DEPARTMENT OF ECONOMICS
AND
INSTITUTE FOR POLICY ANALYSIS

WORKING PAPER SERIES

NUMBER 8702

THE RATIONALITY OF ACCEPTING
COMPOUNDS OF UNATTRACTIVE GAMBLES*

by

Soo Hong Chew

and

Larry G. Epstein

ISSN 0829-4909

UNIVERSITY OF TORONTO
TORONTO, ONTARIO, CANADA
M5S 1A1

Department of Economics
and
Institute for Policy Analysis
University of Toronto
150 St. George Street
Toronto, Ontario
M5S 1A1

January 1987

WORKING PAPER

NUMBER 8702

THE RATIONALITY OF ACCEPTING COMPOUNDS OF UNATTRACTIVE GAMBLES*

by

Soo Hong Chew

and

Larry G. Epstein

* The financial support of the National Science Foundation is gratefully acknowledged.

Working papers are preliminary versions of papers circulated on a limited basis for information and discussion. They have not undergone any editorial or refereeing process.

ABSTRACT

A compound of many independent replicas of a gamble may be chosen by a risk averter even if the single gamble would be rejected given any initial wealth level. Samuelson has shown that such choices are impossible if the expected utility theory of preference is adopted. But they are consistent with more general theories of preference. Consequently, the intuition suggested by the law of large numbers can be correct.

I

If a gamble is declined, could a sequence of independent replicas of the gamble possibly be accepted? Samuelson (1963) argues that such behaviour would be 'irrational.' His argument is based on the following theorem: "If at each income or wealth level within a range, the expected utility of a certain investment or bet is worse than abstention, then no sequence of such independent ventures (that leaves one within the specified range of income) can have a favourable expected utility." Contrary arguments, suggests Samuelson, involve a fallacious application of the law of large numbers.

In this paper we do not dispute the validity of his theorem but we argue that Samuelson's interpretation of the theorem is inappropriate as it identifies 'rational' choice amongst gambles with expected utility theory. Evidence against the empirical validity of expected utility theory has accumulated in the behavioural experimentation literature originating with the Allais paradox. Several alternative theories have been proposed which can account for the behavioural paradoxes and which still retain normatively appealing properties such as consistency with stochastic dominance and risk aversion. (See Machina (1983) for a survey.) We employ two such theories to demonstrate that the behaviour described in the opening sentence can be 'rational'. Indeed, for the preference specifications described below and for a large class of gambles having favourable odds, the decision maker will **necessarily** accept any sufficiently long sequence of independent replicas of the gamble. Thus the intuition suggested by the law of large numbers can be correct.

It should be noted that the wealth-range qualification in Samuelson's theorem is crucial for both its validity and its interpretation. Nielsen (1985) shows that there exist expected utility orderings such that sufficiently long sequences of any actuarially fair gamble are acceptable even if the single gamble would be rejected at the initial wealth level. The "explanation" for such behaviour is that attitudes towards risk vary with wealth and for some wealth that can be reached at an intermediate stage in the sequence, the single gamble would be attractive. This explanation is distinct from one based on the law of large numbers where the attractiveness of long sequences is due to the virtual (though not complete) certainty of a positive gain. The variation of risk aversion with wealth plays no role below since for the preference specifications that we consider the attractiveness of any gamble does not depend upon the level of initial wealth.

II

Consider the space of cumulative distribution functions (c.d.f.'s) on the real line. For simplicity focus on the subset $D^c(R)$ consisting of c.d.f.'s having compact support.

The random variable $\tilde{x}$, with c.d.f. $F_{\tilde{x}} \varepsilon D^c(R)$, represents a gamble. Denote by $\tilde{s}_n$ the sum of n independent replicas of $\tilde{x}$ and by $F_{\tilde{s}_n}$ its c.d.f. Only gambles with favourable odds can be acceptable to a risk averter. Thus suppose that

$$E\tilde{x} > 0 \ . \tag{1}$$

Initial wealth is nonstochastic and is denoted by w. The corresponding degenerate c.d.f. is denoted by δ_w.

