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Interpolation On Contours

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Abstract

In this thesis we consider the problem of finding an optimal interpolant based on contour data in the Euclidean plane. The problem is viewed as a natural extension of a similar univariate problem. We begin with an introduction to the theory of Sobolev spaces. We then give a brief discussion of univariate polynomial splines. The subsequent chapters are devoted to extending the linear and cubic splines to a bivariate setting. We follow with some graphical examples.

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Chapter 1

Introduction

The goal of this manuscript is to extend some common and well understood theory of univariate interpolants to the bivariate case. A common interpolation problem in one real variable is to find a function $f(x)$ subject to optimality constraints, with the additional property that $f(x_i) = y_i$, $i = 0, \ldots, n$, where $\{x_i\}$ and $\{y_i\}$ are specified sets of real numbers. There are two natural ways of generalizing this sort of problem to higher dimensions. One common generalization is to simply increase the number of variables, while still working with isolated points. Another is to view the abscissae from the univariate problem as manifolds of dimension 0, and when working in n dimensional space to consider data given on k dimensional manifolds, where $0 < k < n$.

In the first case, there are many bivariate interpolation schemes that are based on a finite number of isolated points in the plane. For example, the thin-plate spline discussed by Meinguet in [22] minimizes a certain functional related to bending energy, while interpolating specified values at isolated points. While this procedure seems natural enough, and the resulting interpolants are generally well behaved, the defining equations are unstable, and as we shall see in Chapter 7, numerical problems tend to develop when the thin-plate spline is used with large numbers of data.

Whether bivariate point-based interpolation methods encounter numerical difficulties or not, they all have one short-coming in common. They generally make no attempt to be faithful to data given along curves. Although some schemes, like the one developed by J. Grabenstetter in his thesis [15] makes a point of making it easier for the interpolant to be faithful to the data.

In the second case, we encounter a lack of algorithms and theoretical results. The problem of finding a reasonably smooth, minimal energy interpolant to data given on contours is essentially unsolved.

Motivation for a solution to this sort of problem stems from the methods by which one might acquire topographical data. Often, maps are digitally sampled so that one has a very high density of data along contour lines. The fact that the terrain is constant along these curves is ignored by any method based on isolated points. The only way of ensuring that a point-based interpolant will be faithful to the contours is to maintain this high density information. The result can be disastrous – global interpolation schemes will be overloaded by data, and local ones may develop instabilities due to the locally almost collinear data.

We consider the theoretical problem, proving the existence of certain optimal interpolants. These interpolants preserve the continuum of information stored in contour data, and hopefully pave the road to overcoming some of the numerical difficulties posed by otherwise large data sets. The dense information should no longer be necessary, since what we really need are curves. Contours can, in most cases, be accurately reproduced by a univariate interpolation scheme, and only in the case of very convoluted curves would a large amount of data need to be stored.

The second chapter contains introductory material needed for a discussion of splines. This includes some definitions, and a basis for an introduction, in Chapter 3, to the theory of Sobolev spaces.

Chapter 4 contains an introduction to univariate splines. While the subject is not exhausted, we treat, in detail, the univariate problem that we hope to extend to the bivariate case.

In the fifth we develop a theory for extending the linear splines introduced in Chapter 4 from piecewise linear functions to piecewise harmonic functions. This involves making the connection between minimizing the functional

$$\iint_{\Omega} \left(f_x^2 + f_y^2\right)\, dxdy \tag{1.1}$$

over some domain and associated class of functions subject to boundary conditions, and finding a harmonic function subject to the same boundary conditions. The equivalence of the solutions of these two problems, often refered to as Dirichlet's principle has a long and remarkable history. Perhaps most amazing is that the principle was believed to be true by such notable mathematicians as Gauss, Riemann and Dirichlet as early as 1857, but it took 13 years before Weierstrass pointed out some flaws. One of his major objections to the equivalence was, in fact, its dependence on the existence of a solution to the minimization problem. Without too much difficulty, it is possible to construct an example where, for an appropriate class of functions, (1.1) can be brought arbitrarily close to zero, without the existence of a function in that class for which (1.1) is zero. We consider the following example.

Let Ω be the unit disk

$$\left\{(x,y) \,|\, \sqrt{(x^2+y^2)} \leq 1\right\}.$$

We shall look for a minimizer of (1.1) in the space of continuous functions which attain the value of 1 at $(0,0)$ and the value 0 for $\left\{(x,y) \,|\, \sqrt{(x^2+y^2)} = 1\right\}$. While at first glance the problem seems easy enough, we introduce the following sequence of continuous functions. Let

$$u_n(x,y) = -\frac{\ln(n^2r^2+1)}{\ln(n^2+1)} + 1, \text{ where } r = \sqrt{(x^2+y^2)}, \text{ for } n = 1, 2, \ldots.$$

We notice that for each n, $u_n(0,0) = 1$, and $u_n(x,y) = 0$ if $r = 1$. A brief calculation shows that

$$\iint\limits_{\Omega} \left\{ \left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial y}\right)^2 \right\} dxdy \to 0 \text{ as } n \to \infty.$$

The sequence $\{u_n\}$, however, fails to converge to a continuous function. In fact, there is no continuous function that both satisfies the given boundary conditions and minimizes (1.1).

This example brings forth the fact that the minimization problem does not always have a solution. One of the earliest references to such an example where the boundary of the domain contains an isolated point dates back to Zaremba (see page 35 in Monna [23]) in 1911. An example due to Hadamard (see for example page 9 in Courant [8]), forces the point home by showing that there are cases where the problem of finding a harmonic function has a solution while the variational problem of minimizing (1.1) does not.

By the time Weierstrass voiced his concerns, Riemann had already spawned a great number of works which involved arguments inspired by this principle. Unfortunately, Riemann was no longer around to assist in the justification of Dirichlet's principle. It was not until 1900 that Hilbert was able to specify sufficient conditions for the solutions to both problems to exist and be equivalent.

In the sixth chapter we offer a proof of the existence of a solution extending the notion of cubic splines to piecewise biharmonic functions in the plane. Here we make use of Hilbert space theory, specifically the Riesz Representation Theorem. The explicit evaluation of such interpolants is unfortunately limited to one straightforward example. Slightly more complex configurations of the contours already make the calculation of an optimal interpolant considerably more difficult.

This extension is analogous to the thin-plate spline mentioned earlier. There are, however, two important differences. To begin with, the thin-plate spline is designed for interpolating scattered data in the plane, not contours. As well, the thin-plate minimizes the energy over all of R^2, whereas the piecewise biharmonic splines only minimize the energy in a bounded domain.

The idea of finding an energy minimizing function on a closed domain with some boundary conditions is not a new one. This work, however, gives what we believe to be the first unified treatment of the existence and uniqueness of an extension of the notion of a cubic spline to an interpolant based on contours. Specifically, the characterization of piecewise biharmonic splines as globally C^1 functions with continuous second order normal derivatives across the contours is a new and interesting result.

In the seventh chapter we actually compute some examples and display the results graphically. Using the limited example developed at the end of Chapter 6, we compare the results of thin-plate spline interpolants with the two schemes we discussed in Chapters 5 and 6.

This thesis does not, fortunately, answer all of the questions that it raises, there is room yet for work on the problem of computing optimal interpolants for contours.

Chapter 2

Introductory Material

2.1 Notation

2.1.1 Multi-index Notation

For convenience and conciseness, we use **multi-index notation** when working in higher dimensions. A **multi-index** is a vector of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The absolute value of a multi-index α is $|\alpha| = \sum_{i=1}^{n} \alpha_i$. We have convenient notation for exponentiation and factorial for multi-indices: if $x = (x_1, x_2, \ldots, x_n) \in R^n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N^n$ then

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

and

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!.$$

When differentiating, we write

$$\frac{\partial^l f}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n} = \frac{\partial^{|\alpha|} f}{\partial^\alpha x} = D^\alpha f$$

provided f is a sufficiently smooth, real-valued function on R^n, and $\alpha \in N^n$ with $|\alpha| = l$. At times it is convenient to address only one index in a given multi-index.

To this end, we let $e_i = (\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{n-i})$.

2.2 Some Sets in R^n

In what follows we have to place some restrictions on the subsets of R^n which we use. We call a subset Ω of R^n a **domain** if Ω is open and connected. Due to the nature of the problem we address here, we also assume that a domain Ω is bounded, unless otherwise specified. The problems we would like to solve involve finding functions f that satisfy conditions given on the boundary $\partial\Omega$ of Ω. This means that we have to impose some conditions on Ω so that its boundary is 'nice enough'.

A domain Ω is said to have a **locally Lipschitz boundary** if each point x on the boundary of Ω has a neighbourhood U_x such that $\partial\Omega \cap U_x$ is the graph of a Lipschitz continuous function $f_x(\xi_1,\ldots,\xi_{n-1})$ in some Cartesian coordinate system $(\xi_1,\ldots,\xi_n)$. Moreover, we insist that $\Omega \cap U_x$ is determined by

$$U_x \cap \{\xi \in R^n | \, \xi_n < f_x(\xi_1,\ldots,\xi_{n-1})\}.$$

This last condition merely stipulates that Ω should (locally) lie on only one side of its boundary.

2.2.1 Curves in R^2

The boundary of a bounded domain Ω is called **piecewise smooth** if it is comprised of a finite number of **smooth** arcs. A smooth arc or curve γ is one for which a well-defined unit-length tangent vector γ' exists and is continuous at each point $x \in \gamma$, that is

$$\gamma'(x) = \lim_{\substack{y \to x \\ y \in \gamma}} \frac{x-y}{\|x-y\|} \text{ and } \lim_{\substack{y \to x \\ y \in \gamma}} \gamma'(y) = \gamma'(x)$$

At times we expect to parametrize curves, so that a curve Γ is the image of a piecewise C^1 mapping $\sigma : R \to R^2$. In general, the fact that σ is C^1 is not sufficient for Γ to be C^1. As an example, consider the cycloid generated by $\sigma(t) = (t - \sin(t), 1 - \cos(t))$. σ is C^1, but the tangent vector as defined above fails to exist whenever $t = 2k\pi$, $k \in Z$. Because we generally have some control over these curves and their descriptions, it is not unreasonable to insist that σ does not describe the same point of Γ more than once, so whenever we discuss parametrizations we assume that $\sigma(t)$ is one-to-one. This restriction also implies that Γ is **simple,** that is, Γ does not cross itself. In addition, for the sake of brevity, we assume that the parametrization of the curve is done with respect to arc-length. The result of this is that the derivative of the mapping will be identical to the unit tangent vector of the curve. In this case, the symbol used to denote the curve is also used to denote the mapping which generates it.

At times it is necessary to work with curves that have a higher order of smoothness, for example, we could insist that $\gamma'(\gamma(t))$ be C^1, as a mapping from R to R^2. We can generalize and say that a curve γ is of class C^m, or **m-smooth** if the arc-length parametrization $\gamma(t)$ is of class C^m.

2.2.2 Contour Maps

The title of this thesis is 'Interpolation on Contours', and so it seems only fitting to define what we mean by contours, and to indicate how complicated we are willing to let them be.

If Γ is the piecewise m-smooth and locally Lipschitz boundary of a simply connected bounded domain, we call Γ a **m−contour.** We omit the m if the degree of smoothness is clear from context, or if the statement applies for all m. We say that a pair of contours is **nested** if the closure of the region interior to one is contained within the region interior to the other. We note that this means that the two curves do not intersect. If Γ_1 and Γ_2 are two nested contours such that the region interior to Γ_1 is a subset of the region interior to Γ_2, then we say that Γ_1 is **inside** Γ_2.

If a set of non-intersecting contours $\{\Gamma_i\}$ has associated with each member a real number C_i, we call the set $M = \{(\Gamma_i, C_i)\}$ a **contour map.** We relax the condition that the contours not intersect provided that if Γ_i and Γ_j intersect for some i and j, then $C_i = C_j$, and the intersection does not result in any zero angles, i.e. Γ_i and Γ_j are not tangent in any way at the point of intersection.

2.3 Generalized Functions

Generalized or **singular** functions have long been used by physicists, although they do not fit well with classical function theory. For example, one commonly used generalized function is the Dirac delta 'function' $\delta(x)$. The standard physicist's definition for $\delta(x)$ is that for $x \neq 0$, $\delta(x) = 0$, $\delta(0) = \infty$, and $\int_{-\infty}^{\infty} \delta(x)dx = 1$. This is clearly inconsistent with the classical definition of a function. Others might describe the univariate delta 'function' by merely stating that

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx = \phi(x_0) \tag{2.1}$$

for any sufficiently nice function ϕ. We begin by describing what we mean by a 'sufficiently nice function', and then we look at a more rigorous description of generalized functions.

2.3.1 Test Functions

The set of 'sufficiently nice' functions which we consider is the set $C_0^\infty(\Omega)$ of real-valued functions with compact support on a domain $\Omega \subseteq R^n$ and possessing derivatives of all orders. We call $C_0^\infty(\Omega)$ the **space of test functions**, and since any finite linear combination of test functions is still a test function, $C_0^\infty(\Omega)$ is a linear space.

We also require some notion of convergence in $C_0^\infty(\Omega)$. We say that a sequence of functions $f_k(x)$ in $C_0^\infty(\Omega)$ converges to zero if the functions along with all their derivatives converge uniformly to zero, and provided that there exists a single bounded region containing the supports of all the $f_k(x)$. This leads us to a definition for convergence in general – we say that the sequence $f_k(x)$ converges to the function $f(x)$ provided that $f(x) - f_k(x)$ converges to zero.

2.3.2 Linear Functionals

If for every element ϕ in a linear topological space L there is a corresponding number $l(\phi)$, we say that $l(\phi)$ is a **functional** on L. **Linear functionals** are those blessed with the additional property of distributivity:

$$l(a\phi_1 + b\phi_2) = al(\phi_1) + bl(\phi_2) \tag{2.2}$$

for any pair of real constants a and b, and any ϕ_1, $\phi_2 \in L$. A linear functional $l(\phi)$ is **continuous** if whenever $\phi_k \to \phi$ as $k \to \infty$ then $l(\phi_k) \to l(\phi)$ for $\{\phi_k\} \subset L$. If L is a normed space, then we can also refer to **bounded** linear functionals. By bounded we mean that there exists a constant such that for any $\phi \in L$,

$$|l(\phi)| \leq M \|\phi\| .$$

It is not hard to see that continuity and boundedness are equivalent for linear functionals on a normed space.

Now, if f is any locally integrable function, we can define a functional on $C_0^\infty(\Omega)$ by

$$l(\phi) = \langle f, \phi \rangle := \int_\Omega f(x)\phi(x)dx. \tag{2.3}$$

That this functional is linear follows directly from the linearity of the integral, while its continuity follows from our definition of convergence in $C_0^\infty(\Omega)$.In fact, suppose $\{\phi_i\}_{i=1}^\infty \subset C_0^\infty(\Omega)$ and that $\phi_i \to 0$ as $i \to \infty$. This convergence is uniform, so $|\phi_i(x)|$ is uniformly bounded by some constant M for all x. The supports of the ϕ_i are all contained within a single bounded closed region. If we let C be this bounded region, then

$$\lim_{i\to\infty} \langle f, \phi_i \rangle = \lim_{i\to\infty} \int_C f(x)\phi_i(x)dx = \int_C f(x) \lim_{i\to\infty} \phi_i(x)dx = 0$$

by Lebesgue's Dominated Convergence Theorem. This proves continuity of the functional at $\phi \equiv 0$ and continuity in general follows from this and linearity.

Not all continuous linear functionals on $C_0^\infty(\Omega)$ can be represented as in (2.3).

Take, for example, the delta functional

$$l(\phi) = \phi(0), \ \phi \in C_0^\infty(\Omega). \tag{2.4}$$

This functional is linear since if $\phi, \psi \in C_0^\infty(\Omega)$ and $a, b \in R$ then

$$l(a\phi + b\psi) = a\phi(0) + b\psi(0) = al(\phi) + bl(\psi),$$

and continuous since if $\phi_i \to 0$ in $C_0^\infty(\Omega)$ then $l(\phi_i) = \phi_i(0) \to 0$.

Let us suppose, for the sake of a contradiction, that there is a locally integrable function f such that

$$l(\phi) = \langle f, \phi \rangle = \int_\Omega f(x)\phi(x)dx. \tag{2.5}$$

This rule has to hold for all possible functions in $C_0^\infty(\Omega)$. In particular, consider the family of test functions

$$\chi(x; a) = \begin{cases} \exp\left(\dfrac{-a^2}{a^2 - |x|^2}\right) & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a. \end{cases} \tag{2.6}$$

If we look at $\langle f, \chi \rangle$, we find that

$$\langle f, \chi \rangle = \int_\Omega f(x)\chi(x; a)dx = \chi(0; a) = e^{-1}.$$

Now, as $a \to 0$, the support of χ shrinks while $|\chi(x; a)|$ is bounded by e^{-1}, so the integral tends to zero, giving us a contradiction and thus ruling out the existence of f.

Since some of these functionals are in fact representable by true functions, this whole set of continuous linear functionals on $C_0^\infty(\Omega)$ is referred to as the space of **generalized functions**. The functionals which can be written as in (2.3) are referred to as **regular**, and the others as **singular**. Since we would like to view generalized functions as extensions of classical functions we extend the notation as well, so that if f is any generalized function then we write

$$\langle f, \phi \rangle = \int_\Omega f(x)\phi(x)dx \text{ for all } \phi \in C_0^\infty(\Omega),$$

even though the integration may be merely symbolic. Thus we might represent the functional (2.4) by

$$\langle \delta, \phi \rangle = \int_\Omega \delta(x)\phi(x)dx.$$

2.3.3 Generalized Derivatives

Some functions with which we deal are clearly not differentiable in the classical sense. For example, the Heavyside step-function

$$H(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0, \end{cases} \tag{2.7}$$

has no derivative at the origin. We can, however, extend the notion of differentiablity of such functions if we allow the derivatives to be generalized functions.

Consider the integral $\int_{-\infty}^{\infty} f'(x)\phi(x)dx$ where f is a smooth function with derivatives of all orders, and ϕ is any test function. Applying integration by parts, and noting that ϕ has bounded support, we find that

$$\int_{-\infty}^{\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx.$$

Using this rule, we can write

$$\int_{-\infty}^{\infty} f^{(l)}(x)\phi(x)dx = -\int_{-\infty}^{\infty} f^{(l-1)}(x)\phi'(x)dx = \cdots = (-1)^l \int_{-\infty}^{\infty} f(x)\phi^{(l)}(x)dx.$$

Following this example, we define, using multi-index notation, the α $^{\text{th}}$ **generalized derivative** of a generalized function f by

$$\langle D^\alpha f, \phi\rangle = (-1)^{|\alpha|} \langle f, D^\alpha \phi\rangle$$

or,

$$\int_\Omega D^\alpha f(x)\phi(x)dx = (-1)^l \int_\Omega f(x)D^\alpha\phi(x)dx, \text{ for } \alpha \in N^n \text{ and } |\alpha| = l.$$

As an example, consider the function H defined in (2.7). Since H is locally integrable, we can think of it as a generalized function and for any ϕ in $C_0^\infty(R)$, we apply integration by parts to get

$$\int_{-\infty}^{\infty} H'(x)\phi(x)dx = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_0^{\infty} \phi'(x)dx = \phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)dx.$$

We extend the standard notation for differentiation and write $H'(x) = \delta(x)$ or $\frac{dH}{dx}(x) = \delta(x)$.

Some of the standard rules for differentiation apply to distributional derivatives.

For example, for any pair of (generalized) functions f and g,

$$\langle D^{\alpha}(f+g), \phi\rangle = -\langle f+g, D^{\alpha}\phi\rangle = -\langle f, D^{\alpha}\phi\rangle - \langle g, D^{\alpha}\phi\rangle = \langle D^{\alpha}f, \phi\rangle + \langle D^{\alpha}g, \phi\rangle,$$

so that $D^{\alpha}(f+g) = D^{\alpha}f + D^{\alpha}g$. As well, when the generalized functions are true functions, with some additional restrictions, the product rule applies, as does Green's theorem. We discuss these in the next section.

Generalized functions are often refered to as **distributions**, and there are also different notations for the sets of test functions and distributions. For example, in his treatise on distributions [30], Schwartz uses the symbol $\mathcal{D}$ to denote $C_0^{\infty}(\Omega)$, while $\mathcal{D}'$ is used to represent the dual space of distributions. Similarly, generalized derivatives are often referred to as distributional derivatives.

Because we encounter, in later chapters, functions with discontinuities along curves, we introduce the two-dimensional step function

$$S(x,y) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases},$$

which we define for all points (x,y) in the rectangle $Q = (-a,a) \times (-b,b)$, for some pair of positive numbers a and b.

Suppose $f \in C^1(Q)$. Of interest to us is the generalized partial derivative of fS in the horizontal direction – we claim that $\dfrac{\partial(fS)}{\partial x}$ is not, in general, a function. If the distributional derivative of fS is to be a function, then the functional

$$\left\langle \frac{\partial(fS)}{\partial x}, \phi \right\rangle$$

must be representable by a classical function for any test function ϕ. In particular, for ϕ with support strictly within Q we compute

$$\left\langle \frac{\partial (fS)}{\partial x}, \phi \right\rangle = -\left\langle fS, \frac{\partial \phi}{\partial x} \right\rangle$$

$$= -\int_{-b}^{b} \int_{-a}^{a} fS \frac{\partial \phi}{\partial x} dxdy = -\int_{-b}^{b} \int_{0}^{a} f \cdot \frac{\partial \phi}{\partial x} dxdy = -\int_{-b}^{b} \left[(f\phi)\,|_0^a - \int_0^a \frac{\partial f}{\partial x} \phi \, dx \right] dy$$

$$= \int_{-b}^{b} \left[f(0,y)\,\phi(0,y) + \int_0^a \frac{\partial f}{\partial x} \phi \, dx \right] dy = \int_{-b}^{b} f(0,y)\,\phi(0,y)\,dy + \int_{-b}^{b} \int_0^a \frac{\partial f}{\partial x} \phi \, dxdy$$

$$= \int_{-b}^{b} f(0,y)\,\phi(0,y)\,dy + \left\langle \frac{\partial f}{\partial x} S, \phi \right\rangle. \tag{2.8}$$

The second part of (2.8) can clearly be represented by a classical function, but we claim that if f is not identically zero on the vertical axis, then the first part cannot be represented in such a way.

