2012), i.e.,

$$\boldsymbol{\epsilon}^C = \mathbb{S} : \boldsymbol{\epsilon}^*, \qquad \epsilon^C_{ij} = \mathrm{S}_{ijkl}\, \epsilon^*_{kl}, \quad \text{in } \mathcal{D}, \tag{2.19}$$

where, due to the fact that $\boldsymbol{\epsilon}^*$ and $\boldsymbol{\epsilon}^C$ are uniform in the inclusion, it follows that also $\mathbb{S}$ is such.

We remark that, even for the case of an ellipsoidal inclusion, in which the cancelling strain $\boldsymbol{\epsilon}^C$ is uniform in the inclusion $\mathcal{D}$, the cancelling strain $\boldsymbol{\epsilon}^C$ is *never* uniform in the matrix $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$, as it is obviously non-zero on $\partial\mathcal{D}$ and, in general, must decrease towards $\partial\mathcal{B}$, and must be zero on $\partial\mathcal{B}$ under the hypothesis of an *infinite* matrix.

Moreover, we remark that, for an ellipsoidal inclusion, the transformation strain causes only a change in the length of the semi-axes and *not* in their directions. Therefore, the transformation displacement $\boldsymbol{u}^*$ corresponding to the transformation strain $\boldsymbol{\epsilon}^*$ causes *no* rotation. This means that the skew-symmetric part $\boldsymbol{\varphi}^* = \frac{1}{2}(\boldsymbol{h}^* - (\boldsymbol{h}^*)^T)$ of the transformation displacement gradient $\boldsymbol{h}^*$ vanishes identically, so that $\boldsymbol{h}^*$ and its symmetric part $\boldsymbol{\epsilon}^*$ coincide, i.e.,

$$\boldsymbol{\varphi}^* = \frac{1}{2}\left(\boldsymbol{h}^* - (\boldsymbol{h}^*)^T\right) = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{h}^* \equiv \frac{1}{2}\left(\boldsymbol{h}^* + (\boldsymbol{h}^*)^T\right) = \boldsymbol{\epsilon}^*. \tag{2.20}$$

This assumption has been used extensively, e.g., by (Mura, 1987), and turns out to be crucial in the subsequent derivations.

Finally, we assume to deal with an inclusion in an *infinite matrix*, i.e., with an inclusion $\mathcal{D}$ that is much smaller than the whole body $\mathcal{B}$, and that is far away from the boundary $\partial\mathcal{B}$ of the body, so that the effects of *image stresses* (so called from the *image method*, used to deal with inclusions or defects near an interface; see, e.g., Section 3.8 in the book by Balluffi (2012)) can be neglected.

2.1.3 Classification of Inclusions

An inclusion can be characterised by five attributes: shape, volume, misfit with the matrix in terms of geometry, misfit with the matrix in terms of material properties, and level of bonding (also called level of coherence) at the matrix-inclusion interface. In the literature, inclusions with perfect bonding at the interface with the matrix are subdivided into three categories, as reported below.

(1) Inclusion with Misfitting Geometry (Homogeneous Inclusion)

In this case, the inclusion has the same elastic properties of the matrix, described by the elasticity tensor $\mathbb{L}$. Therefore, there is only the geometrical misfit caused by the transformation strain $\boldsymbol{\epsilon}^*$. This type of inclusion is studied directly by means of Eshelby's procedure, which was outlined in Section 2.1.2. In fact, this type of inclusion, called "homogeneous inclusion" by Eshelby (1957) and in the literature, is the case to which all others can be reduced, with suitable equivalence methods.

(2) Inclusion with Misfitting Properties (Inhomogeneous Inclusion)

The inclusion has different elastic properties than the matrix, and thus its elasticity tensor $\mathbb{L}'$ is different from that of the matrix $\mathbb{L}$. However, the transformation strain $\boldsymbol{\epsilon}^*$ is equal to zero and therefore there is no geometrical misfit. In the absence of a stress field caused by external tractions, the stress and the strain in the matrix and inclusion are identically zero. However, when external forces are applied, the misfit in material properties of the inclusion causes a perturbation in the stress and strain fields. Eshelby's method of solution for this type of inclusion, which he called "inhomogeneous inclusion" (Eshelby, 1957), consists in reducing its effect to that of an "equivalent homogeneous inclusion", i.e., in finding that fictitious transformation strain that causes the same perturbation as the difference in material properties (Eshelby, 1961).

(3) Inclusion with Misfitting Geometry and Properties (General Inclusion)

This is the most general case, in which the inclusion has a geometrical misfit, described by a transformation strain $\boldsymbol{\epsilon}^*$, as well as a misfit in material properties, i.e., $\mathbb{L}' \neq \mathbb{L}$. This case too is solved by means of Eshelby's method of the "equivalent homogeneous inclusion" (Eshelby, 1961; Balluffi, 2012). We call this type of inclusion "general inclusion" because cases (1) and (2) are obtained by setting $\mathbb{L}' = \mathbb{L}$ and $\boldsymbol{\epsilon}^* = \mathbf{0}$, respectively. We remark that Eshelby (1957) called "general inclusion" the inclusion of *arbitrary shape*, but this should not be a source of confusion in this context, in which we limit ourselves to the case of *ellipsoidal* inclusions.

2.1.4 The Material Force on a Defect and the Eshelby Stress

In his seminal works, Eshelby studied, for the case of small displacements (Eshelby, 1951) and for the case of large displacements (Eshelby, 1975), the material force $\mathcal{F}^{\mathrm{conf}}$ exerted on a region $\mathcal{D}$ including a defect. The definition of defect also includes (the pun is inevitable) the case of an inclusion, which is obtained when the defect is the whole region $\mathcal{D}$ itself. The material force on the defect is also called *configurational force* because it is the "price to pay" to displace the defect within the body, i.e., to cause a change in *configuration*. Eshelby showed that the configurational force can be computed as the integral of the tractions on the boundary $\partial\mathcal{D}$ of the region $\mathcal{D}$ due to a stress tensor $\mathfrak{p}$ that Eshelby called, in accordance with the jargon of Classical Field Theory, *Maxwell tensor* or *energy-momentum tensor*:

$$\mathcal{F}^{\mathrm{conf}} = \int_{\partial\mathcal{D}} \boldsymbol{\mathfrak{p}}\, \boldsymbol{n} = \int_{\mathcal{D}} \operatorname{div} \boldsymbol{\mathfrak{p}}, \qquad \mathcal{F}_i^{\mathrm{conf}} = \int_{\partial\mathcal{D}} \mathfrak{p}_{ik}\, n_k = \int_{\mathcal{D}} \mathfrak{p}_{ik,k}. \tag{2.21}$$

The stress tensor $\mathfrak{p}$, called *Eshelby stress* by Maugin and Trimarco (1992), is expressed (in the small-displacement theory) by

$$\boldsymbol{\mathfrak{p}} = w\, \boldsymbol{i} - \boldsymbol{h}^T \boldsymbol{\sigma}, \qquad \mathfrak{p}_{ik} = w\, \delta_{ik} - h_{ji}\, \sigma_{jk}, \tag{2.22}$$

where $\boldsymbol{h}$ is the (completely spatial) displacement gradient, $\boldsymbol{\sigma}$ is the (Cauchy) stress and w is the elastic energy density per unit (current) volume of linear elasticity of Eq. (2.6). The configurational force is also often called *inhomogeneity force* (see. e.g., (Epstein and Maugin, 2000)) because, as it can be shown easily in the non-linear theory, it vanishes over a region $\mathcal{D}$ in which the elastic energy is homogeneous, i.e., does not depend on the point (Eshelby, 1975). Under these conditions, Eq. (2.21) implies that the divergence of the Eshelby stress vanishes. In linear elasticity, homogeneity of the elastic energy reduces to homogeneity of the elasticity tensor. A direct proof of the vanishing of the Eshelby stress on a region in which the elasticity tensor is homogeneous is given in Appendix A.1. In passing, we also note that Eq. (2.21) can be interpreted as the J-integral defined by Rice (1968) in fracture mechanics, as pointed out by, e.g., Smelser and Gurtin (1977), Epstein and Elżanowski (2007) (Section 5.6.4) and Gupta and Markenscoff (2012). Indeed, Smelser and Gurtin (1977), following Eshelby (1956) and Rice (1968), call Eq. (2.21), with $\mathfrak{p}$ expressed as in Eq. (2.22), "the Eshelby-Rice conservation law".

Remark. The configurational force $\mathcal{F}^{\text{conf}}$ is said to be *material* because, in the large-deformation theory, when we ought to distinguish the reference from the current configuration, the configurational force is a *material covector* and the Eshelby stress is a completely *material* tensor with the first leg being a covector (i.e., the first index being covariant) and the second leg being a vector (i.e., the second index being contravariant). Just within this remark, we use the modern Continuum Mechanics notation, with uppercase symbols and indices for the reference configuration and lowercase symbols and indices for the current configuration (see, e.g., (Marsden and Hughes, 1983)). In index notation, the configurational force is given by

$$\mathcal{F}^{\text{conf}}_I = \int_{\partial\mathcal{D}} \mathfrak{P}_I{}^K N_K = \int_{\mathcal{D}} \mathfrak{P}_I{}^K{}_{|K}, \tag{2.23}$$

where the vertical bar denotes covariant differentiation. The large-deformation energy-momentum tensor (which is now called Eshelby stress) $\mathfrak{P}$ was obtained by Eshelby (1975) using a variational approach in which the independent variable of the Lagrangian density per unit reference volume is the *spatial* displacement $\boldsymbol{u}$ considered as a function of the *material* points X, and is given by

$$\mathfrak{P}_I{}^K = W\,\delta_I{}^K - H^j{}_I\,P_j{}^K, \tag{2.24}$$

where W is the elastic energy density per unit *reference* volume, $H^j{}_I = u^j{}_{,I}$ is the *two-point* displacement gradient, and $P_j{}^K$ is the first Piola-Kirchhoff stress. The component expression of the completely spatial Eshelby stress $\mathfrak{p}$ of Eq. (2.22) can be obtained by a push-forward on the first leg and forward Piola transformation on the second leg of $\mathfrak{P}$, i.e.,

$$\begin{aligned} \mathfrak{p}_i{}^k &= J^{-1}\, W\, (\boldsymbol{F}^{-1})^I{}_i\, \delta_I{}^K\, F^k{}_K - H^j{}_I\, (\boldsymbol{F}^{-1})^I{}_i\, J^{-1}\, P_j{}^K\, F^k{}_K \\ &= w\, \delta_i{}^k - h^j{}_i\, \sigma_j{}^k, \end{aligned} \tag{2.25}$$

where tensor $\boldsymbol{F}$ (with components $F^i{}_K$) is the deformation gradient, $w = J^{-1}\, W$ is the energy density per unit *current* volume, $h^j{}_i = H^j{}_I\, (\boldsymbol{F}^{-1})^I{}_i = u^j{}_{,i}$ is the *spatial* displacement gradient, and $\sigma_i{}^k = J^{-1}\, P_i{}^K\, F^k{}_K$ is the Cauchy stress. The fully spatial Eshelby tensor $\mathfrak{p}$ of Eq. (2.25) becomes that of the linear theory if the energy density w per unit current volume is the quadratic one of Eq. (2.6).

2.2 Variation of the Interaction Energy and $\mathfrak{p}$-$\mathbb{S}$ Relation

In the *weak* formulation (Eshelby, 1951, 1956), the configurational force $\mathcal{F}^{\text{conf}}$ is defined as the negative of the variation of the total energy $\mathcal{W}$ (considered as a functional, see Eq. (2.42) for the case of the "homogeneous inclusion") evaluated at the identity map $\mathcal{X}$ (defined by $\mathcal{X}(x) = x$, for every $x \in \mathcal{B}$), and performed with respect to the uniform displacement field $\boldsymbol{\xi}$, such that $\boldsymbol{\xi}(x) = \boldsymbol{\xi}_0$, for every $x \in \mathcal{D}$, i.e.,

$$\mathcal{F}^{\text{conf}} . \boldsymbol{\xi}_0 = -\frac{\partial \mathcal{W}}{\partial \boldsymbol{\xi}}(\mathcal{X}) = -\frac{\partial(\mathcal{W}_{\text{el}} + \mathcal{W}_{\text{ext}})}{\partial \boldsymbol{\xi}}(\mathcal{X}), \tag{2.26}$$

where $\mathcal{W}_{\text{el}}$ refers to elastic energy, and $\mathcal{W}_{\text{ext}}$ is the work of the external forces.

This variation is sometimes referred to as a "variational gradient", but it can be performed as the regular directional derivative of the functional $\mathcal{W}$ defined in the space of all admissible configurations. The *configuration space* can be regarded as an affine space (which is a trivial differentiable manifold) if such is the physical space $\mathcal{S}$ in which the body $\mathcal{B}$ dwells. Thus, we have

$$-\frac{\partial \mathcal{W}}{\partial \boldsymbol{\xi}}(\mathcal{X}) = \lim_{h \to 0} \frac{\mathcal{W}(\mathcal{X} - h\, \boldsymbol{\xi}) - \mathcal{W}(\mathcal{X})}{h}, \tag{2.27}$$

where the displacement $-h\,\boldsymbol{\xi}$ is an infinitesimal variation of the identity map $\mathcal{X}$. This defines the map

$$\mathcal{Y} = \mathcal{X} - h\,\boldsymbol{\xi} : x \mapsto x - h\,\boldsymbol{\xi}(x) = x - h\,\boldsymbol{\xi}_0, \tag{2.28}$$

which maps the domain $\mathcal{D}$ of the inclusion into the displaced configuration $\mathcal{Y}(\mathcal{D})$ (Figure 2.1). It is easy to show that, if the displacement field $\boldsymbol{\xi}$ is uniform, i.e., $\boldsymbol{\xi}(x) = \boldsymbol{\xi}_0$ for every x, the determinant of the transformation described by $\mathcal{Y}$ is identically equal to one.

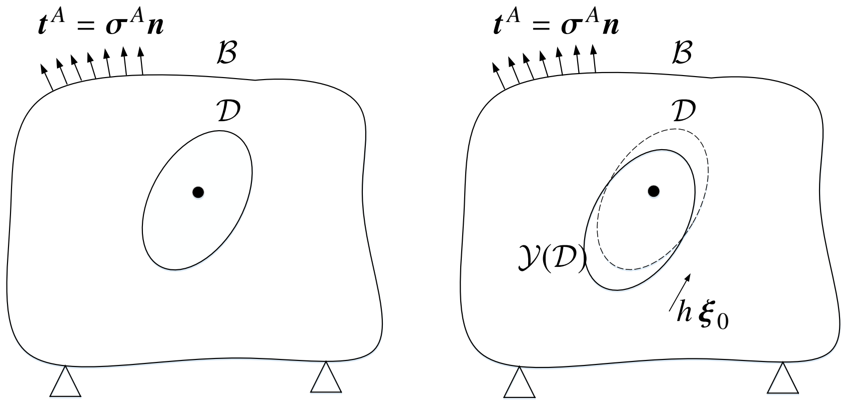

Figure 2.1: Determination of the force on a defect contained in a region $\mathcal{D}$ of a body $\mathcal{B}$. By means of the uniform displacement $\boldsymbol{\xi}(x) = \boldsymbol{\xi}_0$, for every $x \in \mathcal{D}$, we define the map $\mathcal{Y} = \mathcal{X} - h\,\boldsymbol{\xi}$ which maps the region $\mathcal{D}$ into the displaced one $\mathcal{Y}(\mathcal{D})$. The traction field $\boldsymbol{t}^A$ is applied on the boundary and gives rise to the applied stress $\boldsymbol{\sigma}^A$, which is such that $\boldsymbol{t}^A = \boldsymbol{\sigma}^A \boldsymbol{n}$ on the boundary.

Balluffi (2012) showed that it is equivalent to calculate the configurational force as the variation of the total energy $\mathcal{W} = \mathcal{W}_{\text{el}} + \mathcal{W}_{\text{ext}}$ or of the *interaction energy*, which describes the interaction between the applied stress field $\boldsymbol{\sigma}^A$, caused by the external traction field $\boldsymbol{t}^A$ applied on the boundary $\partial\mathcal{B}$ of the body $\mathcal{B}$, and the stress field $\boldsymbol{\sigma}^B$ caused by the presence of the inclusion. The interaction energy $\mathcal{W}^{\text{int}}$ is defined as the total energy $\mathcal{W}$ of the superimposed fields A and B minus the sum of the energies $\mathcal{W}^A$ and $\mathcal{W}^B$ taken individually, i.e.,

$$\mathcal{W}^{\text{int}} = \mathcal{W} - \mathcal{W}^A - \mathcal{W}^B, \tag{2.29a}$$

$$\mathcal{W} = \mathcal{W}_{\text{el}} + \mathcal{W}_{\text{ext}}, \tag{2.29b}$$

$$\mathcal{W}^A = \mathcal{W}^A_{\text{el}} + \mathcal{W}^A_{\text{ext}}, \tag{2.29c}$$

$$\mathcal{W}^B = \mathcal{W}^B_{\text{el}}, \tag{2.29d}$$

where we note that $\mathcal{W}^B_{\text{ext}}$ is identically zero, as the inclusion field B applies no tractions on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, i.e., $\boldsymbol{\sigma}^B \boldsymbol{n} = \mathbf{0}$ on $\partial\mathcal{B}$. Moreover, $\mathcal{W}^A$ is completely independent of the presence of the inclusion, and is always given by

$$\mathcal{W}^A = \mathcal{W}^A_{\text{el}} + \mathcal{W}^A_{\text{ext}} = \int_{\mathcal{B}} \frac{1}{2} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A - \int_{\partial\mathcal{B}} \boldsymbol{t}^A . \boldsymbol{u}^A, \tag{2.30}$$

where $\boldsymbol{t}^A = \boldsymbol{\sigma}^A \boldsymbol{n}$ on $\partial\mathcal{B}$. Balluffi's argument (Balluffi, 2012) is that the variation of $\mathcal{W}^A$ and $\mathcal{W}^B$ is identically zero because a change in the configuration does not affect their functional expression. Therefore, the variation of $\mathcal{W}$ equals that of $\mathcal{W}^{\text{int}}$. The configurational force can therefore also be written as

$$\mathcal{F}^{\text{conf}} . \boldsymbol{\xi}_0 = -\frac{\partial \mathcal{W}^{\text{int}}}{\partial \boldsymbol{\xi}} (\mathcal{X}). \tag{2.31}$$

The goal of this paper is to obtain the relation between the Eshelby stress $\mathfrak{p}$ and the Eshelby fourth-order tensor $\mathbb{S}$ inside an inclusion $\mathcal{D}$ in an infinite matrix $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$. In order do so, we shall exploit the material force $\mathcal{F}^{\text{conf}}$ acting on the inclusion, as it is directly related to the Eshelby stress $\mathfrak{p}$ and to the cancelling strain $\boldsymbol{\epsilon}^C$, which, in turn, can be expressed as a function of the fourth-order Eshelby tensor $\mathbb{S}$.

2.3 Inclusion with Misfitting Geometry (Homogeneous Inclusion)

The problem of the inclusion with misfitting geometry, called "homogeneous inclusion" in the literature, is established via the imaginary procedure described in Section 2.1.2 and assuming that the matrix $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$ and the inclusion $\mathcal{D}$ have the same material properties, described by the same elasticity tensor $\mathbb{L}$, but also have a geometrical misfit described by the transformation strain $\boldsymbol{\epsilon}^*$. Furthermore, we assume that the applied traction field $\boldsymbol{t}^A$ acts on the boundary $\partial\mathcal{B}$ of the body $\mathcal{B}$. The traction field $\boldsymbol{t}^A$ gives rise to the applied stress field $\boldsymbol{\sigma}^A$, which is such that $\boldsymbol{t}^A = \boldsymbol{\sigma}^A \boldsymbol{n}$ on $\partial\mathcal{B}$. The total strain $\boldsymbol{\epsilon}$ in the inclusion or the matrix is equal to sum of the applied strain $\boldsymbol{\epsilon}^A = \mathbb{L}^{-1} : \boldsymbol{\sigma}^A$ and the strain resulting from the geometrical misfit in the inclusion $\boldsymbol{\epsilon}^B$, given by Eq. (2.17a) (which had been written in the absence of external applied tractions). Thus, we have

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^B = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*, \tag{2.32a}$$

$$\boldsymbol{\epsilon}^* \neq \mathbf{0}, \quad \text{in } \mathcal{D}, \tag{2.32b}$$

$$\boldsymbol{\epsilon}^* = \mathbf{0}, \quad \text{in } \mathcal{M} = \mathcal{B} \setminus \mathcal{D}, \tag{2.32c}$$

$$[\![\boldsymbol{\epsilon}^C]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}, \tag{2.32d}$$

$$[\![\boldsymbol{\epsilon}]\!] = [\![\boldsymbol{\epsilon}^A]\!] + [\![\boldsymbol{\epsilon}^B]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}. \tag{2.32e}$$

2.3.1 "Homogeneous" Case: Interaction Energy

The superimposed elastic strain energy of fields A and B is defined by

$$\begin{aligned} \mathcal{W}_{\text{el}} &= \frac{1}{2}\int_{\mathcal{B}} \left(\boldsymbol{\sigma}^A + \boldsymbol{\sigma}^B\right) : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*\right) \\ &= \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^C - \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* \\ &+ \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^C - \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^*. \end{aligned} \tag{2.33}$$

It is possible to prove that four out of the six integrals in Eq. (2.33) vanish identically, and precisely the second, third, fourth, and fifth. Vanishing of the fourth is shown via integration by parts, i.e., if $\boldsymbol{n}$ is the unit vector normal to $\partial\mathcal{B}$,

$$\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A = \int_{\partial\mathcal{B}} \boldsymbol{u}^A \, \boldsymbol{\sigma}^B \, \boldsymbol{n} - \int_{\mathcal{B}} \boldsymbol{u}^A \, . \operatorname{div} \boldsymbol{\sigma}^B = 0. \tag{2.34}$$

Indeed, $\operatorname{div} \boldsymbol{\sigma}^B = \mathbf{0}$ in $\mathcal{B}$ as, by virtue of the equilibrium condition given in Eq. (2.3), each stress field has null divergence individually. We remark again that $\boldsymbol{\sigma}^B \boldsymbol{n} = \mathbf{0}$, as the stress field $\boldsymbol{\sigma}^B$ causes no tractions on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$. Then, the sum of the second and third integrals vanishes as it equals the fourth; indeed, by virtue of the relation between corresponding fields, the sum of the arguments of the second and third integrals equals the argument of the fourth integral, i.e.,

$$\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^C - \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* = \boldsymbol{\sigma}^A : (\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*) = \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^B = \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A. \tag{2.35}$$

Vanishing of the fifth integral is shown via integration by parts, similarly to the case of the second integral. Eventually, the elastic strain energy of superimposed fields A and B reduces to

$$\mathcal{W}_{\text{el}} = \int_{\mathcal{B}} \frac{1}{2} \, \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A - \int_{\mathcal{D}} \frac{1}{2} \, \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^*, \tag{2.36}$$

where the last integral has been restricted to $\mathcal{D}$ because the argument vanishes in $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$ (Eq. (2.32c)). The total energy of the superimposed fields B and A is then

$$\mathcal{W} = \mathcal{W}_{\text{el}} + \mathcal{W}_{\text{ext}} = \int_{\mathcal{B}} \frac{1}{2} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A - \int_{\mathcal{D}} \frac{1}{2} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^* - \int_{\partial\mathcal{B}} \boldsymbol{t}^A . \left(\boldsymbol{u}^C + \boldsymbol{u}^A \right), \tag{2.37}$$

where $\boldsymbol{t}^A = \boldsymbol{\sigma}^A \boldsymbol{n}$ on $\partial\mathcal{B}$. The total energy $\mathcal{W}^B$ of the field B is evaluated by setting $\boldsymbol{\sigma}^A = \mathbf{0}$ everywhere in $\mathcal{B}$, and therefore $\boldsymbol{t}^A = \mathbf{0}$ on $\partial\mathcal{B}$ in Eq. (2.37), i.e.,

$$\mathcal{W}^B = \mathcal{W}^B_{\text{el}} + \mathcal{W}^B_{\text{ext}} = \mathcal{W}^B_{\text{el}} = - \int_{\mathcal{D}} \frac{1}{2} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^*, \tag{2.38}$$

where we recall again that $\mathcal{W}^B_{\text{ext}}$ is identically zero, as the inclusion field applies no tractions on the boundary of $\mathcal{B}$. The total energy $\mathcal{W}^A$ of the field A is given by Eq. (2.30) or, equivalently, can be obtained from Eq. (2.37) by setting $\boldsymbol{\epsilon}^* = \mathbf{0}$ in $\partial\mathcal{B}$, which implies that also $\boldsymbol{u}^C$ vanishes. Then, from Eq. (2.29) the interaction energy between the applied stress field $\boldsymbol{\sigma}^A$ and the stress field $\boldsymbol{\sigma}^B$ caused by the presence of the inclusion is then equal to

$$\mathcal{W}^{\text{int}} = \mathcal{W} - \mathcal{W}^A - \mathcal{W}^B = -\int_{\partial\mathcal{B}} \boldsymbol{t}^A.\boldsymbol{u}^C. \tag{2.39}$$

By exploiting the equilibrium condition $\operatorname{div}\boldsymbol{\sigma}^A = \mathbf{0}$ and the vanishing of the second and third integrals in Eq. (2.33), the integral in Eq. (2.39) can be reduced to

$$\begin{aligned}\int_{\partial\mathcal{B}} \boldsymbol{t}^A.\boldsymbol{u}^C &= \int_{\partial\mathcal{B}} \boldsymbol{u}^C\,\boldsymbol{\sigma}^A\,\boldsymbol{n} = \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{h}^C + \int_{\mathcal{B}} \boldsymbol{u}^C \operatorname{div}\boldsymbol{\sigma}^A \\ &= \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*\right) + \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* = \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*,\end{aligned} \tag{2.40}$$

where, in the penultimate passage, we exploited the fact that the symmetry of the stress $\boldsymbol{\sigma}^A$ "filters" the skew-symmetric part of the displacement gradient $\boldsymbol{h}^C$, which can be thus be replaced by the corresponding strain $\boldsymbol{\epsilon}^C$. In this way, the final form of the interaction energy is (Eshelby, 1957; Mura, 1987)

$$\mathcal{W}^{\text{int}} = -\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*. \tag{2.41}$$

An alternative approach for the determination of the interaction energy is shown in Appendix A.2.

2.3.2 "Homogeneous" Case: $\mathfrak{p}$-$\mathbb{S}$ Relation

The configurational force $\mathcal{F}^{\text{conf}}$ is obtained from Eq. (2.31) by computing the variation of the interaction energy in Eq. (2.41), which is defined as

$$-\frac{\partial \mathcal{W}^{\text{int}}}{\partial \boldsymbol{\xi}}(\mathcal{X}) = = \lim_{h\to 0} \frac{\mathcal{W}^{\text{int}}(\mathcal{X} - h\,\boldsymbol{\xi}) - \mathcal{W}^{\text{int}}(\mathcal{X})}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[-\int_{\mathcal{D}} (\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*) \circ (\mathcal{X} - h\,\boldsymbol{\xi}) + \int_{\mathcal{D}} (\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*) \circ \mathcal{X}\right]. \tag{2.42}$$

The variation is performed by expanding the integrand of the first integral to the first order, and considering that, for an ellipsoidal inclusion, the transformation strain $\boldsymbol{\epsilon}^*$ is uniform:

$$\left[(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*) \circ (\mathcal{X} - h\,\boldsymbol{\xi})\right](x) = (\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*)(x - h\,\boldsymbol{\xi}(x))$$
$$= \left[\boldsymbol{\sigma}^A(x - h\,\boldsymbol{\xi}(x))\right] : \boldsymbol{\epsilon}^*$$
$$= \left[\boldsymbol{\sigma}^A(x) - h\,[(\operatorname{grad}\boldsymbol{\sigma}^A)(x)]\boldsymbol{\xi}(x)\right] : \boldsymbol{\epsilon}^*$$
$$= \left[(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*) \circ \mathcal{X} - h\,\boldsymbol{\epsilon}^* : [(\operatorname{grad}\boldsymbol{\sigma}^A)\,\boldsymbol{\xi}]\right](x). \tag{2.43}$$

Substituting Eq. (2.43) into Eq. (2.42) and considering that the displacement $\boldsymbol{\xi}(x) = \boldsymbol{\xi}_0$ is uniform, we have

$$\mathcal{F}^{\text{conf}}.\,\boldsymbol{\xi}_0 = -\frac{\partial \mathcal{W}^{\text{int}}}{\partial \boldsymbol{\xi}}(\mathcal{X}) = \int_{\mathcal{D}} \left[\boldsymbol{\epsilon}^* : \left[(\operatorname{grad}\boldsymbol{\sigma}^A)\,\boldsymbol{\xi}\right]\right] = \left[\int_{\mathcal{D}} \boldsymbol{\epsilon}^* : \operatorname{grad}\boldsymbol{\sigma}^A\right].\boldsymbol{\xi}_0, \tag{2.44}$$

from which, for the arbitrariness of $\boldsymbol{\xi}_0$, we obtain

$$\mathcal{F}^{\text{conf}} = \int_{\mathcal{D}} \boldsymbol{\epsilon}^* : \operatorname{grad}\boldsymbol{\sigma}^A = \int_{\mathcal{D}} \boldsymbol{h}^* : \operatorname{grad}\boldsymbol{\sigma}^A, \tag{2.45}$$

where the symmetry of the stress $\boldsymbol{\sigma}^A$ allows for replacing $\boldsymbol{\epsilon}^*$ with $\boldsymbol{h}^*$. Indeed, regardless of Eq. (2.20), the double contraction with the symmetric stress $\boldsymbol{\sigma}^A$ would in any case "filter" the skew-symmetric part $\boldsymbol{\varphi}^*$ of the transformation displacement gradient $\boldsymbol{h}^* = \boldsymbol{\epsilon}^* + \boldsymbol{\varphi}^*$. It is now convenient to proceed in components, so that Eq. (2.45) reads

$$\mathcal{F}_i^{\mathrm{conf}} = \int_{\mathcal{D}} \sigma^A_{rs,i}\, h^*_{rs} = \int_{\mathcal{D}} \sigma^A_{rs,i}\, u^*_{r,s} \tag{2.46}$$

and, using Leibnitz' rule, we have

$$\mathcal{F}_i^{\mathrm{conf}} = \int_{\mathcal{D}} \left(\sigma^A_{rs,i}\, u^*_r\right)_{,s} - \int_{\mathcal{D}} \left(\sigma^A_{rs,is}\, u^*_i\right). \tag{2.47}$$

The second term vanishes because of equilibrium (i.e., $\operatorname{div} \boldsymbol{\sigma}^A = \mathbf{0}$, from which also $\sigma^A_{rs,is} = \sigma^A_{rs,si} = (\sigma^A_{rs,s})_{,i} = 0$). Thus, use of Gauss' theorem yields

$$\mathcal{F}_i^{\mathrm{conf}} = \int_{\partial\mathcal{D}} \sigma^A_{rs,i}\, u^*_r\, n_s. \tag{2.48}$$

Substituting the particular form of Stokes' theorem in Eq. (2.15) into Eq. (2.48), and employing the relation between configurational force and Eshelby stress in Eq. (2.21), we obtain

$$\begin{aligned} \mathcal{F}_i^{\mathrm{conf}} &= \int_{\partial\mathcal{D}} \left(\sigma^A_{kl}\, u^*_{k,l}\, \delta_{si} - \sigma^A_{rs}\, u^*_{r,i}\right) n_s = \int_{\partial\mathcal{D}} \left(\sigma^A_{kl}\, \epsilon^*_{kl}\, \delta_{si} - \sigma^A_{rs}\, u^*_{r,i}\right) n_s \\ &= \int_{\partial\mathcal{D}} \mathfrak{p}_{is}\, n_s. \end{aligned} \tag{2.49}$$

In coordinate-free notation, this reads

$$\begin{aligned} \mathcal{F}^{\mathrm{conf}} &= \int_{\partial\mathcal{D}} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{h}^*\right) \boldsymbol{i} - (\boldsymbol{h}^*)^T \boldsymbol{\sigma}^A\right] \boldsymbol{n} = \int_{\partial\mathcal{D}} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*\right) \boldsymbol{i} - (\boldsymbol{h}^*)^T \boldsymbol{\sigma}^A\right] \boldsymbol{n} \\ &= \int_{\partial\mathcal{D}} \boldsymbol{\mathfrak{p}}\, \boldsymbol{n}. \end{aligned} \tag{2.50}$$

Finally, using again Gauss' theorem to revert to volume integrals, we get to

$$\mathcal{F}^{\mathrm{conf}} = \int_{\mathcal{D}} \operatorname{div} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*\right) \boldsymbol{i} - (\boldsymbol{h}^*)^T \boldsymbol{\sigma}^A\right] = \int_{\mathcal{D}} \operatorname{div} \boldsymbol{\mathfrak{p}}. \tag{2.51}$$

By localising Eq. (2.51), we obtain

$$\operatorname{div}\left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*\right)\boldsymbol{i} - (\boldsymbol{h}^*)^T\,\boldsymbol{\sigma}^A\right] = \operatorname{div}\mathfrak{p}, \qquad \text{in } \mathcal{D}, \tag{2.52}$$

which can be integrated into

$$\mathfrak{p} = \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*\right)\boldsymbol{i} - (\boldsymbol{h}^*)^T\,\boldsymbol{\sigma}^A\right] + \boldsymbol{c}, \qquad \text{in } \mathcal{D}, \tag{2.53}$$

where $\boldsymbol{c}$ is a tensor field with vanishing divergence, i.e., $\operatorname{div}\boldsymbol{c} = \boldsymbol{0}$. We shall show that we must have

$$\boldsymbol{c} = \mathfrak{p}^A + \mathfrak{p}^B, \qquad \text{in } \mathcal{D}, \tag{2.54}$$

where $\mathfrak{p}^A$ and $\mathfrak{p}^B$ are the Eshelby stresses due to the fields A and B, respectively. Each of $\mathfrak{p}^A$ and $\mathfrak{p}^B$ is given by the standard definition of Eq. (2.22) and, under the hypothesis of homogeneity of the elasticity tensor over $\mathcal{D}$, has vanishing divergence (see Appendix A.1). With Eq. (2.54), the Eshelby stress obtained by substituting Eq. (2.54) into Eq. (2.53),

$$\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*\right)\boldsymbol{i} - (\boldsymbol{h}^*)^T\,\boldsymbol{\sigma}^A\right], \qquad \text{in } \mathcal{D}, \tag{2.55}$$

reduces to $\mathfrak{p}^A$ when $\boldsymbol{\epsilon}^*$ (and thus $\mathfrak{p}^B$) vanishes, and reduces to $\mathfrak{p}^B$ when $\boldsymbol{\sigma}^A$ (and thus $\mathfrak{p}^A$) vanishes. In particular, each of the terms $\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*\right)\boldsymbol{i}$ and $(\boldsymbol{h}^*)^T\,\boldsymbol{\sigma}^A$ vanishes when *either* $\boldsymbol{\epsilon}^*$ or $\boldsymbol{\sigma}^A$ vanishes.

For an ellipsoidal inclusion, we recall that Eq. (2.20) states that $\boldsymbol{h}^* = \boldsymbol{\epsilon}^*$. Therefore, substitution in Eq. (2.55) yields

$$\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*\right)\boldsymbol{i} - (\boldsymbol{\epsilon}^*)^T\,\boldsymbol{\sigma}^A\right], \qquad \text{in } \mathcal{D}. \tag{2.56}$$

At this point, it is possible to relate, within the inclusion $\mathcal{D}$, the Eshelby stress $\mathfrak{p}$ to the fourth-order Eshelby tensor $\mathbb{S}$ by substituting the expression of the transformation strain $\boldsymbol{\epsilon}^*$ as a function of the cancelling strain $\boldsymbol{\epsilon}^C$ from Eq. (2.19) into Eq. (2.56), i.e.,

$$\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \left[\left(\boldsymbol{\sigma}^A : \mathbb{S}^{-1} : \boldsymbol{\epsilon}^C\right)\boldsymbol{i} - \left(\mathbb{S}^{-1} : \boldsymbol{\epsilon}^C\right)^T \boldsymbol{\sigma}^A\right], \qquad \text{in } \mathcal{D}. \tag{2.57}$$

2.4 Inclusion with Misfitting Properties (Inhomogeneous Inclusion)

In Section 2.3, we have dealt with a homogeneous body, where the matrix and inclusion have the same elastic constants. Here, we consider a body $\mathcal{B}$ comprised of a homogeneous matrix with elasticity tensor $\mathbb{L}$, embedding a homogeneous inclusion $\mathcal{D}$ with an elasticity tensor $\mathbb{L}'$ *different* from that of the matrix. However, there is no geometrical misfit (transformation strain) between inclusion and matrix. We note that, although the matrix and the inclusion are both individually homogeneous, in the literature this case is often called “inhomogeneous inclusion”, because the inclusion has different elastic properties than the matrix.

The boundary $\partial\mathcal{B}$ of the body is subjected to the traction force $\boldsymbol{\sigma}^A\boldsymbol{n}$, which imposes the stress field $\boldsymbol{\sigma}^A$ on the body $\mathcal{B}$. Although the inclusion has no geometrical misfit with the surrounding matrix, and therefore does not cause stress in the absence of external tractions, it perturbs the imposed stress field $\boldsymbol{\sigma}^A$. Eshelby (1957) proposed the method of the *equivalent “homogeneous” inclusion*, in which the inclusion with no geometrical misfit but properties different from those of the matrix is replaced by an inclusion with geometrical misfit and same properties of the matrix, that gives rise to the same perturbation of the applied stress field that the original inclusion would cause. In order to describe the fictitious inclusion with elastic properties $\mathbb{L}$, which has a geometrical misfit that makes it equivalent to the real inclusion with properties $\mathbb{L}'$, Eshelby (1957) introduces the *fictitious* or *equivalent* transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ (Figure 2.2). In this case (inclusion with different elastic properties than the matrix, so-called inhomogeneous inclusion), the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ plays the role of the real transformation strain $\boldsymbol{\epsilon}^*$ of the case of inclusion with geometrical misfit only (the so-called homogeneous inclusion). Thus, by analogy

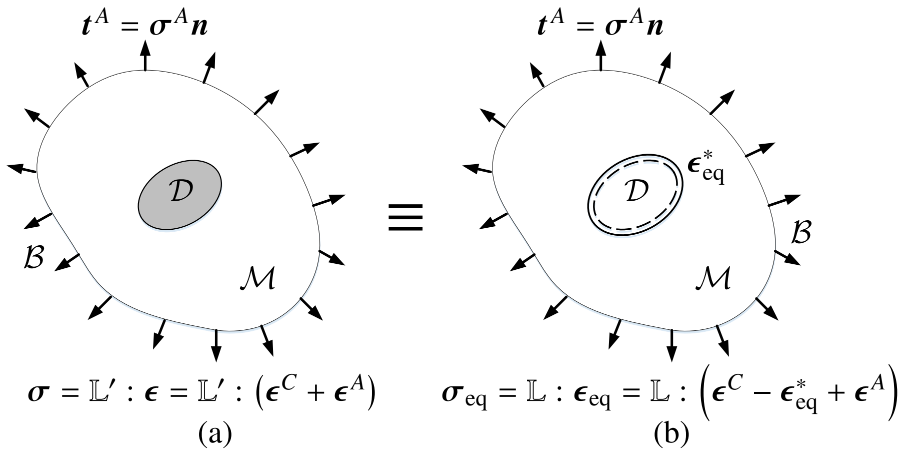

Figure 2.2: (a) the real system, and (b) the equivalent system.

with Eqs. (2.32), we can write the strain in the *equivalent system* as

$$\boldsymbol{\epsilon}_{\text{eq}} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^B_{\text{eq}} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}}, \tag{2.58a}$$

$$\boldsymbol{\epsilon}^*_{\text{eq}} \neq \mathbf{0}, \quad \text{in } \mathcal{D}, \tag{2.58b}$$

$$\boldsymbol{\epsilon}^*_{\text{eq}} = \mathbf{0}, \quad \text{in } \mathcal{M} = \mathcal{B} \setminus \mathcal{D}, \tag{2.58c}$$

$$[\![\boldsymbol{\epsilon}^C]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}, \tag{2.58d}$$

$$[\![\boldsymbol{\epsilon}_{\text{eq}}]\!] = [\![\boldsymbol{\epsilon}^A]\!] + [\![\boldsymbol{\epsilon}^B_{\text{eq}}]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}. \tag{2.58e}$$

In the *real system*, the perturbation in the strain field caused by the fact that the inclusion has different properties than the matrix will be equal to the cancelling strain $\boldsymbol{\epsilon}^C$ of the equivalent system. Thus, the strain in the real system is

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^B = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C, \tag{2.59a}$$

$$[\![\boldsymbol{\epsilon}^C]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}, \tag{2.59b}$$

$$[\![\boldsymbol{\epsilon}]\!] = [\![\boldsymbol{\epsilon}^A]\!] + [\![\boldsymbol{\epsilon}^B]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}. \tag{2.59c}$$

The stresses in the real and equivalent systems must be equal, which implies

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{eq}}, \quad \text{in } \mathcal{D} \qquad \Rightarrow$$
$$\mathbb{L}' : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C\right) = \mathbb{L} : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}}\right), \quad \text{in } \mathcal{D}, \tag{2.60}$$

where we have used the linear constitutive equation with $\mathbb{L}'$ for the real inclusion and with $\mathbb{L}$ for the equivalent inclusion. In the equivalent system, the definition (2.19) of the Eshelby fourth-order tensor becomes

$$\boldsymbol{\epsilon}^C = \mathbb{S} : \boldsymbol{\epsilon}^*_{\text{eq}}, \quad \text{in } \mathcal{D}. \tag{2.61}$$

By substituting Eq. (2.61) into the right-hand side of Eq. (2.60) (the right-hand side represents the *equivalent* system), we obtain

$$\mathbb{L}' : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C\right) = \mathbb{L} : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \mathbb{S}^{-1} : \boldsymbol{\epsilon}^C\right), \quad \text{in } \mathcal{D}, \tag{2.62}$$

which can be rearranged into

$$\mathbb{L} : \mathbb{S}^{-1} : \boldsymbol{\epsilon}^C + (\mathbb{L}' - \mathbb{L}) : \boldsymbol{\epsilon}^C = -(\mathbb{L}' - \mathbb{L}) : \boldsymbol{\epsilon}^A, \quad \text{in } \mathcal{D}. \tag{2.63}$$

We now double-contract both sides of Eq. (2.63) on the left by $\mathbb{S} : \mathbb{L}^{-1}$ and get to

$$\left[\mathbb{I} + \mathbb{S} : [\mathbb{L}^{-1} : \mathbb{L}' - \mathbb{I}]\right] : \boldsymbol{\epsilon}^C = -\mathbb{S} : \left[\mathbb{L}^{-1} : \mathbb{L}' - \mathbb{I}\right] : \boldsymbol{\epsilon}^A, \quad \text{in } \mathcal{D}, \tag{2.64}$$

where $\mathbb{I}$ is the symmetric fourth-order identity tensor with components $\mathbb{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ (see, e.g., (Walpole, 1981)). Next, by introducing the strain concentration tensor $\mathbb{A}$ (Walpole, 1966a,b, 1969; Weng, 1984, 1990) defined by

$$\mathbb{A} = \left[\mathbb{I} + \mathbb{S} : [\mathbb{L}^{-1} : \mathbb{L}' - \mathbb{I}]\right]^{-1}, \quad \text{in } \mathcal{D}, \tag{2.65}$$

and substituting into Eq. (2.64), we obtain

$$\mathbb{A}^{-1} : \boldsymbol{\epsilon}^C = (\mathbb{I} - \mathbb{A}^{-1}) : \boldsymbol{\epsilon}^A, \quad \text{in } \mathcal{D}, \tag{2.66}$$

and the cancelling strain $\boldsymbol{\epsilon}^C$ becomes

$$\boldsymbol{\epsilon}^C = (\mathbb{A} - \mathbb{I}) : \boldsymbol{\epsilon}^A, \quad \text{in } \mathcal{D}. \tag{2.67}$$

The transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ in the equivalent inclusion can be determined by substituting Eq. (2.61) into Eq. (2.67), i.e,

$$\boldsymbol{\epsilon}^*_{\text{eq}} = \mathbb{S}^{-1} : (\mathbb{A} - \mathbb{I}) : \boldsymbol{\epsilon}^A, \quad \text{in } \mathcal{D}. \tag{2.68}$$

We notice that the strain concentration tensor $\mathbb{A}$ defined in Eq. (2.65) reduces to the identity $\mathbb{I}$ when $\mathbb{L}' = \mathbb{L}$, in which case the cancelling strain $\boldsymbol{\epsilon}^C$ and the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ vanish identically (Eqs. (2.67) and (2.68), respectively).

The most important difference between the “homogeneous inclusion” and the “inhomogeneous inclusion” is that, while in the former case the transformation strain causes a stress also in the absence of external tractions, in the latter the *equivalent* transformation strain is *proportional* to the strain arising from the imposed external stress, as shown in Eq. (2.68). Because of its linearity, Eq. (2.68) agrees with the fact that, for an inclusion with no geometrical misfit, the stress is zero in the absence of external tractions. This aspect arises also in terms of energy, as we shall see in Section 2.4.1, and has been emphasised also by Eshelby himself (Eshelby, 1957) (see the last paragraph of page 388).

Finally, if we add $\boldsymbol{\epsilon}^A$ to both sides of Eq. (2.67), we obtain

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C = \mathbb{A} : \boldsymbol{\epsilon}^A, \quad \text{in } \mathcal{D}, \tag{2.69}$$

where $\boldsymbol{\epsilon}$ is the (real) total strain. This justifies the meaning of *strain concentration tensor* for $\mathbb{A}$: tensor $\mathbb{A}$, indeed, captures the change in strain due to the presence of an inclusion with different material properties than the matrix. Clearly, since in the limit $\mathbb{L}' \to \mathbb{L}$ we have that $\mathbb{A} \to \mathbb{I}$, it follows that $\boldsymbol{\epsilon} \to \boldsymbol{\epsilon}^A$.

Remark. The derivation presented above starts from the stress equivalence condition (2.60). Traditionally, one obtains Eq. (2.69) with the strain concentration tensor under traction boundary conditions and a dual relation with a *stress concentration tensor* in terms of stresses under displacement boundary conditions. As noted by Weng (1984), the two formulations (strain concentration tensor and stress concentration tensor) imply each other. A detailed proof of this equivalence is presented in Appendix A.3.

2.4.1 "Inhomogeneous" Case: Interaction Energy

The interaction energy $\mathcal{W}^{\text{int}}$ is determined as the difference between the total energy $\mathcal{W}$ of the superimposed fields A and B and the energy of the individual field $\mathcal{W}^A$. We must keep in mind that, in the real system (Fig. 2.2a), inclusion and the matrix are both stress-free in the absence of external tractions and, therefore, the elastic energy $\mathcal{W}^B = \mathcal{W}^B_{\text{el}}$ due to the field B alone (i.e., in the absence of field A caused by the external tractions) is identically zero. Thus, the interaction energy in Eqs. (2.29) reduces to

$$\mathcal{W}^{\text{int}} = \mathcal{W} - \mathcal{W}^A - \mathcal{W}^B = \mathcal{W} - \mathcal{W}^A, \tag{2.70a}$$

$$\mathcal{W} = \mathcal{W}_{\text{el}} + \mathcal{W}_{\text{ext}}, \tag{2.70b}$$

$$\mathcal{W}^A = \mathcal{W}^A_{\text{el}} + \mathcal{W}^A_{\text{ext}}, \tag{2.70c}$$

$$\mathcal{W}^B = \mathcal{W}^B_{\text{el}} = 0. \tag{2.70d}$$

The superimposed elastic strain energy of the field A due to the external tractions and of the field B due to the presence of the inclusion is given by

$$\begin{aligned}\mathcal{W}_{\text{el}} &= \frac{1}{2}\int_{\mathcal{B}} \left(\boldsymbol{\sigma}^A + \boldsymbol{\sigma}^B\right) : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C\right), \\ &= \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^C + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^C. \end{aligned} \tag{2.71}$$

The third and fourth integrals vanish as seen in Eq. (2.34). For the second integral, we need to consider that, contrary to the case of the "homogeneous" inclusion, $\boldsymbol{\sigma}^A$ and $\boldsymbol{\sigma}^B$ are *not* corresponding fields, because the inclusion region $\mathcal{D}$ has elasticity tensor $\mathbb{L}'$ different from the elasticity tensor $\mathbb{L}$ of the matrix $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$. Thus, we need to exploit Eshelby's method of the equivalent homogeneous inclusion. The second integral can be manipulated into

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^C &= \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}}\right) + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \\ &= \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\epsilon}^A : \mathbb{L} : \left(\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}}\right) + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \\ &= \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}}, \end{aligned} \tag{2.72}$$

where we used $\mathbb{L} : \boldsymbol{\epsilon}^A = \boldsymbol{\sigma}^A$ and the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ of the method of the equivalent homogeneous inclusion (which has elasticity tensor $\mathbb{L}$), so that

$$\boldsymbol{\sigma}^B \equiv \boldsymbol{\sigma}^B_{\text{eq}} = \left(\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}}\right) : \mathbb{L} = \mathbb{L} : \left(\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}}\right) \tag{2.73}$$

is the stress field B in the real system, which is identically equal to that in the equivalent system by construction. The first integral on the right-hand side of Eq. (2.73) equals the third of Eq. (2.71) and therefore vanishes as shown in Eq. (2.34). Therefore, Eq. (2.72) becomes

$$\frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^C = \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}}, \tag{2.74}$$

Substitution of Eq. (2.74) into Eq. (2.71) yields

$$\mathcal{W}_{\text{el}} = \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \tag{2.75}$$

The total superimposed elastic strain energy is finally evaluated as

$$\mathcal{W} = \mathcal{W}_{\text{el}} + \mathcal{W}_{\text{ext}} = \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} - \int_{\partial\mathcal{B}} \boldsymbol{t}^A . \left(\boldsymbol{u}^C + \boldsymbol{u}^A\right). \tag{2.76}$$

The total energy $\mathcal{W}^A$ of the field A is given by Eq. (2.30) or, similarly as it has been done in Section 2.3.1, it can be obtained from Eq. (2.76) by setting $\boldsymbol{\epsilon}^*_{\text{eq}} = \boldsymbol{0}$ in $\partial\mathcal{B}$, which implies that $\boldsymbol{u}^C$ vanishes too. Consequently, from Eq. (2.70), the interaction energy between the applied stress field $\boldsymbol{\sigma}^A$ and the stress field $\boldsymbol{\sigma}^B$ caused by the presence of the inclusion is

$$\mathcal{W}^{\text{int}} = \mathcal{W} - \mathcal{W}^A = \frac{1}{2}\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} - \int_{\partial\mathcal{B}} \boldsymbol{t}^A . \boldsymbol{u}^C. \tag{2.77}$$

Now, we substitute the second term with Eq. (2.40), using $\boldsymbol{\epsilon}^*_{\text{eq}}$ in place of $\boldsymbol{\epsilon}^*$, and finally obtain (Eshelby, 1957)

$$\mathcal{W}^{\text{int}} = -\frac{1}{2}\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}}. \tag{2.78}$$

As noticed by Eshelby (see the comment following Equation (4-10) in (Eshelby, 1957)), the interaction energy for the "inhomogeneous" case is half of that of the "homogeneous" case, which we have reported in Eq. (2.41). This is because the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ of Eq. (2.78) is not given as $\boldsymbol{\epsilon}^*$ in Eq. (2.41), but is linearly related to the applied strain $\boldsymbol{\epsilon}^A = \mathbb{L}^{-1} : \boldsymbol{\sigma}^A$, via Eq. (2.68), from which the usual factor $\frac{1}{2}$ of linear elasticity. We emphasised the difference between the "real" transformation strain of the "homogeneous inclusion" and the fictitious or equivalent one of the "inhomogeneous inclusion" by assigning a different symbol to each. We believe that keeping the distinction also helps keeping track of the various strains in the case of the "general inclusion", in which $\boldsymbol{\epsilon}^*$ and $\boldsymbol{\epsilon}^*_{\text{eq}}$ coexist and are completely distinct.