Let V be a utility functional on $D^c(R)$. The question we consider is whether it is possible to have

$$V(F_{w+\tilde{x}}) < V(\delta_w) \qquad \forall w, \text{ and} \qquad (2)$$

$$V(F_{w+\tilde{s}_n}) > V(\delta_w) \qquad \forall n \text{ greater than some } N(w),\ \forall w. \qquad (3)$$

To show that (2) and (3) are indeed possible, we first take V to be a member of the implicit weighted class which has been axiomatized by Chew (1985b) and Dekel (1986). (The functional V below also belongs to the class of M-estimators in robust statistics (Huber (1964).) Define V on $D^c(R)$ implicitly by

$$V(F) = m, \text{ where } \int \phi(x-m)dF(x) = 0. \qquad (4)$$

The function $\phi:R \to R$ is assumed to be increasing and concave with $\phi(0)=0$. Then V is monotonic in the sense of first degree stochastic dominance and is risk averse in the sense of second degree stochasic dominance. Moreover, for any random variable $\tilde{y}$ and for any w,

$$V(F_{w+\tilde{y}}) = w + V(F_{\tilde{y}}). \qquad (5)$$

Thus (2) holds at **all** w if it holds at **some** w. In other words, the attractiveness of the gamble $\tilde{x}$ does not depend on the level of initial wealth.

In light of (5), the inequalities (2) and (3) take the form

$$V(F_{\tilde{x}}) < V(\delta_0) \qquad \text{and} \qquad (2')$$

$$V(F_{\tilde{s}_n}) > V(\delta_0) \qquad \text{for all sufficiently large } n. \qquad (3')$$

When the functional structure (4) is applied, the latter inequalities are further transformed into

$$\int \phi(x)\ dF_{\tilde{x}}(x) < 0 \qquad \text{and} \qquad (6)$$

$$\int \phi(x)\, dF_{\tilde{s}_n}(x) > 0 \qquad \forall \text{ sufficiently large } n. \qquad (7)$$

These inequalities admit the following useful interpretation: View ϕ as a von Neumann-Morgenstern utility index. Then the corresponding expected utility ordering would reject $\tilde{x}$ at initial wealth level 0 and would accept long sequences of replicas of $\tilde{x}$. Nielsen (1985, Proposition 1) has characterized indices ϕ which satisfy the latter condition. In the differentiable case the characterizing property of ϕ is that

$$\lim_{x \to -\infty} -\phi''(x)/\phi'(x) = 0. \qquad (8)$$

Thus for any function ϕ which satisfies (8) (and the regularity conditions stated earlier), the corresponding functional V validates the intuition suggested by the law of large numbers.

III

To demonstrate the robustness of our argument, we show that it can be based alternatively on another (and axiomatically very different) preference theory, called rank-dependent utility theory. (See Quiggin (1982), Segal (1984), Chew (1985a) and Yaari (1987) for relevant axiomatizations.) The rank-dependent functional has the form

$$V(F) = v^{-1} \left[\int v(z)\, d(g \circ F)(z) \right], \qquad F \in D^c(R), \qquad (9)$$

where $g:[0,1] \to [0,1]$ is continuous, increasing and onto and $v:R \to R$ is continuous and increasing. In this case, V is increasing in the sense of first degree stochastic dominance. If v and g are each concave, then V is risk averse in the sense of second degree stochastic dominance (Chew, Karni and Safra (1985)).

If g is the identity function, (9) reduces to an expected utility certainty equivalent functional. We maintain a nonlinear g and also that v is an exponential function, so that (9) takes the form

$$V(F) = \frac{-1}{A} \log \left[\int e^{-Az} \, d(g \circ F)(z) \right]; \qquad (9')$$

A is a positive constant. Note that V satisfies (5) and so the attractiveness of a gamble is independent of the level of initial wealth.

Inequalities (2') and (3') take the form

$$\int -e^{-Az} \, d(g \circ F_{\tilde{x}})(z) < -1 \qquad \text{and} \qquad (10)$$

$$\int -e^{-Az} \, d(g \circ F_{\tilde{s}_n})(z) > -1 \qquad \forall \text{ sufficiently large } n. \, (11)$$

Assume that g is Lipschitz continuous with Lipschitz constant L>0. (For example, suppose that g is differentiable and $g'(0)<\infty$.) Then (see appendix)

$$\int -e^{-Az} \, d(g \circ F_{\tilde{s}_n})(z) \geq L \int -e^{-Az} \, dF_{\tilde{s}_n}(z) = -L \left[\int e^{-Az} \, dF_{\tilde{x}}(z) \right]^n .$$

Thus (11) is satisfied for all gambles for which

$$\int -e^{-Az} \, dF_{\tilde{x}}(z) > -1 \, , \qquad (12)$$

which requirement is stronger than (1).