Suppose there is a function $g(x,y)$ such that $\langle g, \phi \rangle = \int\limits_{-b}^{b} f(0,y)\,\phi(0,y)\,dy$. Then

$$\int_{-b}^{b} f(0,y)\,\phi(0,y)\,dy = \int_{-b}^{b} \int_{-a}^{a} g(x,y)\,\phi(x,y)\,dxdy \tag{2.9}$$

for any test function ϕ with support in Q, and in particular, for the function

$$\phi_c(x,y) = \chi(x;\epsilon)\,\chi(y-d;c),$$

where χ is the function defined in (2.6), for some positive $c < b$ and $\epsilon < a$ and some d in R.

Equation (2.9) then becomes

$$\frac{1}{e}\int_{-c+d}^{c+d} f(0,y)\,\chi(y;c)\,dy = \int_{-c+d}^{c+d}\int_{-\epsilon}^{\epsilon} g(x,y)\,\phi_c(x,y)\,dxdy. \tag{2.10}$$

At this point we place some restrictions on c and d. By assumption, we can pick a value for d such that $f(0,d) \neq 0$, and since f is assumed to be continuous, for some sufficiently small c, f does not change sign on the interval from $-c+d$ to $c+d$. If we now let ϵ tend to zero, the left-hand side of (2.10) remains constant while the right-hand side clearly tends to zero. This contradiction rules out the existence of g.

2.4 Function Spaces

2.4.1 Spaces of Continuous Functions

Along with the space $C_0^\infty(\Omega)$ of test functions, we use a number of other spaces of continuous functions. Let Ω be any domain in R^n. We let $C^m(\Omega)$ be the space of functions defined on Ω and having continuous partial derivatives of all orders $\leq m$. We denote the subspace of $C^m(\Omega)$ of functions with compact support in Ω by $C_0^m(\Omega)$.

Now, supposing Ω is bounded, if $\phi \in C^0(\Omega)$ is uniformly continuous then ϕ has a unique continuous extension to $\bar{\Omega}$, the closure of Ω. If ϕ and its partial derivatives of order $\leq m$ are uniformly continuous on Ω we will say that $\phi \in C^m(\bar{\Omega})$. With the norm

$$\|\phi\|_{C^m(\bar{\Omega})} = \max_{|\alpha|\leq m}\sup_{x\in\Omega}|D^\alpha\phi(x)|$$

$C^m(\bar{\Omega})$ is a Banach space. To see that $C^m(\bar{\Omega})$ is complete, suppose $\{\phi_j\} \subset C^m(\bar{\Omega})$ is a Cauchy sequence. Then $\|\phi_i - \phi_j\|_{C^m(\bar{\Omega})} \to 0$ as $i,j \to \infty$, which means that for all $x \in \Omega$, and some M_{ij}^{α}, $|D^{\alpha}(\phi_i(x) - \phi_j(x))| < M_{ij}^{\alpha} \to 0$, for $|\alpha| \leq m$. So $D^{\alpha}(\phi_i(x))$ tends uniformly to some continuous function $\psi_{\alpha}(x)$ for each $|\alpha| \leq m$. $\psi_{\alpha}(x)$ is in fact uniformly continuous, since it is the uniform limit of uniformly continuous functions on a compact set. What we need to show now is that $\psi_{\alpha}(x) = D^{\alpha}(\psi_0(x))$ for $|\alpha| \leq m$. We use induction on α. We have equality for $|\alpha| = 0$, so we proceed with the induction step.

Supposing $\psi_{\alpha}(x) = D^{\alpha}(\psi_0(x))$ holds for $|\alpha| < m$, let $\beta = \alpha + e_j$ for some $j = 1, \ldots, n$. $D^{\beta}\phi_i$ converges uniformly to ψ_{β}, and $D^{\beta}\phi_i = D^{e_j}(D^{\alpha}\phi_i)$. Let x be in Ω, then by the Fundamental Theorem of Calculus,

$$\int_0^h D^{\beta}\phi_i(x + te_j)\,dt = D^{\alpha}\phi_i(x + he_j) - D^{\alpha}\phi_i(x)$$

for sufficiently small $h \neq 0$. Dividing this equation by h, we get

$$\frac{1}{h}\int_0^h D^{\beta}\phi_i(x + te_j)\,dt = \frac{D^{\alpha}\phi_i(x + he_j) - D^{\alpha}\phi_i(x)}{h},$$

which, as $i \to \infty$, becomes

$$\frac{1}{h}\int_0^h \psi_{\beta}(x + te_j)\,dt = \frac{\psi_{\alpha}(x + he_j) - \psi_{\alpha}(x)}{h} = \frac{D^{\alpha}\psi_0(x + he_j) - D^{\alpha}\psi_0(x)}{h}.$$

Convergence of the integral on the left is guaranteed by the uniform convergence of the integrand. By the Mean Value Theorem for Integrals it follows that

$$\psi_{\beta}(x + h_1 e_j) = \frac{D^{\alpha}\psi_0(x + he_j) - D^{\alpha}\psi_0(x)}{h},$$

for some h_1 between h and 0. Letting h tend to zero, $h_1 \to 0$ as well, so that

$$\psi_\beta(x) = D^{e_j}\left(D^\alpha \psi_0(x)\right) = D^\beta \psi_0(x),$$

which is exactly what we wanted to show. $C^m\left(\bar{\Omega}\right)$ is, therefore, complete.

2.4.2 L_p Spaces

Given a bounded domain $\Omega \subseteq R^n$, and $1 \leq p < \infty$ we denote by $L_p(\Omega)$ the linear space of measurable real-valued functions f on Ω for which the Lebesgue integral

$$\int_\Omega |f(x)|^p \, dx < \infty.$$

We define a norm on $L_p(\Omega)$ by

$$\|f\|_p = \left\{ \int_\Omega |f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

This satisfies all the properties of a norm except that if $\|f - g\| = 0$ for a pair of functions $f, g \in L_p(\Omega)$, it does not necessarily follow that $f(x) = g(x)$ for all $x \in \Omega$. For this reason, we make no distinction between functions that differ only on a set of measure zero.

At times it is desirable to talk about spaces of functions defined on a manifold S of dimension k, $1 \leq k < n$. In this case, we define $L_p(S)$ to be the space of measurable real-valued functions on S for which the norm

$$\|f\|_{L_p(S)} = \left\{ \int_S |f(x)|^p \, d\mu \right\}^{\frac{1}{p}}$$

is bounded. Here μ is the k-dimensional Lebesgue measure.

An important result which make these spaces particularly attractive is that L_p spaces are complete (Riesz-Fischer).

The space $L_2(\Omega)$ is somewhat special – for any pair of functions f and g in $L_2(\Omega)$, we define their inner-product by

$$(f,g) = \int_\Omega f(x)g(x)dx,$$

which makes $L_2(\Omega)$ a Hilbert space.

An useful inequality in L_p spaces is the **Hölder inequality.** Suppose $p \geq 1$ and $q \geq 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p(\Omega)$, and $g \in L_q(\Omega)$ then not only is $fg \in L_1(\Omega)$, but

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q$$

as well. This inequality delivers some elegant results, for example, if we assume Ω is bounded, $1 \leq p < q < \infty$, and let $f \equiv 1$, and $g \in L_q(\Omega)$ then

$$\| |g|^p \|_1 = \| |g|^p f \|_1 \leq \left\{ \int_\Omega |g|^q \, dx \right\}^{\frac{p}{q}} \left\{ \int_\Omega |f|^{\frac{1}{1-\frac{p}{q}}} \, dx \right\}^{1-\frac{p}{q}}$$

$$= \left\{ \int_{\Omega} |g|^q \, dx \right\}^{\frac{p}{q}} \left\{ \int_{\Omega} 1 dx \right\}^{1-\frac{p}{q}} = \|g\|_q^p \cdot (\mathrm{vol}\,\Omega)^{1-\frac{p}{q}}.$$

Thus g is in L_p, and $\|g\|_p \leq \|g\|_q \cdot (\mathrm{vol}\,\Omega)^{\left(\frac{1}{p}-\frac{1}{q}\right)}$. For generality, we note that this last inequality is obviously also true and reduces to equality whenever $p = q$.

2.4.3 Mollifiers

Before we continue to describe additional function spaces, we pause and introduce a method for approximating L^p functions with smooth functions. The technique involves replacing the value of a function at any point with a weighted average of all values in a neighbourhood of that point. While there are a number of ways of doing this, we chose the following approach.

Let K_a be a function in $C_0^\infty(R^n)$ with support inside the ball of radius a centered at the origin, and $\int_{R^n} K_a(x)dx = 1$. For example, let $K_a(x) = k\chi(x;a)$, where $\chi(x;a)$ is the function introduced in (2.6), and $k = \dfrac{1}{\int_{R^n} \chi(x;a)dx}$. Functions of this sort are called **mollifiers** or **mollifying kernels**, because the function (when it makes sense)

$$f_a(x) = \int_{R^n} K_a(x-y)f(y)dy$$

is often much nicer to work with than the original function f. The function f_a has a number of convenient properties, which result in its being called a **mollification** or **regularization** of f.

Lemma 2.1 *If f is defined on R^n with $f \in L_p(\Omega)$, $1 \leq p < \infty$, with Ω bounded, and $f \equiv 0$ outside of Ω, then*

1. $f_a \in C^\infty(R^n)$,

2. $f_a \in L_p(\Omega)$,

3. $\|f_a\|_p \leq \|f\|_p$ *and* $\|f_a - f\|_p \to 0$ *as* $a \to 0$.

Proof. Consider, for some $h \in R$, and $1 \leq k \leq n$, the quotient

$$Q_h = \frac{f_a(x + he_k) - f_a(x)}{h}$$

$$= \int\limits_{R^n} \left(\frac{K_a(x + he_k - y) - K_a(x - y)}{h} \right) f(y)dy.$$

The limiting value of Q_h as $h \to 0$, if it exists, would be $D^{e_k}(f_a)$. If we assume $h < 1$, then applying the Mean Value Theorem and Hölder's inequality, the difference

$$\left| Q_h - \int\limits_{R^n} \left(D^{e_k} K_a(x - y) \right) f(y)dy \right|$$

$$= \left| \int\limits_{R^n} \left(\frac{K_a(x + he_k - y) - K_a(x - y)}{h} - D^{e_k} K_a(x - y) \right) f(y)\, dy \right|$$

$$= \left| \int\limits_{R^n} \left(D^{e_k} K_a(x - y + h_1 e_k) - D^{e_k} K_a(x - y) \right) f(y)\, dy \right|$$

$$= \left| \int\limits_{R^n} h_1 \left(\frac{D^{e_k} K_a(x - y + h_1 e_k) - D^{e_k} K_a(x - y)}{h_1} \right) f(y)\, dy \right|$$

$$= \left| \int\limits_{R^n} h_1 \left(D^{2e_k} K_a(x - y + h_2 e_k) \right) f(y)\, dy \right|$$

$$\leq \left| \left(\int\limits_{|y|<a+1} (|h_1|\, M)^q\, dy \right)^{\frac{1}{q}} \left(\int\limits_{\Omega} |f(y)|^p\, dy \right)^{\frac{1}{p}} \right| \to 0,$$

as $h \to 0$, where h_1 is some number between h and 0, and similarly h_2 is between h_1 and 0, $M \geq \max_{x \in R^n} |D^{2e_k} K_a(x)|$, and $\frac{1}{q} + \frac{1}{p} = 1$. So

$$\frac{\partial f_a}{\partial x_k} = \int_{R^n} \left(D^{e_k} K_a(x-y)\right) f(y) dy.$$

Since the preceding argument required neither the positivity of K_a nor the fact that $\int_{R^n} K_a dx = 1$, a simple induction step wherein K_a is replaced by its partial derivatives proves (1).

Statement (2) follows trivially from (1) since Ω is bounded. For the proof of (3), see Adams [1] p. 30. $\square$

Usually, we don't specify exactly which mollifier to apply to a function, but in general we use the symbol f_a to denote the mollification of f generated by a mollifier with support contained in the ball of radius a.

Chapter 3

Sobolev Spaces

The term 'Sobolev Space' is used to describe a variety of spaces. Most of these spaces qualify as generalizations of the spaces of continuous functions described in the previous chapter. Instead of classical differentiation, however, we use distributional differentiation. Membership in a Sobolev space then implies the integrability of distributional derivatives of various orders. Many of the Banach spaces now associated with the name of the famous Soviet mathematician S. L. Sobolev were actually well known before his major work in the subject, and as a result we still find references to Beppo Levi spaces and others in the literature, even though these all fall under the umbrella of Sobolev spaces. For the sake of brevity, we only introduce a small subset of these spaces, although we keep notation which hints at a broader class of functions.

3.1 The Sobolev Spaces $W_2^l(\Omega)$

We denote by $W_2^l(\Omega)$ the linear space of functions on a bounded domain $\Omega \subset R^n$ which, along with their generalized derivatives of order less than or equal to l, are members of $L_2(\Omega)$.

We define a norm on $W_2^l(\Omega)$ by

$$\|\psi\|_{l,2} = \left\{ \sum_{|\alpha| \leq l} \int_\Omega (D^\alpha \psi)^2 \, dx \right\}^{\frac{1}{2}}$$

The spaces $W_2^l(\Omega)$ are complete provided functions which agree almost everywhere are considered identical. Normally, although the integrals are based on the domain Ω, we will see later that $\phi \in W_2^l(\Omega)$ can be extended in a meaningful way to $\partial\Omega$ or beyond, provided $\partial\Omega$ is sufficiently smooth. For some calculations, however, it is convenient to assume that $\phi \equiv 0$ outside of $\bar{\Omega}$.

At times it is advantageous to regularize functions in $W_2^l(\Omega)$ so that the generalized derivatives are true derivatives. Suppose $\bar{\Omega}' \subset \Omega$, and let $a < \mathrm{dist}\,(\Omega', \partial\Omega)$, if $\psi \in W_2^l(\Omega)$ then

$$D^\alpha [\psi_a] = [D^\alpha \psi]_a$$

on Ω'. In addition, $\psi_a \to \psi$ in $W_2^l(\Omega')$ (see Adams [1] p. 52). From this follows the fact that the space $C^\infty(\Omega) \cap W_2^l(\Omega)$ is dense in $W_2^l(\Omega)$. As well, with some restrictions on the boundary of Ω, for example if Ω has a piecewise smooth locally Lipschitz boundary, the smaller class of functions $C_0^\infty(R^n)$ is also dense in $W_2^l(\Omega)$.

We introduce a semi-norm on $W_2^l(\Omega)$, which resembles the above norm, but involves only the l^{th} order derivatives:

$$|\psi|_l = \left(\sum_{|\alpha|=l} \int_\Omega (D^\alpha \psi)^2 \, dx \right)^{\frac{1}{2}},$$

and an associated semi-inner product

$$(\psi, \phi)_l = \sum_{|\alpha|=l} \int_\Omega (D^\alpha \psi)(D^\alpha \phi) dx. \tag{3.1}$$

The semi-norm vanishes when and only when ψ is a polynomial of total degree less than l (or differs from one on a set of measure zero). The first case is clear, since if ψ is a polynomial of degree less than l, then $D^\alpha\psi \equiv 0$ for $|\alpha| = l$. On the other hand, suppose $|\psi|_l = 0$. This means that $D^\alpha\psi = 0$ (almost everywhere) for $|\alpha| = l$, which implies that the regularized derivatives $(D^\alpha\psi)_a$, $a > 0$, are also zero on the subset $\Omega_a = \{x \in \Omega |\ \mathrm{dist}\,(\Omega_a, \partial\Omega) < a\}$. If a is small enough, then Ω_a is connected, in which case ψ_a is a polynomial of degree less than l.

To see this, suppose $B_x \subset \Omega_a$ is an open ball centered at $x \in \Omega_a$ and y is any other point in B_x. By Taylor's theorem

$$\psi_a(y) = \sum_{|\alpha|<l} \frac{D^\alpha\psi_a(x)}{\alpha!}(y-x)^\alpha + R_l(y,x)$$

$$= T_x(y) + R_l(y,x)$$

where the remainder term $R_l(y,x) = \sum_{|\alpha|=l} \frac{(y-x)^\alpha}{\alpha!} D^\alpha\psi_a(x^*)$, and x^* is some point on the line segment joining x and y. Of course, $R_l(y,x) = 0$ by the hypotheses, so ψ_a is a polynomial of degree not exceeding $l-1$ on B_x. Since x is essentially arbitrary, it follows that ψ_a is a polynomial of order less than l on each open ball contained in Ω_a. The question now is whether $T_x(z) = T_y(z)$ for any $x, y, z \in \Omega_a$ – we need to know whether these polynomials are all the same. Suppose B_x and B_y have a non-empty intersection A. Then T_x and T_y are the same polynomials since they agree on an open set. Suppose, on the other hand, $B_x \cap B_y$ is empty. Since Ω_a is connected and open, there is a piecewise smooth curve Γ connecting x to y. Every point z on such a path has a neighbourhood $B_z \subset \Omega_a$, and since the curve is compact, we only need a finite number N of these neighbourhoods to cover Γ. Let

$\{B_{z_i}\}_{i=1}^{N}$ denote this set, indexed so that $B_{z_i} \cap B_{z_{i+1}}$ is non-empty. By the preceding argument, T_{z_i} and $T_{z_{i+1}}$ are the same polynomials, so the same is true for T_{z_j} and T_{z_i}, for any $1 \leq i, j \leq N$. B_x and B_y each intersect at least one B_{z_i}, hence T_x is the same polynomial as T_y. Thus ψ_a is a polynomial on Ω_a. Now, around any point x in Ω there is a neighbourhood N_x such that $\bar{N}_x \subset \Omega$. For all $a < \text{dist}\,(N_x, \partial\Omega)$, ψ_a is a polynomial on N_x of degree less that l. Since $\psi_a \to \psi$ on N_x, and the set of polynomials of degree less that l is a finite dimensional, and hence closed, subspace of $L_2\,(\Omega)$, ψ is a polynomial on N_x. It follows then that ψ is a polynomial of degree $< l$, or at least differs from one only on a set of measure zero.

3.2 Imbedding Theorems

Some questions that arise when one deals with Sobolev spaces and generalized derivatives of functions are not easily answered. For example, we might wonder how bad a function must be in order that its generalized derivative not be an integrable function. Also of concern is the boundary behaviour of any such function – what does it mean to say that a function in a Sobolev space interpolates data given on a contour? Is it meaningful to refer to the restriction of a function to some curve? To answer some of these questions, we now introduce some theorems regarding **imbeddings** of Sobolev spaces into other, perhaps more natural, spaces.

Before we proceed, we should say a few things about imbeddings. Given two normed spaces X and Y we say that X is **imbedded** in Y if X is a subspace of Y, and the identity mapping $I : X \to Y$ defined by $Ix = x$ is continuous. If X is imbedded in Y, then we write $X \to Y$. The continuity of the imbedding operator is essentially a statement about the relationship between the two norms in X and Y. More precisely, since I is linear, there exists a constant M, such that

$$\|x\|_Y \leq M \|x\|_X \text{, for all } x \in X.$$

There are times, as we shall see, when we would like to relax the condition that X be a subspace of Y. Specifically, we might attempt to define a linear transformation $L : W_2^1(\Omega) \to L_2(\partial\Omega)$, by $Lf = f|_{\partial\Omega}$. If there exists an M such that

$$\|Lf\|_{L_2(\partial\Omega)} \leq M \|f\|_{2,2}$$

for all $f \in W_2^2(\Omega)$, then the transformation is well defined, and we can feel comfortable about considering the restrictions of functions in Sobolev spaces to the boundaries of their domains. If a linear transformation has the property that it maps every bounded set in its domain onto a **precompact** set in its range, then we say that the transformation is **compact**. A precompact set is one whose closure is compact. If the linear transformation that induces an imbedding is compact, then we say that the imbedding is compact. For one accustomed to working in finite dimensional spaces, the definition of a compact linear transformation may seem somewhat vacuous. We note however, that in an infinite dimensional Banach space, the closed unit sphere is bounded but not compact.

If a bounded domain Ω has a locally Lipschitz boundary then it follows that Ω also has the **cone property**. For bounded domains, the cone property is the requirement that each point x on the boundary should be the vertex of a finite and non-degenerate cone C_x contained in $\bar{\Omega}$. If $l > \frac{n}{2}$, and $\frac{n}{2} < m \leq l$, then by the Rellich-Kondrašov theorem (see Adams [1] p. 144), the imbedding $W_2^l(\Omega) \to C^{l-m}(\bar{\Omega})$ exists and is compact. In general, the existence of an imbedding means that the first space be a subspace of the second. In this case, however, $W_2^l(\Omega)$ consists of equivalence classes of functions, while $C^{l-m}(\bar{\Omega})$ consists of functions, so such a containment is not meaningful. Instead, the imbedding actually guarantees that in each equivalence class in $W_2^l(\Omega)$ there is a function which is in $C^{l-m}(\bar{\Omega})$. In other words, any function for which $\|\cdot\|_{l,2}$ is bounded can be modified on a set of measure zero to be in $C^{l-m}(\bar{\Omega})$. The imbedding is a continuous, linear operator, so there is a constant M such that for all $f \in W_2^m(\bar{\Omega})$, $\|f\|_{C^{l-m}(\bar{\Omega})} \leq M \|f\|_{l,2}$. The compactness of the imbedding means that a bounded set A in $W_2^l(\Omega)$ corresponds to a precompact set B in $C^{l-m}(\bar{\Omega})$.

