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A SHEAF THEORETICAL DEVELOPMENT OF DE RHAM'S THEOREM

by

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ABSTRACT

A complete and detailed sheaf theoretical proof of deRham's theorem is presented.

CONTENTS

INTRODUCTION

The object of this thesis is to present a complete and rigorous sheaf theoretical development of De Rham's Theorem for cohomology.

Sheaf theory provides an excellent basis for proving theorems in topology, differentiable manifolds, complex manifolds, differentiable and algebraic geometry. Sheaf theory is important in topology in that it can give relationships between local and global properties of a topological space.

At the start of chapter one, the definition of a sheaf of K-modules, where K is a principal ideal domain, is stated as well as the definition of a presheaf of K-modules. These two concepts are basic to the whole thesis. The reason for choosing principal ideal domains instead of arbitrary domains is that if K is a principal ideal domain, A', A and A'' are K-modules,

$$0 \longrightarrow A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \longrightarrow 0$$

is a short exact sequence of K-modules and C is a K-module then

$$0 \longrightarrow A' \otimes C \xrightarrow{\alpha \otimes Id} A \otimes C \xrightarrow{\beta \otimes Id} A'' \otimes C \longrightarrow 0$$

is a short exact sequence of K-modules if A'' or C is a torsion-free K-module.

Now there is a sheaf of K-modules associated to every presheaf of K-modules called the sheaf of germs of the presheaf. Also there is a presheaf of K-modules associated to every sheaf of K-modules called the presheaf of sections of the sheaf. These two concepts are very fundamental to the whole of this thesis.

There is a relationship between the sheaf of germs of the presheaf of sections of a sheaf S and the sheaf S. There is also a relationship between the presheaf of sections of the sheaf of germs of a presheaf T and the presheaf T which leads to the concept of a complete presheaf. These relationships are important in sheaf cohomology theory and are explored also in chapter one.

At the start of chapter two, exact sequences of sheaves and presheaves are discussed. Next paracompact spaces and fine sheaves are introduced. Fine sheaves have the useful property that if

$$0 \longrightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S'' \longrightarrow 0$$

is a short exact sequence of sheaves of K-modules over a paracompact space X then

$$0 \longrightarrow \Gamma(S') \xrightarrow{\Gamma(\alpha)} \Gamma(S) \xrightarrow{\Gamma(\beta)} \Gamma(S'') \longrightarrow 0$$

is a short exact sequence of K-modules where $\Gamma(S')$, $\Gamma(S)$ and $\Gamma(S'')$ are the K-modules of sections over X .

In chapter three, axiomatic sheaf cohomology theory is first introduced. Then given a torsion-free fine resolution of the constant sheaf $\mathcal{K} = X \times K$, where X is a paracompact space, it is proved that there exists an axiomatic sheaf cohomology theory. The idea of homomorphism of cohomology theories is then introduced. It is shown that given two axiomatic sheaf cohomology theories $\mathcal{H}$ and $\tilde{\mathcal{H}}$ there exists a unique isomorphism from $\mathcal{H}$ to $\tilde{\mathcal{H}}$.

In chapter four singular cohomology theory is presented and it is proved that the sheaves of germs of singular cochains over a HLC paracompact space X provide a torsion-free fine resolution of the constant sheaf $\mathcal{K}$.

The de Rham cohomology theory is dealt with in chapter five. It is shown, through the aid of Poincare's lemma, that the sheaves of germs of differential forms provide a torsion-free fine resolution of the constant sheaf $\mathcal{R} = X \times R$, where X is a differentiable manifold and R is the field of real numbers.

At the beginning of chapter six, the properties of the differentiable singular cohomology theory on a differentiable manifold X are examined. It is the isomorphism between the differentiable singular cohomology real vector spaces and de Rham cohomology real vector spaces over a differentiable manifold X, two of the most important cohomology vector spaces in applications, that is the main, final result (de Rham's theorem) of this thesis.

CHAPTER I

PRESHEAVES AND SHEAVES

§1. Sheaves of K-Modules

We first recall the definition of a principal ideal domain.

DEFINITION 1.1. A commutative integral domain K with 1 is called a principal ideal domain if every ideal of K is a principal ideal.

1.2 EXAMPLES

(i) The ring Z of integers

(ii) Any Euclidean domain, in particular the polynomial ring $F[X]$ in an indeterminate X over a field F are well known examples of principal ideal domains.

Throughout this chapter K denotes a fixed principal ideal domain and X is a fixed topological space.

DEFINITION 1.3. A sheaf of K-modules over X consists of a topological space S and an onto mapping $\pi : S \to X$ such that

(i) π is a local homeomorphism (that is, for any $a \in S$ there exists an open set $U \ni a$, $\pi(U)$ open in X and $\pi|_U : U \to \pi(U)$ a homeomorphism).

(ii) For any $x \in X$, $\pi^{-1}(x)$ is a K-module.

(iii) The operations on $\pi^{-1}(x)$ are continuous.

Condition (iii) needs an explanation.

Let $E = \{(u,v) \in S \times S \mid \pi(u) = \pi(v)\}$ and λ be any element of K. Then we require that the mappings $\varphi : E \to S$, $\theta_\lambda : S \to S$ given by $\varphi(u,v) = u + v$, $\theta_\lambda(a) = \lambda a$ for any $(u,v) \in E$, $a \in S$ be continuous.

The map $\pi : S \to X$ will be referred to as the projection map of the sheaf S. $\pi^{-1}(x)$ is called the stalk over x of S.

1.4 EXAMPLE. Let A be any K-module taken with the discrete topology, $S = X \times A$ with the product topology and $\pi : S \to X$ the mapping $\pi(x,a) = x$ for all $x \in X$. It is clear that S is a sheaf of K-modules with π as its projection. This sheaf will be referred to as the constant sheaf A.

Various examples of sheaves of K-modules will be given later on.

In what follows S denotes a fixed sheaf of K-modules with projection $\pi : S \to X$.

DEFINITION 1.5. An open subset U of S will be referred to as a special subset of S if $\pi(U)$ is open in X and $\pi|_U : U \to \pi(U)$ is a homeomorphism.

LEMMA 1.6. $\pi : S \to X$ is a continuous open map.

Proof. Since π is a local homeomorphism, it follows that π is a continuous open map.

LEMMA 1.7. For any $x \in X$, S induces the discrete topology on $\pi^{-1}(x)$.

Proof. Let $a \in \pi^{-1}(x)$. By (1.3(i)) there exists a special open set $U \ni a$ of S. Then for any $c \neq a$ in U we have $\pi(c) \neq \pi(a) = x$. Thus $c \notin \pi^{-1}(x)$ for any $c \neq a$ in U. In other words $\pi^{-1}(x) \cap U = \{a\}$. Hence $\{a\}$ is open in $\pi^{-1}(x)$.

DEFINITION 1.8. Let V be open in X.
A continuous map $\sigma : V \to S$ such that $\pi \circ \sigma(x) = x$ for all $x \in V$ is called a section of S over V.

For any $x \in X$ we denote the zero element of $\pi^{-1}(x)$ by 0_x.

LEMMA 1.9 Let V be any open set in X and $h : V \to S$ be given by $h(x) = 0_x$ for all $x \in V$. Then h is a continuous mapping.

Proof. Let $v \in V$ be arbitrary. By (1.3(i)) there exists a special open set U of S with $0_v \in U$. We can assume that $\pi(U) \subset V$. Let $i_U : U \to S$ denote the inclusion map. Then $\sigma : \pi(U) \to S$ given by $\sigma = i_U \circ (\pi|_U)^{-1}$ is continuous. By (1.3(III)) the map $\theta_0 : S \to S$ is continuous, where 0 denotes the zero element of K. Now the map $\theta_0 \circ \sigma : \pi(U) \to S$ carries any $y \in \pi(U)$ into 0_y. Thus $h|_{\pi(U)} = \theta_0 \circ \sigma$. This proves the continuity of h.

Let $T(X)$ be the family of open sets for any topological space X. For any $V \in T(X)$ let $\Gamma(S,V)$ denote the set of sections of S over V. For any σ, τ in $\Gamma(S,V)$ and $\lambda \in K$, it follows from (1.3(iii)) that the maps $\sigma+\tau : V \to S$, $\lambda\sigma : V \to S$ given by $(\sigma+\tau)(x) = \sigma(x) + \tau(x)$, $(\lambda\sigma)(x) = \lambda\sigma(x)$ are continuous. Also $\pi \circ (\sigma+\tau)(x) = x$, $\pi \circ (\lambda\sigma)(x) = x$ for all $x \in V$. Thus $\sigma+\tau$ and $\lambda\sigma$ are sections of S over V. With the above addition and multiplication by elements of K it is clear that the set $\Gamma(S,V)$ acquires the structure of a K-module.

DEFINITION 1.10 The above K-module $\Gamma(S,V)$ is called the K-module of sections of the sheaf S over V.

The zero element of $\Gamma(S,V)$ is the section $h : V \to S$ carrying any $x \in V$ to 0_x.

THEOREM 1.11 Let $V \in T(X)$ and $\sigma \in \Gamma(S,V)$. Then $\sigma : V \to S$ is an open map.

Proof. Let V' be any open subset of V. Any element of $\sigma(V')$ is of the form $\sigma(b)$ for some $b \in V'$. By (1.3(i)) there exists a special open set U of S with $\sigma(b) \in U$. Since σ is continuous it follows that $\sigma^{-1}(U)$ is open in V. Hence $V'' = V' \cap \sigma^{-1}(U)$ is open in V and hence open in X. Also $b \in V''$ and $\sigma(V'') \subset U$. Let $\alpha = \pi|_U : U \to \pi(U)$. Now α is a homeomorphism and $\alpha(\sigma(V'')) = V''$. Thus $\sigma(V'') = \alpha^{-1}(V'')$ is open in U and hence open in S. Clearly $\sigma(b) \in \sigma(V'') \subset \sigma(V')$. This proves that $\sigma(V')$ is open in S.

REMARK 1.12 Let $V \in T(X)$ and $\sigma \in \Gamma(S,V)$. By (1.11) we see that $\sigma(V) \in T(S)$. Let $a \in \sigma(V)$. From (1.3(i)) we see that there exists a special open set N of S with $a \in N \subset \sigma(V)$. Let β be the homeomorphism $\pi|_N : N \to \pi(N)$ and denote the open set $\pi(N)$ of X by W. It follows that $\beta^{-1} = \sigma|_W : W \to N$.

LEMMA 1.13. Let $V \in T(X)$, $\sigma, \tau \in \Gamma(S,V)$ and $b \in V$. Suppose $\sigma(b) = \tau(b)$. Then there exists an open set W of X with $b \in W \subset V$ and $\sigma|_W = \tau|_W$.

Proof. By (1.11) we have $\sigma(V)$ and $\tau(V)$ open in S. Hence $\sigma(V) \cap \tau(V)$ is open in S. If N is a special open set in S with $\sigma(b) = \tau(b) \in N \subset \sigma(V) \cap \tau(V)$ then from (1.12) we have $\sigma|_W = \beta^{-1} = \tau|_W$ where $W = \pi(N)$ and $\beta = \pi|_N : N \to W$.

COROLLARY 1.14 Let $V \in T(X)$, $\sigma \in \Gamma(S,V)$ and $b \in V$. Suppose $\sigma(b) = 0_b$. Then there exists a $W \in T(X)$ with $b \in W \subset V$ and $\sigma(x) = 0_x$ for all $x \in N$.

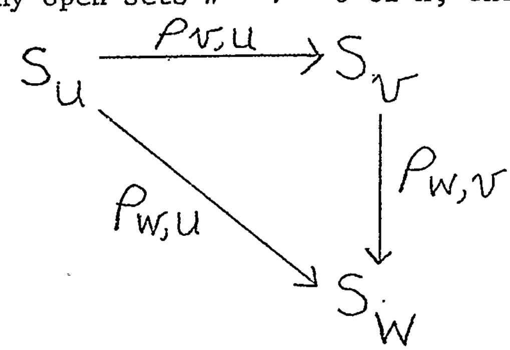
Proof. Take τ to be the zero element of $\Gamma(S,V)$ and apply (1.13).

§2. Presheaves of K-Modules

DEFINITION 2.1. A presheaf $P = \{S_U, \rho_{V,U}\}$ of K-modules over X consists of a K-module S_U for each open set U of X and a K-module homomorphism $\rho_{V,U} : S_U \to S_V$ whenever $V \subset U$ are open sets in X satisfying the following conditions.

(i) $\rho_{U,U} = Id_{S_U}$.

(ii) For any open sets $W \subset V \subset U$ of X, the diagram



commutes (i.e. $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$).

2.2 EXAMPLES.

(1) Let A be any K-module. For every $U \in T(X)$ let $S_U = A$ and for U,V in $T(X)$ satisfying $V \subset U$ let $\rho_{V,U} : A \to A$ be the identity map of A. This is clearly a presheaf of K-modules over X. This presheaf will be referred to as the constant presheaf A.

(2) Let S be any sheaf of K-modules over X. Let $P(S)_U = \Gamma(S,U)$ and $\rho_{V,U}(\sigma) = \sigma|_V$ for any U,V in $T(X)$ satisfying $V \subset U$ and any $\sigma \in \Gamma(S,U)$. It is clear that $P(S) = \{P(S)_U, \rho_{V,U}\}$ is a presheaf of K-modules over X.

This presheaf $P(S)$ will be referred to as the presheaf of sections of S.

(3) Let M be a C^∞ differentiable manifold. For any $U \in T(M)$ let $C^\infty(U)$ be the vector space of real differentiable functions on U and $B(U)$ be the vector space of vector fields on U. Let $V, U \in T(M)$ and $V \subset U$.

Let $\rho_{V,U}(f) = f|_V$ for any $f \in C^\infty(U)$ and $\rho'_{V,U}(g) = g|_V$ for any $g \in B(U)$. Then $C^\infty = \{C^\infty(U), \rho_{V,U}\}$ and $B = \{B(U), \rho'_{V,U}\}$ are presheaves of real vector spaces over M.

(4) Let N be a complex analytic manifold. For any $U \in T(N)$ let $H(U)$ be the vector space of complex holomorphic functions on U. When V,U in $T(N)$ satisfy $V \subset U$ let $\rho_{V,U}f = f|_V$ for any $f \in H(U)$. $H = \{H(U), \rho_{V,U}\}$ is a presheaf of complex vector spaces over N.

§3. The Sheaf Associated to a Presheaf

For any $x \in X$ let Q_x denote the family of open neighbourhoods of x in X. We partially order Q_x by defining $V > U$ whenever $U,V \in Q_x$ and $V \subset U$. Then for any U_1, U_2 in Q_x we have $U_1 \cap U_2 \in Q_x$ and $U_1 \cap U_2 > U_j$ $(j = 1,2)$. Thus Q_x is a directed set.

Let $P = \{S_U, \rho_{V,U}\}$ be a presheaf of K-modules over X. For any $x \in X$ let $S_x = \varinjlim \{S_U, \rho_{V,U}\}$ taken over the directed set Q_x. We briefly recall the definition of S_x.

Let E_x be the disjoint union of the sets S_U for $U \in Q_x$. For $U,V \in Q_x$ and $a \in S_U$, $b \in S_V$ we define a to be equivalent to b if there exists a $W \in Q_x$ with $W \subset U$ and $W \subset V$ (i.e. $W \subset U \cap V$) and also $\rho_{W,U}(a) = \rho_{W,V}(b)$. This is easily seen to be an equivalence relation on E_x and S_x is the set of equivalence classes. We write $\rho_{x,U} : S_U \to S_x$ for the map which assigns to any $a \in S_U$ the equivalence class containing a. If $s_1, s_2 \in S_x$ and $\lambda \in K$ the elements $s_1 + s_2$ and λs_1 of S_x are defined as follows: If $a_1 \in S_U$, $a_2 \in S_V$ such that $s_1 = \rho_{x,U} a_1$ and $s_2 = \rho_{x,V} a_2$, choose any $W \in Q_x$ satisfying $W \subset U \cap V$ and set

$$s_1 + s_2 = \rho_{x,W} (\rho_{W,U} (a_1) + \rho_{W,V}(a_2))$$

and
$$\lambda s_1 = \rho_{x,U} (\lambda a_1)$$

Then it is easily checked that the elements $s_1 + s_2$, λs_1 are independent of the choices made and that S_x is a K-module. Also each map $\rho_{x,U} : S_U \to S_x$ is a K-module homomorphism.