2.4.2 "Inhomogeneous" Case: $\mathfrak{p}$-$\mathbb{S}$ Relation

As for the "homogeneous" case, the configurational force $\mathcal{F}^{\text{conf}}$ is obtained from Eq. (2.31) or Eq. (2.27) by computing the variation of the interaction energy in Eq. (2.78). Since the interaction energy Eq. (2.78) is exactly half of that of the "homogeneous" case of Eq. (2.41), with the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ in place of the "real" transformation strain $\boldsymbol{\epsilon}^*$, the configurational force can be promptly found by analogy with Eq. (2.51) as

$$\mathcal{F}^{\text{conf}} = \int_{\mathcal{D}} \frac{1}{2} \operatorname{div} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \right) \boldsymbol{i} - \left(\boldsymbol{h}^*_{\text{eq}} \right)^T \boldsymbol{\sigma}^A \right] = \int_{\mathcal{D}} \operatorname{div} \mathfrak{p}. \tag{2.79}$$

By localising, we have

$$\tfrac{1}{2} \operatorname{div} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \right) \boldsymbol{i} - \left(\boldsymbol{h}^*_{\text{eq}} \right)^T \boldsymbol{\sigma}^A \right] = \operatorname{div} \mathfrak{p}, \quad \text{in } \mathcal{D}, \tag{2.80}$$

which can be integrated into

$$\mathfrak{p} = \tfrac{1}{2} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \right) \boldsymbol{i} - \left(\boldsymbol{h}^*_{\text{eq}} \right)^T \boldsymbol{\sigma}^A \right] + \boldsymbol{c}, \quad \text{in } \mathcal{D}, \tag{2.81}$$

where, as in the "homogeneous" case, $\boldsymbol{c}$ is a tensor field with vanishing divergence. In this case, however, the constant equals the Eshelby stress of the applied field A, i.e.,

$$\boldsymbol{c} = \mathfrak{p}^A, \quad \text{in } \mathcal{D}. \tag{2.82}$$

Indeed, since the real system of Fig. 2.2a is stress-free in the absence of external tractions, the Cauchy stress $\boldsymbol{\sigma}^B$ and the elastic energy $\frac{1}{2}\boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^B$ vanish identically, and so does the corresponding Eshelby stress $\mathfrak{p}^B$. Thus, the Eshelby stress in the inclusion is evaluated by substituting the Eq. (2.82) into

Eq. (2.81), as

$$\mathfrak{p} = \mathfrak{p}^A + \left[\frac{1}{2} \left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \right) \boldsymbol{i} - \frac{1}{2} \left(\boldsymbol{h}^*_{\text{eq}} \right)^T \boldsymbol{\sigma}^A \right], \quad \text{in } \mathcal{D}. \tag{2.83}$$

Eq. (2.83) shows that, in the limiting case $\mathbb{L}' \to \mathbb{L}$, i.e., no "inhomogeneous" inclusion, the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ vanishes according to Eq. (2.68)) and, therefore, the Eshelby stress reduces to $\mathfrak{p}^A$. Moreover, if the stress field $\boldsymbol{\sigma}^A$ vanishes, the associated Eshelby stress $\mathfrak{p}^A$ vanishes, and so does the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ (Eq. (2.68), with $\boldsymbol{\epsilon}^A = \mathbb{L}^{-1} : \boldsymbol{\sigma}^A$).

By applying Eq. (2.20) to the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$, we obtain $\boldsymbol{h}^*_{\text{eq}} \equiv \boldsymbol{\epsilon}^*_{\text{eq}}$, and Eq. (2.83) becomes

$$\mathfrak{p} = \mathfrak{p}^A + \left[\frac{1}{2} \left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \right) \boldsymbol{i} - \frac{1}{2} \left(\boldsymbol{\epsilon}^*_{\text{eq}} \right)^T \boldsymbol{\sigma}^A \right], \quad \text{in } \mathcal{D}. \tag{2.84}$$

Now, the Eshelby stress $\mathfrak{p}$ and the fourth-order Eshelby tensor $\mathbb{S}$ can be related by substituting the expression of the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$ from Eq. (2.68) into Eq. (2.84), i.e.,

$$\mathfrak{p} = \mathfrak{p}^A + \left[\frac{1}{2} \left(\boldsymbol{\sigma}^A : \mathbb{S}^{-1} : (\mathbb{A} - \mathbb{I}) : \boldsymbol{\epsilon}^A \right) \boldsymbol{i} - \frac{1}{2} \left(\mathbb{S}^{-1} : (\mathbb{A} - \mathbb{I}) : \boldsymbol{\epsilon}^A \right)^T \boldsymbol{\sigma}^A \right], \quad \text{in } \mathcal{D}, \tag{2.85}$$

Note that Eq. (2.85) expresses the Eshelby stress $\mathfrak{p}$ as a function of the applied stress $\boldsymbol{\sigma}^A$, the fourth-order Eshelby $\mathbb{S}$, of the elasticity tensor $\mathbb{L}$ of the matrix (via the applied strain $\boldsymbol{\epsilon}^A$ and the strain concentration tensor $\mathbb{A}$), and of the elasticity tensor $\mathbb{L}'$ of the inclusion (via $\mathbb{A}$).

2.5 Inclusion with Misfitting Geometry and Properties (General Inclusion)

We now study the case of an inclusion $\mathcal{D}$ having elasticity tensor $\mathbb{L}'$, different from the elasticity tensor $\mathbb{L}$ of the matrix, and having *also* has a geometrical misfitting represented by the transformation strain $\boldsymbol{\epsilon}^*$. We refer to this case as to the "general inclusion", as the "homogeneous" and "inhomogeneous" cases can be obtained in the limit $\mathbb{L}' \to \mathbb{L}$ and $\boldsymbol{\epsilon}^* \to \mathbf{0}$, respectively. As in the previous two cases, we assume that the boundary $\partial\mathcal{B}$ of the body $\mathcal{B}$ is subjected to the external tractions $\boldsymbol{t}^A$, giving rise to the applied stress field $\boldsymbol{\sigma}^A$, such that $\boldsymbol{t}^A = \boldsymbol{\sigma}^A \boldsymbol{n}$ on the boundary.

As for the case of the "inhomogeneous inclusion", we employ Eshelby's method of the "equivalent homogeneous inclusion" in order to represent the misfit in material properties. This involves the construction of an equivalent system in which a transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$, equivalent to the misfit in material properties, adds up to the *real* transformation strain $\boldsymbol{\epsilon}^*$ due to the geometrical misfit. Thus, the total strain in the *equivalent system* is defined by

$$\boldsymbol{\epsilon}_{\text{eq}} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^B_{\text{eq}} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}} - \boldsymbol{\epsilon}^*, \tag{2.86a}$$

$$\boldsymbol{\epsilon}^*_{\text{eq}}, \boldsymbol{\epsilon}^* \neq \mathbf{0}, \quad \text{in } \mathcal{D}, \tag{2.86b}$$

$$\boldsymbol{\epsilon}^*_{\text{eq}}, \boldsymbol{\epsilon}^* = \mathbf{0}, \quad \text{in } \mathcal{M} = \mathcal{B} \setminus \mathcal{D}, \tag{2.86c}$$

$$[\![\boldsymbol{\epsilon}^C]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}, \tag{2.86d}$$

$$[\![\boldsymbol{\epsilon}_{\text{eq}}]\!] = [\![\boldsymbol{\epsilon}^A]\!] + [\![\boldsymbol{\epsilon}^B_{\text{eq}}]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}. \tag{2.86e}$$

In the *real system*, the total strain is evaluated as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^B = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*, \tag{2.87a}$$

$$\boldsymbol{\epsilon}^* \neq \mathbf{0}, \quad \text{in } \mathcal{D}, \tag{2.87b}$$

$$\boldsymbol{\epsilon}^* = \mathbf{0}, \quad \text{in } \mathcal{M} = \mathcal{B} \setminus \mathcal{D}, \tag{2.87c}$$

$$[\![\boldsymbol{\epsilon}^C]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}, \tag{2.87d}$$

$$[\![\boldsymbol{\epsilon}]\!] = [\![\boldsymbol{\epsilon}^A]\!] + [\![\boldsymbol{\epsilon}^B]\!] \neq \mathbf{0}, \quad \text{on } \partial\mathcal{D}. \tag{2.87e}$$

Thus, by imposing the equivalence of stresses in the real and equivalent system, we obtain

$$\begin{gathered} \boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{eq}}, \quad \text{in } \mathcal{D} \qquad \Rightarrow \\ \mathbb{L}' : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*\right) = \mathbb{L} : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\text{eq}} - \boldsymbol{\epsilon}^*\right), \quad \text{in } \mathcal{D}. \end{gathered} \tag{2.88}$$

As usual, for an ellipsoidal inclusion, $\boldsymbol{\epsilon}^*$ and $\boldsymbol{\epsilon}^*_{\text{eq}}$ are uniform fields. Thus, the cancelling strain $\boldsymbol{\epsilon}^C$ in the inclusion (Eq. (2.19)) is redefined in terms of the sum of the two uniform transformation strains $\boldsymbol{\epsilon}^*$ and $\boldsymbol{\epsilon}^*_{\text{eq}}$ and Eshelby's tensor $\mathbb{S}$ (Mura, 1987) as

$$\boldsymbol{\epsilon}^C = \mathbb{S} : \left(\boldsymbol{\epsilon}^*_{\text{eq}} + \boldsymbol{\epsilon}^*\right), \quad \text{in } \mathcal{D}. \tag{2.89}$$

Substituting Eq. (2.89) into the right-hand side of Eq. (2.88) (again, the right-hand side represents the *equivalent* system), we obtain

$$\mathbb{L}' : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*\right) = \mathbb{L} : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \mathbb{S}^{-1} : \boldsymbol{\epsilon}^C\right), \quad \text{in } \mathcal{D}, \tag{2.90}$$

which we rearrange into

$$\mathbb{L} : \mathbb{S}^{-1} : \boldsymbol{\epsilon}^C + (\mathbb{L}' - \mathbb{L}) : \boldsymbol{\epsilon}^C = -(\mathbb{L}' - \mathbb{L}) : \boldsymbol{\epsilon}^A + \mathbb{L}' : \boldsymbol{\epsilon}^*, \quad \text{in } \mathcal{D}, \tag{2.91}$$

Note that Eq. (2.91) is identical to Eq. (2.61) of the "inhomogeneous" case, except for the additional term $\mathbb{L}' : \boldsymbol{\epsilon}^*$ on the right-hand side, due to the "real" transformation strain $\boldsymbol{\epsilon}^*$ in the "real" system.

Now, similarly to what we have done in Section 2.4, we double-contract both sides of Eq. (2.91) by $\mathbb{S} : \mathbb{L}^{-1}$ on the left, use the definition of strain concentration tensor $\mathbb{A}$ given in Eq. (2.65), and solve for $\boldsymbol{\epsilon}^C$:

$$\boldsymbol{\epsilon}^C = (\mathbb{A} - \mathbb{I}) : (\boldsymbol{\epsilon}^A - \boldsymbol{\epsilon}^*) + \mathbb{A} : \mathbb{S} : \boldsymbol{\epsilon}^*, \quad \text{in } \mathcal{D}. \tag{2.92}$$

Then, according to Eq. (2.89), the uniform transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}} + \boldsymbol{\epsilon}^*$ in the inclusion of the equivalent system is simply

$$\boldsymbol{\epsilon}^*_{\text{eq}} + \boldsymbol{\epsilon}^* = \mathbb{S}^{-1} : \left[(\mathbb{A} - \mathbb{I}) : (\boldsymbol{\epsilon}^A - \boldsymbol{\epsilon}^*) + \mathbb{A} : \mathbb{S} : \boldsymbol{\epsilon}^*\right], \quad \text{in } \mathcal{D}. \tag{2.93}$$

The (real) total strain $\boldsymbol{\epsilon}$ in the inclusion is obtained by adding $\boldsymbol{\epsilon}^A - \boldsymbol{\epsilon}^*$ to both sides of Eq. (2.92):

$$\boldsymbol{\epsilon} = \mathbb{A} : [\boldsymbol{\epsilon}^A + (\mathbb{S} - \mathbb{I}) : \boldsymbol{\epsilon}^*], \quad \text{in } \mathcal{D}. \tag{2.94}$$

To the best of our knowledge, the relations (2.92) and (2.94), providing the cancelling and total strains, respectively, in the "general" inclusion, have not been given explicitly in the literature.

Clearly, if the "real" transformation strain $\boldsymbol{\epsilon}^*$ vanishes in Eq. (2.92), we obtain the cancelling strain of the "inhomogeneous" inclusion (Eq. (2.67)). Moreover, in the limit $\mathbb{L}' \to \mathbb{L}$, we have $\mathbb{A} \to \mathbb{I}$, from which it follows that $\boldsymbol{\epsilon}^*_{\text{eq}} \to \mathbf{0}$ from Eq. (2.93), and therefore that the cancelling strain $\boldsymbol{\epsilon}^C$ reduces to that of the "homogeneous" inclusion (Eq. (2.19)). Finally, if the applied stress $\boldsymbol{\sigma}^A$ and thus the resulting strain $\boldsymbol{\epsilon}^A$ vanish, we have, manipulating Eq. (2.92),

$$\boldsymbol{\epsilon}^C = [\mathbb{A} : (\mathbb{S} - \mathbb{I}) + \mathbb{I}] : \boldsymbol{\epsilon}^*, \quad \text{in } \mathcal{D}. \tag{2.95}$$

2.5.1 "General" Case: Interaction Energy

We shall show that the interaction energy for the "general" inclusion equals the sum of the interaction energies of the "homogeneous" and "inhomogeneous" cases. The elastic strain energy of the superposed applied field (A) and field caused by the inclusion (B) is given by

$$\begin{aligned}
\mathcal{W}_{\text{el}} &= \frac{1}{2}\int_{\mathcal{B}} \left(\boldsymbol{\sigma}^A + \boldsymbol{\sigma}^B\right) : \left(\boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*\right), \\
&= \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^C - \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* + \\
&+ \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A + \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^C - \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^*.
\end{aligned} \tag{2.96}$$

Eq. (2.96) looks identical to Eq. (2.33) of the "homogeneous" case. However, as in the "inhomogeneous" case, the difference with the "homogeneous" case is that A and B are *not* corresponding fields, because the inclusion has a *different* elasticity tensor than the matrix. Thus, while the fourth and fifth integrals vanish as shown in Eq. (2.34), the sum of the second and third integrals does *not* vanish, and is given by

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{B}}\left(\boldsymbol{\epsilon}^{C}-\boldsymbol{\epsilon}^{*}\right):\boldsymbol{\sigma}^{A} &= \frac{1}{2}\int_{\mathcal{B}}\left(\boldsymbol{\epsilon}^{C}-\boldsymbol{\epsilon}^{*}\right):\mathbb{L}:\boldsymbol{\epsilon}^{A},\\ &= \frac{1}{2}\int_{\mathcal{B}}\left(\boldsymbol{\epsilon}^{C}-\boldsymbol{\epsilon}^{*}-\boldsymbol{\epsilon}^{*}_{\text{eq}}+\boldsymbol{\epsilon}^{*}_{\text{eq}}\right):\mathbb{L}:\boldsymbol{\epsilon}^{A},\\ &= \frac{1}{2}\int_{\mathcal{B}}\boldsymbol{\sigma}^{B}:\boldsymbol{\epsilon}^{A}+\frac{1}{2}\int_{\mathcal{B}}\boldsymbol{\sigma}^{A}:\boldsymbol{\epsilon}^{*}_{\text{eq}},\end{aligned}\tag{2.97}$$

where $\mathbb{L}:\boldsymbol{\epsilon}^{A}=\boldsymbol{\sigma}^{A}$, $\boldsymbol{\epsilon}^{*}_{\text{eq}}$ is the equivalent transformation strain introduced in the method of the equivalent homogeneous inclusion, and

$$\boldsymbol{\sigma}^{B}\equiv\boldsymbol{\sigma}^{B}_{\text{eq}}=\left(\boldsymbol{\epsilon}^{C}-\boldsymbol{\epsilon}^{*}_{\text{eq}}-\boldsymbol{\epsilon}^{*}\right):\mathbb{L}=\mathbb{L}:\left(\boldsymbol{\epsilon}^{C}-\boldsymbol{\epsilon}^{*}_{\text{eq}}-\boldsymbol{\epsilon}^{*}\right),\tag{2.98}$$

is the stress field B in the real system, which by construction coincides with that in the equivalent system. The first integral on the right-hand side of Eq. (2.98) equals the fourth of Eq. (2.96) and therefore vanishes as shown in Eq. (2.34). Therefore, Eq. (2.97) becomes

$$\frac{1}{2}\int_{\mathcal{B}}\left(\boldsymbol{\epsilon}^{C}-\boldsymbol{\epsilon}^{*}\right):\boldsymbol{\sigma}^{A}=\frac{1}{2}\int_{\mathcal{B}}\boldsymbol{\sigma}^{A}:\boldsymbol{\epsilon}^{*}_{\text{eq}},\tag{2.99}$$

which, substituted into Eq. (2.96), gives

$$\mathcal{W}_{\text{el}}=\frac{1}{2}\int_{\mathcal{B}}\boldsymbol{\sigma}^{A}:\boldsymbol{\epsilon}^{A}+\frac{1}{2}\int_{\mathcal{D}}\boldsymbol{\sigma}^{A}:\boldsymbol{\epsilon}^{*}_{\text{eq}}-\frac{1}{2}\int_{\mathcal{D}}\boldsymbol{\sigma}^{B}:\boldsymbol{\epsilon}^{*},\tag{2.100}$$

where the integrals containing $\boldsymbol{\epsilon}^{*}$ and $\boldsymbol{\epsilon}^{*}_{\text{eq}}$ have been restricted to $\mathcal{D}$ because $\boldsymbol{\epsilon}^{*}$ and $\boldsymbol{\epsilon}^{*}_{\text{eq}}$ vanish outside of $\mathcal{D}$.

The total energy of superimposed fields A and B is then

$$\begin{aligned}\mathcal{W}&=\mathcal{W}_{\text{el}}+\mathcal{W}_{\text{ext}}\\ &=\frac{1}{2}\int_{\mathcal{B}}\boldsymbol{\sigma}^{A}:\boldsymbol{\epsilon}^{A}+\frac{1}{2}\int_{\mathcal{D}}\boldsymbol{\sigma}^{A}:\boldsymbol{\epsilon}^{*}_{\text{eq}}-\frac{1}{2}\int_{\mathcal{D}}\boldsymbol{\sigma}^{B}:\boldsymbol{\epsilon}^{*}-\int_{\partial\mathcal{B}}\boldsymbol{t}^{A}.\left(\boldsymbol{u}^{C}+\boldsymbol{u}^{A}\right).\end{aligned}\tag{2.101}$$

The total energy $\mathcal{W}^A$ of the field A is determined from Eq. (2.101) by setting $\boldsymbol{\epsilon}^*_{\mathrm{eq}} = \mathbf{0}$ and $\boldsymbol{\epsilon}^* = \mathbf{0}$ in $\partial\mathcal{B}$, i.e., in the absence of the inclusion. Therefore,

$$\mathcal{W}^A = \mathcal{W}^A_{\mathrm{el}} + \mathcal{W}^A_{\mathrm{ext}} = \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^A - \int_{\partial\mathcal{B}} \boldsymbol{t}^A.\boldsymbol{u}^A. \tag{2.102}$$

The total energy $\mathcal{W}^B$ of the field B is obtained by setting $\boldsymbol{\sigma}^A = \mathbf{0}$ in Eq. (2.101), i.e., absence of external tractions. Thus,

$$\mathcal{W}^B = -\frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^*. \tag{2.103}$$

Then, from Eq. (2.29), the interaction energy is

$$\mathcal{W}^{\mathrm{int}} = \mathcal{W} - \mathcal{W}^A - \mathcal{W}^B = \frac{1}{2}\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\mathrm{eq}} - \int_{\partial\mathcal{B}} \boldsymbol{t}^A.\boldsymbol{u}^C. \tag{2.104}$$

Recalling that $\operatorname{div}\boldsymbol{\sigma}^A = \mathbf{0}$ and using again Eq. (2.98), the surface integral can be written

$$\begin{aligned}
\int_{\partial\mathcal{B}} \boldsymbol{t}^A.\boldsymbol{u}^C &= \int_{\partial\mathcal{B}} \boldsymbol{u}^C \boldsymbol{\sigma}^A \boldsymbol{n} = \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{h}^C = \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^C \\
&= \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\mathrm{eq}} - \boldsymbol{\epsilon}^*\right) + \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^*_{\mathrm{eq}} + \boldsymbol{\epsilon}^*\right) \\
&= \int_{\mathcal{B}} \left(\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^*_{\mathrm{eq}} - \boldsymbol{\epsilon}^*\right) : \mathbb{L} : \boldsymbol{\epsilon}^A + \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^*_{\mathrm{eq}} + \boldsymbol{\epsilon}^*\right) \\
&= \int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A + \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^*_{\mathrm{eq}} + \boldsymbol{\epsilon}^*\right) \\
&= \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^*_{\mathrm{eq}} + \boldsymbol{\epsilon}^*\right),
\end{aligned} \tag{2.105}$$

where, again, the integral of $\boldsymbol{\sigma}^B : \boldsymbol{\epsilon}^A$ vanishes as seen in Eq. (2.34). Finally, the interaction energy takes the form

$$\mathcal{W}^{\mathrm{int}} = -\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* - \frac{1}{2}\int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\mathrm{eq}}, \tag{2.106}$$

which equals the sum of the energies calculated for the “homogeneous” case (Eq. (2.41)) and the "inhomogeneous" case (Eq. (2.78)).

Remark. The expressions of the interaction energy obtained by Mura (1987) and Balluffi (2012) are slightly different (and also different with respect to each other). Indeed, while we considered the interaction energy to be given by $\mathcal{W}^{\text{int}} = \mathcal{W} - \mathcal{W}^A - \mathcal{W}^B$, as in Eq. (2.29), they used $\mathcal{W}^{\text{int}} = \mathcal{W} - \mathcal{W}^A$. That is, they set $\mathcal{W}^B = 0$, as in the case of the "inhomogeneous" inclusion. We believe that this is incorrect, because the "general" inclusion *does* have a non-zero, *real* transformation strain $\boldsymbol{\epsilon}^*$, and thus *does* have a non-zero stress field $\boldsymbol{\sigma}^B$ and energy $\mathcal{W}^B$ in the absence of the applied stress field $\boldsymbol{\sigma}^A$, where $\boldsymbol{\sigma}^B$ arises from the cancelling strain of Eq. (2.95).

2.5.2 "General" Case: $\mathfrak{p}$-$\mathbb{S}$ Relation

Since the interaction energy for the "general" case equals the sum of the interaction energies of the "homogeneous" and "inhomogeneous" cases, so does the configurational force, which can be evaluated by summing Eq. (2.51) and Eq. (2.79), as

$$\begin{aligned} \mathcal{F}^{\text{conf}} &= \int_{\mathcal{D}} \frac{1}{2} \operatorname{div} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \right) \boldsymbol{i} - \left(\boldsymbol{h}^*_{\text{eq}} \right)^T \boldsymbol{\sigma}^A \right] \\ &+ \int_{\mathcal{D}} \operatorname{div} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* \right) \boldsymbol{i} - \left(\boldsymbol{h}^* \right)^T \boldsymbol{\sigma}^A \right] = \int_{\mathcal{D}} \operatorname{div} \mathfrak{p}. \end{aligned} \tag{2.107}$$

With considerations similar to those made in the "homogeneous" and "inhomogeneous" cases, Eq. (2.107) can be integrated into

$$\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \tfrac{1}{2} \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^*_{\text{eq}} \right) \boldsymbol{i} - \left(\boldsymbol{h}^*_{\text{eq}} \right)^T \boldsymbol{\sigma}^A \right] + \left[\left(\boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* \right) \boldsymbol{i} - \left(\boldsymbol{h}^* \right)^T \boldsymbol{\sigma}^A \right]. \tag{2.108}$$

where, as we have already seen, $\mathfrak{p}^A$ and $\mathfrak{p}^B$ have zero divergence. By applying Eq. (2.20) to both the "real" transformation strain $\boldsymbol{\epsilon}^*$ and the equivalent transformation strain $\boldsymbol{\epsilon}^*_{\text{eq}}$, and rearranging, we obtain

$$\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \left[\boldsymbol{\sigma}^A : \left(\tfrac{1}{2} \boldsymbol{\epsilon}^*_{\text{eq}} + \boldsymbol{\epsilon}^* \right) \boldsymbol{i} - \left(\tfrac{1}{2} \boldsymbol{\epsilon}^*_{\text{eq}} + \boldsymbol{\epsilon}^* \right)^T \boldsymbol{\sigma}^A \right], \quad \text{in } \mathcal{D}, \tag{2.109}$$

which is convenient to write as

$$\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \left[\frac{1}{2}\,\boldsymbol{\sigma}^A : \left(\boldsymbol{\epsilon}^*_{\text{eq}} + \boldsymbol{\epsilon}^* + \boldsymbol{\epsilon}^*\right)\boldsymbol{i} - \frac{1}{2}\left(\boldsymbol{\epsilon}^*_{\text{eq}} + \boldsymbol{\epsilon}^* + \boldsymbol{\epsilon}^*\right)^T \boldsymbol{\sigma}^A\right]. \quad \text{in } \mathcal{D}. \tag{2.110}$$

The relationship between the Eshelby stress $\mathfrak{p}$ and the fourth-order Eshelby tensor $\mathbb{S}$ is obtained by use of Eq. (2.93) as

$$\begin{aligned}\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \tfrac{1}{2}\Big[\boldsymbol{\sigma}^A : \big[\mathbb{S}^{-1} : \big[(\mathbb{A} - \mathbb{I}) : (\boldsymbol{\epsilon}^A - \boldsymbol{\epsilon}^*) + \mathbb{A} : \mathbb{S} : \boldsymbol{\epsilon}^*\big] + \boldsymbol{\epsilon}^*\big]\Big]\,\boldsymbol{i} \\ - \tfrac{1}{2}\Big[\,\big[\mathbb{S}^{-1} : \big[(\mathbb{A} - \mathbb{I}) : (\boldsymbol{\epsilon}^A - \boldsymbol{\epsilon}^*) + \mathbb{A} : \mathbb{S} : \boldsymbol{\epsilon}^*\big] + \boldsymbol{\epsilon}^*\big]^T \boldsymbol{\sigma}^A\Big], \quad \text{in } \mathcal{D}.\end{aligned} \tag{2.111}$$

It is easy to verify that, in the limiting case $\mathbb{L}' \to \mathbb{L}$, for which $\mathbb{A} \to \mathbb{I}$, Eq. (2.111) reduces to the expression of the Eshelby stress in Eq. (2.56) for the "homogeneous inclusion" and, in the limiting case $\boldsymbol{\epsilon}^* \to \mathbf{0}$, for which $\mathfrak{p}^B \to \mathbf{0}$, Eq. (2.111) reduces to the expression of the Eshelby stress in Eq. (2.85) of the "inhomogeneous inclusion".

Chapter 3

Eshelby's Inclusion Theory in the Light of Noether's Theorem

This chapter is based on Federico et al. (2019)

In a classical paper, Eshelby (1951) introduced the concept of *configurational force* as the force required for a region containing a defect in a material body to undergo a *material* virtual displacement. This idea led to the mechanical Maxwell energy-momentum tensor that has been subsequently termed *Eshelby stress* in continuum mechanics (Maugin and Trimarco, 1992). The procedure followed by Eshelby (1951) is comprised of a set of operations in which the elastic energy in the interior of a region and the net work that the surface tractions exert of the region are evaluated individually. In another work, Eshelby (1975) used Hamilton's standard variational approach of field theory and found his energy momentum-stress directly, using the components of the regular spatial displacement and of the displacement gradient as the entities called *fields* in the jargon of field theory. In the same paper, Eshelby (1975) also sketched the procedure for the case in which the *fields* are the components of the configuration map, which is the common choice in modern continuum mechanics.

Although initially conceived for a single inclusion or for a discrete set of inclusions, Eshelby's theory naturally applies to *inhomogeneous materials* or materials with continuous distributions of defects. Epstein and Maugin (1990) obtained the Eshelby stress using the concepts of material uniformity and material isomorphism introduced by Noll (1967) for inhomogeneous materials. Gurtin (1995, 1999) reformulated and generalised Eshelby's approach with the method of the varying control volumes and considered the Eshelby stress as the appropriate stress of an *independent* material balance law. The Eshelby stress has been seen as the object capturing inhomogeneities and singularities (e.g., Epstein and Maugin, 1990; Gurtin, 1995, 1999; Epstein and Maugin, 2000; Epstein and Elżanowski, 2007; Verron et al., 2009; Weng and Wong, 2009; Maugin, 2011), or the *driving force* of phenomena of material evolution such as plasticity and growth-remodelling (e.g., Maugin and Epstein, 1998; Epstein and Maugin, 2000; Cermelli et al., 2001; Epstein, 2002; Imatani and Maugin, 2002; Grillo et al., 2003, 2005; Epstein, 2009, 2015; Grillo et al., 2016, 2017; Hamedzadeh et al., 2019), or phase transitions, or evolution of the interfaces among phases (e.g., Gurtin, 1986, 1993; Gurtin and Podio-Guidugli, 1996; Fried and Gurtin, 1994, 2004).

In a didactic spirit, the aim of this work is to reproduce the results of Eshelby (1951, 1975) *directly* by means of the classical Noether's theorem (for a translation into English of Noether's original 1918 paper, see Noether, 1971) for continuum systems, as presented by Hill (1951). The derivation is made using the components of the configuration map as the "fields" and those of the deformation gradient as the "gradients of the fields", while an appropriate "topological" transformation represents the material virtual displacement on the region containing the defect. We would like to emphasise that this work is more than a mere rewrite of Eshelby's findings in a more modern notation. While the relation between Eshelby's work and Noether's theorem has been highlighted in several papers (e.g., Knowles and Sternberg, 1972; Eshelby, 1975; Fletcher, 1976; Edelen, 1981; Golebiewska Herrmann, 1982; Olver, 1984a,b; Huang and Batra, 1996; Kienzler and Herrmann, 2000; Maugin, 2011), to the best of our knowledge, no work in the literature establishes an explicit relation between Eshelby's inclusion theory (and, specifically, the procedure to deal with the presence of the inclusion; Eshelby, 1951, 1975) and Noether's theorem.

In Section 3.1, we introduce the notation and give some basic definitions. In particular, we introduce standard and Eshelbian configurations and their *variations*, i.e., displacement fields. The setting is declaredly differential geometrical, although we avoid using differentiable manifolds for simplicity. In Section 3.2, we review, with our notation and within a suitable geometrical setting, Eshelby's original derivation (Eshelby, 1951) of configurational forces. Similarly, in Section 3.3, we review Eshelby's variational derivation (Eshelby, 1975). Finally, in Section 3.4, which is the core of the work, we introduce Noether's theorem, and show how its application renders directly the results of both the previous derivations.

3.1 Theoretical Background

In this section, we illustrate the notation that we employ and report some fundamental results relevant to this work. We generally use index-free notation but sometimes it is useful to show the corresponding expression in index notation. Therefore, we present most expressions in both notations. In index notation, the customary Einstein's summation convention for repeated indices is enforced throughout and a subscript preceded by a comma, as in $f_{,i}$, denotes partial differentiation with respect to its i-th argument.

3.1.1 General Notation and Basic Definitions

Here we review some basic definitions of continuum mechanics, in order to elucidate the notation that we employ. The notation is essentially that of Truesdell and Noll (1965) and Marsden and Hughes (1983), with some modifications (Federico, 2012; Federico et al., 2016). We work in a simplified setting based on the use of affine spaces, whose rigorous definition can be found, e.g., in the treatise by Epstein (2010). We could use a presentation in terms of differentiable manifolds (Noll, 1967; Marsden and Hughes, 1983; Epstein, 2010; Segev, 2013), but using affine spaces avoids many of the intricacies of higher-level differential geometry and makes the presentation more intuitive.

An affine space is a set $\mathcal{S}$, called the point space, considered together with a vector space $\mathcal{V}$, called the modelling space, and a mapping $\mathcal{S} \times \mathcal{S} \to \mathcal{V} : (x, y) \mapsto y - x = \boldsymbol{u}$. This means that, at every point $x \in \mathcal{S}$, it is possible to univocally attach the vector given by $\boldsymbol{u} = y - x$, for every point $y \in \mathcal{S}$. The set of all vectors emanating from point x is a vector space denoted $T_x\mathcal{S} = \{\boldsymbol{u} \in \mathcal{V} : \boldsymbol{u} = y - x,\ \forall y \in \mathcal{S}\}$ and called *tangent space* to $\mathcal{S}$ at x. In the differential geometrical definition, the tangent space $T_x\mathcal{S}$ is the set of the vectors that are each *tangent* at x to one of the infinite possible regular curves $c : [a, b] \to \mathcal{S} : s \mapsto c(s)$ such that $c(s_0) = x$, where $s_0 \in]a, b[$, i.e., the vectors (see Figure 3.1)

$$\boldsymbol{u} = \lim_{h\to 0} \frac{c(s_0 + h) - c(s_0)}{h} = c'(s_0) \in T_x\mathcal{S}. \tag{3.1}$$

For the case of an affine space $\mathcal{S}$, this definition of tangent space $T_x\mathcal{S}$ coincides with that given by the expression $\boldsymbol{u} = y - x$. Indeed, by varying the curve c passing by x, we obtain all possible "tip points" y of the tangent vectors defined as $\boldsymbol{u} = y - x$. The dual space of $T_x\mathcal{S}$, i.e., the vector space of all linear maps $\boldsymbol{\varphi} : T_x\mathcal{S} \to \mathbb{R}$, is denoted $T_x^\star\mathcal{S}$ and is called the *cotangent space* to $\mathcal{S}$ at x. The disjoint unions of all tangent and cotangent spaces are called *tangent bundle* $T\mathcal{S}$ and *cotangent bundle* $T^\star\mathcal{S}$, respectively.

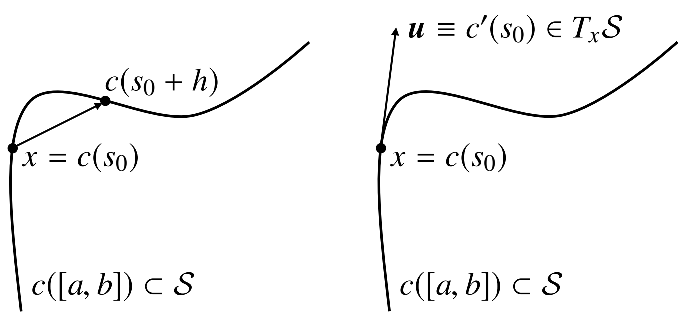

Figure 3.1: Differential geometrical definition of tangent vector at a point $x \in \mathcal{S}$. Left: the *secant* vector $c(s_0 + h) - c(s_0)$ passing by $x = c(s_0)$. Right: the tangent vector $\boldsymbol{u} = c'(s_0)$ at $x = c(s_0)$, obtained as the limit of the secant.

Vector fields and covector fields (or fields of one-forms) on an open set $\mathcal{A} \subseteq \mathcal{S}$ are maps

$$\boldsymbol{u} : \mathcal{A} \subseteq \mathcal{S} \to T\mathcal{S} : x \mapsto \boldsymbol{u}(x) \in T_x\mathcal{S}, \tag{3.2a}$$

$$\boldsymbol{\varphi} : \mathcal{A} \subseteq \mathcal{S} \to T^\star\mathcal{S} : x \mapsto \boldsymbol{\varphi}(x) \in T_x^\star\mathcal{S}, \tag{3.2b}$$

and tensor fields of higher order are defined analogously. Rather than speaking of contractions of vectors and covectors in a specific tangent and cotangent space, we can directly speak of the contractions of vector fields and covector fields in the tangent and cotangent bundle, and we denote the contraction by means of simple juxtaposition, i.e.,

$$\boldsymbol{\varphi}\, \boldsymbol{u} = \boldsymbol{u}\, \boldsymbol{\varphi} = \varphi_a\, u^a. \tag{3.3}$$

The physical space $\mathcal{S}$ is equipped with a metric tensor $\boldsymbol{g}$, a symmetric and positive definite second-order tensor field defining the scalar product of two vector fields as

$$\boldsymbol{g} : T\mathcal{S} \times T\mathcal{S} \to \mathbb{R} : (\boldsymbol{u}, \boldsymbol{v}) \mapsto \langle \boldsymbol{u}, \boldsymbol{v} \rangle \equiv \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = u^a\, g_{ab}\, v^b. \tag{3.4}$$

We always assume to employ the Levi-Civita connection, i.e., the covariant derivative associated with the metric tensor $\boldsymbol{g}$ via the Christoffel symbols given by (see, e.g., Marsden and Hughes, 1983)

$$\gamma^a_{bc} = \tfrac{1}{2}\, g^{ad}(g_{cd,b} + g_{bd,c} - g_{bc,d}), \tag{3.5}$$

which are symmetric in their lower indices, i.e., $\gamma^a_{bc} = \gamma^a_{cb}$. The covariant derivative $\nabla_{\boldsymbol{u}} \boldsymbol{v}$ of the vector field $\boldsymbol{v}$ in the direction of the vector field $\boldsymbol{u}$ has the component expression

$$[\nabla_{\boldsymbol{u}} \boldsymbol{v}]^a \equiv v^a{}_{|b}\, u^b = v^a{}_{,b}\, u^b + \gamma^a_{bc}\, v^c\, u^b. \tag{3.6}$$

and defines the *gradient* $\operatorname{grad} \boldsymbol{v}$ as the tensor field such that its definition as a linear map is $(\operatorname{grad} \boldsymbol{v})\boldsymbol{u} \equiv \nabla_{\boldsymbol{u}} \boldsymbol{v}$, with components $[\operatorname{grad} \boldsymbol{v}]^a{}_b = v^a{}_{|b}$. The covariant derivative and the gradient of a tensor field of arbitrary order are defined analogously.

Remark. A scalar is a *tensor of order zero* and thus we find more natural to use the convention adopted by, e.g., Epstein (2010, see page 116) and to consider the gradient of a scalar field f as the *covector* field (or one-form) $\operatorname{grad} f$ such that $(\operatorname{grad} f)(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} f$, as for a tensor of any other order.

Accordingly, the components of grad f are $f_{,a}$. The other possible convention is that adopted by Marsden and Hughes (1983, see page 69), according to which the gradient of f is the vector field with components $g^{ab} f_{,b}$. Note that, in either case, since f is a tensor of order zero, the Christoffel symbols of the connection are *not* involved in the gradient, which is thus connection-independent. There are several advantages in defining the gradient as a covector. First, this definition is *metric-independent*, whereas the vector definition clearly necessitates that a metric tensor $\boldsymbol{g}$ be defined. Second, the covector definition accommodates the analytical mechanical definition of force as a covector field: indeed, an integrable force is the negative of the gradient of a potential energy and is thus consistently represented as a covector field. Finally, with the covector definition of grad f, we have the remarkable chain of identities

$$\nabla f \equiv \operatorname{grad} f \equiv \mathrm{d}f \equiv \mathrm{D}f, \tag{3.7}$$

where $\mathrm{d}f$ is the *exterior derivative* of f, when seen as a zero-form (see, e.g., Epstein, 2010, page 116), and $\mathrm{D}f$ is the *Fréchet derivative* (or tangent map) of f, when seen as a point map from $\mathcal{A} \subset \mathcal{S}$ into $\mathbb{R}$.

In the following, the physical space $\mathcal{S}$ is identified with the affine space $\mathbb{E}^3$, which is $\mathbb{R}^3$ considered both as the point space and as the modelling vector space.

3.1.2 Bodies, Configurations and the Deformation Gradient

In the simplified presentation that we adopt, a deformable continuous body $\mathcal{B}$ is identified with one of its placements in the physical space $\mathcal{S}$, and this particular placement is called reference configuration. The body is assumed to be endowed with the material metric $\boldsymbol{G}$, which induces the corresponding Levi-Civita connection, similarly to what seen for the spatial metric $\boldsymbol{g}$.

A *configuration*, or *deformation*, of the body is an *embedding*

$$\phi : \mathcal{B} \to \mathcal{S} : X \mapsto x = \phi(X), \tag{3.8}$$

i.e., a map such that its codomain-restriction $\phi : \mathcal{B} \to \phi(\mathcal{B})$ is a diffeomorphism, i.e., a continuos and differentiable map, which is invertible, with continuous and differentiable inverse $\Phi \equiv \phi^{-1} : \phi(\mathcal{B}) \to \mathcal{B}$. The configuration ϕ maps *material* points $X = (X^1, X^2, X^3)$ in the body $\mathcal{B}$ into *spatial* points $x = (x^1, x^2, x^3)$ in $\mathcal{S}$, i.e., $\phi(X) = x$.

Since we are going to introduce another class of configurations, called *Eshelbian*, we shall refer to the standard definition of configuration given above as to a *conventional configuration*. The set of all k-times differentiable conventional configuration maps (with $k \in \mathbb{N}$) constitutes the *conventional* configuration space $\mathcal{C}$ of the body $\mathcal{B}$. Since $\mathcal{S}$ is an affine space, the space $C^k(\mathcal{B}, \mathcal{S})$ of the k-times differentiable maps from $\mathcal{B}$ into $\mathcal{S}$ is an infinite-dimensional affine space. Thus, considering $\mathcal{C}$ as an open set in $C^k(\mathcal{B}, \mathcal{S})$ (Marsden and Hughes, 1983) makes $\mathcal{C}$ an infinite-dimensional trivial manifold. A tangent vector $\boldsymbol{\eta}$ in the functional tangent space $T_\phi\mathcal{C}$ can be thought of as the tangent at ϕ to a curve of maps in $\mathcal{C}$ (i.e., a one-parameter family of maps in $\mathcal{C}$), and is a vector field *covering* the configuration ϕ, i.e.,

$$\boldsymbol{\eta} : \mathcal{B} \to T\mathcal{S} : X \mapsto \boldsymbol{\eta}(X) \in T_{\phi(X)}\mathcal{S} = T_x\mathcal{S}. \tag{3.9}$$

The vector field $\boldsymbol{\eta}$ is called a (conventional) *displacement field* (and, when compatible with the constraints, but not necessarily attained by the body, it is called a *virtual displacement*). Figure 3.2 shows the displacement $\boldsymbol{\eta}(X) = \boldsymbol{\eta}(\Phi(x))$ as a tangent vector at $T_x\mathcal{S}$ and an illustration of the configuration space with the displacement field $\boldsymbol{\eta}$ as a tangent vector at $T_\phi\mathcal{C}$.

The deformation gradient at point X is the *tangent map* of ϕ, i.e., the tensor

$$(T\phi)(X) = \boldsymbol{F}(X) : T_X\mathcal{B} \to T_x\mathcal{S}, \tag{3.10}$$

with $x = \phi(X)$, expressing the Fréchet derivative of ϕ at X. Since the existence of the Fréchet derivative of ϕ implies the existence of its Gâteaux derivative (or directional derivative), $\boldsymbol{F}(X)$ can

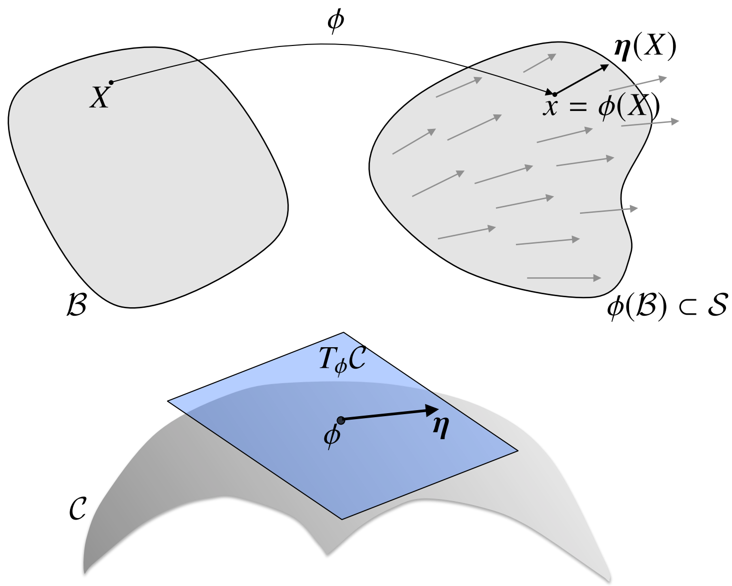

Figure 3.2: A conventional displacement field. Top: The displacement $\boldsymbol{\eta}(X) = \boldsymbol{\eta}(\Phi(x)) \in T_x\mathcal{S}$ as a tangent vector attached at $x = \phi(X)$. Bottom: The displacement field $\boldsymbol{\eta}$ as a tangent vector attached at the configuration ϕ, which is a point in the configuration space $\mathcal{C}$, here depicted as a surface, for the sake of an intuitive graphical representation.

be defined through the limit

$$(\partial_{\boldsymbol{M}}\phi)(X) := \lim_{h\to 0} \frac{\phi(X + h\,\boldsymbol{M}) - \phi(X)}{h} = [(T\phi)(X)]\boldsymbol{M} = [\boldsymbol{F}(X)]\boldsymbol{M}, \tag{3.11}$$

and the Gâteaux derivative $\partial_{\boldsymbol{M}}\phi(X)$ of ϕ with respect to *any* tangent vector $\boldsymbol{M} \in T_X\mathcal{B}$ equals the Fréchet derivative $\boldsymbol{F}(X)\boldsymbol{M}$, which is linear in $\boldsymbol{M}$. In components, Equation (3.11) reads

$$(\partial_{\boldsymbol{M}}\phi)^a(X) = (T\phi)^a{}_B(X)\,M^B = F^a{}_B(X)\,M^B = \phi^a{}_{,B}(X)\,M^B, \tag{3.12}$$

where we recall that the comma denotes partial differentiation. Note that $\boldsymbol{F}(X)$ is a two-point tensor as it has the *domain leg* in $T_X\mathcal{B}$ and the *codomain leg* in $T_x\mathcal{S}$. As a tensor field, the deformation gradient is

$$\boldsymbol{F} : \mathcal{B} \to T\mathcal{S} \otimes T^\star\mathcal{B}. \tag{3.13}$$

The deformation gradient $\boldsymbol{F}$ pushes-forward material vector fields $\boldsymbol{M}$ with components M^A into spatial vector fields $\phi_*\boldsymbol{M} = (\boldsymbol{F} \circ \Phi)\,(\boldsymbol{M} \circ \Phi)$ with components $(F^a{}_A \circ \Phi)\,(M^A \circ \Phi)$. The inverse $\boldsymbol{F}^{-1}$ pulls-back spatial vector fields $\boldsymbol{m}$ with components m^a into material vector fields $\phi^*\boldsymbol{m} = (\boldsymbol{F}^{-1} \circ \phi)\,(\boldsymbol{m} \circ \phi)$ with components $((\boldsymbol{F}^{-1})^A{}_a \circ \phi)\,(m^a \circ \phi)$. The transpose $\boldsymbol{F}^T$ pulls-back spatial covector fields $\boldsymbol{\pi}$ with components π_a into material covector fields $\phi^*\boldsymbol{\pi} = (\boldsymbol{F}^T \circ \phi)\,(\boldsymbol{\pi} \circ \phi)$ with components $((\boldsymbol{F}^T)_A{}^a \circ \phi)\,(\pi_a \circ \phi) = F^a{}_A\,(\pi_a \circ \phi)$. The inverse transpose $\boldsymbol{F}^{-T}$ pushes-forward material covector fields $\boldsymbol{\Pi}$ with components Π_A into spatial covector fields $\phi_*\boldsymbol{\Pi} = (\boldsymbol{F}^{-T} \circ \Phi)\,(\boldsymbol{\Pi} \circ \Phi)$ with components $((\boldsymbol{F}^{-T})_a{}^A \circ \Phi)\,(\Pi_A \circ \Phi) = (\boldsymbol{F}^{-1})^A{}_a\,(\Pi_A \circ \Phi)$.

The determinant $J = \det \boldsymbol{F}$ has the meaning of *volume ratio*, in the spirit of the theorem of the change of variables applied to the transformation from the spatial region $\phi(\mathcal{R}) \subset \mathcal{S}$ to the corresponding material region $\mathcal{R} \subset \mathcal{B}$.

3.1.3 Eshelbian Configurations and Their Tangent Maps

Grillo et al. (2003) introduced the concept of *admissible reference configuration set* of a body as the set of all reference configurations obtained by appl ying a diffeomorphism to the reference configuration $\mathcal{B}$ representing the body (which has some similarities with the idea of boundary reparametresations introduced by Gurtin, 1995). Here, we make use of this concept in a slightly different way.

An *Eshelbian configuration* $\mathcal{Y}$ is a diffeomorphism on the body $\mathcal{B}$. Since we define the body $\mathcal{B}$ as a trivial manifold, i.e., an open subset of the physical space $\mathcal{S}$, the codomain of an Eshelbian configuration $\mathcal{Y}$ should be the whole space $\mathcal{S}$ and the image would be an open set $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B}) \subset \mathcal{S}$. However, if the body $\mathcal{B}$ were a non-trivial manifold, the image $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B})$ would be another non-trivial manifold. To keep the notation as general as possible, we prefer to avoid declaring $\mathcal{S}$ as the codomain of $\mathcal{Y}$. Rather, we consider all admissible diffeomorphisms $\mathcal{Y}$, each with its image $\tilde{\mathcal{B}}$, and we obtain the collection of all admissible reference configurations $\tilde{\mathcal{B}}$, which clearly also contains $\mathcal{B}$ itself (see also Grillo et al., 2003). Then, we consider the union $\mathcal{N} = \cup_{\mathcal{Y}} \tilde{\mathcal{B}}$ of these mutually diffeomorphic sets $\tilde{\mathcal{B}}$, and define the generic Eshelbian configuration as

$$\mathcal{Y} : \mathcal{B} \to \mathcal{N} : X \mapsto \tilde{X} = \mathcal{Y}(X), \tag{3.14}$$

which has the further notational advantage of not tying $\mathcal{Y}$ to its specific image $\tilde{\mathcal{B}}$.

Analogously to the case of a conventional configuration, the tangent map of an Eshelbian configuration at point X is the tensor

$$(T\mathcal{Y})(X) : T_X\mathcal{B} \to T_{\tilde{X}}\mathcal{N}, \tag{3.15}$$

with $\tilde{X} = \mathcal{Y}(X)$. Again, $(T\mathcal{Y})(X)$ is the Fréchet derivative of $\mathcal{Y}$ at X and, since $\mathcal{Y}$ is a diffeomorphism, $(T\mathcal{Y})(X)$ can be computed by means of the Gâteaux derivative of $\mathcal{Y}$ at X, i.e.,

$$(\partial_{\boldsymbol{M}}\mathcal{Y})(X) = \lim_{h\to 0} \frac{\mathcal{Y}(X + h\,\boldsymbol{M}) - \mathcal{Y}(X)}{h} = [(T\mathcal{Y})(X)]\boldsymbol{M}. \tag{3.16}$$

The *material identity map* is the particular case of Eshelbian configuration obtained by considering that $\mathcal{B} \subset \mathcal{N}$, and is defined as

$$\mathcal{X} : \mathcal{B} \to \mathcal{B} : X \mapsto X = \mathcal{X}(X), \tag{3.17}$$

with the component representation

$$\mathcal{X}^A : \mathcal{B} \to \mathbb{R} : X \mapsto X^A = \mathcal{X}^A(X) \equiv \mathcal{X}^A(X^1, X^2, X^3). \tag{3.18}$$

Its tangent map is clearly the (material) identity tensor in $T\mathcal{B}$, i.e.,

$$T\mathcal{X} = \boldsymbol{I} : T\mathcal{B} \to T\mathcal{B}, \qquad (T\mathcal{X})^A{}_B = \mathcal{X}^A{}_{,B} = \delta^A{}_B. \tag{3.19}$$

Also in the case of Eshelbian configurations, we can exploit the affine structure of $\mathcal{S}$: since all sets

$\tilde{\mathcal{B}}$ are open subsets of $\mathcal{S}$, also $\mathcal{N} = \cup_{\mathcal{Y}}\tilde{\mathcal{B}} \subseteq \mathcal{S}$ is an open set, and thus we can define the space of all Eshelbian configurations as an open subset $\mathcal{M}$ of the infinite-dimensional affine space $C^k(\mathcal{B},\mathcal{N})$, which makes $\mathcal{M}$ an infinite-dimensional trivial manifold.

Remark. In our setting, in which the physical space $\mathcal{S}$ is an affine space and a body $\mathcal{B}$ is a subset of $\mathcal{S}$, the distinction between a conventional configuration $\phi : \mathcal{B} \to \mathcal{S}$ and an Eshelbian configuration $\mathcal{Y} : \mathcal{B} \to \mathcal{N}$ seems to fade out, because $\mathcal{N} = \cup_{\mathcal{Y}}\tilde{\mathcal{B}} \subseteq \mathcal{S}$. However this is not the case, as will become clear from the explanation given in Section 3.2 (see also Figures 3.3 and 3.4). Moreover, when $\mathcal{B}$ is a general manifold, the distinction is fundamental. In this case, while a conventional configuration ϕ remains an embedding of $\mathcal{B}$ in $\mathcal{S}$, i.e., it gives $\mathcal{B}$ a placement $\phi(\mathcal{B}) \subset \mathcal{S}$, an Eshelbian configuration transforms the manifold $\mathcal{B}$ into *a different* manifold $\tilde{\mathcal{B}}$, which – like $\mathcal{B}$ – is *not* necessarily a subset of $\mathcal{S}$.