If on the other hand, $l \leq \frac{n}{2}$, then the imbedding is not quite as strong. By Sobolev's Imbedding Theorem, Theorem 4.26 in Adams [1], we have, if Ω^k is the intersection of Ω with a hyperplane of dimension k with $n - 2l < k \leq n$,

$$W_2^l(\Omega) \to L_q(\Omega^k).$$

Here q is allowed to take on any value from 2 to $2k/(n-2l)$, the upper bound being increased to ∞ in the event that $l = \frac{n}{2}$.

The meaning of this embedding is somewhat opaque without further justification. In general, the restrictions of two functions in an equivalence class of $W_2^l(\Omega)$ to some hyperplanar section will not correspond in $L_q\left(\Omega^k\right)$, since the points of Ω^k for $k < n$ constitute a set of zero measure in R^n. We could therefore modify $\phi \in W_2^l(\Omega)$ to be almost anything in Ω^k without changing $\|\phi\|_{l,2}$ but drastically changing

$\|\phi\|_{L_q(\Omega^k)}$. We remedy this by recalling that $C^\infty(\Omega)$ is dense in $W_2^l(\Omega)$. Suppose that $\{f_j\} \subset C^\infty(\Omega)$ converges in $W_2^l(\Omega)$ to a function ϕ. The imbedding tells us that the restrictions $f_j^k = f_j|_{\Omega^k}$, which are in $C^\infty\left(\Omega^k\right)$, converge in $L_q\left(\Omega^k\right)$ to some function $\tilde{\phi}$ (independent of the choice of sequence), and that there exists a constant K, independent of ϕ such that

$$\left\|\tilde{\phi}\right\|_{L_q\left(\Omega^k\right)} \leq K \|\phi\|_{l,2}.$$

This imbedding can be used to show that, under certain conditions on $\partial\Omega$, $W_2^l(\Omega) \rightarrow L_q(\partial\Omega)$ provided of course that l, q, and n satisfy the appropriate inequalities. Here we first introduce the concept of an **extension operator.** For our purposes, an extension operator E is a linear operator mapping $W_2^l(\Omega)$ into $W_2^l(R^n)$, with the properties that $\|E\phi\|_{l,2,R^n} \leq K \|\phi\|_{l,2,\Omega}$ for some K (independent of ϕ), and $E\phi = \phi$ almost everywhere in Ω, for every $\phi \in W_2^l(\Omega)$. For general domains Ω (bounded or unbounded) the existence of such an operator depends heavily on the nature of the boundary. In Chapter 3 of Stein [32] we find a fairly extensive treatment of extension operators for a variety of spaces. In particular, Section 3 contains a constructive proof of the existence of an operator of exactly the type we require. This existence requires that the boundary of Ω is **minimally smooth.** By this we mean that there exists an $\epsilon > 0$, an $M > 0$ and a sequence $\{U_i\}_{i=1}^{\infty}$ of open sets such that

1. for each $x \in \partial\Omega$, the ball of radius ϵ centred at x is contained in at least one set U_i,

2. $\{U_i\}$ is a **locally finite cover** of $\partial\Omega$, that is, $\bigcap_{i=1}^{\infty} U_i = \emptyset$, but $\bigcup_{i=1}^{\infty} U_i \supset \partial\Omega$, and

3. within each U_j, there exists a Cartesian coordinate system within which $\partial\Omega$ is the graph of a Lipschitz continuous function $f_j(\xi_1, ..., \xi_{n-1})$ for which the Lipschitz constant does not exceed M. Moreover, Ω should lie on only one side of its boundary, that is, for each j, $U_j \cap \Omega = \{\xi \in R^n | \xi_n < f_j(\xi_1, \ldots, \xi_{n-1})\}$.

It is clear that for a bounded domain Ω the insistence that $\partial\Omega$ is locally Lipschitz and piecewise l-smooth implies that $\partial\Omega$ is minimally smooth. Thus armed with an extension operator E and an associated constant K, we proceed to sketch the proof of the imbedding of $W_2^l(\Omega)$ into $L_q(\partial\Omega)$ which we give specifically for $\Omega \subset R^2$ with a locally Lipschitz, piecewise l-smooth boundary.

Let $\{V_i\}_{i=1}^{P}$ be a finite collection of open balls which almost cover $\partial\Omega$. Specifically, let them be chosen so that the only points of $\partial\Omega$ not covered are those at which the tangent fails to exist. In addition, we insist that each V_i is small enough that $V_i \cap \partial\Omega$ is connected. There exists a constant M such that for each V_i there exists a l–diffeomorphism T_i mapping V_i to a neighbourhood N_i of the origin such that

1. $T_i(V_i \cap \partial\Omega) = N_i \cap \{x \,|\, x = (0, x_2) \in R^2\} = \{0\} \times (a_i, b_i)$, for appropriate a_i and b_i,

2. $T_i(V_i \cap \Omega) = N_i \cap \{x | x_1 > 0\}$, and

3. the partial derivatives up to order l of the components of T_i and its inverse are bounded in absolute value by M.

The construction of such diffeomorphisms is performed for $l = 1$ in the proofs of Lemmas (5.4) and (5.5). An analogous construction yields the result for $l > 1$.

The boundary of each ball V_i is smooth, so the boundary of its image under T_i will be at least l-smooth, which allows us to use the imbedding theorem as stated above for lines intersecting N_i.

We rely on the extension operator E to give us values for ϕ on R^2. We can write,

$$\int_{\partial\Omega} |E\phi(x)|^q \, d\sigma \leq \sum_{i=1}^{P} \int_{V_i \cap \partial\Omega} |E\phi(x)|^q \, d\sigma.$$

Letting $y = T_i(x) = (y_1, y_2)$, for $x \in V_i$, and $y' = T_i(x) = (0, y_2)$ for $x \in V_i \cap \partial\Omega$, we obtain

$$\sum_{i=1}^{P} \int_{V_i \cap \partial\Omega} |E\phi(x)|^q \, d\sigma \leq \sum_{i=1}^{P} \int_{a_i}^{b_i} \left|E\phi\left(T_i^{-1}(y')\right)\right|^q \cdot \left\| \left(\frac{\partial x_1}{\partial y_2}, \frac{\partial x_2}{\partial y_2} \right) \right\| dy_2.$$

Since the partial derivatives up to order l of the components of the transformations T_i^{-1} are bounded by M,

$$\int_{\partial\Omega} |E\phi(x)|^q \, d\sigma \leq M \sum_{i=1}^{P} \int_{a_i}^{b_i} \left|E\phi\left(T_i^{-1}(y')\right)\right|^q dy_2 \leq M \sum_{i=1}^{P} \left\|E\phi \circ T_i^{-1}\right\|_{L_q(a_i,b_i)}^q$$

Using the embedding of $W_2^l(N_i)$ into $L_q(a_i, b_i)$ for $2 \leq q \leq 2k/(n-2l)$ and the bounds on the partial derivatives of T_i, there exist constants K_1 and K_2 such that for $i = 1, \ldots, P$,

$$\left\|E\phi \circ T_i^{-1}\right\|_{L_q(a_i,b_i)}^q \leq K_1 \left(\left\|E\phi \circ T_i^{-1}\right\|_{l,2,N_i}^2 \right)^{\frac{q}{2}} \leq K_2 \left(\|E\phi\|_{l,2,V_i}^2 \right)^{\frac{q}{2}}.$$

Now, since $\frac{q}{2} \geq 1$,

$$\int_{\partial\Omega} |E\phi(x)|^q \, d\sigma \leq K_2 \sum_{i=1}^{P} \left(\|E\phi\|_{l,2,V_i}^2 \right)^{\frac{q}{2}} \leq K_2 \left(\sum_{i=1}^{P} \|E\phi\|_{l,2,V_i}^2 \right)^{\frac{q}{2}}$$

$$\leq K_2 \left(P \, \|E\phi\|_{l,2,R^n}^2 \right)^{\frac{q}{2}} \leq K \, \|E\phi\|_{l,2,\Omega}^q \, .$$

We note that the corners of $\partial\Omega$ constitute a set of zero measure in the integral over $\partial\Omega$, so the fact that we exclude them does not disturb the process. This proof is a modification of the proof of Theorem 5.22 in Adams [1]. The main difference being a relaxation of the requirements on $\partial\Omega$.

3.3 Other Results in Sobolev Spaces

A brief search through the literature concerning Sobolev spaces turns up a wide variety of notations and definitions. For example, in texts about the theory of finite elements such as Ciarlet [7] and Oden and Reddy [25], the space $W_2^l(\Omega)$ is usually denoted by $H^l(\Omega)$. On the other hand, in [21], Maz'ja uses the symbol $W_p^l(\Omega)$ to denote the space of functions in L_p whose generalized derivatives of exactly order l are also in L_p. Even for $p = 2$, this is not necessarily equivalent to our space $W_2^l(\Omega)$, since in general the integrability of higher order generalized derivatives does not necessarily imply the integrability of lower order ones (see Example 2, Section 1.5.1 in Sobolev [31]). We wish to use some of the results presented in Maz'ja [21], so we introduce the notation used therein, except that whenever symbols do not coincide in definition with those we have already defined, we use a calligraphic symbol instead.

By $L_2^l(\Omega)$ we denote the space of distributions on Ω whose derivatives of order

l are members of $L_2(\Omega)$. Some authors refer to this space as a 'Beppo-Levi space'. We also define a subspace of $L_2^l(\Omega)$, namely $\mathcal{W}_2^l(\Omega) = L_2^l(\Omega) \cap L_2(\Omega)$ and equip it with the norm

$$\|\phi\|_{\mathcal{W}_2^l(\Omega)} = |\phi|_l + \|\phi\|_2 .$$

If Ω is bounded and has the cone property, then the three spaces $L_2^l(\Omega)$, $\mathcal{W}_2^l(\Omega)$, and $W_2^l(\Omega)$ coincide. That is, they consist of exactly the same functions. This correspondence is taken without regard to the equivalence classes generated by the norms. This follows from the Generalized Poincaré Inequality (Maz'ja [21] pp. 22), which is stated as follows.

Lemma 3.1 *Let Ω be a bounded domain with the cone property, and let ω be an arbitrary open subset with closure contained in Ω. Then for any $u \in L_2^l(\Omega)$, there exists a polynomial*

$$\Pi(x) = \sum_{|\alpha|<l} (u, \phi_\alpha)\, x^\alpha,$$

such that

$$\sum_{k=0}^{l-1} |u - \Pi|_k \leq C\, |u|_l .$$

Here, ϕ_α are fixed test functions with support restricted to ω, and C is a constant independent of u. The inner product $(.,.)$ is the standard inner product in $L_2(\omega)$.

Proof. For a proof, see Maz'ja [21]. □

The inclusions $W_2^l(\Omega) \subseteq \mathcal{W}_2^l(\Omega) \subseteq L_2^l(\Omega)$, are already clear from the definitions of the spaces. Thus to show that these spaces coincide, it suffices to show that $L_2^l(\Omega) \subseteq W_2^l(\Omega)$. Suppose $u \in L_2^l(\Omega)$, then not only is $|u|_l$ finite, but by the Lemma, so are each of $|u - \Pi|_k$, for $0 \leq k < l$. Since Ω is bounded, $\Pi \in L_2^k(\Omega)$, so

$|u|_k = |u - \Pi + \Pi|_k \leq |u - \Pi|_k + |\Pi|_k < \infty$, and thus $\|u\|_{W_2^l(\Omega)} < \infty$. We conclude therefore that $u \in W_2^l(\Omega)$.

In addition, we note that $\mathcal{W}_2^l(\Omega)$ is complete (Theorem 1.1.12 Maz'ja [21])

We recall that any two functions differing by a polynomial of degree less than l will not be distinguishable by the semi-norm $|\cdot|_l$. Consider, therefore, the factor space $\mathcal{L}_2^l(\Omega) = L_2^l(\Omega)/\mathcal{P}_{l-1}$, where $\mathcal{P}_{l-1}$ is the space of polynomials of degree less that l. On this new space, the elements of which are equivalence classes of the form $\mathbf{u} = \left\{u + \Pi \mid u \in L_2^l(\Omega)\text{, and } \Pi \in \mathcal{P}_{l-1}\right\}$, $|\cdot|_l$ is a norm. By Theorem 1.1.13 in Maz'ja [21] $\mathcal{L}_2^l(\Omega)$ with the norm $|\cdot|_l$ is complete. This leads us to a theorem on equivalent norms on Sobolev spaces.

Theorem 3.2 *Let Ω be a bounded domain with the cone property, and $\mathcal{F}$ a continuous functional on $\mathcal{W}_2^l(\Omega)$ with the following properties*

1. *F does not vanish for any non-zero polynomials of degree less than l,*
2. *for any constant a, and any $\phi \in \mathcal{W}_2^l(\Omega)$, $F(a\phi) = aF(\phi)$, and*
3. *for any $\phi, \psi \in \mathcal{W}_2^l(\Omega), |F(\phi + \psi)| \leq |F(\phi)| + |F(\psi)|$*

The norm generated by

$$|\cdot|_l + |\mathcal{F}| \tag{3.2}$$

is equivalent to the standard norm in $\mathcal{W}_2^l(\Omega)$.

The statement of this proof is essentially identical to that of Theorem 1.1.15 in Maz'ja, with a few technical clarifications.

Proof. Let $B(\Omega)$ denote the completion of $\mathcal{W}_2^l(\Omega)$ with respect to the new norm (3.2). Now, suppose $\{u_i\}_{i=0}^{\infty}$ is a Cauchy sequence in $B(\Omega)$. This means that $|u_i - u_j|_l + |\mathcal{F}(u_i - u_j)| \to 0$ as $i, j \to \infty$, which implies that $|u_i - u_j|_l \to 0$. Since $\mathcal{L}_2^l(\Omega)$ is complete, u_i converges to some element u in an equivalence class of $\mathcal{L}_2^l(\Omega)$, that is, $u \in L_2^l(\Omega)$. Since the domain has the cone property, we already know that if $u \in \mathcal{W}_2^l(\Omega)$ then $u \in L_2^l(\Omega)$, so $B \subseteq L_2^l(\Omega) = \mathcal{W}_2^l(\Omega)$. We conclude that $\mathcal{W}_2^l(\Omega)$ is complete with respect to the new norm. The Identity mapping I : $\left(\mathcal{W}_2^l(\Omega), \|\cdot\|_{\mathcal{W}_2^l(\Omega)}\right) \to \left(\mathcal{W}_2^l(\Omega), |\cdot|_{2,l} + |\mathcal{F}|\right)$ is therefore a linear bijection between Banach spaces. By the continuity of the functional $\mathcal{F}$,

$$|u|_l + |\mathcal{F}(u)| \le C \|u\|_{\mathcal{W}_2^l(\Omega)},$$

for all $u \in \mathcal{W}_2^l(\Omega)$, and some constant $C > 0$, so I is also continuous. We apply the Open Mapping Theorem (cf. Zeidler [34] page 777), to conclude that the inverse mapping I^{-1} is also continuous, so that there exists a $D > 0$ such that

$$\|u\|_{\mathcal{W}_2^l(\Omega)} \le D\left(|u|_l + |\mathcal{F}(u)|\right).$$

The two norms are therefore equivalent. □

Since the two spaces $W_2^1(\Omega)$ and $\mathcal{W}_2^1(\Omega)$ are, in fact, the same space with the same norm, we conclude that any norm of the form (3.2) is equivalent to the standard norm on $W_2^1(\Omega)$ as well. We will make good use of this equivalence in subsequent chapters.

3.4 Equivalent Spaces

In some cases, problems posed in terms of Sobolev spaces can be approached by an indirect method, more specifically through a density argument of some sort. In such a case we might show that a property holds for a class of smooth functions, show that this space is dense in the Sobolev space in question, and conclude from this that the property holds in the latter as well. To this end we introduce some spaces which are equivalent to certain Sobolev spaces. In the following, we assume that Ω satisfies the cone condition.

Let $H_2^m(\Omega)$ denote the completion of $C^m(\Omega) \cap W_2^m(\Omega)$ in W_2^m. Clearly $H_2^m(\Omega) \subseteq W_2^m(\Omega)$ by virtue of the completeness of $W_2^m(\Omega)$. The reverse containment follows from the density of $C_0^\infty(R^n)$ in $W_2^m(\Omega)$, and the density of $C^m\left(\bar{\Omega}\right)$ in $C_0^\infty(R^n)$ with respect to the norm $\|\cdot\|_{m,p}$. We have, therefore, $H_2^m(\Omega) = W_2^m(\Omega)$.

One might wonder whether it is reasonable, for any $f \in W_2^1(\Omega)$, to expect the existence of a sequence of functions say $\phi_i \in C^\infty\left(\bar{\Omega}\right)$ all having the same boundary values on $\partial\Omega$ as f that converges to f. While this might at first appear obvious, we hasten to point out that the density of the subspace of smooth functions says nothing about what boundary values an approximating sequence must have. Luckily, some conclusions can be drawn in this regard.

We denote by $W_0^{m,2}(\Omega)$ the completion of $C_0^\infty(\Omega) \cap W_2^m(\Omega)$ with respect to the norm $\|\cdot\|_{m,2}$. We shall let $W_0(\Omega) = \{f \in W_2^1(\Omega) \mid f|_{\partial\Omega} = 0\}$, remembering that by $f|_{\partial\Omega} = 0$ we mean that $f(x) = 0$ for almost all $x \in \partial\Omega$. We have the following theorem which is found in Kufner et al. [13], Section 6.6.

Theorem 3.3 *For a domain Ω having a locally Lipschitz boundary, the spaces $W_0(\Omega)$*

and $W_0^{1,2}(\Omega)$ *are equivalent.*

Proof. We first show that $W_0^{1,2}(\Omega) \subseteq W_0(\Omega)$. By virtue of the definition of the space , there exists, for any $\phi \in W_0^{1,2}(\Omega)$ and any $\epsilon > 0$, a function $\psi \in C_0^\infty$, such that $\psi|_{\partial\Omega} = 0$, and

$$\|\phi - \psi\|_{1,2} < \epsilon.$$

This, along with the imbedding of $W_0^{1,2}(\Omega) \to L_2(\partial\Omega)$ gives us

$$\|\phi|_{\partial\Omega} - \psi|_{\partial\Omega}\|_{L_2(\partial\Omega)} = \|\phi|_{\partial\Omega}\|_{L_2(\partial\Omega)} \leq K\,\|\phi - \psi\|_{1,2} < K\epsilon,$$

for some positive constant K. The arbitrariness of ϵ allows us to conclude that ϕ vanishes almost everywhere on $\partial\Omega$, so $\phi \in W_0(\Omega)$.

We sketch the reverse containment $W_0(\Omega) \subseteq W_0^{1,2}(\Omega)$ which can be shown through an argument that takes advantage of the properties of the boundary. We first pick any particular $\psi \in W_0(\Omega)$. Given that we can find a finite number of bounded regions U_j covering $\partial\Omega$, each with the property that $U_j \cap \partial\Omega$ can be described by a Lipschitz continuous function in a local coordinate system, we consider the less formidable task of finding, for each U_j, a sequence of functions in $\phi_{ji} \in W_0^{1,2}(\Omega)$ such that $\operatorname{supp}\phi_{ji} \subset U_j \cap \bar{\Omega}$. Using a partition of unity $\{u_j\}$ subordinate to the sets U_j, we can combine these sequences to obtain another function

$$\phi_i = \sum u_j \phi_{ji},$$

which converges to ψ. For the detailed proof, see Kufner et al. [13], p. 326. □

3.5 Rules of Differentiation

As mentioned in Section (2.3.3) the product rule applies to generalized derivatives. We must proceed with caution, because even if we do not restrict ourselves to considering regular distributions, the product of two distributions is not, in general, a distribution. However, if $f, g \in W_2^l(\Omega)$ and $l > 0$, then any first order generalized derivative of their product is in $W_2^{l-1}(\Omega)$, moreover

$$D^\alpha (fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^\beta f \cdot D^\gamma g, \text{ for } |\alpha| \leq l.$$

To see this, consider the mollifications of f and g, namely f_ϵ and g_ϵ for $\epsilon > 0$, and suppose $|\alpha| = 1$. Recalling that $f_\epsilon \to f$ and $g_\epsilon \to g$ in $L_2(\Omega')$, where Ω' is such that $\bar{\Omega}' \subset \Omega$, we notice that for any $\phi \in C_0^\infty(\Omega)$, bounded in absolute value, along with all of its partial derivatives, by 1,

$$\begin{aligned}
\left| \int_\Omega (fg - f_\epsilon g_\epsilon) D^\alpha \phi dx \right| &\leq \int_{\mathrm{supp}(\phi)} |fg - f_\epsilon g_\epsilon| \, dx \\
&= \int_{\mathrm{supp}(\phi)} |f(g - g_\epsilon) + g_\epsilon (f - f_\epsilon)| \, dx \\
&\leq \int_{\mathrm{supp}(\phi)} |f| \cdot |g - g_\epsilon| \, dx + \int_{\mathrm{supp}(\phi)} |g_\epsilon| \cdot |f - f_\epsilon| \, dx \\
&\leq \|f\|_{L_2(\mathrm{supp}(\phi))} \cdot \|g - g_\epsilon\|_{L_2(\mathrm{supp}(\phi))} \\
&\quad + \|g_\epsilon\|_{L_2(\mathrm{supp}(\phi))} \cdot \|f - f_\epsilon\|_{L_2(\mathrm{supp}(\phi))},
\end{aligned}$$

which tends to 0 as $\epsilon \to 0^+$. Since, for $|\alpha| \leq l$, $D^\alpha f$, and $D^\alpha g$ are also in $L_2(\Omega)$, a similar result holds. In general, if $|\beta + \gamma| \leq l$, then as $\epsilon \to 0^+$,

$$\int_\Omega D^\beta (f_\epsilon) \cdot D^\gamma (g_\epsilon)\, \phi dx \to \int_\Omega \phi D^\beta f \cdot D^\gamma g dx.$$

Applying the classical product rule, we obtain,

$$D^\alpha (f_\epsilon \cdot g_\epsilon) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^\beta (f_\epsilon) \cdot D^\gamma (g_\epsilon),$$

so that

$$\int_\Omega \phi D^\alpha (f_\epsilon \cdot g_\epsilon)\, dx = \int_\Omega \phi \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^\beta (f_\epsilon) \cdot D^\gamma (g_\epsilon)\, dx.$$

Letting $\epsilon \to 0^+$, we get

$$\int_\Omega \phi D^\alpha (f \cdot g)\, dx = \int_\Omega \phi \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^\beta f \cdot D^\gamma g dx.$$

In the following, we make use of integrals around the boundary of the domain Ω. So that these make sense, we shall insist that $\partial\Omega$ is piecewise 1-smooth and locally Lipschitz.