Let $S(P) = \bigcup_{x \in X} S_x$ (disjoint union).

For any $U \in T(X)$ and any $a \in S_U$ let $N_a = \{\rho_{x,U}(a) \mid x \in U\} \subset S(P)$. The map $x \longrightarrow \rho_{x,U}(a)$ is a set theoretic injection of U in N_a.

THEOREM 3.1 The family $\mathcal{B} = \{N_a \mid a \in S_U, U \in T(X)\}$ is a base for a topology on $S(P)$.

Proof. Let $s \in S(P)$. Then $s \in S_x$ for some (actually unique) $x \in X$. Thus $s = \rho_{x,U}(a)$ for some $a \in S_U$ where $U \in Q_x$. So $s \in N_a$ for a $N_a \in \mathcal{B}$. This proves that $S(P) = \bigcup_{N_a \in \mathcal{B}} N_a$.

Suppose N_a, N_b are any sets belonging to $\mathcal{B}$ and $s \in N_a \cap N_b$. If $s \in S_x$ then $a \in S_U$, $b \in S_V$ for some U,V in Q_x. Also $\rho_{x,U}(a) = s = \rho_{x,V}(b)$. Hence there exists some $W \in Q_x$ with $W \subset U \cap V$ and $\rho_{W,U}(a) = \rho_{W,V}(b)$. If we denote the element $\rho_{W,U}(a) = \rho_{W,V}(b)$ of S_W by c, then $s \in N_c \subset N_a \cap N_b$.

Thus the conditions that $\mathcal{B}$ be a base for a topology on $S(P)$ are satisfied.

THEOREM 3.2 Let $\pi : S(P) \to X$ be given by $\pi(s) = x$ for any $s \in S_x$. With the topology determined by the base $\mathcal{B}$, $S(P)$ is a sheaf of K-modules with π as its projection.

Proof. For any basic open set N_a in $S(P)$ with $a \in S_U$ and U open in X, it is clear that $\pi(N_a) = U$. Thus π carries basic open sets of $S(P)$ onto open sets of X and so π is an open map.

It is clear that $\pi|N_a : N_a \to U$ is a one-to-one map of N_a onto U. For any open set $V \subset U$, it is clear that $(\pi|_{N_a})^{-1}(V) = N_b$ where $b = \rho_{V,U}(a)$. Thus $\pi|_{N_a}$ is a continuous map. It follows that $\pi|_{N_a} : N_a \to U$ is a one-to one, open and continuous map and so is a homeomorphism. This proves $\pi : S(P) \to X$ is a local homeomorphism.

Clearly $\pi^{-1}(x) = S_x$ is a K-module.

Let $E = \{(s,t) \in S(P) \times S(P) | \pi(s) = \pi(t)\}$ and $\varphi : E \to S(P)$ be the map $\varphi(s,t) = s + t$. Let $(s,t) \in E$ be arbitrary. Let $\pi(s) = \pi(t) = x \in X$. Any basic open set in $S(P)$ containing $s + t$ is of the form N_d with $d \in S_W$, $W \in Q_x$ and $\rho_{x,W}(d) = s + t$. Let $s = \rho_{x,U}(a)$ and $t = \rho_{x,V}(b)$ with U,V in Q_x and $a \in S_U$, $b \in S_V$. Since $\rho_{x,U}(a) + \rho_{x,V}(b) = \rho_{x,W}(d)$ there exists $N \in Q_x$ such that $N \subset U \cap V \cap W$ and $\rho_{N,W}(d) = \rho_{N,U}(a)+\rho_{N,V}(b)$. Let $d' = \rho_{N,W}(d)$, $a' = \rho_{N,U}(a)$ and $b' = \rho_{N,V}(b)$. Then $d' = a' + b'$. Hence $\rho_{y,N}(d') = \rho_{y,N}(a') + \rho_{y,N}(b')$ for every $y \in N$. It follows that $\varphi((N_{a'} \times N_{b'}) \cap E) \subset N_{d'}$. But $N_{d'} \subset N_d$, $s \in N_{a'}$ and $t \in N_{b'}$. So $(s,t) \in (N_{a'} \times N_{b'}) \cap E$ and $\varphi((N_{a'} \times N_{b'}) \cap E) \subset N_d$. This proves the continuity of φ.

Let $\lambda \in K$ and $\theta_\lambda : S(P) \to S(P)$ be given by $\theta_\lambda(s) = \lambda s$. Let

$s \in S(P)$ be arbitrary. Then $\pi(s) = \pi(\lambda s)$. Let $x = \pi(s) = \pi(\lambda s)$. Any basic open set containing λs is of the form N_d with $d \in S_U$ for some $U \in Q_x$ and $\rho_{x,U}(d) = \lambda s$. We can write $s = \rho_{x,V}(a)$ for some $a \in S_V$ with $V \in Q_x$. From $\rho_{x,V}(\lambda a) = \lambda s = \rho_{x,U}(d)$ it follows that there exists a $W \in Q_x$ such that $W \subset U \cap V$ and $\rho_{W,V}(\lambda a) = \rho_{W,U}(d)$. If $a' = \rho_{W,V}(a)$ then $s = \rho_{x,W}(a')$. Therefore $\lambda\rho_{y,W}(a') = \lambda\rho_{y,W}(d')$ for all $y \in W$, where $d' = \rho_{W,U}(d)$. From this we get $\theta_\lambda(N_{a'}) \subset N_{d'} \subset N_d$. Clearly $s \in N_{a'}$. This proves the continuity of θ_λ.

DEFINITION 3.3. The sheaf $S(P)$ is called the sheaf associated to the presheaf P or the sheaf of germs of the presheaf P.

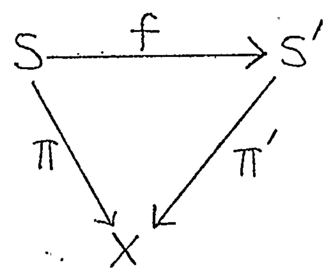
REMARK 3.4. The choice of the topology for $S(P)$ is important. Each S_x depends only on the K-modules S_u for $U \in Q_x$. Choose $\mathcal{B} = \{N_a | a \in S_U, U \in T(X)\}$ as a base for a topology on $S(P)$ then we can try to relate the various S_x's.

§4. Homomorphisms of Sheaves and Presheaves.

All the sheaves and presheaves considered here are of K-modules.

DEFINITION 4.1. Let S and S' be sheaves over X with projections $\pi : S \to X$ and $\pi' : S' \to X$. A sheaf homomorphism $f : S \to S'$ is a continuous map such that

(i) is commutative and



(ii) for each $x \in X$, $f_x = f|_{S_x}$ is a K-module homomorphism of S_x into S'_x where $S_x = \pi^{-1}(x)$ and $S'_x = \pi'^{-1}(x)$.

DEFINITION 4.2. A sheaf isomorphism is a sheaf homomorphism with an inverse which is also a sheaf homomorphism.

DEFINITION 4.3. Let S be a sheaf over X with projection $\pi : S \to X$. A sheaf S' with projection $\pi' : S' \to X$ is called a subsheaf of S if

(i) S' is a subspace of S

(ii) $\pi' = \pi|_{S'}$ and

(iii) S'_x is a submodule of S_x for every $x \in X$.

LEMMA 4.4 If S' is a subsheaf of S then S' is open in S.

Proof. Let $s' \in S'$. By (1.3(i)) there exists a special open set U' of S' with $s' \in U'$. Then $\pi'|_{U'} = \pi|_{U'}$ is a homeomorphism of U' onto an open set V' in X. Let $j : S' \to S$ denote the inclusion. Then $\sigma = j \circ (\pi|_{U'})^{-1} : V' \to S$ is a section of S. Since V' is open in X it follows that $\sigma(V')$ is open in S. But $\sigma(V') = U'$. Thus U' is open in S. Thus given any $s' \in S'$ there exists an open set U' of S with $s' \in U' \subset S'$. This proves that S' is open in S.

Conversely, if S' is an open subset of S such that $S'_x = S' \cap S_x$ is a K-submodule of S_x for every $x \in X$, it is easily seen that S' is a subsheaf of S.

THEOREM 4.5. Let R be a subsheaf of S. For each $x \in X$, let $n_x : S_x \to \frac{S_x}{R_x}$ be the canonical quotient map. Let $\frac{S}{R}$ denote the disjoint union $\bigcup_{x \in X} \frac{S_x}{R_x}$ and $n : S \to \frac{S}{R}$ be given by $n|_{S_x} = n_x$. Let $\frac{S}{R}$ be provided with the quotient topology (i.e. a set T of $\frac{S}{R}$ is open if and only if $n^{-1}(T)$ is open in S). Then $n : S \to \frac{S}{R}$ is an open map.

Proof. Let H be any open subset of S. To show that $n(H)$ is open, we have to prove that $n^{-1}(n(H))$ is open in S. Let $s \in n^{-1}(n(H))$. Then there exists a $t \in H$ such that $n(s) = n(t)$. Let $\pi(s) = x$. Then $\pi(t) = x$ and $n_x(s-t) = 0$. Hence $(s-t) \in R_x$. Thus there exists an $r \in R_x$ with $s = t + r$. Also H is an open set in S with $t \in H$. By (1.3(i)) applied to S and R respectively, we get a special open set M in S with $t \in M \subset H$ and a special open set N of R with $r \in N$. By (4.4), N is open in S as well. Let $\alpha = \pi|_M$ and $\beta = \pi|_N$. Then $\alpha(M)$, $\beta(N)$ are open in X and $\alpha : M \to \alpha(M)$, $\beta : N \to \beta(N)$ are homeomorphisms. If $P = \alpha(M) \cap \beta(N)$ then $P \in Q_x$. Let $\gamma = \alpha^{-1}|_P$, $\delta = \beta^{-1}|_P$. Then γ and δ are elements of $\Gamma(S,P)$. From (1.11) we see that $(\gamma+\delta)(P)$ is open in S since $\gamma + \delta \in \Gamma(S,P)$. For any $y \in P$, $\gamma(y) \in M \subset H$ and $\delta(y) \in R$. Hence $n(\gamma(y) + \delta(y)) = n_y(\gamma(y) \in n(H))$. This implies $\gamma(y) + \delta(y) \in n^{-1}(n(H))$ for all $y \in P$. Clearly $\gamma(x) + \delta(x) = t + r = s$. Hence $s \in (\gamma+\delta)(P) \subset n^{-1}(n(H))$. This proves that $n^{-1}(n(H))$ is open in S.

THEOREM 4.6 Let R be a subsheaf of S and $n : S \to \frac{S}{R}$ the map introduced in (4.5). Let $\pi' : \frac{S}{R} \to X$ be given by $\pi'\left(\frac{S_x}{R_x}\right) = x$. With the quotient topology on $\frac{S}{R}$, it is a sheaf over X with π' as its projection.

Proof. For any open set V in X, $n^{-1}(\pi'^{-1}(V)) = \pi^{-1}(V)$ is open in S. Hence $\pi'^{-1}(V)$ is open in $\frac{S}{R}$. This means $\pi' : \frac{S}{R} \to X$ is continuous. Also if H is open in $\frac{S}{R}$ then $n^{-1}(H)$ is open in S, (by the definition of quotient topology) and $\pi'(H) = \pi(n^{-1}(H))$. Since π is open it follows that $\pi(n^{-1}(H))$ is open in X. Hence $\pi' : \frac{S}{R} \to X$ is an open map. Let $c \in \frac{S}{R}$. There exists a $s \in S$ satisfying $n(s) = c$. Let U be a special open set in S with $s \in U$. Then $n(U)$ is open in $\frac{S}{R}$ by (4.5). From $\pi' \circ n = \pi$ and the fact that $\pi|_U$ is a one-to-one map we see that $\pi' : n(U) \to \pi(U)$ is a one-to-one map. Now π' is continuous and open, so $\pi'|n(U) : n(U) \to \pi(U)$ is a homeomorphism. This proves that π' is a local homeomorphism.

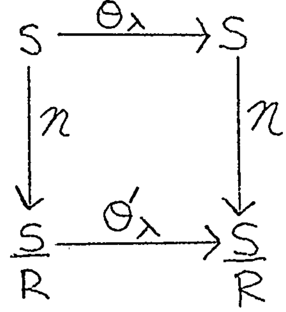
Let $E(S) = \{(s,t) \in S \times S | \pi(s) = \pi(t)\}$ and $E\left(\frac{S}{R}\right) = \{(c,d) \in \frac{S}{R} \times \frac{S}{R} | \pi'(c) = \pi'(d)\}$. Let $\varphi : E(S) \to S$, $\varphi' : E\left(\frac{S}{R}\right) \to \frac{S}{R}$ be given by $\varphi(s,t) = s+t$, $\varphi'(c,d) = c+d$. Let A be any open subset of $\frac{S}{R}$ containing $c_0 + d_0$ where $(c_0,d_0) \in E\left(\frac{S}{R}\right)$. Let $(s_0,t_0) \in S \times S$ satisfy

$$n(s_0) = c_0,\ n(t_0) = d_0. \text{ Then } \pi(s_0) = \pi'(c_0) = \pi'(d_0) = \pi(t_0).$$

Hence $(s_0,t_0) \in E(S)$. The set $B = n^{-1}(A)$ is open in S and $s_0 + t_0 \in B$. The continuity of φ give s open sets U_0, V_0 in S with $s_0 \in U_0$, $t_0 \in V_0$ and $\varphi((U_0 \times V_0) \cap E(S)) \subset B$. Then $C_0 = n(U_0)$, $D_0 = n(V_0)$ are open in $\frac{S}{R}$, $c_0 \in C_0$, $d_0 \in D_0$ and $\varphi'((C_0 \times D_0) \cap E(\frac{S}{R})) \subset A$. This proves the continuity of φ'.

Let $\lambda \in K$, $\theta_\lambda : S \to S$, $\theta'_\lambda : \frac{S}{R} \to \frac{S}{R}$ be the maps defined by $\theta_\lambda(s) = \lambda s$, $\theta'_\lambda(c) = \lambda c$ for all $s \in S$, $c \in \frac{S}{R}$.

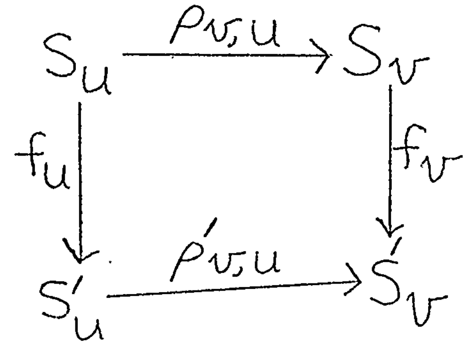
Clearly the diagram



commutes.

If C is any open set in $\frac{S}{R}$, $n^{-1}(\theta'^{-1}_\lambda(C)) = \theta_\lambda^{-1}(n^{-1}(C))$ is open in S since θ_λ and n are continuous. Hence $\theta'^{-1}_\lambda(C)$ is open in $\frac{S}{R}$. This proves the continuity of θ'_λ.

DEFINITION 4.7. Let $P = \{S_U, \rho_{V,U}\}$ and $P' = \{S'_U, \rho'_{V,U}\}$ be two presheaves over X. A presheaf homomorphism $f : P \to P'$ consists of a K-module homomorphism $f_U : S_U \to S'_U$ for each $U \in T(X)$ satisfying the condition that



is commutative for all open sets $V \subset U$ of X.