A tangent vector $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$ is a vector field

$$\boldsymbol{U} : \mathcal{B} \to T\mathcal{B} : X \mapsto \boldsymbol{U}(X) \in T_X\mathcal{B}, \tag{3.20}$$

and is called a *material displacement* field. When an Eshelbian configuration $\mathcal{Y} : \mathcal{B} \to \mathcal{N}$, is defined as a *perturbation* of the material identity $\mathcal{X}$, i.e.,

$$\mathcal{Y}(X) = \mathcal{X}(X) + h\,\boldsymbol{U}(X) = X + h\,\boldsymbol{U}(X), \quad \mathcal{Y}^A(X) = \mathcal{X}^A(X) + h\,U^A(X) = X^A + h\,U^A(X), \tag{3.21}$$

where $h \in \mathbb{R}$ is a smallness parameter and $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$, it is called an "infinitesimal transformation of the coordinates", in the language of field theory. Omitting the argument X, we can write

$$\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}, \qquad \mathcal{Y}^A = \mathcal{X}^A + h\,U^A. \tag{3.22}$$

The tangent map of $\mathcal{Y}$ in Equation (3.22) is expressed by

$$T\mathcal{Y} = T\mathcal{X} + h\,\mathrm{Grad}\,\boldsymbol{U} = \boldsymbol{I} + h\,\mathrm{Grad}\,\boldsymbol{U}, \quad (T\mathcal{Y})^A{}_B = (T\mathcal{X})^A{}_B + h\,U^A{}_{|B} = \delta^A{}_B + h\,U^A{}_{|B}, \tag{3.23}$$

where $\boldsymbol{I}$ is the material identity tensor and $\operatorname{Grad}\boldsymbol{U}$, with components $U^A{}_{|B}$, is the gradient (or covariant derivative) of $\boldsymbol{U}$. For $h \to 0$, the Jacobian determinant of $T\mathcal{Y}$ is

$$\begin{aligned} \det(T\mathcal{Y}) &= \det(\boldsymbol{I} + h \operatorname{Grad}\boldsymbol{U}) = 1 + h \operatorname{Tr}(\operatorname{Grad}\boldsymbol{U}) + o(h) \\ &= 1 + h \operatorname{Div}\boldsymbol{U} + o(h) = 1 + h\, U^A{}_{|A} + o(h). \end{aligned} \tag{3.24}$$

3.1.4 Conventions on Forces and Stresses

As mentioned in Remark 3.1.1, in the analytical mechanics / field theory approach, followed by, e.g., Hill (1951) and Eshelby (1975), forces are regarded as *covector* fields, acting on velocity or displacement vector fields. Thus, the contraction of a force with a velocity or displacement is given precisely by (3.3). Consequently, the first leg of the stress (the "force leg") is a *covector*, while the second leg (the "area leg") is a *vector*. Indeed, in the expression of Cauchy's theorem, the traction vectors relative to the spatial and material elements of area are given by

$$\boldsymbol{t_n} = \boldsymbol{\sigma}\, \boldsymbol{n}, \quad \mathbf{t}_{\boldsymbol{N}} = \boldsymbol{P}\boldsymbol{N}, \qquad (\boldsymbol{t_n})_a = \sigma_a{}^b\, n_b, \quad (\mathbf{t}_{\boldsymbol{N}})_a = P_a{}^B\, N_B. \tag{3.25}$$

In Equation (3.25), $\boldsymbol{n}$ is the normal covector to a surface element at the spatial point $x = \phi(X)$ in the current configuration, $\boldsymbol{N}$ is the normal covector to the corresponding surface element at the material point X in the reference configuration and the first Piola-Kirchhoff stress is related to Cauchy stress by means of the backward Piola transformation

$$\boldsymbol{P} = J\,(\boldsymbol{\sigma} \circ \phi)\, \boldsymbol{F}^{-T}, \qquad P_a{}^B = J\,(\sigma_a{}^b \circ \phi)\,(\boldsymbol{F}^{-T})_b{}^B. \tag{3.26}$$

Equations (3.25) and (3.26) show that the tractions $\boldsymbol{t_n}$ and $\mathbf{t}_{\boldsymbol{N}}$ are indeed *covectors* if the Cauchy stress $\boldsymbol{\sigma}$ and the first Piola-Kirchhoff stress $\boldsymbol{P}$, respectively, are treated as "mixed" tensors (we remark that $\mathbf{t}_{\boldsymbol{N}} \neq \boldsymbol{t_n}$, since $\boldsymbol{N}$ is related to $\boldsymbol{n}$ by the formula of the change of area, also known as Nanson's formula; see, e.g., Bonet and Wood, 2008).

3.2 Eshelby's Original Derivation of the Weak Form

Eshelby (1951) derived the weak form of the expression of the configurational force balance by means of a thought experiment subdivided in several steps. This form is weak as it is an integral equation expressing a *virtual work*. We note that, in this section, we define the total energy $\mathcal{E}_{\mathcal{D}}$ in a region $\mathcal{D}$ of the body as a functional on the manifold $\mathcal{M}$, the Eshelbian configuration space.

Eshelby (1951) considered a body $\mathcal{B}$, subjected to constraints and external loads, and in whose interior is located a *defect* of any kind: a point defect, a dislocation, an inclusion, or even a region in which the material properties are inhomogeneous. To fix ideas, we follow Eshelby's graphical example with a point defect, as shown in Figure 3.3. The left panel in Figure 3.3 shows what Eshelby called the *original* body, in which a region $\mathcal{D}$ (highlighted in dark grey), bounded by the smooth material surface $\Sigma = \partial\mathcal{D}$, is selected such that the defect is contained in $\mathcal{D}$. The right panel in Figure 3.3 represents a *replica* of the original body, in which a different region $\tilde{\mathcal{D}}$ (also highlighted in dark grey), bounded by the smooth material surface $\tilde{\Sigma} = \partial\tilde{\mathcal{D}}$, is selected so that the defect is contained in $\tilde{\mathcal{D}}$ (see also Kienzler and Herrmann, 2000). Since Σ and $\tilde{\Sigma}$ are both smooth, it is always possible to find an Eshelbian configuration $\mathcal{Y}$ transforming $\mathcal{D}$ into $\tilde{\mathcal{D}}$, i.e., $\mathcal{Y}(\mathcal{D}) = \tilde{\mathcal{D}}$. Moreover, if Σ and $\tilde{\Sigma}$ are "close enough", then $\tilde{\mathcal{D}}$ is obtainable from $\mathcal{D}$ through a perturbation of the form defined in Equation (3.21), whose domain restriction to $\mathcal{D}$ is

$$\mathcal{Y} : \mathcal{D} \to \mathcal{B} : X \mapsto \mathcal{Y}(X) = \mathcal{X}(X) + h\,\boldsymbol{U}(X), \tag{3.27}$$

where we recall that h is a smallness parameter. Note that Eshelby (1951) chose $h\,\boldsymbol{U}$ to be a *uniform* (i.e., *rigid*) material displacement field $h\,\boldsymbol{U}(X) = -h\,\boldsymbol{U}_0$ over $\mathcal{D}$. Eshelby's choice makes the procedure easier to illustrate and yields directly the *strong form* of the inclusion problem. Here, we derive the *weak form* first and then obtain the strong form by adding Eshelby's assumption, $h\,\boldsymbol{U}(X) = -h\,\boldsymbol{U}_0$, at the very end. However, it is helpful to keep the uniform displacement $-h\,\boldsymbol{U}_0$ in one's mind and, to this end, we chose to represent this uniform displacement in Figure 3.3, following Eshelby's original thought experiment.

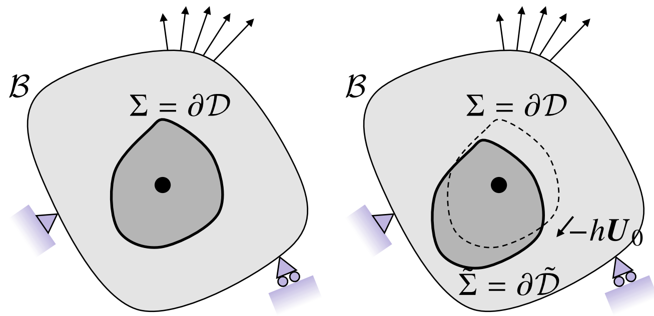

Figure 3.3: Determination of the force on a defect (the solid black circle). Left: original body, with the defect contained in a region $\mathcal{D}$, bounded by the smooth surface $\Sigma = \partial\mathcal{D}$. Right: replica body, with the defect contained in a different region $\tilde{\mathcal{D}}$, bounded by the smooth surface $\tilde{\Sigma} = \partial\tilde{\mathcal{D}}$. As in Eshelby's original scheme (Eshelby, 1975), here we depict the material displacement $h\,\boldsymbol{U}$ as being *uniform* over the material region $\mathcal{D}$ enclosed by the surface Σ, i.e., $h\,\boldsymbol{U}(X) = -h\,\boldsymbol{U}_0$ for every $X \in \mathcal{D}$.

We remark that, since the map $\mathcal{Y}$ of Equation (3.27) is Eshelbian, the body is undergoing *no* deformation, in the sense that it is *not* changing its shape, but only its configuration. Indeed, one chooses the surface Σ enclosing the region $\mathcal{D}$ and the surface $\tilde{\Sigma}$ enclosing the region $\tilde{\mathcal{D}}$ *independently* and then finds a suitable $\mathcal{Y}$ mapping $\mathcal{D}$ into $\tilde{\mathcal{D}}$. Clearly, this mere fact does *not* displace the defect at all, but simply represents a different choice of enclosing surface. The displacement of the defect in the reference configuration actually takes place when we *replace* the region $\mathcal{D}$ in the original body with the region $\tilde{\mathcal{D}}$ cut from the replica body (which is straightforward in the case of a Eshelby's rigid displacement $-h\,\boldsymbol{U}_0$), where $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are related by the *material transformation* $\mathcal{Y}$ described by (3.27). Note that, in this replacement, the defect is moved together with the region $\tilde{\mathcal{D}}$ (see point (iii) below).

Our goal is to determine the *variation in energy* accompanying this *change in reference configuration*. In order to achieve this, we perform the thought experiment proposed by Eshelby (1951, 1975) and described below.

(i) In the original body, cut out the material in the region $\mathcal{D}$. If the body is pre-stressed for any reason, then apply traction forces to the boundary $\Sigma = \partial\mathcal{D}$ of the cavity that has been created, in order to avoid relaxation.

(ii) Similarly, in the replica body, cut out the material in the region $\tilde{\mathcal{D}} = \mathcal{Y}(\mathcal{D})$ and apply suitable tractions to the boundary $\tilde{\Sigma} = \partial\tilde{\mathcal{D}} = \partial[\mathcal{Y}(\mathcal{D})] \equiv \mathcal{Y}(\partial\mathcal{D})$ to prevent relaxation. Let us denote the total elastic energy $\mathcal{E}^{\mathrm{el}}_{\mathcal{D}} : \mathcal{M} \to \mathbb{R}$ in $\mathcal{Y}(\mathcal{D})$ by

$$\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{Y}) = \int_{\mathcal{Y}(\mathcal{D})} W = \int_{\mathcal{D}} \det(T\mathcal{Y})\, W \circ \mathcal{Y}, \tag{3.28}$$

where we used the theorem of the change of variables to transform the integral over the displaced region $\mathcal{Y}(\mathcal{D})$ into an integral over the original region $\mathcal{D}$. Similarly, in the original region, the total elastic energy would be

$$\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X}) = \int_{\mathcal{D}} W = \int_{\mathcal{D}} W \circ \mathcal{X}, \tag{3.29}$$

where we exploited the identity $\mathcal{X}(X) = X$ in writing $W = W \circ \mathcal{X}$. Therefore, the difference in energy due to the perturbation $\mathcal{Y}$ (i.e., due to the different selection of the surfaces $\tilde{\Sigma}$ and Σ) is

$$\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{Y}) - \mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X}) = \int_{\mathcal{D}} \det(T\mathcal{Y})\, W \circ \mathcal{Y} - \int_{\mathcal{D}} W \circ \mathcal{X} = \int_{\mathcal{D}} [\det(T\mathcal{Y})\, W \circ \mathcal{Y} - W \circ \mathcal{X}]. \tag{3.30}$$

By expressing the map $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$ (see Equation (3.21)), considering that, for $h \to 0$, $\det T\mathcal{Y} = 1 + h\,\mathrm{Div}\,\boldsymbol{U} + o(h)$ (see Equation (3.24)) and

$$W \circ \mathcal{Y} = W \circ (\mathcal{X} + h\,\boldsymbol{U}) = W \circ \mathcal{X} + h\,[(\mathrm{Grad}\,W) \circ \mathcal{X}]\,\boldsymbol{U} + o(h), \tag{3.31}$$

Equation (3.30) becomes

$$\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X} + h\,\boldsymbol{U}) - \mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X}) = \int_{\mathcal{D}} \big[h\,(W \circ \mathcal{X})\,\mathrm{Div}\,\boldsymbol{U} + h\,[(\mathrm{Grad}\,W) \circ \mathcal{X}]\,\boldsymbol{U} + o(h)\big]. \tag{3.32}$$

Now, we can divide both sides of Equation (3.32) by h and take the limit for $h \to 0$ so that, on the left-hand side, we have the *variational* Gâteaux derivative of $\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}$ with respect to the material displacement field $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$, evaluated at the identity map $\mathcal{X}$, i.e.,

$$(\partial_{\boldsymbol{U}} \mathcal{E}^{\mathrm{el}}_{\mathcal{D}})(\mathcal{X}) = \lim_{h \to 0} \frac{\mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X} + h\,\boldsymbol{U}) - \mathcal{E}^{\mathrm{el}}_{\mathcal{D}}(\mathcal{X})}{h} = \int_{\mathcal{D}} \big[(W \circ \mathcal{X})\,\mathrm{Div}\,\boldsymbol{U} + [(\mathrm{Grad}\,W) \circ \mathcal{X}]\,\boldsymbol{U}\big]. \quad (3.33)$$

By using the identities $(\mathrm{Grad}\,W) \circ \mathcal{X} = \mathrm{Grad}\,W$ and $W \circ \mathcal{X} = W$, we can write

$$(\partial_{\boldsymbol{U}} \mathcal{E}^{\mathrm{el}}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \big[W\,\mathrm{Div}\,\boldsymbol{U} + [\mathrm{Grad}\,W]\,\boldsymbol{U}\big], \quad (3.34)$$

which, using by Leibniz' rule and the identity $\mathrm{Div}(W\,\boldsymbol{U}) = \mathrm{Div}(W\,\boldsymbol{I}\,\boldsymbol{U})$ (where $\boldsymbol{I}$ is the material identity tensor), becomes

$$(\partial_{\boldsymbol{U}} \mathcal{E}^{\mathrm{el}}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big[W\,\boldsymbol{I}\,\boldsymbol{U}\big]. \quad (3.35)$$

(iii) Before the deformation ϕ occurs, the region $\tilde{\mathcal{D}} = \mathcal{Y}(\mathcal{D})$ that had been isolated from the replica body could be *"transplanted"*[1] into the cavity (resulting from the elimination of the original region $\mathcal{D}$) in the original body by simply applying the opposite displacement field $-h\,\boldsymbol{U}$. In Eshelby's choice of a uniform displacement, this would be the *rigid translation* $h\,\boldsymbol{U}_0$, as shown in Figure 3.4. This is as if the defect had been displaced of the amount $h\,\boldsymbol{U}_0$.

However, after the deformation ϕ occurs, $\phi(\tilde{\mathcal{D}}) = \phi(\mathcal{Y}(\mathcal{D}))$ from the replica and $\phi(\mathcal{D})$ from the original body are *different* in general, and thus $\phi(\tilde{\mathcal{D}}) = \phi(\mathcal{Y}(\mathcal{D}))$ *may not* fit the cavity with deformed surface $\partial[\phi(\mathcal{D})] \equiv \phi(\partial\mathcal{D}) = \phi(\Sigma)$ in the original body. Indeed, the points of the deformed surface $\partial[\phi(\mathcal{D})] \equiv \phi(\partial\mathcal{D}) = \phi(\Sigma)$ in the original body and the points of the deformed surface $\partial[\phi(\mathcal{Y}(\mathcal{D}))] \equiv \phi(\partial(\mathcal{Y}(\mathcal{D}))) = \phi(\partial\tilde{\mathcal{D}}) = \phi(\tilde{\Sigma})$ in the replica body generally differ by the (conventional spatial) displacement

$$\phi(X + h\,\boldsymbol{U}(X)) - \phi(X) = [\boldsymbol{F}(X)](h\,\boldsymbol{U}(X)) + o(h), \quad (3.36)$$

[1]We are borrowing the term "transplant" from Epstein and Maugin (2000) and Imatani and Maugin (2002), but with a more strictly "surgical" meaning.

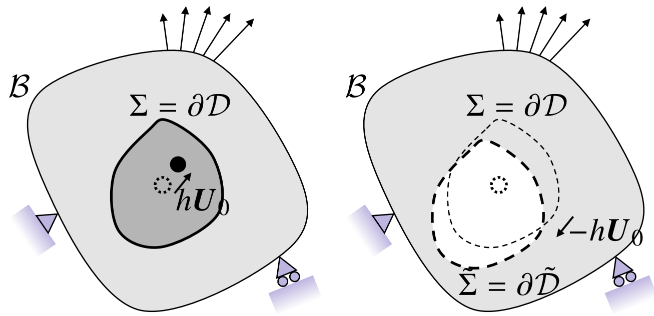

Figure 3.4: *Before* the deformation ϕ takes place, the region $\tilde{\mathcal{D}} = \mathcal{Y}(\mathcal{D})$ could be transplanted from the replica (right panel) to the original body (left panel), into the cavity resulting from the removal of the original region $\mathcal{D}$, by simply applying the negative of the displacement $-h\,\boldsymbol{U}_0$. This procedure effectively displaces the defect by the amount $h\,\boldsymbol{U}_0$ in the original body. We remark that this no longer holds *after* deformation has taken place.

which, recalling that $X + h\,\boldsymbol{U}(X) = \mathcal{Y}(X)$ and $X = \mathcal{X}(X)$, omitting the argument X and using the linearity of $\boldsymbol{F}$, can be written as

$$\phi \circ \mathcal{Y} - \phi \circ \mathcal{X} = h\,\boldsymbol{F}\boldsymbol{U} + o(h). \tag{3.37}$$

In order to deform the surface $\phi(\Sigma) = \phi(\partial\mathcal{D})$ of the cavity in the original body in such a way that $\phi(\tilde{\mathcal{D}})$ from the replica body can exactly fit in it, we must *adjust* the deformation. This can be achieved, in fact, by introducing a new deformation, $\bar{\phi}$, which, applied to $\mathcal{Y}(\mathcal{D}) = \tilde{\mathcal{D}}$, is such that the overall displacement is null, i.e.,

$$\bar{\phi}(\mathcal{Y}(X)) - \phi(X) = \mathbf{0}. \tag{3.38}$$

Since $\bar{\phi}$ has to adjust ϕ in order to eliminate the mismatch generated by the combined effect of $\mathcal{Y}$ and ϕ (note how the composition $\phi \circ \mathcal{Y}$ is, in fact, the mathematical representation of the "combined effect"), it is natural to define $\bar{\phi}$ as a perturbation of ϕ. Hence, we set

$$\bar{\phi} = \phi + h\,\boldsymbol{\eta}, \tag{3.39}$$

where, without loss of generality, the same smallness parameter, h, is used as that defining

$\mathcal{Y} = \mathcal{X} + h\boldsymbol{U}$. With the aid of (3.39), and in the limit $h \to 0$, Equation (3.38) becomes

$$
\begin{aligned}
&\phi \circ (\mathcal{X} + h\,\boldsymbol{U}) + h\,\boldsymbol{\eta} \circ (\mathcal{X} + h\,\boldsymbol{U}) - \phi \circ \mathcal{X} \\
&= h\,\boldsymbol{F}\boldsymbol{U} + o(h) + h\,\boldsymbol{\eta} + h^2[\boldsymbol{\eta} \circ \mathcal{X}]\boldsymbol{U} + o(h^2) \\
&= h[\boldsymbol{F}\boldsymbol{U} + \boldsymbol{\eta}] + o(h) = \boldsymbol{0}. \qquad (3.40)
\end{aligned}
$$

At the lowest order, Equation (3.40) gives the condition sought for $\boldsymbol{\eta}$, i.e., that it has to compensate for $\boldsymbol{U}$, thereby yielding

$$
\boldsymbol{F}\boldsymbol{U} + \boldsymbol{\eta} = \boldsymbol{0} \qquad \Rightarrow \qquad -h\,\boldsymbol{\eta} = h\,\boldsymbol{F}\boldsymbol{U}. \qquad (3.41)
$$

This interpretation of the displacement $\boldsymbol{\eta}$ is the core of Noether's Theorem, which will be addressed in Section 3.4.

The work necessary to adjust the deformation of $\mathcal{B} \setminus \mathcal{D}$ according to (3.39) is exerted by the first Piola-Kirchhoff surface traction $\boldsymbol{P}(-\boldsymbol{N}) = -\boldsymbol{P}\boldsymbol{N}$, where the minus sign comes from the fact that we regard $\boldsymbol{N}$ as the *outward* normal to the boundary $\Sigma = \partial\mathcal{D}$ of $\mathcal{D}$, which is *inward* with respect to the remainder $\mathcal{B} \setminus \mathcal{D}$ of the body. The integral of this work per unit referential area over the surface $\Sigma = \partial\mathcal{D}$ gives what Cermelli et al. (2001) called the "*net work*"

$$
\begin{aligned}
\mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}(\mathcal{Y}) &= \int_{\partial\mathcal{D}} (-\boldsymbol{P}\boldsymbol{N})\,(-h\,\boldsymbol{\eta}) + o(h) \\
&= -h \int_{\partial\mathcal{D}} (\boldsymbol{P}\boldsymbol{N})\,(\boldsymbol{F}\boldsymbol{U}) + o(h) = -h \int_{\partial\mathcal{D}} [(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}]\,\boldsymbol{N} + o(h), \qquad (3.42)
\end{aligned}
$$

where we rewrote the covector-vector contraction $(\boldsymbol{F}\boldsymbol{U})\,(\boldsymbol{P}\boldsymbol{N})$ by using the definition of transpose, i.e.,

$$
\begin{aligned}
(\boldsymbol{F}\boldsymbol{U})\,(\boldsymbol{P}\boldsymbol{N}) &= F^a{}_A\, U^A\, P_a{}^B\, N_B = (\boldsymbol{P}^T)^B{}_a\, F^a{}_A\, U^A\, N_B = [(\boldsymbol{F}^T\,\boldsymbol{P})^T]^B{}_A\, U^A\, N_B \\
&= [(\boldsymbol{F}^T\boldsymbol{P})^T\boldsymbol{U}]\,\boldsymbol{N}. \qquad (3.43)
\end{aligned}
$$

Note that, for the sake of a lighter notation, we are writing $\boldsymbol{F}^T$ and $\boldsymbol{P}^T$ for $\boldsymbol{F}^T \circ \phi$ and $\boldsymbol{P}^T \circ \phi$. Rigorously speaking, the composition by ϕ would be necessary, since $\boldsymbol{F}^T$ and $\boldsymbol{P}^T$ are defined in the current configuration $\phi(\mathcal{B})$ (Marsden and Hughes, 1983). Since $\boldsymbol{N}$ is the *outward* normal to $\Sigma = \partial\mathcal{D}$, the net work (3.42) is the *negative* of the work that the Piola tractions $\boldsymbol{P}\boldsymbol{N}$ would exert over the displacement $-h\,\boldsymbol{\eta}$ of Equation (3.41) on the referential surface $\Sigma = \partial\mathcal{D}$, seen as the boundary of the referential region $\mathcal{D}$. This observation allows us to apply the divergence theorem to (3.42), which yields

$$\mathcal{E}^{\text{nw}}_{\mathcal{D}}(\mathcal{Y}) = -h \int_{\mathcal{D}} \operatorname{Div}\left[(\boldsymbol{F}^T \boldsymbol{P})^T \boldsymbol{U}\right] + o(h). \tag{3.44}$$

This can be made into an increment by expressing the map $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$, and considering that $\mathcal{E}^{\text{nw}}_{\mathcal{D}}(\mathcal{X}) = 0$, i.e.,

$$\mathcal{E}^{\text{nw}}_{\mathcal{D}}(\mathcal{X} + h\,\boldsymbol{U}) - \mathcal{E}^{\text{nw}}_{\mathcal{D}}(\mathcal{X}) = -h \int_{\mathcal{D}} \operatorname{Div}\left[(\boldsymbol{F}^T \boldsymbol{P})^T \boldsymbol{U}\right] + o(h). \tag{3.45}$$

Now, dividing by h and passing to the limit $h \to 0$, we obtain the functional directional derivative

$$(\partial_{\boldsymbol{U}} \mathcal{E}^{\text{nw}}_{\mathcal{D}})(\mathcal{X}) = \lim_{h \to 0} \frac{\mathcal{E}^{\text{nw}}_{\mathcal{D}}(\mathcal{X} + h\,\boldsymbol{U}) - \mathcal{E}^{\text{nw}}_{\mathcal{D}}(\mathcal{X})}{h} = -\int_{\mathcal{D}} \operatorname{Div}\left[(\boldsymbol{F}^T \boldsymbol{P})^T \boldsymbol{U}\right]. \tag{3.46}$$

(iv) The deformed transformed region $\phi(\tilde{\mathcal{D}}) = \phi(\mathcal{Y}(\mathcal{D}))$ from the replica body can finally be exactly suited into the cavity left by the removal of $\mathcal{D}$ in the original body and we are able to weld together across the interface. We note that Eshelby (1975) needs to make considerations on the infinitesimals of order greater than h. In our approach, these are automatically taken care of (and eliminated) by the limit operation in Equation (3.46). To cite Eshelby (1975) verbatim, except for using our notation for the displacement,

> "We are now left with the system as it was to begin with, except that the defect has been shifted by $-h\,\boldsymbol{U} = h\,\boldsymbol{U}_0$, as required."

The associated variation in the total energy $\mathcal{E}_{\mathcal{D}} : \mathcal{M} \to \mathbb{R}$ of the system is obtained as $\mathcal{E}_{\mathcal{D}} = \mathcal{E}^{\mathrm{el}}_{\mathcal{D}} + \mathcal{E}^{\mathrm{nw}}_{\mathcal{D}}$, i.e., by summing Equations (3.35) and (3.46), i.e.,

$$(\partial_{\boldsymbol{U}} \mathcal{E}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big[W \boldsymbol{I}\, \boldsymbol{U}\big] - \int_{\mathcal{D}} \mathrm{Div}\,[(\boldsymbol{F}^T \boldsymbol{P})^T \boldsymbol{U}], \tag{3.47}$$

which can be written as

$$(\partial_{\boldsymbol{U}} \mathcal{E}_{\mathcal{D}})(\mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big[\mathfrak{E}^T \boldsymbol{U}\big] = \int_{\partial\mathcal{D}} (\mathfrak{E}\boldsymbol{N})\, \boldsymbol{U}. \tag{3.48}$$

Equation (3.48) quantifies the variation in energy necessary to obtain a new reference configuration in which the defect is displaced in direction $\boldsymbol{U}$ with respect to the original one. In the context of the theory of defects, Eshelby (1951) called the tensor $\mathfrak{E}$, with the expression

$$\mathfrak{E} = W \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{P}, \qquad \mathfrak{E}_A{}^B = W\, \delta_A{}^B - F^a{}_A\, P_a{}^B, \tag{3.49}$$

the *Maxwell tensor of elasticity* and later (Eshelby, 1975) the *energy-momentum tensor*, in analogy with Maxwell's terminology from field theory. This analogy will be completely clear in Section 3.3. Later, Maugin and Trimarco (1992) gave $\mathfrak{E}$ the name of *Eshelby stress* in his honour.

At the end of Eshelby's thought experiment, we have the expression in Equation (3.48), which can be thought of as the *virtual work* exerted by the Eshelby tractions $\mathfrak{E}\boldsymbol{N}$ on the material displacement field $\boldsymbol{U}$ on the boundary $\partial\mathcal{D}$ of the region $\mathcal{D}$. Using Eshelby's assumption $\boldsymbol{U}(X) = -\boldsymbol{U}_0$ for every $X \in \mathcal{D}$, we can write Equation (3.48) as

$$(\partial_{-\boldsymbol{U}_0} \mathcal{E}_{\mathcal{D}})(\mathcal{X}) = -\int_{\mathcal{D}} (\mathrm{Div}\, \mathfrak{E})\, \boldsymbol{U}_0 = -\int_{\partial\mathcal{D}} (\mathfrak{E}\boldsymbol{N})\, \boldsymbol{U}_0. \tag{3.50}$$

In order to obtain (in our notation) Equation (17) in the paper by Eshelby (1951), we use Cartesian coordinates, so that it is legitimate to rewrite the integral as

$$\mathfrak{F}\,\boldsymbol{U}_0 = (\partial_{-\boldsymbol{U}_0}\mathcal{E}_{\mathcal{D}})(\mathcal{X}) = -\left(\int_{\mathcal{D}} \operatorname{Div}\mathfrak{C}\right)\boldsymbol{U}_0 = -\left(\int_{\partial\mathcal{D}} \mathfrak{C}\,\boldsymbol{N}\right)\boldsymbol{U}_0, \tag{3.51}$$

where $\mathfrak{F}$ was *defined* by Eshelby as the *total* inhomogeneity force, producing work over the uniform virtual displacement $\boldsymbol{U}_0$. We remark that the total inhomogeneity force $\mathfrak{F}$ can only be defined in the case of Cartesian coordinates, which is the only particular case in which integration of a vector field makes sense (see warning at page 134 in the text by Marsden and Hughes, 1983).

3.3 Eshelby's Variational Derivation of the Strong Form

In his seminal paper, Eshelby (1975) used a variational approach and wrote the Euler-Lagrange equations for a generic system with a potential energy depending – in the language of classical field theory – on fields, "gradients" of fields and coordinates. In this quite general framework, Elasticity can be seen as a particular case. Here, we follow Eshelby's derivation (Eshelby, 1975) step by step, using our notation and adding our comments. Then, we shall show how this specialises to the case of large- and small-deformation Elasticity. The only difference with Eshelby's procedure is that, whenever we look at the variational problem as an elasticity problem, our fields are the components of the configuration map, rather than the components of the displacement. Note that, in contrast with Section 3.2, here we define the total energy $\mathcal{E}_{\mathcal{D}}$ in a region $\mathcal{D}$ of the body as a functional on the manifold $\mathcal{C}$, the conventional configuration space.

Let us assume a potential energy density W, defined per unit referential volume, given by

$$W(X) = \hat{W}(\phi(X), \boldsymbol{F}(X), X), \tag{3.52}$$

where ϕ is a collection of scalar fields (in the case of continuum mechanics, the configuration map, with components ϕ^a), $\boldsymbol{F}$ is the collection of the gradients of the fields (in our case, the deformation gradient, with components $F^a{}_A = \phi^a{}_{,A}$), and X is the collection of the independent variables (in our case, the material coordinates X^A). Note that we *distinguish* between the *scalar field* W (function of the coordinates X^A) and the *associated constitutive function* $\hat{W}$ (function of the fields ϕ^a, the gradients $F^a{}_A = \phi^a{}_{,A}$ and the coordinates X^A). By using the material identity map $\mathcal{X}$ of Equation (3.17) (such that $X = \mathcal{X}(X)$, in components, $X^A = \mathcal{X}^A(X)$)), the potential energy can be rewritten in the form

$$W(X) = \hat{W}(\phi(X), \boldsymbol{F}(X), \mathcal{X}(X)) = [\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})](X). \tag{3.53}$$

Thus, by dropping the argument X on the far left and the far right sides, we have

$$W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}). \tag{3.54}$$

In order to find the Euler-Lagrange equations associated with $W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})$, we need to consider the total energy $\mathcal{E}_{\mathcal{B}} : \mathcal{C} \to \mathbb{R}$ over the whole body $\mathcal{B}$, given by

$$\mathcal{E}_{\mathcal{B}}(\phi) = \int_{\mathcal{B}} W = \int_{\mathcal{B}} \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \tag{3.55}$$

and calculate its variation with respect to a conventional displacement $\boldsymbol{\eta}$, which is given by the Gâteaux derivative

$$\begin{aligned}(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) &= \lim_{h \to 0} \frac{\mathcal{E}_{\mathcal{B}}(\phi + h\,\boldsymbol{\eta}) - \mathcal{E}_{\mathcal{B}}(\phi)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \int_{\mathcal{B}} \big[\hat{W} \circ (\phi + h\,\boldsymbol{\eta}, \boldsymbol{F} + h \operatorname{Grad} \boldsymbol{\eta}, \mathcal{X}) - \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\big],\end{aligned} \tag{3.56}$$

with $\boldsymbol{\eta}$ chosen in a suitable subset of $T_{\phi}\mathcal{C} \cap C^1(\mathcal{B}, T\mathcal{S})$, as will be clarified later in this section. In the jargon of field theory, this is called a "variation on the fields, with frozen coordinates", i.e., we are going to calculate the integral on the *fixed* domain $\mathcal{B}$. The transformation on the configuration map ϕ (the "fields" ϕ^a) is given by

$$\phi \mapsto \bar{\phi} = \phi + h\,\boldsymbol{\eta}, \tag{3.57a}$$

$$\phi^a \mapsto \bar{\phi}^a = \phi^a + h\,\eta^a, \tag{3.57b}$$

and the transformation on the tangent map $T\phi = \boldsymbol{F}$ (the "gradients" $F^a{}_A = \phi^a{}_{,A}$) is

$$T\phi = \boldsymbol{F} \mapsto T\bar{\phi} = \bar{\boldsymbol{F}} = T(\phi + h\,\boldsymbol{\eta}) = \boldsymbol{F} + h\,\mathrm{Grad}\,\boldsymbol{\eta}, \tag{3.58a}$$

$$\phi^a{}_{,A} = F^a{}_A \mapsto \bar{F}^a{}_A = \bar{\phi}^a{}_{,A} = \phi^a{}_{,A} + h\,\eta^a{}_{|A} = F^a{}_A + h\,\eta^a{}_{|A}, \tag{3.58b}$$

where $\mathrm{Grad}\,\boldsymbol{\eta}$, with components $\eta^a{}_{|A}$, is the covariant derivative of the displacement $\boldsymbol{\eta}$.

We follow the standard derivation by expanding the argument of the integral as

$$\begin{aligned}\hat{W} \circ (\phi + h\,\boldsymbol{\eta}, \boldsymbol{F} + h\,\mathrm{Grad}\,\boldsymbol{\eta}, \mathcal{X}) - \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \\ = \frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\ h\,\eta^a + \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\ h\,\eta^a{}_{|A} + o(h),\end{aligned} \tag{3.59}$$

substituting in (3.56) and performing the limit, which results in

$$\begin{aligned}(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) &= \int_{\mathcal{B}} \left[\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\ \eta^a + \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\ \eta^a{}_{|A} \right] \\ &= \int_{\mathcal{B}} \big[-f_a\,\eta^a + P_a{}^A\,\eta^a{}_{|A} \big] = \int_{\mathcal{B}} \big[-\boldsymbol{f}\,\boldsymbol{\eta} + \boldsymbol{P} : \mathrm{Grad}\,\boldsymbol{\eta} \big],\end{aligned} \tag{3.60}$$

where $\boldsymbol{f}$ and $\boldsymbol{P}$ are given by

$$f_a = -\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \boldsymbol{f} = -\frac{\partial \hat{W}}{\partial \phi} \circ (\phi, \boldsymbol{F}, \mathcal{X}) \tag{3.61a}$$

$$P_a{}^A = \frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \boldsymbol{P} = \frac{\partial \hat{W}}{\partial \boldsymbol{F}} \circ (\phi, \boldsymbol{F}, \mathcal{X}). \tag{3.61b}$$

In the case of elasticity in continuum mechanics, when the potential is given as the sum of an elastic potential and a potential of the external body forces, i.e.,

$$\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \hat{W}_{\text{el}} \circ (\boldsymbol{F}, \mathcal{X}) + \hat{W}_{\text{ext}} \circ (\phi, \mathcal{X}), \tag{3.62}$$

the covector field $\boldsymbol{f}$ and the tensor field $\boldsymbol{P}$ take the meaning of external body force per unit volume and first Piola-Kirchhoff stress, respectively. Now, considering that

$$\boldsymbol{P} : \operatorname{Grad} \boldsymbol{\eta} = P_a{}^A \, \eta^a{}_{|A} = (P_a{}^A \, \eta^a)_{|A} - P_a{}^A{}_{|A} \, \eta^a = \operatorname{Div}(\boldsymbol{P}^T \boldsymbol{\eta}) - (\operatorname{Div} \boldsymbol{P}) \, \boldsymbol{\eta}, \tag{3.63}$$

the variation becomes

$$\begin{aligned} (\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) &= \int_{\mathcal{B}} \big[- \boldsymbol{f} \, \boldsymbol{\eta} + \operatorname{Div}(\boldsymbol{\eta} \, \boldsymbol{P}) - (\operatorname{Div} \boldsymbol{P}) \, \boldsymbol{\eta} \big] \\ &= - \int_{\mathcal{B}} (\boldsymbol{f} + \operatorname{Div} \boldsymbol{P}) \, \boldsymbol{\eta} + \int_{\mathcal{B}} \operatorname{Div}(\boldsymbol{\eta} \, \boldsymbol{P}) \end{aligned} \tag{3.64}$$

and, by applying Gauss' divergence theorem,

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = - \int_{\mathcal{B}} (\boldsymbol{f} + \operatorname{Div} \boldsymbol{P}) \, \boldsymbol{\eta} + \int_{\partial \mathcal{B}} (\boldsymbol{P} \boldsymbol{N}) \, \boldsymbol{\eta}, \tag{3.65}$$

where $\boldsymbol{N}$ is the normal to the boundary $\partial \mathcal{B}$ and $(\boldsymbol{P} \boldsymbol{N}) \, \boldsymbol{\eta} = \boldsymbol{\eta} \, (\boldsymbol{P} \boldsymbol{N})$.

We now look for a configuration ϕ at which $\mathcal{E}_{\mathcal{B}}(\phi)$ is stationary. For this purpose, we impose the condition $(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = 0$, in which ϕ is unknown, and we study it under the restriction that $\boldsymbol{\eta}$ vanish on $\partial \mathcal{B}$ (Hill, 1951). This choice annihilates the surface integral on the right-hand-side of (3.65), so that the stationarity condition becomes

$$(\partial_{\boldsymbol{\eta}} \mathcal{E}_{\mathcal{B}})(\phi) = - \int_{\mathcal{B}} (\boldsymbol{f} + \operatorname{Div} \boldsymbol{P}) \, \boldsymbol{\eta} = 0, \qquad \boldsymbol{\eta} \in \mathcal{V}, \tag{3.66}$$

where $\mathcal{V} := \{ \boldsymbol{\eta} \in T_\phi \mathcal{C} \cap C^1(\mathcal{B}, T\mathcal{S}) : \boldsymbol{\eta}(X) = \boldsymbol{0}, \, \forall \, X \in \partial \mathcal{B} \}$. We require now that (3.66) be satisfied for all $\boldsymbol{\eta} \in \mathcal{V}$, which leads to the Euler-Lagrange equations

$$\boldsymbol{f} + \operatorname{Div} \boldsymbol{P} = \boldsymbol{0}, \qquad f_a + P_a{}^A{}_{|A} = 0. \tag{3.67}$$

If the external body forces acting on $\mathcal{B}$ are only those given by $\boldsymbol{f}$, which admit the potential density $\hat{W}_{\text{ext}} \circ (\phi, \mathcal{X})$, Equation (3.67) represents, in continuum mechanics, the Lagrangian (static) equilibrium equations, i.e., spatial equations described in terms of the material coordinates. If ϕ is a solution to (3.67), and the boundary of $\mathcal{B}$ can be written as the disjoint union of a Dirichlet part and a Neumann part, i.e., $\partial\mathcal{B} = \partial_D\mathcal{B} \sqcup \partial_N\mathcal{B}$, then the variation $(\partial_{\boldsymbol{\eta}}\mathcal{E}_{\mathcal{B}})(\phi)$ in Equation (3.65) becomes

$$(\partial_{\boldsymbol{\eta}}\mathcal{E}_{\mathcal{B}})(\phi) = \int_{\partial\mathcal{B}} (\boldsymbol{P}\,\boldsymbol{N})\,\boldsymbol{\eta} = \int_{\partial_N\mathcal{B}} (\boldsymbol{P}\,\boldsymbol{N})\,\boldsymbol{\eta}, \tag{3.68}$$

where the surface integral is restricted to the Neumann boundary, $\partial_N\mathcal{B}$, because the displacement $\boldsymbol{\eta}$, although being arbitrary, has to vanish on the Dirichlet boundary, $\partial_D\mathcal{B}$. In this case, the stationarity condition on $\mathcal{E}_{\mathcal{B}}$ requires the vanishing of the surface integral on the far right-hand-side of Equation (3.65) . This can be obtained if $\partial_N\mathcal{B}$ is a set of null measure, or if no contact forces are applied onto $\partial_N\mathcal{B}$. On the contrary, when contact forces are present, the stationarity condition on $\mathcal{E}_{\mathcal{B}}$ must be corrected by requiring that $(\partial_{\boldsymbol{\eta}}\mathcal{E}_{\mathcal{B}})(\phi)$ be balanced by the work performed by the contact forces on $\boldsymbol{\eta}$. This result follows from the extended Hamilton's Principle (dell'Isola and Placidi, 2011).

If Equation (3.65) is referred to a set $\mathcal{D} \subset \mathcal{B}$, and is evaluated for a configuration ϕ solving (3.67), the volume integral vanishes by virtue of the Euler-Lagrange equations, while internal contact forces are exchanged through $\partial\mathcal{D}$. In this case, $\boldsymbol{\eta}$ is not required to vanish on $\partial\mathcal{D}$, and the variational procedure leads to

$$(\partial_{\boldsymbol{\eta}}\mathcal{E}_{\mathcal{D}})(\phi) = \int_{\partial\mathcal{D}} (\boldsymbol{P}\boldsymbol{N})\,\boldsymbol{\eta}, \tag{3.69}$$

thereby returning the virtual work exerted by the contact forces acting on $\partial\mathcal{D}$.

Let us now assume that ϕ satisfies the Euler-Lagrange equations (3.67), and let us take the material gradient Grad W of the energy density W, i.e., the partial derivatives of W with respect to X^B,

$$\begin{aligned} W_{,B} &= \left[\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right]_{,B} \\ &= \left[\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right] \phi^a{}_{,B} + \left[\frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right] F^a{}_{A|B} + \frac{\partial \hat{W}}{\partial \mathcal{X}^B} \circ (\phi, \boldsymbol{F}, \mathcal{X}) \\ &= -f_a\, F^a{}_B + P_a{}^A\, F^a{}_{A|B} + \frac{\partial \hat{W}}{\partial \mathcal{X}^B} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \end{aligned} \tag{3.70}$$

where we used the definitions of the components of the deformation gradient, $F^a{}_A = \phi^a{}_{,A}$, of the body force and the first Piola-Kirchhoff stress, and $F^a{}_{A|B}$ are the components of the third-order two-point tensor Grad $\boldsymbol{F}$. The last term in Equation (3.70) is usually called "explicit" gradient of the field W and denoted $(\partial W/\partial X^B)|_{\text{expl}}$ in the literature (e.g., Eshelby, 1975; Epstein and Maugin, 1990), whereas we regard it as the collection of the partial derivatives of the constitutive function $\hat{W}$ with respect to $\mathcal{X}^B$ (which, we recall, are the functions such that $\mathcal{X}^B(X) = X^B$). The negative of the "explicit" gradient defines the *material inhomogeneity force* or *configurational force*

$$\boldsymbol{\mathfrak{F}} = -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \mathfrak{F}_A = -\frac{\partial \hat{W}}{\partial \mathcal{X}^A} \circ (\phi, \boldsymbol{F}, \mathcal{X}). \tag{3.71}$$

Substituting the expressions of the Lagrangian force $\boldsymbol{f}$, the Piola-Kirchhoff stress $\boldsymbol{P}$, and the configurational force $\boldsymbol{\mathfrak{F}}$ into Equation (3.70), we obtain

$$\text{Grad}\, W = -\boldsymbol{F}^T \boldsymbol{f} + \boldsymbol{P} : \text{Grad}\, \boldsymbol{F} - \boldsymbol{\mathfrak{F}}, \tag{3.72}$$

where the double contraction ":" in the second term is of the two legs of $\boldsymbol{P}$ with the first two legs of Grad $\boldsymbol{F}$. By invoking the symmetry of the Christoffel symbols Γ^A_{BC} associated with the Levi-Civita Connection induced by the material metric $\boldsymbol{G}$, so that $F^a{}_{A|B} = F^a{}_{B|A}$, we work out the second term on the right-hand-side of (3.72) in components, i.e.,

$$P_a{}^A F^a{}_{A|B} = P_a{}^A F^a{}_{B|A} = (P_a{}^A F^a{}_B)_{|A} - P_a{}^A{}_{|A} F^a{}_B, \tag{3.73}$$

which, in component-free notation, reads

$$\boldsymbol{P} : \mathrm{Grad}\boldsymbol{F} = \mathrm{Div}(\boldsymbol{F}^T\boldsymbol{P}) - \boldsymbol{F}^T\mathrm{Div}\boldsymbol{P}. \tag{3.74}$$

By substituting this result into (3.72), we obtain

$$\begin{aligned}\mathrm{Grad}\,W &= -\boldsymbol{F}^T\boldsymbol{f} + \mathrm{Div}(\boldsymbol{F}^T\boldsymbol{P}) - \boldsymbol{F}^T\mathrm{Div}\boldsymbol{P} - \boldsymbol{\mathfrak{F}} \\ &= -\boldsymbol{F}^T[\boldsymbol{f} + \mathrm{Div}\boldsymbol{P}] + \mathrm{Div}(\boldsymbol{F}^T\boldsymbol{P}) - \boldsymbol{\mathfrak{F}}.\end{aligned} \tag{3.75}$$

Moreover, using the Euler-Lagrange equation (3.67) yields

$$\mathrm{Grad}\,W = \mathrm{Div}(\boldsymbol{F}^T\boldsymbol{P}) - \boldsymbol{\mathfrak{F}}. \tag{3.76}$$

Finally, by virtue of the identity $\mathrm{Grad}\,W = \mathrm{Div}\left(W\boldsymbol{I}^T\right)$, where $\boldsymbol{I}$ is the material identity tensor, Equation (3.76) becomes

$$\boldsymbol{\mathfrak{F}} + \mathrm{Div}\,\boldsymbol{\mathfrak{E}} = \boldsymbol{0}, \qquad \mathfrak{F}_A + \mathfrak{E}_A{}^B{}_{|B} = 0, \tag{3.77}$$

where $\boldsymbol{\mathfrak{E}}$ is the Eshelby stress defined as in Equation (3.49).

Similarly to other field theories, like Electromagnetism or General Relativity, the tensor $\boldsymbol{\mathfrak{E}}$ defined in (3.49) plays the role of the ("spatial" part of the) energy-momentum tensor of the theory under study. However, we emphasise that, while $\boldsymbol{\mathfrak{E}}$ has been obtained with the aid of a variational argument in the present framework, more general approaches exist, in which $\boldsymbol{\mathfrak{E}}$ is introduced as a primary dynamical quantity (Gurtin, 1995). Equation (3.77) is called *material equilibrium equation* or *configurational equilibrium equation* (Gurtin, 1995), by analogy with the equilibrium equation (3.67) described by the Euler-Lagrange equations.

According to Equation (3.71), if the body $\mathcal{B}$ is homogeneous, then we have

$$\mathfrak{F}_A(X) = -\left[\frac{\partial \hat{W}}{\partial \mathcal{X}^A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\right](X) = 0, \qquad \forall X \in \mathcal{B}, \tag{3.78}$$

and Equation (3.77) implies the vanishing of the divergence of the Eshelby stress. On the contrary, if there is *any* inhomogeneity in $\mathcal{D}$ (i.e., the derivative $\partial\hat{W}/\partial\mathcal{X}^A$ is non-vanishing), this will be captured by the integral of the traction forces $\mathfrak{E}\,\boldsymbol{N}$ of the Eshelby stress over the boundary $\partial\mathcal{D}$.

We now show that Equation (3.48) yields the *weak formulation* of the *strong form* described in Equation (3.77). This is easy to see by referring to Equation (3.51), which we obtained from Equation (3.48) (or Equation (3.50)) by working in Cartesian coordinates and using Eshelby's displacement $\boldsymbol{U} = -\boldsymbol{U}_0$, constant over $\mathcal{D}$. Indeed, by solving the material equilibrium equation (3.77) for $\mathfrak{F}$, using Cartesian coordinates, integrating over $\mathcal{D}$, applying Gauss' theorem and contracting both sides with $\boldsymbol{U}_0$, we obtain the *total configurational force* on the region $\mathcal{D}$ as the covector $\mathcal{F}$ such that

$$\mathcal{F}\,\boldsymbol{U}_0 = -\left(\int_{\mathcal{D}} \operatorname{Div}\mathfrak{E}\right)\boldsymbol{U}_0 = -\left(\int_{\partial\mathcal{D}} \mathfrak{E}\,\boldsymbol{N}\right)\boldsymbol{U}_0 = \left(\int_{\mathcal{D}} \mathfrak{F}\right)\boldsymbol{U}_0, \tag{3.79}$$

i.e., $\mathcal{F}$ is the integral of the inhomogeneity force density $\mathfrak{F}$, as we see by comparing with Equation (3.51). Note that, if the body $\mathcal{D}$ is homogeneous, Equations (3.77) and (3.78) imply the vanishing of the divergence of the Eshelby stress, and therefore the vanishing of the volume integral and the equivalent surface integral on the right-hand-side of Equation (3.79).

3.4 Derivation of the Weak Form with Noether Theorem

In Noether's Theorem, we need to contemporarily transform the domain and perform a variation on the arguments of the Lagrangian. In the jargon of classical field theory, these are called a *transformation of the coordinates* (material coordinates, in our case) and a *variation of the fields*, respectively. Together, these give the *total variation*. We have already shown the transformation of the material coordinates in Section 3.1.3 and the variation on the fields in Section 3.3 and we turn now to the total variation. Then, we apply Noether's theorem to *directly* obtain Eshelby's results. In the application of Noether's Theorem, we define the total energy $\mathcal{E}_{\mathcal{D}}$ of a region $\mathcal{D}$ as a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$.

3.4.1 Total Variation

In the language of field theory, the *total variation* is obtained by evaluating the *variation of the fields at frozen coordinates* given in (3.57) and (3.58) at the transformed points $\tilde{X} = \mathcal{Y}(X)$, where $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U} : \mathcal{B} \to \tilde{\mathcal{B}}$ is the *infinitesimal transformation of the coordinates* defined in (3.21), with $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$. In order to avoid confusion, some care must be exercised.

We recall that the manifold $\mathcal{C}$ is the configuration space of the body $\mathcal{B}$, a configuration ϕ is an element of $\mathcal{C}$ and a displacement field $\boldsymbol{\eta}$ is a tangent vector of $T_\phi\mathcal{C}$. Let us denote by $\tilde{\mathcal{C}}$ the configuration space of the "perturbed" body $\tilde{\mathcal{B}} = \mathcal{Y}(\mathcal{B})$, to which the points $\tilde{X} = \mathcal{Y}(X)$ belong. Consider the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$ and the *restriction* of the configuration ϕ and the displacement field $\boldsymbol{\eta}$ defined in a subset $\mathcal{D} \subset \mathcal{B} \cap \tilde{\mathcal{B}}$ (see Figure 3.5). In this restriction, it is legitimate to evaluate ϕ and $\boldsymbol{\eta}$ at $\tilde{X}$.

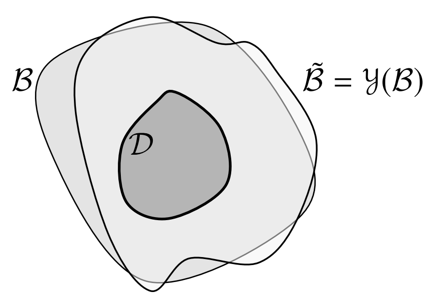

Figure 3.5: A domain $\mathcal{D}$ (dark grey) in the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$ between the body $\mathcal{B}$ (solid grey) and the perturbed body $\tilde{\mathcal{B}}$ (transparent grey).