For any $f, g \in W_2^1(\Omega)$, the integration by parts formula

$$\int_\Omega \frac{\partial f}{\partial x_i} g dx = \int_{\partial\Omega} f g n_i d\sigma - \int_\Omega f \frac{\partial g}{\partial x_i} dx,$$

where n_i are the components of $\bar{n}$, the outward facing normal vector to $\partial\Omega$, is valid. This follows readily from the Divergence Theorem (see, for example [20]). Similarly, if $f \in W_2^2(\Omega)$, then

$$\int_{\Omega} \frac{\partial^2 f}{\partial x_i^2} g dx = \int_{\partial\Omega} \frac{\partial f}{\partial x_i} g n_i d\sigma - \int_{\Omega} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} dx,$$

so that, summing over $i = 1, \ldots, n$, we obtain one of Green's fundamental formulae,

$$\int_{\Omega} g \Delta f dx = \int_{\partial\Omega} g \nabla f \cdot n d\sigma - \int_{\Omega} \nabla g \cdot \nabla f dx.$$

Here Δ symbolizes the differential operator

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

In the same way it can be shown that if $f \in W_2^4(\Omega)$ and $g \in W_2^2(\Omega)$ then

$$\begin{aligned}\int_{\Omega} g \Delta^2 f dx &= \int_{\partial\Omega} \left(g\left(\nabla\left(\Delta f\right)\right) - Hf\nabla g\right) \cdot \vec{n} d\sigma \\ &+ \int_{\Omega} \left(\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 g}{\partial x_1^2} + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial^2 g}{\partial x_1 \partial x_2} + \frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 g}{\partial x_2^2}\right) dx.\end{aligned}$$

Here, Hf is the Hessian matrix of f, that is,

$$Hf = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix}.$$

Chapter 4

Univariate Splines

The first question that arises is 'What is a **spline?**'. The answer is two-fold. The original splines were thin elastic rods used in drafting. If a draftsman wished to trace out a curve, he would simply lay down some weights in appropriate places, and use them to hold a spline in place while he traced along it. This system intimates some notion of the drawn curve having minimal bending energy. As mathematical objects, **cubic** splines are models of the original mechanical spline, minimizing a linearized form of the true bending energy. The linearization is based on the assumption that the deflections are 'small'. The resulting functional is the square of the semi-norm $|\cdot|_2$. In general, splines of different orders are functions which minimize functionals that resemble this 'energy' functional.

In the univariate case, the resulting curves are, perhaps surprisingly, the graphs of **piecewise-polynomial** functions, subject to some constraints. The constraints come in a variety of forms, but here the splines we consider are interpolants of supplied data, subject to continuity, smoothness, and end conditions.

4.1 Polynomial splines

We use the following definitions. Given a sequence of real abscissae $\{x_i\}_{i=0}^{n}$, where for convenience we assume $x_i < x_j$ whenever $i < j$, we say that any function that is a polynomial of degree m on each interval (x_i, x_{i+1}), $i = 0, \ldots, n-1$ is **piecewise-**

polynomial of degree m.

Definition 4.1 *An* **interpolating polynomial spline of degree m** *is a piecewise-polynomial function S of degree m with the additional properties that S is continuous with its derivatives up to order $m-1$ on (x_0, x_n), and that for some specified real values $\{y_i\}_{i=0}^n$, $S(x_i) = y_i$, $i = 0, \ldots, n$.*

Where it causes no confusion, interpolating polynomial splines of degree 1 are called **linear splines**, splines of degree 3 are called **cubic splines**, etc. It turns out that the optimality conditions which we impose exclude the splines of even degree, and so we focus our attention on those of odd degree. It should be noted that while the even degree splines do not fit in with the theory presented here, they do play a part in the theory of **weighted** splines [6].

Having defined splines, we should verify their existence and, with some additional constraints, uniqueness.

The existence of interpolating polynomial splines is guaranteed by the following theorem.

Theorem 4.2 *Given a set of abscissae $\{x_i\}_{i=0}^n$ with $x_i < x_j$ whenever $i < j$, and a set of ordinates $\{y_i\}_{i=0}^n$ there always exists a polynomial spline $S(x)$ of degree m that satisfies $S(x_i) = y_i$, $i = 0, \ldots, n$.*

Proof. Associate with each interval $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, the Taylor polynomial

$$p_i(x) = \sum_{j=0}^{m} a_{ij}(x - x_i)^j,$$

defined on the real line, where $a_{ij} = \dfrac{p_i^{(j)}(x_i)}{j!}$. If we specify the values of the derivatives $p_i^{(j)}(x_i)$, $j = 0, \ldots, m-1$, and $p_i(x_{i+1}) = y_{i+1}$, this uniquely determines p_i, since

$$p_i^{(m)}(x_i) = m! \frac{y_{i+1} - \displaystyle\sum_{j=0}^{m-1} \frac{p_i^{(j)}(x_i)}{j!}(x_{i+1} - x_i)^j}{(x_{i+1} - x_i)^m}.$$

For the moment, let us assign arbitrary values to $p_0^{(j)}(x_0)$, $j = 1, \ldots, m-1$. These, along with the values y_0 and y_1 uniquely determine p_1, and hence $p_0^{(j)}(x_1)$, $j = 1, \ldots, m-1$, are uniquely determined. Using these derivatives and $p_1(x_2) = y_2$, we can construct p_1, and so on all the way to p_{n-1}. We can now define our spline to be

$$S(x) = p_i(x), \text{ if } x_i \leq x < x_{i+1},\ i = 0, \ldots, n-1 \text{ and}$$
$$S(x_n) = y_n.$$

□

The arbitrariness of our choice for the derivatives at the left-hand end of the spline clearly rules out uniqueness unless we impose more conditions. In the mechanical spline, we might expect that the free ends which are no longer constrained by ducks should in fact be straight, that is, they should have zero curvature. This, for the cubic spline, translates into $S''(x) = 0$, for $x \notin (x_0, x_n)$. This leads us to the **natural end-conditions.**

Definition 4.3 *A polynomial spline of degree $2k-1$ satisfying the natural end-conditions*

$$S^{(j)}(x_0) = S^{(j)}(x_n) = 0,\ j = k, \ldots, 2k-2$$

is called a **natural spline.**

Once again, we encounter the question of existence. Fortunately, natural splines do exist, and they are unique provided that the data satisfy certain conditions.

4.2 Optimality and Existence of Natural Splines

We begin by showing that the natural splines, provided they exist and are of appropriate degree, are **optimal** in that they minimize the semi-norm $|\cdot|_k$ over a subspace of $W_2^k(x_0, x_n)$. The particular subspace is the set of functions in $W_2^k(x_0, x_n)$ which satisfy the given set of interpolation conditions. It is at this point useful to note that the rather opaque definition of the space $W_2^k(\Omega)$ can be simplified considerably when $\Omega \subset R$. The Rellich–Kondrašov embedding theorem guarantees that if $f \in W_2^k(x_0, x_n)$ then f differs on a set of measure zero from a function in $C^{k-1}[x_0, x_n]$. In addition, as is pointed out by Sobolev on page 32 of [31], for the generalized derivative of a function ϕ on a subset Ω of R to be in $L_2(\Omega)$, ϕ must be absolutely continuous. This leads us to the alternate definition

$$W_2^k(x_0, x_n) = \left\{ f | f \in C^{k-1}[x_0, x_n],\, f^{(k-1)} \text{ is absolutely continuous, and } f^{(k)} \in L_2 \right\}.$$

This is exactly the space which has been associated with splines since their invention, although the symbols used are often different. For example, in [2], Ahlberg et. al. use the symbol $\mathcal{K}^n(x_0, x_n)$ to denote this class of functions.

The following lemma will prove useful in showing the optimality of splines.

Lemma 4.4 *Given the abscissae* $\{x_i\}_{i=0}^n$ *with* $x_i < x_j$ *whenever* $i < j$, *and ordinates* $\{y_i\}_{i=0}^n$, *if* $S(x)$ *is a natural spline of degree* $2k-1$ *satisfying the interpolation conditions*

$$S(x_i) = y_i, \text{ for } i = 0, \ldots, n,$$

then if $z \in W_2^k(x_0, x_n)$ *and* z *is an* **interpolant of zero data**, *i.e.* $z(x_i) = 0$, $i = 0, \ldots, n$, *then the semi-inner product*

$$(S, z)_k = \int_{x_0}^{x_n} \left(S^{(k)}(x) z^{(k)}(x)\right) dx \tag{4.1}$$

vanishes.

Proof. We first verify that $S(x) \in W_2^k(x_0, x_n)$. Since $S(x)$ is a spline of degree $2k-1$, $S(x) \in C^{2k-2}[x_0, x_n] \subseteq C^k[x_0, x_n]$ if $k > 2$, and if $k = 1$, then $S'(x)$ is piecewise constant. Thus $S(x) \in W_2^k(x_0, x_n)$.

With regards to the semi-inner product, we apply integration by parts $k-1$ times. In the first application, we see

$$(S, z)_k = \int_{x_0}^{x_n} \left(S^{(k)}(x) z^{(k)}(x)\right) dx \tag{4.2}$$

$$= S^{(k)}(x) z^{(k-1)}(x)\Big|_{x_0}^{x_n} - \int_{x_0}^{x_n} \left(S^{(k+1)}(x) z^{(k-1)}(x)\right) dx \tag{4.3}$$

$$= -\int_{x_0}^{x_n} \left(S^{(k+1)}(x) z^{(k-1)}(x)\right) dx. \tag{4.4}$$

This integration by parts is justified since $S^{(k)}(x)$ is differentiable, and $z^{(k)}(x)$ is locally integrable. The left-hand portion of (4.3) vanishes since it involves the k^{th}

initial and terminal derivatives of S, which were assumed to be zero. This pattern repeats itself so that after integrating by parts $k-1$ times, (4.2) reduces to

$$\int_{x_0}^{x_n} \left(S^{(k)}(x)z^{(k)}(x)\right) dx = (-1)^k \int_{x_0}^{x_n} \left(S^{(2k-1)}(x)z'(x)\right) dx$$

$$\begin{aligned} &= (-1)^k \sum_{i=0}^{n-1} S^{(2k-1)}\Big|_{(x_i,x_{i+1})} \int_{x_i}^{x_{i+1}} (z'(x))\,dx \\ &= (-1)^k \sum_{i=0}^{n-1} S^{(2k-1)}(x_i^*)\,(z(x))_{x_i}^{x_{i+1}} = 0 \end{aligned} \tag{4.5}$$

since $S^{(2k-1)}(x)$ is piecewise constant ($x_i^* \in (x_i, x_{i+1})$) and $z(x_i) = 0$, $i = 0, \ldots, n$. □

Theorem 4.5 *Given the abscissae $\{x_i\}_{i=0}^n$ with $x_i < x_j$ whenever $i < j$, and ordinates $\{y_i\}_{i=0}^n$, if $1 \le k \le n+1$, then the natural spline $S(x)$ of degree $2k-1$ satisfying the interpolation conditions*

$$S(x_i) = y_i, \textit{ for } i = 0, \ldots, n$$

exists, is unique, and is optimal in that it minimizes the semi-norm $|\cdot|_k$ over the subspace of $W_2^k(x_0, x_n)$ of functions which satisfy the interpolation conditions.

Proof. Perhaps the most natural way to show the existence and uniqueness of natural splines is to write down a system of defining equations, and show that the system has a unique solution. We are searching, as before, for a set of n polynomials $p_i(x)$, which we hope to join together in a smooth fashion. We begin by writing down the $n+1$ interpolation conditions

$$p_i(x_i) = y_i, \text{ for } i = 0, \ldots, n.$$

The remaining equations arise from the $(2k-1)(n-1)$ smoothness conditions

$$p_i^{(j)}(x_i) - p_{i+1}^{(j)}(x_i) = 0, \text{ for } i = 0, \ldots, n-2 \text{ and } j = 0, \ldots, 2k-2,$$

and the $2k-2$ natural end-conditions

$$S^{(j)}(x_0) = S^{(j)}(x_n) = 0,\ j = k, \ldots, 2k-2.$$

This gives us a total of $2kn$ equations, and since we want n polynomials, each of degree $2k-1$, we wish to determine $2kn$ coefficients.

Now we have a square system of linear equations which we could write as

$$A\mathbf{x} = \mathbf{b}.$$

Where the vector $\mathbf{x}$ consists of the coefficients a_{ij} of the polynomials $p_i(x)$, the matrix A depends only on the abscissae x_i, and only the vector $\mathbf{b}$ depends on the ordinates y_i. Solving the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ is therefore equivalent to finding a natural spline which interpolates zero data at the given abscissae. This system clearly has a solution, namely the trivial one with associated spline $S(x) \equiv 0$. Now for a contradiction, suppose it has other solutions. Let $Z(x)$ be the spline associated with any one of those solutions. Then since Z is a polynomial spline and an interpolant of zero data, Lemma (4.4) tells us that $|Z|_k^2 = (Z,Z)_k = 0$. This means that Z is itself a polynomial of degree less than k. Since there are $n+1 \geq k$ abscissae at which Z is zero, Z must be identically zero. This is the contradiction we seek, proving the existence and uniqueness of the solution of the system $A\mathbf{x} = \mathbf{0}$, and hence that of $A\mathbf{x} = \mathbf{b}$.

Suppose f is any other function in $W_2^{(k)}(x_0, x_n)$ that satisfies the interpolation conditions. Then $|f|_k^2 = (f,f)_k = (f - S + S, f - S + S)_k = (f - S, f - S)_k + 2\,(f - S, S)_k + (S,S)_k$. Since $f - S$ is an interpolant of zero data, $(f - S, S)_k = 0$, and so $(f,f)_k = (f - S, f - S)_k + (S,S)_k$. This leads us to the conclusion that

$$|f|_k \geq |S|_k$$

so S is optimal. □

Chapter 5

Piecewise Harmonic Splines

We propose to generalize the univariate linear splines to the bivariate case. In this generalization, we view the interpolation conditions established in the preceding chapter as constraints on level curves. This extends nicely to the bivariate case where the notion of curves in the plane is much more natural. The increase in dimension also increases the complexity of the problem of finding an interpolant, and questions of existence and optimality are no longer as easily answered.

The problem we seek to solve is as follows. Given a contour map

$$\mathcal{M} = \{(\Gamma_i, C_i)\,|\,0 \le i \le N\}$$

and a bounded domain $\Omega \in R^2$, is there a function $\phi \in W_2^1(\Omega)$, such that

$$\phi(x) = C_i \text{ whenever } x \in \Gamma_i \cap \Omega, \tag{5.1}$$

and such that $|\phi|_1 \le |\psi|_1$ for any other $\psi \in W_2^1(\Omega)$ that satisfies the interpolation conditions (5.1)?

In the univariate case, we were able to work with fairly basic tools to show that the function which minimized $|\cdot|_k$ existed and was merely piecewise polynomial. The one property of polynomials that carries over to the higher dimensional problem is that if $S(x)$ is a polynomial spline of degree $2k-1$, then $S^{(2k)}(x) = 0$ for $x \in (x_i, x_{i+1})$,

$i = 0, \ldots, n-1$. In the bivariate case, when $k = 1$, this property is known as harmonicity, and instead of $\dfrac{d^2}{dx^2}$ the differential operator is the Laplacian:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

5.1 Dirichlet's Principle

In its original form, according to Courant in [[8]], 'Dirichlet's Principle' quite simply states that the variational problem of minimizing $|\cdot|_1$ over the subspace of functions having finite potential energy in $C^0(\Omega)$ with values on $\partial\Omega$ prescribed by a continuous function ψ is equivalent to solving the differential equation

$$\begin{gathered} \Delta\phi = 0, \text{ subject to} \\ \phi|_{\partial\Omega} = \psi. \end{gathered}$$

Unfortunately, this equivalence comes with several caveats. It turns out that in general, existence of a solution to either of the above problems is not enough to guarantee existence of a solution to the other. Moreover, the problem of minimization does not always have a solution.

These difficulties are overcome by the insistence that there exists at least one function which satisfies the boundary conditions.

A more precise statement of Dirichlet's Principle is as follows.

Theorem 5.1 *Given a bounded domain Ω in R^2 with a piecewise smooth and locally Lipschitz boundary $\partial\Omega$, and a real-valued, measurable function ψ on $\partial\Omega$, the problem of finding a function $\phi \in W_2^1(\Omega)$ for which $|\phi|_1$ attains a minimum value d and*

$\phi|_{\partial\Omega} = \psi$, *has a unique solution provided that there exists at least one function* $u \in W_2^1(\Omega)$ *with the property that* $u|_{\partial\Omega} = \psi$. *Moreover, this solution is* **harmonic** *in* Ω, *i.e.,* $\Delta\phi = 0$ *on* Ω.

Proof.

1. Existence of a solution

   Let W denote the set of functions in $W_2^1(\Omega)$ that have the boundary value ψ. By the hypotheses this set is not empty, and since each $\phi \in W$ is in $W_2^1(\Omega)$, $0 \le |\phi|_1 < \infty$. Let $d = \inf_{\phi \in W} |\phi|_1$. From the set W let $\{\phi_i\}_{i=0}^{\infty}$ be any sequence such that

   $$\lim_{i\to\infty} |\phi_i|_1 = d.$$

   We claim that this sequence converges in $W_2^1(\Omega)$. To show this, we first construct a norm, equivalent to the usual norm on $W_2^1(\Omega)$, and then we show that $\{\phi_i\}$ converges with respect to this new norm. Define a projector $\Pi_1 : W_2^1(\Omega) \to \mathcal{P}_0(\Omega)$, where $\mathcal{P}_0(\Omega)$ is the space of zero degree polynomials (constants) on Ω, by

   $$\Pi_1\phi = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \phi \, d\sigma.$$

   This integral makes sense because of the imbedding of $W_2^1(\Omega)$ into $L_2(\partial\Omega)$, and since $|\partial\Omega| < \infty$, $L_2(\Omega) \subset L_1(\partial\Omega)$. Here we use $|\cdot|$ to denote the length of $\partial\Omega$ or the surface area of Ω, respectively. Π_1 is clearly a projector since if $\Pi_1\phi = K_\phi$, a constant, then

   $$\Pi_1(\Pi_1\phi) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} K_\phi \, d\sigma = K_\phi.$$

The functional $(K)_0 = |\partial\Omega|\,|K|$ is a norm on $\mathcal{P}_0(\Omega)$, and so it follows that

$$((\phi))_1 = (\Pi_1\phi)_0 + |\phi|_1 = \left| \int_{\partial\Omega} \phi\, d\sigma \right| + |\phi|_1$$

is a norm on $W_2^1(\Omega)$. It remains to be shown that this norm is equivalent to the standard norm.

The imbedding $W_2^1(\Omega) \to L_2(\partial\Omega)$ holds, so there exists a constant C such that

$$\|\phi\|_{L_2(\partial\Omega)} \leq C\,\|\phi\|_{1,2}\,.$$

By the Hölder inequality

$$\|\phi\|_{L_1(\partial\Omega)} \leq |\partial\Omega|^{\frac{1}{2}}\,\|\phi\|_{L_2(\partial\Omega)}\,.$$

Absorbing the length of the boundary into the constant, we get

$$\left| \int_{\partial\Omega} \phi\, d\sigma \right| \leq C\,\|\phi\|_{1,2}\,.$$

Thus $\int_{\partial\Omega} \phi\, d\sigma$ is a continuous functional on $W_2^1(\Omega)$. By Theorem (3.2), it follows that $((\cdot))_1$ is equivalent to $\|\cdot\|_{l,2}$ on $W_2^1(\Omega)$.