Given any homomorphism $\varphi : S \to S'$ of sheaves over X, it induces a homomorphism $P(\varphi): P(S) \to P(S')$ of the presheaves of sections of S and S' respectively. For any $U \in T(X)$ and any section $\sigma : U \to S$ of S over U the composite $\varphi \circ \sigma: U \to S'$ is a section of S' over U. We define $P(\varphi)_U : \Gamma(S,U) \to \Gamma(S',U)$ by $P(\varphi)_U(\sigma) = \varphi \circ \sigma$ for any $\sigma \in \Gamma(S,U)$. It is easy to see that each $P(\varphi)_U$ is a K-module homomorphism. Also for any $\sigma \in \Gamma(S,U)$ and any open $V \subset U$ it is clear that $\varphi \circ (\sigma|V) = (\varphi \circ \sigma)|V$. This shows that the diagram

$$\begin{array}{ccc} \Gamma(S,U) & \xrightarrow{\rho_{V,U}} & \Gamma(S,V) \\ {\scriptstyle P(\varphi)_U}\downarrow & & \downarrow{\scriptstyle P(\varphi)_V} \\ \Gamma(S',U) & \xrightarrow{\rho'_{V,U}} & \Gamma(S',V) \end{array}$$

is commutative. Hence $P(\varphi) = \{P(\varphi)_U\}$ is a homomorphism of presheaves.

DEFINITION 4.8. $P(\varphi)$ is called the homomorphism of presheaves of sections induced by φ.

Let $f = \{f_U\} : P = \{S_U, \rho_{V,U}\} \to P' = \{S'_U, \rho'_{V,U}\}$ be a homomorphism of presheaves. Let $S(P)$ and $S(P')$ denote the sheaves associated to P and P'. Let $s \in S_x$. Let $a \in S_U$ with $U \in Q_x$ be such that $\rho_{x,U}(a) = s$. Define $f_x(s)$ to be the element $\rho'_{x,U}(f_U(a))$. First we will show that f_x is well defined. If $s = \rho_{x,V}(b)$ with $V \in Q_x$ and $b \in S_V$, then there exists a $W \in Q_x$ such that $W \subset U \cap V$ and $\rho_{W,U}(a) = \rho_{W,V}(b)$. Then

$$\begin{aligned} \rho'_{x,U}(f_U(a)) &= \rho'_{x,W}(\rho'_{W,U}(f_U(a))) \\ &= \rho'_{x,W}(f_W\rho_{W,U}(a)) \\ &= \rho'_{x,W}(f_W\rho_{W,V}(b)) \\ &= \rho'_{x,W}(\rho'_{W,V}(f_V(b))) \\ &= \rho'_{x,V}(f_V(b)). \end{aligned}$$

This proves that f_x is well defined.

Let s,t be elements of S_x and $\lambda \in K$. Then we can find a $U \in Q_x$ and b,c in S_U such that $\rho_{x,U}(b) = s$ and $\rho_{x,U}(c) = t$. Then $s+t = \rho_{x,U}(b+c)$. Hence $f_x(s+t) = \rho'_{x,U}(f_U(b+c)) = \rho'_{x,U}f_U(b) + \rho'_{x,U}f_U(c) = f_x(s) + f_x(t)$ and $f_x(\lambda s) = \rho'_{x,U}(f_U(\lambda b)) = \lambda\rho'_{x,U}(b) = \lambda f_x(s)$. This shows that $f_x : S_x \to S_x$ is a K-module homomorphism for each $x \in X$.

LEMMA 4.9. The map $S(f) : S(P) \to S(P')$ defined by $S(f)|_{S_x} = f_x$ is a homomorphism of sheaves.

Proof. To prove this lemma the only fact that remains to be checked is the continuity of $S(f)$. Let $s \in S_x$ and $s' = f_x(s)$. A basic open set containing s' is of the form $N_{a'}$, where $a' \in S'_U$, $U \in Q_x$ and $\rho'_{x,U}(a') = s'$. Now there exists some $V \in Q_x$ and $c \in S_V$ such that $s = \rho_{x,V}\, c$. Then by the definition of f_x we get $\rho'_{x,U}(a') = \rho'_{x,V}f_V(c)$ Hence there exists a $W \in Q_x$ such that $W \subset U \cap V$ and $\rho'_{W,U}(a') = \rho'_{W,V}f_V(c)$. Let $d = \rho_{W,V}(c) \in S_W$. Then $N_d = \{\rho_{y,W}(d) \mid y \in W\}$ is an open set in S with $s = \rho_{x,W}(d) \in N_d$. Also

$$\begin{aligned} f_y\rho_{y,W}(d) &= \rho'_{y,W}f_W(d) = \rho'_{y,W}f_W\rho_{W,V}(c) \\ &= \rho'_{y,W}\rho'_{W,V}f_V(c) = \rho'_{y,W}\rho'_{W,U}(a') \\ &= \rho'_{y,U}(a') \in N_{a'}. \end{aligned}$$

Thus $S(f)(N_d) \subset N_{a'}$. This proves the continuity of $S(f)$.

LEMMA 4.10. Any sheaf homomorphism $f : S \to S'$ is an open map.

Proof. From (1.3(i)) it is clear that special open sets form a base for the topology of S. If U is any special open set of S, $\pi(U) = V$ is open in X and $\alpha = \pi|_U : U \to V$ is a homeomorphism. Thus $\gamma : V \to S$

defined by $\gamma(x) = \alpha^{-1}(x)$ is a section of S over V. It follows that $f \circ \gamma : V \to S'$ is a section of S' over V. From $f(U) = f(\gamma(V))$ and (1.11) we see that $f(U)$ is open in S'. This proves that f is open.

Let S,T be sheaves over X with $\pi : S \to X$, $p : T \to X$ as the projection maps. We denote the stalks over x of S and T by S_x and T_x respectively. Let $f : S \to T$ be a homomorphism of sheaves. Let f_x denote the K-module homomorphism $f|_{S_x} : S_x \to T_x$. Let $S'_x = \text{Ker}(f_x)$ and $T'_x = \text{Image}(f_x)$. Then S'_x is a submodule of S_x, T'_x is a submodule of T_x for all $x \in X$. Let $T' = \bigcup_{x \in X} T'_x$. As an immediate consequence of (4.10) we see that T' is an open subset of T. Hence T' is a subsheaf of T with $p' = p|_{T'}$ as its projection. If 0_T denotes the zero section of T then the image $0_T(X)$ of 0_T is open in T by (1.11). Hence $f^{-1}(0_T(X))$ is open in S. But $f^{-1}(0_T(X)) = S'$ where $S' = \bigcup_{x \in X} S'_x$. It follows that S' is a subsheaf of S with $\pi' = \pi|_{S'}$ as its projection.

DEFINITION 4.11. Let $f : S \to T$ be a sheaf homomorphism. The subsheaf S' of S defined above is called the kernel of f and is denoted $\text{Ker}(f)$. The subsheaf T' of T defined above is called the image of f and is denoted by $\text{Im}(f)$.

The constant sheaf over X determined by the zero module will be denoted by 0.

§5. Complete Presheaves.

Let S be any sheaf over X and $P(S)$ the associated presheaf. Then for any $U \in T(X)$, let $P(S)_U = \Gamma(S,U)$ the module of sections of S over U. Also for any U, V in $T(X)$ satisfying $V \subset U$ let the map $\rho_{V,U} : P(S)_U \to P(S)_V$ be given by $\rho_{V,U}(\sigma) = \sigma|_V$ for any $\sigma \in \Gamma(S,U)$.

Let S_x be the stalk over x of S. Let $S(P(S))$ be the sheaf associated to the presheaf $P(S)$. The stalk $S(P(S))_x$ over x of $S(P(S))$ is $\varinjlim \{P(S)_U, \rho_{V,U}\}$ taken over the directed set Q_x.

Let $s \in S(P(S))_x$ and $\sigma \in \Gamma(S,U) = P(S)_U$ a representative of s. Define $\theta_x : S(P(S))_x \to S_x$ by $\theta_x(s) = \sigma(x)$. To see that θ_x is well defined suppose $\tau \in \Gamma(S,V)$ also represents s. Then there exists a $W \in Q_x$ with $W \subset U \cap V$ and $\rho_{W,U}(\sigma) = \rho_{W,V}(\tau)$. Thus $\sigma|_W = \tau|_W$. In particular $\sigma(x) = \tau(x)$. If S_1 and S_2 are elements of $S(P(S))_x$ we can pick a $U \in Q_x$ such that there exists σ_1, σ_2 in $\Gamma(S,U)$ representing S_1 and S_2 respectively. Then $\sigma_1+\sigma_2$ represents S_1+S_2 and for any $\lambda \in K$ the element $\lambda\sigma_1$ of $\Gamma(S,U)$ represents λS_1. From $(\sigma_1+\sigma_2)(x) = \sigma_1(x) + \sigma_2(x)$, $(\lambda\sigma_1)(x) = \lambda\sigma_1(x)$ it follows that $\theta_x : S(P(S))_x \to S_x$ is a K-module homomorphism.

LEMMA 5.1 $\theta_x : S(P(S))_x \to S_x$ is an isomorphism for each $x \in X$.

Proof. Let $a \in S_x$. Then by (1.3(i)) there exists a special open set $N \ni a$ in S. If $V = \pi(N)$ then $\sigma = j \circ (\pi|_N)^{-1} : V \to S$ is a section of S over V, where $j : N \to S$ denotes the inclusion. Since $\sigma(x) = a$, we see that $\theta_x(s) = a$ where $s \in S(P(S))_x$ is represented by $\sigma \in \Gamma(S,V)$. Thus $\theta_x : S(P(S))_x \to S_x$ is an onto mapping.

Suppose $s \in S(P(S))_x$ satisfies $\theta_x(s) = 0$. Let $\tau \in \Gamma(S,U)$ represents s. By the definition of θ_x we have $\tau(x) = 0_x$. By (1.14) there exists a $W \in Q_x$ such that $\tau(y) = 0_y$ for all $y \in W$. In other words $\rho_{W,U}(\tau) = 0$ in $\Gamma(S,W)$. Since s is also represented by $\rho_{W,U}(\tau)$, it follows that $s = 0$. This proves that $\theta_x : S(P(S))_x \to S_x$ is a K-module monomorphism.

THEOREM 5.2. The map $\theta : S(P(S)) \to S$ defined by $\theta|_{S(P(S))_x} = \theta_x$ is an isomorphism of sheaves of K-modules.

Proof. Because of lemma (4.10) and (5.1) we need only prove the continuity of θ. Let N be any special open set in S and $V = \pi(N)$. Then $\sigma = j \circ (\pi|_N)^{-1} : V \to S$ defined in (5.1) is an element of $\Gamma(S,V)$ and the open set N_σ of $S(P(S))$ given by $N_\sigma = \{\rho_{y,V}(\sigma) | y \in V\}$ is precisely the inverse image $\theta^{-1}(N)$ of N by θ. This proves the continuity of θ.

The essence of theorem (5.2) is that if we started with a sheaf S over X and passed to the sheaf associated to the presheaf of sections of S then we get a sheaf which is canonically isomorphic to S.

Let $P = \{P_U, \rho_{V,U}\}$ be a presheaf over X, $S(P)$ the sheaf associated to P and $P(S(P))$ the presheaf of sections of $S(P)$. Let $a \in P_U$ with $U \in T(X)$. Then $N_a = \{\rho_{x,U}(a) | x \in U\}$ is open in $S(P)$ and $\pi : N_a \to U$ defined by $\pi(\rho_{x,U}(a)) = x$ is a homeomorphism of N_a onto U. (see the proof of 3.2). Hence $\tilde{a} : U \to S(P)$ defined by $\tilde{a}(x) = \rho_{x,U}(a)$ is a section of $S(P)$ over U. Consider the map $h_U : P_U \to \Gamma(S(P),U)$ given by $h_U(a) = \tilde{a}$. It is clear that h_U is a K-module homomorphism. Also if U,V are any two open sets in X with $V \subset U$ and $\rho_{V,U}(a) = b \in P_V$ then for any $x \in V$ we have $\rho_{x,V}(b) = \rho_{x,U}(a)$. Thus $\tilde{b} = \tilde{a}|_V$. This proves that $h = \{h_U\} : P \to P(S(P))$ is a homomorphism of presheaves.

DEFINITION 5.3. The presheaf homomorphism $h : P \to P(S(P))$ will be referred to as the canonical homomorphism of P in $P(S(P))$.

DEFINITION 5.4. A sheaf homomorphism $f : S \to S'$ is called a monomorphism if $f_x : S_x \to S'$ is a K-module monomorphism for all $x \in X$. f is called an epimorphism if $f_x : S_x \to S'_x$ is a K-module epimorphism for all $x \in X$. Thus f is a monomorphism if and only if Ker(f) is the zero subsheaf of S and f is an epimorphism if and only if Im(f) = S'.

DEFINITION 5.5. A presheaf homomorphism $f = \{f_U\} : P = \{P_U, \rho_{V,U}\} \to P' = \{P'_U, \rho_{V,U}\}$ is called a monomorphism if $f_U : P_U \to P'_U$ is a K-module monomorphism for each $U \in T(X)$. f is called an epimorphism if $f_U : P_U \to P'_U$ is a K-module epimorphism.

Let P be a presheaf and $h : P \to P(S(P))$ be the canonical homomorphism. (See 5.3). In general h will not be an isomorphism of presheaves. We will now determine conditions which will ensure that h is an isomorphism of presheaves.

THEOREM 5.6. Suppose $h : P \to P(S(P))$ is an isomorphism of presheaves. Let $U \in T(X)$ and $\{U_\alpha\}_{\alpha \in J}$ be any family of open subsets of X satisfying $U = \bigcup_{\alpha \in J} U_\alpha$. Then the following two conditions hold.

(1) Whenever a, b are elements of P_U such that $\rho_{U_\alpha, U}(a) = \rho_{U_\alpha, U}(b)$ for all $\alpha \in J$ then $a = b$.

(2) If $a_\alpha \in P_{U_\alpha}$ are elements satisfying $\rho_{U_\alpha \cap U_\beta, U_\alpha}(a_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(a_\beta)$ for all α, β in J then there exists an element $a \in P_U$ such that $\rho_{U_\alpha, U}(a) = a_\alpha$.

Proof. Let a, b be elements of P_U satisfying $\rho_{U_\alpha, U}(a) = \rho_{U_\alpha, U}(b)$ for all $\alpha \in J$. Let $\tilde{a} = h_U(a) \in \Gamma(S(P), U)$ and $\tilde{b} = h_U(b) \in \Gamma(S(P), U)$. By assumption h_V is a K-module isomorphism for each $V \in T(X)$. Hence to prove that $a = b$ it is sufficient to prove that $h_U(a) = h_U(b)$. Let $x \in U$. Then $x \in U_\alpha$ for some $\alpha \in J$. Hence

$$\tilde{a}(x) = \rho_{x,U}(a) = \rho_{x,U_\alpha}(\rho_{U_\alpha,U}(a)) = \rho_{x,U_\alpha}(\rho_{U_\alpha,U}(b)) = \rho_{x,U}(b) = \tilde{b}(x).$$

Thus $\tilde{a} = \tilde{b}$ and (1) is proved.

(2) Let $a_\alpha \in P_{U_\alpha}$ and $a_\beta \in P_{U_\beta}$ be such that $\rho_{U_\alpha \cap U_\beta, U_\alpha}(a_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(a_\beta)$ for all α, β in J. Let $\tilde{a}_\alpha = h_{U_\alpha}(a_\alpha) \in \Gamma(S(P), U_\alpha)$. Then $h_{U_\alpha \cap U_\beta}(\rho_{U_\alpha \cap U_\beta, U_\alpha}(a_\alpha)) = \tilde{a}_\alpha|_{U_\alpha \cap U_\beta}$. Thus $\tilde{a}_\alpha : U_\alpha \to S(P)$ are sections of the sheaf $S(P)$ satisfying $\tilde{a}_\alpha|_{U_\alpha \cap U_\beta} = \tilde{a}_\beta|_{U_\alpha \cap U_\beta}$. Hence there exists a unique section $\sigma : U \to S(P)$ over U such that $\sigma|_{U_\alpha} = \tilde{a}_\alpha$. Since $h_U : P_U \to \Gamma(S(P), U)$ is a K-module isomorphism, there exists an element $a \in P_U$ such that $h_U(a) = \sigma$. Then $h_{U_\alpha}(\rho_{U_\alpha, U}(a)) = \sigma|_{U_\alpha} = \tilde{a}_\alpha = h_{U_\alpha}(a_\alpha)$. Since each h_{U_α} is a K-module monomorphism, we get $\rho_{U_\alpha, U}(a) = a_\alpha$.