We now define the total variation $\mathcal{C} \to \tilde{\mathcal{C}} : \phi \mapsto \bar{\phi}$ by evaluating the *variations of the fields at* *frozen coordinates* of Equations (3.57) and (3.58) at $\tilde{X} \in \tilde{\mathcal{B}} \cap \mathcal{B}$, i.e., we define

$$\bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h\,\boldsymbol{\eta}(\tilde{X}), \qquad \bar{\phi}^a(\tilde{X}) = \phi^a(\tilde{X}) + h\,\eta^a(\tilde{X}), \tag{3.80}$$

and

$$\bar{\boldsymbol{F}}(\tilde{X}) = \boldsymbol{F}(\tilde{X}) + h\,(\mathrm{Grad}\,\boldsymbol{\eta})(\tilde{X}), \qquad \bar{F}^a{}_A(\tilde{X}) = F^a{}_A(\tilde{X}) + h\,\eta^a{}_{|A}(\tilde{X}), \tag{3.81}$$

where h is, with no loss of generality, the same smallness parameter as $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$. To obtain the final form of the total variation, we substitute the transformation (3.21) of the coordinates into the variations on the configuration (3.80) and on the tangent (3.81) of the configuration, respectively, and use Taylor expansion. For the configuration, we have

$$\begin{aligned}\bar{\phi}(\tilde{X}) &= \phi(X + h\,\boldsymbol{U}(X)) + h\,\boldsymbol{\eta}(X + h\,\boldsymbol{U}(X)) \\ &= \phi(X) + h\,\boldsymbol{F}(X)\,\boldsymbol{U}(X) + h\,\boldsymbol{\eta}(X) + o(h), \end{aligned} \tag{3.82a}$$

$$\begin{aligned}\bar{\phi}^a(\tilde{X}) &= \phi^a(X + h\,\boldsymbol{U}(X)) + h\,\eta^a(X + h\,\boldsymbol{U}(X)) \\ &= \phi^a(X) + h\,F^a{}_B(X)\,U^B(X) + h\,\eta^a(X) + o(h), \end{aligned} \tag{3.82b}$$

from which, using $\bar{\phi}(\tilde{X}) = \bar{\phi}(\mathcal{Y}(X)) = (\bar{\phi} \circ \mathcal{Y})(X)$ and omitting the argument X, we have

$$\bar{\phi} \circ \mathcal{Y} = \phi + h\,(\boldsymbol{\eta} + \boldsymbol{F}\,\boldsymbol{U}) + o(h) = \phi + h\,\boldsymbol{w} + o(h), \tag{3.83a}$$

$$\bar{\phi}^a \circ \mathcal{Y} = \phi^a + h\,(\eta^a + F^a{}_B\,U^B) + o(h) = \phi^a + h\,w^a + o(h), \tag{3.83b}$$

where

$$\boldsymbol{w} = \boldsymbol{\eta} + \boldsymbol{F}\,\boldsymbol{U}, \qquad w^a = \eta^a + F^a{}_B\,U^B. \tag{3.84}$$

For the tangent map, we have

$$\begin{aligned}\bar{\boldsymbol{F}}(\tilde{X}) &= \boldsymbol{F}(X + h\,\boldsymbol{U}(X)) + h\,(\operatorname{Grad}\boldsymbol{\eta})(X + h\,\boldsymbol{U}(X)) \\ &= \boldsymbol{F}(X) + h\,(\operatorname{Grad}\boldsymbol{F})(X)\,\boldsymbol{U}(X) + h\,(\operatorname{Grad}\boldsymbol{\eta})(X) + o(h), \qquad (3.85a)\\ \bar{F}^a{}_A(\tilde{X}) &= F^a{}_A(X + h\,\boldsymbol{U}(X)) + h\,\eta^a{}_{|A}(X + h\,\boldsymbol{U}(X)) \\ &= F^a{}_A(X) + h\,F^a{}_{A|B}(X)\,U^B(X) + h\,\eta^a{}_{|A}(X) + o(h), \qquad (3.85b)\end{aligned}$$

and thus,

$$\bar{\boldsymbol{F}} \circ \mathcal{Y} = \boldsymbol{F} + h\,(\operatorname{Grad}\boldsymbol{\eta} + (\operatorname{Grad}\boldsymbol{F})\,\boldsymbol{U}) + o(h) = \boldsymbol{F} + h\,\boldsymbol{Y} + o(h), \tag{3.86a}$$

$$\bar{F}^a{}_A \circ \mathcal{Y} = F^a{}_A + h\,(\eta^a{}_{|A} + F^a{}_{A|B}\,U^B) + o(h) = F^a{}_A + h\,Y^a{}_A + o(h), \tag{3.86b}$$

where

$$\boldsymbol{Y} = \operatorname{Grad}\boldsymbol{\eta} + (\operatorname{Grad}\boldsymbol{F})\,\boldsymbol{U}, \qquad Y^a{}_A = \eta^a{}_{|A} + F^a{}_{A|B}\,U^B. \tag{3.87}$$

3.4.2 Variation of the Total Energy

Since we are working in the static case, we replace the action functional and the Lagrangian density with the total energy functional $\mathcal{E}$ and the potential energy density W. The total energy in a subset $\mathcal{D} \subset \mathcal{B} \cap \tilde{\mathcal{B}}$ is a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$, i.e.,

$$\mathcal{E}_{\mathcal{D}} : \mathcal{C} \times \mathcal{M} \to \mathbb{R} : (\phi, \mathcal{Y}) \mapsto \mathcal{E}_{\mathcal{D}}(\phi, \mathcal{Y}) = \int_{\mathcal{Y}(\mathcal{D})} \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \tag{3.88}$$

where the integration domain $\mathcal{Y}(\mathcal{D})$ must belong to the intersection $\mathcal{B} \cap \tilde{\mathcal{B}}$. We now consider the coordinate transformation $\mathcal{Y} = \mathcal{X} + h\,\boldsymbol{U}$, where $\boldsymbol{U} \in T_{\mathcal{X}}\mathcal{M}$ is a tangent vector at the identity $\mathcal{X}$, and the field transformation is $\bar{\phi} = \phi + h\,\boldsymbol{\eta}$, where $\boldsymbol{\eta} \in T_{\phi}\mathcal{C}$ is a tangent vector at the configuration ϕ.

The variation of the energy is given by the directional derivative

$$
\begin{aligned}
(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) &= \lim_{h\to 0}\frac{\mathcal{E}_{\mathcal{D}}(\bar{\phi},\mathcal{Y})-\mathcal{E}_{\mathcal{D}}(\phi,\mathcal{X})}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_{\mathcal{Y}(\mathcal{D})}\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})-\int_{\mathcal{D}}\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\right],
\end{aligned} \tag{3.89}
$$

evaluated at the conventional configuration ϕ and Eshelbian configuration $\mathcal{X}$, with respect to the pair of tangent vectors $(\boldsymbol{\eta},\boldsymbol{U})\in T_{(\phi,\mathcal{X})}(\mathcal{C}\times\mathcal{M})$ in the product manifold $\mathcal{C}\times\mathcal{M}$. Note also that, in the second integral, we used $\mathcal{X}(\mathcal{D})=\mathcal{D}$.

Application of the theorem of the change of variables on the first integral in (3.89) yields

$$
\int_{\mathcal{Y}(\mathcal{D})}\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X}) = \left[\int_{\mathcal{D}}(1+h\,\mathrm{Div}\,\boldsymbol{U})\;\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})\circ\mathcal{Y}\right]+o(h), \tag{3.90}
$$

where the determinant $\det(T\mathcal{Y})=1+h\,\mathrm{Div}\,\boldsymbol{U}+o(h)$ follows from Equation (3.24). We now notice that

$$
\begin{aligned}
\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})\circ\mathcal{Y} &= \hat{W}\circ(\bar{\phi}\circ\mathcal{Y},\bar{\boldsymbol{F}}\circ\mathcal{Y},\mathcal{X}\circ\mathcal{Y}) \\
&= \hat{W}\circ(\phi+h\,\boldsymbol{w}+o(h),\boldsymbol{F}+h\,\boldsymbol{Y}+o(h),\mathcal{X}+h\,\boldsymbol{U}),
\end{aligned} \tag{3.91}
$$

where we made use of the total variations (3.83) and (3.86), as well as of the identity $\mathcal{X}\circ\mathcal{Y}=\mathcal{Y}=\mathcal{X}+h\,\boldsymbol{U}$. Now, we expand in Taylor series up to the first order, and obtain

$$
\begin{aligned}
\hat{W}\circ(\bar{\phi},\bar{\boldsymbol{F}},\mathcal{X})\circ\mathcal{Y} = \hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X}) &+ \frac{\partial\hat{W}}{\partial\phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\;h\,w^a \\
&+ \frac{\partial\hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\;h\,Y^a{}_A \\
&+ \frac{\partial\hat{W}}{\partial\mathcal{X}^B}\circ(\phi,\boldsymbol{F},\mathcal{X})\;h\,U^B+o(h).
\end{aligned} \tag{3.92}
$$

Using Equations (3.90), (3.91) and (3.92) in the variation of the energy (3.89), we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \lim_{h\to 0}\frac{1}{h}\left[\int_{\mathcal{D}} h\left(\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\; U^B{}_{|B} + \right.\right.$$
$$\left.\left. + \frac{\partial \hat{W}}{\partial \phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\; w^a + \frac{\partial \hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\; Y^a{}_A + \frac{\partial \hat{W}}{\partial \mathcal{X}^B}\circ(\phi,\boldsymbol{F},\mathcal{X})\; U^B\right) + o(h)\right]. \quad (3.93)$$

The smallness parameter cancels out and the term $o(h)$ disappears in the limit $h \to 0$. Thus, we write

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}}\left(\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\; U^B{}_{|B} + \right.$$
$$\left. + \frac{\partial \hat{W}}{\partial \phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\; w^a + \frac{\partial \hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\; Y^a{}_A + \frac{\partial \hat{W}}{\partial \mathcal{X}^B}\circ(\phi,\boldsymbol{F},\mathcal{X})\; U^B\right), \quad (3.94)$$

and we use the explicit expressions (3.84) and (3.87) of the total variations $\boldsymbol{w}$ and $\boldsymbol{Y}$:

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}}\left(\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\; U^B{}_{|B} + \frac{\partial \hat{W}}{\partial \phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\;(\eta^a + F^a{}_B\, U^B) + \right.$$
$$\left. + \frac{\partial \hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\;(\eta^a{}_{|A} + F^a{}_{A|B}\, U^B) + \frac{\partial \hat{W}}{\partial \mathcal{X}^B}\circ(\phi,\boldsymbol{F},\mathcal{X})\; U^B\right). \quad (3.95)$$

Since

$$(\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X}))_{,B} = \frac{\partial \hat{W}}{\partial \phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\; F^a{}_B + \frac{\partial \hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\; F^a{}_{A|B} + \frac{\partial \hat{W}}{\partial \mathcal{X}^A}\circ(\phi,\boldsymbol{F},\mathcal{X})\;\delta^A{}_B, \quad (3.96)$$

we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}}\left(\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X})\; U^B{}_{|B} + (\hat{W}\circ(\phi,\boldsymbol{F},\mathcal{X}))_{,B}\, U^B + \right.$$
$$\left. + \frac{\partial \hat{W}}{\partial \phi^a}\circ(\phi,\boldsymbol{F},\mathcal{X})\;\eta^a + \frac{\partial \hat{W}}{\partial F^a{}_A}\circ(\phi,\boldsymbol{F},\mathcal{X})\;\eta^a{}_{|A}\right). \quad (3.97)$$

Using Leibniz' rule in the first two terms and in the last two terms and separating the integrals, we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \left[\big(\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})\, U^B\big)_{|B} + \left(\frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X})\, \eta^a \right)_{|A} \right] + \int_{\mathcal{D}} \left[\frac{\partial \hat{W}}{\partial \phi^a} \circ (\phi, \boldsymbol{F}, \mathcal{X}) - \left(\frac{\partial \hat{W}}{\partial F^a{}_A} \circ (\phi, \boldsymbol{F}, \mathcal{X}) \right)_{|A} \right] \eta^a. \tag{3.98}$$

Now we use the definitions (3.61), which, in the context of continuum mechanics, give the body force $\boldsymbol{f}$ and the first Piola-Kirchhoff stress $\boldsymbol{P}$, use $W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})$ and change index A into B in the first integral. So, we have

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \left[\big(W\, U^B\big)_{|B} + \big(\eta^a\, P_a{}^B\big)_{|B} \right] - \int_{\mathcal{D}} \big(f_a + P_a{}^A{}_{|A}\big)\, \eta^a, \tag{3.99}$$

which corresponds to Equation (17) in the paper by Hill (1951). In the first integral, we use $U^B = U^A\ \delta_A{}^B$ in the first term and the definition (3.83) of the total variation $\boldsymbol{w}$ to eliminate $\eta^a = w^a - F^a{}_A\, U^A$ in the second term, and then we split the first integral into two, to obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \left[U^A\ \big(W\ \delta_A{}^B - F^a{}_A\, P_a{}^B\big)\right]_{|B} + + \int_{\mathcal{D}} \big(w^a\, P_a{}^B\big)_{|B} - \int_{\mathcal{D}} \big(f_a + P_a{}^A{}_{|A}\big)\, \eta^a, \tag{3.100}$$

where we recognise the Eshelby stress $\mathfrak{E}_A{}^B = W\ \delta_A{}^B - F^a{}_A\, P_a{}^B$ defined in Equation (3.49). Finally, we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \big(U^A\ \mathfrak{E}_A{}^B\big)_{|B} + \int_{\mathcal{D}} \big(w^a\, P_a{}^B\big)_{|B} - \int_{\mathcal{D}} \big(f_a + P_a{}^A{}_{|A}\big)\, \eta^a, \tag{3.101}$$

which, in component-free formalism, reads

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big(\boldsymbol{\mathfrak{E}}^T \boldsymbol{U}\big) + \int_{\mathcal{D}} \mathrm{Div}\big(\boldsymbol{P}^T \boldsymbol{w}\big) - \int_{\mathcal{D}} (\boldsymbol{f} + \mathrm{Div}\, \boldsymbol{P})\, \boldsymbol{\eta}. \tag{3.102}$$

If the variation (3.102) is evaluated for a configuration ϕ solving the Euler-Lagrange equations (3.67), we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big(\boldsymbol{\mathfrak{C}}^T \boldsymbol{U}\big) + \int_{\mathcal{D}} \mathrm{Div}\big(\boldsymbol{P}^T \boldsymbol{w}\big), \tag{3.103}$$

where the first two integrals contain the contributions to the *Noether current density* $\boldsymbol{\mathfrak{C}}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w}$. The extension of the result (3.103) to the case of the presence of non-integrable body forces $\boldsymbol{f}$ is treated in Appendix A.4.

3.4.3 Eshelby's Results and Conservation of Noether's Current

The variational procedure followed in Section 3.4.2 was conducted by introducing the one-parameter families of transformations $\mathcal{Y}(X) = X + h\,\boldsymbol{U} = \tilde{X}$ and $\bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h\,\boldsymbol{\eta}(\tilde{X})$, which allowed to compute the Gâteaux derivative of total energy $\mathcal{E}_{\mathcal{D}}$ along the pair of directions $(\boldsymbol{\eta}, \boldsymbol{U})$. Transformations of this kind are said to be *symmetries* if they do not alter the numerical value of $\mathcal{E}_{\mathcal{D}}$, i.e., if it holds true that $\mathcal{E}_{\mathcal{D}}(\bar{\phi}, \mathcal{Y}) = \mathcal{E}_{\mathcal{D}}(\phi, \mathcal{X})$ for sufficiently small values of h. Following an argument reported by Hill (1951), a condition ensuring the compliance with this equality and the form-invariance of the Euler-Lagrange equations is obtained by means of what in field theory is called a *divergence transformation* (Hill, 1951; Maugin, 1993). For the case of an infinitesimal symmetry transformation, the divergence transformation reads

$$\int_{\mathcal{D}} (1 + h\,\mathrm{Div}\,\boldsymbol{U})\,\hat{W} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) \circ \mathcal{Y} = \int_{\mathcal{D}} \big[\hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + h\,\mathrm{Div}\,\boldsymbol{\Omega}\big], \tag{3.104}$$

where $\boldsymbol{\Omega} = \hat{\boldsymbol{\Omega}} \circ \mathcal{X}$ is a vector field to be determined. Note that, in order to leave the Euler-Lagrange equations (3.67) invariant, $\hat{\boldsymbol{\Omega}}$ must *not* depend on $\boldsymbol{F}$ (Hill, 1951). By dividing Equation (3.104) by h and taking the limit for $h \to 0$, we obtain

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi,\mathcal{X}) - \int_{\mathcal{D}} \mathrm{Div}\,\boldsymbol{\Omega} = \int_{\mathcal{D}} \big[\mathrm{Div}\big(\boldsymbol{\mathfrak{C}}^T \boldsymbol{U}\big) + \mathrm{Div}\big(\boldsymbol{P}^T \boldsymbol{w}\big) - \mathrm{Div}\,\boldsymbol{\Omega}\big] = 0. \tag{3.105}$$

According to this result, to a given pair $\boldsymbol{U}$ and $\boldsymbol{w}$ there corresponds the conservation law

$$\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) + \mathrm{Div}\big(\boldsymbol{P}^T \boldsymbol{w}\big) - \mathrm{Div}\,\boldsymbol{\Omega} = 0, \tag{3.106}$$

which allows to determine $\boldsymbol{\Omega}$. In several circumstances of interest, such as the one related to the conservation of momentum or angular momentum, one can take $\boldsymbol{\Omega}$ to be zero from the outset and look for transformations $\boldsymbol{U}$ and $\boldsymbol{w}$ leading to conservation laws of the form

$$\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) + \mathrm{Div}\big(\boldsymbol{P}^T \boldsymbol{w}\big) = 0. \tag{3.107}$$

In the remainder of our work, we specialise to this case in order to retrieve Eshelby's result in the light shed by Noether's theorem. Some remarks on divergence transformations are reported in Appendix A.5.

Eshelby (1975) imposed $\boldsymbol{\eta} = -\boldsymbol{F}\,\boldsymbol{U}$, i.e., that the conventional displacement $\boldsymbol{\eta}$ be equal to the negative of the push-forward of the material displacement $\boldsymbol{U}$, as shown in Equation (3.41), in order to preserve compatibility. This condition, in turn, imposes the vanishing of the total variation, i.e., $\boldsymbol{w} = \boldsymbol{\eta} + \boldsymbol{F}\,\boldsymbol{U} = \boldsymbol{0}$. With this hypothesis, the integral of $\mathrm{Div}\big(\boldsymbol{P}^T \boldsymbol{w}\big)$ in Equation (3.103) vanishes identically and the variation reduces to

$$(\partial_{(\boldsymbol{\eta},\boldsymbol{U})}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = \int_{\mathcal{D}} \mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big). \tag{3.108}$$

which coincides with the result shown in Equation (3.48).

Now we can exploit Noether's theorem to obtain Eshelby's final result. Noether's Theorem states that

> *For every continuous symmetry under which the integral $\mathcal{E}_{\mathcal{D}}$ is invariant, there is a conserved current density.*

In this case, the Noether current density is $\mathfrak{E}^T \boldsymbol{U}$. For it to be conserved, the divergence $\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big)$ has to vanish and, in fact, a direct computation, in which the configurational force balance (3.77) is used, yields the condition

$$\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) = \mathfrak{E} : \mathrm{Grad}\,\boldsymbol{U} + (\mathrm{Div}\mathfrak{E})\,\boldsymbol{U} = \mathfrak{E} : \mathrm{Grad}\,\boldsymbol{U} - \mathfrak{F}\,\boldsymbol{U} = 0. \tag{3.109}$$

Equation (3.109) is known as *Noetherian identity* (Podio-Guidugli, 2001), and places restrictions on the class of transformations $\boldsymbol{U}$ that comply with the requirement $\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) = 0$, which can thus be said to be *symmetry transformations*. Indeed, a field $\boldsymbol{U}$ is a symmetry transformation (i.e., it leaves $\mathcal{E}_{\mathcal{D}}$ invariant) if, and only if, it satisfies (3.109) (for a similar result in a different context, see also Grillo et al., 2003, 2019). Looking at (3.109), we notice that, when the inhomogeneity force, $\mathfrak{F}$, vanishes identically i.e., when the body is *materially homogeneous* and, thus, the energy density $\hat{W}$ *does not* depend on the material points, the Noetherian identity reduces to

$$\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big) = \mathfrak{E} : \mathrm{Grad}\,\boldsymbol{U} = 0. \tag{3.110}$$

This result implies that *any* arbitrary uniform displacement field $\boldsymbol{U}$, for which $\mathrm{Grad}\,\boldsymbol{U} = \boldsymbol{0}$, annihilates the divergence of the Noether current density and is, thus, a symmetry transformation. A body endowed with this property is said to enjoy the symmetry of *material homogeneity*. We notice, however, that, when $\mathfrak{F}$ is not null, $\boldsymbol{U}$ may no longer be uniform. This means that $\mathfrak{F}$ *breaks* the symmetry of material homogeneity and a new class of transformations $\boldsymbol{U}$ has to be determined.

We also note that, under the hypothesis of homogeneous material, Equation (3.108) implies the vanishing of the divergence of $\mathfrak{E}^T \boldsymbol{U}$, and not of $\mathfrak{E}$. In order to obtain the vanishing of the divergence of the Eshelby stress $\mathfrak{E}$, we implement the last of Eshelby's hypotheses, namely the fact that the material displacement $\boldsymbol{U}$ is uniform on $\mathcal{D}$ and given by $\boldsymbol{U}(X) = -\boldsymbol{U}_0$, for every $X \in \mathcal{D}$. This implies that in the integral of $\mathrm{Div}\big(\mathfrak{E}^T \boldsymbol{U}\big)$ in Equation (3.111), the displacement $\boldsymbol{U} = -\boldsymbol{U}_0$ can be brought out of the divergence, i.e.,

$$(\partial_{(\boldsymbol{\eta},-\boldsymbol{U}_0)}\mathcal{E}_{\mathcal{D}})(\phi, \mathcal{X}) = -\int_{\mathcal{D}} (\mathrm{Div}\,\mathfrak{E})\,\boldsymbol{U}_0. \tag{3.111}$$

which coincides with Equation (3.50) obtained using Eshelby's original procedure. Now, the vanishing of the variation due to the homogeneity of the material implies the vanishing of Div $\mathfrak{C}$, as in the strong form (3.77) considered with condition (3.78).

Chapter 4

Eshelby Force and Power for Uniform Bodies

This chapter is based on Alhasadi et al. (2019)

In his pioneering study on the mechanics of materials with dislocations, Eshelby (1951) introduced the mechanical counterpart of Maxwell's energy-momentum tensor, which is now commonly referred to as the Eshelby stress. The integral of the divergence of the Eshelby stress over a region containing a defect represents the *material* or *configurational* force acting on the defect necessary to cause the *change in configuration* needed to displace the defect. In a subsequent work, Eshelby (1975) showed that, for an elastic body, the Eshelby stress arises naturally from a variational approach, as the proper measure of stress in the *material* balance of linear momentum.

In another fundamental work, Noll (1967) defined a uniform body as a body in which every point is made of the same material: in this definition, the energy function at each pair of points can be related by a linear isomorphism (i.e., an invertible linear map) $\boldsymbol{P}$, called *material isomorphism*. Noll (1967) also showed that the torsion of the linear connection induced by the material isomorphisms captures the presence of inhomogeneities in the body. We recall that, given a connection (also referred to as covariant derivative) with Christoffel symbols Γ^A_{BC}, the torsion is the third-order tensor **Tor** with components $\mathrm{Tor}^A{}_{BC} = \Gamma^A_{BC} - \Gamma^A_{CB}$. In the case of Noll's material isomorphism $\boldsymbol{P}$, the Christoffel symbols are given by $\Gamma^A_{CB} = P^A{}_\alpha\,(\boldsymbol{P}^{-1})^\alpha{}_{C,B}$.

Later, Epstein and Maugin (1990) succeeded in finding a natural relation between Noll's and Eshelby's theories by deriving an identity involving a "modified" Eshelby stress and the torsion of the material connection, for a uniform body with inhomogeneities, in the static case, in the absence of body forces and without growth.

The objective of this work is to use Eshelby's variational approach (Eshelby, 1975) to obtain an analogous differential identity in a uniform body in a general thermoelastic framework, involving inertial effects, body forces and volumetric growth. To this end, it is convenient to introduce the terminology *Eshelby force* for the divergence of the Eshelby stress. In the classic static case, the Eshelby force is simply the negative of the configurational force. The identity that we obtain involves the $\boldsymbol{P}$-divergence of the modified Eshelby stress, which we call *modified Eshelby force*, the torsion of the material connection, thermal, growth and linear momentum terms. Moreover, always using Eshelby's variational approach (Eshelby, 1975), we define the *Eshelby power* as the temporal analogue of the Eshelby force for a body whose Lagrangian explicitly depends on time. Then, an equivalent construction is employed to derive the differential identity of the *modified Eshelby power* for a uniform thermoelastic body. We also conclude that the driving force of the processes of growth-remodelling is the *Mandel stress*. Eventually, we derive a relation between the differential identities for the *modified Eshelby power* and the *modified Eshelby force* in the dynamical case for a uniform body.

Remark. Initially, Eshelby only limited his study to linear elasticity (Eshelby, 1951) and treated the case of non-linear elasticity in a later paper (Eshelby, 1975), in which he used the displacement gradient $H^i{}_J = u^i{}_{|J}$ (in our notation) as the kinematical descriptor of deformation, but also pointed out how the formulation in the deformation gradient $F^i{}_J = \phi^i{}_{,J}$, which we employ, would look like. As pointed out by Ericksen (1995, 1998), the fact that Eshelby used the linear theory in his first works on defects and in most of his applicative works may have caused confusion in some authors.

4.1 Theoretical Framework

In this section, we start by briefly reporting the notation employed throughout the work, which essentially follows the works by Marsden and Hughes (1983) and Epstein and Maugin (1990). Then, we report the material balance equations for a continuum, following Epstein and Maugin (Epstein and Maugin, 2000). Subsequently, we discuss the Lagrangian of a thermoelastic body, following in some respect the work by Grillo et al. (2003). Finally, we recall the linear concepts of connection and covariant derivative.

4.1.1 Notation

The physical space $\mathcal{S}$ is here considered, for simplicity, to coincide with the affine space $\mathbb{E}^3$ modelled on $\mathbb{R}^3$. A continuum body is identified with one placement $\mathcal{B}$, chosen as the reference configuration, in the physical space $\mathcal{S}$. A motion of the body is a time-parametrised mapping

$$\phi(\,\cdot\,,t) : \mathcal{B} \to \mathcal{S} : X \mapsto x = \phi(X,t) \tag{4.1}$$

which, at every time t, maps *material* points $X = (X^1, X^2, X^3)$ in the body $\mathcal{B}$ into *spatial* points $x = (x^1, x^2, x^3)$ in $\mathcal{S}$. The map ϕ is called configuration map, and is such that its codomain-restriction $\phi(\,\cdot\,,t) : \mathcal{B} \to \phi(\mathcal{B},t)$ is a diffeomorphism, i.e., a continuous and differentiable map, which is invertible, with a continuous and differentiable inverse $\phi^{-1}(\,\cdot\,,t) \equiv [\phi(\,\cdot\,,t)]^{-1}$.

The *Lagrangian velocity* is defined as the partial derivative $\dot{\phi}(\,\cdot\,,t) \equiv (\partial_t\phi)(\,\cdot\,,t) : \mathcal{B} \to T\mathcal{S}$ and is a spatial vector field *over* the configuration ϕ (Marsden and Hughes, 1983). The *Eulerian velocity* is defined as the spatial vector field $\boldsymbol{v}(\,\cdot\,,t) : \phi(\mathcal{B}) \to T\mathcal{S}$ such that $\boldsymbol{v}(x,t) = \dot{\phi}(X,t)$.

At each material point $X \in \mathcal{B}$, the tangent map of the configuration is the Fréchet differential of ϕ, i.e., the linear map

$$\boldsymbol{F}(X,t) \equiv (T\phi)(X,t) : T_X\mathcal{B} \to T_x\mathcal{S}, \tag{4.2}$$

called *deformation gradient*, mapping material vectors in the tangent space $T_X\mathcal{B}$ into spatial vectors in the tangent space $T_x\mathcal{S}$ and such that, in coordinate charts $\{X^A\}$ in $\mathcal{B}$ and $\{x^a\}$ in $\mathcal{S}$, it is represented by $F^a{}_A = \phi^a{}_{,A}$.

The body $\mathcal{B}$ and the space $\mathcal{S}$ are equipped with the metric tensors $\boldsymbol{G}$ and $\boldsymbol{g}$, which induce inner products in the tangent bundles $T\mathcal{B}$ and $T\mathcal{S}$, respectively. We denote the inner product with a simple "low dot", in both cases. For two material vectors, $\boldsymbol{W}.\boldsymbol{Y} \equiv \boldsymbol{W}\,\boldsymbol{G}\,\boldsymbol{Y} = W^A\,G_{AB}\,Y^B$ and, for two spatial vectors, $\boldsymbol{w}.\boldsymbol{y} \equiv \boldsymbol{w}\,\boldsymbol{g}\,\boldsymbol{y} = w^a\,g_{ab}\,y^b$. The contraction of a vector and a covector is denoted by Dirac's *bra-ket*, e.g., for the case of material vectors and covectors, $\langle \boldsymbol{\Psi} \,|\, \boldsymbol{W} \rangle = \langle \boldsymbol{W} \,|\, \boldsymbol{\Psi} \rangle = \Psi_A\,W^A$.

4.1.2 Connections and Covariant Derivatives

A *linear connection* in a manifold $\mathcal{B}$ is a map that, to every pair of smooth vector fields $\boldsymbol{U}, \boldsymbol{V} : \mathcal{B} \to T\mathcal{B}$, associates the vector field $\nabla_{\boldsymbol{U}}\boldsymbol{V}$, called the *covariant derivative* of $\boldsymbol{V}$ in the direction of $\boldsymbol{U}$. The covariant derivative has three fundamental properties, for every smooth vector fields $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W} : \mathcal{B} \to T\mathcal{B}$ and for every differentiable scalar fields $f, g : \mathcal{B} \to \mathbb{R}$,

$$\nabla_{\boldsymbol{U}}(\boldsymbol{V} + \boldsymbol{W}) = \nabla_{\boldsymbol{U}}\boldsymbol{V} + \nabla_{\boldsymbol{U}}\boldsymbol{W}, \qquad \text{additivity in the argument,} \tag{4.3a}$$

$$\nabla_{\boldsymbol{U}}(f\,\boldsymbol{V}) = f\,\nabla_{\boldsymbol{U}}\boldsymbol{V} + (\partial_{\boldsymbol{U}} f)\boldsymbol{V}, \qquad \text{Leibniz' rule in the argument,} \tag{4.3b}$$

$$\nabla_{f\,\boldsymbol{U}+g\,\boldsymbol{V}}\boldsymbol{W} = f\,\nabla_{\boldsymbol{U}}\boldsymbol{W} + g\,\nabla_{\boldsymbol{V}}\boldsymbol{W}, \qquad \text{linearity in the direction of differentiation,} \tag{4.3c}$$

where $\partial_{\boldsymbol{U}} f$ is the directional derivative of f with respect to $\boldsymbol{U}$. Because of the linearity in the direction of differentiation, property (4.3c), the covariant derivative $\nabla_{\boldsymbol{U}}\boldsymbol{V}$ defines the linear map

$$\operatorname{Grad}\boldsymbol{V} : T\mathcal{B} \to T\mathcal{B} : \boldsymbol{U} \mapsto [\operatorname{Grad}\boldsymbol{V}]\boldsymbol{U} \equiv \nabla_{\boldsymbol{U}}\boldsymbol{V}, \tag{4.4}$$

called the *gradient* of $\boldsymbol{V}$. The gradient of a vector field is a tensor often denoted by $\nabla\boldsymbol{V}$ (Marsden and Hughes, 1983). Other possible notations are $\Gamma\boldsymbol{V}$ (Noll, 1967) and $D\boldsymbol{V}$ (Bishop and Goldberg, 1968).

In a coordinate chart $\{X^A\}$ with associated basis vectors $\boldsymbol{E}_A$, we define the *Cristoffel symbols* of the connection ∇ via

$$\nabla_{\boldsymbol{E}_B}\boldsymbol{E}_A = [\mathrm{Grad}\,\boldsymbol{E}_A]\,\boldsymbol{E}_B = \Gamma^C_{AB}\boldsymbol{E}_C. \tag{4.5}$$

Remark. The convention $\nabla_{\boldsymbol{E}_B}\boldsymbol{E}_A = \Gamma^C_{AB}\boldsymbol{E}_C$ of Eq. (4.5) for the ordering of the lower indices of the Christoffel symbols is the same adopted, e.g., by Noll (1967), Bishop and Goldberg (1968) and Epstein and Maugin (1990). Other Authors, e.g., Kobayashi and Nomizu (1963) and Marsden and Hughes (1983), use the convention $\nabla_{\boldsymbol{E}_B}\boldsymbol{E}_A = \Gamma^C_{BA}\boldsymbol{E}_C$.

Using Christoffel symbols (with the convention of Eq. (4.5)) and the properties (4.3) of connections, the covariant derivative $\nabla_{\boldsymbol{U}}\boldsymbol{V}$ has component expression

$$\begin{aligned}\nabla_{\boldsymbol{U}}\boldsymbol{V} &= U^B\,\nabla_{\boldsymbol{E}_B}(V^A\,\boldsymbol{E}_A) = U^B\,(V^A{}_{,B}\,\boldsymbol{E}_A + V^A\,\Gamma^C_{AB}\,\boldsymbol{E}_C)\\ &= U^B\,(V^A{}_{,B} + \Gamma^A_{CB}\,V^C)\,\boldsymbol{E}_A,\end{aligned} \tag{4.6}$$

from which, using the definition $U^B = \boldsymbol{E}^B(\boldsymbol{U}) = \langle \boldsymbol{E}^B \,|\, \boldsymbol{U}\rangle$ of basis covector, we obtain the representation of the gradient as

$$\mathrm{Grad}\,\boldsymbol{V} = (V^A{}_{,B} + \Gamma^A_{CB}\,V^C)\,\boldsymbol{E}_A \otimes \boldsymbol{E}^B = V^A{}_{;B}\;\boldsymbol{E}_A \otimes \boldsymbol{E}^B, \tag{4.7}$$

where

$$V^A{}_{;B} = V^A{}_{,B} + \Gamma^A_{CB}\,V^C \tag{4.8}$$

is the component form of the covariant derivative, and is often simply called *covariant derivative* itself.

The *Lie bracket* of two vector fields $\boldsymbol{U}$ and $\boldsymbol{V}$ is defined as the vector field $[\boldsymbol{U},\boldsymbol{V}]$ such that, for every differentiable scalar field f,

$$\partial_{[\boldsymbol{U},\boldsymbol{V}]} f = \partial_{\boldsymbol{U}} \partial_{\boldsymbol{V}} f - \partial_{\boldsymbol{V}} \partial_{\boldsymbol{U}} f. \tag{4.9}$$

In components, using the differentiability of f, i.e., $\partial_{\boldsymbol{U}} f = (\text{Grad}\, f)\, \boldsymbol{U} = U^A f_{,A}$, Eq. (4.9) reads

$$\begin{aligned}
\partial_{[\boldsymbol{U},\boldsymbol{V}]} f &= U^A (V^B f_{,B})_{,A} - V^A (U^B f_{,B})_{,A} \\
&= U^A V^B{}_{,A} f_{,B} + U^A V^B f_{,BA} - V^A U^B{}_{,A} f_{,B} - V^A U^B f_{,BA} \\
&= U^A V^B{}_{,A} f_{,B} - V^A U^B{}_{,A} f_{,B},
\end{aligned} \tag{4.10}$$

from which, using again the differentiability of f,

$$[\boldsymbol{U}, \boldsymbol{V}]^B = U^A V^B{}_{,A} - V^A U^B{}_{,A}. \tag{4.11}$$

Note that Eq. (4.11) *does not* involve covariant differentiation of the vector fields $\boldsymbol{U}$ and $\boldsymbol{V}$.

The *torsion* of a connection ∇ is defined as the third-order tensor

$$\text{Tor} : T\mathcal{B} \times T\mathcal{B} \to T\mathcal{B} : (\boldsymbol{U}, \boldsymbol{V}) \mapsto \nabla_{\boldsymbol{U}} \boldsymbol{V} - \nabla_{\boldsymbol{V}} \boldsymbol{U} - [\boldsymbol{U}, \boldsymbol{V}]. \tag{4.12}$$

The component expression of the torsion is found using Eqs. (4.1.2) and (4.11), as

$$\text{Tor}^A{}_{BC} = \Gamma^A_{BC} - \Gamma^A_{CB}, \tag{4.13}$$

which shows that, when the torsion vanishes, the Christoffel symbols are symmetric, i.e., $\Gamma^A_{BC} = \Gamma^A_{CB}$.

A metric tensor $\boldsymbol{G}$ is said to be compatible with a connection ∇ if, and only if

$$\text{Grad}\, \boldsymbol{G} = \mathbf{O}, \qquad G_{AB;C} = 0. \tag{4.14}$$

Levi-Civita's fundamental theorem of Differential Geometry (see, e.g., (Marsden and Hughes, 1983)) ensures that, given a metric tensor $\boldsymbol{G}$, there is a unique connection that is torsion-free and compatible with $\boldsymbol{G}$. In this case, the connection is called a *metric connection* and, more specifically, the connection associated by Levi-Civita's theorem to the metric $\boldsymbol{G}$ is called $\boldsymbol{G}$-connection, and its Christoffel symbols can be shown to be equal to (Marsden and Hughes, 1983)

$$\mathrm{K}^{A}_{BC} = \tfrac{1}{2}\, G^{AD}(G_{CD,B} + G_{BD,C} - G_{BC,D}), \tag{4.15}$$

which are clearly symmetric in the low indices, from which follows the vanishing of the $\boldsymbol{G}$-torsion $\mathrm{K}^{A}_{BC} - \mathrm{K}^{A}_{CB}$.

Note that we use K as the kernel symbol for the Christoffel symbols of a metric connection, not to confuse them with those of a generic connection, for which the kernel symbol is Γ. Moreover, we indicate covariant differentiation with respect to the $\boldsymbol{G}$-connection by a vertical bar, i.e., we write

$$V^{A}{}_{|B} = V^{A}{}_{,B} + \mathrm{K}^{A}_{CB}\, V^{C}. \tag{4.16}$$

4.1.3 Material Balance Equations

Here we report the material form of the balance equations as presented by Epstein and Maugin (2000) for a body with growth. All integral balance equations are written in an arbitrary open subset $\mathcal{R} \subset \mathcal{B}$. We remark that, as noted, e.g., by Marsden and Hughes (1983), the balance of linear momentum in the integral form (4.19) and the balance of angular momentum in the integral form (4.22) only make sense if the space $\mathcal{S}$ is affine *and* if Cartesian coordinates are employed, so that one can integrate the components one by one. One way to avoid the problem of the integration of a covector (or vector) field is to use the equivalent approach by Noll (1974) or the equivalent one by Green and Rivlin (1964) (both reported in the text by Marsden and Hughes (1983)), in which requiring invariance of the equation of balance of energy (4.25) with respect to translations and rotations yields the differential equations of balance of linear momentum (4.21) and angular momentum (4.23), respectively. Here, under the tacit assumption of Cartesian coordinates in an affine space, we write the balance of linear momentum and the balance of angular momentum in the integral forms (4.19) and (4.22), involving the integration of covector/vector quantities.

The integral form of the balance of mass reads

$$\partial_t \int_{\mathcal{R}} \varrho_R = \int_{\mathcal{R}} \Pi + \int_{\partial\mathcal{R}} \langle \boldsymbol{M} \mid \boldsymbol{N} \rangle, \tag{4.17}$$

where ϱ_R is the referential mass density, Π is the mass source density per unit referential volume and $\boldsymbol{M}$ is the mass flux density per unit referential area. Localising, we have

$$\dot{\varrho}_R = \Pi + \operatorname{Div} \boldsymbol{M} \tag{4.18}$$

where Div is the material divergence (with respect to the material coordinates X^A).

The integral form of the material balance of linear momentum is expressed by

$$\partial_t \int_{\mathcal{R}} \varrho_R \, \boldsymbol{g} \, \dot{\phi} = \int_{\mathcal{R}} [\boldsymbol{f} + \Pi \, \boldsymbol{g} \, \dot{\phi} + \boldsymbol{z}] + \int_{\partial\mathcal{R}} [\boldsymbol{T} + (\boldsymbol{g} \, \dot{\phi}) \otimes \boldsymbol{M} + \boldsymbol{A}] \, \boldsymbol{N}, \tag{4.19}$$

where $\varrho_R \, \boldsymbol{g} \, \dot{\phi}$ (abuse of notation for $\varrho_R \, (\boldsymbol{g} \circ \phi) \, \dot{\phi}$) is the linear momentum density, obtained by lowering the contravariant index of the vector $\varrho_R \, \dot{\phi}$ by means of the metric tensor $\boldsymbol{g}$, $\boldsymbol{f}$ is the body force per unit referential volume, $\Pi \, \boldsymbol{g} \, \dot{\phi}$ is the momentum rate associated with the volumetric growth, $\boldsymbol{z}$ is the “irreversible” momentum rate, $\boldsymbol{T}$ is the first Piola-Kirchhoff stress, $(\boldsymbol{g} \, \dot{\phi}) \otimes \boldsymbol{M}$ is the momentum flux associated with the mass flux and $\boldsymbol{A}$ is the irreversible momentum flux. Localising, we obtain

$$\partial_t (\varrho_R \, \boldsymbol{g} \, \dot{\phi}) = \boldsymbol{f} + \Pi \, \boldsymbol{g} \, \dot{\phi} + \boldsymbol{z} + \operatorname{Div}(\boldsymbol{T} + (\boldsymbol{g} \, \dot{\phi}) \otimes \boldsymbol{M} + \boldsymbol{A}) \tag{4.20}$$

and, using the balance of mass (4.18), we get to

$$\varrho_R\, \boldsymbol{g}\, \ddot{\phi} = \boldsymbol{f} + \boldsymbol{z} + \operatorname{Div} \boldsymbol{T} + \boldsymbol{g}\, \dot{\boldsymbol{F}} \boldsymbol{M} + \operatorname{Div} \boldsymbol{A}, \tag{4.21}$$

where we used $\operatorname{Grad} \dot{\phi} = \dot{\boldsymbol{F}}$ and $\boldsymbol{F}$ is the deformation gradient.

The material balance of angular momentum reads

$$\partial_t \int_{\mathcal{R}} \boldsymbol{r} \times (\varrho_R\, \boldsymbol{g}\, \dot{\phi}) = \int_{\mathcal{R}} \boldsymbol{r} \times [\boldsymbol{f} + \Pi\, \boldsymbol{g}\, \dot{\phi} + \boldsymbol{z}] + \int_{\partial\mathcal{R}} \boldsymbol{r} \times \big[[\boldsymbol{T} + (\boldsymbol{g}\, \dot{\phi}) \otimes \boldsymbol{M} + \boldsymbol{A}]\, \boldsymbol{N}\big], \tag{4.22}$$

and, when localised using balance of mass and linear momentum, yields the symmetry

$$\boldsymbol{g}^{-1} \widetilde{\boldsymbol{T}}\, \boldsymbol{F}^T = \boldsymbol{F}\, \widetilde{\boldsymbol{T}}^T \boldsymbol{g}^{-1} \tag{4.23}$$

of the Kirchhoff-like stress $\widetilde{\boldsymbol{\tau}} = \boldsymbol{g}^{-1} \widetilde{\boldsymbol{T}} \boldsymbol{F}^T$ (in components, $\widetilde{\tau}^{ac} = g^{ab}\, \widetilde{T}_b{}^C\, F^c{}_C$), where the Piola-like stress $\widetilde{\boldsymbol{T}}$ is

$$\widetilde{\boldsymbol{T}} = \boldsymbol{T} + (\boldsymbol{g}\, \dot{\phi}) \otimes \boldsymbol{M} + \boldsymbol{A}. \tag{4.24}$$

The integral form of the material balance of energy reads

$$\begin{aligned} \partial_t \int_{\mathcal{R}} (\tfrac{1}{2}\, \varrho_R\, \dot{\phi} \,.\, \dot{\phi} + \varrho_R\, \mathcal{E}) = \int_{\mathcal{R}} (\tfrac{1}{2}\, \Pi\, \dot{\phi} \,.\, \dot{\phi} + \Pi\, \mathcal{E} + \langle \boldsymbol{f} \mid \dot{\phi} \rangle + \mathcal{H} + \mathcal{U} + \langle \boldsymbol{z} \mid \dot{\phi} \rangle) \\ + \int_{\partial\mathcal{R}} \big\langle [\dot{\phi}\, (\boldsymbol{T} + \boldsymbol{A}) + (\mathcal{E} + \tfrac{1}{2}\, \dot{\phi} \,.\, \dot{\phi}) \boldsymbol{M} + \boldsymbol{Q}] \,\big|\, \boldsymbol{N} \big\rangle, \end{aligned} \tag{4.25}$$

where $\mathcal{E}$ is the internal energy per unit mass, $\mathcal{H}$ represents the rate of non-mechanical energy supply per unit mass, $\mathcal{U}$ is the non-compliant volumetric contribution to the internal energy, and

$\boldsymbol{Q} = J\,\boldsymbol{q} \circ (\phi, \tau)\,\boldsymbol{F}^{-T}$ denotes the material heat flux. The local form of Eq. (4.25) takes the form

$$\partial_t \left(\tfrac{1}{2} \varrho_R\, \dot{\phi} \,.\, \dot{\phi} + \varrho_R\, \mathcal{E}\right) = \tfrac{1}{2}\, \Pi\, \dot{\phi} \,.\, \dot{\phi} + \Pi\, \mathcal{E} + \langle \boldsymbol{f} \mid \dot{\phi} \rangle + \mathcal{H} + \mathcal{U} + \langle \boldsymbol{z} \mid \dot{\phi} \rangle + \\ + \operatorname{Div}\big[\dot{\phi}\,(\boldsymbol{T} + \boldsymbol{A}) + (\mathcal{E} + \tfrac{1}{2}\, \dot{\phi} \,.\, \dot{\phi})\boldsymbol{M} + \boldsymbol{Q}\big], \tag{4.26}$$

By exploiting the balance of mass (4.18) and linear momentum (4.21), we obtain

$$\varrho_R\, \dot{\mathcal{E}} = \mathcal{H} + \mathcal{U} + (\boldsymbol{T} + \boldsymbol{A}) : \dot{\boldsymbol{F}} + \langle \operatorname{Grad} \mathcal{E} \mid \boldsymbol{M} \rangle + \operatorname{Div} \boldsymbol{Q}. \tag{4.27}$$

The material form of the Clausius-Duhem entropy inequality, which expresses the second law of thermodynamics, is

$$\partial_t \int_{\mathcal{R}} \varrho_R\, \mathcal{S} \geq \int_{\mathcal{R}} \left[\Pi\, \mathcal{S} + \frac{\mathcal{H}}{\Theta} + \frac{\mathcal{U}}{\Theta} + \mathcal{Z}\right] + \int_{\partial\mathcal{R}} \left\langle \left[\frac{\boldsymbol{Q}}{\Theta} + \mathcal{S}\, \boldsymbol{M}\right] \,\middle|\, \boldsymbol{N} \right\rangle, \tag{4.28}$$

where $\mathcal{S}$ is the entropy per unit mass, Θ is the absolute temperature and $\mathcal{Z}$ is the volumetric source of non-compliant entropy. Localising and multiplying by the temperature Θ, we obtain

$$\Theta\, \partial_t(\varrho_R\, \mathcal{S}) \geq \Pi\, \Theta\, \mathcal{S} + \mathcal{H} + \mathcal{U} + \Theta\, \mathcal{Z} \\ - \frac{1}{\Theta} \langle \operatorname{Grad} \Theta \mid \boldsymbol{Q} \rangle + \operatorname{Div} \boldsymbol{Q} + \Theta\, \langle \operatorname{Grad} \mathcal{S} \mid \boldsymbol{M} \rangle + \Theta\, \mathcal{S} \operatorname{Div} \boldsymbol{M}. \tag{4.29}$$

Using balance of mass (4.18) and balance of energy (4.27), and grouping the terms in $\boldsymbol{M}$, we get

$$\varrho_R \Theta\, \dot{\mathcal{S}} \geq \varrho_R\, \dot{\mathcal{E}} - (\boldsymbol{T} + \boldsymbol{A}) : \dot{\boldsymbol{F}} + \Theta\, \mathcal{Z} - \frac{1}{\Theta} \langle \operatorname{Grad} \Theta \mid \boldsymbol{Q} \rangle - \langle \operatorname{Grad} \mathcal{E} - \Theta \operatorname{Grad} \mathcal{S} \mid \boldsymbol{M} \rangle. \tag{4.30}$$

We now introduce the material Helmholtz free energy per unit mass as

$$\Psi = \mathcal{E} - \Theta\, \mathcal{S}, \tag{4.31}$$

the time derivative of which is

$$\dot{\Psi} = \dot{\mathcal{E}} - \dot{\Theta}\,\mathcal{S} - \Theta\,\dot{\mathcal{S}}, \tag{4.32}$$

and substitute in Eq. (4.30), to obtain

$$-\varrho_R\dot{\Psi} - \varrho_R\mathcal{S}\,\dot{\Theta} - \Theta\,\mathcal{Z} + (\boldsymbol{T} + \boldsymbol{A}) : \dot{\boldsymbol{F}} + \frac{1}{\Theta}\,\langle \boldsymbol{Q}\,|\,\mathrm{Grad}\,\Theta\rangle + \langle \mathrm{Grad}\,\mathcal{E} - \Theta\,\mathrm{Grad}\,\mathcal{S}\,|\,\boldsymbol{M}\rangle \geq 0. \tag{4.33}$$

As shown by Epstein and Maugin (2000), a first-grade material *does not* admit a non-zero mass flux. Thus, we can impose $\boldsymbol{M} = \boldsymbol{0}$ in all balance equations. Furthermore, we shall make the simplifying assumption of zero non-compliant terms, i.e., $\boldsymbol{z} = \boldsymbol{0}$, $\boldsymbol{A} = \boldsymbol{0}$, $\mathcal{U} = 0$ and $\mathcal{Z} = 0$. This amounts to assuming that, "miraculously" (to use the same adverb used by Epstein and Maugin (2000)), the new mass generated via Π enters the system at *exactly* the same linear momentum, stress (non-compliant momentum flux), internal energy and entropy as the mass already in the system. Thus, the balance equations and the entropy inequality reduce to

$$\dot{\varrho}_R = \Pi, \tag{4.34a}$$

$$\varrho_R\,\boldsymbol{g}\,\ddot{\phi} = \boldsymbol{f} + \mathrm{Div}\,\boldsymbol{T}, \tag{4.34b}$$

$$\boldsymbol{g}^{-1}\boldsymbol{T}\,\boldsymbol{F}^T = \boldsymbol{F}\,\boldsymbol{T}^T\boldsymbol{g}^{-1}, \tag{4.34c}$$

$$\varrho_R\,\dot{\mathcal{E}} = \boldsymbol{T} : \dot{\boldsymbol{F}} + \mathcal{H} + \mathrm{Div}\,\boldsymbol{Q}, \tag{4.34d}$$

$$0 \leq -\varrho_R\dot{\Psi} - \varrho_R\mathcal{S}\,\dot{\Theta} + \boldsymbol{T} : \dot{\boldsymbol{F}} + \frac{1}{\Theta}\,\langle \boldsymbol{Q}\,|\,\mathrm{Grad}\,\Theta\rangle, \tag{4.34e}$$

which are identical to the standard equations in the absence of growth, except for the volumetric source of mass Π in the right-hand side of the balance of mass. Note also that the balance of angular momentum reduces to the standard one (with the symmetry of the standard Kirchhoff stress $\boldsymbol{\tau} = \boldsymbol{g}^{-1}\boldsymbol{T}\boldsymbol{F}^T$, with components $\tau^{ac} = g^{ab}\,T_b{}^C\,F^c{}_C$), since the Piola-like stress $\widetilde{\boldsymbol{T}}$ of Eq. (4.24) reduces to the standard Piola-Kirchhoff stress $\boldsymbol{T}$.