Now, since any pair of functions $\phi_i,\ \phi_j \in W$ were chosen to agree on $\partial\Omega$, we have

$$((\phi_i - \phi_j))_1 = |\phi_i - \phi_j|_1\,.$$

From the definition of W, we already have, for any $\epsilon > 0$, an $N > 0$ such that

$$|\phi_i|_1^2 \leq d^2 + \epsilon, \text{ whenever } i > N.$$

Now, for $i, j > N$,

$$|\phi_i - \phi_j|_1^2 + |\phi_i + \phi_j|_1^2 = 2\int_\Omega \left(\frac{\partial \phi_i}{\partial x_1}\right)^2 + \left(\frac{\partial \phi_i}{\partial x_2}\right)^2 + \left(\frac{\partial \phi_j}{\partial x_1}\right)^2 + \left(\frac{\partial \phi_j}{\partial x_2}\right)^2 dx$$

$$= 2(|\phi_i|_1^2 + |\phi_j|_1^2) \leq 2\left(d^2 + \epsilon + d^2 + \epsilon\right) = 4d^2 + 4\epsilon.$$

Since $\dfrac{\phi_i + \phi_j}{2} \in W$, $\left|\dfrac{\phi_i + \phi_j}{2}\right|_1^2 \geq d^2$, so that

$$|\phi_i - \phi_j|_1^2 \leq 4d^2 + 4\epsilon - |\phi_i + \phi_j|_1^2$$

$$\leq 4d^2 + 4\epsilon - 4d^2$$

$$= 4\epsilon,$$

from which we conclude that $((\phi_i - \phi_j))_1 \to 0$ as $i, j \to \infty$. Since $W_2^1(\Omega)$ is complete, the sequence ϕ_i converges to some function $\phi_0 \in W_2^1(\Omega)$. We also require that $\phi_0 \in W$. From the imbedding again, and the equivalence of norms, it follows that there exists a constant K, depending only on Ω, such that

$$\int_{\partial\Omega} (\phi_i - \phi_0)^2 \, d\sigma \leq K\left((\phi_i - \phi_0)\right)_1^2.$$

This means that $\|\phi_i - \phi_0\|_{L_2(\partial\Omega)} \to 0$. Since $\phi_i|_{\partial\Omega} = \psi$, it follows from the completeness of $L_2(\partial\Omega)$ that $\phi_0|_{\partial\Omega} = \psi$, so $\phi_0 \in W$.

2. Smoothness and Harmonicity of the Solution

The underlying principle of this proof is reminiscent of the method we used to prove that cubic splines are optimal. First we claim that if $\xi \in W_2^1(\Omega)$ with $\xi|_{\partial\Omega} = 0$ then

$$(\phi_0, \xi)_1 = 0. \tag{5.2}$$

We notice that for $\epsilon \in R$, $\phi_i + \epsilon\xi \in W$, so

$$0 \leq |\phi_i + \epsilon\xi|_1^2 - d^2 = \epsilon^2 |\xi|_1^2 + 2\epsilon(\phi_i, \xi)_1 + |\phi_i|_1^2 - d^2.$$

The discriminant of this quadratic in ϵ is

$$4(\phi_i, \xi)_1^2 - 4|\xi|_1^2 \left(|\phi_i|_1^2 - d^2\right),$$

which is non-positive since the quadratic itself is non-negative. Thus,

$$(\phi_i, \xi)_1^2 \leq |\xi|_1 \left(|\phi_i|_1^2 - d^2\right) \to 0, \text{ as } i \to \infty,$$

proving our claim.

We now construct a particular function ξ such that the Laplacian of ξ can be written as the sum of two mollifiers, each dependent on a different parameter. The idea is to apply integration by parts, and using (5.2) we show that when the two mollifiers are applied to ϕ_0 the results are equal, independent of the choice of the parameters. The conclusion of course, is that ϕ_0 is equal to its mollifications, and is therefore in $C^\infty(\Omega)$.

Now, let $\psi \in C^\infty[0,\infty)$ be a monotone function such that $\psi(x) = 1$ for $0 \le x \le \frac{1}{2}$, $\psi(x) = 0$ for $x \ge 1$. Fix $\delta > 0$, and pick h_1 and h_2 such that $0 < h_1 < h_2 < \delta$. We now set, for $y \in \Omega_\delta = \{x | x \in \Omega \text{ and } \operatorname{dist}(x,\Omega) > \delta\}$ and $x \in R^2$,

$$\xi(x,y) = \ln\frac{1}{r}\left[\psi\left(\frac{r}{h_1}\right) - \psi\left(\frac{r}{h_2}\right)\right].$$

Here $r = \|x-y\|$. Figure (5.1) shows an example of a pair of functions ψ and ξ, $h_1 = 1$ and $h_2 = 2$. Since $\xi(x,y) = 0$ whenever $r < \frac{h_1}{2}$ or $r > \frac{h_2}{2}$, we have $\xi(x,y) \in C_0^\infty\left(\bar{\Omega}\right)$ for each y. This of course means that for each y, (5.2) holds, and so

$$(\phi_0(\cdot), \xi(\cdot,y))_1 = 0.$$

Since the partial derivatives in x_1 and x_2 of ξ are still in $C_0^\infty\left(\bar{\Omega}\right)$ for each y, the definition of the generalized derivative of ϕ_0 gives us

$$0 = (\phi_0(x), \xi(x,y))_1 = -\int_\Omega \phi_0 \Delta_x \xi \, dx. \tag{5.3}$$

Here we emphasize that the Laplacian of ξ is taken with y fixed, and thus write Δ_x instead of Δ. Let us examine this Laplacian in more detail.

$$\Delta_x\xi = \Delta_x\left(\psi\left(\frac{r}{h_1}\right)\ln\frac{1}{r}\right) - \Delta_x\left(\psi\left(\frac{r}{h_2}\right)\ln\frac{1}{r}\right)$$

$$= \Delta_x\left(\psi\left(\frac{r}{h_1}\right)\ln\frac{h_1}{r} - \psi\left(\frac{r}{h_1}\right)\ln h_1\right)$$

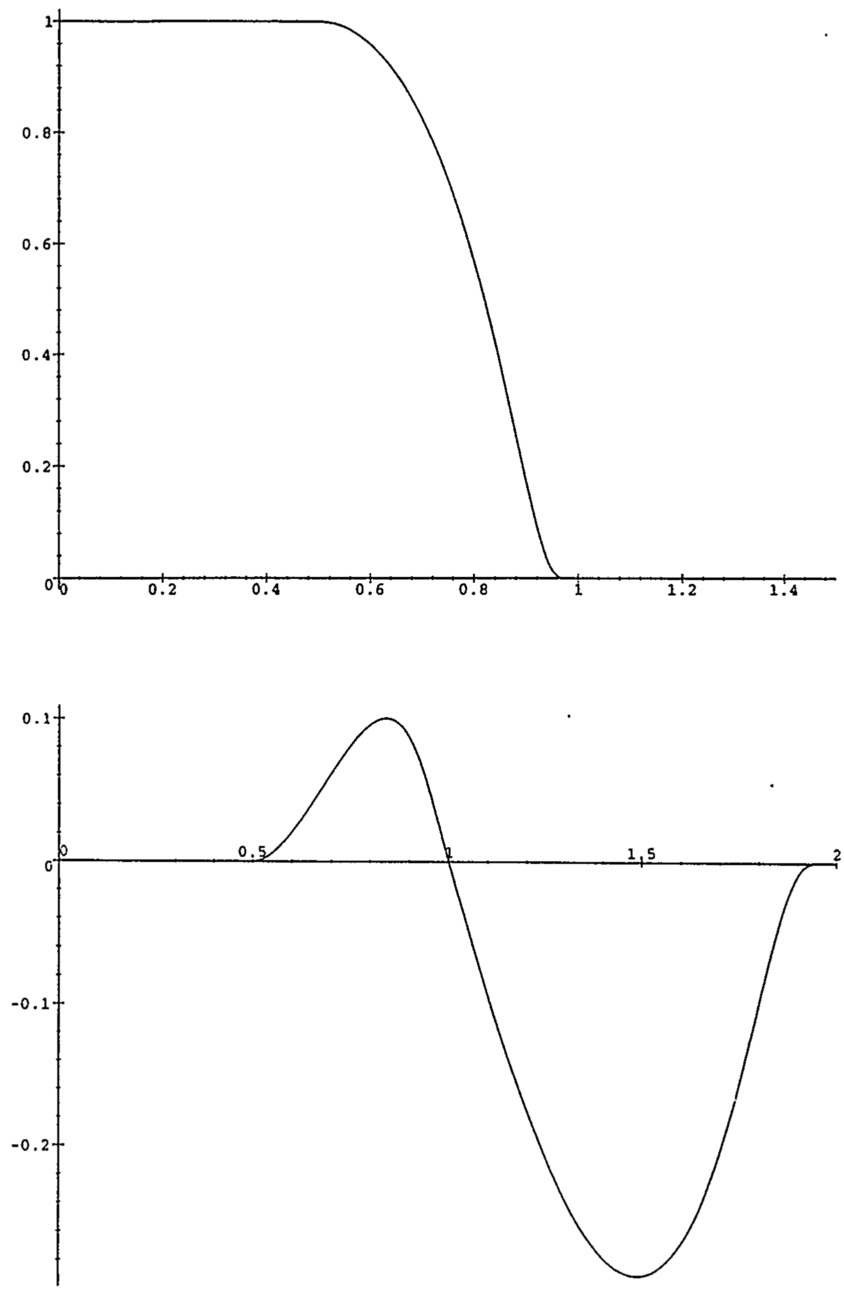

Figure 5.1: A monotone function ψ and associated function ξ

$$-\Delta_x\left(\psi\left(\frac{r}{h_2}\right)\ln\frac{h_2}{r}-\psi\left(\frac{r}{h_2}\right)\ln h_2\right).$$

Setting $\omega(r,h) = \Delta_x\left(\psi\left(\frac{r}{h}\right)\ln\frac{h}{r}-\psi\left(\frac{r}{h}\right)\ln h\right)$, we get $\Delta_x\xi = \omega(r,h_1) - \omega(r,h_2)$. We now quantify the smoothness of ω. Since $\psi\left(\frac{r}{h}\right)$ is constant for either $r < \frac{h}{2}$ or $r > h$, and

$$\Delta_x(\ln r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\ln r\right) = 0,$$

$\omega(r,h) = 0$ when $0 < r < \frac{h}{2}$ or $r > h$. If we define $\omega(0,h) = 0$, we see that ω has continuous derivatives of all orders, and has compact support in Ω_δ. Now by (5.3),

$$\int_\Omega \phi_o\omega(r,h_1)\,dx = \int_\Omega \phi_o\omega(r,h_2)\,dx. \tag{5.4}$$

A straightforward calculation shows that $\int_\Omega \omega(r,h)\,dx = 2\pi$ so we divide (5.4) by 2π to get

$$\frac{1}{2\pi}\int_\Omega \phi_o\omega(r,h_1)\,dx = \frac{1}{2\pi}\int_\Omega \phi_o\omega(r,h_2)\,dx. \tag{5.5}$$

Recalling the properties of mollifiers from section (2.4.3), we note that for each y and h, $\frac{\omega(r,h)}{2\pi}$ is of class C^∞ with compact support and

$$\int_\Omega \frac{\omega(r,h)}{2\pi}dx = 1.$$

Each side of (5.5) can therefore be considered to be a mollification of ϕ_0 on Ω_δ. Since the mollification is evidently independent of the choice of h_1 and h_2, we conclude that ϕ_0 is identical to either of the mollifications, and is therefore of class C^∞ in Ω_δ. Reducing δ to include more of Ω in Ω_δ does not change any of the above constructions, so we conclude that $\phi_0 \in C^\infty(\Omega)$.

If we now let ξ be any test function, (5.2) and a simple integration by parts give us

$$(\phi_0, \xi)_1 = \int_\Omega \xi \Delta \phi_0 dx = 0.$$

Which means at least that $\Delta\phi_0 = 0$ in the distributional sense, but since ϕ_0 is differentiable in the ordinary sense, it is truly harmonic.

3. Uniqueness of the Solution

ϕ_0 is the unique (up to a set of measure zero) minimizer of $|\cdot|_1$ in W. If $\psi_0 \in W$, is such that $|\psi_0|_1 = d$, then we could construct a minimizing sequence by alternating ϕ_0 and ψ_0. This is contrary to the previously verified convergence of a minimizing sequence.

We now assert the uniqueness of the solution to Laplace's equation, $\Delta\phi = 0$, in W.

Suppose that there exists some other function $\psi_0 \in W$ such that $\Delta\psi_0 = 0$, possibly in only the distributional sense. It is clear that $|\psi_0|_1 > d$, since equality would bring into question the optimality of ϕ_0. Letting ξ be any test function on Ω we have

$$0 = \langle \Delta\psi_0, \xi \rangle = \left\langle \frac{\partial^2 \psi_0}{\partial x_1^2}, \xi \right\rangle + \left\langle \frac{\partial^2 \psi_0}{\partial x_2^2}, \xi \right\rangle$$

$$= -\left\langle \frac{\partial \psi_0}{\partial x_1}, \frac{\partial \xi}{\partial x_1} \right\rangle - \left\langle \frac{\partial \psi_0}{\partial x_2}, \frac{\partial \xi}{\partial x_2} \right\rangle = -(\psi_0, \xi)_1 .$$

Suppose instead that $\xi \in W_2^1(\Omega)$ and that $\xi|_{\partial\Omega} = 0$, by Theorem (3.3), $\xi \in W_0^{1,2}(\Omega)$. Therefore, there exists a sequence of test functions ξ_i converging to ξ in $W_2^1(\Omega)$, and we conclude that

$$0 = \lim_{i \to \infty} (\psi_0, \xi_i)_1 = (\psi_0, \xi)_1 .$$

This leads us to consider the squared semi-norm

$$|\psi_0 + \xi|_1^2 = |\psi_0|_1^2 + 2(\psi_0, \xi)_1 + |\xi|_1^2 = |\psi_0|_1^2 + |\xi|_1^2 .$$

Since ϕ_0 and ψ_0 are both elements of W, their difference $\xi_0 = \phi_0 - \psi_0$ satisfies $\xi_0|_{\partial\Omega} = 0$. We therefore have

$$(\psi_0, \xi_0)_1 = 0,$$

so that

$$d^2 < |\psi_0 + \xi_0|_1^2 = |\phi_0|_1^2$$

This contradicts the underlying assumption that $|\phi_0|_1 = d$, so ϕ_0 is the only harmonic function in W.

□

The framework for this proof comes mostly from Sobolev [31].

We now have a very smooth function ϕ_0, defined on Ω, which minimizes $|\cdot|_1$ and is also harmonic. Unfortunately, ϕ_0 is presently extended to $\partial\Omega$ only in the sense of $L_2(\partial\Omega)$. We desire a little more, namely, we would like ϕ_0 to be in $C^0\left(\bar{\Omega}\right)$. At this point we would like to rely on more traditional solutions of the boundary value problem, $\Delta\phi = 0$ in R^2, for example the one given in Axler et al. [5], where a continuous harmonic function is shown to exist on the closure of the domain provided that the data supplied on the boundary is continuous, and the boundary satisfies an **external cone condition**. This condition requires that each point on the boundary be the vertex of a fixed non-degenerate cone contained in the exterior of the domain. Our assumption that $\partial\Omega$ is locally Lipschitz and piecewise smooth guarantees that $\partial\Omega$ satisfies such an external cone condition. Unfortunately, the bounds on the first order partial derivatives are not sufficient to guarantee that this solution lies in the space $W_2^1(\Omega)$.

In proving Theorem (5.1), we used an argument which assumed very little about the space of functions wherein we base our search for a solution. We have, accordingly, arrived at a solution about which we know very little. In [8], Courant proves Dirichlet's Principle without the generality of Sobolev spaces. In fact, his proof is based on the space of functions, continuous on $\bar{\Omega}$, piecewise smooth on Ω and for which $|\cdot|_1$ is finite. He finds a solution, let us call it u, which is harmonic, and continuous up to and including the boundary. Such a solution is clearly in the subspace W in which we sought an optimal function. This means that u qualifies as a solution to our problem. By the uniqueness of our solution, however, this means that $u = \phi_0$, so we can safely conclude that ϕ_0 is continuous on $\bar{\Omega}$. We do not include a sketch of Courant's proof of the principle, since he supplies several, one of which follows essentially the same lines as ours.

The problem we encounter when applying Dirichlet's principle is determining whether a given set of boundary conditions is satisfied by at least one function in the space. In the next section we deal with this question in the context of contour data.

5.2 Existence and Optimality

Suppose we are given a contour map, $\mathcal{M} = \{(\Gamma_i, C_i)\,|i = 1, \ldots, N\}$, in which each pair of contours is nested. For example, the level curves of $f(x,y) = x^2 + y^2$, given by $f(x,y) = n$, $n = 1, \ldots, N$, form such a contour map. For simplicity of indexing let us assume that if $i < j$, then Γ_i is inside Γ_j. Let U_1 be the region interior to Γ_1, and in general, let U_i be the region bounded by Γ_{i-1} and Γ_i, $i = 2, ..., N$. If the bounded domain $\Omega \supset \bigcup_{i=1}^{n} \bar{U}_i$ has a locally Lipschitz boundary, then we have the following theorem.

Theorem 5.2 *If each Γ_i is piecewise smooth and locally Lipschitz, there exists a function $\phi \in W_2^1(\Omega) \cap C^0\left(\bar{\Omega}\right)$ satisfying $\phi\,|_{\Gamma_i} = C_i$, $i = 1, \ldots, N$, for which $|\phi|_1 \leq |\psi|_1$ for any other interpolating function $\psi \in W_2^1(\Omega)$, moreover, this function ϕ is harmonic in each region U_i, $i = 1, \ldots, N$. We call ϕ a* **piecewise harmonic spline.**

The proof of Theorem (5.2) relies on Dirichlet's principle, so we supply a general theorem indicating the permissibility of our boundary conditions.

Theorem 5.3 *Suppose $\{\gamma_i\}_{i=1}^{N}$ is a set of 1-smooth arcs in a bounded domain in R^2 with only pair-wise intersections, and that defined on each γ_i there is a C^1 function ψ_i. If $\{\gamma_i\}_{i=1}^{N}$ and $\{\psi_i\}_{i=1}^{N}$ are such that if γ_i and γ_j intersect for some $i, j \leq N$,*

$i \neq j$ at a point p, then $\psi_i(p) = \psi_j(p)$ and γ_i' is not parallel to γ_j' at p, then there exists a function $u \in C^1(R^2)$ such that $u|_{\gamma_i} = \psi_i$, $i = 1, \ldots, N$.

Before we prove Theorem (5.3) we supply a few technical details, all pertaining to functions defined on 1-smooth arcs.

Lemma 5.4 *Suppose $\gamma_1, \gamma_2 \subset R^2$ are smooth arcs intersecting at a point p and γ_1' and γ_2' are not parallel at p. If the functions ψ_1 defined on γ_1 and ψ_2 defined on γ_2 are C^1 and $\psi_1(p) = \psi_2(p)$, then there exists, on a neighbourhood N_p of p, a function $\Psi \in C^1(N_p)$ such that $\Psi(q) = \psi_i(q)$ whenever $q \in N_p \cap \gamma_i$, $i = 1, 2$.*

Proof. Let us assume that γ_1 and γ_2 are parametrized by arc-length such that $\gamma_1(0) = \gamma_2(0) = p$, and that neither curve ends at p. This does not result in a loss of generality for if either of the curves ends at p, they can be extended in a C^1 fashion with a straight line, as can the values of the functions defined on them.

Let $T(u, v) := \gamma_1(u) + \gamma_2(v) - p$, for all values of u and v for which the evaluation makes sense. We note that T is C^1 and that $T(u, 0) = \gamma_1(u)$, and $T(0, v) = \gamma_2(v)$. The Jacobian

$$JT(u, v) = \det\left((\gamma_1'(u))^{\mathrm{T}} \mid (\gamma_2'(v))^{\mathrm{T}}\right)$$

is non-zero provided the two tangent vectors are not parallel. This condition is guaranteed by the hypotheses when $u = v = 0$, so $JT(0, 0) \neq 0$. Thus, by the Inverse Function Theorem there exist neighbourhoods M of $(0, 0)$ and N_p of $p = T(0, 0)$ such that T has a C^1 inverse $T^{-1} : N_p \to M$.

Now, let $G(u, v) := \psi_1(T(u, 0)) + \psi_2(T(0, v)) - \psi_1(p) = \psi_1(\gamma_1(u)) + \psi_2(\gamma_2(v)) - \psi_1(p)$. This function has several desirable properties: $G(u, 0) = \psi_1(\gamma_1(u))$, $G(0, v) = \psi_2(\gamma_2(v))$, and $G \in C^1(M)$.

We now let $\Psi(x,y) := G(T^{-1}(x,y))$ for $(x,y) \in N_p$. As a composition of two C^1 functions, Ψ is also C^1, and satisfies the required interpolation conditions. □

Lemma 5.5 *If γ is a smooth arc, and ψ is a C^1 function defined on γ, then for each point p of γ, there exists, on a neighbourhood N_p of p, a function $\psi \in C^1(N_p)$ such that $\Psi(q) = \psi(q)$ whenever $q \in N_p \cap \gamma$.*

Proof. Let $\gamma_1 = \gamma$ and set $\gamma_2(t) = p + t\vec{n}(p)$, where $\vec{n}(p)$ is a unit normal to γ at p. In addition, let $\psi_1 = \psi$ and $\psi_2 = \psi(p)$. Apply Lemma (5.4). □

We can now proceed with the proof of Theorem (5.3)

Proof. At each point p on γ_i, $i = 1, \ldots, N$, that is not on some other curve γ_j, we apply Lemma (5.5) and construct a function $F_p(x,y)$ which is C^1 in some neighbourhood N_p of p, and satisfies the interpolation conditions. At any point p that is the result of an intersection, we apply Lemma (5.4), to similarly obtain a C^1 function $F_p(x,y)$ defined on some neighbourhood N_p of p. Since we assume no tangency at points of intersection, we can ensure that $N_p \cap \{\gamma_i\}_{i=1}^N$ is connected for each p.

We now have an uncountable number of neighbourhoods covering $\{\gamma_i\}_{i=1}^N$. Since $\{\gamma_i\}_{i=1}^N$ is closed and bounded, it is compact and we therefore need only a finite number M of these neighbourhoods to cover the boundary. Let us denote these neighbourhoods by C_i, and the functions associated with them by F_i, $i = 1, \ldots, M$.

If we add the set $C_0 = R^2 \setminus \{\gamma_i\}_{i=1}^N$ to the collection $\{C_i\}$ then we can construct a partition of unity $\{\phi_i\}_{i=0}^M \subset C^\infty(R^2)$ with $\operatorname{supp}\phi_i \subset C_i$, for $i = 0, \ldots, M$, such that $\sum_{i=0}^M \phi_i \equiv 1$ (for a proof of this, see Gel'fand and Shilov, vol. 1 p. 142 [14]).