REMARK 5.7 In the proof of (1) all that we needed was that the map h_V be a K-module monomorphism, for any $V \in T(X)$. Thus (1) should hold whenever $h : P \to P(S(P))$ is a K-module monomorphism. However in the proof of (2) we need the full strength of the assumption that each h_V is a K-module isomorphism. If we assumed that each h_V is an onto mapping we would get an element $a \in P_U$ such that $h_U(a) = \sigma$. But this will not imply that $\rho_{U_\alpha, U}(a) = a_\alpha$.

Also, because of (1) the element $a \in P_U$ satisfying the requirements in (2) is unique.

DEFINITION 5.8. A presheaf $P = \{P_U, \rho_{V,U}\}$ is said to be complete if the following two conditions are satisfied.

(1) If $U_\alpha \in T(X)$ for all $\alpha \in J$, $U = \bigcup_{\alpha \in J} U_\alpha$, $a \in P_U$, $b \in P_U$ and $\rho_{U_\alpha, U}(a) = \rho_{U_\alpha, U}(b)$. for all $\alpha \in J$ then $a = b$.

(2) If $U_\alpha \in T(X)$ for all $\alpha \in J$, $U = \bigcup_{\alpha \in J} U_\alpha$ $a_\alpha \in P_{U_\alpha}$ and

$\rho_{U_\alpha \cap U_\beta, U_\alpha}(a_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(a_\beta)$ for all α, β in J then there exists a unique $a \in P_U$ such that $\rho_{U_\alpha, U}(a) = a_\alpha$ for all $\alpha \in J$.

THEOREM 5.9 The canonical homomorphism $h : P \to P(S(P))$ is an isomorphism of presheaves if and only if P is a complete presheaf.

Proof. If the canonical homomorphism $h : P \to P(S(P))$ is an isomorphism of presheaves then from theorem (5.6) P is a complete presheaf.

Suppose P is a complete presheaf.

(a) $h_U : P_U \to \Gamma(S(P), U)$ is a K-module monomorphism for each $U \in T(X)$. Suppose c, d in P_U satisfy $h_U(c) = h_U(d)$. Let $x \in U$. Then $\rho_{x,U}(c) = \rho_{x,U}(d)$ in $S(P)_x$. Since $S(P)_x = \varinjlim \{P_V, \rho_{W,V}\}_{V \in Q_x}$ over Q_x, we see that for a certain $W_x \in Q_x$ with $W_x \subset U$ we have $\rho_{W_x,U}(c) = \rho_{W_x,U}(d)$. Now $\bigcup_{x \in U} W_x = U$. By condition (1) in the definition of completeness we get $c = d$. This proves (a).

(b) $h_U : P_U \to \Gamma(S(P), U)$ is a K-module epimorphism for all $U \in T(X)$.

Let $\sigma : U \to S(P)$ be any section of $S(P)$ over U. For any $x \in U$ we have $\sigma(x) \in S(P)_x$. Since $S(P)_x = \varinjlim \{P_V, \rho_{W,V}\}_{V \in Q_x}$, it follows that there exists a $V_x \in Q_x$ and a $c_x \in P_{V_x}$ such that $\rho_{x,V_x}(c_x) = \sigma(x)$ and $V_x \subset U$. Let $\tau_x = h_{V_x}(c_x) \in \Gamma(S(P), V_x)$. Then $\tau_x(x) = \sigma(x)$. It follows from (1.13), that there exists a $W_x \in Q_x$ with $W_x \subset V_x$ and $\sigma|_{W_x} = \tau|_{W_x}$. Let $d_x \in P_{W_x}$ be given by $d_x = \rho_{W_x, V_x}(c_x)$. Then $h_{W_x}(d_x) = \tau_x|_{W_x} = \sigma|_{W_x}$.

Let x and y be any two elements of U. Then

$h_{W_x \cap W_y}(\rho_{W_x \cap W_y, W_x}(d_x)) = \sigma|_{W_x \cap W_y} = h_{W_x \cap W_y}(\rho_{W_x \cap W_y, W_y}(d_y))$. By (a) each h_V is a K-module monomorphism. Hence $\rho_{W_x \cap W_y, W_x}(d_x) = \rho_{W_x \cap W_y, W_y}(d_y)$.

Since $\bigcup_{x \in U} W_x = U$, condition (2) in the definition of completeness yields an element $d \in P_U$ such that $\rho_{W_x,U}(d) = d_x$ for all $x \in U$.

Consider the element $h_U(d) \in \Gamma(S(P),U)$. For any $x \in U$, we have

$$\{h_U(d)\}(x) = \rho_{x,U}(d) = \rho_{x,W_x}(\rho_{W_x,U}(d)) = \rho_{x,W_x}(d_x) = \{h_{W_x}(d_x)\}(x) = \sigma(x).$$

Thus $h_U(d) = \sigma$. This proves (b).

In later chapters, examples of noncomplete and complete presheaves will be given.

CHAPTER II

FINE SHEAVES

§1. Exact Sequence of Sheaves and Presheaves.

All sheaves and presheaves considered are of K-modules over a fixed space X, where K is a principal ideal domain.

DEFINITION 1.1. A sequence of sheaves

$$\cdots \longrightarrow S_{i-1} \xrightarrow{\varphi_{i-1}} S_i \xrightarrow{\varphi_i} S_{i+1} \xrightarrow{\varphi_{i+1}} \cdots$$

(where each φ_i is a sheaf homomorphism) is called exact if $\mathrm{Ker}(\varphi_i) = \mathrm{Im}(\varphi_{i-1})$ for all i. This is equivalent to the condition that $\mathrm{Ker}(\varphi_{i,x}) = \mathrm{Im}(\varphi_{i-1,x})$ for all $x \in X$, where $\varphi_{i,x} = \varphi_i|S_{i,x}$ with $S_{i,x}$ the stalk of S_i over x.

DEFINITION 1.2. A sequence of presheaves

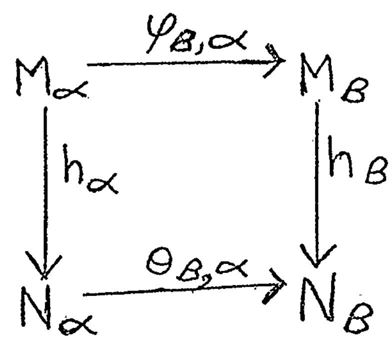
$$\cdots \longrightarrow P_{i-1} \xrightarrow{f_{i-1}} P_i \xrightarrow{f_i} P_{i+1} \xrightarrow{f_{i+1}} \cdots$$

(where each f_i is a homomorphism of presheaves) is called exact if for each $U \in T(X)$ the sequence

$$\cdots \longrightarrow P_{i-1,U} \xrightarrow{f_{i-1,U}} P_{i,U} \xrightarrow{f_{i,U}} P_{i+1,U} \xrightarrow{f_{i+1,U}} \cdots$$

is an exact sequence of K-modules.

DEFINITION 1.3. Let Ω be any directed set, $\{M_\alpha, \varphi_{\beta,\alpha}\}$ and $\{N_\alpha, \theta_{\beta,\alpha}\}$ be direct families of K-modules indexed by Ω. By a homomorphism of the direct family $\{N_\alpha, \theta_{\beta,\alpha}\}$, we mean a family $\{h_\alpha\}_{\alpha\in\Omega}$ where $h_\alpha : M_\alpha \to N_\alpha$ is a K-module homomorphism for each $\alpha \in \Omega$ such that



is commutative whenever $\alpha < \beta$ in Ω.

Let $M = \varinjlim\{M_\alpha,\phi_{\beta,\alpha}\}_{\alpha\in\Omega}$ and $N = \varinjlim\{N_\alpha,\theta_{\beta,\alpha}\}_{\alpha\in\Omega}$. Any homomorphism $\{h_\alpha\}$ of $\{M_\alpha,\phi_{\beta,\alpha}\}_{\alpha\in\Omega}$ into $\{N_\alpha,\theta_{\beta,\alpha}\}_{\alpha\in\Omega}$ induces a unique homomorphism $h:M \to N$. Every element x of M is the equivalence class of an element x_α of some M_α. We define $h(x)$ to be the equivalence class of $h_\alpha(x_\alpha)$. If x_α and x_β both represent the equivalence class of x then there exists a $\gamma \in \Omega$ with $\alpha < \gamma$, $\beta < \gamma$ and $\phi_{\gamma,\alpha}(x_\alpha) = \phi_{\gamma,\beta}(x_\beta)$. Then

$$\theta_{\gamma,\alpha}(h_\alpha(x_\alpha)) \quad = \quad h_\gamma \circ \phi_{\gamma,\alpha}(x_\alpha)$$

$= h_\gamma \circ \phi_{\gamma,\beta}(x_\beta) = \theta_{\gamma,\beta}(h_\beta(x_\beta))$. Thus $h_\alpha(x_\alpha)$ and $h_\beta(x_\beta)$ represent the same element in N. This shows that h is well-defined. It is easily checked that h is a K-module homomorphism. Let $[x_\alpha]$ denote the equivalence class of x_α in M. Let $\rho^M_\alpha:M_\alpha \to M$ such that $\rho^M_\alpha(x_\alpha) = [x_\alpha]$ for any $x_\alpha \in M_\alpha$. Let $\rho^N_\alpha:N_\alpha \to N$ such that $\rho^N_\alpha(y_\alpha) = [y_\alpha]$ for any $y_\alpha \in N_\alpha$. It is clear that ρ^M_α and ρ^N_α are K-module homomorphisms.

THEOREM 1.4 Let $\{E'_\alpha,\phi'_{\beta,\alpha}\}$, $\{E_\alpha,\phi_{\beta,\alpha}\}$ and $\{E''_\alpha,\phi''_{\beta,\alpha}\}$ be direct families of K-modules indexed by the directed set Ω. Let $\{h_\alpha\}$ be a homomorphism $\{E'_\alpha,\phi'_{\beta,\alpha}\} \to \{E_\alpha,\phi_{\beta,\alpha}\}$ and $\{g_\alpha\}$ a homomorphism $\{E_\alpha,\phi_{\beta,\alpha}\} \to \{E''_\alpha,\phi''_{\beta,\alpha}\}$. Suppose for each $\alpha \in \Omega$ the sequence

$$E'_\alpha \xrightarrow{h_\alpha} E_\alpha \xrightarrow{g_\alpha} E''_\alpha$$

is exact. Let $E' = \varinjlim_{\alpha\in\Omega}\{E'_\alpha, \phi'_{\beta,\alpha}\}$, $E = \varinjlim_{\alpha\in\Omega}\{E_\alpha, \phi_{\beta,\alpha}\}$ and $E'' = \varinjlim_{\alpha\in\Omega}\{E''_\alpha, \phi''_{\beta,\alpha}\}$. Let $h:E' \to E$ and $g:E \to E''$ denote the homomorphisms induced by $\{h_\alpha\}$ and $\{g_\alpha\}$ respectively. Then $E' \xrightarrow{h} E \xrightarrow{g} E''$ is an exact sequence of K-modules.

Proof. Let $x' \in E'$. There exists an $\alpha \in \Omega$ and a $x'_\alpha \in E'_\alpha$ such that $x' = [x'_\alpha]$. By definition $g \circ h(x') = [g_\alpha \circ h_\alpha(x'_\alpha)]$. But $g_\alpha \circ h_\alpha = 0$ so $g \circ h\,(x') = 0$. Thus $\mathrm{Ker}(g) \supset \mathrm{Im}(h)$ (i).

Let $y \in \mathrm{Ker}(g)$. Let $y_\alpha \in E_\alpha$ represent y. Then $g_\alpha(y_\alpha)$ represents zero in E''. Hence there exists a $\gamma > \alpha$ in Ω such that $\phi''_{\gamma,\alpha}(g_\alpha(y_\alpha)) = 0$ in E''_γ. Let $y_\gamma = \phi_{\gamma,\alpha}(y_\alpha) \in E_\gamma$. Then

$$\begin{aligned} g_\gamma(y_\gamma) &= g_\gamma(\phi_{\gamma,\alpha}(y_\alpha)) \\ &= \phi''_{\gamma,\alpha}(g_\alpha(y_\alpha)) = 0 . \end{aligned}$$

The exactness of $E'_\gamma \xrightarrow{h_\gamma} E_\gamma \xrightarrow{g_\gamma} E''_\gamma$ yields an element $u'_\gamma \in E'_\gamma$ such that $h_\gamma(u'_\gamma) = y_\gamma$.

Let $u' \in E'$ be represented by u'_γ. Then

$$\begin{aligned} h(u') &= [h_\gamma(u'_\gamma)] = [y_\gamma] = [\phi_{\gamma,\alpha}(y_\alpha)] = [y_\alpha] \\ &= y . \end{aligned}$$

This gives $\mathrm{Ker}(g) \subset \mathrm{Im}(h)$.

We can thus conclude $\mathrm{Ker}(g) = \mathrm{Im}(h)$.

COROLLARY 1.5. Let

$$\cdots \to P_{i-1} \xrightarrow{f_{i-1}} P_i \xrightarrow{f_i} P_{i+1} \xrightarrow{f_{i+1}} \cdots$$

be an exact sequence of presheaves. Then the sequence

$$\cdots \longrightarrow S(P_{i-1}) \xrightarrow{S(f_{i-1})} S(P_i) \xrightarrow{S(f_i)} S(P_{i+1}) \longrightarrow \cdots$$

is an exact sequence of sheaves.

Proof. We have only to prove the exactness of the stalks,

$$\cdots \longrightarrow S(P_{i-1})_x \xrightarrow{S(f_{i-1})_x} S(P_i)_x \xrightarrow{S(f_i)_x} S(P_{i+1})_x \longrightarrow \cdots .$$

This is an immediate consequence of Theorem (1.4) applied to the homomorphisms

$$\{f_{i-1,u}\} : \{P_{i-1,u}, \rho_{v,u}\} \longrightarrow \{P_{i,u}, \rho_{v,u}\}$$

and $$\{f_{i,u}\} : \{P_{i,u}, \rho_{v,u}\} \longrightarrow \{P_{i+1,u}, \rho_{v,u}\}$$

of direct families over the directed set Q_x.

THEOREM 1.6. Suppose $0 \longrightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S''$ is an exact sequence of sheaves. Then $0 \longrightarrow P(S') \xrightarrow{P(\alpha)} P(S) \xrightarrow{P(\beta)} P(S'')$ is an exact sequence of presheaves.

Proof. Let $U \in T(X)$ and $\sigma' \in \Gamma(S',U)$ be such that $P(\alpha)(\sigma') = 0$ in $\Gamma(S,U)$. Then for any $x \in U$ we have $\alpha_x(\sigma'(x)) = 0$. Since $\alpha_x : S'_x \to S_x$ is a K-module monomorphism for each x, we get $\sigma'(x) = 0$ for all $x \in U$. Thus $\sigma' = 0$ in $\Gamma(S',U)$. Then we can conclude $0 \longrightarrow P(S')_U \xrightarrow{P(\alpha)_U} P(S)_U$ is exact.

Let $\tau' \in \Gamma(S',U)$ be arbitrary. Then for any $x \in U$, $\beta_x \circ \alpha_x(\tau'(x)) = 0$. Thus $P(\beta)_U \circ P(\alpha)_U = 0$ in $\Gamma(S'',U)$.