For our purposes, it is convenient to define the internal energy, entropy and Helmholtz free energy per unit reference volume as

$$E = \varrho_R\, \mathcal{E}, \tag{4.35a}$$

$$S = \varrho_R\, \mathcal{S}, \tag{4.35b}$$

$$W = \varrho_R\, \Psi, \tag{4.35c}$$

which are related by the analogue of Eq. (4.31), i.e.,

$$W = E - \Theta\, S, \tag{4.36}$$

whose time derivatives are, using conservation of mass (4.34a),

$$\dot{E} = \varrho_R\, \dot{\mathcal{E}} + \Pi\, \mathcal{E} = \varrho_R\, \dot{\mathcal{E}} + \Pi\, E/\varrho_R, \tag{4.37a}$$

$$\dot{S} = \varrho_R\, \dot{\mathcal{S}} + \Pi\, \mathcal{S} = \varrho_R\, \dot{\mathcal{S}} + \Pi\, S/\varrho_R, \tag{4.37b}$$

$$\dot{W} = \varrho_R\, \dot{\Psi} + \Pi\, \Psi = \varrho_R\, \dot{\Psi} + \Pi\, W/\varrho_R, \tag{4.37c}$$

which are related by

$$\dot{W} = \dot{E} - \dot{\Theta}\, S - \Theta\, \dot{S}, \tag{4.38}$$

With these definitions, we can rewrite the balance of energy and the entropy inequality as

$$\dot{E} - \Pi\, \frac{E}{\varrho_R} = \boldsymbol{T} : \dot{\boldsymbol{F}} + \mathcal{H} + \operatorname{Div} \boldsymbol{Q}, \tag{4.39a}$$

$$0 \leq -\dot{W} + \Pi\, \frac{W}{\varrho_R} - S\, \dot{\Theta} + \boldsymbol{T} : \dot{\boldsymbol{F}} + \frac{1}{\Theta}\, \langle \boldsymbol{Q} \,|\, \operatorname{Grad} \Theta \rangle. \tag{4.39b}$$

4.1.4 Lagrangian Density of a Thermoelastic Body

We assume to have a thermoelastic body with Lagrangian density function

$$\mathcal{L}(X,t) = \hat{\mathcal{L}}(\phi(X,t), \dot{\phi}(X,t), \boldsymbol{F}(X,t), \Theta(X,t), X, t), \tag{4.40}$$

where $\mathcal{L}$ is the Lagrangian density *field* (per unit reference volume), and $\hat{\mathcal{L}}$ is the corresponding *constitutive function*, whose arguments are the values of the configuration map ϕ (with components ϕ^a), of the velocity $\dot{\phi}$ (with components $\dot{\phi}^a$), of the deformation gradient $\boldsymbol{F}$ (with components ${F^a}_A = {\phi^a}_{,A}$) and of the absolute temperature $\Theta > 0$, as well as of the point X and of time t. If we use the material identity map $\mathcal{X}$ (such that $\mathcal{X}(X,t) = X$, in components, $\mathcal{X}^A(X,t) = X^A$) and the time map τ (such that $\tau(X,t) = t$), the Lagrangian density can be rewritten in the form

$$\begin{aligned}\mathcal{L}(X,t) &= \hat{\mathcal{L}}(\phi(X,t), \dot{\phi}(X,t), \boldsymbol{F}(X,t), \Theta(X,t), \mathcal{X}(X,t), \tau(X,t)) \\ &= [\hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau)](X,t),\end{aligned} \tag{4.41}$$

where "$\circ$" is the function composition symbol. Thus, by dropping the argument (X,t) on both sides, we can refer to the *function* rather than to the *values* and write

$$\mathcal{L} = \hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau). \tag{4.42}$$

This allows us to still express the explicit dependence of the Lagrangian density on the point X and time t. We shall use this notation throughout this work for all constitutive functions.

The Lagrangian density (per unit reference volume) $\mathcal{L}$ is defined as

$$\mathcal{L} = K - W_{\text{tot}}, \quad \Rightarrow \quad \hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau) = \hat{K} \circ (\dot{\phi}, \mathcal{X}, \tau) - \hat{W}_{\text{tot}} \circ (\phi, \Theta, \boldsymbol{F}, \mathcal{X}, \tau), \tag{4.43}$$

In Eq. (4.43), K is the kinetic energy density (per unit reference volume) is

$$K = \frac{1}{2}\, \varrho_R\, \dot{\phi}.\dot{\phi}\,; \quad \varrho_R = \hat{\varrho}_R \circ (\mathcal{X}, \tau) \quad \Rightarrow \quad \hat{K} \circ (\dot{\phi}, \mathcal{X}, \tau) = \frac{1}{2}\, \hat{\varrho}_R \circ (\mathcal{X}, \tau)\, \dot{\phi}.\dot{\phi}, \tag{4.44}$$

where $\varrho_R = \hat{\varrho}_R \circ (\mathcal{X}, \tau)$ denotes the referential mass density (which could depend explicitly not only on the point X, but also on time t, in the case of volumetric growth), and $\hat{W}_{\text{tot}}$ is the total potential energy, given by the sum of the Helmholtz free energy per unit referential volume W and the potential of the external forces W_{ext}, i.e.,

$$W_{\text{tot}} = \hat{W}_{\text{tot}} \circ (\phi, \boldsymbol{F}, \Theta, \mathcal{X}, \tau) = \hat{W} \circ (\boldsymbol{F}, \Theta, \mathcal{X}, \tau) + \hat{W}_{\text{ext}} \circ \phi. \tag{4.45}$$

The external body force density (per unit reference volume) $\boldsymbol{f}$ (with components f_a), the linear momentum density (per unit reference volume) $\varrho_R\, \boldsymbol{g}\, \dot{\phi}$ (with components $\varrho_R\, g_{ab}\, \dot{\phi}^b$), the entropy density (per unit reference volume) S, and the first Piola-Kirchhoff stress $\boldsymbol{T}$ (with components ${T_a}^A$) are given by

$$f_a = \frac{\partial \hat{\mathcal{L}}}{\partial \phi^a} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau), \tag{4.46a}$$

$$\varrho_R\, g_{ab}\, \dot{\phi}^b = \frac{\partial \hat{\mathcal{L}}}{\partial \dot{\phi}^a} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau), \tag{4.46b}$$

$$S = \frac{\partial \hat{\mathcal{L}}}{\partial \Theta} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau), \tag{4.46c}$$

$${T_a}^A = -\frac{\partial \hat{\mathcal{L}}}{\partial {F^a}_A} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau). \tag{4.46d}$$

Remark. Some authors prefer to exclude the configuration ϕ from the arguments of the constitutive function $\hat{W}$ of the potential energy W, in order to treat W strictly as the elastic energy, and this results in having the negative of the pull-back of the body force feature in the material balance. See, for instance, the works by Ericksen (see Eq. (4) in (Ericksen, 1998)) and Maugin (see Eq. (9) in (Maugin, 2006)). While this choice is certainly legitimate, we prefer to follow Eshelby (1975), who in turns follows the standard methods of Field Theory, and include *both* ϕ and $\boldsymbol{F} = T\phi$ among the arguments of our Lagrangian, so that the derivative of the Lagrangian with respect to ϕ is precisely the body force. In our approach, W is the sum of the potential of the external forces and the elastic energy, as shown in Eq. (4.45).

4.2 Material Inhomogeneity

In this section, we start by deriving the *balance of material momentum* and introduce the concept of *Eshelby stress* and of its divergence, which we call *Eshelby force*, in the thermoelastic general framework, following the approach indicated by Eshelby (1975). Then, following an analogous approach, we take the derivative of the Lagrangian with respect to *time* and obtain the *canonical balance of energy*, the analogue of the balance of material momentum, featuring the *Eshelby power*, which is the analogue of the Eshelby force (for further discussions on this analogy, see, e.g., (Epstein, 2015; Epstein and de León, 2016)). We emphasise that the explicit dependence on time of the Lagrangian implies an *evolution phenomenon* (see, e.g., (Wang et al., 1974)).

From now on, we shall mainly work in components, because it would be at times cumbersome to write in component-free notation, for several reasons, e.g., the presence of the third-order tensor **Tor** and the need to distinguish the $\boldsymbol{P}$-connection from the $\boldsymbol{G}$-connection.

4.2.1 Configurational Force and Canonical Balance of Momentum

Here, following the variational approach presented by Eshelby (1975), we introduce the *Eshelby force* as the divergence of the Eshelby stress. In the classical statical case, the Eshelby force would coincide with the negative of the configurational force. We start from the Lagrangian density (per unit reference volume) defined in Section 4.1.4, Eq. (4.42), and, rather than deriving the Euler-Lagrange equations, we exploit the balance equations obtained in Section 4.1.3 to examine the dependence of the Lagrangian on the material point X, i.e., we take its *material gradient*. Indeed, if we were to use the Euler-Lagrange equations, we would stumble into the time derivative of the mass density, for which we would need to invoke the balance of mass in any case.

Let us take the material gradient of the Lagrangian density, whose A-th component is

$$\begin{aligned}
\mathcal{L}_{,A} &= \left[\hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau)\right]_{,A} \\
&= \left[\frac{\partial \hat{\mathcal{L}}}{\partial \phi^a} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] F^a{}_A + \left[\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\phi}^a} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] \dot{F}^a{}_A + \\
&+ \left[\frac{\partial \hat{\mathcal{L}}}{\partial \Theta} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] \Theta_{,A} + \left[\frac{\partial \hat{\mathcal{L}}}{\partial F^a{}_B} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] F^a{}_{B|A} + \\
&+ \left[\frac{\partial \hat{\mathcal{L}}}{\partial \mathcal{X}^A} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right],
\end{aligned} \tag{4.47}$$

where, naturally, $F^a{}_A = \phi^a{}_{,A}$. The last term is usually called "explicit" gradient of $\mathcal{L}$ and denoted $(\partial \mathcal{L}/\partial X^A)\,|_{\text{expl}}$, in the literature (e.g., (Eshelby, 1975; Epstein and Maugin, 1990)), whereas we simply regard it as the collection of the partial derivatives of the constitutive function $\mathcal{L}$ with respect to the $\mathcal{X}^A$ (which, we recall, are the functions such that $\mathcal{X}^A(X) = X^A$). This "explicit" gradient gives rise to the *material inhomogeneity force* or *configurational force*

$$\mathfrak{F} = \frac{\partial \hat{\mathcal{L}}}{\partial \mathcal{X}} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau), \qquad \mathfrak{F}_A = \frac{\partial \hat{\mathcal{L}}}{\partial \mathcal{X}^A} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau). \tag{4.48}$$

Substituting Eqs. (4.46) and the material force (4.48) into Eq. (4.47), we obtain

$$\operatorname{Grad} \mathcal{L} = \boldsymbol{f}\, \boldsymbol{F} + \varrho_R\, \dot{\phi}\, \boldsymbol{g}\, \dot{\boldsymbol{F}} + S \operatorname{Grad} \Theta - \boldsymbol{T} : \operatorname{Grad} \boldsymbol{F} + \mathfrak{F}, \tag{4.49}$$

where the colon ":" in the second term denotes double contraction of the two legs of $\boldsymbol{T}$ with the first two legs of $\operatorname{Grad} \boldsymbol{F}$ (i.e., $(\boldsymbol{T} : \operatorname{Grad} \boldsymbol{F})_B = T_a{}^A\, F^a{}_{A|B}$). Hence, we substitute the balance of linear momentum Eq. (4.34b) (for the case of no non-compliant source and flux of linear momentum and no mass flux) into Eq. (4.49) and obtain

$$\operatorname{Grad} \mathcal{L} = \left(\varrho_R\, \partial_t(\dot{\phi}\, \boldsymbol{g}) - \operatorname{Div} \boldsymbol{T}\right) \boldsymbol{F} + \varrho_R\, \dot{\phi}\, \boldsymbol{g}\, \dot{\boldsymbol{F}} + S \operatorname{Grad} \Theta - \boldsymbol{T} : \operatorname{Grad} \boldsymbol{F} + \mathfrak{F}, \tag{4.50}$$

and, by rearranging,

$$\operatorname{Grad}\mathcal{L} = \varrho_R\,\partial_t(\dot{\phi}\,\boldsymbol{g})\,\boldsymbol{F} + \varrho_R\,\dot{\phi}\,\boldsymbol{g}\,\dot{\boldsymbol{F}} - (\operatorname{Div}\boldsymbol{T})\,\boldsymbol{F} - \boldsymbol{T} : \operatorname{Grad}\boldsymbol{F} + S\,\operatorname{Grad}\Theta + \mathfrak{F}. \tag{4.51}$$

If we define the *material momentum* per unit referential volume $\mathfrak{P}$ as

$$\mathfrak{P} = -\varrho_R\,\dot{\phi}\,\boldsymbol{g}\,\boldsymbol{F}, \tag{4.52}$$

we can express its derivative as

$$\dot{\mathfrak{P}} = -\partial_t(\varrho_R\,\dot{\phi}\,\boldsymbol{g}\,\boldsymbol{F}) = -\Pi\,\dot{\phi}\,\boldsymbol{g}\,\boldsymbol{F} - \varrho_R\,\partial_t(\dot{\phi}\,\boldsymbol{g})\,\boldsymbol{F} - \varrho_R\,\dot{\phi}\,\boldsymbol{g}\,\dot{\boldsymbol{F}}. \tag{4.53}$$

and the sum of the first two terms on the right-hand side of Eq. (4.51) is equal to

$$\varrho_R\,\partial_t(\dot{\phi}\,\boldsymbol{g})\,\boldsymbol{F} + \varrho_R\,\dot{\phi}\,\boldsymbol{g}\,\dot{\boldsymbol{F}} = -\dot{\mathfrak{P}} - \Pi\,\dot{\phi}\,\boldsymbol{g}\,\boldsymbol{F}. \tag{4.54}$$

Also, the third and fourth terms on the right-hand side of Eq. (4.51) can be combined into

$$\operatorname{Div}(\boldsymbol{F}^T\boldsymbol{T}) = (\operatorname{Div}\boldsymbol{T})\,\boldsymbol{F} + \boldsymbol{T} : \operatorname{Grad}\boldsymbol{F}, \tag{4.55}$$

as it can be shown in components, by invoking the symmetry of the Christoffel symbols associated with the Levi-Civita connection induced by the material metric $\boldsymbol{G}$, so that $F^a{}_{A|B} = F^a{}_{B|A}$ (e.g., (Marsden and Hughes, 1983))

$$(F^a{}_B\,T_a{}^A)_{|A} = T_a{}^A{}_{|A}\,F^a{}_B + T_a{}^A\,F^a{}_{B|A} = T_a{}^A{}_{|A}\,F^a{}_B + T_a{}^A\,F^a{}_{A|B}. \tag{4.56}$$

By virtue of Eq. (4.54) and Eq. (4.55) and the identity $\operatorname{Grad}\mathcal{L} = \operatorname{Div}(\mathcal{L}\,\boldsymbol{I}^T)$, where $\boldsymbol{I}$ is the material identity tensor, Eq. (4.51) becomes finally, by reordering the terms,

$$\dot{\mathfrak{P}} = \mathfrak{F} - \Pi\,\dot{\phi}\,\boldsymbol{g}\boldsymbol{F} + S\,\mathrm{Grad}\,\Theta + \mathrm{Div}\,\mathfrak{E}, \tag{4.57}$$

where

$$\mathfrak{E} = -\mathcal{L}\,\boldsymbol{I}^T - \boldsymbol{F}^T\boldsymbol{T}, \tag{4.58}$$

is the *Eshelby stress*, with components

$$\mathfrak{E}_A{}^B = -\mathcal{L}\,\delta_A{}^B - F^a{}_A\,T_a{}^B. \tag{4.59}$$

Eq. (4.57) is called the *balance of material momentum* within the general thermoelastic framework. We call the divergence of the Eshelby stress the *Eshelby force*, i.e.,

$$\mathfrak{N} = \mathrm{Div}\,\mathfrak{E}, \qquad \mathfrak{N}_A = \mathfrak{E}_A{}^B{}_{|B}, \tag{4.60}$$

so that Eq. (4.57) reads

$$\dot{\mathfrak{P}} = \mathfrak{F} - \Pi\,\dot{\phi}\,\boldsymbol{g}\boldsymbol{F} + S\,\mathrm{Grad}\,\Theta + \mathfrak{N}. \tag{4.61}$$

In the statical case, this reduces to $\mathbf{0} = \mathfrak{F} + \mathfrak{N}$, which is the classical equation found by Eshelby.

4.2.2 Energy Release Rate and Canonical Balance of Energy

The time counterpart of the configurational force is the *energy release rate*, which is associated with the expenditure of energy required to displace the inhomogeneity by Eshelby's approach (Eshelby, 1975). Here, we also introduce the time counterpart of the Eshelby force, which we call *Eshelby* *power*. Taking the *time derivative* of the Lagrangian density (4.42), we have

$$\begin{aligned}
\dot{\mathcal{L}} &= \partial_t \big[\hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau)\big] \\
&= \left[\frac{\partial \hat{\mathcal{L}}}{\partial \phi^a} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] \dot{\phi}^a + \left[\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\phi}^a} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] \ddot{\phi}^a + \\
&+ \left[\frac{\partial \hat{\mathcal{L}}}{\partial \Theta} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] \dot{\Theta} + \left[\frac{\partial \hat{\mathcal{L}}}{\partial F^a{}_A} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right] \dot{F}^a{}_A + \\
&+ \left[\frac{\partial \hat{\mathcal{L}}}{\partial \tau} \circ (\phi, \dot{\phi}, \Theta, \boldsymbol{F}, \mathcal{X}, \tau)\right].
\end{aligned} \tag{4.62}$$

The last term could be called the "explicit" time derivative of $\mathcal{L}$ and be denoted $(\partial \mathcal{L}/\partial t)\,|_{\text{expl}}$. However, following our notation, we simply regard it as the derivative of the constitutive function $\hat{\mathcal{L}}$ with respect to the argument τ. This "explicit" time derivative takes the name of *energy release rate*

$$\mathfrak{G} = \frac{\partial \hat{\mathcal{L}}}{\partial \tau} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau), \tag{4.63}$$

and is the temporal counterpart of the configurational force $\boldsymbol{\mathfrak{F}}$. Now, we substitute Eqs. (4.46) and (4.63) into Eq. (4.62), so that we have

$$\dot{\mathcal{L}} = \langle \boldsymbol{f} \mid \dot{\phi} \rangle + \varrho_R \, \dot{\phi} \,.\, \ddot{\phi} + S \, \dot{\Theta} - \boldsymbol{T} : \dot{\boldsymbol{F}} + \mathfrak{G}, \tag{4.64}$$

where $\dot{\phi} \,.\, \ddot{\phi} = \dot{\phi} \, \boldsymbol{g} \, \ddot{\phi} = [(\partial \hat{\mathcal{L}}/\partial \dot{\phi}^a) \circ (\#)] \ddot{\phi}^a$ is related to the time derivative of the kinetic energy. Indeed, using balance of mass (4.34a), we have

$$\dot{K} = \partial_t \left(\tfrac{1}{2} \, \varrho_R \, \dot{\phi} \,.\, \dot{\phi}\right) = \tfrac{1}{2} \, \Pi \, \dot{\phi} \,.\, \dot{\phi} + \varrho_R \, \ddot{\phi} \,.\, \dot{\phi}. \tag{4.65}$$

By substituting the material balance of linear momentum (4.34b), the definition (4.43) of Lagrangian density, $K - \mathcal{L} = W$, and the time derivative (4.65) of the kinetic energy in Eq. (4.64), we obtain

$$\dot{\mathcal{L}} = 2\dot{K} - \Pi\ \dot{\phi}.\dot{\phi} + S\,\dot{\Theta} - \langle \operatorname{Div} \boldsymbol{T} \mid \dot{\phi} \rangle - \boldsymbol{T} : \dot{\boldsymbol{F}} + \mathfrak{G}. \tag{4.66}$$

If we define the *energy flow vector* (Eshelby, 1970) by

$$\mathfrak{R} = -\dot{\phi}\,\boldsymbol{T} = -\boldsymbol{T}^T\dot{\phi}, \tag{4.67}$$

we have

$$\operatorname{Div} \mathfrak{R} = -\langle \operatorname{Div} \boldsymbol{T} \mid \dot{\phi} \rangle - \boldsymbol{T} : \dot{\boldsymbol{F}} = \operatorname{Div}(-\dot{\phi}\,\boldsymbol{T}), \tag{4.68}$$

and thus, after reordering the terms, Eq. (4.66) can be rewritten

$$-2\dot{K} = \mathfrak{G} - \Pi\ \dot{\phi}.\dot{\phi} + S\,\dot{\Theta} + [-\dot{\mathcal{L}} + \operatorname{Div} \mathfrak{R}], \tag{4.69}$$

Eq. (4.69) is the *canonical balance of energy* in the general thermoelastic framework, and is term-by-term analogous to the canonical balance of material momentum of Eq. (4.61). This can be seen more clearly by writing Eq. (4.69) as

$$-2\dot{K} = \mathfrak{G} - \Pi\ \dot{\phi}.\dot{\phi} + S\,\dot{\Theta} + \mathfrak{Y}. \tag{4.70}$$

where

$$\mathfrak{Y} = -\dot{\mathcal{L}} + \operatorname{Div} \mathfrak{R} \tag{4.71}$$

is called *Eshelby power* and represents the temporal counterpart of $\mathcal{N} = \operatorname{Div}\mathfrak{E}$ in Eq. (4.61). Indeed, if we decompose the divergence of the Eshelby stress into

$$\operatorname{Div}\mathfrak{E} = \operatorname{Div}(-\mathcal{L}\,\boldsymbol{I}^T - \mathfrak{M}) = -\operatorname{Grad}\mathcal{L} - \operatorname{Div}(\mathfrak{M}), \tag{4.72}$$

where

$$\mathfrak{M} = \boldsymbol{F}^T\boldsymbol{T} \tag{4.73}$$

is the *Mandel stress*, we have that $-\dot{\mathcal{L}}$ corresponds to $-\operatorname{Grad}\mathcal{L}$ and $\operatorname{Div}\mathfrak{R} = \operatorname{Div}(-\dot{\phi}\,\boldsymbol{T})$ corresponds to $-\operatorname{Div}(\mathfrak{M}) = -\operatorname{Div}(\boldsymbol{F}^T\boldsymbol{T})$.

4.3 Material Uniformity

Here, we relate the Eshelby force and Eshelby power defined in Section 4.2 to the theory of material uniformity (Noll, 1967; Epstein and Maugin, 1990) to obtain two differential identities. The first involves a "modified" Eshelby force, which we shall show to be the quantity that captures the *presence* of the inhomogeneities. The second involves a "modified" Eshelby power, which captures the *evolution* of the inhomogeneities.

4.3.1 Definition of Materially Uniform Body

A body is said to be *materially uniform* if all points are made of the same material, i.e., if all points are pairwise materially isomorphic (Noll, 1967; Epstein and Maugin, 1990). In other words, the body is uniform if there exists a material isomorphism $\boldsymbol{P}(X,t) : \mathcal{A} \to T_X\mathcal{B}$ from a *fixed* vector space $\mathcal{A}$, called the *archetype*, to the tangent space $T_X\mathcal{B}$ of each point X of the body. Clearly, the archetype $\mathcal{A}$ and the body manifold $\mathcal{B}$ must have the *same dimension*. The isomorphism between any two points X and Y of the body $\mathcal{B}$ descends from the definition of archetype by transitivity, i.e., given the values $\boldsymbol{P}(X,t)$ and $\boldsymbol{P}(Y,t)$ of the material isomorphism at points X and Y, the isomorphism between $T_X\mathcal{B}$ and $T_Y\mathcal{B}$ is given by $\boldsymbol{P}(Y,t)\,\boldsymbol{P}^{-1}(X,t)$. Note that $\boldsymbol{P}(\,\cdot\,,t)$, as a tensor field, is defined in $\mathcal{B}$, i.e., it is a function of X. The concept is illustrated in Figure 4.1.

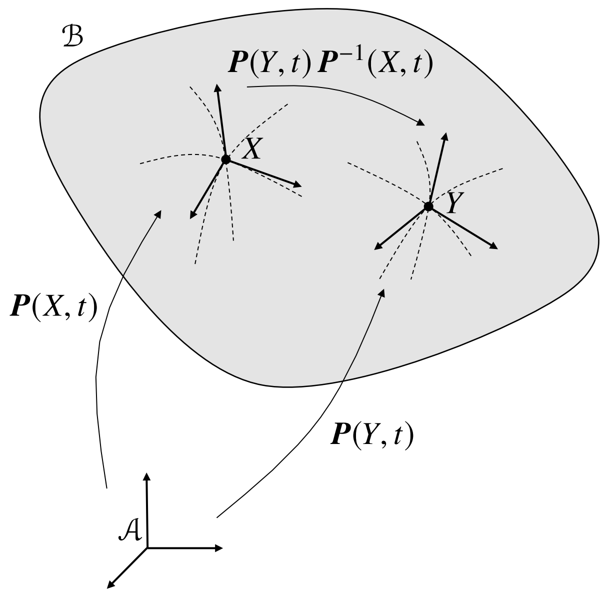

Figure 4.1: The representation of a materially uniform body $\mathcal{B}$ and the isomorphism between two points X and Y. The vector space $\mathcal{A}$ is the archetype.

As we shall see in Section 4.3.4, the material isomorphism between any two material points of the body $\mathcal{B}$ implies that the two points have exactly the same material behaviour. For now, let us focus on the geometrical properties of the uniformity tensor field $\boldsymbol{P}(\,\cdot\,,t)$. Let us assume that there is a metric tensor $\mathfrak{g}$ in the archetype $\mathcal{A}$; since $\mathcal{A}$ is a fixed vector space, $\mathfrak{g}$ is a fixed tensor. The material metric $\boldsymbol{G}$ in the body $\boldsymbol{B}$ is the tensor field whose values $\boldsymbol{G}(X)$ are the metric tensors in the tangent spaces $T_X\mathcal{B}$, for every point $X \in \mathcal{B}$. The Levi-Civita connection of the material metric $\boldsymbol{G}$ has symmetric Christoffel symbols given by Eq. (4.15) and thus zero torsion.

Following the definition given by Noll (1967), a vector field $\boldsymbol{V}(\,\cdot\,,t) : \mathcal{B} \to T\mathcal{B}$ is called parallel with respect to $\boldsymbol{P}$, or $\boldsymbol{P}$-parallel, if, and only if, for every material point X,

$$\boldsymbol{V}(X,t) = \boldsymbol{P}(X,t)\,\mathfrak{v}, \tag{4.74}$$

where $\mathfrak{v}$ is a *fixed* vector in the archetype $\mathcal{A}$. It follows that, for every material points X and Y,

$$\boldsymbol{P}^{-1}(X,t)\boldsymbol{V}(X,t) = \boldsymbol{P}^{-1}(Y,t)\boldsymbol{V}(Y,t) = \mathfrak{v} \quad \Rightarrow \quad \boldsymbol{V}(Y,t) = \boldsymbol{P}(Y,t)\boldsymbol{P}^{-1}(X,t)\boldsymbol{V}(X,t). \tag{4.75}$$

The $\boldsymbol{P}$-connection is defined as that connection such that the covariant derivative of a $\boldsymbol{P}$-parallel vector field vanishes identically. In components, if we indicate the $\boldsymbol{P}$-covariant derivative by a semi-colon and the associated Christoffel symbols by Γ^A_{BC}, we have

$$V^A{}_{;B} = V^A{}_{,B} + \Gamma^A_{CB}\, V^C = 0 \quad \Rightarrow \quad \Gamma^A_{CB}\, V^C = -V^A{}_{,B}. \tag{4.76}$$

Using the definition (4.74) of $\boldsymbol{P}$-parallel vector field and the fact that $\mathfrak{v}$ is fixed in $\mathcal{A}$, we obtain

$$-V^A{}_{,B} = -P^A{}_{\alpha,B}\, \mathfrak{v}^\alpha = -P^A{}_{\alpha,B}\, (\boldsymbol{P}^{-1})^\alpha{}_C\, V^C \tag{4.77}$$

from which, comparing with Eq. (4.76) and for the arbitrariness of $\boldsymbol{V}$, we obtain the Christoffel symbols of the $\boldsymbol{P}$-connection as

$$\Gamma^A_{CB} = -(\boldsymbol{P}^{-1})^\alpha{}_C\, P^A{}_{\alpha,B} = P^A{}_\alpha\, (\boldsymbol{P}^{-1})^\alpha{}_{C,B}, \tag{4.78}$$

where, in the alternative expression $\Gamma^A_{CB} = P^A{}_\alpha\, (\boldsymbol{P}^{-1})^\alpha{}_{C,B}$, we used $P^A{}_\alpha\, (\boldsymbol{P}^{-1})^\alpha{}_B = \delta^A{}_B$. As in the general case, we define the torsion of the $\boldsymbol{P}$-connection as in Eq. (4.1.2) and denote it by **Tor**.

4.3.2 Properties of the Uniformity Field

We shall now study the fundamental properties of the uniformity field $\boldsymbol{P}$.

Trivially, $\boldsymbol{P}$ is $\boldsymbol{P}$-parallel, i.e., the $\boldsymbol{P}$-covariant derivative of $\boldsymbol{P}$ is zero. Indeed, considering that the basis vector $\boldsymbol{\epsilon}_\alpha$ in the archetype $\mathcal{A}$ is a *constant* vector, we have

$$P^A{}_{\alpha;B} = P^A{}_{\alpha,B} + \Gamma^A_{CB} P^C{}_\alpha = P^A{}_{\alpha,B} - (\boldsymbol{P}^{-1})^\beta{}_C\, P^A{}_{\beta,B}\, P^C{}_\alpha = P^A{}_{\alpha,B} - P^A{}_{\alpha,B} = 0. \tag{4.79}$$

If we take the $\boldsymbol{G}$-covariant derivative of $\boldsymbol{P}$, we have

$$P^A{}_{\alpha|B} = P^A{}_{\alpha,B} + \mathrm{K}^A_{CB}\, P^C{}_\alpha, \tag{4.80}$$

which can be written

$$P^A{}_{\alpha|B} = P^A{}_{\beta,B}\, (\boldsymbol{P}^{-1})^\beta{}_C\, P^C{}_\alpha + \mathrm{K}^A_{CB}\, P^C{}_\alpha = [\mathrm{K}^A_{CB} - \Gamma^A_{CB}]\, P^C{}_\alpha. \tag{4.81}$$

Considering that $\boldsymbol{P}$ is $\boldsymbol{P}$-parallel (i.e., $P^A{}_{\alpha;B} = 0$, see Eq. (4.79)), we can rewrite identity (4.81) as

$$P^A{}_{\alpha|B} = P^A{}_{\alpha;B} + [\mathrm{K}^A_{CB} - \Gamma^A_{CB}]\, P^C{}_\alpha. \tag{4.82}$$

It is convenient to calculate the gradient of the determinant $J_{\boldsymbol{P}} = \det \boldsymbol{P}$. This can be done directly, via a standard but tedious procedure, as we show in Appendix A.6, or in a single passage, using Eq. (4.82). Indeed, we have

$$(J_{\boldsymbol{P}})_{,B} = \left[\frac{\partial \det}{\partial P^A{}_\alpha} \circ \boldsymbol{P}\right] P^A{}_{\alpha|B} = J_{\boldsymbol{P}}\, (\boldsymbol{P}^{-1})^\alpha{}_A\, [\mathrm{K}^A_{CB} - \Gamma^A_{CB}]\, P^C{}_\alpha, \tag{4.83}$$

from which we obtain

$$(J_{\boldsymbol{P}})_{,B} = J_{\boldsymbol{P}}\, \left[\mathrm{K}^C_{CB} - \Gamma^C_{CB}\right]. \tag{4.84}$$

The time derivative of $J_{\boldsymbol{P}} = \det \boldsymbol{P}$ is more straightforward, as it does not involve any covariant derivative. Indeed,

$$\dot{J}_{\boldsymbol{P}} = \left[\frac{\partial \det}{\partial P^A{}_\alpha} \circ \boldsymbol{P}\right] \dot{P}^A{}_\alpha = J_{\boldsymbol{P}}\, (\boldsymbol{P}^{-1})^\alpha{}_A\, \dot{P}^A{}_\alpha = J_{\boldsymbol{P}}\, \dot{P}^A{}_\alpha\, (\boldsymbol{P}^{-1})^\alpha{}_A = J_{\boldsymbol{P}}\, (\boldsymbol{L}_{\boldsymbol{P}})^A{}_A, \tag{4.85}$$

from which,

$$\dot{J}_{\boldsymbol{P}} = J_{\boldsymbol{P}}\,\mathrm{tr}(\boldsymbol{L}_{\boldsymbol{P}}), \tag{4.86}$$

where

$$\boldsymbol{L}_{\boldsymbol{P}} = \dot{\boldsymbol{P}}\boldsymbol{P}^{-1} \tag{4.87}$$

is the *inhomogeneity rate tensor* (often called "inhomogeneity velocity gradient", which is an abuse of terminology since $\boldsymbol{P}$ is, in general, *non-integrable*).

From Eq. (4.84), we see that, when $\boldsymbol{P}$ is *unimodular*, i.e., $J_{\boldsymbol{P}} = \det \boldsymbol{P} = 1$,

$$\Gamma^{C}_{CB} = \mathrm{K}^{C}_{CB}, \tag{4.88}$$

which, for the case of Cartesian coordinates (when the Christoffel symbols K^{A}_{BC} vanish), implies that the *trace* Γ^{C}_{CB} vanishes. Moreover, from Eq. (4.86), we have that

$$\mathrm{tr}(\boldsymbol{L}_{\boldsymbol{P}}) = 0, \tag{4.89}$$

i.e., $\boldsymbol{L}_{\boldsymbol{P}}$ is traceless.

For later use, we calculate the relation between the $\boldsymbol{P}$-covariant derivative and the $\boldsymbol{G}$-covariant derivative for the case of mixed two-point tensors, such as the deformation gradient $\boldsymbol{F}$ (first index contravariant, second index covariant) and the first Piola-Kirchhoff stress $\boldsymbol{T}$ (first index covariant, second index contravariant). The $\boldsymbol{G}$-covariant derivative of the deformation gradient $\boldsymbol{F}$ is

$$F^{a}{}_{A|B} = F^{a}{}_{A,B} + F^{c}{}_{A}\,(\gamma^{a}_{cb} \circ \phi)\,F^{b}{}_{B} - F^{a}{}_{C}\,\mathrm{K}^{C}_{AB}, \tag{4.90}$$

and the $\boldsymbol{P}$-covariant derivative is

$$F^a{}_{A;B} = F^a{}_{A,B} + F^c{}_A\,(\gamma^a_{cb} \circ \phi)\; F^b{}_B - F^a{}_C\;\Gamma^C_{AB}. \tag{4.91}$$

Thus,

$$F^a{}_{A|B} = F^a{}_{A;B} - [\mathrm{K}^C_{AB} - \Gamma^C_{AB}]\; F^a{}_C, \tag{4.92}$$

where we notice the analogy with Eq. (4.82). Since its only large index is contravariant, it is easy to verify that the $\boldsymbol{G}$-covariant divergence of the first Piola-Kirchhoff stress $\boldsymbol{T}$ is related to its $\boldsymbol{P}$-covariant divergence by means of the same type of relation for $\boldsymbol{P}$ seen in Eq. (4.82), i.e.,

$$T_a{}^A{}_{|A} = T_a{}^A{}_{;A} + [\mathrm{K}^A_{CA} - \Gamma^A_{CA}]\; T_a{}^C. \tag{4.93}$$

Finally, we note that, since the Lagrangian velocity $\dot{\phi}$ is a *spatial* quantity (indeed, it has only one *small* contravariant index), its covariant derivative is *unaffected* by the choice of the connection used in the body, i.e.,

$$\dot{\phi}^a{}_{|A} = \dot{\phi}^a{}_{;A} = \dot{\phi}^a{}_{,A} + (\gamma^a_{cb} \circ \phi)\; \dot{\phi}^c\; F^b{}_A = \dot{F}^a{}_A. \tag{4.94}$$

4.3.3 Lagrangian of a Uniform Body

For a uniform body, the *explicit* dependence of the Lagrangian density $\hat{\mathcal{L}}$ on the point X and the time t is only through $\boldsymbol{P}$, and the archetypal Lagrangian density $\hat{\Lambda}$ does not depend on X and t, and instead depends on $\boldsymbol{F}\boldsymbol{P}$ rather than on $\boldsymbol{P}$, i.e., the material leg of $\boldsymbol{F}$ is transformed by $\boldsymbol{P}$. Thus,

$$\mathcal{L} = \hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \boldsymbol{F}, \Theta, \mathcal{X}, \tau) = J_{\boldsymbol{P}}^{-1}\; \hat{\Lambda} \circ (\phi, \dot{\phi}, \boldsymbol{F}\boldsymbol{P}, \Theta), \tag{4.95}$$

where $J_{\boldsymbol{P}}^{-1}$ arises from the theorem of the change of variables. Figure 4.2 illustrates the map $\boldsymbol{F}\boldsymbol{P}$, with components $F^a{}_A \, P^A{}_\alpha$.

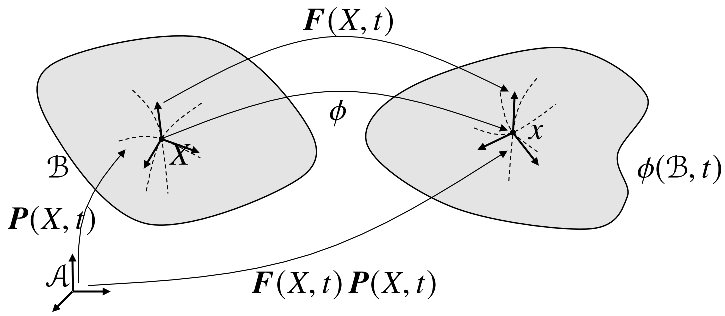

Figure 4.2: Mappings between the archetype $\mathcal{A}$ and the tangent spaces in the body and in the current configuration for a uniform body $\mathcal{B}$.

We differentiate $\hat{\mathcal{L}}$ with respect to $\boldsymbol{F}$ and obtain

$$\frac{\partial \hat{\mathcal{L}}}{\partial \boldsymbol{F}} \circ (\phi, \dot{\phi}, \boldsymbol{F}\boldsymbol{P}, \Theta) = J_{\boldsymbol{P}}^{-1} \frac{\partial \hat{\Lambda}}{\partial(\boldsymbol{F}\boldsymbol{P})} \circ (\phi, \dot{\phi}, \boldsymbol{F}\boldsymbol{P}, \Theta)\, \boldsymbol{P}^T, \tag{4.96}$$

from which the identity

$$\frac{\partial \hat{\Lambda}}{\partial(\boldsymbol{F}\boldsymbol{P})} \circ (\phi, \dot{\phi}, \boldsymbol{F}\boldsymbol{P}, \Theta) = J_{\boldsymbol{P}} \, \frac{\partial \hat{\mathcal{L}}}{\partial \boldsymbol{F}} \circ (\phi, \dot{\phi}, \boldsymbol{F}\boldsymbol{P}, \Theta)\, \boldsymbol{P}^{-T} = -J_{\boldsymbol{P}} \, \boldsymbol{T}\, \boldsymbol{P}^{-T}. \tag{4.97}$$

In the following, we are going to study the gradient and time derivative of the archetypical Lagrangian. Note that, following Epstein and Maugin (1990), we refer to a certain quantity rescaled by the determinant $J_{\boldsymbol{P}}$ of the uniformity field $\boldsymbol{P}$ as to the "modified" version of the quantity and we denote it with a superposed tilde. For instance, the modified Eshelby stress is $\widetilde{\mathfrak{E}} = J_{\boldsymbol{P}} \, \mathfrak{E}$. The only two exceptions to this convention are precisely the core of this work: the modified Eshleby force and the modified Eshelby power.

4.3.4 Gradient of the Lagrangian of a Uniform Body

We take the material gradient of the archetypal Lagrangian density $\hat{\Lambda}$, whose A-th component is the partial derivative

$$\begin{aligned}\left[\hat{\Lambda}\circ(\phi,\dot{\phi},\boldsymbol{F}\boldsymbol{P},\Theta)\right]_{,A} = &\left[\frac{\partial\hat{\Lambda}}{\partial\phi^a}\circ(\phi,\dot{\phi},\boldsymbol{F}\boldsymbol{P},\Theta)\right] F^a{}_A + \left[\frac{\partial\hat{\Lambda}}{\partial\dot{\phi}^a}\circ(\phi,\dot{\phi},\boldsymbol{F}\boldsymbol{P},\Theta)\right] \dot{F}^a{}_A +\\ &+ \left[\frac{\partial\hat{\Lambda}}{\partial(F^a{}_B P^B{}_\alpha)}\circ(\phi,\dot{\phi},\boldsymbol{F}\boldsymbol{P},\Theta)\right] (F^a{}_C\, P^C{}_\alpha)_{|A} +\\ &+ \left[\frac{\partial\hat{\Lambda}}{\partial\Theta}\circ(\phi,\dot{\phi},\boldsymbol{F}\boldsymbol{P},\Theta)\right]\, \Theta_{,A}.\end{aligned} \tag{4.98}$$

We consider Eq. (4.95) and substitute Eqs. (4.46) and (4.97) into Eq. (4.98), we obtain

$$\begin{aligned}[\hat{\Lambda}\circ(\#)]_{,A} = &J_{\boldsymbol{P}}\, f_a\, F^a{}_A + J_{\boldsymbol{P}}\, \varrho_R\, \dot{\phi}^b\, g_{ab}\, \dot{F}^a{}_A\\ &- J_{\boldsymbol{P}}\, T_a{}^B\, F^a{}_{B|A} - J_{\boldsymbol{P}}\, F^a{}_B\, T_a{}^C\, (\boldsymbol{P}^{-1})^\alpha{}_C\, P^B{}_{\alpha|A} + J_{\boldsymbol{P}}\, S\, \Theta_{,A},\end{aligned} \tag{4.99}$$

where from now on we use (#) in place of the appropriate arguments, for the sake of brevity. Note that, since the Christoffel symbols K^A_{BC} and $\gamma^a{}_{bc}$ of the Levi-Civita connections induced by the material and spatial metric tensors $\boldsymbol{G}$ and $\boldsymbol{g}$ are symmetric, we have that

$$\begin{aligned}F^a{}_{A|B} &= F^a{}_{A,B} + F^c{}_A\, (\gamma^a_{cb}\circ\phi)\, F^b{}_B - F^a{}_C\, \mathrm{K}^C_{AB}\\ &= F^a{}_{B,A} + F^c{}_A\, (\gamma^a_{bc}\circ\phi)\, F^b{}_B - F^a{}_C\, \mathrm{K}^C_{BA} = F^a{}_{B|A},\end{aligned} \tag{4.100}$$

where we also used Schwarz' Theorem in $F^a{}_{A,B} = \phi^a{}_{,AB} = \phi^a{}_{,BA} = F^a{}_{B,A}$. Using this result and Eqs. (4.82) and (4.92) to replace $P^B{}_{\alpha|A}$ and $F^a{}_{B|A} = F^a{}_{A|B}$, respectively, we obtain

$$\begin{aligned}[\hat{\Lambda}\circ(\#)]_{,A} = &J_{\boldsymbol{P}}\, f_a\, F^a{}_A + J_{\boldsymbol{P}}\, \varrho_R\, \dot{\phi}^b\, g_{ab}\, \dot{F}^a{}_A\\ &- \widetilde{T}_a{}^B\, F^a{}_{A;B} - \widetilde{\mathfrak{M}}_C{}^B\, \Gamma^C_{AB} + \widetilde{\mathfrak{M}}_C{}^B\, \Gamma^C_{BA} + \widetilde{S}\, \Theta_{,A},\end{aligned} \tag{4.101}$$

where $\widetilde{T}_a{}^B = J_{\boldsymbol{P}}\, T_a{}^B$ is the *modified first Piola-Kirchhoff stress*, $\mathfrak{M}_C{}^B = F^a{}_C\, T_a{}^B$ is the *Mandel stress*, $\widetilde{\mathfrak{M}}_C{}^B = J_{\boldsymbol{P}}\, \mathfrak{M}_C{}^B = F^a{}_C\, (J_{\boldsymbol{P}}\, T_a{}^B) = F^a{}_C\, \widetilde{T}_a{}^B$ is the *modified Mandel stress*, and $\widetilde{S} = J_{\boldsymbol{P}}\, S$ is the *modified material entropy density*. Note that the Christoffel symbols K^A_{BC} of the $\boldsymbol{G}$-connection cancelled out and disappeared from the equation, showing that Eq. (4.101) is *independent* of the choice of the material metric $\boldsymbol{G}$.

The third term on the right-hand side of Eq. (4.101) can be factorised into

$$\widetilde{T}_a{}^B\, F^a{}_{A;B} = (F^a{}_A\, \widetilde{T}_a{}^B)_{;B} - \widetilde{T}_a{}^B{}_{;B}\, F^a{}_A = (\widetilde{\mathfrak{M}}_A{}^B)_{;B} - \widetilde{T}_a{}^B{}_{;B}\, F^a{}_A. \tag{4.102}$$

Also, the fourth and fifth terms on the right-hand side of Eq. (4.101), by exploiting the definition (4.1.2) of torsion of a connection, can be combined into

$$\widetilde{\mathfrak{M}}_C{}^B\, \Gamma^C_{BA} - \widetilde{\mathfrak{M}}_C{}^B\, \Gamma^C_{AB} = \widetilde{\mathfrak{M}}_C{}^B\, \mathrm{Tor}^C{}_{BA}. \tag{4.103}$$

By virtue of Eqs. (4.102) and (4.103), Eq. (4.101) becomes

$$\begin{aligned}[\hat{\Lambda} \circ (\#)]_{;A} + [\widetilde{\mathfrak{M}}_A{}^B]_{;B} = J_{\boldsymbol{P}}\, f_a\, F^a{}_A + J_{\boldsymbol{P}}\, \varrho_R\, \dot{\phi}^b\, g_{ab}\, \dot{F}^a{}_A \\ + \widetilde{T}_a{}^B{}_{;B}\, F^a{}_A + \widetilde{\mathfrak{M}}_C{}^B\, \mathrm{Tor}^C{}_{BA} + \widetilde{S}\, \Theta_{,A}.\end{aligned} \tag{4.104}$$

Now, we take the $\boldsymbol{P}$-covariant derivative of the *modified* Piola-Kirchhoff stress $\widetilde{\boldsymbol{T}}$, i.e.,

$$\widetilde{T}_a{}^A{}_{;A} = (J_{\boldsymbol{P}}\, T_a{}^A)_{;A} = \widetilde{T}_a{}^A\, [\mathrm{K}^C_{CA} - \Gamma^C_{CA}] + J_{\boldsymbol{P}}\, T_a{}^A{}_{;A}, \tag{4.105}$$

and, by exploiting Eqs. (4.34b) and (4.93), Eq. (4.105) finally becomes

$$\widetilde{T}_a{}^A{}_{;A} = \widetilde{T}_a{}^A\, \mathrm{Tor}^B{}_{AB} + J_{\boldsymbol{P}}\, \varrho_R\, g_{ab}\, \ddot{\phi}^b - J_{\boldsymbol{P}}\, f_a. \tag{4.106}$$

Now in Eq. (4.104), we replace $\widetilde{T}_a{}^B{}_{;B}$ by Eq. (4.106), and change sign to both sides, to obtain

$$[-\hat{\Lambda} \circ (\#)\, \delta_A{}^B - \widetilde{\mathfrak{M}}_A{}^B]_{;B} = -\widetilde{\mathfrak{M}}_A{}^C \, \mathrm{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_C{}^B \, \mathrm{Tor}^C{}_{BA} \\ - J_{\boldsymbol{P}}\, \varrho_R\, \dot{\phi}^b\, g_{ab}\, \dot{F}^a{}_A - J_{\boldsymbol{P}}\, \varrho_R\, \ddot{\phi}^b\, g_{ab}\, F^a{}_A - \widetilde{S}\, \Theta_{,A}, \quad (4.107)$$

where we have exploited the identity $\mathrm{Grad}\, \hat{\Lambda} = \mathrm{Div}(\hat{\Lambda}\, \boldsymbol{I}^T)$ (with components $[\hat{\Lambda} \circ (\#)]_{,A} = [\hat{\Lambda} \circ (\#)\, \delta_A{}^B]_{,A}$), with $\boldsymbol{I}^T$ (with components $\delta_A{}^B$) being the transpose of the material identity tensor.

If we use the definition (4.52) of the material momentum density $\boldsymbol{\mathcal{P}}$ and the expression (4.53) of its time derivative in Eq. (4.107), we obtain

$$[-\hat{\Lambda} \circ (\#)\, \delta_A{}^B - \widetilde{\mathfrak{M}}_A{}^B]_{;B} = -\widetilde{\mathfrak{M}}_A{}^C \, \mathrm{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_C{}^B \, \mathrm{Tor}^C{}_{BA} \\ + J_{\boldsymbol{P}}\, \dot{\mathcal{P}}_A + J_{\boldsymbol{P}}\, \Pi\, \dot{\phi}^a\, g_{ab}\, F^b{}_A - \widetilde{S}\, \Theta_{,A}. \quad (4.108)$$

Also, the mass source term Π in the term $\Pi\, \dot{\phi}\, \boldsymbol{g}\, \boldsymbol{F}$ in Eq. (4.108) can be expressed in terms of $\boldsymbol{P}$ via the balance of mass (4.34) (Epstein and Maugin, 2000). Indeed, if ϱ_{arch} is the mass density in the archetype, which is *a single fixed value* $\varrho_{\mathrm{arch}} = J_{\boldsymbol{P}}(X)\, \varrho_R(X)$, for every X in $\mathcal{B}$, we have

$$\dot{\varrho}_R = (J_{\boldsymbol{P}}^{-1})^{\cdot}\, \varrho_{\mathrm{arch}} = -J_{\boldsymbol{P}}^{-2}\, \dot{J}_{\boldsymbol{P}}\, \varrho_{\mathrm{arch}} = -J_{\boldsymbol{P}}^{-2}\, J_{\boldsymbol{P}}\, \mathrm{tr}(\boldsymbol{L}_{\boldsymbol{P}})\, \varrho_{\mathrm{arch}} = -\varrho_R\, \mathrm{tr}(\boldsymbol{L}_{\boldsymbol{P}}), \quad (4.109)$$

where we used the definition (4.87) of the inhomogeneity rate tensor. From this and balance of mass (4.34), we obtain

$$\dot{\varrho}_R = \Pi = -\varrho_R\, \mathrm{tr}(\dot{\boldsymbol{P}}\boldsymbol{P}^{-1}) = -\varrho_R\, \mathrm{tr}(\boldsymbol{L}_{\boldsymbol{P}}), \quad (4.110)$$

and, from Eq. (4.110), the fourth term on the right-hand side of Eq. (4.108) can be re-written in the form

$$J_{\boldsymbol{P}}\,\Pi\,\dot{\phi}^a\,g_{ab}F^b{}_A = -J_{\boldsymbol{P}}\,\varrho_R\,\mathrm{tr}(\boldsymbol{L_P})\,\dot{\phi}^a\,g_{ab}F^b{}_A = J_{\boldsymbol{P}}\,\mathrm{tr}(\boldsymbol{L_P})\,(-\varrho_R\,\dot{\phi}^a\,g_{ab}F^b{}_A)$$
$$= \mathrm{tr}(\boldsymbol{L_P})\,(J_{\boldsymbol{P}}\,\mathcal{P}_A) = (\boldsymbol{L_P})^B{}_B\,\widetilde{\mathcal{P}}_A, \tag{4.111}$$

where $\widetilde{\mathcal{P}}_A = J_{\boldsymbol{P}}\,\mathcal{P}_A$ is the *modified material linear momentum*. Therefore, Eq. (4.108) becomes

$$[-\hat{\Lambda}\circ(\#)\,\delta_A{}^B - \widetilde{\mathfrak{M}}_A{}^B]_{;B} = -\widetilde{\mathfrak{M}}_A{}^C\,\mathrm{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_C{}^B\,\mathrm{Tor}^C{}_{BA} + \widetilde{\dot{\mathcal{P}}}_A + (\boldsymbol{L_P})^B{}_B\,\widetilde{\mathcal{P}}_A\; - \widetilde{S}\,\Theta_{,A}, \tag{4.112}$$

where $\widetilde{\dot{\mathcal{P}}}_A = J_{\boldsymbol{P}}\,\dot{\mathcal{P}}_A$ is modified time derivative of the material linear momentum.