The function $u = \sum_{i=1}^{M} \phi_i F_i$ is in $C^1(R^2)$, and takes on the values of ψ_i on each γ_i, $i = 1, \ldots, N$. $\square$

We now have the necessary tools for the proof of Theorem (5.2).

Proof. On each of the regions U_i, $i = 1, \ldots, N$, Theorem (5.3) guarantees us the existence of a function u_i which satisfies the boundary conditions $u_i|_{\Gamma_i} = C_i$ and $u_i|_{\Gamma_{i-1}} = C_{i-1}$ and is in $C^1\left(\bar{U}_i\right)$. This of course means that the first order partial derivatives are bounded and thus each function u_i is in $W_2^1(U_i)$. By Dirichlet's principle, there exists a unique function $\phi_i \in C^\infty(U_i)$, harmonic in U_i, which minimizes $|\cdot|_1$ over all interpolating functions in the space $W_2^1(U_i)$.

By the remarks following Theorem (5.1) each $\phi_i \in C^0\left(\bar{U}_i\right)$. If we let

$$\phi(x) = \begin{cases} \phi_i(x) & \text{if } x \in U_i,\ i = 1, \ldots, N, \\ C_i & \text{if } x \in \Gamma_i,\ i = 1, \ldots, N, \\ C_N & \text{otherwise,} \end{cases}$$

then $\phi \in C^0\left(\bar{\Omega}\right)$, and we claim that $\|\phi\|_{1,2}^2 = \sum_{i=1}^{N} \|\phi_i\|_{1,2,U_i}^2 < \infty$, so that $\phi \in W_2^1(\Omega)$. This is essentially Theorem (2.1.1) of Ciarlet [7]. For the equality, it is sufficient to show that $D^{e_j}\phi|_{U_i} = D^{e_j}\phi_i$, since each ϕ_i is an element of $W_2^1(U_i)$. Let ξ be any test function on Ω. For $i = 1, \ldots, N$, and $j = 1, 2$ we have

$$\int_{U_i} \xi D^{e_j}\phi_i dx = \int_{\Gamma_i} \phi_i \xi n_j d\sigma - \int_{\Gamma_{i-1}} \phi_i \xi n_j d\sigma - \int_{U_i} \phi_i D^{e_j}\xi dx.$$

For Ω', the portion of Ω not covered by the U_i we have

$$0 = \int_{\Omega'} \xi D^{e_j} C_N dx = \int_{\Gamma_N} C_N \xi n_j d\sigma - \int_{\partial\Omega} \phi C_N \xi n_j d\sigma - \int_{\Omega'} C_N D^{e_j} \xi dx$$

Summing over all of the subregions and rearranging the integrals, we get

$$\int_{\Omega} \phi D^{e_j} \xi dx = \sum_{i=1}^{N} \int_{\Gamma_i} (\phi_{i+1} - \phi_i) \xi n_j d\sigma - \sum_{i=1}^{N} \int_{U_i} \xi D^{e_j} \phi_i dx,$$

which, accounting for the fact that $\phi \in C^0\left(\bar{\Omega}\right)$, is

$$\int_{\Omega} \phi D^{e_j} \xi dx = -\sum_{i=1}^{N} \int_{U_i} \xi D^{e_j} \phi_i dx.$$

By the definition of the generalized partial derivative, it follows then that $D^{e_j} \phi|_{U_i} = D^{e_j} \phi_i$, whereby $\phi \in W_2^1(\Omega)$.

Now, of course, the fact that ϕ is optimal on each region U_i is not obviously sufficient to guarantee that ϕ will be the optimal function on Ω. To show that it is optimal, we resort to a sequence of arguments similar to those we used to prove the optimality of the univariate splines.

Recalling (5.2), we claim that if $\psi \in W_2^1(\Omega)$ is an interpolant of zero data (i.e. $\psi|_{\Gamma_i} = 0$, $i = 1, \ldots, N$), then $(\phi, \psi)_1 = 0$. In fact,

$$(\phi, \psi)_1 = \int_{\Omega} \frac{\partial \phi}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} \frac{\partial \psi}{\partial x_2} dx \tag{5.6}$$

$$= \sum_{i=1}^{N} \int_{U_i} \frac{\partial \phi}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} \frac{\partial \psi}{\partial x_2} dx + 0. \tag{5.7}$$

$$= \sum_{i=1}^{N} (\phi, \psi)_{1,U_i} = 0.$$

Now, suppose $\psi \in W_2^1(\Omega)$ satisfies the same interpolation conditions as ϕ. Then

$$(|\psi|_1)^2 = (\psi, \psi)_1 = (\phi + (\psi - \phi), \phi + (\psi - \phi))_1$$

$$= (\phi, \phi)_1 + 2(\phi, \psi - \phi)_1 + (\psi - \phi, \psi - \phi)_1$$

$$= (|\phi|_1)^2 + 0 + (|\psi - \phi|_1)^2 \geq (|\phi|_1)^2 .$$

So ϕ is optimal. □

5.3 A More Realistic Case

The problem posed and solved above assumed more of the contour map than can be expected from an actual topographical map. We propose a way of extending the preceding methods to real maps. For example, a typical topographical map is bounded by a rectangle, with contours crossing that boundary. Unfortunately, there is usually little or no information available regarding what happens outside of the rectangular region, and so we have to base our search for an interpolant solely upon pieces of contours.

Suppose now, that we are given a contour map, $\mathcal{M} = \{(\Gamma_i, C_i) | i = 1, \ldots, N\}$, and a closed rectangular region Q. Let us denote by M_Q the collection of n curves γ_j and associated constants C_j, generated by $\Gamma_i \cap Q$, $i = 1, \ldots, N$. Note that in general, $N \neq n$. The n curves γ_i divide Q into regions U_j, $j = 1, \ldots, P$ for some $P > 0$. Since there is no guarantee that any of the curves are nested contours, there is no longer any convenient correspondence between the number of curves and the number of regions.

At this point we recall that most of the main theorems so far have required that the regions involved have locally Lipschitz boundaries. This presents us with a restriction on our contour map. If γ_i meets the curve ∂Q at a point x_0, we assume that the tangents of the two curves are not parallel at x_0. If they were, then one of the U_j for which $x_0 \in \partial U_j$ will not have a locally Lipschitz boundary (if it does have a locally Lipschitz boundary, then the neighbouring region sharing x_0 will not). In addition, we insist that the only intersections that occur are pairwise – this means that points in Γ_i at which there is no tangent vector should not lie on ∂Q.

For any U_j, not necessarily all of ∂U_j consists of pieces of contours – there are likely to be portions of ∂U_j which are part of ∂Q. If this is the case, then there are no boundary values specified on some portions of ∂U_j. This leaves us with several options. We could look for solutions to the minimization problem leaving those portions with free boundary conditions, or we could specify values and use Dirichlet's principle as stated in (5.1).

In the second case, we notice that if we extend the boundary conditions along ∂Q so that the specified values result in a C^1 function being associated with all of ∂U_j, then the proof of Theorem (5.2) applies, and there exists an optimal function satisfying those new boundary conditions. The optimality of the function is limited in that it is not likely to be optimal when compared to all other functions which only satisfy the interpolation conditions on the contours.

Chapter 6

Piecewise Biharmonic Splines

In the previous chapter we discussed a sort of 'linear spline' for contour data in the plane. This interpolant has some of the same problems that linear splines for data on the line have. For one thing, the resulting surface is not, in general, C^1. This of course means that for contour data, the spline will have visible creases. The linear spline is also not a model of a physical spline: it doesn't minimize bending energy – it minimizes potential energy. If we assume that the data values on the contours describe a distribution of charges, then the resulting piecewise harmonic function describes the potential in the interior of each region.

We now propose to construct an interpolant which does minimize the bending energy. As in the univariate case, we linearize the true bending energy by assuming that deflections are 'small'. The result is that we aim to minimize

$$|f|_2 .$$

This leads us to look for minimizers in a Sobolev space, particularly $f \in W_2^2(\Omega)$.

We begin with by applying some Hilbert space theory, thereby showing the existence and uniqueness of a solution to the problem, and then we characterize some of its properties. In particular, the solution will be biharmonic on subregions delimited by contours, that is

$$\Delta^2 f = 0,$$

of class $C^1\left(\bar{\Omega}\right)$, and it will have continuous second order normal derivatives across each contour. We compare this with the cubic splines, for which the same properties hold, the normal derivatives, of course, reduce to the second order derivatives, so the univariate spline is in fact C^2.

6.1 A Subspace of $W_2^2(\Omega)$

Suppose $\mathcal{M} = \{(C_i, \Gamma_i)\}_{i=1}^N$is a contour map with $N > 1$, and that for each $i < N$, $\Gamma_i \subset \Omega$, and $\Gamma_N = \partial\Omega$. Unless we specify otherwise, we assume in this chapter that each Γ_i is 2-smooth. Since we are concerned primarily with searching for a function for which the Γ_i are level curves, we consider the space

$$X_{\mathcal{M}} = \left\{\phi \in W_2^2(\Omega) \mid \phi|_{\Gamma_N} = 0 \text{ and } \phi|_{\Gamma_i} \text{ is constant for } i = 1, \ldots, N-1\right\}.$$

We notice that $|\cdot|_2$ is a norm on $X_{\mathcal{M}}$ provided that the contours are not all parallel straight lines. By the same reasoning, $(\cdot,\cdot)_2$ is a true inner-product on $X_{\mathcal{M}}$. We claim that $X_{\mathcal{M}}$, with the norm $|\cdot|_2$, is complete. To show this, we apply Theorem (3.2), with the functional $\mathcal{F} = \|\cdot\|_{L_2(\Gamma_N)}$ which vanishes on $X_{\mathcal{M}}$. We notice that $\mathcal{F}$ has all of the properties required so that $|\cdot|_2 + \mathcal{F}$ is a norm equivalent to $\|\cdot\|_{2,2}$ on $W_2^2(\Omega)$. Because $\mathcal{F}$ vanishes on $X_{\mathcal{M}}$, $|\cdot|_2$ and $\|\cdot\|_{2,2}$ are therefore equivalent on $X_{\mathcal{M}}$. We combine this equivalence with the following lemma to obtain the completeness.

Lemma 6.1 *$X_{\mathcal{M}}$ is a Hilbert space with the norm $\|\cdot\|_{2,2}$.*

Proof. Suppose $\{\phi_i\} \subset X_{\mathcal{M}}$ is a Cauchy sequence. By the completeness of $W_2^2(\Omega)$, $\phi_i \to \phi \in W_2^2(\Omega)$. In addition, Sobolev's imbedding theorem gives us

$W_2^2(\Omega) \to C^0\left(\bar{\Omega}\right)$, so for all $\psi \in W_2^2(\Omega)$ there exists some constant $K > 0$ such that

$$\|\psi\|_{C^0(\bar{\Omega})} \le K\,\|\psi\|_{2,2}\,.$$

From this we conclude that

$$\sup_{x\in\bar{\Omega}} |\phi_i - \phi| = \|\phi_i - \phi\|_{C^0(\bar{\Omega})} \le K\,\|\phi_i - \phi\|_{2,2} \to 0,$$

and particularly, for any Γ_j

$$\sup_{x\in\Gamma_j} |\phi_i - \phi| \to 0,$$

so $\phi \in X_{\mathcal{M}}$. Therefore $X_{\mathcal{M}}$ is a complete inner-product space. □

We now introduce a collection of linear functionals on $X_{\mathcal{M}}$. For any $\phi \in X_{\mathcal{M}}$ and for each Γ_i, let

$$L_i\phi = \phi|_{\Gamma_i}.$$

These functionals are clearly linear, and they are, in fact, bounded.

Lemma 6.2 *The* **evaluation functionals** $L_i\phi = \phi|_{\Gamma_i}$, $i = 1, \ldots, N$ *are bounded on* $X_{\mathcal{M}}$.

Proof. For some constants K and K_1 we have, for any $\phi \in X_{\mathcal{M}}$,

$$|L_i\phi| = |\,\phi|_{\Gamma_i}\,| \le \sup_{x\in\Omega} |\phi| \le K\,\|\phi\|_{2,2} \le K_1\,|\phi|_2\,.$$

The second inequality follows from the imbedding theorem, and the last inequality holds because of the equivalence of the two norms. □

Because of the boundedness, the Riesz representation theorem guarantees the existence of **representers** $f_i \in X_{\mathcal{M}}$, $i = 1, \ldots, N$, such that for any $\psi \in X_{\mathcal{M}}$

$$L_i\psi = (f_i, \psi)_2 \, .$$

The first $N-1$ of these functions are linearly independent, since if $\sum_{i=1}^{N-1} a_i f_i = 0$, then

$$0 = \left(\sum_{i=1}^{N-1} a_i f_i, \psi\right)_2 = \sum_{i=1}^{N-1} a_i (f_i, \psi)_2 = \sum_{i=1}^{N-1} a_i \psi|_{\Gamma_i}, \text{ for all } \psi \in X_{\mathcal{M}}.$$

From the arbitrariness of ψ, it follows that $a_i = 0$, for $i = 2, \ldots, N$.

We now consider the function

$$\phi = C_N + \sum_{i=1}^{N-1} b_i f_i.$$

where the b_i's are determined by solving the system

$$\left(\sum_{i=1}^{N-1} b_i f_i, f_i\right)_2 = C_i - C_N, \, i = 1, \ldots, N-1,$$

which, in matrix form, is

$$\begin{pmatrix} (f_1, f_1)_2 & (f_2, f_1)_2 & \cdot \;\; \cdot & (f_{N-1}, f_1)_2 \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & (f_{N-1}, f_{N-2})_2 \\ (f_1, f_{N-1})_2 & \cdot & \cdot \;\; \cdot & (f_{N-1}, f_{N-1})_2 \end{pmatrix} \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{N-1} \end{pmatrix} = \begin{pmatrix} C_1 - C_N \\ \cdot \\ \cdot \\ \cdot \\ C_{N-1} - C_N \end{pmatrix}.$$

Compressing the notation, we write this as

$$F\mathbf{b} = \mathbf{c}.$$

The matrix F is positive definite since for any vector $\mathbf{a} \in R^{N-1}$,

$$\mathbf{a}^{\mathsf{T}} F \mathbf{a} = \mathbf{a}^{\mathsf{T}} \begin{pmatrix} \sum_{i=1}^{N-1} a_i \left(f_i, f_1\right)_2 \\ \vdots \\ \sum_{i=1}^{N-1} a_i \left(f_i, f_{N-1}\right)_2 \end{pmatrix} = \left(\sum_{i=1}^{N-1} a_i f_i, \sum_{i=1}^{N-1} a_i f_i \right)_2 \geq 0,$$

with equality only if $a_i = 0$ for all i. The system therefore has a unique solution, and the function ϕ is well defined. Moreover, ϕ satisfies the interpolation conditions $\phi|_{\Gamma_i} = C_i$, $i = 1, \ldots, N$. It should be noted that since ϕ does not necessarily vanish on Γ_N, ϕ is not likely to be in $X_{\mathcal{M}}$, but $\phi - C_N$ is in $X_{\mathcal{M}}$. We can now sample the fruits of our labours in the form of the following theorem.

Theorem 6.3 *The function ϕ described above is optimal in that if $\psi \in X_{\mathcal{M}}$ is any other function such that $\psi + C_N$ satisfies the interpolation conditions then $|\phi - C_N|_2 < |\psi|_2$. ϕ is, moreover, the unique optimal function.*

Proof. We begin by assuming that ψ differs from $\phi - C_N$ on a set of positive measure so that the norm

$$|\psi - \phi + C_N|_2 > 0.$$

We consider the inner-product

$$(\phi - C_N, \phi - C_N - \psi)_2 = \sum_{i=1}^{N-1} b_i \left(f_i, \phi - C_N - \psi\right)_2 = \sum_{i=1}^{N-1} b_i \left(\phi - C_N - \psi\right) |_{\Gamma_i} = 0,$$

and find that the function ϕ is orthogonal to all interpolants of zero data. We take advantage of this to show

$$(|\psi|_2)^2 = (\psi, \psi)_2 = (\phi - C_N + \psi - \phi + C_N, \phi - C_N + \psi - \phi + C_N)_2$$

$$= (\phi - C_N, \phi - C_N)_2 + 2(\phi - C_N, \psi - \phi + C_N)_2 + (\psi - \phi + C_N, \psi - \phi + C_N)_2$$

$$= (|\phi - C_N|_2)^2 + 0 + (|\psi - \phi + C_N|_2)^2 > (|\phi - C_N|_2)^2 .$$

So that $|\psi|_2 > |\phi - C_N|_2$, whence we conclude that ϕ is optimal and unique. □

We now hypothesize about the order of continuity of ϕ. Recalling our previous notation, we assume that the contour map divides Ω into regions U_j, $j = 1, \ldots, N$, such that $\partial U_j = \Gamma_j \cup \Gamma_{j-1}$, where, for convenience, we set $\Gamma_0 = \emptyset$. For other reasons, which will become clear as we progress, we also assume from now on that the contours are all at least 6-smooth.

Theorem 6.4 *If we denote by ϕ_i the restriction of ϕ to U_i, then ϕ is optimal if and only if $\Delta^2 \phi_i = 0$ on U_i, $\phi \in C^1\left(\bar{\Omega}\right)$, ϕ has continuous second order normal derivatives on Γ_i for $i = 1, \ldots, N-1$, and the second order normal derivative of ϕ vanishes on Γ_N.*

Before we continue with the proof of this theorem, we have to address some technical details. For example, since we only know that the generalized derivatives up to order 2 of ϕ are locally integrable functions, we will find it inconvenient to work with the fourth order differential operator Δ^2. We deal with this problem first.

Suppose for the moment that Ω is a bounded domain with an m-smooth boundary, and that $u \in C^4\left(\bar{\Omega}\right)$ and $f \in L_1(\Omega)$. Then the differential equation

$$\Delta^2 u = f \tag{6.1}$$

makes sense whether it has a solution or not. Let us assume that it does have a solution on Ω. If we let ψ be any function in $C_0^\infty(\Omega)$, then we can multiply both sides of (6.1) by ψ, and integrate over Ω to get

$$\int_\Omega \psi \Delta^2 u \, dx = \int_\Omega \psi f \, dx. \tag{6.2}$$

Applying integration by parts to the left-hand side of (6.2) we obtain

$$\int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \psi}{\partial x_1^2} + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \psi}{\partial x_2^2} \right) dx = \int_\Omega \psi f \, dx. \tag{6.3}$$

The boundary terms from the integration by parts drop out since ψ has compact support. If $u \in W_2^2(\Omega)$ is a solution of (6.3) then we say that u is a **strong solution** of $\Delta^2 u = f$. Now, let $g \in C^4\left(\bar{\Omega}\right)$ be any function that is constant on $\partial\Omega$. We could now consider solving the equation

$$\int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \psi}{\partial x_1^2} + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \psi}{\partial x_2^2} \right) dx = \int_\Omega \psi(-\Delta^2 g) \, dx \tag{6.4}$$

for $u \in W_0^{2,2}(\Omega)$. If u is smooth enough, then $v = u + g$ will be biharmonic, and satisfy $v|_{\partial\Omega} = g|_{\partial\Omega}$ and $\nabla v|_{\partial\Omega} = \nabla g|_{\partial\Omega}$.

We will show in the proof of Theorem (6.4) that on U_i, ϕ_i qualifies as a candidate for v as defined above with $f = -\Delta^2 g$. So we now state a result which applies to solutions to problems in the form of (6.3).

The second part of Theorem 17.2 in Friedman [12] tells us the following. If $\partial\Omega$ is $(4+l)$-smooth for $l \geq 0$, and $f \in W_2^l(\Omega)$ then if $u \in W_0^{2,2}(\Omega)$ is a strong solution of $\Delta^2 u = f$, then $u \in W_2^{4+l}(\Omega)$. This, together with some of the imbedding theorems of Chapter 3 tells us some useful things about the behaviour of u on $\partial\Omega$. We will discuss these further in the proof of Theorem (6.4).

We now have the necessary tools to procede with the proof of Theorem (6.4).

Proof. First, let us assume that ϕ is optimal. To show that ϕ_i is biharmonic on U_i, we recall from the proof of Theorem (6.3) that if $\psi \in C_0^\infty(U_i)$, then

$$(\phi_i, \psi)_2 = (\phi, \psi)_2 = 0.$$

Now, since each Γ_i is 6-smooth, there exists a function $g_i \in C^6\left(\bar{\Omega}\right)$ such that $g_i|_{\Gamma_i} = C_i$ and $g|_{\Gamma_{i-1}} = C_{i-1}$. If we let $v_i = \phi_i - g_i$ then

$$(v_i, \psi)_2 = (\phi_i, \psi)_2 - (g_i, \psi)_2 = (g_i, \psi)_2 .$$

Applying integration by parts, we obtain

$$\begin{aligned}(g_i, \psi)_2 &= \int\limits_{U_i} \left(\frac{\partial^2 g_i}{\partial x_1^2} \frac{\partial^2 \psi}{\partial x_1^2} + 2 \frac{\partial^2 g_i}{\partial x_1 \partial x_2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 g_i}{\partial x_2^2} \frac{\partial^2 \psi}{\partial x_2^2} \right) dx \\ &= -\int\limits_{U_i} \psi \Delta^2 g_i \, dx.\end{aligned}$$

Thus,

$$(v_i\psi)_2 = \int\limits_{U_i} \left(\frac{\partial^2 v_i}{\partial x_1^2}\frac{\partial^2 \psi}{\partial x_1^2} + 2\frac{\partial^2 v_i}{\partial x_1 \partial x_2}\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 v_i}{\partial x_2^2}\frac{\partial^2 \psi}{\partial x_2^2} \right) dx$$
$$= -\int\limits_{U_i} \psi \Delta^2 g_i \, dx.$$

The function v_i is, therefore, a strong solution of (6.1) with $f = \Delta^2 g_i \in C^2(U_i)$. Since $\phi_i \in W_2^2(U_i)$, $v_i \in W_0^{2,2}(U_i)$. From our previous remarks, $v_i \in W_2^{4+2}(U_i)$. Since $g \in C_2^6\left(\bar{\Omega}\right)$, it follows that $\phi_i \in W_2^6(U_i)$. By the Rellich-Kondrašov Theorem $W_2^6(U_i) \to C^4\left(\bar{U}_i\right)$, so $\Delta^2 v_i = \Delta^2\phi_i - \Delta^2 g_i$, and $\Delta^2 v_i = \Delta^2 g_i$. Thus, $\Delta^2\phi_i = 0$. This argument applies for $i = 1, \ldots, N$, thus proving the first property of ϕ.