Let $\sigma \in \Gamma(S,U)$ be such that $P(\beta)_U(\sigma) = 0$ in $\Gamma(S'',U)$. Then

$\beta_x(\sigma(x)) = 0$ for all $x \in U$. Since $0 \longrightarrow S'_x \xrightarrow{\alpha_x} S_x \xrightarrow{\beta_x} S''_x \longrightarrow 0$ is exact, it follows that $\sigma(x) \in \alpha_x(S'_x)$. Hence $\sigma(U) \subset \alpha(S')$. If $\alpha(S')$ is taken with the subspace topology induced from that on S, we see that $\sigma: U \to \alpha(S')$ is continuous. From (I (4.10)) we see that $\alpha: S' \to S$ is an open map. Hence $\alpha(S')$ is an open subspace of S and $\alpha: S' \to \alpha(S')$ is a one-to-one, continuous, open map. It follows that $\alpha: S' \to \alpha(S)$ is a homeomorphism. Thus there exists a unique continuous map $\sigma': U \to S'$ with $\alpha \circ \sigma' = \sigma$. Then for any $x \in U$, we have $\sigma'(x) \in S'_x$. Thus $\sigma' \in \Gamma(S', U)$. Also $P(\alpha)_U(\sigma') = \alpha \circ \sigma' = \sigma$. It follows that $P(S')_U \xrightarrow{P(\alpha)_U} P(S)_U \xrightarrow{P(\beta)_U} P(S'')_U$ is exact. This completes the proof of Theorem (1.6).

DEFINITION 1.7. The K-module $\Gamma(S,X)$ will be referred to as the K-module of global cross sections of S. We will denote this K-module by $\Gamma(S)$.

When $f: S' \to S$ is a homomorphism of sheaves, the map $P(f)_X: \Gamma(S', X) = \Gamma(S') \to \Gamma(S) = \Gamma(S,X)$ will be denoted by $\Gamma(f)$. For any $\sigma' \in \Gamma(S')$, $\Gamma(f)(\sigma')$ is the element $f \circ \sigma'$ of $\Gamma(S)$.

As a corollary to theorem (1.6) we get the following:

COROLLARY 1.8. Let $0 \longrightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S''$ be any exact sequence of sheaves. Then $0 \longrightarrow \Gamma(S') \xrightarrow{\Gamma(\alpha)} \Gamma(S) \xrightarrow{\Gamma(\beta)} \Gamma(S'')$ is an exact sequence of K-modules.

EXAMPLE 1.9. Let $X = S^1$ and let $c = (1,0) \in S^1$ and $d = (-1,0) \in S^1$. Let $R_x = 0$ for $x \in S^1 - \{\{c\} \cup \{d\}\}$, $R_c = K$ and $R_d = K$. Let

$R = \bigcup_{x \in S^1} R_x$ (disjoint union). For any $U \in T(X)$ let $U_0 = \{0_u \in R_u | u \in U\}$. For any $V \in Q_c$ and any $\lambda \neq 0$ in K let $V_\lambda^c = \{0_x\}_{x \in V-\{c\}} \cup \{\lambda_c\}$ where λ_c denotes the element λ of R_c and 0_x denotes the zero element of R_x (which incidentally is the only element of R_x whenever $x \notin \{c\} \cup \{d\}$). For any $W \in Q_d$ and $\lambda \neq 0$ in K let $W_\lambda^d = \{0_x\}_{x \in W-\{d\}} \cup \{\lambda_d\}$ where λ_d denotes the element λ of R_d. The family $\mathcal{B}$ constituted by $\{U_0\}_{U \in T(X)}$, $\{V_\lambda^c\}_{V \in Q_c}$ and $\{W_\lambda^d\}_{W \in Q_d}$ is a base for a topology on R. If $\pi : R \to X$ is the map $\pi(R_x) = x$ for all $x \in S^1$ then R with projection π is a sheaf of K-modules.

Let S be the constant sheaf $X \times K$ on X. If we define $f : S \to R$ by $f_x = 0$ for $x \notin \{c\} \cup \{d\}$ and $f_c = Id_K$, $f_d = Id_K$ then it is clear that f is a sheaf epimorphism. So $S \xrightarrow{f} R \longrightarrow 0$ is exact. Any section σ of S over the whole of X is given by $\sigma(x) = (x, \lambda)$ for a fixed $\lambda \in K$. Given any μ, ν in K there exists a global section $\tau_{\mu,\nu} : X \to R$ of R defined by

$$\tau_{\mu,\nu}(x) = \begin{cases} \mu & \text{if } x = c \\ 0 & \text{if } x \notin \{c\} \cup \{d\} \\ \nu & \text{if } x = d \end{cases}.$$

Whenever $\mu \neq 0$ or $\nu \neq 0$ or $\mu \neq \nu$ there is no global section of S which gets mapped onto $\tau_{\mu,\nu}$ by $\Gamma(f)$. Thus $\Gamma(S) \xrightarrow{\Gamma(f)} \Gamma(R) \longrightarrow 0$ is NOT exact.

§2. Paracompact Spaces

DEFINITION 2.1. Let $\mathcal{E} = \{E_\alpha\}_{\alpha \in I}$ and $\mathcal{F} = \{F_\beta\}_{\beta \in J}$ be coverings of X. We say that $\mathcal{E}$ is a refinement of $\mathcal{F}$ if there exists a map $\lambda : I \to J$ such that $E_\alpha \subset F_{\lambda(\alpha)}$ for all $\alpha \in I$.

DEFINITION 2.2. A family $\{E_\alpha\}_{\alpha\in I}$ of subsets of X is said to be locally finite if for any $x \in X$ there exists an open set U_x of X with $x \in U_x$ such that $I_x = \{\alpha \in I | E_\alpha \cap U_x \neq \emptyset\}$ is finite.

DEFINITION 2.3. X is called paracompact if

(i) X is Hausdorff and

(ii) every open covering of X has a locally finite open refinement.

It is well-known that a paracompact space is normal. In this section X will be assumed throughout to be paracompact. All the sheaves are of K-modules over X.

DEFINITION 2.4. Let $f:S \to T$ be a sheaf homomorphism. Then the closure in X of the set $\{x \in X | f(S_x) \neq 0\}$ is called the support of the homomorphism f. We denote the support of f by Supp(f).

DEFINITION 2.5. A sheaf S is called fine, if given any locally finite open covering $\{U_\alpha\}_{\alpha\in J}$ of X, there exist sheaf endomorphisms h_α of S so that

(i) $\text{Supp}(h_\alpha) \subset U_\alpha$ and

(ii) $\sum_{\alpha\in J} h_\alpha = Id_S$.

2.6 EXAMPLES.

(1) Let X be a C^∞ differentiable manifold. Let $\mathcal{E}^\infty(X)$ be the sheaf associated to the presheaf of C^∞ functions of X. Then $\mathcal{E}^\infty(X)$ is a fine sheaf over X. Let $\{U_\alpha\}_{\alpha\in J}$ be any locally finite open covering

of X. Then there exist C^∞ functions $f_\alpha : X \to R$ with support $(f_\alpha) \subset U_\alpha$ and $\sum_{\alpha \in J} f_\alpha = 1$ on X. For any open U in X let $g_{\alpha,U} : C^\infty(U) \to C^\infty(U)$ be the map $g_{\alpha,U}(\phi) = f_\alpha \cdot \phi$ (pointwise product). Then $g_\alpha = \{g_{\alpha,U}\}$ is a presheaf homomorphism of the presheaf of C^∞ functions of X. If $h_\alpha : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ are induced by g_α then it is easily seen that $\mathrm{Supp}(h_\alpha) \subset U_\alpha$ and that $\sum_{\alpha \in J} h_\alpha = Id_{\mathcal{C}^\infty(X)}$.

(2) Without proof we will mention that the sheaf of germs of holomorphic functions on the complex plane $\mathbb{C}$ is an example of a sheaf which is not fine. The sheaf of germs of holomorphic functions is the sheaf associated to the presheaf of holomorphic functions over $\mathbb{C}$.

THEOREM 2.7. Let $\phi : S \to T$ be a sheaf epimorphism with $R = \mathrm{Ker}(\phi)$ a fine sheaf. Then $\Gamma(\phi) : \Gamma(S) \to \Gamma(T)$ is an onto map.

Proof. Let $\tau \in \Gamma(T)$. Let $x_0 \in X$. Since ϕ is onto, there exists $u_0 \in S_{x_0}$ such that $\phi(u_0) = \tau(x_0)$. Let π and p denote the projections of the sheaves S and T, respectively. Since π is a local homeomorphism, there exists an open set $U \ni x_0$ in X and an element $\sigma \in \Gamma(S,U)$ with $\sigma(x_0) = u_0$. Then $\phi \circ \sigma$ is a section of T over U with $\phi \circ \sigma(x_0) = \tau(x_0)$. By (I (1.13)) there exists an open set U' of X with $x_0 \in U' \subset U$ and $\phi \circ \sigma|_{U'} = \tau|_{U'}$. Thus for any $x_0 \in X$, there exists an open set U' of X with $x_0 \in U'$ and a section σ of S over U' such that $\phi \circ \sigma = \tau|_{U'}$.

Since X is paracompact, there exists a locally finite open covering $\{U_\alpha\}_{\alpha \in J}$ of X and sections $\sigma_\alpha \in \Gamma(S,U_\alpha)$ such that $\phi \circ \sigma_\alpha = \tau|_{U_\alpha}$.

Now on $U_\alpha \cap U_\beta$ let $\sigma_{\alpha\beta} = \sigma_\alpha - \sigma_\beta$, from $\phi \circ \sigma_\alpha|_{U_\alpha \cap U_\beta} = \tau|_{U_\alpha \cap U_\beta}$ $= \phi \circ \sigma_\beta|_{U_\alpha \cap U_\beta}$ we see that $\sigma_{\alpha\beta}(U_\alpha \cap U_\beta) \subset R$. Thus $\sigma_{\alpha\beta} \in \Gamma(R, U_\alpha \cap U_\beta)$ for any α, β in J. Since R is a fine sheaf there exist sheaf endomorphisms $h_\alpha : R \to R$ with $\mathrm{Supp}(h_\alpha) \subset U_\alpha$ and $\sum_{\alpha \in J} h_\alpha = Id_R$. From $\mathrm{Supp}(h_\beta) \subset U_\beta$ we see that $g_{\alpha\beta} : U_\alpha \to R$ defined by $g_{\alpha\beta}|_{U_\alpha \cap U_\beta} = h_\beta \circ \sigma_{\alpha\beta}$ and $g_{\alpha\beta} = 0$ on $U_\alpha - U_\beta$ is a section of R over U_α. Let $S^*_\alpha : U_\alpha \to R$ be defined by $S^*_\alpha = \sum_{\beta \in J} g_{\alpha\beta}$. Since $\sum_{\beta \in J}$ is a locally finite sum it follows that S^*_α is a section of R over U_α. Then using the identity $\sigma_{\mu\gamma} - \sigma_{\nu\gamma} = (\sigma_\mu - \sigma_\gamma) - (\sigma_\nu - \sigma_\gamma) = \sigma_\mu - \sigma_\nu = \sigma_{\mu\nu}$ on $U_\mu \cap U_\nu \cap U_\gamma$ for any μ, ν and γ in J and that $\sum_{\beta \in J} h_\beta = Id_R$ we can prove $S^*_\mu - S^*_\nu = \sigma_\mu - \sigma_\nu$ on $U_\mu \cap U_\nu$ for any μ, ν in J. Let $\theta_\alpha = \sigma_\alpha - S^*_\alpha \in \Gamma(S, U_\alpha)$. Then $\theta_\alpha = \theta_\beta$ on $U_\alpha \cap U_\beta$ for any α, β in J. Also $\phi \circ \theta_\alpha = \tau|_{U_\alpha}$. It follows that $\theta : X \to S$ defined by $\theta|_{U_\alpha} = \theta_\alpha$, is a well-defined section of S satisfying $\phi \circ \theta = \tau$.

COROLLARY 2.8. Let $0 \longrightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S'' \longrightarrow 0$ be an exact sequence of sheaves over X with S' fine. Then $0 \longrightarrow \Gamma(S') \xrightarrow{\Gamma(\alpha)} \Gamma(S) \xrightarrow{\Gamma(\beta)} \Gamma(S'') \to 0$ is exact.

Proof. Immediate from theorem (2.7) and corollary (1.8).

CHAPTER III

SHEAF COHOMOLOGY

§1. Axiomatic Sheaf Cohomology

DEFINITION 1.1. A sheaf cohomology theory $\mathcal{H}$ for X with coefficients in sheaves of K-modules consists of

(I) a K-module $H^q(X,S)$ for each sheaf S of K-modules over X and each integer q,

(II) for each sheaf homomorphism $f:S \to S'$, a K-module homomorphism $f_*^q:H^q(X,S) \to H^q(X,S')$ for each integer q and

(III) for each short exact sequence of sheaves

$$0 \longrightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S'' \longrightarrow 0 \quad ,$$

a K-module homomorphism $\partial^q:H^q(X,S'') \longrightarrow H^{q+1}(X,S')$ for each integer q such that (a), (b), (c), (d), (e), and (f) mentioned below are all satisfied.

(a) $H^q(X,S) = 0$ for $q < 0$ and there exists a K-module isomorphism $\theta_S:\Gamma(S) \to H^0(X,S)$ such that if $f:S \to S'$ is a sheaf homomorphism then

$$\begin{array}{ccc} \Gamma(S) & \xrightarrow{\theta_S} & H^0(X,S) \\ \downarrow{\scriptstyle \Gamma(f)} & & \downarrow{\scriptstyle f_*^0} \\ \Gamma(S') & \xrightarrow{\theta_{S'}} & H^0(X,S') \end{array}$$

commutes.

(b) If S is a fine sheaf then $H^q(X,S) = 0$ for $q > 0$.

(c) If the sequence of sheaves $0 \longrightarrow S' \xrightarrow{\alpha} S \xrightarrow{\beta} S'' \longrightarrow 0$ is exact then the sequence of K-modules

$$\cdots \to H^q(X,S') \xrightarrow{\alpha_*^q} H^q(X,S) \xrightarrow{\beta_*^q} H^q(X,S'') \xrightarrow{\partial^q} H^{q+1}(X,S') \xrightarrow{\alpha_*^{q+1}} \cdots$$

is exact.

(d) If $Id:S \to S$ is the identity sheaf homomorphism then $Id_*^q:H^q(X,S) \longrightarrow H^q(X,S)$ is the identity K-module homomorphism.

(e) Let S, S' and S'' be sheaves of K-modules. Let f, g and h be sheaf homomorphisms such that

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ & \searrow^{g} & \downarrow h \\ & & S'' \end{array}$$

commutes

then

$$\begin{array}{ccc} H^q(X,S) & \xrightarrow{f_*^q} & H^q(X,S') \\ & \searrow^{g_*^q} & \downarrow h_*^q \\ & & H^q(X,S'') \end{array}$$

commutes.

for all q.

(f) If

$$\begin{array}{ccccccccc} 0 & \to & S' & \xrightarrow{f} & S & \xrightarrow{g} & S'' & \to & 0 \\ & & \downarrow \phi' & & \downarrow \phi & & \downarrow \phi'' & & \\ 0 & \to & T' & \xrightarrow{F} & T & \xrightarrow{G} & T'' & \to & 0 \end{array}$$

is a commutative diagram where the mappings are sheaf homomorphisms and the rows are short exact sequences of sheaves then the following diagram commutes

$$\begin{array}{ccc} H^q(X,S'') & \xrightarrow{\partial_1^q} & H^{q+1}(X,S') \\ \Big\downarrow \phi_*''^q & & \Big\downarrow \phi_*'^q \\ H^q(X,T'') & \xrightarrow{\partial_2^q} & H^{q+1}(X,T') \end{array}$$

DEFINITION 1.2. The K-module $H^q(X,S)$ is the qth cohomology K-module of X with coefficients in the sheaf S with respect to the cohomology $\mathcal{H}$.

DEFINITION 1.3. A sheaf of K-modules C is called torsion-free if each stalk C_x for $x \in X$ is a torsion-free K-module.

DEFINITION 1.4. An exact sequence of sheaves

$$0 \longrightarrow A \xrightarrow{j} C_0 \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} \cdots$$

is called a resolution of the sheaf A.

If further each sheaf C_i is a fine sheaf, it is called a fine resolution. Also if each sheaf C_i is a torsion-free sheaf, it is called a torsion-free resolution.

REMARK 1.5. With each sheaf S and with each resolution

$$0 \longrightarrow A \xrightarrow{j} C_0 \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} \cdots$$

of A we associate the cochain complex of K-modules

$$0 \longrightarrow \Gamma(C_0 \otimes S) \xrightarrow{\Gamma(\alpha_0 \otimes Id)} \Gamma(C_1 \otimes S) \xrightarrow{\Gamma(\alpha_1 \otimes Id)} \cdots .$$

We denote this cochain complex by $\Gamma(C^* \otimes S)$.