Now, consider the term $-\hat{\Lambda}\circ(\#)\,\delta_A{}^B - \widetilde{\mathfrak{M}}_A{}^B$, whose $\boldsymbol{P}$-divergence is the left-hand side of Eq. (4.112). Using the definition (4.95) of the material and archetypal constitutive functions of the Lagrangian $\mathcal{L}$ and the definition (4.59) of Eshelby stress, we recognise that this term is the *modified Eshelby stress* $\widetilde{\boldsymbol{E}}$, i.e.,

$$[-\hat{\Lambda}\circ(\#)\,\delta_A{}^B - \widetilde{\mathfrak{M}}_A{}^B]_{;B} = [J_{\boldsymbol{P}}\,(-\mathcal{L}\,\delta_A{}^B - F^a{}_A\,T_a{}^B)]_{;B} = [J_{\boldsymbol{P}}\,\mathfrak{E}_A{}^B]_{;B} = \widetilde{\mathfrak{E}}_A{}^B{}_{;B}. \tag{4.113}$$

We call the $\boldsymbol{P}$-divergence of the modified Eshelby stress the *modified Eshelby force*, i.e.,

$$\widetilde{\mathcal{N}}_A = \widetilde{\mathfrak{E}}_A{}^B{}_{;B} = [-\hat{\Lambda}\circ(\#)\,\delta_A{}^B - \widetilde{\mathfrak{M}}_A{}^B]_{;B}, \tag{4.114}$$

and we remark that this is an *exception* to the definition of modified quantities and indeed $\widetilde{\boldsymbol{\mathcal{N}}} \neq J_{\boldsymbol{P}}\,\boldsymbol{\mathcal{N}}$. With definition (4.114), Eq. (4.112) becomes the sought differential identity:

$$\widetilde{\mathcal{N}}_A = -\widetilde{\mathfrak{M}}_A{}^C\,\mathrm{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_C{}^B\,\mathrm{Tor}^C{}_{BA} + \widetilde{\dot{\mathcal{P}}}_A + (\boldsymbol{L_P})^B{}_B\,\widetilde{\mathcal{P}}_A\; - \widetilde{S}\,\Theta_{,A}. \tag{4.115}$$

Here we need two important remarks. First, in the homogeneous case, when $P^A{}_\alpha = \mathbf{1}^A{}_\alpha$ (tensor $\boldsymbol{P}$ equals the *shifter*), the $\boldsymbol{P}$-torsion **Tor** and the inhomogeneity rate tensor $\boldsymbol{L_P}$ vanish identically, and we recover the equation found by Maugin (2006). Second, when there is no growth, the body is isothermal and we consider only the static case, the trace $(\boldsymbol{L_P})^B{}_B$, the gradient $\Theta_{,A}$ and the inertial term $\dot{\widetilde{\mathfrak{P}}}_A$ vanish, and we recover the differential identity for the Eshelby stress reported by Epstein and Maugin (1990).

However, the differential identity in the work by Epstein and Maugin (1990) featured the modified Eshelby stress on the right-hand side, whereas we have the modified Mandel stress. In order to show that

$$\widetilde{\mathfrak{E}}_A{}^C \operatorname{Tor}^B{}_{CB} + \widetilde{\mathfrak{E}}_C{}^B \operatorname{Tor}^C{}_{BA} = -\widetilde{\mathfrak{M}}_A{}^C \operatorname{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_C{}^B \operatorname{Tor}^C{}_{BA}, \tag{4.116}$$

we look at the spherical terms $-J_{\boldsymbol{P}} \mathfrak{L}\, \boldsymbol{I}^T$ (in components, $-J_{\boldsymbol{P}} \mathfrak{L}\, \delta_A{}^C$), and we write

$$\begin{aligned}
-J_{\boldsymbol{P}} \mathfrak{L}\, \delta_A{}^C \operatorname{Tor}^B{}_{CB} - J_{\boldsymbol{P}} \mathfrak{L}\, \delta_B{}^C \operatorname{Tor}^B{}_{CA} &= - J_{\boldsymbol{P}} \mathfrak{L}\, \delta_A{}^C (\Gamma^B{}_{CB} - \Gamma^B{}_{BC}) - \\
&\quad - J_{\boldsymbol{P}} \mathfrak{L}\, \delta_B{}^C (\Gamma^B{}_{CA} - \Gamma^B{}_{AC}) \\
&= - J_{\boldsymbol{P}} \mathfrak{L}\, \Gamma^B{}_{AB} + J_{\boldsymbol{P}} \mathfrak{L}\, \Gamma^B{}_{BA} - \\
&\quad - J_{\boldsymbol{P}} \mathfrak{L}\, \Gamma^B{}_{BA} + J_{\boldsymbol{P}} \mathfrak{L}\, \Gamma^B{}_{AB} = 0,
\end{aligned} \tag{4.117}$$

which shows that the spherical term of the Eshelby stress, representing the arbitrary zero-level value of the Lagrangian, are *filtered* and do *not* affect balance of linear momentum. They in fact only appear under the sign of divergence on the right-hand side of Eq. (4.115).

We remark that Eq. (4.110) states that, if Π vanishes, $\boldsymbol{L_P}$ is *traceless* and describes *pure remodelling*. In this case, the differential identity Eq. (4.115) reduces to the form

$$\widetilde{\mathfrak{N}}_A = -\widetilde{\mathfrak{M}}_A{}^C \operatorname{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_B{}^C \operatorname{Tor}^B{}_{CA} + \dot{\widetilde{\mathfrak{P}}}_A - \widetilde{S}\, \Theta_{,A}. \tag{4.118}$$

In the isothermal and no-growth case, we recover the equation of Epstein and Maugin (1990), except that the Eshelby stress is replaced by the negative of the Mandel stress on the right-hand side and for the presence of the additional dynamical term (modified time derivative of the material linear momentum), which was absent in the work by Epstein and Maugin (1990), since they considered the static case:

$$\widetilde{\mathfrak{N}}_A = -\widetilde{\mathfrak{M}}_A{}^C \,\mathrm{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_B{}^C \,\mathrm{Tor}^B{}_{CA} + \widetilde{\dot{\mathfrak{P}}}_A. \tag{4.119}$$

We also note that there is an interesting analogue of Eq. (4.110). If we take the material gradient of the archetypical density $\varrho_{\mathrm{arch}} = J_{\boldsymbol{P}}\, \varrho_R$, we obtain

$$0 = (\varrho_{\mathrm{arch}})_{,B} = (J_{\boldsymbol{P}})_{,B}\; \varrho_R + J_{\boldsymbol{P}}\; (\varrho_R)_{,B} \tag{4.120}$$

from which the gradient of the material density ϱ_R is given by

$$(\varrho_R)_{,B} = -J_{\boldsymbol{P}}^{-1}(J_{\boldsymbol{P}})_{,B}\; \varrho_R. \tag{4.121}$$

Substituting Eq. (4.84) into Eq. (4.121), we obtain

$$(\varrho_R)_{,B} = \varrho_R\; \left(\Gamma^C_{CB} - \mathrm{K}^C_{CB}\right). \tag{4.122}$$

Thus, the dependence of the material density on the point X, i.e., the fact that its gradient does not vanish, is directly related to the connection of the uniformity field $\boldsymbol{P}$. In the homogeneous case, when $P^A{}_\alpha = \mathbf{1}^A{}_\alpha$ (tensor $\boldsymbol{P}$ equals the *shifter*), $J_{\boldsymbol{P}} = 1$ identically, the right-hand side of Eq. (4.119) reduces to $\dot{\mathfrak{P}}_A$, and the right-hand side of Eq. (4.122) goes to zero via the vanishing of the difference of the two Christoffel symbols, according to Eq. (4.81).

4.3.5 Time Derivative of the Lagrangian of a Uniform Body

Let us consider the Lagrangian density (4.42) and calculate the time derivative of the archetypal Lagrangian density $\hat{\Lambda}$ of Eq. (4.95), i.e.,

$$\begin{aligned}\partial_t[\hat{\Lambda} \circ (\phi, \dot{\phi}, \boldsymbol{FP}, \Theta)] = &\left[\frac{\partial \hat{\Lambda}}{\partial \phi^a} \circ (\phi, \dot{\phi}, \boldsymbol{FP}, \Theta)\right] \dot{\phi}^a + \left[\frac{\partial \hat{\Lambda}}{\partial \dot{\phi}^a} \circ (\phi, \dot{\phi}, \boldsymbol{FP}, \Theta)\right] \ddot{\phi}^a + \\ &+ \left[\frac{\partial \hat{\Lambda}}{\partial (F^a{}_B P^B{}_\alpha)} \circ (\phi, \dot{\phi}, \boldsymbol{FP}, \Theta)\right] \partial_t (F^a{}_C\, P^C{}_\alpha) + \\ &+ \left[\frac{\partial \hat{\Lambda}}{\partial \Theta} \circ (\phi, \dot{\phi}, \boldsymbol{FP}, \Theta)\right] \dot{\Theta},\end{aligned} \tag{4.123}$$

Now, we consider Eq. (4.95) and substitute Eqs. (4.46) and Eq. (4.97) into Eq. (4.123), and obtain

$$\partial_t[\hat{\Lambda} \circ (\#)] = J_{\boldsymbol{P}}\, f_a\, \dot{\phi}^a + J_{\boldsymbol{P}}\, \varrho_R\, \dot{\phi}^b\, g_{ab}\, \ddot{\phi}^a - J_{\boldsymbol{P}}\, T_a{}^B\, \dot{F}^a{}_B - J_{\boldsymbol{P}}\, F^a{}_B\, T_a{}^C\, (\boldsymbol{P}^{-1})^\alpha{}_C\, \dot{P}^B{}_\alpha + \widetilde{S}\, \dot{\Theta}. \tag{4.124}$$

According to Eqs. (4.95) and (4.87), the left-hand side of Eq. (4.124) takes the form

$$\begin{aligned}\partial_t[\hat{\Lambda} \circ (\#)] &= \partial_t[J_{\boldsymbol{P}}\, \hat{\mathcal{L}} \circ (\#)] = \partial_t[J_{\boldsymbol{P}}\, \mathcal{L}] = J_{\boldsymbol{P}}\, (\boldsymbol{P}^{-1})^\alpha{}_B\, \dot{P}^B{}_\alpha\, \mathcal{L} + J_{\boldsymbol{P}}\, \dot{\mathcal{L}} \\ &= J_{\boldsymbol{P}}\, (\boldsymbol{L}_{\boldsymbol{P}})^B{}_B\, \mathcal{L} + J_{\boldsymbol{P}}\, \dot{\mathcal{L}}.\end{aligned} \tag{4.125}$$

We substitute this result back Eq. (4.125) into Eq. (4.124) and obtain

$$J_{\boldsymbol{P}}\, \dot{\mathcal{L}} = J_{\boldsymbol{P}}\, f_a\, \dot{\phi}^a + J_{\boldsymbol{P}}\, \varrho_R\, \dot{\phi}^b\, g_{ab}\, \ddot{\phi}^a - J_{\boldsymbol{P}}\, T_a{}^B\, \dot{F}^a{}_B + \left[-J_{\boldsymbol{P}}\, \mathcal{L}\, \delta_B{}^C - J_{\boldsymbol{P}}\, F^a{}_B\, T_a{}^C\right] (\boldsymbol{L}_{\boldsymbol{P}})^B{}_C + \widetilde{S}\, \dot{\Theta}, \tag{4.126}$$

where the term in the square brackets on the right-hand side Eq. (4.126) is recognisable as the *modified Eshebly stress* $\widetilde{\mathfrak{E}}_A{}^B = J_{\boldsymbol{P}}\, \mathfrak{E}_A{}^B$. Thus

$$J_{\boldsymbol{P}}\, \dot{\mathcal{L}} = J_{\boldsymbol{P}}\, f_a\, \dot{\phi}^a + J_{\boldsymbol{P}}\, \varrho_R\, \dot{\phi}^b\, g_{ab}\, \ddot{\phi}^a - J_{\boldsymbol{P}}\, T_a{}^B\, \dot{F}^a{}_B + \widetilde{\mathfrak{E}}_B{}^C (\boldsymbol{L}_{\boldsymbol{P}})^B{}_C + \widetilde{S}\, \dot{\Theta}, \tag{4.127}$$

Using Eq. (4.106) to eliminate $J_{\boldsymbol{P}}\, f_a$ in the first term on the right-hand side of Eq. (4.127), we get

$$J_{\boldsymbol{P}}\,\dot{\mathfrak{L}} = -\widetilde{T}_a{}^A{}_{;A}\,\dot{\phi}^a + \widetilde{T}_a{}^A\,\dot{\phi}^a\,\mathrm{Tor}^B{}_{AB} + 2J_{\boldsymbol{P}}\,\varrho_R\,\dot{\phi}^b\,g_{ab}\,\ddot{\phi}^a - J_{\boldsymbol{P}}\,T_a{}^B\,\dot{F}^a{}_B + \widetilde{\mathfrak{E}}_B{}^C(\boldsymbol{L_P})^B{}_C + \widetilde{S}\,\dot{\Theta}, \quad (4.128)$$

Then, using Eq. (4.65) in order to replace $\dot{\phi}\,.\,\ddot{\phi} = \dot{\phi}\,\boldsymbol{g}\,\ddot{\phi}$ in the second term on the right-hand side of Eq. (4.128), we obtain

$$J_{\boldsymbol{P}}\,\dot{\mathfrak{L}} + \widetilde{T}_a{}^A{}_{;A}\,\dot{\phi}^a + \widetilde{T}_a{}^B\,\dot{F}^a{}_B = -\widetilde{\mathfrak{R}}^A\,\mathrm{Tor}^B{}_{AB} + \widetilde{\mathfrak{E}}_B{}^C(\boldsymbol{L_P})^B{}_C + 2J_{\boldsymbol{P}}\,\dot{K} - J_{\boldsymbol{P}}\,\Pi\,\dot{\phi}.\dot{\phi} + \widetilde{S}\,\dot{\Theta}, \quad (4.129)$$

where $\widetilde{\mathfrak{R}}^A = -J_{\boldsymbol{P}}\,T_a{}^A\,\dot{\phi}^a = -\widetilde{T}_a{}^A\,\dot{\phi}^a$ is called the *modified energy flow vector*. By virtue of Eq. (4.94), the second and the third terms on the left-hand side of Eq. (4.129) can be combined into

$$-\widetilde{\mathfrak{R}}^A{}_{;A} = (\widetilde{T}_a{}^A\,\dot{\phi}^a)_{;A} = \widetilde{T}_a{}^A{}_{;A}\,\dot{\phi}^a + \widetilde{T}_a{}^B\,\dot{\phi}^a{}_{;B} = \widetilde{T}_a{}^A{}_{;A}\,\dot{\phi}^a + \widetilde{T}_a{}^B\,\dot{F}^a{}_B. \quad (4.130)$$

Also, we note that Eqs. (4.110) and (4.44) imply

$$\Pi\,\dot{\phi}.\dot{\phi} = -2K\,\mathrm{tr}(\boldsymbol{L_P}). \quad (4.131)$$

We replace these results back into Eqs. (4.130) and (4.131) and change sign to both sides of Eq. (4.129), so we obtain

$$-J_{\boldsymbol{P}}\,\dot{\mathfrak{L}} + \widetilde{\mathfrak{R}}^A{}_{;A} = \widetilde{\mathfrak{R}}^A\,\mathrm{Tor}^B{}_{AB} - \widetilde{\mathfrak{E}}_B{}^C(\boldsymbol{L_P})^B{}_C - 2\,\widetilde{\dot{K}} - 2\,\widetilde{K}\,(\boldsymbol{L_P})^B{}_B - \widetilde{S}\,\dot{\Theta}, \quad (4.132)$$

where $\widetilde{K} = J_{\boldsymbol{P}}\,K$ and $\widetilde{\dot{K}} = J_{\boldsymbol{P}}\,\dot{K}$ are the modified kinetic energy and kinetic energy rate, respectively. The term

$$\widetilde{\mathfrak{Y}} = -J_{\boldsymbol{P}}\,\dot{\mathfrak{L}} + \widetilde{\mathfrak{R}}^A{}_{;A}, \quad (4.133)$$

is the *modified Eshelby power*, which is analogous to the modified Eshelby force $\widetilde{\mathfrak{N}}_A = \widetilde{\mathfrak{E}}_A{}^B{}_{;B}$ of Eq. (4.115). We remark that this is the only other exception to the definition of modified quantities and indeed $\widetilde{\mathfrak{Y}} \neq J_{\boldsymbol{P}}\, \mathfrak{Y}$. Thus

$$\widetilde{\mathfrak{Y}} = \widetilde{\mathfrak{R}}^A\, \mathrm{Tor}^B{}_{AB} - \widetilde{\mathfrak{E}}_A{}^B\, (\boldsymbol{L_P})^A{}_B - 2\, \widetilde{\dot{K}} - 2\, \widetilde{K}\, (\boldsymbol{L_P})^B{}_B - \widetilde{S}\, \dot{\Theta}. \tag{4.134}$$

In the case of pure remodelling and *no* growth (i.e., when $\boldsymbol{L_P}$ is traceless), we note that Eq. (4.134) reduces to the form

$$\widetilde{\mathfrak{Y}} = \widetilde{\mathfrak{R}}^A\, \mathrm{Tor}^B{}_{AB} - \widetilde{\mathfrak{E}}_A{}^B\, (\boldsymbol{L_P})^A{}_B - 2\, \widetilde{\dot{K}} - \widetilde{S}\, \dot{\Theta}. \tag{4.135}$$

In order to further handle the second term on the right-hand side of Eq. (4.135), we recall the definition (4.59) of the *Eshelby stress* $\boldsymbol{\mathfrak{E}}$, the relation (4.110) between the volumetric growth Π and $\boldsymbol{L_P} = \dot{\boldsymbol{P}}\boldsymbol{P}^{-1}$, and the definition (4.43) of Lagrangian density and obtain

$$\begin{aligned}\widetilde{\mathfrak{E}}_A{}^B\, (\boldsymbol{L_P})^A{}_B &= J_{\boldsymbol{P}}\, [-\mathcal{L}\, \delta_A{}^B - F^a{}_A\, T_a{}^B]\, \dot{P}^A{}_\alpha (\boldsymbol{P}^{-1})^\alpha{}_B \\ &= -J_{\boldsymbol{P}}\, \mathcal{L}\, (\boldsymbol{L_P})^A{}_A - J_{\boldsymbol{P}}\, \mathfrak{M}_A{}^B\, (\boldsymbol{L_P})^A{}_B = -\widetilde{\mathfrak{M}}_A{}^B\, (\boldsymbol{L_P})^A{}_B,\end{aligned} \tag{4.136}$$

where, in the last passage, we considered the condition of no growth, $\mathrm{tr}\, \boldsymbol{L_P} = (\boldsymbol{L_P})^A{}_A = 0$. Thus, with Eq. (4.136), we can write Eq. (4.135) as the temporal counterpart of Eq. (4.118) (which was written for the case of a uniform material under pure remodelling), i.e.,

$$\widetilde{\mathfrak{Y}} = \widetilde{\mathfrak{R}}^A\, \mathrm{Tor}^B{}_{AB} + \widetilde{\mathfrak{M}}_A{}^B\, (\boldsymbol{L_P})^A{}_B - 2\, \widetilde{\dot{K}} - \widetilde{S}\, \dot{\Theta}. \tag{4.137}$$

In the purely dynamical case (no growth, isothermal process), the modified Eshelby power of Eq. (4.137) reads

$$\widetilde{\mathfrak{Y}} = \widetilde{\mathfrak{R}}^A\, \mathrm{Tor}^B{}_{AB} + \widetilde{\mathfrak{M}}_A{}^B\, (\boldsymbol{L_P})^A{}_B - 2\, \widetilde{\dot{K}}. \tag{4.138}$$

If we recall Eq. (4.110), and write it

$$\dot{\varrho}_R = \Pi = -\varrho_R \, \mathrm{tr}(\boldsymbol{L_P}), \tag{4.139}$$

we have that Eqs. (4.138) and (4.139) are the counterparts of Eqs. (4.119) and (4.122), respectively. In the homogeneous case, when $P^A{}_\alpha = \mathbf{1}^A{}_\alpha$, the right-hand side of Eq. (4.138) reduces to $-2\,\dot{K}$, and the right-hand side of Eq. (4.139) goes to zero via the vanishing of the trace of the inhomogeneity rate tensor.

4.3.6 Entropy Inequality

In this section, we handle the entropy inequality (Clausisus-Duhem inequality) in Eq. (4.39b) in terms of the theory of uniformity. We start by recalling that, by virtue of Eq. (4.45), the thermoelastic Helmholtz free energy $W = \hat{W} \circ (\boldsymbol{F}, \Theta, \mathcal{X}, \tau)$ is given by

$$W = \hat{W} \circ (\boldsymbol{F}, \Theta, \mathcal{X}, \tau) = \hat{W}_{\mathrm{tot}} \circ (\phi, \boldsymbol{F}, \Theta, \mathcal{X}, \tau) - \hat{W}_{\mathrm{ext}} \circ \phi, \tag{4.140}$$

in terms of the total energy W_{tot} and the potential W_{ext} of the external forces. Its time the derivative is

$$\dot{W} = \boldsymbol{T} : \dot{\boldsymbol{F}} - S\,\dot{\Theta} + \frac{\partial \hat{W}}{\partial \tau} \circ (\#), \tag{4.141}$$

where we used Eq. (4.46). Since $\hat{\mathcal{L}} \circ (\#) = J_{\boldsymbol{P}}^{-1}\,\hat{\Lambda} \circ (\#)$ (uniformity condition (4.95)), also the time derivatives are equal, i.e.,

$$\partial_t \big[\hat{\mathcal{L}} \circ (\#)\big] = \partial_t \big[J_{\boldsymbol{P}}^{-1}\,\hat{\Lambda} \circ (\#)\big] \tag{4.142}$$

Thus, the "explicit" time derivative of $\mathcal{L}$, i.e., the *energy release rate* defined in Eq. (4.63), is evaluated as

$$\mathfrak{G} = \frac{\partial \hat{\mathcal{L}}}{\partial \tau} \circ (\#) = [-J_{\boldsymbol{P}}^{-1} \, \hat{\Lambda} \circ (\#) \, \boldsymbol{I}^T - \boldsymbol{F}^T \, \frac{\partial \hat{\Lambda}}{\partial (\boldsymbol{F}\boldsymbol{P})} \circ (\#)] \, \dot{\boldsymbol{P}} \, \boldsymbol{P}^{-1}, \tag{4.143}$$

by exploiting Eqs. (4.97), (4.58) and (4.87), Eq. (4.143) becomes

$$\frac{\partial \hat{\mathcal{L}}}{\partial \tau} \circ (\#) = [-\mathcal{L} \, \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{T}] : \boldsymbol{L_P} = \mathfrak{E} : \boldsymbol{L_P}. \tag{4.144}$$

Eq. (4.144) is a result that was also found in Grillo et al. (2003). In order to further manipulate Eq. (4.141), we need to notice that, by virtue of the definition (4.44) of $K = \hat{K} \circ (\dot{\phi}, \mathcal{X}, \tau)$ and of the balance of mass (4.34a), we have

$$\frac{\partial \hat{K}}{\partial \tau} \circ (\dot{\phi}, \mathcal{X}, \tau) = \frac{1}{2} \, \Pi \, \dot{\phi} \, . \, \dot{\phi}. \tag{4.145}$$

Also, we observe that, by exploiting the definition (4.43) of Lagrangian density, $\hat{\mathcal{L}} = \hat{K} - \hat{W}_{\text{tot}}$, and Eq. (4.145), the explicit time derivative $(\partial \hat{W}/\partial \tau) \circ (\#)$ is given by

$$\frac{\partial \hat{W}_{\text{tot}}}{\partial \tau} \circ (\#) = \frac{1}{2} \, \Pi \, \dot{\phi} \, . \, \dot{\phi} - \mathfrak{E} : \boldsymbol{L_P}. \tag{4.146}$$

Moreover, we note that, since $\hat{W}_{\text{ext}}$ does *not* depend explicitly on time,

$$\frac{\partial \hat{W}}{\partial \tau} \circ (\#) \equiv \frac{\partial \hat{W}_{\text{tot}}}{\partial \tau} \circ (\#). \tag{4.147}$$

Substituting Eqs. (4.146) and (4.147) into Eq. (4.141) and using Eq. (4.131), we get

$$\dot{W} = -K \, \text{tr}(\boldsymbol{L_P}) + \boldsymbol{T} : \dot{\boldsymbol{F}} - S \, \dot{\Theta} - \mathfrak{E} : \boldsymbol{L_P}. \tag{4.148}$$

Finally, substituting Eq. (4.148) into Eq. (4.39b), we obtain an expression of the entropy inequality (Clausisus-Duhem inequality) in the general framework of a uniform body:

$$\mathfrak{E} : \boldsymbol{L_P} + \Pi \, \frac{W}{\varrho_R} + K \, \mathrm{tr}(\boldsymbol{L_P}) + \frac{1}{\Theta} \langle \mathrm{Grad}\, \Theta \,|\, \boldsymbol{Q} \rangle \geq 0. \tag{4.149}$$

Recalling the definitions (4.58), (4.110), (4.87) and (4.43) of the Eshelby stress $\mathfrak{E}$, volumetric growth Π, inhomogeneity rate tensor $\boldsymbol{L_P}$ and Lagrangian $\mathcal{L}$, we observe that the sum of the first two terms of the inequality (4.149) is given by

$$\left[F^a{}_A \, \frac{\partial \hat{\mathcal{L}}}{\partial F^a{}_B} \circ (\#) - \mathcal{L} \, \delta_A{}^B \right] \dot{P}^A{}_\alpha (\boldsymbol{P}^{-1})^\alpha{}_B - (\boldsymbol{L_P})^A{}_A \, W =$$

$$= F^a{}_A \, \frac{\partial \hat{\mathcal{L}}}{\partial F^a{}_B} \circ (\#) \, (\boldsymbol{L_P})^A{}_B - K \, (\boldsymbol{L_P})^A{}_A, \tag{4.150}$$

where the term

$$-F^a{}_A \, \frac{\partial \hat{\mathcal{L}}}{\partial F^a{}_B} \circ (\#) = \mathfrak{M}_A{}^B \tag{4.151}$$

can be recognised, by comparison with Eqs. (4.46d) and (4.73), the Mandel stress. Thus, Eq. (4.149) finally becomes

$$-\mathfrak{M} : \boldsymbol{L_P} + \frac{1}{\Theta} \langle \mathrm{Grad}\, \Theta \,|\, \boldsymbol{Q} \rangle \geq 0, \tag{4.152}$$

which shows that the Mandel stress is the *driving force* of the growth-remodelling process, whose kinematics is described by $\boldsymbol{L_P}$.

4.3.7 Relation Between the Two Identities

Here, for the general, dynamical case, we derive a relation between the identity involving the modified Eshelby force $\widetilde{\mathfrak{N}}_A = \widetilde{\mathfrak{E}}_A{}^B{}_{;B}$ and that involving the modified Eshelby power $\widetilde{\mathfrak{Y}} = -J_{\boldsymbol{P}} \, \dot{\mathcal{L}} + \widetilde{\mathfrak{R}}^A{}_{;A}$, namely, Eq. (4.119) and Eq. (4.138).

Let us first multiply the time derivative of material momentum density $\dot{\mathcal{P}}_A$ Eq. (4.53) by the negative of the *inverse velocity* $\dot{\Phi}^A$ (where $\Phi(\,\cdot\,,t) = [\phi(\,\cdot\,,t)]^{-1}$ is the inverse motion; see (Epstein and Maugin, 2000)), to obtain

$$-\dot{\mathcal{P}}_A\,\dot{\Phi}^A = -\Pi\,\dot{\phi}^a\,g_{ab}\,(-F^b{}_A\,\dot{\Phi}^A) - \varrho_R\,\partial_t(\dot{\phi}^a\,g_{ab})\,(-F^b{}_A\,\dot{\Phi}^A) - \varrho_R\,\dot{\phi}^a\,g_{ab}\,(-\dot{F}^b{}_A\,\dot{\Phi}^A). \quad (4.153)$$

Now, we exploit the relation between the material inverse velocity and the spatial Lagrangian velocity (Epstein and Maugin, 2000), i.e.,

$$\dot{\phi}^b = -F^b{}_A\,\dot{\Phi}^A, \quad (4.154)$$

and consider the expression of the kinetic energy in Eq. (4.65). Thus, Eq. (4.153) yields

$$-\dot{\mathcal{P}}_A\,\dot{\Phi}^A = -2\dot{K} + \mathcal{P}_A\,\ddot{\Phi}^A. \quad (4.155)$$

We then multiply Eq. (4.155) by $J_{\boldsymbol{P}}$ and use the two identities Eq. (4.119) and Eq. (4.138) to eliminate $2\,\widetilde{\dot{K}} = 2\,J_{\boldsymbol{P}}\,\dot{K}$ and $\widetilde{\dot{\mathcal{P}}}_A = J_{\boldsymbol{P}}\dot{\mathcal{P}}_A\,\dot{\Phi}^A$, respectively, and obtain

$$\begin{aligned}
&-\widetilde{\mathcal{N}}_A\,\dot{\Phi}^A - \widetilde{\mathfrak{M}}_A{}^C\,\mathrm{Tor}^B{}_{CB}\,\dot{\Phi}^A - \widetilde{\mathfrak{M}}_B{}^C\,\mathrm{Tor}^B{}_{CA}\,\dot{\Phi}^A = \\
&\qquad = \widetilde{\mathcal{Y}} - \widetilde{\mathfrak{R}}^C\,\mathrm{Tor}^B{}_{CB} - \widetilde{\mathfrak{M}}_A{}^B\,(\boldsymbol{L_P})^A{}_B + \widetilde{\mathcal{P}}_A\,\ddot{\Phi}^A. \quad (4.156)
\end{aligned}$$

By utilising Eqs. (4.67) and (4.154), we observe that the second term on the left-hand side of Eq. (4.156) can be factorised into

$$\widetilde{\mathfrak{M}}_A{}^C\,\dot{\Phi}^A\,\mathrm{Tor}^B{}_{CB} = J_{\boldsymbol{P}}\,T_a{}^C\,(F^a{}_A\,\dot{\Phi}^A)\,\mathrm{Tor}^B{}_{CB} \quad = J_{\boldsymbol{P}}(-\dot{\phi}^a\,T_a{}^C)\,\mathrm{Tor}^B{}_{CB} = \widetilde{\mathfrak{R}}^C\,\mathrm{Tor}^B{}_{CB}. \quad (4.157)$$

Finally, we substitute (4.157) into Eq. (4.156) and obtain

$$\widetilde{\mathcal{Y}} = -\widetilde{\mathfrak{N}}_A \, \dot{\Phi}^A + \widetilde{\mathfrak{M}}_B{}^C \left[(\boldsymbol{L_P})^B{}_C - \mathrm{Tor}^B{}_{CA} \, \dot{\Phi}^A \right] - \widetilde{\mathcal{P}}_A \, \ddot{\Phi}^A, \tag{4.158}$$

which shows that the modified Eshelby power represents a measure of the mechanical power expended in a uniform body to make the inhomogeneity evolve, and thus carries information about inhomogeneity and its evolution.

Chapter 5

Eshelby's Inclusion Problem in Large Deformations

This chapter is based on Alhasadi and Federico (2019)

The theory of inclusions in an elastic matrix has been extensively studied by many researchers, particularly in the framework of the linear elastic theory under small deformations. In Eshelby's seminal work (Eshelby, 1957), an inclusion in a material body is seen as a region that generally has a *geometrical misfit* with the surrounding material, which causes the arising of *residual stresses*. This geometrical misfit has been represented by Eshelby in terms of a *transformation strain* (or *eigenstrain*) $\boldsymbol{\epsilon}^T$. The transformation strain is an imagined strain transforming the shape of a cavity in a body into a different shape; this new shape is then assigned material properties and implanted back into the body. Because of the geometrical misfit, the implant will only be possible by first applying surface tractions causing a strain exactly equal to $-\boldsymbol{\epsilon}^T$ and by subsequently releasing these surface tractions, which will cause a strain $\boldsymbol{\epsilon}^C$, called *cancelling strain* in both the inclusion and the surrounding material, giving rise to residual stresses. Eshelby called the strain attained by the inclusion at the end of this procedure the *total (elastic) strain* $\boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^T$, where $\boldsymbol{\epsilon}^T$ is non-zero *only* in the inclusion. It is important to emphasise that, in Eshelby's procedure (see also Mura, 1987; Balluffi, 2012), *all* strains are *piecewise compatible*, i.e., integrable separately in the inclusion and in the matrix, to obtain the corresponding transformation displacement $\boldsymbol{u}^T$ and cancelling displacement $\boldsymbol{u}^C$. It has been shown that the only case for which a uniform transformation strain results in uniform stress and strain fields inside the inclusion is that of an *ellipsoidal inclusion* (Sendeckyj, 1970; Rodin, 1996; Ru and Schiavone, 1996; Markenscoff, 1997; Kim et al., 2008).

For the case of an ellipsoidal inclusion in an isotropic matrix, Eshelby (1957) showed that the cancelling strain is related to the transformation strain via $\boldsymbol{\epsilon}^C = \mathbb{S} : \boldsymbol{\epsilon}^T$, where the fourth-order tensor $\mathbb{S}$ is now commonly referred to as *Eshelby's (fourth-order) tensor* in his honour. Since the ellipsoidal inclusion can describe several cases, ranging from cracks (extremely oblate ellipsoid with negligible elastic properties) to fibres (extremely prolate ellipsoid), this approach has been used widely in literature on linear composite material with inclusions (e.g., Walpole, 1966a,b; Tandon and Weng, 1984; Weng, 1984, 1990; Federico et al., 2004), in order to find the overall properties of a composite material starting from the properties of the individual constituents.

In the framework of large-deformation mechanics, Eshelby's inclusion problem has been formulated by means of a multiplicative decomposition of the deformation gradient in the form $\boldsymbol{F} = \boldsymbol{F}^e\boldsymbol{F}^*$ (in the notation of Diani and Parks, 2000; Yavari and Goriely, 2013), where $\boldsymbol{F}^*$ has been termed the "eigenstrain deformation gradient" (corresponding to Eshelby's $-\boldsymbol{\epsilon}^T$), and $\boldsymbol{F}_e$ the "elastic deformation gradient" (corresponding to Eshelby's $\boldsymbol{\epsilon}^C$).

We note that this approach is conceptually based on the classical Bilby-Kröner-Lee (Bilby et al., 1957; Kröner, 1959; Lee, 1969) decomposition of the deformation gradient, originally devised for plasticity, i.e., $\boldsymbol{F} = \boldsymbol{F}_e\boldsymbol{F}_a$, where $\boldsymbol{F}_a$ is the anelastic part of the deformation, which, in the case of plasticity, represents precisely the *plastic* deformation. In the equivalent approach to anelasticity stemming from Noll's theory of material uniformity (Noll, 1967), Epstein and Maugin (1990) defined the *implant* tensor $\boldsymbol{P}$ as the tensor mapping an *archetypal* point into the body, generally causing a non-integrable distortion in the body. In this formulation, the purely elastic deformation of the theory of plasticity reads $\boldsymbol{F}_e = \boldsymbol{F}\boldsymbol{P}$. Comparison with the Bilby-Kröner-Lee decomposition yields that the implant tensor is the inverse of the plastic deformation, i.e., $\boldsymbol{P} = \boldsymbol{F}_a^{-1}$.

We are aware of only one attempt to obtain the large-deformation counterpart of Eshelby's relation $\boldsymbol{\epsilon}^C = \mathbb{S} : \boldsymbol{\epsilon}^T$ between the transformation and the cancelling strain in terms of the fourth-order Eshelby tensor. Nemat-Nasser (1999), used an approach in terms of the time derivative of the deformation gradient $\boldsymbol{F}$ and proposed an additive expression of these $\dot{\boldsymbol{F}}$-like tensors as the large-deformation counterpart of the additive decomposition of the infinitesimal strain. However, this must be considered as an approximation, since the various time derivatives of the $\boldsymbol{F}$-like tensors that Nemat-Nasser (1999) considered map between different spaces and thus, rigorously speaking, cannot be added.

The goal of this work is threefold. First, we shall show that Eshelby's inclusion problem admits a multiplicative decomposition of the deformation gradient that is *piecewise* (matrix/inclusion) compatible, i.e., each resulting deformation gradient is in fact the tangent map of a certain configuration map in the matrix or the inclusion. Second, we shall determine the multiplicative decomposition of the deformation gradient corresponding to the additive decomposition of the infinitesimal strain used by Eshelby (1957) in the case of small deformations; in particular, we shall show that Eshelby's original additive decomposition of the infinitesimal strain is most naturally obtained via a *mixed* formulation, in which the transformation strain (which we denote $\boldsymbol{\epsilon}_t$) is the linearisation of an $\boldsymbol{F}$-type tensor of the Bilby-Kröner-Lee decomposition, while the cancelling strain (which we denote $\boldsymbol{\epsilon}_c$) is the linearisation of a $\boldsymbol{P}$-type tensor of the Noll-Epstein-Maugin decomposition. Third, we shall show how Eshelby's fourth-order tensor $\mathbb{S}$ can be recast in an incremental formulation in the case of large deformations and that the cancelling deformation outside the inclusion can be obtained via a suitable evolution law.

5.1 Theoretical Framework

We identify a continuum body with one placement $\mathcal{B}$ in the physical space $\mathcal{S}$, called the reference configuration. For the sake of simplicity, we regard the physical space $\mathcal{S}$ to coincide with the affine space $\mathbb{E}^3$ modelled on $\mathbb{R}^3$ (Epstein, 2010). A *configuration* of the body is a mapping

$$\phi : \mathcal{B} \to \mathcal{S} : X \mapsto x = \phi(X) \tag{5.1}$$

which maps *material* points $X = (X^1, X^2, X^3)$ in the body $\mathcal{B}$ into *spatial* points $x = (x^1, x^2, x^3)$ in $\mathcal{S}$. Note that, for the sake of a lighter notation, we omitted the parameter time (rigorously, the ϕ in Eq. (5.1) is $\phi(\,\cdot\,, t)$ at a given time t). The map ϕ is called configuration map, and is an *embedding*, i.e., it is such that its codomain-restriction $\phi : \mathcal{B} \to \phi(\mathcal{B})$ is a diffeomorphism: a continuous and differentiable map, which is invertible, with a continuous and differentiable inverse $\Phi \equiv \phi^{-1}$. The tangent map of the configuration at each material point $X \in \mathcal{B}$ is the Fréchet differential of ϕ, i.e., the linear map

$$\boldsymbol{F}(X) \equiv (T\phi)(X) : T_X\mathcal{B} \to T_x\mathcal{S}, \tag{5.2}$$

called *deformation gradient*, for every material point $X \in \mathcal{B}$, $\boldsymbol{F}(X)$ maps vectors of the tangent space $T_X\mathcal{B}$ into spatial vectors of the tangent space $T_x\mathcal{S}$ (with $x = \phi(X)$) and such that, in coordinate charts $\{X^A\}$ in $\mathcal{B}$ and $\{x^a\}$ in $\mathcal{S}$, it is represented by $F^a{}_A = \phi^a{}_{,A}$. The determinant $J = \det \boldsymbol{F}$ is the volume rate.

5.2 Anelastic Phenomena

Anelastic phenomena have been represented in two alternative ways: the Bilby-Kröner-Lee (Bilby et al., 1957; Kröner, 1959; Lee, 1969) decomposition of the deformation gradient and Noll's theory of uniformity (Noll, 1967), further developed by Epstein and Maugin (e.g., Epstein and Maugin, 1990). In the Bilby-Kröner-Lee multiplicative decomposition of the deformation gradient,

$$\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_a, \tag{5.3}$$

the tensors $\boldsymbol{F}_a$ and $\boldsymbol{F}_e$ describe the anelastic and elastic parts of the deformation, respectively. In general, they are both *non-integrable*, i.e., they are not the tangent of any configuration map. This means that $\boldsymbol{F}_a(X)$ maps the tangent space $T_X\mathcal{B}$ at X into a vector space $N_X\mathcal{B}$ describing the *natural state* (i.e., stress-free state) of point X (Stefano et al., 2019). The non-integrability of $\boldsymbol{F}_a$ implies that the natural states $N_X\mathcal{B}$ and $N_Y\mathcal{B}$ of two neighbouring points X and Y will in general *not* be compatible. The elastic deformation $\boldsymbol{F}_e(X)$ maps $N_X\mathcal{B}$ into the tangent space $T_x\mathcal{S} \equiv T_{\phi(X)}\mathcal{S}$ in the current deformation and *restores compatibility*. Note that, while $\boldsymbol{F}_e$ and $\boldsymbol{F}_a$ are individually not integrable, their product is, since it coincides with the deformation gradient $\boldsymbol{F} = T\phi$. Figure 5.1 illustrates the Bilby-Kröner-Lee decomposition and the lack of compatibility in the *collection of natural states* $N\mathcal{B}$ (Stefano et al., 2019).

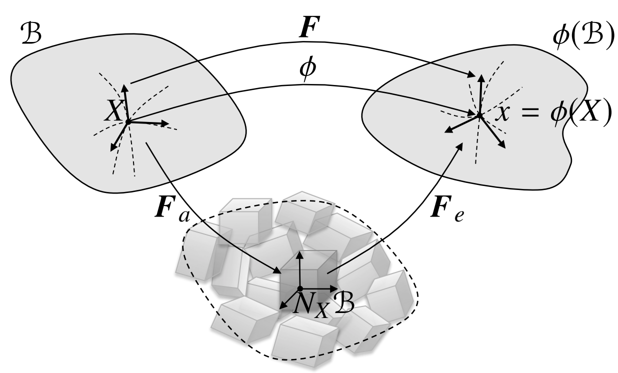

Figure 5.1: Bilby-Kröner-Lee decomposition, with the generally incompatible collection of natural states $N\mathcal{B}$, given by the disjoint union of the vector spaces $N_X\mathcal{B}$.

In the definition by Noll (1967), a body is *uniform* if all of its points are made of the same material, even if, in the chosen reference configuration, they are distorted in such a way that they exhibit *different* material properties. In the approach followed by Epstein and Maugin (1990), this means that, for every point X, there exists a *uniformity map* $\boldsymbol{P}(X)$ mapping a *fixed* vector space $\mathcal{A}$, called the *archetype*, into the tangent space $T_X\mathcal{B}$. Figure 5.2 shows the archetype and the uniformity maps $\boldsymbol{P}(X)$ and $\boldsymbol{P}(Y)$ of two distinct points X and Y. Since all points are assumed to be made of the same material, it is possible to map $T_X\mathcal{B}$ into $T_Y\mathcal{B}$ by means of $\boldsymbol{P}(Y)\,\boldsymbol{P}^{-1}(X)$.

For the case of a uniform body, the Bilby-Kröner-Lee (BKL) decomposition and the Noll-

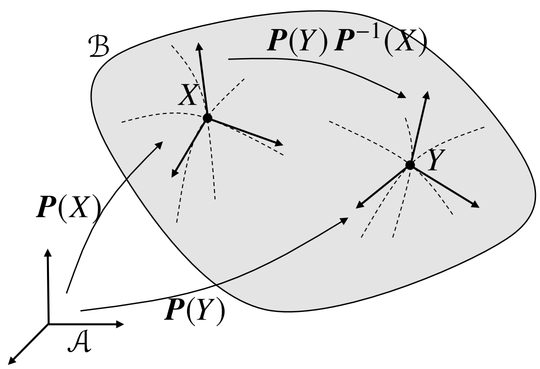

Figure 5.2: The representation of a materially uniform body $\mathcal{B}$ and the isomorphism between two points X and Y. The vector space $\mathcal{A}$ is the archetype.

Epstein-Maugin (NEM) decomposition become equivalent once we set

$$\boldsymbol{F}_e = \boldsymbol{F}\boldsymbol{P}, \qquad \boldsymbol{P} = \boldsymbol{F}_a^{-1}, \tag{5.4}$$

as illustrated in Figure 5.3.

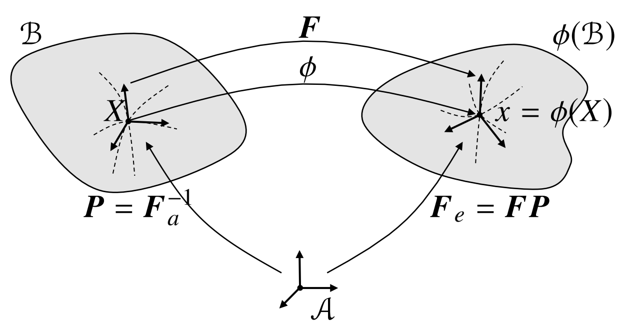

Figure 5.3: Equivalence of the BKL and NEM decompositions for a uniform material.

5.3 Eshelby's Inclusion under Finite Deformations

In small deformations, Eshelby's imaginary procedure for the characterisation of the inclusion problem is introduced in terms of infinitesimal strains, which can be decomposed additively. Under finite deformations, we need to refer to the corresponding multiplicative decomposition of the deformation gradient. We note that the Bilby-Kröner-Lee and the Noll-Epstein-Maugin decompositions were developed with *incompatible* deformations and *continuous distributions of* *defects* in mind. In the case of an inclusion, these decompositions give rise to deformation tensors that are *piecewise compatible*, i.e., compatible in the matrix and inclusion *separately*. This means that the incompatibility is *discrete*, in the sense that there are two bodies (the matrix and the inclusion) each undergoing compatible deformations, but whose shapes are *incompatible with each other*.

We also note that Eshelby considered, as the reference configuration, the incompatible stress-free state. For this reason, his "total strain $\boldsymbol{\epsilon}$" would correspond, in the spirit of the Bilby-Kröner-Lee or Noll-Epstein-Maugin decompositions, to an *elastic strain* $\boldsymbol{\epsilon}_e$, linearisation of the elastic deformation $\boldsymbol{F}_e$ of Eqs. (5.3) and (5.4). Similarly, Eshelby's "applied strain $\boldsymbol{\epsilon}^A$" corresponds to the *globally integrable* deformation gradient $\boldsymbol{F} = T\phi$. In order to avoid confusion, we follow the notation dictated by the decompositions (5.3) and (5.4), and then report, in Table 5.1 the equivalence among several different notations found in the literature, starting from that of Eshelby (1957). Our description of Eshelby's procedure is similar to the original one (Eshelby, 1957, 1975) and to that given in a previous work (Alhasadi and Federico, 2017), and describes the case of an inclusion made *of the same material* of the matrix, but with a misfitting shape, causing residual stresses. Eshelby (1957) called this type of inclusion a "homogeneous inclusion".

5.3.1 Eshelby's Procedure

With reference to Figure 5.4, let $\tilde{\mathcal{M}}$ be an elastic and stress-free body with a cavity described by the open region $\tilde{\mathcal{R}}$ in its interior. Let $\mathcal{D}^*$ be another elastic and stress-free body, called *inclusion*, which does *not* fit the cavity $\tilde{\mathcal{R}}$. For the moment, let us limit ourselves to the case in which the material contained in $\mathcal{D}^*$ is *the same* as in $\tilde{\mathcal{M}}$ (Eshelby's "homogeneous inclusion"). Together, $\tilde{\mathcal{M}}$ and $\mathcal{D}^*$ constitute an *incompatible stress-free configuration* $\mathcal{B}^*$ of some body $\mathcal{B}$, which we are now going to construct. We note that in the procedure originally presented by Eshelby (1957), the first passage is slightly different, although equivalent. We shall clarify this in Section 5.3.2.

In order to fit $\mathcal{D}^*$ into the cavity $\tilde{\mathcal{R}}$ of $\tilde{\mathcal{M}}$, we apply tractions on the boundary $\partial\mathcal{D}^*$ of the inclusion. As a result, the inclusion $\mathcal{D}^*$ will undergo a deformation described by the map $\wp_t^{(i)} : \mathcal{D}^* \to \tilde{\mathcal{D}}$ with tangent map $\boldsymbol{P}_t^{(i)} \equiv T\wp_t^{(i)}$, transforming $\mathcal{D}^*$ precisely into $\tilde{\mathcal{D}} \equiv \tilde{\mathcal{R}}$. We call the tangent map $\boldsymbol{P}_t^{(i)}$ the *transformation implant*, which is *by definition* compatible (i.e., integrable). At the same time, we note that nothing has happened to $\tilde{\mathcal{M}}$. Thus, everywhere in $\tilde{\mathcal{M}}$, we have that $\wp_t^{(m)} : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ coincides with the identity map $\widetilde{\mathrm{Id}}$ in $\tilde{\mathcal{M}}$ and $\boldsymbol{P}_t^{(m)} = \tilde{\boldsymbol{I}}$ is the identity tensor in $T\tilde{\mathcal{M}}$, which reflects the fact that the matrix $\tilde{\mathcal{M}}$ is undergoing *no* deformation. We denote $\tilde{\mathcal{B}} = \tilde{\mathcal{M}} \cup \tilde{\mathcal{D}}$ the configuration of the sought body $\mathcal{B}$ in which the matrix $\tilde{\mathcal{M}}$ is still stress-free but the inclusion $\tilde{\mathcal{D}}$ is subjected to the stresses caused by the transformation map $\wp_t^{(i)}$ and transformation implant $\boldsymbol{P}_t^{(i)}$. We can define the *overall* transformation map $\wp_t$ and transformation implant $\boldsymbol{P}_t$ *piecewise* as

$$\wp_t = \begin{cases} \wp_t^{(i)} & : \mathcal{D}^* \to \tilde{\mathcal{D}}, \quad \text{in } \mathcal{D}^*, \\ \wp_t^{(m)} \equiv \widetilde{\mathrm{Id}} & : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}, \quad \text{in } \tilde{\mathcal{M}}, \end{cases} \tag{5.5}$$

and

$$\boldsymbol{P}_t = \begin{cases} \boldsymbol{P}_t^{(i)} & : T\mathcal{D}^* \to T\tilde{\mathcal{D}}, \quad \text{in } \mathcal{D}^*, \\ \boldsymbol{P}_t^{(m)} \equiv \tilde{\boldsymbol{I}} & : T\tilde{\mathcal{M}} \to T\tilde{\mathcal{M}}, \quad \text{in } \tilde{\mathcal{M}}. \end{cases} \tag{5.6}$$

Once the transformed inclusion $\tilde{\mathcal{D}}$ has been fit into the cavity $\tilde{\mathcal{R}}$, we weld (i.e., impose *perfect bonding between matrix and inclusion*) and relax the surface tractions on $\partial\tilde{\mathcal{D}}$. This causes *both* the inclusion $\tilde{\mathcal{D}}$ and the matrix $\tilde{\mathcal{M}}$ to relax elastically with a deformation

$$\wp_c : \tilde{\mathcal{B}} \to \mathcal{B}, \tag{5.7}$$

called *cancelling map*, which is everywhere continuous (in order to respect the perfect bonding condition) but is non-differentiable on $\partial\tilde{\mathcal{D}}$, i.e., its tangent map $\boldsymbol{P}_c \equiv T\wp_c$ is not defined on $\partial\tilde{\mathcal{D}}$ and presents a *jump condition* on $\partial\tilde{\mathcal{D}}$. Thus, the *cancelling implant* $\boldsymbol{P}_c$ is defined piecewise as

$$\boldsymbol{P}_c = \begin{cases} \boldsymbol{P}_c^{(i)} & : T\tilde{\mathcal{D}} \to T\mathcal{D}, \quad \text{in } \tilde{\mathcal{D}}, \\ \boldsymbol{P}_c^{(m)} & : T\tilde{\mathcal{M}} \to T\mathcal{M}, \quad \text{in } \tilde{\mathcal{M}}. \end{cases} \tag{5.8}$$

The configuration resulting from $\wp_c$ is the sought reference configuration $\mathcal{B}$ of our body with a misfitting inclusion, and is subjected to *residual stresses*, which are *continuous* on the matrix-inclusion interface $\partial\mathcal{D}$. When $\mathcal{B}$ is subjected to a deformation described by the configuration map ϕ and its tangent map $\boldsymbol{F} = T\phi$, it attains the *current configuration* $\phi(\mathcal{B})$.

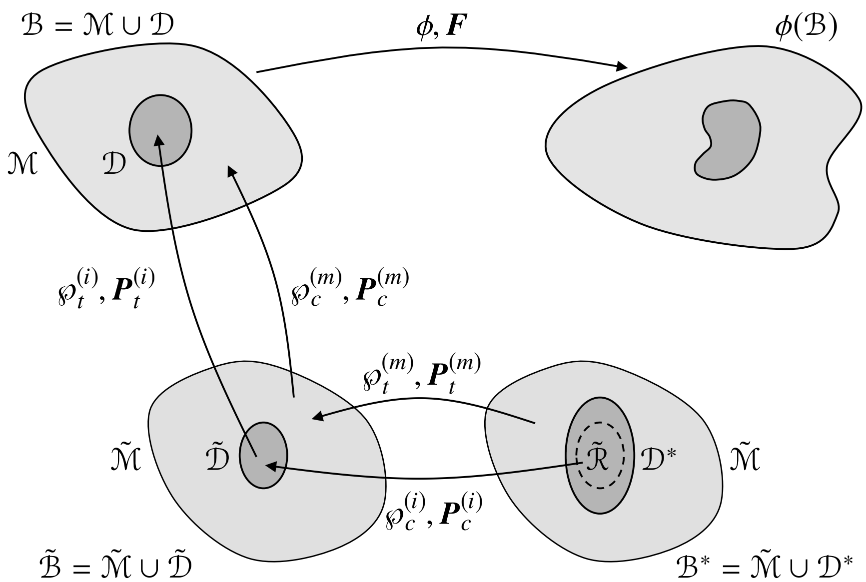

Figure 5.4: Eshelby's procedure to *implant* an inclusion in a homogeneous matrix.

We conclude this section by remarking that the continuity of the cancelling map $\wp_c$ and the discontinuity of the cancelling implant $\boldsymbol{P}_c$ are the large-deformation analogues of the continuity of the cancelling displacement $\boldsymbol{u}_c$ and the discontinuity of the cancelling strain $\boldsymbol{\epsilon}_c$ on the matrix-inclusion interface in the original work by Eshelby (1957) (in which they were denoted $\boldsymbol{u}^C$ and $\boldsymbol{\epsilon}^C$, respectively).