To show the second property, we again make use of the imbedding theorems from Chapter 3. Since $\phi \in W_2^2(\Omega)$ we already have $\phi \in C^0\left(\bar{\Omega}\right)$. We now verify that the gradient must be continuous. On Γ_N, ϕ clearly has a unique C^1 extension, specifically that of ϕ_N, so we consider the case when i is less than N. Since ϕ is in $X_{\mathcal{M}}$, both ϕ_i and ϕ_{i+1} are constant on Γ_i, from this we conclude that the tangential derivatives coincide. We therefore concentrate on the normal derivatives.

Since $\dfrac{\partial \phi_i}{\partial \vec{n}}$ and $\dfrac{\partial \phi_{i+1}}{\partial \vec{n}}$ are continuous functions on Γ_i, their difference is also continuous. If the two derivatives are not equal at some point p of Γ_i, then there is, in fact, a segment of Γ_i including p on which the difference $g(p) = \dfrac{\partial \phi_i}{\partial \vec{n}}(p) - \dfrac{\partial \phi_{i+1}}{\partial \vec{n}}(p)$, $p \in \Gamma_i$, is either positive or negative. We suppose, without loss of generality that the difference is positive. Since Γ_i is 2-smooth, for some $a > 0$ there exists a diffeomorphism $T : R^2 \to R^2$ mapping the offending segment of Γ_i to the interval $\{0\} \times (-a, a)$. Since $\phi \in W_2^2(\Omega)$, $D^\alpha \phi \in L_2(\Omega)$ for $|\alpha| = 2$, and hence the same is true of second order derivatives of $f = \phi(T^{-1})$. In particular, $\dfrac{\partial^2 f}{\partial x^2}$ is in L_2 on the rectangle $Q = (-a, a) \times (-b, b)$ for sufficiently small $a > 0$. We claim, however, that this derivative (taken of course in the distributional sense) is not even a function in the classical sense.

If we set $f_1 = f$ for $x < 0$ and $f_2 = f$ for $x > 0$, then on their respective domains, f_1 and f_2 are both of class (at least) C^2, with bounded second order partials. They therefore have C^2 extensions $\tilde{f}_1$ and $\tilde{f}_2$ to the entire rectangle. Recalling the two-dimensional step function $S(x,y)$ from Section (2.3.3), let

$$\tilde{f} = f - \tilde{f}_1 = \begin{cases} 0 & x < 0, \\ f_2 - \tilde{f}_1 & x > 0 \end{cases} = \left(\tilde{f}_2 - \tilde{f}_1\right) S.$$

Since $\tilde{f}_2 - \tilde{f}_1$ is C^2, $\frac{\partial}{\partial x}\left(\tilde{f}_2 - \tilde{f}_1\right)$ is C^1, and so the results of Section (2.3.3) apply, and $\frac{\partial^2}{\partial x^2}\left(\tilde{f}\right)$ is not a classical function. From this it follows that $\frac{\partial^2}{\partial x^2}(f) = \frac{\partial^2}{\partial x^2}\left(\tilde{f}\right) + \frac{\partial^2}{\partial x^2}\left(\tilde{f}_1\right)$ is not in $L_2(Q)$.

This contradicts our assumption that $\phi \in W_2^2(\Omega)$, therefore $\phi \in C^1\left(\bar{\Omega}\right)$.

Finally, we consider the second order normal derivatives of ϕ on each Γ_i. We now assume $\psi \in C^2\left(\bar{\Omega}\right)$, with $\psi|_{\Gamma_i} = 0$, for $i = 1, \ldots, N$, and apply integration by parts to $(\phi, \psi)_2$. Doing so we obtain

$$\begin{aligned} \sum_{i=1}^{N} \int_{U_i} \psi \Delta^2 \phi_i \, dx \quad = \quad & \sum_{i=2}^{N} \left(\int_{\Gamma_i} \left(\psi \left(\nabla \left(\Delta \phi_i\right)\right) - H\phi_i \nabla \psi\right) \cdot \vec{n} \, d\sigma \right. \\ & \left. - \int_{\Gamma_{i-1}} \left(\psi \left(\nabla \left(\Delta \phi_i\right)\right) - H\phi_i \nabla \psi -\right) \cdot \vec{n} \, d\sigma \right) \\ & + \sum_{i=1}^{N} \int_{U_i} \left(\frac{\partial^2 \phi_i}{\partial x_1^2} \frac{\partial^2 \psi}{\partial x_1^2} + 2 \frac{\partial^2 \phi_i}{\partial x_1 \partial x_2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_i}{\partial x_2^2} \frac{\partial^2 \psi}{\partial x_2^2} \right) dx. \end{aligned} \tag{6.5}$$

Since each ϕ_i has bounded third order derivatives up to ∂U_i, the product $\psi \nabla \left(\Delta \phi_i\right)$ makes sense on Γ_i and Γ_{i-1} and in fact vanishes there. By virtue of the fact that for each i, $\Delta^2\phi_i = 0$, the left side of (6.5) vanishes. Since ψ is an interpolant of zero data, so does the last integral on the right-hand side. As a result, (6.5) reduces to

$$\sum_{i=1}^{N-1} \int_{\Gamma_i} (H\phi_i - H\phi_{i+1}) \nabla\psi \cdot \vec{n}\, d\sigma = 0. \tag{6.6}$$

If we restrict ψ further so that its support lies within $U_i \cup U_{i+1} \cup \Gamma_i$, for some particular i, then

$$\int_{\Gamma_i} (H\phi_i - H\phi_{i+1}) \nabla\psi \cdot \vec{n}\, d\sigma = 0. \tag{6.7}$$

Since ψ is constant on Γ_i, it follows that $\nabla\psi$ is a vector parallel to $\vec{n}$, that is, for some real valued function $\tilde{\psi}_i$ defined on Γ_i, $\nabla\psi = \tilde{\psi}_i \vec{n}$. Using this, (6.7) becomes

$$\int_{\Gamma_i} \left(\vec{n}^{\mathsf{T}} H\phi_i \vec{n} - \vec{n}^{\mathsf{T}} H\phi_{i+1}\vec{n}\right) \tilde{\psi}_i\, d\sigma = 0.$$

From the arbitrariness of ψ, and hence of $\tilde{\psi}_i$, we conclude that

$$\left(\vec{n}^{\mathsf{T}} H\phi_i \vec{n} - \vec{n}^{\mathsf{T}} H\phi_{i+1}\vec{n}\right) = 0$$

almost everywhere on Γ_i. This is the result we seek, indicating that the second order normal derivatives of ϕ_i and ϕ_{i+1} coincide on Γ_i, for $i = 1, \ldots, N-1$. Finally, we consider Γ_N. Suppose ψ is a zero-interpolant of class $C^2\left(\bar{\Omega}\right)$, with support strictly within $U_N \cup \Gamma_N$. Now (6.6) becomes

$$\int_{\Gamma_N} (-H\phi_N)\, \nabla\psi \cdot \vec{n}\, d\sigma = 0,$$

which tells us that $\vec{n}^{\mathrm{T}} H\phi_N \vec{n} = 0$ on Γ_N.

The converse of this theorem is trivial. If each patch ϕ_i is biharmonic, and $\phi \in C^1\left(\bar{\Omega}\right)$, with continuous second order normal derivatives on each Γ_i, $i = 1, \ldots, N-1$, and $\left.\frac{\partial^2 \phi}{\partial \vec{n}^2}\right|_{\Gamma_N} = 0$, we obtain, for $\psi \in X_{\mathcal{M}}$ any interpolant of zero data,

$$\begin{aligned}
(\phi, \psi)_2 &= \sum_{i=1}^{N} \int_{U_i} \left(\frac{\partial^2 \phi_i}{\partial x_1^2} \frac{\partial^2 \psi_i}{\partial x_1^2} + 2 \frac{\partial^2 \phi_i}{\partial x_1 \partial x_2} \frac{\partial^2 \psi_i}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_i}{\partial x_2^2} \frac{\partial^2 \psi}{\partial x_2^2} \right) dx \\
&= \sum_{i=2}^{N} \int_{\Gamma_i} \left(\left(H\phi_i - H\phi_{i-1} \right) \nabla\psi - \psi \left(\nabla\left(\Delta\phi_i\right) - \nabla\left(\Delta\phi_{i-1}\right) \right) \right) \cdot \vec{n}\, d\sigma \\
&\quad + \sum_{i=1}^{N} \int_{U_i} \psi \Delta^2 \phi_i\, dx \\
&= 0.
\end{aligned}$$

Applying, what is by now, a familiar argument, we obtain the result that if ψ is any other function in $X_{\mathcal{M}}$, such that $\psi + C_N$ satisfies the interpolation conditions, then $|\phi|_2 \leq |\psi|_2$. □

The requirement that $\partial\Omega$ be at least 6-smooth is by no means guaranteed to be a strict bound. It is due to the nature of the theorems quoted that we are forced to use such a high degree of smoothness. It is quite possible that given the simple nature of the differential operator Δ^2 some of the conditions of these theorems could be relaxed.

6.2 Some Examples

At this point a reasonable question to ask is whether it is possible to explicitly compute a piecewise biharmonic spline. Given representers, the problem is trivial, but finding the representers is not a trivial task. Following is an example wherein it is possible, and not extraordinarily difficult, to compute the interpolating spline. We make no effort, however, to find representers.

In our first example, we consider a contour map M comprised of concentric circles S_i with radii R_i, and associated data values C_i for $i = 1, \ldots, N$. In this case, we would expect the resulting interpolant to be radially symmetric and piecewise biharmonic. We suppose that all the contours in M are in fact centered at the origin.

The first step in looking for an interpolant is finding a general solution to the biharmonic equation $\Delta^2 u = 0$. Since we are looking for a solution which will be radially symmetric, we can change variables to polar coordinates, and discount any derivatives taken with respect to angle. In polar coordinates, therefore, we have

$$\Delta^2 u = u_{rrrr} + \frac{2}{r} u_{rrr} - \frac{1}{r^2} u_{rr} + \frac{1}{r^3} u_r = 0.$$

It is easily verified that the general solution to this differential equation is

$$u\,(r) = a\,r^2 \ln r + b \ln r + c\,r^2 + d\,,$$

for any constants a, b, c, and d. On each region U_i delimitted by the contours S_i and S_{i-1}, $i = 2, \ldots N$, we consider the patches $u_i\,(r) = a_i\,r^2 \ln r + b_i \ln r + c_i\,r^2 + d_i$. On Ω_1 we require that $u_1\,(r) = a_1\,r^2 \ln r + b_1 \ln r + c_1\,r^2 + d_1$ be of class at least C^2. This implies that special attention is required at the origin. Our first concern is that u_1 should be continuous at the origin, so $b_1 = 0$, and since $\lim_{r\to 0^+} r^2 \ln r = 0$, we can safely set $u_1(0) = d_1$. Next, we require that

$$\frac{\partial u_1}{\partial r} = a_1 (2r \ln r + r) + 2c_1 r$$

be continuous at the origin. Since this is already the case, we proceed to

$$\frac{\partial^2 u_1}{\partial r^2} = a_1 (2 \ln r + 3) + 2c_1,$$

which is unbounded as $r \to 0^+$. We therefore conclude that $a_1 = 0$ so that $u_1(r) = c_1 r^2 + d_1$.

Our goal is to find coefficients $a_i, \ldots, d_i$, for $i = 1, \ldots, N$ such that the function comprised of these patches interpolates the given values on each S_i, is of class C^2 on Ω, and such that $\left.\frac{\partial^2 u_N}{\partial r^2}\right|_{r=R_N} = 0$. The equations for interpolation and continuity are

$$\begin{aligned} u_i &= u_{i+1}, \text{ on } S_i, \ i = 1, \ldots, N-1 \\ u_i &= C_i \text{ on } S_i, \ i = 1, \ldots, N, \end{aligned} \tag{6.8}$$

which reduce to

$$(a_{i+1} - a_i) R_i^2 \ln R_i + (b_{i+1} - b_i) \ln R_i + (c_{i+1} - c_i) R_i^2 + (d_{i+1} - d_i) = 0,$$

for $i = 1, \ldots, N-1$, and

$$a_i R_i^2 \ln R_i + b_i \ln R_i + c_i R_i^2 + d_i = C_i, \text{ for } i = 1, \ldots, N.$$

The equations governing first-order differentiability are

$$(a_{i+1} - a_i)\,(2R_i \ln R_i + R_i) + (b_{i+1} - b_i)\,\frac{1}{R_i} + (c_{i+1} - c_i)\,2R_i = 0, \text{ for } i = 1, \ldots, N-1,$$

and those for second-order differentiability are

$$(a_{i+1} - a_i)\,(2\ln R_i + 3) - (b_{i+1} - b_i)\,\frac{1}{R_i^2} + (c_{i+1} - c_i)\,2 = 0, \text{ for } i = 1, \ldots, N-1.$$

The last equation comes from the 'natural' boundary condition $\left.\frac{\partial^2 u_N}{\partial r^2}\right|_{r=R_N} = 0$, or more explicitly,

$$a_N\,(2\ln R_N + 3) - b_N \frac{1}{R_N^2} + 2c_N = 0.$$

This gives us $4N - 2$ equations, and since a_1 and c_1 are already determined to be zero, we have only $4N - 2$ unknown values. To see that this system has a unique solution, we start by considering the homogeneous system. This system clearly has solutions, since solving the homogeneous system corresponds to finding an optimal interpolant of zero-data, which is trivial. Any solution of the homogeneous system is optimal by Lemma (6.4), and by Theorem (6.3) it is also unique. The non-homogeneous system of equations therefore has a unique solution, and we may, without excessive computation, find an interpolating piecewise biharmonic spline for the contour map M. We present some graphical examples of this spline in the next chapter.

Any attempt at a more general example turns out to be thwarted quite early in the game. Suppose, for example, that we wished to find an interpolant on some more arbitrary domains. While it would be difficult enough to find a solution if we actually knew all of the boundary conditions for each biharmonic 'patch', the problem at hand is a little more devious. We have to leave the normal derivatives on the contours of each patch as unknowns. This could work in two ways.

A method which is often applied to cubic splines is to write down a system of equations which determines the required slopes in terms of the necessarily matching curvatures of neighbouring segments. We could attempt the same here, seeking to find the first-order normal derivatives which would make the second-order ones continuous. The difficulty which arises is that general solutions to the biharmonic problem usually involve computing the Fourier series of the boundary values. This is easy enough for the zero-order conditions, but for the unknown first order conditions, we encounter a problem with far more unknowns than we would like. Perhaps some sort of iteration wherein an initial guess is made could be applied here. Despite the lack of a slick algorithm, such an approach may actually be quite feasible. An advantage would be that at each step of such an iteration, the interpolant would already be C^1. This means that for reasonable contours one could easily concoct an ad hoc scheme for 'guessing' the gradients along the contours, and already have a very well behaved interpolant. Some future work on an iteration scheme could bring this interpolant closer and closer to an optimal one. We should note that relying on local information to generate slopes has been used by many people in the context of piecewise cubic interpolants. For a reasonably broad description of such techniques see Chapter 3 of Lancaster and Šalkauskas [19].

Another approach may be to consider the opposite problem. Instead of looking for the first-order derivatives, we could look for the second order derivatives. Our search criteria would be that the first-order normal derivatives should be continuous across each contour. This could lead us to solving a pair of differential equations as follows. On a given domain Ω with sufficiently smooth boundary, we know that there exists a solution to the problem

$$\Delta g = 0,$$

subject to

$$g|_{\partial\Omega} = h$$

for some sufficiently smooth function h defined on $\partial\Omega$. Similarly, it can be shown that there exists a solution to the problem

$$\Delta\phi = g$$

subject to

$$\phi|_{\partial\Omega} = f$$

for some sufficiently smooth function f on $\partial\Omega$. The function ϕ would therefore have the properties

$$\Delta^2\phi = 0,$$

$$\phi|_{\partial\Omega} = f, \text{ and}$$

$$\Delta\phi|_{\partial\Omega} = h.$$

If we could describe patches ϕ_i on neighbouring regions U_i by

$$\Delta^2\phi_i = 0,$$
$$\phi_i|_{\Gamma_i} = \phi_{i+1}|_{\Gamma_i}, \text{ and}$$
$$\Delta\phi_i|_{\Gamma_i} = \Delta\phi_{i+1}|_{\Gamma_i}, \text{ for } i = 1, \ldots, N-1,$$

then perhaps it could be shown that such an interpolant is optimal whenever

$$\left.\frac{\partial\phi_i}{\partial\vec{n}}\right|_{\Gamma_i} = \left.\frac{\partial\phi_i}{\partial\vec{n}}\right|_{\Gamma_i} \quad \text{for } i = 1, \ldots, N-1,$$

with some appropriate value given for $\Delta\phi_n|_{\Gamma_n}$. We hasten to stress that the existence of such a solution, while appearing likely, is still hypothetical. This could be a subject for further research.

Chapter 7

Some Graphical Examples

While the technical descriptions and proofs of the previous chapters are necessary to ensure that we don't try to evaluate a non-existent interpolant, we must keep in mind that such descriptions tend not to give one an accurate impression of what the resultant interpolant will look like. The interpolants we described are 'optimal' in a mathematical sense, which gives a mathematician great comfort. In the end, however, the appearance of the interpolant, the time required for calculation and the ease of calculation are often much more important than whether a mathematician says "It's an optimal interpolant" or not.

The piecewise harmonic spline developed in Chapter 5 serves mainly as an example – the resulting surface will have creases at almost every contour line. This may not, in every case, be a bad thing. For someone wishing to get a reasonable approximation to a surface, while remaining faithful to contour data, such a spline would do the job. There are a few notable quirks of course, for example if a contour does not enclose other contours, then the harmonic patch inside that contour will necessarily be constant.

The piecewise biharmonic spline discussed in Chapter 6 should have wider applications than the harmonic spline. It is intended as an alternative to the thin-plate spline. The thin-plate spline is a fairly effective interpolant provided one does not attempt to interpolate at too many points. As mentioned in the introduction, this causes problems when one attempts to enforce contour data. One major advantage

Radius	Height
1.0	0.0
2.0	1.0
3.0	2.0
4.0	3.0
5.0	4.0
6.0	5.0

Table 7.1: Data Set 1

of the thin-plate spline is that the further the point of evaluation is from data, the lower the curvature of the surface at that point. The spline conveniently produces a plane when given planar data (unlike some other methods), and is very smooth except at the data points where the second order derivatives fail to exist. Generally, while a thin-plate spline, being a global interpolant, is difficult to compute, it gives reasonable results. Lacking a variety of examples for the piecewise biharmonic spline, we can not make any widespread claims about it giving reasonable results. We can however give some examples of how well it does work with the simple example we constructed in the previous chapter.

7.1 The Sample Data

We begin by introducing our test data. We have, so far, only explicitly computed the piecewise biharmonic splines for contours which are concentric circles. By varying the configuration of these, we can get an impression of how well all three techniques mentioned above compare.

Our first data set is quite straight forward. We essentially sample a cone on uniformly spaced radii (see Table (7.1)).

Radius	Height
1.0	0.0
2.0	1.0
3.0	2.0
4.0	3.0
5.0	4.0
6.0	0.0

Table 7.2: Data Set 2

Our second data set is a bit more demanding. We modify the first data set by setting the value on the outer contour to zero, as a test to see what effect a rapid variation has on the surface.

7.2 The Results

For the first data set, the results are fairly competitive. If we sample each contour at 10 points, the thin-plate spline is well-behaved, and is almost conical as Figure (7.1) shows.

The piecewise biharmonic spline in Figure (7.2) is already guaranteed to have circular contours, so this is not a concern.

Figure (7.3) shows a cross-section where we see how well the spline recovers from the ninety degree corner at the origin.

The piecewise harmonic spline in Figure (7.4) already shows its creases even with such a simple data set. A brief calculation shows that all radially symmetric harmonic functions are of the form $a + b\ln(r)$ for some real constants a and b. This fact is evident when one looks at the cross-section in Figure (7.5).