If $\theta : S \to S'$ is a sheaf homomorphism then it easily follows that $\Gamma(Id \otimes \theta) : \Gamma(C^* \otimes S) \longrightarrow \Gamma(C^* \otimes S')$ is a cochain mapping.

THEOREM 1.6. For each torsion-free fine resolution of the constant sheaf $\mathcal{K} = X \times K$, there exists an associated cohomology theory $\mathcal{H}$ for X with coefficients in sheaves of K-modules over X.

Proof. Let $\mathcal{K} = X \times K$ be the constant sheaf and

(i) $$0 \longrightarrow \mathcal{K} \xrightarrow{\alpha} C_0 \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} \cdots$$

be a torsion-free fine resolution of $\mathcal{K}$. For each sheaf S there exists an associated cochain complex $\Gamma(C^* \otimes S)$.

(ii) $$0 \longrightarrow \Gamma(C_0 \otimes S) \xrightarrow{\Gamma(\alpha_0 \otimes Id)} \Gamma(C_1 \otimes S) \xrightarrow{\Gamma(\alpha_1 \otimes Id)} \cdots$$

For each integer q let $H^q(X,S) = H^q(\Gamma(C^* \otimes S))$.

Let $\emptyset : S \to T$ be a sheaf homomorphism. By remark (1.5), there exists a cochain mapping $\Phi : \Gamma(Id \otimes \emptyset) : \Gamma(C^* \otimes S) \to \Gamma(C^* \otimes T)$. Let $\emptyset_*^q : H^q(X,S) \to H^q(X,T)$ be the mapping induced by Φ in cohomology.

Let $0 \to S' \xrightarrow{\gamma} S \xrightarrow{\delta} S'' \to 0$ be a short exact sequence of sheaves. For $i \geq 0$, C_i is a torsion-free sheaf, so the sequence of sheaves

(iii) $$0 \longrightarrow C_i \otimes S' \xrightarrow{Id \otimes \gamma} C_i \otimes S \xrightarrow{Id \otimes \delta} C_i \otimes S'' \longrightarrow 0$$

is exact for each i.

For $i \geq 0$, C_i is a fine sheaf so the sequence

(iv) $$0 \to \Gamma(C_i \otimes S') \xrightarrow{\Gamma(Id \otimes \gamma)} \Gamma(C_i \otimes S) \xrightarrow{\Gamma(Id \otimes \delta)} \Gamma(C_i \otimes S'') \to 0$$

is exact by (II 2.8).

Since (iv) is an exact sequence of K-modules there exists an associated K-module homomorphism to it, $\partial^q : H^q(X,S'') \to H^{q+1}(X,S')$. We take this ∂^q to be the K-module homomorphism associated to

$$0 \longrightarrow S' \xrightarrow{\gamma} S \xrightarrow{\delta} S'' \longrightarrow 0 \quad .$$

We will now prove that the axioms for a sheaf cohomology theory are all satisfied.

Axiom (a). Since $\Gamma(C_q \otimes S) = 0$ for $q < 0$, $H^q(X,S) = 0$ for $q < 0$. Let $L_q = \text{Kernel}(\alpha_q) = \text{Image}(\alpha_{q-1})$. By (I(4.10)) L_q is a subsheaf of C_q. We know (i) is exact so

(v) $$0 \longrightarrow \mathcal{K} \xrightarrow{\alpha} C_0 \xrightarrow{\beta_0} L_1 \longrightarrow 0$$

is an exact sequence of sheaves, where $\beta_0(e) = \alpha_0(e)$ for $e \in C_0$. We know also that C_1 is a torsion-free sheaf and L_1 is a subsheaf of C_1 so L_1 is a torsion-free sheaf.

Thus

(vi) $$0 \longrightarrow \mathcal{K} \otimes S \xrightarrow{\alpha \otimes Id} C_0 \otimes S \xrightarrow{\beta_0 \otimes Id} L_1 \otimes S \longrightarrow 0$$

is an exact sequence of sheaves. Hence by (II (1.8))

(vii) $$0 \to \Gamma(\mathcal{K} \otimes S) \xrightarrow{\Gamma(\alpha \otimes Id)} \Gamma(C_0 \otimes S) \xrightarrow{\Gamma(\beta_0 \otimes Id)} \Gamma(L_1 \otimes S)$$

is an exact sequence of K-modules. Hence

$$\begin{aligned} \text{Kernel}(\Gamma(\alpha_0 \otimes Id)) &= \text{Kernel}(\Gamma(\beta_0 \otimes Id)) \\ &= \text{Image}\ (\Gamma(\alpha \otimes Id)) \quad . \end{aligned}$$

Since (vii) is exact, $\Gamma(\alpha \otimes Id)$ is a K-module monomorphism so

$$\text{Image}(\Gamma(\alpha \otimes Id)) \simeq \Gamma(\mathcal{K} \otimes S) \quad .$$

Now $\mathcal{K} \otimes S \simeq S$ so $\Gamma(S) \simeq \Gamma(\mathcal{K} \otimes S) \simeq \text{Image}(\Gamma(\alpha \otimes Id)) = \text{Kernel}(\Gamma(\alpha_0 \otimes Id))$. Thus $H^0(X,S) = \text{Kernel}(\Gamma(\alpha_0 \otimes Id)) \simeq \Gamma(S)$. Let $\theta_S : H^0(X,S) \to \Gamma(S)$ be the above K-module isomorphism for the sheaf S.

Let $\emptyset : S \to T$ be a sheaf homomorphism. Let $\Gamma(\emptyset):\Gamma(S) \to \Gamma(T)$ be such that if $\gamma \in \Gamma(S)$ then $\{\Gamma(\emptyset)\}(\gamma) = \emptyset \circ \gamma$. It is easily seen that the diagram

$$\begin{array}{ccc} \Gamma(S) & \xrightarrow{\Theta_S} & H^{\circ}(X,S) \\ \downarrow{\scriptstyle \Gamma(\emptyset)} & & \downarrow{\scriptstyle \emptyset^{\circ}_{*}} \\ \Gamma(T) & \xrightarrow{\Theta_T} & H^{\circ}(X,T) \end{array}$$

commutes.

Hence axiom (a) is satisfied.

Axiom (b). Let F be a fine sheaf of K-modules over X. We know (i) is exact so Image(α_{q-1}) = Kernel $(\alpha_q) = L_q$ for $q \geqslant 1$ and Image(α) = Kernel$(\alpha_0) = L_1$. It follows that the sequence

(viii) $$0 \longrightarrow L_q \xrightarrow{i_q} C_q \xrightarrow{\beta_q} L_{q+1} \longrightarrow 0$$

is exact, where i_q is the inclusion map and $\beta_q(e) = \alpha_q(e)$ for $e \in C_q$. Since L_{q+1} is a subsheaf of C_{q+1} and C_{q+1} is a torsion-free sheaf, it follows L_{q+1} is a torsion-free sheaf. Thus

(ix) $$0 \to L_q \otimes F \xrightarrow{i_q \otimes Id} C_q \otimes F \xrightarrow{\beta_q \otimes Id} L_{q+1} \otimes F \to 0$$

is an exact sequence of sheaves.

Since F is a fine sheaf,

(x) $$0 \to \Gamma(L_q \otimes F) \xrightarrow{\Gamma(i_q \otimes Id)} \Gamma(C_q \otimes F) \xrightarrow{\Gamma(\beta_q \otimes Id)} \Gamma(L_{q+1} \otimes F) \to 0$$

is an exact sequence of K-modules. So $\Gamma(i_q \otimes Id)$ is a K-module monomorphism. Since $\alpha_q = i_{q+1} \circ \beta_q$ we get $\Gamma(\alpha_q \otimes Id) = \Gamma(i_{q+1} \otimes Id) \circ \Gamma(\beta_q \otimes Id)$. Therefore Kernel$(\Gamma(\alpha_q \otimes Id))$ = Kernel$(\Gamma(\beta_q \otimes Id))$ = Image$(\Gamma(i_q \otimes Id))$ $\simeq \Gamma(L_q \otimes F)$. We know $\alpha_{q-1} = i_q \circ \beta_{q-1}$ so $\Gamma(\alpha_{q-1} \otimes Id) = \Gamma(i_q \otimes Id) \circ \Gamma(\beta_{q-1} \otimes Id)$.

Thus

$$\begin{aligned}&\text{Image}(\Gamma(\alpha_{q-1}\otimes Id))\\ &\qquad = \text{Image}(\Gamma(\beta_{q-1}\otimes Id)) = \Gamma(L_q \otimes F) \quad ,\end{aligned}$$

since $\Gamma(\beta_{q-1}\otimes Id)$ is a K-module epimorphism. Hence

$$H^q(X,F) = \frac{\text{Kernel}(\Gamma(\alpha_q \otimes Id))}{\text{Image } (\Gamma(\alpha_{q-1}\otimes Id))} = \frac{\Gamma(L_q \otimes F)}{\Gamma(L_q \otimes F)} = 0 \quad .$$

Axiom (c). Let

$$0 \longrightarrow S' \xrightarrow{\gamma} S \xrightarrow{\delta} S'' \longrightarrow 0$$

be a short exact sequence of sheaves. Since each C_i for $i \geq 0$ is a fine sheaf, it follows

$$0 \to \Gamma(C_i \otimes S') \xrightarrow{\Gamma(Id\otimes\gamma)} \Gamma(C_i \otimes S) \xrightarrow{\Gamma(Id\otimes\delta)} \Gamma(C_i \otimes S'') \to 0$$

is an exact sequence of K-modules. Axiom (c) follows immediately.

Axiom (d). Let $Id{:}S \to S$ be the identity sheaf homomorphism. Then $\Gamma(Id \otimes Id)$ is the identity K-module homomorphism and $Id_*^q{:}H^q(X,S) \to H^q(X,S)$ is the identity homomorphism.

Axiom (e). Let

$$\begin{array}{ccc} T & \xrightarrow{\sigma} & T' \\ & \searrow^{\psi} & \downarrow \tau \\ & & T'' \end{array}$$

commute

where σ,τ and ψ are sheaf homomorphisms. Then

$$\begin{array}{ccc} \Gamma(T) & \xrightarrow{\Gamma(\sigma)} & \Gamma(T') \\ & \searrow^{\Gamma(\psi)} & \downarrow{\scriptstyle \Gamma(\tau)} \\ & & \Gamma(T'') \end{array}$$

commutes.

It follows immediately that

$$\begin{array}{ccc} \Gamma(C^*\otimes T) & \xrightarrow{\Gamma(I_D\otimes\sigma)} & \Gamma(C^*\otimes T') \\ & \searrow^{\Gamma(I_D\otimes\psi)} & \downarrow{\scriptstyle \Gamma(I_D\otimes\tau)} \\ & & \Gamma(C^*\otimes T'') \end{array}$$

commutes.

Thus

$$\begin{array}{ccc} H^q(X,T) & \xrightarrow{\sigma^q_*} & H^q(X,T') \\ & \searrow^{\psi^q_*} & \downarrow{\scriptstyle \tau^q_*} \\ & & H^q(X,T'') \end{array}$$

commutes.

Axiom (f). Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S' & \xrightarrow{\gamma} & S & \xrightarrow{\delta} & S'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\ 0 & \longrightarrow & T' & \xrightarrow{\mu} & T & \xrightarrow{\nu} & T'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of sheaves. It follows from (iii) and (iv) on p.34 that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(C^*\otimes S') & \xrightarrow{\Gamma(I_D\otimes\gamma)} & \Gamma(C^*\otimes S) & \xrightarrow{\Gamma(I_D\otimes\delta)} & \Gamma(C^*\otimes S'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Gamma(I_D\otimes\phi')} & & \downarrow{\scriptstyle \Gamma(I_D\otimes\phi)} & & \downarrow{\scriptstyle \Gamma(I_D\otimes\phi'')} & & \\ 0 & \longrightarrow & \Gamma(C^*\otimes T') & \xrightarrow{\Gamma(I_D\otimes\mu)} & \Gamma(C^*\otimes T) & \xrightarrow{\Gamma(I_D\otimes\nu)} & \Gamma(C^*\otimes T'') & \longrightarrow & 0 \end{array}$$

is commutative with exact rows.

The existence of torsion-free fine resolutions will be demonstrated in later chapters.

LEMMA 1.7. Let A, B, C and D be sheaves over X. Let $\alpha : A \to B$, $\beta : B \to C$, $\gamma : A \to D$ and $\delta : D \to C$ be sheaf homomorphisms such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ D & \xrightarrow{\delta} & C \end{array}$$

commutes.

Then

$$\begin{array}{ccc} H^q(X,A) & \xrightarrow{\alpha_*^q} & H^q(X,B) \\ \downarrow{\scriptstyle\gamma_*^q} & & \downarrow{\scriptstyle\beta_*^q} \\ H^q(X,D) & \xrightarrow{\delta_*^q} & H^q(X,C) \end{array}$$

commutes.

Proof. By (e) of (1.1) we have $\beta_*^q \circ \alpha_*^q = (\beta \circ \alpha)_*^q$ and $\delta_*^q \circ \gamma_*^q = (\delta \circ \gamma)_*^q$. Since $\beta \circ \alpha = \delta \circ \gamma$ we get $(\beta \circ \alpha)_*^q = (\delta \circ \gamma)_*^q$. This proves lemma (1.7).

DEFINITION 1.8. Let S be a sheaf of K-modules of X. A discontinuous section of S over an open set U of X is a mapping $f : U \to S$ such that $\pi \circ f = Id$ on S. Let $DP(U,S)$ be the set of discontinuous sections on U. Let $g,h \in DP(U,S)$ and $\lambda \in K$. Define

$$(g+h)(e) = g(e) + h(e)$$

and

$$(\lambda \cdot g)(e) = \lambda \cdot (g(e))$$

for all e in U. It is easy to see $DP(U,S)$ with these operations is a K-module. Let $V \in \mathcal{T}(X)$ and $V \subset U$. Let $\alpha \in DP(U,S)$ and

$$\rho_{V,U}: DP(U,S) \longrightarrow DP(V,S)$$

such that $\rho_{V,U}(a) = a|_V$. $\{DP(U,S), \rho_{V,U}\}$ is trivially a presheaf of K-modules over S. The associated sheaf S_0 is called the sheaf of germs of discontinuous sections.

LEMMA 1.9. S_0 is a fine sheaf.

Proof. Let $\{U_\alpha\}_{\alpha\in J}$ be a locally finite open cover of X. Since X is paracompact there exists a refinement $\{V_\beta\}_{\beta\in J}$ of $\{U_\alpha\}_{\alpha\in J}$ such that $\overline{V}_\alpha \subset U_\alpha$ and $\bigcup_{\beta\in J} V_\beta = X$.

Let $\Gamma: X \to J$ such that $x \in V_{\Gamma(x)}$ for $x \in X$.

Let $\phi_\alpha: X \to K$ such that

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } \Gamma(x) = \alpha \\ 0 & \text{when } \Gamma(x) \neq \alpha \quad \text{for } x \in X \end{cases} .$$

Since $\bigcup_{\beta\in J} V_\beta = X$ it follows that $\sum_{\beta\in J} \phi_\beta = Id$ on X. Now $\Gamma^{-1}(\alpha) \subset V_\alpha$ so $\overline{\Gamma^{-1}(\alpha)} \subset \overline{V}_\alpha \subset U_\alpha$. Hence support $(\phi_\alpha) \subset U_\alpha$.