5.3.2 BKL, NEM, Eshelby's and Proposed Representations

We show how Ehselby's procedure (Eshelby, 1957) can be represented by the BKL and NEM multiplicative decompositions.

For the case of the BLK decomposition, with reference to Figure 5.5, we have

$$\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_t \boldsymbol{F}_c, \tag{5.9}$$

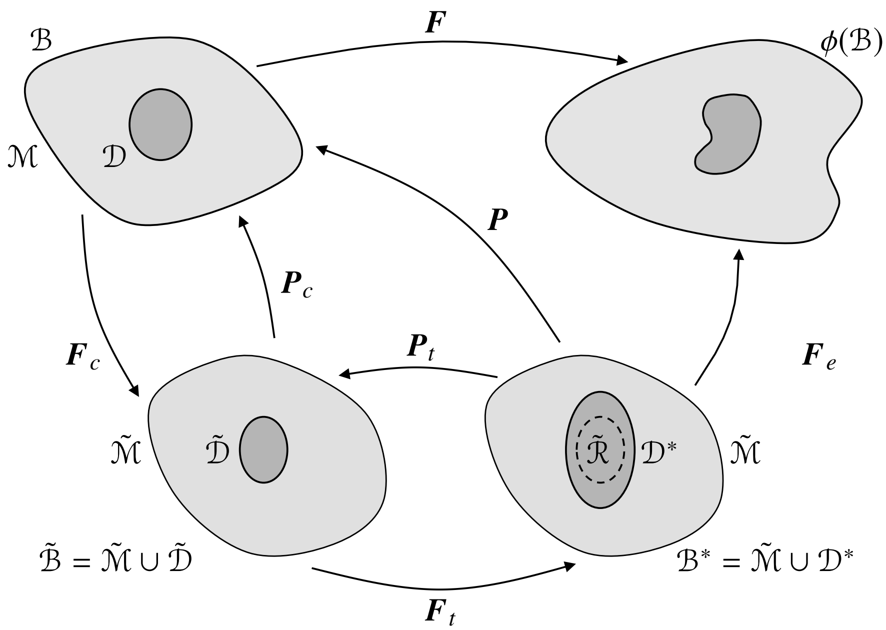

Figure 5.5: Mixed BKL-NEM decomposition to describe Eshelby's inclusion problem in large deformations.

where $\boldsymbol{F}_t\boldsymbol{F}_c$, given by the *transformation deformation gradient* $\boldsymbol{F}_t$ and the *cancelling deformation gradient* $\boldsymbol{F}_c$, replaces the anelastic deformation $\boldsymbol{F}_a$ of the classical BLK decomposition (5.3). We remark that, while the $\boldsymbol{F}_a$ in (5.3) is generally *non-integrable*, both the transformation and cancelling deformation gradients $\boldsymbol{F}_t$ and $\boldsymbol{F}_c$ are.

Always with reference to Figure 5.5, for the case of the NEM multiplicative decomposition, we have

$$\boldsymbol{F}_e = \boldsymbol{F}\boldsymbol{P}_c\boldsymbol{P}_t, \tag{5.10}$$

where the implant tensor $\boldsymbol{P}$ has been replaced by the product $\boldsymbol{P}_c\boldsymbol{P}_t$ of the cancelling implant $\boldsymbol{P}_c$ and transformation implant $\boldsymbol{P}_t$ described in Section 5.3.1. Comparing with Eq. (5.9), we have that

$$\boldsymbol{P}_c = \boldsymbol{F}_c^{-1}, \qquad \boldsymbol{P}_t = \boldsymbol{F}_t^{-1}. \tag{5.11}$$

In order to match Eshelby's original additive decomposition of the infinitesimal strain, we propose a *mixed* BKL-NEM formulation, in which we consider as primary objects the transformation deformation $\boldsymbol{F}_t = \boldsymbol{P}_t^{-1}$ and the cancelling implant $\boldsymbol{P}_c$, and write the decomposition as

$$\boldsymbol{F}_e = \boldsymbol{F}\boldsymbol{P}_c\boldsymbol{F}_t^{-1}, \tag{5.12}$$

in which we must keep in mind that $\boldsymbol{F}_t^{-1} = \boldsymbol{P}_t$ is defined *piecewise* in the matrix and in the inclusion, according to Eq. (5.6).

As we shall show later (see Eq. (5.26)) the linearised form of the decomposition (5.12) is

$$\boldsymbol{\epsilon}_e = \boldsymbol{\epsilon} + \boldsymbol{\epsilon}_c - \boldsymbol{\epsilon}_t, \tag{5.13}$$

where $\boldsymbol{\epsilon}_e$, $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}_c$ and $\boldsymbol{\epsilon}_t$ are the linearisations of $\boldsymbol{F}_e$, $\boldsymbol{F}$, $\boldsymbol{P}_c$ and $\boldsymbol{F}_t$, respectively. The perfect match with Eshelby's formulation Eshelby (1957) can be seen by writing Eq. (5.13) in Eshelby's notation (see also Table 5.1) as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^A + \boldsymbol{\epsilon}^C - \boldsymbol{\epsilon}^T. \tag{5.14}$$

We remark that, in his original presentation Eshelby (1957) actually deformed the shape of the cavity $\tilde{\mathcal{R}}$ into the shape of the misfitting inclusion $\mathcal{D}^*$ by means of the transformation strain $\boldsymbol{\epsilon}_t$, corresponding, in our framework, to $\boldsymbol{P}_t = \boldsymbol{F}_t^{-1}$. Precisely for this reason, Eqs. (5.13) and (5.14) show that Eshelby's formulation was in fact a *mixed* formulation, with the (*implant*) cancelling strain $\boldsymbol{\epsilon}_c$, corresponding to our cancelling implant $\boldsymbol{P}_c$, and the transformation strain $\boldsymbol{\epsilon}_t$, corresponding to our transformation deformation $\boldsymbol{F}_t$.

deformation	inf. strain	Eshelby (1957)	Mura (1987)	Alhasadi and Federico (2017)	Diani and Parks (2000); Yavari and Goriely (2013)
$\boldsymbol{F}$	$\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}^A$	$\boldsymbol{\epsilon}^0$	$\boldsymbol{\epsilon}^A$	$[\mathbf{1}]$
$\boldsymbol{F}_t$	$\boldsymbol{\epsilon}_t$	$\boldsymbol{\epsilon}^T$	$\boldsymbol{\epsilon}^*$	$\boldsymbol{\epsilon}^*$	$(\boldsymbol{F}^*)^{-1}$
$\boldsymbol{F}_c$	$-\boldsymbol{\epsilon}_c$	$-\boldsymbol{\epsilon}^C$	$-\boldsymbol{\epsilon}$	$-\boldsymbol{\epsilon}^C$	$(\boldsymbol{F}^e)^{-1}$
$\boldsymbol{P}_t$	$-\boldsymbol{\epsilon}_t$	$-\boldsymbol{\epsilon}^T$	$-\boldsymbol{\epsilon}^*$	$-\boldsymbol{\epsilon}^*$	$\boldsymbol{F}^*$
$\boldsymbol{P}_c$	$\boldsymbol{\epsilon}_c$	$\boldsymbol{\epsilon}^C$	$\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}^C$	$\boldsymbol{F}^e$
$\boldsymbol{P}=\boldsymbol{P}_c\boldsymbol{F}_t^{-1}$	$\boldsymbol{\epsilon}_c-\boldsymbol{\epsilon}_t$	$\boldsymbol{\epsilon}^C-\boldsymbol{\epsilon}^T$	$\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^*$	$\boldsymbol{\epsilon}^B$	$\boldsymbol{F}$
$\boldsymbol{F}_e$	$\boldsymbol{\epsilon}_e$	$\boldsymbol{\epsilon}$	$\boldsymbol{e}$	$\boldsymbol{\epsilon}$	$[\mathbf{1}\boldsymbol{F}]$

Table 5.1: Comparison of the notation presented here for large deformations ("deformation" column) and for small deformation ("inf. strain" column) and the various notations used in the literature (Eshelby, 1957; Mura, 1987; Diani and Parks, 2000; Yavari and Goriely, 2013; Alhasadi and Federico, 2017). For the case of Diani and Parks (2000) and Yavari and Goriely (2013), note that *no* deformation is applied after the implantation of the inclusion and this is why we indicated the symbols corresponding to our $\boldsymbol{F} = T\phi$ and $\boldsymbol{F}_e$ in brackets. In the notation of Diani and Parks (2000) and Yavari and Goriely (2013), the deformation gradient $\boldsymbol{F} = T\phi$ would be represented by a simple shifter $\mathbf{1}$ (see, e.g., Eringen, 1980; Marsden and Hughes, 1983, for a definition of shifter) and, consequently, the elastic deformation $\boldsymbol{F}_e$ would be represented as $\mathbf{1}\boldsymbol{F}$. For the case of the book by Mura (1987), the transformation strain $\boldsymbol{\epsilon}^*$ is called an *eigenstrain*.

5.3.3 Rate Form of the Proposed Decomposition

In order to obtain the large deformation analogue of Eshelby's expression $\boldsymbol{\epsilon}_c = \mathbb{S} : \boldsymbol{\epsilon}_t$ (in our notation), let us start by taking the time derivative of Eq. (5.12), i.e.,

$$\begin{aligned}\dot{\boldsymbol{F}}_e &= \dot{\boldsymbol{F}}\boldsymbol{P}_c\boldsymbol{F}_t^{-1} + \boldsymbol{F}\dot{\boldsymbol{P}}_c\boldsymbol{F}_t^{-1} + \boldsymbol{F}\boldsymbol{P}_c(\boldsymbol{F}_t^{-1})^{\cdot} \\ &= \dot{\boldsymbol{F}}\boldsymbol{P}_c\boldsymbol{F}_t^{-1} + \boldsymbol{F}\dot{\boldsymbol{P}}_c\boldsymbol{F}_t^{-1} - \boldsymbol{F}\boldsymbol{P}_c\boldsymbol{F}_t^{-1}\dot{\boldsymbol{F}}_t\boldsymbol{F}_t^{-1} \\ &= \dot{\boldsymbol{F}}\boldsymbol{P}_c\boldsymbol{F}_t^{-1} + \boldsymbol{F}\dot{\boldsymbol{P}}_c\boldsymbol{F}_t^{-1} - \boldsymbol{F}_e\dot{\boldsymbol{F}}_t\boldsymbol{F}_t^{-1}.\end{aligned} \tag{5.15}$$

where we used $(\boldsymbol{F}_t\,\boldsymbol{F}_t^{-1})^{\cdot} = \boldsymbol{0}$ in the first passage and Eq. (5.12) in the last passage. By analogy with the velocity gradient $\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, the elastic rate is defined as

$$\begin{aligned}\boldsymbol{l}_e = \dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1} = \dot{\boldsymbol{F}}\boldsymbol{P}_c\boldsymbol{F}_t^{-1}\boldsymbol{F}_t\boldsymbol{P}_c^{-1}\boldsymbol{F}^{-1} \\ + \boldsymbol{F}\dot{\boldsymbol{P}}_c\boldsymbol{F}_t^{-1}\boldsymbol{F}_t\boldsymbol{P}_c^{-1}\boldsymbol{F}^{-1} \\ - \boldsymbol{F}_e\dot{\boldsymbol{F}}_t\boldsymbol{F}_t^{-1}\boldsymbol{F}_e^{-1},\end{aligned} \tag{5.16}$$

from which

$$\begin{aligned}\boldsymbol{l}_e = \dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1} &= \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} + \boldsymbol{F}\dot{\boldsymbol{P}}_c\boldsymbol{P}_c^{-1}\boldsymbol{F}^{-1} - \boldsymbol{F}_e\dot{\boldsymbol{F}}_t\boldsymbol{F}_t^{-1}\boldsymbol{F}_e^{-1}, \\ &= \boldsymbol{l} + \boldsymbol{F}\boldsymbol{L}_c\boldsymbol{F}^{-1} - \boldsymbol{F}_e\boldsymbol{\Lambda}_t\boldsymbol{F}_e^{-1},\end{aligned} \tag{5.17}$$

where $\boldsymbol{l}_e = \dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1}$ and $\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$ are entirely spatial, $\boldsymbol{L}_c = \dot{\boldsymbol{P}}_c\boldsymbol{P}_c^{-1}$ is entirely in the reference configuration $\mathcal{B}$ with residual stresses, and $\boldsymbol{\Lambda}_t = \dot{\boldsymbol{F}}_t\boldsymbol{F}_t^{-1}$ is entirely in the stress-free configuration (archetypal configuration) $\mathcal{B}^*$. By defining the push-forward $\boldsymbol{l}_c = \boldsymbol{F}\boldsymbol{L}_c\boldsymbol{F}^{-1}$ and the elastic push-forward $\boldsymbol{l}_t = \boldsymbol{F}_e\boldsymbol{\Lambda}_t\boldsymbol{F}_e^{-1}$, the spatial elastic rate can be written as

$$\boldsymbol{l}_e = \boldsymbol{l} + \boldsymbol{l}_c - \boldsymbol{l}_t. \tag{5.18}$$

We remark that $\boldsymbol{l}_t$, like $\boldsymbol{F}_t$, is defined *piecewise*, and we have that $\boldsymbol{l}_t$ is *zero* in the matrix and non-zero only in the inclusion. Using Eq. (5.17), we obtain the $\boldsymbol{F}_e$-*pull-back* of Eq. (5.18) to the stress-free configuration $\mathcal{B}^*$ as

$$\begin{aligned} \boldsymbol{\Lambda}_e &= \boldsymbol{F}_e^{-1}\boldsymbol{l}_e\boldsymbol{F}_e \\ &= \boldsymbol{F}_e^{-1}\boldsymbol{l}\,\boldsymbol{F}_e + \boldsymbol{F}_e^{-1}\boldsymbol{F}\boldsymbol{L}_c\boldsymbol{F}^{-1}\,\boldsymbol{F}_e - \boldsymbol{F}_e^{-1}\boldsymbol{F}_e\boldsymbol{\Lambda}_t\boldsymbol{F}_e^{-1}\,\boldsymbol{F}_e \\ &= \boldsymbol{F}_e^{-1}\boldsymbol{l}\,\boldsymbol{F}_e + \boldsymbol{P}^{-1}\boldsymbol{L}_c\,\boldsymbol{P} - \boldsymbol{\Lambda}_t, \end{aligned} \tag{5.19}$$

where we used $\boldsymbol{F}_e^{-1}\boldsymbol{F} = \boldsymbol{P}$. Thus, by defining $\boldsymbol{\Lambda}_e = \boldsymbol{F}_e^{-1}\boldsymbol{l}\,\boldsymbol{F}_e$ and $\boldsymbol{\Lambda}_c = \boldsymbol{P}^{-1}\ \boldsymbol{L}_c\boldsymbol{P} = \boldsymbol{F}_e^{-1}\boldsymbol{l}_c\,\boldsymbol{F}_e$, we have

$$\boldsymbol{\Lambda}_e = \boldsymbol{\Lambda} + \boldsymbol{\Lambda}_c - \boldsymbol{\Lambda}_t. \tag{5.20}$$

We are going to develop the case of ellipsoidal inclusion in the framework of large deformations in Section 5.3.4 and we are going to show that, despite the fact that the velocity gradient $\boldsymbol{l}$, all $\boldsymbol{l}$-like tensors in Eq. (5.18) and their $\boldsymbol{F}_e$-pull-backs in Eq. (5.20) generally lack symmetry, Eqs. (5.18) and (5.20) can be righteously considered as the large-deformation counterparts of Eshelby's original decomposition (5.13).

5.3.4 Ellipsoidal Inclusion

For the case of the ellipsoidal inclusion in an isotropic matrix, in the small-deformation formulation used by Eshelby (1957) (see also Mura, 1987), the cancelling strain is related to the transformation strain via the fourth-order Eshelby stress, i.e.,

$$\boldsymbol{\epsilon}_c = \mathbb{S} : \boldsymbol{\epsilon}_t. \tag{5.21}$$

In Eshelby's procedure, the transformation strain $\boldsymbol{\epsilon}_t$ and the cancelling strain $\boldsymbol{\epsilon}_c$ are both *integrable* for an ellipsoidal inclusion that is going to be implanted into an ellipsoidal cavity (Balluffi, 2012; Alhasadi and Federico, 2017). Moreover the transformation displacement field $\boldsymbol{u}_t$ and the cancelling displacement field $\boldsymbol{u}_c$ can be both considered to be *irrotational* in the inclusion, i.e., the transformation displacement gradient $\boldsymbol{h}_t = \operatorname{grad} \boldsymbol{u}_t$ and the cancelling displacement gradient $\boldsymbol{h}_c = \operatorname{grad} \boldsymbol{u}_c$ have *null* skew-symmetric parts $\boldsymbol{\varphi}_t$ and $\boldsymbol{\varphi}_c$ and thus coincide with their symmetric parts $\boldsymbol{\epsilon}_t$ and $\boldsymbol{\epsilon}_c$.

In our large-deformation counterpart of Eshelby's procedure, we also imagine the transformation deformation $\boldsymbol{F}_t$ to cause no rotation of the ellipsoidal inclusion, and similarly for the cancelling implant $\boldsymbol{P}_c$. We first note that, in general, the infinitesimal strain $\boldsymbol{\epsilon}$, symmetric part of the displacement gradient $\boldsymbol{h} = \operatorname{grad} \boldsymbol{u}$, is the linearisation of the rate of deformation $\boldsymbol{d}$, symmetric part of the velocity gradient $\boldsymbol{l}$. Indeed, $\boldsymbol{\epsilon}$ can be obtained by defining the displacement field as $\boldsymbol{u} \equiv \boldsymbol{v}\, \delta t$, where $\boldsymbol{v}$ is the velocity and δt is an increment of time. We subsequently extract the symmetric part of Eq. (5.18), which is the large-deformation counterpart of (5.13), then look at the $\boldsymbol{F}_e$ pull-back of the resulting expression, and finally introduce the fourth-order Eshelby tensor, to find the counterpart of Eq. (5.21).

First, let us left-multiply Eq. (5.18) by the spatial metric tensor $\boldsymbol{g}$ in oder to make the velocity gradient $\boldsymbol{l}$ and all $\boldsymbol{l}$-like tensors fully *covariant*, i.e.,

$$\boldsymbol{g}\, \boldsymbol{l}_e = \boldsymbol{g}\, \boldsymbol{l} + \boldsymbol{g}\, \boldsymbol{l}_c - \boldsymbol{g}\, \boldsymbol{l}_t. \tag{5.22}$$

Then, as for any fully covariant (or fully contravariant) second-order tensor, we decompose the covariant velocity gradient $\boldsymbol{l}^\flat \equiv \boldsymbol{g}\, \boldsymbol{l}$ into the sum of its symmetric and skew-symmetric parts, i.e.,

$$\boldsymbol{g}\, \boldsymbol{l} = \boldsymbol{d} + \boldsymbol{w}, \qquad \boldsymbol{d}^T = \boldsymbol{d}, \qquad \boldsymbol{w}^T = -\boldsymbol{w}, \tag{5.23}$$

where, for the case of the velocity gradient,

$$\boldsymbol{d} = \tfrac{1}{2}(\boldsymbol{g}\, \boldsymbol{l} + \boldsymbol{l}^T \boldsymbol{g}), \qquad \boldsymbol{w} = \tfrac{1}{2}(\boldsymbol{g}\, \boldsymbol{l} - \boldsymbol{l}^T \boldsymbol{g}), \tag{5.24}$$

take the meaning of rate of deformation and spin tensor, respectively. Using the same decomposition (5.24) in all terms in Eq. (5.22), we obtain

$$\boldsymbol{d}_e + \boldsymbol{w}_e = \boldsymbol{d} + \boldsymbol{w} + \boldsymbol{d}_c + \boldsymbol{w}_c - \boldsymbol{d}_t - \boldsymbol{w}_t. \tag{5.25}$$

Extracting the symmetric part of (5.25), we obtain

$$\boldsymbol{d}_e = \boldsymbol{d} + \boldsymbol{d}_c - \boldsymbol{d}_t, \tag{5.26}$$

which is the large-deformation counterpart of Eshelby's relation (5.13). In passing, we note that, considering that the transformation rate $\boldsymbol{l}_t$ and the cancelling rate $\boldsymbol{l}_c$ are *irrotational* by hypothesis (i.e., $\boldsymbol{w}_t = \boldsymbol{0}$ and $\boldsymbol{w}_c = \boldsymbol{0}$), we have, for the skew-symmetric part,

$$\boldsymbol{w}_e = \boldsymbol{w}. \tag{5.27}$$

Now, we evaluate the $\boldsymbol{F}_e$-pull-back to the archetype $\mathcal{B}^*$ of the sym-skew decomposition (5.23) of the covariant velocity gradient $\boldsymbol{l}^\flat \equiv \boldsymbol{g}\,\boldsymbol{l}$ as

$$\boldsymbol{C}_e \boldsymbol{\Lambda} = \boldsymbol{\Delta} + \boldsymbol{\Omega}, \qquad \boldsymbol{\Delta}^T = \boldsymbol{\Delta}, \qquad \boldsymbol{\Omega}^T = -\boldsymbol{\Omega}, \tag{5.28}$$

where $\boldsymbol{C}_e = \boldsymbol{F}_e^T \boldsymbol{g}\, \boldsymbol{F}_e$ is the $\boldsymbol{F}_e$-pull-back metric in the archetype $\mathcal{B}^*$ and the symmetric and skew-symmetric parts $\boldsymbol{\Delta}$ and $\boldsymbol{\Omega}$ are intended to be taken with respect to the metric $\boldsymbol{C}_e$, i.e.,

$$\boldsymbol{\Delta} = \tfrac{1}{2}(\boldsymbol{C}_e \boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T \boldsymbol{C}_e), \qquad \boldsymbol{\Omega} = \tfrac{1}{2}(\boldsymbol{C}_e \boldsymbol{\Lambda} - \boldsymbol{\Lambda}^T \boldsymbol{C}_e). \tag{5.29}$$

Applying the decomposition (5.29) to Eq. (5.20), and with consideration similar to those that led to Eq. (5.26), we have

$$\boldsymbol{\Delta}_e = \boldsymbol{\Delta} + \boldsymbol{\Delta}_c - \boldsymbol{\Delta}_t. \tag{5.30}$$

We are now ready to introduce the *archetypal* fourth-order *Eshelby tensor* $\mathbb{H}$, which relates the symmetric parts $\mathbf{\Delta}_c$ and $\mathbf{\Delta}_t$ of the cancelling and transformation rates *in the inclusion* via the relation

$$\mathbf{\Delta}_c = \mathbb{H} : \mathbf{\Delta}_t, \qquad (\Delta_c)_{\alpha\beta} = \mathrm{H}_{\alpha\beta}{}^{\pi\varrho}\,(\Delta_t)_{\pi\varrho}, \qquad \text{in } \mathcal{D}^*, \tag{5.31}$$

which the large-deformation counterpart of Eshelby's classical relation $\boldsymbol{\epsilon}_c = \mathbb{S} : \boldsymbol{\epsilon}_t$. The large-deformation Eshelby tensor $\mathbb{H}$ has the same symmetries of the classical $\mathbb{S}$, i.e., both minor symmetries and *no* major symmetry.

By noting that the implant rate tensor $\mathbf{\Delta}_{\boldsymbol{P}}$ is given by the last two terms of the right-hand side of Eq. (5.20), i.e.,

$$\mathbf{\Delta}_{\boldsymbol{P}} = \mathbf{\Delta}_c - \mathbf{\Delta}_t, \tag{5.32}$$

we can substitute (5.32) into (5.31) and obtain

$$\mathbf{\Delta}_{\boldsymbol{P}} = [\mathbb{H} - \mathbb{I}^T] : \mathbf{\Delta}_t, \qquad \text{in } \mathcal{D}^*, \tag{5.33}$$

where $\mathbb{I}$ is the (symmetric) fourth-order identity tensor with components

$$\mathrm{I}^{\pi\varrho}{}_{\alpha\beta} = \tfrac{1}{2}\big(\delta^{\pi}{}_{\alpha}\,\delta^{\varrho}{}_{\beta} + \delta^{\pi}{}_{\beta}\,\delta^{\varrho}{}_{\alpha}\big). \tag{5.34}$$

Because of the analogy between the large-deformation Eshelby tensor $\mathbb{H}$ and Eshelby's classical tensor $\mathbb{S}$, the components of $\mathbb{H}$ can be taken to be equal to those of $\mathbb{S}$, which are tabulated in various works (e.g., Qiu and Weng, 1990; Federico et al., 2004), with a *caveat*. Eshelby's tensor $\mathbb{S}$ is thought for an isotropic, linear elastic matrix with Poisson's ratio ν_0. We can think of this Poisson's ratio to be taken in the stress-free incompatible configuration $\mathcal{B}^*$. Thus, once an isotropic hyperelastic constitutive equation is chosen for the matrix, the corresponding Poisson's ratio can be obtained and used to evaluate the components of tensor $\mathbb{H}$, modelling them on those of $\mathbb{S}$.

5.3.5 Solution Outside of the Inclusion

The large-deformation Eshelby's relation (5.31) solves the problem *inside* the inclusion and determines the boundary conditions of position and stress on the inclusion-matrix interface. The solution outside of the inclusion could be achieved via an appropriate evolution law for the implant $\boldsymbol{P}$. We recall that, outside the inclusion, the transformation implant $\boldsymbol{P}_t = \boldsymbol{F}_t^{-1}$ equals the identity $\tilde{\boldsymbol{I}}$: thus, for all practical purposes, we only need to find *one* $\boldsymbol{P}$, which coincides with the cancelling implant $\boldsymbol{P}_c$ outside the inclusion.

Epstein and Maugin (2000) (see also Epstein and Elżanowski, 2007) postulated an evolution law for the implant tensor $\boldsymbol{P}$ describing an anelastic phenomenon as

$$\dot{\boldsymbol{P}} = \boldsymbol{P}\,\hat{f}(J_{\boldsymbol{P}}\,\boldsymbol{P}^T\,\boldsymbol{M}\,\boldsymbol{P}^{-T}) \;\Rightarrow\; \boldsymbol{\Lambda}_{\boldsymbol{P}} = \hat{f}(J_{\boldsymbol{P}}\,\boldsymbol{P}^T\,\boldsymbol{M}\,\boldsymbol{P}^{-T}), \tag{5.35}$$

where $J_{\boldsymbol{P}} = \det \boldsymbol{P}$, $\boldsymbol{M} = \boldsymbol{F}^T\boldsymbol{T}$ is the *Mandel stress* (with $\boldsymbol{T}$ being the *first Piola-Kirchhoff stress*) and $\hat{f}$ is a function valued in the space of archetypal "mixed" tensors in $\mathcal{B}^*$, such as $\boldsymbol{\Lambda}_{\boldsymbol{P}} = \boldsymbol{P}^{-1}\dot{\boldsymbol{P}}$ (which has components $(\boldsymbol{\Lambda}_{\boldsymbol{P}})^{\alpha}{}_{\beta}$) and not depending explicitly on the point X. The simplest possible form of $\hat{f}$ is

$$\hat{f}(J_{\boldsymbol{P}}\,\boldsymbol{P}^T\,\boldsymbol{M}\,\boldsymbol{P}^{-T}) = -k\,J_{\boldsymbol{P}}\,\boldsymbol{C}_e^{-1}\,\boldsymbol{P}^T\,\boldsymbol{M}\,\boldsymbol{P}^{-T}\,\boldsymbol{C}_e, \tag{5.36}$$

where k is a positive constant and the $\boldsymbol{F}_e$-pull-back metric $\boldsymbol{C}_e$ and its inverse $\boldsymbol{C}_e^{-1}$ serve to lower the second index and raise the first index of $\boldsymbol{P}^T\boldsymbol{M}\,\boldsymbol{P}^{-T}$, respectively.

By substituting Eq. (5.36) into Eq. (5.35), we obtain

$$\boldsymbol{\Lambda}_{\boldsymbol{P}} = -k\,J_{\boldsymbol{P}}\,\boldsymbol{C}_e^{-1}\boldsymbol{P}^T\,\boldsymbol{M}\,\boldsymbol{P}^{-T}\boldsymbol{C}_e, \tag{5.37}$$

and, by left-multiplying Eq. (5.37) by $\boldsymbol{C}_e$ and extracting the symmetric part of $\boldsymbol{C}_e\,\boldsymbol{\Lambda}_{\boldsymbol{P}}$, we get

$$\begin{aligned}\boldsymbol{\Delta}_{\boldsymbol{P}} &= \tfrac{1}{2}\big(\boldsymbol{C}_e\,\boldsymbol{\Lambda}_{\boldsymbol{P}} + \boldsymbol{\Lambda}_{\boldsymbol{P}}^T\boldsymbol{C}_e\big)\\ &= -k\,J_{\boldsymbol{P}}\,\tfrac{1}{2}\big(\boldsymbol{P}^T\,\boldsymbol{M}\,\boldsymbol{P}^{-T}\boldsymbol{C}_e + \boldsymbol{C}_e\boldsymbol{P}^{-1}\,\boldsymbol{M}^T\boldsymbol{P}\big).\end{aligned} \tag{5.38}$$

where we recall again that, outside the inclusion, the implant rate tensor $\boldsymbol{\Delta}_{\boldsymbol{P}}$ coincides with the cancelling implant rate $\boldsymbol{\Delta}_c$ (exactly like $\boldsymbol{P}$ reduces to $\boldsymbol{P}_c$).

Chapter 6

Discussion

This thesis contains four major parts, developed in Chapters 2 to 5 and discussed here.

Chapter 2 is concerned with obtaining a relation between Eshelby stress and fourth-order Eshelby tensor in the three fundamental types of inclusion, within the theory of small deformations. In Chapter 3, the classical results of the Eshelbian inclusion problem are obtained by implementing Noether's theorem, based on the framework of modern differential geometry. In Chapter 4 is dedicated to showing how the use of the general framework of material uniformity, a differential identity for the *modified Eshelby stress* and an analogous one for a *modified Eshelby power* can be obtained for uniform bodies. Furthermore, a relation is obtained between the two differential identities (*Eshelby force* and *Eshelby power*) in a dynamical setting. In Chapter 5, the problem of the Eshelby inclusion in large deformations is addressed by introducing a multiplicative decomposition of deformation gradient analogous to the additive decomposition of the infinitesimal strain used by Eshelby in his imagined construction of the inclusion problem.

6.1 Eshelby Stress and Tensor within an Ellipsoidal Inclusion

The multifaceted genius of Eshelby has given rise to fundamental developments in modern Continuum Mechanics. Among the problems that Eshelby tackled, we focussed on the effect of inhomogeneities, which is captured by the stress that Eshelby (1951) called energy-momentum tensor or Maxwell stress and that Maugin and Trimarco (1992) called Eshelby stress, and the problem of the ellipsoidal inclusion, in which the "cancelling strain" arising from the geometrical misfit of the inclusion with the matrix is captured by the fourth-order Eshelby tensor (Eshelby, 1957).

The objective of this work (Alhasadi and Federico, 2017, see Chapter 2) was to derive the mathematical relation between the Eshelby stress $\mathfrak{p}$ and the fourth-order Eshelby tensor $\mathbb{S}$ within an ellipsoidal inclusion, under the hypothesis of small displacements and linear elasticity. We based our derivation on the fact that both the Eshelby stress $\mathfrak{p}$ and the fourth-order Eshelby tensor $\mathbb{S}$ are related to the net material force acting on a region $\mathcal{D}$ containing a defect. In the case at study, the "defect" occupies the whole region $\mathcal{D}$ and coincides with the *inclusion*. We studied the case of inclusion with geometrical misfit and same elastic properties of the matrix, known in the literature as "homogeneous inclusion", the case of inclusion with no geometrical misfit but different properties than the matrix, called "inhomogeneous inclusion", and finally the case with both misfit in geometry and material properties, which we call "general inclusion", and found the final expressions

$$\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \left[\left(\boldsymbol{\sigma}^A : \mathbb{S}^{-1} : \boldsymbol{\epsilon}^C\right) \boldsymbol{i} - \left(\mathbb{S}^{-1} : \boldsymbol{\epsilon}^C\right)^T \boldsymbol{\sigma}^A\right], \qquad \text{in } \mathcal{D}, \qquad (2.57\text{R})$$

$$\mathfrak{p} = \mathfrak{p}^A + \left[\tfrac{1}{2}\left(\boldsymbol{\sigma}^A : \mathbb{S}^{-1} : (\mathbb{A} - \mathbb{I}) : \boldsymbol{\epsilon}^A\right) \boldsymbol{i} - \tfrac{1}{2}\left(\mathbb{S}^{-1} : (\mathbb{A} - \mathbb{I}) : \boldsymbol{\epsilon}^A\right)^T \boldsymbol{\sigma}^A\right], \qquad \text{in } \mathcal{D}, \qquad (2.85\text{R})$$

$$\begin{aligned}\mathfrak{p} = \mathfrak{p}^A + \mathfrak{p}^B + \tfrac{1}{2}\Big[\boldsymbol{\sigma}^A : \big[\mathbb{S}^{-1} : \big[(\mathbb{A} - \mathbb{I}) : (\boldsymbol{\epsilon}^A - \boldsymbol{\epsilon}^*) + \mathbb{A} : \mathbb{S} : \boldsymbol{\epsilon}^*\big] + \boldsymbol{\epsilon}^*\big]\Big]\, \boldsymbol{i} \\ - \tfrac{1}{2}\Big[\big[\mathbb{S}^{-1} : \big[(\mathbb{A} - \mathbb{I}) : (\boldsymbol{\epsilon}^A - \boldsymbol{\epsilon}^*) + \mathbb{A} : \mathbb{S} : \boldsymbol{\epsilon}^*\big] + \boldsymbol{\epsilon}^*\big]^T \boldsymbol{\sigma}^A\Big], \qquad \text{in } \mathcal{D}. \qquad (2.111\text{R})\end{aligned}$$

It is clear that Eq. (2.57) for the "homogeneous inclusion" is trivial, as the cancelling strain $\boldsymbol{\epsilon}^C$ must in any case be determined from the assigned transformation strain $\boldsymbol{\epsilon}^*$ via Eq. (2.19), which brings Eq. (2.57) back to the form of Eq. (2.56), in which the Eshelby stress $\mathfrak{p}$ is exclusively function of the applied stress $\boldsymbol{\sigma}^A$ and the transformation strain $\boldsymbol{\epsilon}^*$. However, it is necessary to work out the case of the "homogeneous inclusion" to derive the other two.

In contrast, we find Eq. (2.85) for the "inhomogeneous inclusion" to be very appealing, as the absence of a "real" transformation strain $\boldsymbol{\epsilon}^*$ makes it possible to express the Eshelby stress $\mathfrak{p}$ as a function of exclusively the applied stress $\boldsymbol{\sigma}^A$ and of quantities that depend on the shape of the inclusion and the material properties of inclusion and matrix. Indeed: *i*) the fourth-order Eshelby tensor $\mathbb{S}$ depends on the shape of the ellipsoidal inclusion, i.e., on the ratios among the semi-axes, and, for an isotropic matrix, on the Poisson's coefficient of the matrix (Eshelby, 1957; Qiu and Weng, 1990); *ii*) the fourth-order strain concentration tensor $\mathbb{A}$ depends on $\mathbb{S}$ and the elasticity tensors $\mathbb{L}$ and $\mathbb{L}'$ (Eq. (2.65)); *iii*) the applied strain $\boldsymbol{\epsilon}^A$ is given by $\boldsymbol{\epsilon}^A = \mathbb{L}^{-1} : \boldsymbol{\sigma}^A$.

This result could be very helpful to directly find the net material force on the "inhomogeneous" inclusion, by employing tabulated values of the components of $\mathbb{S}$ and the elastic constants of matrix and inclusion. For instance, Qiu and Weng (1990) report the components of $\mathbb{S}$ for the case of a spheroidal inclusion, i.e., an inclusion that is a revolution ellipsoid, with two of the semi-axes being equal. The case of Eq. (2.111) for the "general inclusion" has the same advantages of Eq. (2.85) for the "inhomogeneous inclusion", although it obviously needs the "real" transformation strain $\boldsymbol{\epsilon}^*$ as an additional input. We remark that the "inhomogeneous" inclusion is perhaps the most relevant in the treatment of composite materials, as it has been used as the basis of the description of composites with ellipsoidal and fibre-like inclusions (Walpole, 1966a,b, 1969; Tandon and Weng, 1984; Weng, 1984, 1990; Ru et al., 2001; Federico et al., 2004; Kim et al., 2008), in which the geometrical misfit is null or negligible with respect to the misfit in material properties.

From a more general perspective, we have attempted to relate these two aspects of Eshelby's pioneering research, which we find important from the epistemological point of view. We have also tried to present an organic and hopefully clear formulation for the known problem of the ellipsoidal inclusion, which could serve as a didactic tool in micromechanics.

6.2 Eshelby's Inclusion and Noether's Theorem

In this work (Federico et al., 2019, see Chapter 3) we systematically reviewed the two procedures proposed by Eshelby to study the effect of inhomogeneity in an elastic body, in the differential geometric picture of Continuum Mechanics. The first procedure (Eshelby, 1951) involves the classical cutting-replacing-welding operations and is mathematically represented by defining the energy as a functional on the manifold $\mathcal{M}$ of the *Eshelbian configurations* $\mathcal{Y}$ (which transform the domain $\mathcal{D}$ containing the inclusion/defect), and performing a variation *on the coordinates*, i.e., a variational derivative made with respect to a *material* displacement field $\boldsymbol{U}$, seen as a variation of the identity Eshelbian configuration $\mathcal{X}$. The second procedure (Eshelby, 1975) follows Hamilton's principle of stationary action. Accordingly, the energy is defined as a functional on the manifold $\mathcal{C}$ of the *conventional configurations* ϕ, and a variation is performed *on the fields*, i.e., a variational derivative is calculated with respect to a *spatial* displacement, seen as a variation of the configuration map ϕ.

The natural manner to unify the two procedures is the use of *Noether's Theorem*, in which a variation on both fields and coordinates (total variation) is used. Indeed, to obtain this result, we defined the energy as a functional on the *product manifold* $\mathcal{C} \times \mathcal{M}$ of the conventional configurations ϕ and the Eshelbian configurations $\mathcal{Y}$, and performed a variational derivative with respect to the pair $(\boldsymbol{\eta}, \boldsymbol{U})$, which is a variation with respect to the pair $(\phi, \mathcal{X})$. While certainly no additional proof was needed to demonstrate the beauty and generality of Noether's Theorem, we find that it is insightful to look at Eshelby's theory of defects from the point of view of Noether's conservation laws.

6.3 Eshelby Force and Power for Uniform Bodies

The theory of configurational forces can be presented within the general framework of material uniformity (Alhasadi et al., 2019, see Chapter 4). Eshelby (1975) used a variational approach, which we could call *with respect to the material point X*, to derive the configurational force $\mathfrak{F}$ and the divergence $\mathcal{N} = \mathrm{Div}\, \mathfrak{E}$ of the Eshelby stress, which here we called *Eshelby force*. Following the same approach *with respect to time t*, we obtained the energy release rate $\mathfrak{G}$ and the *Eshelby power* $\mathcal{Y} = -\dot{\mathcal{L}} + \mathrm{Div}\, \mathfrak{R}$, which are the counterparts of the configurational force $\mathfrak{F}$ and the Eshelby force $\mathcal{N} = \mathrm{Div}\, \mathfrak{E}$, respectively. We note that Eshelby had obtained a similar, but less general result (Eshelby, 1970). Indeed, since he did not consider a time-dependent Lagrangiaan, he only obtained the contribution Div $\mathfrak{R}$ to the total Eshelby power of Eq. (4.71).

We showed that not only the $\boldsymbol{P}$-divergence of the modified Eshelby stress (Epstein and Maugin, 1990), which we called here modified Eshelby force $\widetilde{\mathfrak{N}}_A = \widetilde{\mathfrak{E}}_A{}^B{}_{;B} = [-\hat{\Lambda} \circ (\#)\, \delta_A{}^B - \widetilde{\mathfrak{M}}_A{}^B]_{;B}$, but also the modified Eshelby power $\widetilde{\mathfrak{Y}} = -J_{\boldsymbol{P}}\, \dot{\mathcal{L}} + \widetilde{\mathfrak{R}}^A{}_{;A}$ obeys a particular differential identity, based on the torsion of the connection arising from the uniformity isomorphism $\boldsymbol{P}$. Through the study of the dissipation inequality, we deduced that the Mandel stress $\mathfrak{M}$, which is the dual of the inhomogeneity rate tensor $\boldsymbol{L}_{\boldsymbol{P}} = \dot{\boldsymbol{P}}\boldsymbol{P}^{-1}$ (Epstein and Elżanowski, 2007), is the driving force of the processes of growth-remodelling. Finally, we obtained a relation between the two differential identities (for the modified Eshelby force and for the modified Eshelby power) and, using the definition of the inverse velocity (Epstein and Maugin, 2000), we concluded that the Eshelby power is a measure of the mechanical power expended in the evolution of a uniform body.

6.4 Eshelby's Inclusion Problem in Large Deformations

In his classical paper, Eshelby (1957) used a thought experiment to describe the decomposition of the infinitesimal strain for the case of an inclusion in an otherwise homogeneous matrix. We had studied the problem in the small-deformation theory (Alhasadi and Federico, 2017, see Chapter 2) and, in this work (Alhasadi and Federico, 2019, see Chapter 5), we obtained the large-deformation counterpart of Eshelby's additive decomposition of the infinitesimal strain in the form of a multiplicative decomposition of the deformation gradient. We showed that, in order to match Eshelby's procedure for the *implant* of the inclusion into the matrix, the multiplicative decomposition should be a *hybrid* between the classical Bilby-Kröner-Lee decomposition (5.3) and the Noll-Esptein-Maugin decomposition (5.4), which is based on the use of the implant tensor $\boldsymbol{P}$ of the theory of material uniformity. The result is the *mixed* decomposition (5.12), in which the deformation gradient $\boldsymbol{F}$, the transformation deformation gradient $\boldsymbol{F}_t$, the cancelling implant $\boldsymbol{P}_c$ and the elastic deformation $\boldsymbol{F}_e$ correspond exactly to Eshelby's strain $\boldsymbol{\epsilon}$, transformation strain $\boldsymbol{\epsilon}_t$, cancelling strain $\boldsymbol{\epsilon}_c$ and elastic ("total", in Eshelby's terminology) strain $\boldsymbol{\epsilon}_e$. So to speak, the transformation strain

$\boldsymbol{\epsilon}_t$ is a "forward" deformation gradient (linear counterpart of $\boldsymbol{F}_t$) and the cancelling strain $\boldsymbol{\epsilon}_c$ is a "backward" implant (linear counterpart of the implant $\boldsymbol{P}_c$. Furthermore, we suggest a solution outside the inclusion based on the evolution law approach proposed by Epstein and Maugin (2000).

We emphasise that Eq. (5.12) is general and valid for an inclusion of any shape, and that Eshelby's equation (5.13) is the linearisation of the proposed decomposition (5.12) only for the case of the ellipsoidal inclusion, when the transformation displacement field and the cancelling displacement field in the inclusion are *irrotational*. Had Eshelby (1957) studied the general case of inclusion of arbitrary shape, his equation (5.13) would have come from the symmetrisation of the analogous expression

$$\boldsymbol{h}_e = \boldsymbol{h} + \boldsymbol{h}_c - \boldsymbol{h}_t, \tag{6.2}$$

in terms of displacement gradients, which represents the linearisation of our equation (5.18) in terms of velocity gradients.

6.5 Outlook and Future Work

The research in this thesis has focussed on two of Eshelby's classical results: the *configurational force* acting on a defect, which gives rise to the Eshelby stress (Eshelby, 1951), and the study of the inclusion problem, in general and for the case of the ellipsoidal inclusion, which gives rise to Eshelby's fourth-order tensor (Eshelby, 1957). Initially, the goal was to find a relation between the two tensors in the small-deformation regime and then extend to the large-deformation regime.

While it has been possible to achieve this relation in small deformations and the inclusion (Alhasadi and Federico, 2017), an analogous relation in large deformations remains an open problem. However, we successfully generalised the inclusion problem to large deformation, via the mixed Bilby-Kröner-Lee / Noll-Epstein-Maugin decomposition that we proposed (Alhasadi and Federico, 2019). The path to this result was made possible via our study of the inclusion problem in the light of Noether's theorem (Federico et al., 2019) and the theory of continuously distributed defects (Alhasadi et al., 2019).

Our future plans include:

- Further attempts to extend the relation between the two Eshelby tensors to the case of large deformations;
- Extension of the large-deformation inclusion problem to the case of the ellipsoidal "inhomogeneous inclusion", i.e., inclusion with material properties *different* from those of the matrix but no geometrical misfit, and to the case of the ellipsoidal "general inclusion", with both material property mismatch and geometrical misfit;
- Application of the theory that we developed so far, with implementation into numerical schemes such as Finite Elements. As mentioned in Section 6.4, we plan to apply this method to biomechanical problems such as tumour growth.

Appendix A

Appendix

A.1 Vanishing of $\operatorname{div} \mathfrak{p}^{\mathbf{A}}$ and $\operatorname{div} \mathfrak{p}^{\mathbf{B}}$

Here, we give a direct proof of the fact that, for a linear elastic material, the divergence of the Eshelby stress $\boldsymbol{\mathfrak{p}}^X$ arising from an elastic field described by $\boldsymbol{u}^X$, $\boldsymbol{\epsilon}^X$, $\boldsymbol{\sigma}^X$ is identically zero over a region $\mathcal{D}$ in which the elasticity tensor $\mathbb{L}$ is homogeneous. If $\mathcal{D}$ is an inclusion, the elasticity tensor is the same as that of the matrix, $\mathbb{L}$, for the case of "homogeneous inclusion", and a different one $\mathbb{L}'$ for the cases of "inhomogeneous inclusion" and "general inclusion". The proof is easiest to show in components and the elasticity tensor is simply called $\mathbb{L}$ for the sake of a lighter notation.

Given the Eshelby stress

$$\mathfrak{p}_{ij}^X = \frac{1}{2}\left(\sigma_{rs}^X\, h_{rs}^X\right)\delta_{ij} - h_{ri}^X\, \sigma_{rj}^X, \tag{A.1}$$

its divergence is

$$\mathfrak{p}_{ij,j}^X = \frac{1}{2}\left(\sigma_{rs,j}^X\, h_{rs}^X + \sigma_{rs}^X\, h_{rs,j}^X\right)\delta_{ij} - h_{ri,j}^X\, \sigma_{rj}^X - h_{ri}^X\, \sigma_{rj,j}^X. \tag{A.2}$$

Recalling that, in the absence of external body forces, $\sigma_{rj,j}^X = 0$, and with some straightforward manipulations (including use of the symmetry of second partial derivatives, i.e., $u_{r,ij} = u_{r,ji}$, from which $h_{ri,j} = h_{rj,i}$), we have

$$\mathfrak{p}_{ij,j}^X = \frac{1}{2}\left[\sigma_{rs,i}^X\, h_{rs}^X - \sigma_{rs}^X\, h_{rs,i}^X\right]. \tag{A.3}$$

By using the linear elastic constitutive equation (2.7) and assuming *homogeneity* of the elasticity tensor $\mathbb{L}$ within $\mathcal{D}$, we get

$$\mathfrak{p}^X_{ij,j} = \tfrac{1}{2}\left[\mathrm{L}_{rskl}\ h^X_{kl,i} h^X_{rs} - \mathrm{L}_{rskl}\ h^X_{kl}\ h^X_{rs,i}\right]. \tag{A.4}$$

Finally, by virtue of the major symmetry of the elasticity tensor, i.e., $\mathrm{L}_{rskl} = \mathrm{L}_{klrs}$, we finally obtain

$$\mathfrak{p}^X_{ij,j} = \tfrac{1}{2}\left[\mathrm{L}_{klrs}\ h^X_{rs,i} h^X_{kl} - \mathrm{L}_{rskl}\ h^X_{kl}\ h^X_{rs,i}\right] = 0. \tag{A.5}$$

A.2 Alternative Derivation of $\mathcal{W}^{\text{int}}$

An alternative derivation of the interaction energy in Eq. (2.41) was proposed by Balluffi (2012). A rearrangement of Eq. (2.29) yields

$$\mathcal{W}^{\text{int}} = \left(\mathcal{W}_{\text{el}} - \mathcal{W}^A_{\text{el}} - \mathcal{W}^B_{\text{el}}\right) + \left(\mathcal{W}_{\text{ext}} - \mathcal{W}^A_{\text{ext}}\right). \tag{A.6}$$

The first term in Eq. (A.6) is equal to

$$\begin{aligned}\mathcal{W}_{\text{el}} - \mathcal{W}^A_{\text{el}} - \mathcal{W}^B_{\text{el}} &= \frac{1}{2}\int_{\mathcal{B}} \left(\boldsymbol{\sigma}^A + \boldsymbol{\sigma}^B\right) : \left(\boldsymbol{h}^A + \boldsymbol{h}^B\right) - \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{h}^A - \frac{1}{2}\int_{\mathcal{B}} \boldsymbol{\sigma}^B : \boldsymbol{h}^B \\ &= \frac{1}{2}\int_{\mathcal{B}} \left(\boldsymbol{\sigma}^A : \boldsymbol{h}^B + \boldsymbol{\sigma}^B : \boldsymbol{h}^A\right),\end{aligned} \tag{A.7}$$

which, by virtue of the rule of corresponding fields (Eq. (2.9)), reduces to

$$\mathcal{W}_{\text{el}} - \mathcal{W}^A_{\text{el}} - \mathcal{W}^B_{\text{el}} = \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{h}^B. \tag{A.8}$$

The second term in Eq. (A.6) can be simplified as

$$\mathcal{W}_{\text{ext}} - \mathcal{W}^A_{\text{ext}} = \int_{\partial\mathcal{B}} -\boldsymbol{t}^A . \left(\boldsymbol{u}^C + \boldsymbol{u}^A\right) - \int_{\partial\mathcal{B}} -\boldsymbol{t}^A . \boldsymbol{u}^A = -\int_{\partial\mathcal{B}} \boldsymbol{t}^A . \boldsymbol{u}^C, \tag{A.9}$$

where, again, $\boldsymbol{t}^A = \boldsymbol{\sigma}^A \boldsymbol{n}$ on $\partial\mathcal{B}$. Therefore,

$$\mathcal{W}^{\text{int}} = \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^B - \int_{\partial\mathcal{B}} \boldsymbol{t}^A . \boldsymbol{u}^C. \tag{A.10}$$

By applying Gauss' theorem to the second term of Eq. (A.10) and recalling that $\boldsymbol{\epsilon}^* \equiv \boldsymbol{h}^* = \text{grad}\, \boldsymbol{u}^*$ (Eq. (2.20)), we obtain

$$\mathcal{W}^{\text{int}} = \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^B - \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{h}^C. \tag{A.11}$$

Now, we exploit additivity of the integral operation over disjoint subsets (i.e., subsets with zero-measure intersection) and rewrite the second term of Eq. (A.11) as the sum of integrals over the regions $\mathcal{D}$ and $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$:

$$\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{h}^C = \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \left(\boldsymbol{h}^B + \boldsymbol{\epsilon}^*\right) + \int_{\mathcal{M}} \boldsymbol{\sigma}^A : (\boldsymbol{h}^B + \boldsymbol{\epsilon}^*). \tag{A.12}$$

Since the transformation strain $\boldsymbol{\epsilon}^*$ vanishes in $\mathcal{M}$, we have

$$\int_{\mathcal{B}} \boldsymbol{\sigma}^A : \left(\boldsymbol{h}^B + \boldsymbol{\epsilon}^*\right) = \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \left(\boldsymbol{h}^B + \boldsymbol{\epsilon}^*\right) + \int_{\mathcal{M}} \boldsymbol{\sigma}^A : \boldsymbol{h}^B. \tag{A.13}$$

Finally, the total interaction energy (A.11) of the fields A (applied) and B (caused by the inclusion) becomes

$$\mathcal{W}^{\text{int}} = \int_{\mathcal{B}} \boldsymbol{\sigma}^A : \boldsymbol{h}^B - \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \left(\boldsymbol{h}^B + \boldsymbol{\epsilon}^*\right) - \int_{\mathcal{M}} \boldsymbol{\sigma}^A : \boldsymbol{h}^B = - \int_{\mathcal{D}} \boldsymbol{\sigma}^A : \boldsymbol{\epsilon}^* \tag{A.14}$$

which coincides with Eq. (2.41). This same alternative approach could be employed to derive the interaction energy for the cases of the "inhomogeneous inclusion" and of the "general inclusion".