For the second data set, the differences between the thin-plate spline and the

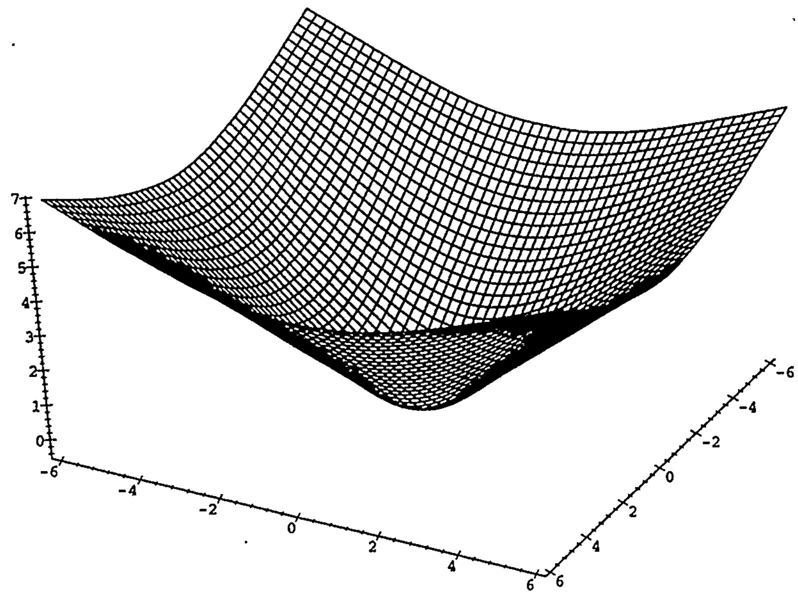

Figure 7.1: A Thin-plate Spline on Data Set 1 - 10 Points per Contour

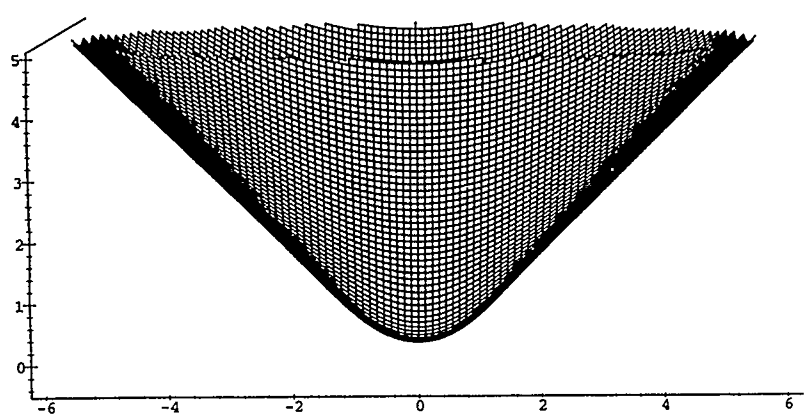

Figure 7.2: A Piecewise Biharmonic Spline on Data Set 1

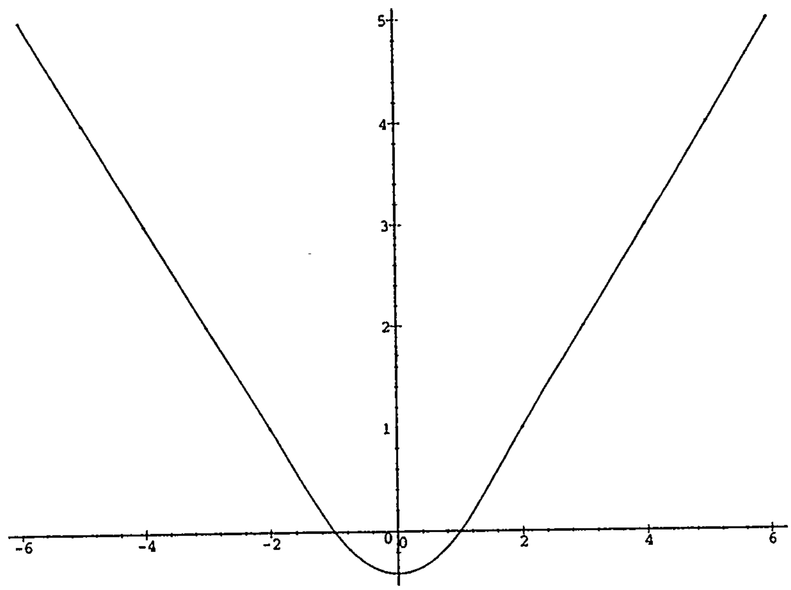

Figure 7.3: A Cross-section of the Spline in Fig. 7.2

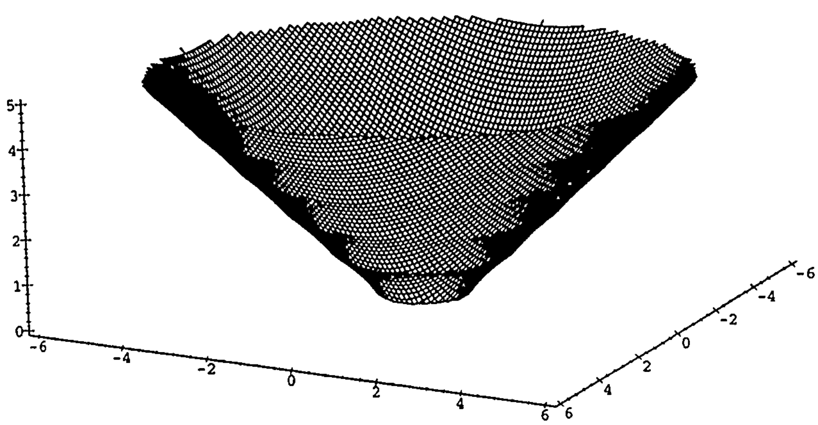

Figure 7.4: A Piecewise Harmonic Spline on Data Set 1

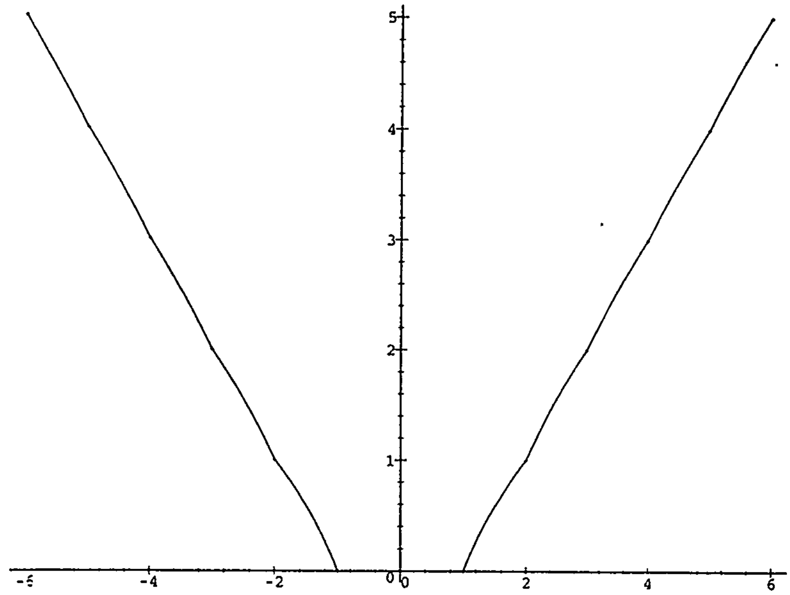

Figure 7.5: A Cross-section of the Spline in Fig. 7.4

piecewise biharmonic spline in Figures (7.6) and (7.7) are already more noticeable.

The change in curvature of the data pulls the thin-plate spline down in between each pair of data points on the 5^{th} contour as we can see in Figure (7.6). The piecewise biharmonic spline displayed in Figure (7.7) naturally has no such problems, and looking at the cross-section in Figure (7.8), we can see that it is not highly oscillatory.

If we try to correct the sagging in the thin-plate spline by sampling each contour at more points, the sagging is less pronounced, but eventually, we are unable to compute the surface accurately. Already at 40 points per contour, evaluation of the spline in Figure (7.9) requires solving 240 equations with 240 unknowns. Some small oscillations visible in the surface point to trouble ahead. If many more data points are added the system will become ill-conditioned.

In fact, to emphasize the instability of the problem, we consider two more exam-

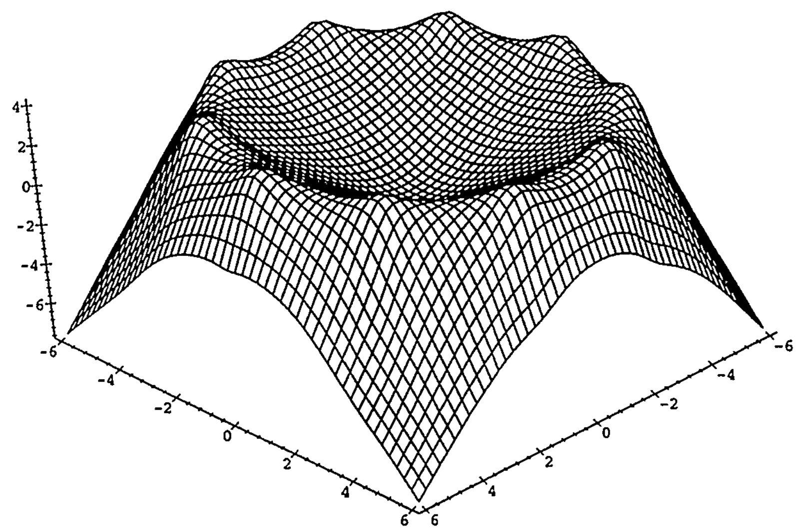

Figure 7.6: A Thin-plate Spline on Data Set 2 - 10 Points per Contour

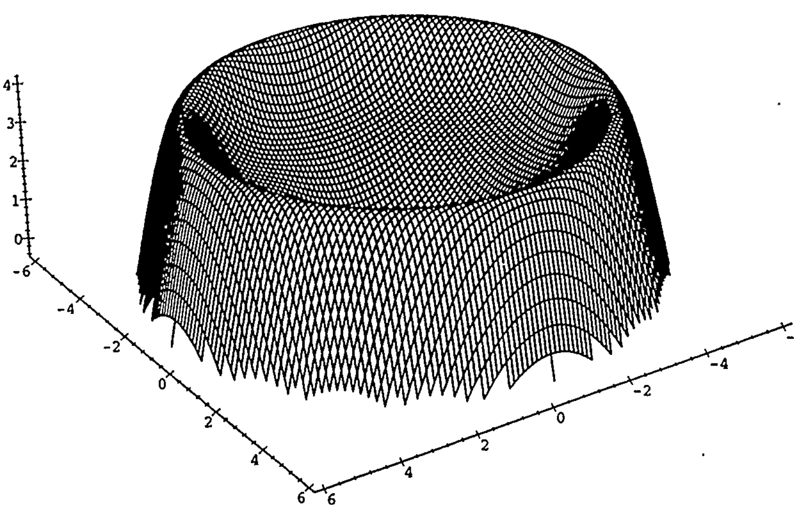

Figure 7.7: A Piecewise Biharmonic Spline on Data Set 2

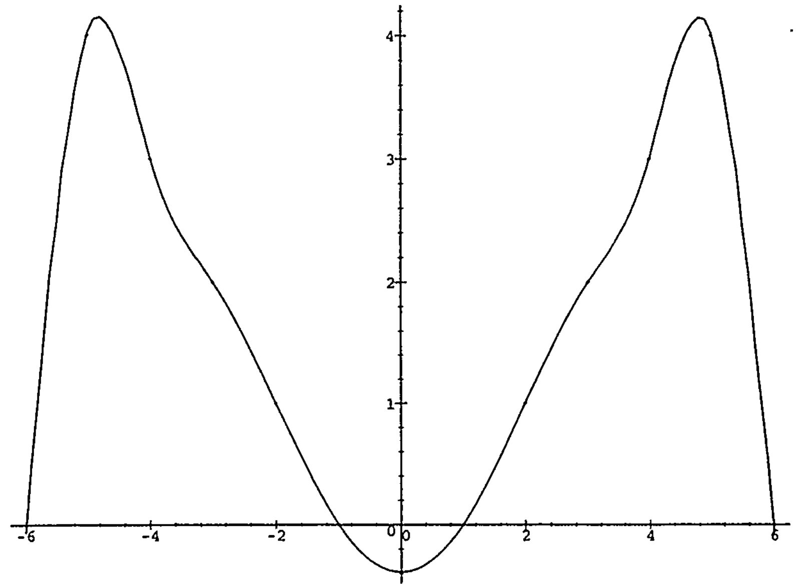

Figure 7.8: A Cross-section of the Spline in Fig. 7.7

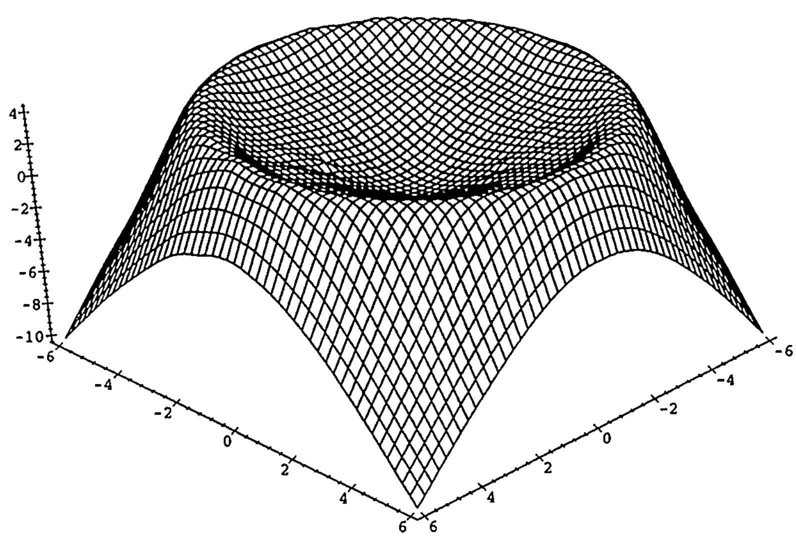

Figure 7.9: A Thin-plate Spline on Data Set 2 - 40 Points per Contour

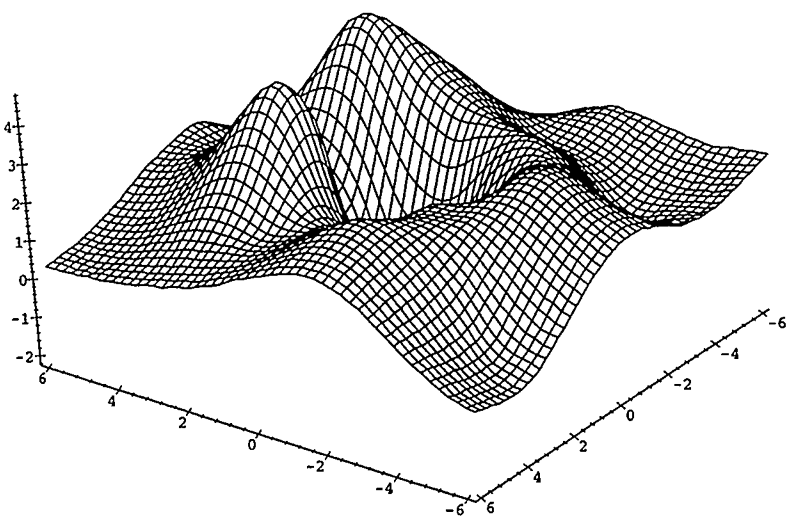

Figure 7.10: A Thin-plate Spline on Data Set 2 – 10 Points per contour, 5 Digits of Precision

ples, where we carry only 5 digits of precision for all calculations. For the thin-plate spline in Figure (7.10), we again sample at 10 uniformly spaces points per contour. The result is reasonable with the high-precision example in Figure (7.6), but when we reduce the precision in Figure (7.10), the surface bears no resemblance to the data.

When we reduce the precision for the piecewise biharmonic spline in Figure (7.11), we notice only that the surface is not as smooth as it should be. This roughness is to be expected given the low precision.

These limited figures indicate that it may be worth-while to search for algorithms for computing the piecewise biharmonic splines on more general domains.

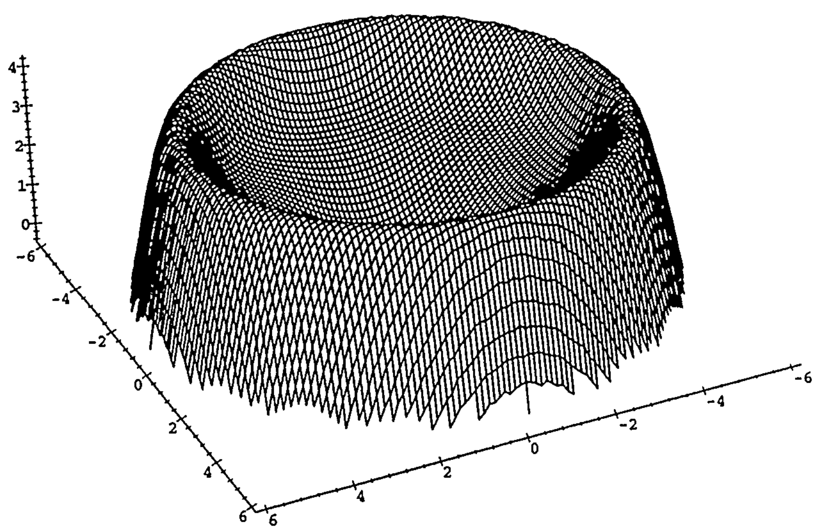

Figure 7.11: A Piecewise Biharmonic Spline on Data Set 2 – 5 Digits of Precision

Bibliography

[1] Adams, R.A., *Sobolev Spaces*, Academic Press, Harcourt Brace Jovanovich, 1975.

[2] Ahlberg, E.N., Nilson E.N., and Walsh J.L., *The Theory of Splines and their Applications*, Academic Press, New York, 1967.

[3] Aronszajn, N., Creese, T.M., and Lipkin, L.J., *Polyharmonic Functions*, Oxford Mathematical Monographs, Clarendon Press Oxford, 1983.

[4] Ash, R.B, *Measure, Integration, and Functional Analysis*, Academic Press, 1972.

[5] Axler, S., Bourdon, P., and Ramey, W., *Harmonic Function Theory*, Graduate Texts in Mathematics, Springer-Verlag New York, 1992.

[6] Bos, L.P., and Salkauskas, K., *Weighted Splines Based on Piecewise Polynomial Weight Functions*, in "Curve and Surface Design", SIAM Series on Geometric Design, Chapter 5, pp. 87-98, 1992.

[7] Ciarlet, P.G., *The Finite Element Methods for Elliptic Problems*, North-Holland Publishing Company, 1978.

[8] Courant, R., *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces* Pure and Applied Mathematics series Volume III, Interscience Publishers, New York, 1950.

[9] Dautray, R., and Lions, J.L., *Mathematical Analysis and Numerical Methods for Science and Technology (English trans.)*, Volume 2, Springer-Verlag Berlin

Heidelberg, 1988.

[10] Dieudonné, *Treatise on Analysis (English Translation)*, Volumes 10-VII and 10-VIII, Academic Press, Harcourt Brace Jovanovich, Publishers, 1988 and 1993.

[11] Friedlander, F.G., *Introduction to the Theory of Distributions* Cambridge University Press, 1982.

[12] Friedman, A., *Partial Differential Equations*, Holt, RineHart and Winston Inc., Robert E. Krieger Pub. Co. Inc., Florida, 1976.

[13] Fučik, S., Kufner, A., and John, O., *Function spaces*, Monographs and Textbooks on Mechanics of Solids and Fluids Number 3, Noordhoff International Publishing, Leyden The Netherlands, 1977.

[14] Gel'fand, I. and Shilov, G., *Generalized Functions*, Volume I, Academic Press, New York and London, 1964.

[15] Grabenstetter, J.E., *Aspects of Finite Element Interpolation with Smoothing*, Masters Thesis, Faculty of Science, University of Calgary, 1988.

[16] Grusa, K.U., *Zweidimensionale, Interpolierende Lg-Splines und ihre Anwendungen*, Lecture Notes in Mathematics Number 916, Springer-Verlag Berlin Heidelberg, 1982.

[17] Kaplan, W., *Advanced Calculus*, Second Edition, Addison-Wesley Publishing Co. Inc., 1973.

[18] Kellogg, O.D., *Foundations of Potential theory*, Frederick Ungar Publishing Company, New York, 1929.

[19] Lancaster, P., K. Salkauskas, *Curve and Surface Fitting*, Academic Press, Harcourt Brace Jovanovich, 1986.

[20] Marsden, J.E., Tromba, A.J., *Vector Calculus 3rd Ed.* W.H. Freeman and Company, 1988.

[21] Maz'ja, V.G., *Sobolev Spaces (English trans.)*, Springer Series in Soviet Mathematics, Springer-Verlag Berlin Heidelberg, 1985.

[22] Meinguet, J., *Multivariate Interpolation at Arbitrary Points Made Simple*, Z. Angew. Math. Phys. 30, 292-304, 1979.

[23] Monna, A.F., *Dirichlet's principle – A mathematical comedy of errors and its influence on the development of analysis*, Oosthoek, Scheltema and Holkema, Utrecht, The Netherlands, 1975.

[24] Nehari, Z., *Conformal Mapping*, Dover Publications Inc. New York, 1952.

[25] Oden, J.T., and Reddy, J.N., *An Introduction to the Mathematical Theory of Finite Elements*, Wiley Interscience, New York, 1976.

[26] Powell, M.J.D., *Approximation theory and Methods*, Cambridge University Press, 1981.

[27] Rektorys, K., *Variational Methods in Science and Engineering*, Reidl, Dordrecht, 1977.

[28] Roach, G.F., *Green's Functions, Introductory Theory with Applications*, Van Nostrand Reinhold Company London, 1970.

[29] Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill Book Company, Inc., New York, Third Edition, 1976.

[30] Schwartz, L., *Théorie des Distributions*, Volume II, Hermann, Paris, 1959.

[31] Sobolev, S.L., *Some Applications of Functional Analysis in Mathmatical Physics 3rd. Ed (English trans.)*, American Mathematical Society, 1991.

[32] Stein, E.M., *Singular Integrals and Differentiability Properties of Functions*, (Princeton Math. Series, Vol. 30), Chapter III, Princeton Univ. Press, Princeton New Jersey, 1970.

[33] Zeidler, E., *Nonlinear Functional Analysis and its Applications I (English trans.)*, Springer-Verlag New York, 1986.

[34] Zeidler, E., *Nonlinear Functional Analysis and its Applications II/A (English trans.)*, Springer-Verlag New York, 1990.