Let $\ell_{\alpha,U}: DP(U,S) \to DP(U,S)$ such that $\{\ell_{\alpha,U}(\sigma)\}(e) = \phi_\alpha(e)\cdot\sigma(e)$ for $e \in U$ and $\sigma \in DP(U,S)$. Now $\{\ell_{\alpha,U}\}_{U \text{ open}}$ is a presheaf endomorphism since

$$\begin{aligned} \{\ell_{\alpha,V}(\rho_{V,U}(\sigma))\}(b) &= \phi_\alpha(b)\cdot(\rho_{V,U}(\sigma))(b) \\ &= (\rho_{V,U}(\phi_\alpha(b)\cdot\sigma))(b) \\ &= \{\rho_{V,U}\,\ell_{\alpha,U}(\sigma)\}(b) \end{aligned} .$$

Let θ_α be the associated sheaf endomorphism. Let $d \in S_{0_x}$ where $x \in X$. Since $S_{0_x} = \varinjlim_{\substack{U \text{ open} \\ U \ni x}} \{DP(U,S), \rho_{V,U}\}$, there exists $U \ni x$ open and

$h \in DP(U,S)$ such that $d = \rho_{x,U}(h)$.

Now

$$\sum_{\alpha \in J} \theta_\alpha (d) = \sum_{\alpha \in J} \theta_\alpha \rho_{x,U}(h)$$
$$= \rho_{x,U} \sum_{\alpha \in J} \ell_{\alpha,U}(h) \quad .$$

Also

$$\left\{\sum_{\alpha \in J} \ell_{\alpha,U}(h)\right\}(c) = \sum_{\alpha \in J} \phi_\alpha(c) \cdot h(c)$$
$$= h(c) \quad .$$

So $\sum_{\alpha \in J} \ell_{\alpha,U}(h) = h$ and

$$\sum_{\alpha \in J} \theta_\alpha(d) = \rho_{x;U} \sum_{\alpha \in J} \ell_{\alpha,U}(h)$$
$$= \rho_{x;U}(h)$$
$$= d \quad .$$

Hence $\sum_{\alpha \in J} \theta_\alpha = Id$ and S_0 is a fine sheaf.

§2. Uniqueness of Sheaf Cohomology

Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ be sheaf cohomology theories for X with coefficients in sheaves of K-modules.

DEFINITION 2.1. A homomorphism $\mathcal{B}$ of the cohomology theory $\mathcal{H}$ to the cohomology theory $\widetilde{\mathcal{H}}$ consists of a K-module homomorphism $\mathcal{B}^q_S : H^q(X,S) \to \widetilde{H}^q(X,S)$ for each sheaf S of K-modules and each integer q such that (i), (ii) and (iii) below are satisfied.

(i) By (a) of (1.1) there exists K-module isomorphisms, $\theta_S : H^0(X,S) \to \Gamma(S)$ and $\widetilde{\theta}_S : \widetilde{H}^0(X,S) \to \Gamma(S)$. The diagram

$$\begin{array}{ccc} H^{\circ}(X,S) & \xrightarrow{\theta_S} & \Gamma(S) \\ \downarrow \beta^{\circ}_S & & \downarrow Id \\ \hat{H}^{\circ}(X,S) & \xrightarrow{\tilde{\theta}_S} & \Gamma(S) \end{array}$$

must be commutative.

(ii) If $f:S \to T$ is a sheaf homomorphism then

$$\begin{array}{ccc} H^q(X,S) & \xrightarrow{f^q_*} & H^q(X,T) \\ \downarrow \beta^q_S & & \downarrow \beta^q_T \\ \hat{H}^q(X,S) & \xrightarrow{\tilde{f}^q_*} & \tilde{H}^q(X,T) \end{array}$$

must commute.

(iii) If $0 \to S' \xrightarrow{\alpha} S \xrightarrow{\beta} S'' \to 0$ is a short exact sequence of sheaves of K-modules then the following diagram

$$\begin{array}{ccc} H^q(X,S'') & \xrightarrow{\partial^q} & H^{q+1}(X,S') \\ \downarrow \beta^q_{S''} & & \downarrow \beta^q_{S'} \\ \hat{H}^q(X,S'') & \xrightarrow{\tilde{\partial}^q} & \hat{H}^{q+1}(X,S') \end{array}$$

must commute for all integers q.

DEFINITION 2.2. A homomorphism $\emptyset$ of the cohomology theory $\mathcal{H}$ to the cohomology theory $\tilde{\mathcal{H}}$ is an isomorphism of cohomology theories if

$$\emptyset^q_S : H^q(X,S) \to \tilde{H}^q(X,S)$$

is a K-module isomorphism for each sheaf S of K-modules and each integer q.

DEFINITION 2.3. The identity isomorphism Id of the cohomology theory $\mathcal{H}$ to $\mathcal{H}$ is the isomorphism of cohomology theories such that

$$Id_S^q : H^q(X,S) \to H^q(X,S)$$

is the identity K-module isomorphism for each S of K-modules and each integer q.

THEOREM 2.4. Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ be cohomology theories on X with coefficients in sheaves of K-modules over X. There exists a unique homomorphism of cohomology theories of $\mathcal{H}$ to $\widetilde{\mathcal{H}}$.

Proof. The first step is proving a homomorphism of cohomology theories of $\mathcal{H}$ to $\widetilde{\mathcal{H}}$ exists.

Let S be a sheaf of K-modules. Then $\overline{S} = \dfrac{S_0}{S}$ is a sheaf of K-modules. Let $\alpha_S : S \to S_0$ be the inclusion map and $\beta_S : S_0 \to \overline{S}$ the quotient map. Then the sequence of sheaves of K-modules

(i) $$0 \longrightarrow S \xrightarrow{\alpha_S} S_0 \xrightarrow{\beta_S} \overline{S} \longrightarrow 0$$

is exact.

By (a) of (1.1) there exists K-module isomorphisms $\mu_S : \Gamma(S) \to H^0(X,S)$ and $\nu_S : \Gamma(S) \to \widetilde{H}^0(X,S)$.

Let $\phi_S^0 : H^0(X,S) \to \widetilde{H}^0(X,S)$ such that $\phi_S^0 = \nu_S \circ \mu_S^{-1}$. We know ν_S and μ_S^{-1} are K-module isomorphisms so ϕ_S^0 is a K-module isomorphism. It follows

$$\begin{array}{ccc} \Gamma(S) & \xrightarrow{\mu_S} & H^\circ(X,S) \\ \big\downarrow I_D & & \big\downarrow \phi_S^\circ \\ \Gamma(S) & \xrightarrow{\nu_S} & \widetilde{H}^\circ(X,S) \end{array}$$

commutes.

So condition (a) of (1.1) is satisfied.

Let $\theta : S \to T$ be any sheaf homomorphism. By (II) of (1.1) there exist K-module homomorphisms

$$\theta_*^p : H^p(X,S) \to H^p(X,T)$$
$$\tilde{\theta}_*^p : \tilde{H}^p(X,S) \to \tilde{H}^p(X,T)$$

for integers p. We know from (a) of (1.1) that

(ii) $\quad \mu_T \circ \Gamma(\theta) = \theta_*^0 \circ \mu_S \quad$ and

(iii) $\quad \nu_T \circ \Gamma(\theta) = \tilde{\theta}_*^0 \circ \nu_S$.

Hence

$$\begin{aligned} \tilde{\theta}_*^0 \circ \phi_S^0 &= \tilde{\theta}_*^0 \circ \nu_S \circ \mu_S^{-1} \\ &= \nu_T \circ \Gamma(\theta) \circ \mu_S^{-1} \\ &= \nu_T \circ \mu_T^{-1} \circ \theta_*^0 \\ &= \phi_T^0 \circ \theta_*^0 . \end{aligned}$$

Thus

$$\begin{array}{ccc} H^0(X,S) & \xrightarrow{\theta_*^0} & H^0(X,T) \\ \downarrow{\scriptstyle \phi_S^0} & & \downarrow{\scriptstyle \phi_T^0} \\ \tilde{H}^0(X,S) & \xrightarrow{\tilde{\theta}_*^0} & \tilde{H}^0(X,T) \end{array}$$

commutes.

Therefore (ii) of (2.1) is true for $q = 0$. Now $\theta : S \to T$ is any sheaf homomorphism so we can conclude

$$\begin{array}{ccc} H^0(X,\bar{S}) & \xrightarrow{\bar{\theta}_*^0} & H^0(X,\bar{T}) \\ \downarrow{\scriptstyle \phi_{\bar{S}}^0} & & \downarrow{\scriptstyle \phi_{\bar{T}}^0} \\ \tilde{H}^0(X,\bar{S}) & \xrightarrow{\tilde{\bar{\theta}}_*^0} & \tilde{H}^0(X,\bar{T}) \end{array}$$

commutes.

From (f) of (1.1) we know that

$$\begin{array}{ccc} H^0(X,\overline{S}) & \xrightarrow{\partial_S^0} & H^1(X,S) \\ \Big\downarrow \tilde{\overline{\theta}}{}_*^0 & & \Big\downarrow \theta_*^1 \\ H^0(X,\overline{T}) & \xrightarrow{\partial_T^0} & H^1(X,T) \end{array}$$

and

$$\begin{array}{ccc} \widetilde{H}^0(X,\overline{S}) & \xrightarrow{\widetilde{\partial}_S^0} & \widetilde{H}^1(X,S) \\ \Big\downarrow \widetilde{\overline{\theta}}{}_*^0 & & \Big\downarrow \widetilde{\theta}_*^1 \\ \widetilde{H}^0(X,\overline{T}) & \xrightarrow{\widetilde{\partial}_T^0} & \widetilde{H}^1(X,T) \end{array}$$

commute.

The K-module homomorphisms ∂_S^0, ∂_T^0, $\widetilde{\partial}_S^0$ and $\widetilde{\partial}_T^0$ exist by (f) of (1.1).

We know by lemma (1.9) that S_0 is a fine sheaf. So by (b) of (1.1) $H^q(X,S_0) = 0$ for $q > 0$. Therefore, using (c) of (1.1), the sequences of K-modules

(v) $$H^0(X,S_0) \xrightarrow{\beta_{S*}^0} H^0(X,\overline{S}) \xrightarrow{\partial_S^0} H^1(X,S) \to 0$$

and

(vi) $$\widetilde{H}^0(X,S_0) \xrightarrow{\widetilde{\beta}_{S*}^0} \widetilde{H}^0(X,\overline{S}) \xrightarrow{\widetilde{\partial}_S^0} \widetilde{H}^1(X,S) \to 0$$

are exact. So ∂_S^0 and $\widetilde{\partial}_S^0$ are K-module epimorphisms. Thus

$$H^1(X,S) \simeq \frac{H^0(X,\overline{S})}{\text{Image}(\beta_{S*}^0)} \simeq \frac{\Gamma(\overline{S})}{\text{Image}(\Gamma(\beta_S))}$$

and

$$\widetilde{H}^1(X,S) \simeq \frac{\widetilde{H}^0(X,\overline{S})}{\text{Image}(\widetilde{\beta}_{S*}^0)} \simeq \frac{\Gamma(\overline{S})}{\text{Image}(\Gamma(\beta_S))} .$$

Thus there exists a K-module isomorphism $\phi_S^1 : H^1(X,S) \to \widetilde{H}^1(X,S)$ such that

$$\begin{array}{ccc} H^0(X,\bar{S}) & \xrightarrow{\partial_S^0} & H^1(X,S) \\ \Big\downarrow \phi_{\bar{S}}^0 & & \Big\downarrow \phi_S^1 \\ \tilde{H}^0(X,\bar{S}) & \xrightarrow{\tilde{\partial}_{\bar{S}}^0} & \tilde{H}^1(X,S) \end{array}$$

commutes.

We know $H^q(X,S_0) = 0$ for $q \geq 1$. So by (c) of (1.1) the sequences

(vii) $$0 \longrightarrow H^p(X,\bar{S}) \xrightarrow{\partial_S^p} H^{p+1}(X,\bar{S}) \longrightarrow 0$$

and

(viii) $$0 \longrightarrow \tilde{H}^p(X,\bar{S}) \xrightarrow{\tilde{\partial}_S^p} \tilde{H}^{p+1}(X,\bar{S}) \longrightarrow 0$$

are exact. It follows ∂_S^p and $\tilde{\partial}_S^p$ are K-module isomorphisms.

Assume for each sheaf of K-modules T over X that there exists a K-module isomorphism

$$\phi_T^p : H^p(X,T) \longrightarrow \tilde{H}^p(X,T) \quad .$$

By induction there exists a K-module isomorphism $\phi_S^{p+1} : H^{q+1}(X,S) \to \tilde{H}^{q+1}(X,S)$ such that

$$\begin{array}{ccc} H^p(X,\bar{S}) & \xrightarrow{\partial_S^p} & H^{p+1}(X,S) \\ \Big\downarrow \phi_{\bar{S}}^p & & \Big\downarrow \phi_S^{p+1} \\ \tilde{H}^p(X,\bar{S}) & \xrightarrow{\tilde{\partial}_S^p} & \tilde{H}^{p+1}(X,S) \end{array}$$

commutes.

Let $\theta : S \to T$ be a sheaf homomorphism. Let $x \in X$. Since $\alpha_S | S_x$ and $\alpha_T | T_x$ are K-module monomorphisms there exists $\theta_{0x} : S_{0x} \to T_{0x}$ such that the diagram

$$\begin{array}{ccc} S_x & \xrightarrow{\alpha_S | S_x} & S_{0x} \\ \downarrow{\scriptstyle \theta|S_x} & & \downarrow{\scriptstyle \theta_{0x}} \\ T_x & \xrightarrow{\alpha_T | T_x} & T_{0x} \end{array}$$

commutes.

Let $\theta_0 = \{\theta_{0x} \mid x \in X\}$. θ_0 is a sheaf homomorphism. It follows easily that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha_S} & S_0 \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta_0} \\ T & \xrightarrow{\alpha_T} & T_0 \end{array}$$

commutes.

We know that

$$\begin{array}{ccc} H^0(X,\bar{S}) & \xrightarrow{\partial^0_S} & H^1(X,S) \\ \downarrow{\scriptstyle \phi^0_{\bar{S}}} & & \downarrow{\scriptstyle \phi^1_S} \\ \widehat{H}^0(X,\bar{S}) & \xrightarrow{\widehat{\partial}^0_S} & \widehat{H}^1(X,S) \end{array}$$

and

$$\begin{array}{ccc} H^0(X,\bar{T}) & \xrightarrow{\partial^0_T} & H^1(X,T) \\ \downarrow{\scriptstyle \phi^0_{\bar{T}}} & & \downarrow{\scriptstyle \phi^1_T} \\ \widehat{H}^0(X,T) & \xrightarrow{\widehat{\partial}^0_T} & \widehat{H}^1(X,\bar{T}) \end{array}$$

commute by the definitions of ϕ^1_S and ϕ^1_T. Therefore if $a \in H^1(X,S)$ since ∂^0_S is a K-module epimorphism there exists $b \in H^0(X,\bar{S})$ such that $\partial^0_S(b) = a$. Thus

$$\begin{aligned} \phi^1_T \cdot (\theta^1_*(a)) &= \phi^1_T \cdot \theta^1_* \cdot \partial^0_S(b) \\ &= \phi^1_T \cdot \partial^0_T \cdot \bar{\theta}^0_*(b) = \widehat{\partial}^0_T \cdot \phi^0_T \cdot \bar{\theta}^0_*(b) \end{aligned}$$

$$= \widetilde{\partial}^0_T \cdot \tilde{\tilde{\theta}}^0_* \cdot \phi^0_S(b) = \widetilde{\theta}^1_* \cdot \widetilde{\partial}^0_S \cdot \phi^0_S(b)$$

$$= \widetilde{\theta}^1_* \cdot \phi^1_S(a) \quad .$$

Hence

$$\begin{array}{ccc} H^1(X,S) & \xrightarrow{\theta^1_*} & H^1(X,T) \\ \Big\downarrow \phi^1_S & & \Big\downarrow \phi^1_T \\ \widetilde{H}^1(X,S) & \xrightarrow{\widetilde{\theta}^1_*} & \widetilde{H}^1(X,T) \end{array}$$

commutes.

We have by (f) of (1.1) that $\partial^p_T \circ \overline{\theta}^p_* = \theta^p_* \circ \partial^p_S$ and $\widetilde{\partial}^p_T \circ \tilde{\overline{\theta}}^p_* = \widetilde{\theta}^p_* \circ \widetilde{\partial}^p_S$. We know also that

$$\widetilde{\partial}^p_S \circ \phi^p_S = \phi^{p+1}_S \circ \partial^p_S$$

and

$$\widetilde{\partial}^p_T \circ \phi^p_T = \phi^{p+1}_T \circ \partial^p_T$$

by the definitions of ϕ^p_S and ϕ^p_T .