A.3 Equivalent Derivation of Tensor $\mathbb{A}$

Following Weng (see Eq. (10) in (Weng, 1984) and Eq. (2.7) in (Weng, 1990)), we define the stress $\boldsymbol{\sigma}$ within the ellipsoidal "inhomogeneous" inclusion as

$$\boldsymbol{\sigma} = \mathbb{B} : \boldsymbol{\sigma}^A, \quad \text{in } \mathcal{D}, \tag{A.15}$$

where $\boldsymbol{\sigma}^A$ is the applied stress, and $\mathbb{B}$ is the *stress concentration tensor* defined by (Weng, 1990),

$$\mathbb{B} = \left[\mathbb{I} + \mathbb{L} : (\mathbb{I} - \mathbb{S}) : [\mathbb{L}'^{-1} - \mathbb{L}^{-1}]\right]^{-1}, \quad \text{in } \mathcal{D}. \tag{A.16}$$

Since $\boldsymbol{\sigma}^A = \mathbb{L} : \boldsymbol{\epsilon}^A$ (stress that would be attained if the inclusion had the same elasticity tensor $\mathbb{L}$ of the matrix) and $\boldsymbol{\sigma} = \mathbb{L}' : \boldsymbol{\epsilon}$ (stress actually attained in the inclusion with elasticity tensor $\mathbb{L}'$), we can write Eq. (A.15) as

$$\boldsymbol{\epsilon} = \mathbb{L}'^{-1} : \mathbb{B} : \mathbb{L} : \boldsymbol{\epsilon}^A. \tag{A.17}$$

By substituting the expression (A.16) of the tensor $\mathbb{B}$ into Eq. (A.17), we obtain

$$\boldsymbol{\epsilon} = \mathbb{L}'^{-1} : \left[\mathbb{I} + \mathbb{L} : (\mathbb{I} - \mathbb{S}) : [\mathbb{L}'^{-1} - \mathbb{L}^{-1}]\right]^{-1} : \mathbb{L} : \boldsymbol{\epsilon}^A. \tag{A.18}$$

If we consider that, for every invertible fourth-order tensors $\mathbb{Y}$ and $\mathbb{Z}$, it holds that $(\mathbb{Y} : \mathbb{Z})^{-1} = \mathbb{Z}^{-1} : \mathbb{Y}^{-1}$, we have

$$\boldsymbol{\epsilon} = \left[\mathbb{L}^{-1} : \mathbb{L}' + (\mathbb{I} - \mathbb{S}) : [\mathbb{L}'^{-1} - \mathbb{L}^{-1}] : \mathbb{L}'\right]^{-1} : \boldsymbol{\epsilon}^A, \tag{A.19}$$

where, after some straightforward manipulation, the fourth-order tensor that double-contracts with $\boldsymbol{\epsilon}^A$ can be recognised to be the strain concentration tensor $\mathbb{A}$ of Eq. (2.65). Therefore Eq. (A.19) coincides with Eq. (2.69).

A.4 Monogenic and Polygenic Forces

The variational setting adopted in our work serves as a basis for the employment of Noether's Theorem (see Section 5), which, for first order theories, is generally enunciated for a Lagrangian density function depending on "fields and gradients of the fields". Hence, the expression of the energy density used so far, i.e., $W = \hat{W} \circ (\phi, \boldsymbol{F}, \mathcal{X})$, is meant to replicate, up to the sign, the standard functional dependence of a generic Lagrangian density function, for which Noether's Theorem is formulated. In principle, however, *neither* the introduction of the Eshelby stress tensor *nor* that of the configurational force density require any variational framework. Indeed, as clearly shown by Gurtin (1995), the existence of these quantities stands on its own, and it necessitates neither the hypothesis of hyperelastic material nor the assumption of body forces descending from a generalised potential density. The Eshelby stress tensor, for instance, is defined also for a generic Cauchy elastic material (for a definition of Cauchy elastic materials, see, e.g., Ogden, 1997), for which the first Piola-Kirchhoff stress tensor, $\boldsymbol{P}$, cannot be determined by differentiating the body's free energy density with respect to its deformation gradient tensor. In this respect, we recall Gurtin's words: "*My derivation of Eshelby's relation is accomplished without recourse to constitutive equations or to a variational principle*" (Gurtin, 1995). Yet, what is referred to as "Eshelby stress tensor" and "configurational force density" within a given theory may well depend on whether or not the body is hyperelastic and the body forces admit a potential.

To focus on the consequences of the existence of such a potential, we consider first a hyperelastic and inhomogeneous material with energy density $W^{\mathrm{el}} := \check{W}^{\mathrm{el}} \circ (\boldsymbol{F}, \mathcal{X})$, and subjected to body forces for which no integrability hypothesis is made. Then, following Gurtin's approach (Gurtin, 1995), the following configurational force balance applies

$$\operatorname{Div} \mathfrak{E}^{\mathrm{el}} + \mathfrak{F}^{\mathrm{el}} = \mathbf{0}, \tag{A.20}$$

where $\mathfrak{E}^{\mathrm{el}} := W^{\mathrm{el}} \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{P}$ is the Eshelby stress tensor obtained by using W^{el} as free energy density, and $\mathfrak{F}^{\mathrm{el}}$ is the configurational force density satisfying Equation (A.20). Note that, for the sake of a lighter notation, we write $\boldsymbol{F}^T$ in lieu of $\boldsymbol{F}^T \circ \phi$ throughout this section.

To identify $\mathfrak{F}^{\text{el}}$ from Equation (A.20), we compute explicitly the divergence of $\mathfrak{E}^{\text{el}}$, while recalling the equilibrium equation $\text{Div}\,\boldsymbol{P} + \boldsymbol{f} = \boldsymbol{0}$. Thus, we find

$$\mathfrak{F}^{\text{el}} = -\text{Div}\,\mathfrak{E}^{\text{el}} = -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^T \boldsymbol{f}, \tag{A.21}$$

thereby reaching the conclusion that $\mathfrak{F}^{\text{el}}$ consists of the sum of two contributions, denoted by

$$\mathfrak{F}^{\text{el,inh}} := -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}), \tag{A.22a}$$

$$\mathfrak{F}^{\text{el,b}} := -\boldsymbol{F}^T \boldsymbol{f}, \tag{A.22b}$$

and ascribable to the inhomogeneity of the material and to the presence of the body force $\boldsymbol{f}$, respectively. We emphasise that Equations (A.21), (A.22a) and (A.22b) are true *regardless* of any prescription on the integrability of $\boldsymbol{f}$. Still, without loss of generality, we may assume the splitting $\boldsymbol{f} = \boldsymbol{f}^{\text{p}} + \boldsymbol{f}^{\text{m}}$, where $\boldsymbol{f}^{\text{m}}$ is assumed to admit the generalised energy potential density $W^{\text{m}} = \check{W}^{\text{m}} \circ (\phi, \mathcal{X})$, such that

$$\boldsymbol{f}^{\text{m}} = -\frac{\partial \check{W}^{\text{m}}}{\partial \phi} \circ (\phi, \mathcal{X}). \tag{A.23}$$

In the terminology of Lanczos (1970, page 30), $\boldsymbol{f}^{\text{p}}$ is said to be "*polygenic*", whereas $\boldsymbol{f}^{\text{m}}$ is referred to as a "*monogenic*" force density, because it is "*generated by a single scalar function*", i.e., $\check{W}^{\text{m}}$.

The splitting $\boldsymbol{f} = \boldsymbol{f}^{\text{p}} + \boldsymbol{f}^{\text{m}}$ and Equation (A.23) permit to rewrite $\mathfrak{F}^{\text{el}}$ as

$$\begin{aligned} \mathfrak{F}^{\text{el}} &= -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^T \boldsymbol{f} \\ &= -\frac{\partial \check{W}^{\text{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) + \boldsymbol{F}^T \left[\frac{\partial \check{W}^{\text{m}}}{\partial \phi} \circ (\phi, \mathcal{X}) \right] - \boldsymbol{F}^T \boldsymbol{f}^{\text{p}}, \end{aligned} \tag{A.24}$$

and, since it holds true that

$$\operatorname{Grad} W^{\mathrm{m}} = \boldsymbol{F}^T \left[\frac{\partial \check{W}^{\mathrm{m}}}{\partial \phi} \circ (\phi, \mathcal{X}) \right] + \frac{\partial \check{W}^{\mathrm{m}}}{\partial \mathcal{X}} \circ (\phi, \mathcal{X}), \tag{A.25}$$

the force density $\boldsymbol{\mathfrak{F}}^{\mathrm{el}}$ takes on the expression

$$\boldsymbol{\mathfrak{F}}^{\mathrm{el}} = -\frac{\partial \check{W}^{\mathrm{el}}}{\partial \mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}) - \frac{\partial \check{W}^{\mathrm{m}}}{\partial \mathcal{X}} \circ (\phi, \mathcal{X}) + \operatorname{Grad} W^{\mathrm{m}} - \boldsymbol{F}^T \boldsymbol{f}^{\mathrm{p}}. \tag{A.26}$$

Moreover, by exploiting the identity $\operatorname{Grad} W^{\mathrm{m}} = \operatorname{Div}(W^{\mathrm{m}} \boldsymbol{I}^T)$, setting

$$W^{\mathrm{el}} = \check{W}^{\mathrm{el}} \circ (\boldsymbol{F}, \mathcal{X}) = \hat{W}^{\mathrm{el}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \text{with } \frac{\partial \hat{W}^{\mathrm{el}}}{\partial \phi} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \boldsymbol{0}, \tag{A.27a}$$

$$W^{\mathrm{m}} = \check{W}^{\mathrm{m}} \circ (\phi, \mathcal{X}) = \hat{W}^{\mathrm{m}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \qquad \text{with } \frac{\partial \hat{W}^{\mathrm{m}}}{\partial \boldsymbol{F}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = \boldsymbol{0}, \tag{A.27b}$$

and defining the overall energy density, $\hat{W} := \hat{W}^{\mathrm{el}} + \hat{W}^{\mathrm{m}}$, we obtain

$$\boldsymbol{\mathfrak{F}}^{\mathrm{el}} = -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + \operatorname{Div}(W^{\mathrm{m}} \boldsymbol{I}^T) - \boldsymbol{F}^T \boldsymbol{f}^{\mathrm{p}}. \tag{A.28}$$

Finally, substituting this result into Equation (A.20) yields

$$\operatorname{Div}\left(W^{\mathrm{el}} \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{P}\right) - \frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + \operatorname{Div}(W^{\mathrm{m}} \boldsymbol{I}^T) - \boldsymbol{F}^T \boldsymbol{f}^{\mathrm{p}} = \boldsymbol{0}, \tag{A.29}$$

which can be recast in the form

$$\operatorname{Div}\left(W \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{P}\right) - \frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^T \boldsymbol{f}^{\mathrm{p}} = \boldsymbol{0}. \tag{A.30}$$

We recognise that the term under divergence in Equation (A.30) is the Eshelby stress tensor used in our work, i.e., $\boldsymbol{\mathfrak{E}} = W \boldsymbol{I}^T - \boldsymbol{F}^T \boldsymbol{P}$, which is constructed with the energy density W. Accordingly, the corresponding configurational force is given by

$$\mathfrak{F} := -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) - \boldsymbol{F}^T \boldsymbol{f}^{\mathrm{p}} = \mathfrak{F}^{\mathrm{el}} - \operatorname{Grad} W^{\mathrm{m}}, \tag{A.31}$$

so that Equation (A.30) returns the configurational force balance $\operatorname{Div} \mathfrak{C} + \mathfrak{F} = \mathbf{0}$. In the absence of polygenic forces, i.e., for $\boldsymbol{f}^{\mathrm{p}} = \mathbf{0}$, the form of the configurational force balance is maintained up to the re-definition of $\mathfrak{F}$, which reduces to

$$\mathfrak{F} := -\frac{\partial \hat{W}}{\partial \mathcal{X}} \circ (\phi, \boldsymbol{F}, \mathcal{X}), \tag{A.32}$$

a result stating that the inhomogeneity force $\mathfrak{F}$ acquires the meaning of an *effective* force accounting for two contributions: the inhomogeneities of the material featuring in the body's hyperelastic behaviour and, thus, represented by W^{el}, and the inhomogeneities of the energy density W^{m}, which describes the interaction of the body with its surrounding world (e.g., via the mass density).

A.5 Divergence Transformation

Let us consider a field theoretical framework and analyse a static problem, described by the Lagrangian density function $\mathcal{L} = \hat{\mathcal{L}} \circ (\varphi, \operatorname{Grad} \varphi, \mathcal{X})$, in which φ is a scalar field (the generalisation to the situation in which φ is a collection of N scalar fields is straightforward). We emphasise that φ is *not* the deformation here, but only a generic scalar field, as it could be the case for temperature or for the scalar potential in Electromagnetism. Consequently, the evaluation $\varphi(X)$, with $X \in \mathcal{B}$, only represents the value taken by φ at X, i.e., it is not the embedding of the material point X into the three-dimensional Euclidean space. Within this setting, the quantity $\operatorname{Grad} \varphi$ need not be the "material gradient" of φ. Still, we maintain the notation introduced so far in our work in order not to generate confusion.

After renaming $\hat{\mathcal{L}} \equiv \hat{\mathcal{L}}_{\text{old}}$, we express the *divergence transformation* as (Hill, 1951)

$$\hat{\mathcal{L}}_{\text{new}} \circ (\varphi, \text{Grad}\,\varphi, \mathcal{X}) = \hat{\mathcal{L}}_{\text{old}} \circ (\varphi, \text{Grad}\,\varphi, \mathcal{X}) + \text{Div}\,\mathbf{\Omega}, \tag{A.33}$$

where $\mathbf{\Omega} = \hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})$ is an arbitrary vector field. Moreover, we notice that the vector-valued function $\hat{\mathbf{\Omega}}$ has to be independent of Grad φ.

A first direct consequence of (A.33) is that the overall Lagrangian[1] associated with the body transforms from

$$L_{\mathcal{B}}^{\text{old}}(\varphi) = \int_{\mathcal{B}} [\hat{\mathcal{L}}_{\text{old}} \circ (\varphi, \text{Grad}\varphi, \mathcal{X})] \tag{A.34}$$

into

$$L_{\mathcal{B}}^{\text{new}}(\varphi) = \int_{\mathcal{B}} [\hat{\mathcal{L}}_{\text{new}} \circ (\varphi, \text{Grad}\,\varphi, \mathcal{X})], \tag{A.35}$$

where $L_{\mathcal{B}}^{\text{old}}(\varphi)$ and $L_{\mathcal{B}}^{\text{new}}(\varphi)$ differ from each other by the boundary term $\int_{\partial\mathcal{B}} \mathbf{\Omega}\boldsymbol{N}$, i.e.,

$$L_{\mathcal{B}}^{\text{new}}(\varphi) = L_{\mathcal{B}}^{\text{old}}(\varphi) + \int_{\partial\mathcal{B}} [\hat{\mathbf{\Omega}} \circ (\varphi, \mathcal{X})]\boldsymbol{N}. \tag{A.36}$$

Since the variational procedure yielding the stationarity conditions for $L_{\mathcal{B}}^{\text{old}}(\varphi)$ and $L_{\mathcal{B}}^{\text{new}}(\varphi)$ requires the fields φ and $\bar{\varphi} = \varphi + h\,\eta$ to coincide with each other on $\partial\mathcal{B}$ (indeed, η is chosen such that it vanishes on $\partial\mathcal{B}$), a field φ for which $L_{\mathcal{B}}^{\text{old}}(\varphi)$ is stationary makes $L_{\mathcal{B}}^{\text{new}}(\varphi)$ stationary too. Moreover, such a field has to satisfy the same set of Euler-Lagrange equations. Indeed, upon recalling the

[1]In a more general – yet conceptually equivalent – framework, we should speak of action functional, rather than "overall Lagrangian", with the former being defined as the time integral of the latter over a given (bounded) time interval. However, since all the quantities introduced in the present work are independent of time because of the hypothesis of static problem, the action and the "overall Lagrangian" are defined up to a multiplicative constant representing the width of the given time interval. For this reason, the formulation used in our work is totally equivalent to the general one.

expression of the covariant divergence of $\boldsymbol{\Omega}$, i.e.,

$$\begin{aligned}\operatorname{Div}\boldsymbol{\Omega} &= \Omega^A{}_{,A} + \Gamma^A_{BA}\Omega^B \\ &= \left[\frac{\partial\hat{\Omega}^A}{\partial\varphi}\circ(\varphi,\mathcal{X})\right]\varphi_{,A} + \frac{\partial\hat{\Omega}^A}{\partial\mathcal{X}^A}\circ(\varphi,\mathcal{X}) + \Gamma^A_{BA}[\hat{\Omega}^B\circ(\varphi,\mathcal{X})], \end{aligned} \tag{A.37}$$

and substituting (A.37) into (A.33), we find that another consequence of Equation (A.33) is given by the identities

$$\begin{aligned}\frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi}\circ(\ldots) &= \frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi}\circ(\ldots) + \left[\frac{\partial^2\hat{\Omega}^A}{\partial\varphi^2}\circ(\varphi,\mathcal{X})\right]\varphi_{,A} \\ &\quad + \frac{\partial^2\hat{\Omega}^A}{\partial\mathcal{X}^A\partial\varphi}\circ(\varphi,\mathcal{X}) + \Gamma^A_{BA}\left[\frac{\partial\hat{\Omega}^B}{\partial\varphi}\circ(\varphi,\mathcal{X})\right], \end{aligned} \tag{A.38a}$$

$$\frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi_{,B}}\circ(\ldots) = \frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi_{,B}}\circ(\ldots) + \frac{\partial\hat{\Omega}^B}{\partial\varphi}\circ(\varphi,\mathcal{X}), \tag{A.38b}$$

$$\begin{aligned}\left[\frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi_{,B}}\circ(\ldots)\right]_{|B} &= \left[\frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi_{,B}}\circ(\ldots)\right]_{|B} + \left[\frac{\partial^2\hat{\Omega}^B}{\partial\varphi^2}\circ(\varphi,\mathcal{X})\right]\varphi_{,B} \\ &\quad + \frac{\partial^2\hat{\Omega}^B}{\partial\varphi\partial\mathcal{X}^B}\circ(\varphi,\mathcal{X}) + \Gamma^B_{DB}\left[\frac{\partial\hat{\Omega}^D}{\partial\varphi}\circ(\varphi,\mathcal{X})\right], \end{aligned} \tag{A.38c}$$

which imply the invariance of the Euler-Lagrange equations under the transformation (A.33), i.e.,

$$\frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi}\circ(\ldots) - \left(\frac{\partial\hat{\mathcal{L}}_{\text{old}}}{\partial\varphi_{,B}}\circ(\ldots)\right)_{|B} = \frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi}\circ(\ldots) - \left(\frac{\partial\hat{\mathcal{L}}_{\text{new}}}{\partial\varphi_{,B}}\circ(\ldots)\right)_{|B} = 0. \tag{A.39a}$$

We emphasise that this result holds true because $\operatorname{Div}[\hat{\boldsymbol{\Omega}}\circ(\varphi,\mathcal{X})]$ solves identically the Euler-Lagrange equations, i.e.,

$$\frac{\partial}{\partial\varphi}\operatorname{Div}[\hat{\boldsymbol{\Omega}}\circ(\varphi,\mathcal{X})] - \operatorname{Div}\left(\frac{\partial}{\partial\operatorname{Grad}\varphi}\operatorname{Div}[\hat{\boldsymbol{\Omega}}\circ(\varphi,\mathcal{X})]\right) = 0. \tag{A.40}$$

If φ is a collection of N independent scalar fields, Equation (A.40) becomes a system of N scalar equations, i.e., in components,

$$\frac{\partial}{\partial \varphi^{\mu}} \mathrm{Div}[\hat{\boldsymbol{\Omega}} \circ (\varphi, \mathcal{X})] - \left(\frac{\partial}{\partial \varphi^{\mu}{}_{,A}} \mathrm{Div}[\hat{\boldsymbol{\Omega}} \circ (\varphi, \mathcal{X})] \right)_{|A} = 0, \quad \mu = 1, \dots, N. \tag{A.41}$$

However, the quantity

$$\frac{\partial}{\partial \varphi^{\mu}{}_{,A}} \mathrm{Div}[\hat{\boldsymbol{\Omega}} \circ (\varphi, \mathcal{X})], \qquad \mu = 1, \dots, N, \quad A = 1, 2, 3, \tag{A.42}$$

is not, in general, the component of a tensor field. Indeed, if it were, for example for $N = 3$, the covariant divergence constituting the second term on the left-hand-side of Equation (A.41) would require to differentiate the tensors $\boldsymbol{e}^{\mu} \otimes \boldsymbol{E}_A$ of a suitable tensor basis, thereby yielding a term, obtained by differentiating $\boldsymbol{e}^{\mu}$, that does not cancel with the first summand of Equation (A.41). Hence, Equation (A.41) would not be satisfied.

The situation just depicted occurs when the "fields" of the triplet $(\varphi^1, \varphi^2, \varphi^3)$ acquire the meaning of the components of the deformation, an object that has the mathematical meaning of an embedding and, thus, that is not truly identifiable with a collection of genuine scalar fields. Indeed, when $(\varphi^1, \varphi^2, \varphi^3)$ is replaced by (ϕ^1, ϕ^2, ϕ^3), the corresponding "gradient" is none other than $\boldsymbol{F}$ and, more importantly, the quantity in (A.42) becomes (with $a \in \{1, 2, 3\}$ and $A \in \{1, 2, 3\}$)

$$\frac{\partial}{\partial \phi^{a}{}_{,A}} \mathrm{Div}[\hat{\boldsymbol{\Omega}} \circ (\phi, \mathcal{X})] = \frac{\partial}{\partial F^{a}{}_{A}} \mathrm{Div}[\hat{\boldsymbol{\Omega}} \circ (\phi, \mathcal{X})], \tag{A.43}$$

which takes on the meaning of a fictitious first Piola-Kirchhoff stress tensor. The consequence of this result is that the covariant divergence of the right-hand-side of Equation (A.43) does not cancel with $\partial \mathrm{Div}[\hat{\boldsymbol{\Omega}} \circ (\phi, \mathcal{X})]/\partial \phi^a$. This leads us to the conclusion, already stated by Maugin (1993, see page 100), that $\hat{\boldsymbol{\Omega}}$ should depend "***at most***" on X "*and **not** on the fields*".

Since we consider a static problem, for which the body's Lagrangian density function coincides with the negative of its total energy density, following Hill (1951), we introduce the functions $W_{\mathrm{old}} = \hat{W}_{\mathrm{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X})$ and $W_{\mathrm{new}} = \hat{W}_{\mathrm{new}} \circ (\phi, \boldsymbol{F}, \mathcal{X})$, and we reformulate the transformation (A.33) as

$$-\hat{W}_{\text{new}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) = -\hat{W}_{\text{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X}) + \operatorname{Div} \boldsymbol{\Omega}, \tag{A.44}$$

with $\boldsymbol{\Omega} \equiv \hat{\boldsymbol{\Omega}} \circ \mathcal{X}$. For the reasons outlined above, the divergence transformation (A.44) is such that the overall energies $\mathcal{E}^{\text{old}}_{\mathcal{D}}(\phi) = \int_{\mathcal{D}} \hat{W}_{\text{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X})$ and $\mathcal{E}^{\text{new}}_{\mathcal{D}}(\phi) = \int_{\mathcal{D}} \hat{W}_{\text{new}} \circ (\phi, \boldsymbol{F}, \mathcal{X})$ are stationary for the same deformation ϕ, which thus satisfies the same Euler-Lagrange equations. Indeed, since $\hat{\boldsymbol{\Omega}}$ is independent of ϕ, it holds true that

$$\frac{\partial \hat{W}_{\text{new}}}{\partial \phi^b} \circ (\ldots) = \frac{\partial \hat{W}_{\text{old}}}{\partial \phi^b} \circ (\ldots), \tag{A.45a}$$

$$\frac{\partial \hat{W}_{\text{new}}}{\partial F^b{}_B} \circ (\ldots) = \frac{\partial \hat{W}_{\text{old}}}{\partial F^b{}_B} \circ (\ldots), \tag{A.45b}$$

$$\frac{\partial \hat{W}_{\text{old}}}{\partial \phi^b} \circ (\ldots) - \left(\frac{\partial \hat{W}_{\text{old}}}{\partial F^b{}_B} \circ (\ldots) \right)_{|B} = \frac{\partial \hat{W}_{\text{new}}}{\partial \phi^b} \circ (\ldots) - \left(\frac{\partial \hat{W}_{\text{new}}}{\partial F^b{}_B} \circ (\ldots) \right)_{|B} = 0. \tag{A.45c}$$

After proving this property, we superimpose the transformations $X \mapsto \tilde{X} = \mathcal{Y}(X) = X + h\,\boldsymbol{U}$ and $\phi(X) \mapsto \bar{\phi}(\tilde{X}) = \phi(\tilde{X}) + h\,\boldsymbol{\eta}(\tilde{X})$ to the divergence transformation (A.44), and we require the invariance of the overall energy under the resulting, global transformation (Hill, 1951). This yields the equality

$$\underbrace{\int_{\mathcal{D}} \{[\hat{W}_{\text{new}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X})] \circ \mathcal{Y}\} \det(T\mathcal{Y})}_{\equiv \mathcal{E}^{\text{new}}_{\mathcal{D}}(\bar{\phi}, \mathcal{Y})} = \underbrace{\int_{\mathcal{D}} \hat{W}_{\text{old}} \circ (\phi, \boldsymbol{F}, \mathcal{X})}_{\equiv \mathcal{E}^{\text{old}}_{\mathcal{D}}(\phi, \mathcal{X})}, \tag{A.46}$$

where $T\mathcal{Y}$ is the tangent map of $\mathcal{Y}$. By applying a "rescaled" divergence transformation to the left-hand-side of Equation (A.46), i.e.,

$$\hat{W}_{\text{new}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) = \hat{W}_{\text{old}} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) - \operatorname{Div}(h\,\boldsymbol{\Omega}), \tag{A.47}$$

we obtain

$$\int_{\mathcal{D}} \{[\hat{W}_{\rm old} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X})] \circ \mathcal{Y} - {\rm Div}(h\,\boldsymbol{\Omega}) \circ \mathcal{Y}\} \det(T\mathcal{Y}) = \int_{\mathcal{D}} \hat{W}_{\rm old} \circ (\phi, \boldsymbol{F}, \mathcal{X})\,. \tag{A.48}$$

We remark that the smallness parameter h, which multiplies $\boldsymbol{\Omega}$ in (A.47) and (A.48), has been introduced in order to make the divergence transformation infinitesimal, as is the case for the transformations on the material points and on the deformation.

By rearranging Equation (A.48), so as to separate the transformations on the material points and on the deformation from the divergence transformation, we find

$$\int_{\mathcal{D}} \{[\hat{W}_{\rm old} \circ (\bar{\phi}, \bar{\boldsymbol{F}}, \mathcal{X}) \circ \mathcal{Y}] \det(T\mathcal{Y}) - \hat{W}_{\rm old} \circ (\phi, \boldsymbol{F}, \mathcal{X})\} = \int_{\mathcal{D}} [{\rm Div}(h\,\boldsymbol{\Omega}) \circ \mathcal{Y}] \det(T\mathcal{Y})\,. \tag{A.49}$$

By using the result reported in (3.103), at the first order in h, Equation (A.49) becomes

$$\int_{\mathcal{D}} {\rm Div}[\mathfrak{E}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w}] = \int_{\mathcal{D}} {\rm Div}\,\boldsymbol{\Omega} \quad \Rightarrow \quad \int_{\mathcal{D}} {\rm Div}[\mathfrak{E}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = 0\,, \tag{A.50}$$

thereby implying that Noether's current density is given by $\mathfrak{J} = \mathfrak{E}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}$ and that, after localisation, the conservation laws should be sought for in the form

$${\rm Div}[\mathfrak{E}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = 0\,. \tag{A.51}$$

The choice of $\boldsymbol{\Omega}$ depends on the type of conservation law and on the associated class of symmetry which one is interested in looking at.

Within the present context, Equation (A.51) constitutes the most general form of conservation law pertaining to Noether's current. This result, however, can be exploited in much deeper detail: indeed, granted the Euler-Lagrange equations $\boldsymbol{f} + {\rm Div}\,\boldsymbol{P} = \boldsymbol{0}$, if, for a given choice of the fields $\boldsymbol{U}$, $\boldsymbol{w}$ and $\boldsymbol{\Omega}$, (A.51) is satisfied as an identity, then a specific physical quantity is conserved and the fields are said to be *symmetries*.

For the problem under investigation, Equation (A.51) can be recast in the equivalent form (Hill, 1951; Grillo et al., 2003, 2019)

$$\begin{aligned}\mathrm{Div}[\boldsymbol{\mathfrak{C}}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] &= (\mathrm{Div}\, \boldsymbol{\mathfrak{C}})\boldsymbol{U} + \boldsymbol{\mathfrak{C}} : \mathrm{Grad}\, \boldsymbol{U} + (\mathrm{Div}\, \boldsymbol{P})\boldsymbol{w} + \boldsymbol{P} : \mathrm{Grad}\, \boldsymbol{w} - \mathrm{Div}\, \boldsymbol{\Omega} \\ &= -\boldsymbol{\mathfrak{F}}\, \boldsymbol{U} + \boldsymbol{\mathfrak{C}} : \mathrm{Grad}\, \boldsymbol{U} - \boldsymbol{f}\, \boldsymbol{w} + \boldsymbol{P} : \mathrm{Grad}\, \boldsymbol{w} - \mathrm{Div}\, \boldsymbol{\Omega} = 0\,. \end{aligned} \tag{A.52}$$

If one is interested in looking at the conservation of linear momentum, one sets $\boldsymbol{U} = \boldsymbol{0}$, $\boldsymbol{\Omega} = \boldsymbol{0}$ and $\boldsymbol{w} = \boldsymbol{w}_0$, with $\boldsymbol{w}_0$ being a uniform displacement field. In this case, Equation (A.52) is not satisfied. Indeed, it occurs that

$$\mathrm{Div}[\boldsymbol{\mathfrak{C}}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = \mathrm{Div}[\boldsymbol{P}^T \boldsymbol{w}_0] = -\boldsymbol{f}\, \boldsymbol{w}_0 \neq 0\,, \tag{A.53}$$

which shows that linear momentum is not conserved because of the body forces $\boldsymbol{f}$.

On the same footing, the presence of the inhomogeneity force, $\boldsymbol{\mathfrak{F}}$, spoils the conservation of the pseudo-momentum (Maugin, 1993), and this is reflected by the fact that uniform translations of material points, hereafter denoted by $\boldsymbol{U} = \boldsymbol{U}_0$, are not symmetry transformations. This is encompassed by Equation (A.52) by setting $\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{\Omega} = \boldsymbol{0}$, thereby obtaining

$$\mathrm{Div}[\boldsymbol{\mathfrak{C}}^T \boldsymbol{U} + \boldsymbol{P}^T \boldsymbol{w} - \boldsymbol{\Omega}] = -\boldsymbol{\mathfrak{F}}\, \boldsymbol{U}_0 \neq 0\,. \tag{A.54}$$

In fact, Hill (1951) presents several examples, from which we largely took inspiration, and, among those, he shows that the only case in which $\boldsymbol{\Omega}$ should be taken different from the null vector is the case in which velocity transformations are applied, a situation referred to as the *centre-of-mass theorem*.

A.6 Derivative of the Determinant of the Material Isomorphism

Here, we calculate the derivative $(J_{\boldsymbol{P}})_{,B}$ directly. Since $\boldsymbol{P}$ is a *two-point tensor*, like the deformation gradient $\boldsymbol{F}$, its determinant $J_{\boldsymbol{P}} = \det \boldsymbol{P}$ *does not* coincide with the determinant $\det[\![P^I{}_\alpha]\!]$ of its representing matrix (Marsden and Hughes, 1983; Federico, 2015; Federico et al., 2016). Indeed, in order to obtain the determinant of the two point tensor $\boldsymbol{P}(X) : \mathcal{A} \to T_X\mathcal{B}$, we need to adjust $\det[\![P^I{}_\alpha]\!]$ by using the coefficients of the volume forms of the archetype $\mathcal{A} \equiv \mathbb{R}^3$, and of the volume form in the tangent bundle $T\mathcal{B}$ of the body $\mathcal{B}$. Let us assume that the volume forms are given metrically, i.e., there is a metric tensor $\mathfrak{g}$ in the archetype (and this tensor is a *single fixed* value) and a metric tensor $\boldsymbol{G}$ and in $T\mathcal{B}$ (which is a *tensor field* on $\mathcal{B}$). Then the expression of the determinant of $\boldsymbol{P}$ reads (cf., e.g., (Marsden and Hughes, 1983))

$$J_{\boldsymbol{P}} = \det \boldsymbol{P} = \sqrt{\det[\![G_{IJ}]\!]}\ \det[\![P^I{}_\alpha]\!]\ \frac{1}{\sqrt{\det[\![\mathfrak{g}_{\alpha\beta}]\!]}}. \tag{A.55}$$

The partial derivative of $J_{\boldsymbol{P}}$ is evaluated as

$$\begin{aligned}(J_{\boldsymbol{P}})_{,B} &= \left(\det[\![P^I{}_\alpha]\!]\right)_{,B} \frac{\sqrt{\det[\![G_{IJ}]\!]}}{\sqrt{\det[\![\mathfrak{g}_{\alpha\beta}]\!]}} + \frac{\det[\![P^I{}_\alpha]\!]}{\sqrt{\det[\![\mathfrak{g}_{\alpha\beta}]\!]}} \left(\sqrt{\det[\![G_{IJ}]\!]}\ \right)_{,B} \\ &= \det[\![P^I{}_\alpha]\!]\ (\boldsymbol{P}^{-T})_C{}^\gamma\ P^C{}_{\gamma,B} \frac{\sqrt{\det[\![G_{IJ}]\!]}}{\sqrt{\det[\![\mathfrak{g}_{\alpha\beta}]\!]}} + \frac{1}{2} \frac{\det[\![P^I{}_\alpha]\!]}{\sqrt{\det[\![\mathfrak{g}_{\alpha\beta}]\!]}} \sqrt{\det[\![G_{IJ}]\!]}\ G^{AC}\ G_{AC,B}\end{aligned} \tag{A.56}$$

which gives

$$(J_{\boldsymbol{P}})_{,B} = -J_{\boldsymbol{P}} \left(\Gamma^C_{CB} - \tfrac{1}{2}\, G^{AC}\, G_{AC,B}\right). \tag{A.57}$$

Using the identity (cf., e.g., (Marsden and Hughes, 1983))

$$G_{AC,B} = G_{DC}\, \mathrm{K}^D_{AB} + G_{DA}\, \mathrm{K}^D_{CB}, \tag{A.58}$$

where K^A_{BC} are the Christoffel symbols of the Levi-Civita connection associated with $\boldsymbol{G}$, we obtain

$$(J_{\boldsymbol{P}})_{,B} = J_{\boldsymbol{P}} \left[\mathrm{K}^C_{CB} - \Gamma^C_{CB}\right] . \tag{A.59}$$

Since the difference of the Christoffel symbols of any two connections are the components of a tensor (as shown, e.g., in Eq. (4.82)), we have that the expression in (4.84) is the B-component of a *true* vector.

Bibliography

Alhasadi, M. F., Epstein, M., and Federico, S. 2019. Eshelby force and power for uniform bodies. *Acta Mechanica*, 230(5):1663–1684.

Alhasadi, M. F. and Federico, S. 2017. Relation between Eshelby stress and Eshelby fourth-order tensor within an ellipsoidal inclusion. *Acta Mechanica*, 228:1045–1069.

Alhasadi, M. F. and Federico, S. 2019. Eshelby's inclusion problem in large deformations. *submitted to the Proceedings of the Royal Society Series A*.

Ammari, H., Capdeboscq, Y., Kang, H., Lee, H., Milton, G. W., and Zribi, H. 2010. Progress on the strong Eshelby's conjecture and extremal structures for the elastic moment tensor. *Journal de Mathématiques Pures et Appliquées*, 94:93–106.

Balluffi, R. W. 2012. *Introduction to Elasticity Theory for Crystal Defects*. Cambridge University Press, New York.

Bilby, B., Gardner, L., and Stroh, A. 1957. Continuous distributions of dislocations and the theory of plasticity. In *Proceedings of the 9th International Congress of Applied Mechanics*, volume 8, pages 35–44. University of Bruxelles.

Bishop, R. L. and Goldberg, S. I. 1968. *Tensor Analysis on Manifolds*. Prentice-Hall, NJ, USA.

Bonet, J. and Wood, R. D. 2008. *Nonlinear Continuum Mechanics for Finite Element Analysis (Second Edition)*. Cambridge University Press, Cambridge, UK.

Cermelli, P., Fried, E., and Sellers, S. 2001. Configurational stress, yield and flow in rate-independent plasticity. *Proceedings of the Royal Society A*, 457:1447–1467.

Curnier, A., He, Q.-C., and Zysset, P. 1995. Conewise linear elastic materials. *Journal of Elasticity*, 37:1–38.

dell'Isola, F. and Placidi, L. 2011. Variational principles are a powerful tool also for formulating field theories. In dell'Isola, F. and Gavrilyuk, S., editors, *Variational Models and Methods in Solid and Fluid Mechanics*, pages 1–15. Springer-Verlag.

Diani, J. L. and Parks, D. 2000. Problem of an inclusion in an infinite body, approach in large deformation. *Mechanics of Materials*, 32(1):43–55.

Edelen, D. G. B. 1981. Aspects of variational arguments in the theory of elasticity: Fact and folklore. *International Journal of Solids and Structures*, 17:729–740.

Epstein, M. 2002. The eshelby tensor and the theory of continuous distributions of inhomogeneities. *Mechanics Research Communications*, 29:501–506.

Epstein, M. 2009. The split between remodelling and aging. *International Journal of Non-Linear Mechanics*, 44:604–609.

Epstein, M. 2010. *The Geometrical Language of Continuum Mechanics*. Cambridge University Press, Cambridge, UK.

Epstein, M. 2015. Mathematical characterization and identification of remodeling, growth, aging and morphogenesis. *Journal of the Mechanics and Physics of Solids*, 84:72–84.

Epstein, M. and de León, M. 2016. Unified geometric formulation of material uniformity and evolution. *Mathematics and Mechanics of Complex Systems*, 4(1):17–29.

Epstein, M. and Elżanowski, M. 2007. *Material Inhomogeneities and Their Evolution*. Springer, Berlin, Germany.

Epstein, M. and Maugin, G. A. 1990. The energy momentum tensor and material uniformity in finite elasticity. *Acta Mechanica*, 83:127–133.

Epstein, M. and Maugin, G. A. 2000. Thermomechanics of volumetric growth in uniform bodies. *International Journal of Plasticity*, 16:951–978.

Ericksen, J. L. 1995. Remarks concerning forces on line defects. In Casey, J. and Crochet, M. J., editors, *Theoretical, Experimental, and Numerical Contributions to the Mechanics of Fluids and Solids*, pages 247–271. Springer.

Ericksen, J. L. 1998. On nonlinear elasticity theory for crystal defects. *International Journal of Plasticity*, 14:9–24.

Eringen, A. C. 1980. *Mechanics of Continua*. Robert E. Krieger Publishing Company, Huntington, NY, USA.

Eshelby, J. D. 1951. The force on an elastic singularity. *Philosophical Transactions of the Royal Society A*, 244A:87–112.

Eshelby, J. D. 1956. The continuum theory of lattice defects. *Solid State Physics*, 3:79–144.

Eshelby, J. D. 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proceedings of the Royal Society A*, 241:376–396.

Eshelby, J. D. 1961. Elastic inclusions and inhomogeneities. *In Vol. 2 of Progress in Solid Mechanics, ed. I. N. Sneddon and R. Hill. Amsterdam: North-Holland*, pages 87–140.

Eshelby, J. D. 1970. Energy relations and the energy-momentum tensor in continuum mechanics. Inelastic Behaviour of Solids*, Kanninen, M. F., et al., Eds., McGraw-Hill, New York*, pages 77–115.

Eshelby, J. D. 1975. The elastic energy-momentum tensor. *Journal of Elasticity*, 5:321–335.

Federico, S. 2012. Covariant formulation of the tensor algebra of non-linear elasticity. *International Journal of Non-Linear Mechanics*, 47:273–284.

Federico, S. 2015. Some remarks on metric and deformation. *Mathematics and Mechanics of Solids*, 20:522–539.

Federico, S., Alhasadi, M. F., and Grillo, A. 2019. Eshelby's inclusion theory in the light of noether's theorem. *in press, Mathematics and Mechanics of Complex Systems*.

Federico, S., Grillo, A., and Herzog, W. 2004. A transversely isotropic composite with a statistical distribution of spheroidal inclusions: a geometrical approach to overall properties. *Journal of the Mechanics and Physics of Solids*, 52:2309–2327.

Federico, S., Grillo, A., and Segev, R. 2016. Material description of fluxes in terms of differential forms. *Continuum Mechanics and Thermodynamics*, 28:379–390.

Fletcher, D. C. 1976. Conservation laws in linear elastodynamics. *Archive for Rational Mechanics and Analysis*, 60:329–353.

Fried, E. and Gurtin, M. E. 1994. Dynamic solid-solid transitions with phase characterized by an order parameter. *Physica D*, 72:287–308.

Fried, E. and Gurtin, M. E. 2004. A unified treatment of evolving interfaces accounting for small deformations and atomic transport with emphasis on grain-boundaries and epitaxy. *Advances in Applied Mechanics*, 40:1–177.

Gibbs, J. W. 1878. On the equilibrium of heterogeneous substances. *Transactions of the Connecticut Academy*, 3:108–248. Reprinted in: The Scientific Papers of J. Willard Gibbs, vol. 1, Dover, New York (1961).

Golebiewska Herrmann, A. 1982. Material momentum tensor and path-independent integrals of fracture mechanics. *International Journal of Solids and Structures*, 18:319–326.

Green, A. E. and Rivlin, R. S. 1964. On Cauchy's equations of motion. *Zeitschrift für Angewandte Mathematik und Physik*, 15(3):290–292.

Grillo, A., Di Stefano, S., and Federico, S. 2019. Growth and remodelling from the perspective of Noether's theorem. *Mechanics Research Communications*, 97:89–95.

Grillo, A., Federico, S., Giaquinta, G., Herzog, W., and La Rosa, G. 2003. Restoration of the symmetries broken by reversible growth in hyperelastic bodies. *Theoretical and Applied Mechanics*, 30:311–331.

Grillo, A., Prohl, R., and Wittum, G. 2015. A generalised algorithm for anelastic processes in elastoplasticity and biomechanics. *Mathematics and Mechanics of Solids*, in press, DOI: 10.1177/1081286515598661.

Grillo, A., Prohl, R., and Wittum, G. 2016. A poroplastic model of structural reorganisation in porous media of biomechanical interest. *Continuum Mechanics and Thermodynamics*, 28:579–601.

Grillo, A., Prohl, R., and Wittum, G. 2017. A generalised algorithm for anelastic processes in elastoplasticity and biomechanics. *Mathematics and Mechanics of Solids*, 22:502–527.

Grillo, A., Zingali, G., Federico, S., Herzog, W., and Giaquinta, G. 2005. The role of material inhomogeneities in biological growth. *Theoretical and Applied Mechanics*, 32:21–38.

Gupta, A. and Markenscoff, X. 2012. A new interpretation of configurational forces. *Journal of Elasticity*, 108:225–228.

Gurtin, M. E. 1986. Two-phase deformations of elastic solids. In *The Breadth and Depth of Continuum Mechanics*, pages 147–175. Springer, Berlin.

Gurtin, M. E. 1993. The dynamics of solid-solid phase transitions 1. Coherent interfaces. *Archive for Rational Mechanics and Analysis*, 123:305–335.

Gurtin, M. E. 1995. The nature of configurational forces. *Archive for Rational Mechanics and Analysis*, 131:67–100.

Gurtin, M. E. 1999. *Configurational Forces as Basic Concepts of Continuum Physics*. Springer.

Gurtin, M. E. and Podio-Guidugli, P. 1996. On configurational inertial forces at a phase interface. *Journal of Elasticity*, 44:255–269.

Hamedzadeh, A., Grillo, A., Epstein, M., and Federico, S. 2019. Remodelling of biological tissues with fibre recruitment and reorientation in the light of the theory of material uniformity. *Mechanics Research Communications*, 96:56–61.

Hill, E. L. 1951. Hamilton's principle and the conservation theorems of mathematical physics. *Reviews of Modern Physics*, 23(3):253.

Hill, R. 1961. Discontinuity relations in mechanics of solids. *Progress in Solid Mechanics*, 2:245–276.

Hill, R. 1965. A self-consistent mechanics of composite materials. *Journal of the Mechanics and Physics of Solids*, 13:213–222.

Huang, Y.-N. and Batra, R. C. 1996. Energy-momentum tensors in nonsimple elastic dielectrics. *Journal of Elasticity*, 42:275–281.

Imatani, S. and Maugin, G. A. 2002. A constitutive model for material growth and its application to three-dimensional finite element analysis. *Mechanics Research Communications*, 29:477–483.

Kang, H. and Milton, G. W. 2008. Solutions to the Pólya-Szegö conjecture and the weak Eshelby conjecture. *Archive for Rational Mechanics and Analysis*, 188:93–116.

Kienzler, R. and Herrmann, G. 2000. *Mechanics in Material Space with Applications to Defect and Fracture Mechanics*. Springer-Verlag, Berlin, Germany, first edition.

Kim, C. I., Vasudevan, M., and Schiavone, P. 2008. Eshelby's conjecture in finite plane elastostatics. *The Quarterly Journal of Mechanics and Applied Mathematics*, 61:63–73.

Knops, R. J. 1964. Further considerations of the elastic inclusion problem. *Proceedings of the Edinburgh Mathematical Society (Series 2)*, 14:61–70.

Knowles, J. K. and Sternberg, E. 1972. On a class of conservation laws in linearized and finite elastostatics. *Archive for Rational Mechanics and Analysis*, 44:187–211.

Kobayashi, S. and Nomizu, K. 1963. *Foundations of Differential Geometry*. John Wiley & Sons, New York, USA.

Kröner, E. 1959. Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen. *Archive for Rational Mechanics and Analysis*, 4:273–334.

Lanczos, C. 1970. *The Variational Principle of Mechanics*. University of Toronto Press, Toronto, Canada, fourth edition.

Landau, L. D. and Lifshitz, E. M. 1939. *The Classical Theory of Fields (Course of Theoretical Physics, Vol. 2)*. Pergamon Press, Oxford, UK, Fourth Revised English (1975) edition.

Lee, E. H. 1969. Elastic-plastic deformation at finite strains. *Journal of Applied Mechanics*, 36(1):1–6.

Liu, L. P. 2008. Solutions to the Eshelby conjectures. *Proceedings of the Royal Society A*, 464:573–594.

Markenscoff, X. 1997. On the shape of the Eshelby inclusions. *Journal of Elasticity*, 49:163–166.

Marsden, J. E. and Hughes, T. J. R. 1983. *Mathematical Foundations of Elasticity*. Prentice-Hall, Englewood Cliff, NJ, USA.

Maugin, G. A. 1993. *Material Inhomogeneities in Elasticity*. CRC Press, Boca Raton, FL, USA.

Maugin, G. A. 2006. On canonical equations of continuum thermomechanics. *Mechanics Research Communications*, 33:705–710.

Maugin, G. A. 2011. *Configurational Forces: Thermomechanics, physics, mathematics, and numerics*. Chapman and Hall, Boca Raton, FL, USA.

Maugin, G. A. and Epstein, M. 1998. Geometrical material structure of elastoplasticity. *International Journal of Plasticity*, 14:109–115.

Maugin, G. A. and Trimarco, C. 1992. Pseudomomentum and material forces in nonlinear elasticity: variational formulations and application to brittle fracture. *Acta Mechanica*, 94:1–28.

Mura, T. 1987. *Micromechanics of Defects in Solids*. The Hague: Martinus Nijhoff.

Nemat-Nasser, S. 1999. Averaging theorems in finite deformation plasticity. *Mechanics of Materials*, 31:493–523.

Noether, E. 1971. Invariant variation problems. *Transport Theory and Statistical Physics*, 1:186–207.

Noll, W. 1967. Materially uniform bodies with inhomogeneities. *Archive for Rational Mechanics and Analysis*, 27:1–32.

Noll, W. 1974. La mécanique classique basée sur un axiome d'objectivité. In *The Foundations of Mechanics and Thermodynamics: selected papers by W. Noll*, pages 47–56. Springer, Berlin.

Ogden, R. W. 1997. *Non-linear Elastic Deformations*. Dover, New York, USA.

Olver, P. J. 1984a. Conservation laws in elasticity: I: general results. *Archive for Rational Mechanics and Analysis*, 85:111–129.

Olver, P. J. 1984b. Conservation laws in elasticity II: linear homogeneous isotropic elastostatics. *Archive for Rational Mechanics and Analysis*, 85:131–160.

Podio-Guidugli, P. 2001. Configurational balances via variational arguments. *Interfaces and Free Boundaries*, 3:223–232.

Podio-Guidugli, P. 2002. Configurational forces: are they needed? *Mechanics Research Communications*, 29:513–519.

Qiu, Y. P. and Weng, G. J. 1990. On the application of Mori-Tanaka's theory involving transversely isotropic spheroidal inclusions. *International Journal of Engineering Science*, 28:1121–1137.

Rice, J. R. 1968. A path independent integral and the approximate analysis of strain concentration by notches and cracks. *Journal of Applied Mechanics*, 35:379–386.

Rodin, G. J. 1996. Eshelby's inclusion problem for polygons and polyhedra. *Journal of the Mechanics and Physics of Solids*, 44:1977–1995.

Ru, C. Q. and Schiavone, P. 1996. On the elliptic inclusion in anti-plane shear. *Mathematics and Mechanics of Solids*, 1:327–333.

Ru, C. Q., Schiavone, P., and Mioduchowski, A. 2001. Elastic fields in two jointed half-planes with an inclusion of arbitrary shape. *Zeitschrift für angewandte Mathematik und Physik*, 52:18–32.

Ru, C. Q., Schiavone, P., Sudak, L. J., and Mioduchowski, A. 2005. Uniformity of stresses inside an elliptic inclusion in finite plane elastostatics. *International Journal of Non-Linear Mechanics*, 40:281–287.

Segev, R. 2013. Notes on metric independent analysis of classical fields. *Mathematical Methods in the Applied Sciences*, 36:497–566.

Sendeckyj, G. P. 1970. Elastic inclusion problems in plane elastostatics. *International Journal of Solids and Structures*, 6:1535–1543.

Smelser, R. E. and Gurtin, M. E. 1977. On the J-integral for bi-material bodies. *International Journal of Fracture*, 13:382–384.

Stefano, S. D., Carfagna, M., Knodel, M. M., Hashlamoun, K., Federico, S., and Grillo, A. 2019. Anelastic reorganisation of fibre-reinforced biological tissues. *Computing and Visualization in Science*, 20:95–109.

Tandon, G. P. and Weng, G. 1984. The effect of aspect ratio of inclusions on the elastic properties of unidirectionally aligned composites. *Polymer composites*, 5:327–333.

Truesdell, C. and Noll, W. 1965. *The Non-Linear Field Theories of Mechanics*, volume III of *S. Flügge, ed., Encyclopedia of Physics*. Springer-Verlag, Berlin, Germany.

Verron, E., Aït-Bachir, M., and Castaing, P. 2009. Some new properties of the Eshelby stress tensor. In *IUTAM Symposium on Progress in the Theory and Numerics of Configurational Mechanics*, pages 27–35.

Walpole, L. J. 1966a. On bounds for the overall elastic moduli of inhomogeneous systems - I. *Journal of the Mechanics and Physics of Solids*, 14:151–162.

Walpole, L. J. 1966b. On bounds for the overall elastic moduli of inhomogeneous systems - II. *Journal of the Mechanics and Physics of Solids*, 14:289–301.

Walpole, L. J. 1967. The elastic field of an inclusion in an anisotropic medium. *Proceedings of the Royal Society A*, 300:270–289.

Walpole, L. J. 1969. On the overall elastic moduli of composite materials. *Journal of the Mechanics and Physics of Solids*, 17:235–251.

Walpole, L. J. 1981. Elastic behavior of composite materials: theoretical foundations. *Advances in Applied Mechanics*, 21:169–242.

Wang, C.-C., F., and Bloom 1974. Material uniformity and inhomogeneity in anelastic bodies. *Archive for Rational Mechanics and Analysis*, 53(3):246–276.

Weng, G. J. 1984. Some elastic properties of reinforced solids, with special reference to isotropic ones containing spherical inclusions. *International Journal of Engineering Science*, 22:845–856.

Weng, G. J. 1990. The theoretical connection between Mori-Tanaka's theory and the Hashin-Shtrikman-Walpole bounds. *International Journal of Engineering Science*, 28:1111–1120.

Weng, G. J. and Wong, D. T. 2009. Thermodynamic driving force in ferroelectric crystals with a rank-2 laminated domain pattern, and a study of enhanced electrostriction. *Journal of the Mechanics and Physics of Solids*, 57:571–597.

Yavari, A. and Goriely, A. 2013. Nonlinear elastic inclusions in isotropic solids. *Proceedings of the Royal Society Series A*, 469(2160):20130415.