To summarize, for the subset of favourable gambles satisfying (12), sufficiently long sequences of independent replicas will necessarily be accepted; and this is true regardless of whether or not the single gamble would be accepted. (Note that (12) does not rule out (10) if g is not linear on [0,1].)

APPENDIX

Lemma: Let $g:[0,1]\to[0,1]$ be increasing, onto and Lipschitz continuous with constant $L>0$. Let $v:R\to(-\infty,0]$ be continuous. Then

$$\int v(z)\, d(g\circ F)(z) \geq L \int v(z)\, dF(z), \qquad \forall F\in D^C(R).$$

Proof: C.d.f.'s having finite supports are dense in $D^C(R)$, where the latter has the weak convergence topology, and the rank-dependent functional is continuous on $D^C(R)$. Thus it suffices to prove that for all $x_1<x_2<\ldots<x_n$ and corresponding probabilities $p_1,\ldots,p_n$,

$$\sum_{i=1}^{n} v(x_i)\ \left[\, g\left(\sum_{0}^{i} p_j\right) - g\left(\sum_{0}^{i-1} p_j\right)\right] \geq L \sum_{i=1}^{n} v(x_i)\, p_i.$$

But this is clearly true given the assumptions on g and v.||

FOOTNOTES

1. If ϕ satisfies (8), then the ordering represented by the expected value of ϕ exhibits risk aversion which increases with wealth at least over some interval of negative wealths. But (5) implies that V, which is the utility functional of primary interest, exhibits constant absolute risk aversion.

REFERENCES

Chew, S.H., 1985a, An axiomatization of the rank-dependent quasilinear mean generalizing the Gini mean and the quasilinear mean, Working Paper 156, Johns Hopkins University.

_________, 1985b, Implicit-weighted and semi-weighted utility theories, M-estimators and non-demand revelation of second price auctions for an uncertain auctioned object, Working Paper 155, Johns Hopkins University.

_________, E. Karni and Z. Safra, 1985, Risk aversion in the theory of expected utility with rank-dependent probabilities, Working Paper 16-85, Tel Aviv University, forthcoming in Journal of Economic Theory.

Dekel, E., 1986, An axiomatic characterization of preferences under uncertainty, Journal of Economic Theory 40, 304-318.

Huber, P.J., 1964, Robust estimation of a location parameter, Annals of Mathematical Statistics 35, 73-101.

Machina, M., 1983, The economic theory of individual behaviour towards risk: Theory, evidence and new directions, IMSSS Technical Report 433, Stanford University.

Nielsen, L.T., 1985, Attractive compounds of unattractive investments and gambles, Scandinavian Journal of Economics 87, 463-473.

Quiggin, J., 1982, Anticipated utility theory, Journal of Economic Behaviour and Organization 3, 323-343.

Samuelson, P.A., 1963, Risk and uncertainty: A fallacy of large numbers, Scientia 6th series, 57th year, April-May, 1-6.

Segal, U., 1984, Nonlinear decision weights with the independence axiom, Working Paper 353, UCLA.

Yaari, M., 1987, The dual theory of choice under risk, Econometrica, forthcoming.

RECENT INSTITUTE WORKING PAPERS

Number	Title	Author	Pages
8612	Unlimited Liability and Free Banking in Scotland	J. Carr S. Glied F. Mathewson	16
8613	Economic Analysis of Fault and No-Fault Liability Systems	S. Rea, Jr.	41
8614	Unemployment Insurance and Male Unemployment Duration in Canada	J. Ham S.A. Rea, Jr.	33
8615	The Response of Interest Rates to the Federal Reserve's Weekly Money Announcements: The "Puzzle" of Anticipated Money	R. Deaves	22
8616	The Size of the Government Sector and Economic Growth	J. Carr	16
8617	Economic Contests with Endogenous Monitoring	M.K. Berkowitz Y. Kotowitz	22
8701	Non-Expected Utility Preferences in a Temporal Framework with an Application to Consumption-Savings Behaviour	S.H. Chew L.G. Epstein	44

Pages
16
41
33
22
16
22
44