Let $c \in H^{p+1}(X,S)$. Since ∂^p_S is a K-module epimorphism there exists $d \in H^p(X,\overline{S})$ such that $\partial^p_S(d) = c$. Hence

$$\begin{aligned} \phi^{p+1}_T \cdot (\theta^{p+1}_*(c)) &= \phi^{p+1}_T \cdot \theta^{p+1}_* \cdot \partial^p_S(d) \\ &= \phi^{p+1}_T \cdot \partial^p_T \cdot \overline{\theta}^p_*(d) = \widetilde{\partial}^p_T \cdot \phi^p_T \cdot \overline{\theta}^p_*(d) \\ &= \widetilde{\partial}^p_T \cdot \tilde{\overline{\theta}}^p_* \cdot \phi^p_S(d) = \widetilde{\theta}^{p+1}_* \cdot \widetilde{\partial}^p_S \cdot \phi^p_S(d) \\ &= \widetilde{\theta}^{p+1}_* \cdot \phi^{p+1}_S \cdot \partial^p_S(d) = \widetilde{\theta}^{p+1}_* \cdot \phi^{p+1}_S(c) \quad . \end{aligned}$$

Hence

$$\begin{array}{ccc} H^{p+1}(X,S) & \xrightarrow{\theta^{p+1}_*} & H^{p+1}(X,T) \\ \Big\downarrow \phi^{p+1}_S & & \Big\downarrow \phi^{p+1}_T \\ \widetilde{H}^{p+1}(X,S) & \xrightarrow{\widetilde{\theta}^{p+1}_*} & \widetilde{H}^{p+1}(X,T) \end{array}$$

commutes

Therefore by induction

$$\begin{array}{ccc} H^q(X,S) & \xrightarrow{\theta^q_*} & H^q(X,T) \\ \downarrow \phi^q_S & & \downarrow \phi^q_T \\ \hat{H}^q(X,S) & \xrightarrow{\hat{\theta}^q_*} & \hat{H}^q(X,T) \end{array}$$

commutes for $q \geq 1$. So (b) of (2.1) is satisfied.

Let $0 \to R \xrightarrow{\alpha} S \xrightarrow{\beta} T \to 0$ be an exact sequence of sheaves of K-modules. Let R_0 and S_0 be the sheaves of germs of R and S, respectively. Thus $\overline{R} = \frac{R_0}{R}$ and $\overline{S} = \frac{S_0}{S}$ are sheaves of K-modules.

Now the sequence $0 \to R \xrightarrow{\rho} R_0 \xrightarrow{\sigma} \overline{R} \to 0$ is exact where ρ is the inclusion map and σ is the quotient map. Also the sequence $0 \to S \xrightarrow{\phi} S_0 \xrightarrow{\psi} \overline{S} \to 0$ is exact where ϕ is the inclusion map and ψ is the quotient map.

Let $\Delta = \phi \circ \alpha$. Now Image$(\Delta)$ is a subsheaf of S_0 so $G = \dfrac{S_0}{\text{Image}(\Delta)}$ is a sheaf. Since Δ is a sheaf monomorphism $0 \to R \xrightarrow{\Delta} S_0 \xrightarrow{\tau} G \to 0$ is an exact sequence of sheaves where τ is the quotient map.

Let $x \in X$. Since $\beta_x : S_x \to T_x$ is a K-module epimorphism, $T_x \simeq \dfrac{S_x}{\text{Image}(\alpha_x)}$. Now $G_x = \dfrac{S_{0x}}{\text{Image}(\Delta_x)}$ so there exists $\gamma_x : T_x \to G_x$ a K-module homomorphism such that

$$\begin{array}{ccc} S_x & \xrightarrow{\beta_x} & T_x \\ \downarrow \phi_x & & \downarrow \gamma_x \\ S_{0x} & \xrightarrow{\tau_x} & G_x \end{array}$$

commutes.

Let $\gamma : T \to G$ such that $\gamma|_{T_x} = \gamma_x$ for all $x \in X$. Then

$$\begin{array}{ccc} S & \xrightarrow{\beta} & T \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \gamma} \\ S_0 & \xrightarrow{\tau} & T_0 \end{array}$$

commutes.

Hence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\alpha} & S & \xrightarrow{\beta} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I_D} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & R & \xrightarrow{\Delta} & S_0 & \xrightarrow{\tau} & G & \longrightarrow & 0 \end{array}$$

commutes.

Since ρ and Δ are sheaf monomorphisms, there exists $\delta : R_0 \to S_0$ a sheaf homomorphism so that

$$\begin{array}{ccc} R & \xrightarrow{\Delta} & S_0 \\ \uparrow{\scriptstyle I_D} & & \uparrow{\scriptstyle \delta} \\ R & \xrightarrow{\rho} & R_0 \end{array}$$

commutes.

Since Δ is a sheaf monomorphism, there exists a sheaf isomorphism $\eta : R \to \text{Image}(\Delta)$. It follows that there exists $\varepsilon : \overline{R} \to G$ a sheaf homomorphism such that

$$\begin{array}{ccc} S_0 & \xrightarrow{\tau} & G \\ \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \epsilon} \\ R_0 & \xrightarrow{\sigma} & \overline{R} \end{array}$$

commutes.

Hence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\Delta} & S_0 & \xrightarrow{\tau} & G & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \epsilon} & & \\
0 & \longrightarrow & R & \xrightarrow{\rho} & R_0 & \xrightarrow{\sigma} & \overline{R} & \longrightarrow & 0
\end{array}$$

commutes.

By lemma (1.9) S_0 and R_0 are fine sheaves so by (b) of (1.1) $H^q(X,S_0) = H^q(X,R_0) = 0$ for $q > 0$. Now by (f) of (1.1) and lemma (1.7) it follows that

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^0(X,R) & \xrightarrow{\alpha^0_*} & H^0(X,S) & \xrightarrow{\beta^0_*} & H^0(X,T) & \xrightarrow{\partial^0} & H^1(X,R) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle Id^0_*} & & \downarrow{\scriptstyle \varphi^0_*} & & \downarrow{\scriptstyle \gamma^0_*} & & \downarrow{\scriptstyle Id^1_*} & & \\
\cdots & \longrightarrow & H^0(X,R) & \xrightarrow{\Delta^0_*} & H^0(X,S_0) & \xrightarrow{\tau^0_*} & H^0(X,G) & \xrightarrow{\partial^0_{/}} & H^1(X,R) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle Id^0_*} & & \uparrow{\scriptstyle \delta^0_*} & & \uparrow{\scriptstyle \epsilon^0_*} & & \uparrow{\scriptstyle Id^1_*} & & \\
\cdots & \longrightarrow & H^0(X,R) & \xrightarrow{\rho^0_*} & H^0(X,R_0) & \xrightarrow{\sigma^0_*} & H^0(X,\overline{R}) & \xrightarrow{\partial^0_{//}} & H^1(X,R) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^p(X,S) & \xrightarrow{\beta^p_*} & H^p(X,T) & \xrightarrow{\partial^p} & H^{p+1}(X,R) & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle \gamma^p_*} & & \downarrow{\scriptstyle Id^{p+1}_*} & & \\
 & & 0 & \longrightarrow & H^p(X,G) & \xrightarrow{\partial^p_{/}} & H^{p+1}(X,R) & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle \epsilon^p_*} & & \uparrow{\scriptstyle Id^{p+1}_*} & & \\
 & & 0 & \longrightarrow & H^p(X,\overline{R}) & \xrightarrow{\partial^p_{//}} & H^{p+1}(X,R) & \longrightarrow & 0
\end{array}$$

commute. By (c) of (1.1) each row in (I) and (II) is exact, where

$\partial^q : H^q(X,T) \longrightarrow H^{q+1}(X,R)$,

$\partial^q_{/} : H^q(X,G) \longrightarrow H^{q+1}(X,R)$, and

$\partial^q_{//} : H^q(X,\overline{R}) \longrightarrow H^{q+1}(X,R)$ for $q \geq 0$, are the K-module homomorphisms described by (II) of (1.1). Also the corresponding sequences in $\widetilde{\mathcal{H}}$ cohomology commute.

By (a) of (1.1) there exist K-module isomorphisms

$$\begin{aligned} \xi &: \Gamma(T) \longrightarrow H^0(X,T) \quad , \\ \chi &: \Gamma(G) \longrightarrow H^0(X,G) \quad , \quad \text{and} \\ \mathcal{K} &: \Gamma(\overline{R}) \longrightarrow H^0(X,\overline{R}) \quad . \end{aligned}$$

Thus from (I),

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Gamma(R) & \xrightarrow{\Gamma(\alpha)} & \Gamma(S) & \xrightarrow{\Gamma(\beta)} & \Gamma(T) & \xrightarrow{\partial^0\circ\xi} & H^1(X,R) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \Gamma(I_D)} & & \downarrow{\scriptstyle \Gamma(\varphi)} & & \downarrow{\scriptstyle \Gamma(\gamma)} & & \downarrow{\scriptstyle ID^1_*} & & \\ 0 & \longrightarrow & \Gamma(R) & \xrightarrow{\Gamma(\Delta)} & \Gamma(S_0) & \xrightarrow{\Gamma(\tau)} & \Gamma(G) & \xrightarrow{\partial^0_{\prime}\circ\chi} & H^1(X,R) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \Gamma(I_D)} & & \uparrow{\scriptstyle \Gamma(\delta)} & & \uparrow{\scriptstyle \Gamma(\epsilon)} & & \uparrow{\scriptstyle ID^1_*} & & \\ 0 & \longrightarrow & \Gamma(R) & \xrightarrow{\Gamma(\rho)} & \Gamma(R_0) & \xrightarrow{\Gamma(\sigma)} & \Gamma(\overline{R}) & \xrightarrow{\partial^0_{\prime\prime}\circ\mathcal{K}} & H^1(X,R) & \longrightarrow & 0 \end{array}$$

commutes, where each row is exact. Also there exists K-module isomorphisms $\widetilde{\xi}:\Gamma(T) \to \widetilde{H}^0(X,T)$, $\widetilde{\chi}:\Gamma(G) \to \widetilde{H}^0(X,G)$ and $\widetilde{\mathcal{K}}:\Gamma(\overline{R}) \to \widetilde{H}^0(X,\overline{R})$ and the corresponding diagram in $\widetilde{\mathcal{H}}$ cohomology commutes.

Now the second row of (III) is exact so there exists a K-module isomorphism $\Omega_a:H^1(X,R) \to \dfrac{\Gamma(G)}{\text{Image}(\Gamma(\tau))}$. Also the third row of (III) is exact so there exists a K-module isomorphism $\Omega_b:H^1(X,R) \to \dfrac{\Gamma(\overline{R})}{\text{Image}(\Gamma(\sigma))}$. From the corresponding diagram in the cohomology theory there exist K-module isomorphisms $\widetilde{\Omega}_a:\widetilde{H}^1(X,R) \to \dfrac{\Gamma(G)}{\text{Image}(\Gamma(\tau))}$ and $\widetilde{\Omega}_b:\widetilde{H}^1(X,R) \to \dfrac{\Gamma(G)}{\text{Image}(\Gamma(\tau))}$.

Now $\Gamma(\gamma):\Gamma(T) \to \Gamma(G)$ so if $\lambda:\Gamma(G) \to \dfrac{\Gamma(G)}{\text{Image}(\Gamma(\tau))}$ is the quotient map then $\lambda \circ \Gamma(\gamma) : \Gamma(T) \to \dfrac{\Gamma(G)}{\text{Image}(\Gamma(\tau))}$. Since Ω_a and Ω_b are K-module isomorphisms there exists a K-module isomorphism

$$\iota : \frac{\Gamma(G)}{\text{Image}(\Gamma(\tau))} \to \frac{\Gamma(\overline{R})}{\text{Image}(\Gamma(\sigma))} .$$

Now $\partial^0 : H^0(X,T) \to H^1(X,R)$ is the composition $\Omega_b^{-1} \circ \iota \circ \lambda \circ \Gamma(\gamma) \circ \xi$ and $\widetilde{\partial} : \widetilde{H}^0(X,T) \to H^1(X,R)$ is the composition $\widetilde{\Omega}_b^{-1} \circ \iota \circ \lambda \circ \Gamma(\gamma) \circ \widetilde{\xi}$.

Consider the diagram

$$\begin{array}{ccccccccc}
H^0(X,T) & \xrightarrow{\xi} & \Gamma(T) & \xrightarrow{\lambda\circ\Gamma(\gamma)} & \dfrac{\Gamma(G)}{\text{Image}(\Gamma(\tau))} & \xrightarrow{\iota} & \dfrac{\Gamma(\overline{R})}{\text{Image}(\Gamma(\sigma))} & \xrightarrow{\Omega_b^{-1}} & H^1(X,R) \\
\downarrow \phi_T^0 & \text{(i)} & \downarrow \Gamma(I_D) & \text{(ii)} & \downarrow \Gamma(I_D) & \text{(iii)} & \downarrow \Gamma(I_D) & \text{(iv)} & \downarrow \phi_T^1 \\
\widetilde{H}^0(X,T) & \xrightarrow{\widetilde{\xi}} & \Gamma(T) & \xrightarrow{\lambda\circ\Gamma(\gamma)} & \dfrac{\Gamma(G)}{\text{Image}(\Gamma(\tau))} & \xrightarrow{\iota} & \dfrac{\Gamma(\overline{R})}{\text{Image}(\Gamma(\sigma))} & \xrightarrow{\widetilde{\Omega}_b^{-1}} & \widetilde{H}^1(X,R)
\end{array}$$

Square (i) commutes by (a) of this theorem, (ii) and (iii) commute trivially, and (iv) commutes by definition of ϕ_T^1.

Hence

$$\begin{array}{ccc}
H^0(X,T) & \xrightarrow{\partial^0} & H^1(X,R) \\
\downarrow \phi_T^0 & & \downarrow \phi_R^1 \\
\widetilde{H}^0(X,T) & \xrightarrow{\widetilde{\partial}^0} & \widetilde{H}^1(X,R)
\end{array}$$

commutes.

It follows from (II) that for $q \geq 1$,

$$\partial^q : H^q(X,T) \to H^{q+1}(X,R) = (\partial^q_{//} \circ (\varepsilon^q_*)^{-1} \circ \gamma^q_*)$$

where $(\varepsilon^q_*)^{-1}$ exists since $\partial^p_{/}$ and $\partial^p_{//}$ are K-module isomorphisms for $p \geq 1$.

Consider the diagram

$$
\begin{array}{ccccccc}
H^q(X,T) & \xrightarrow{\gamma^q_*} & H^q(X,G) & \xrightarrow{(\epsilon^q_*)^{-1}} & H^q(X,\bar{R}) & \xrightarrow{\partial^q_{/\!/}} & H^{q+1}(X,R) \\
\downarrow \phi^q_T & (v) & \downarrow \phi^q_G & (vi) & \downarrow \phi^q_{\bar{R}} & (vii) & \downarrow \phi^{q+1}_R \\
\hat{H}^q(X,T) & \xrightarrow{\hat{\gamma}^q_*} & \hat{H}^q(X,G) & \xrightarrow{(\hat{\epsilon}^q_*)^{-1}} & \hat{H}^q(X,\bar{R}) & \xrightarrow{\hat{\partial}^q_{/\!/}} & \hat{H}^{q+1}(X,R)
\end{array}
$$

Squares (v) and (vi) commute by (b) of this theorem while square (vii) commutes by definition of ϕ^{q+1}_R.

Thus

$$
\begin{array}{ccc}
H^q(X,T) & \xrightarrow{\partial^q} & H^{q+1}(X,R) \\
\downarrow \phi^q_T & & \downarrow \phi^{q+1}_R \\
\hat{H}^q(X,T) & \xrightarrow{\hat{\partial}^q} & \hat{H}^{q+1}(X,R)
\end{array}
$$

commutes.

Thus (c) is proved.

The uniqueness of ϕ for $q = 0$ follows from part (a). For $q = 1$ uniqueness follows from the definition of $\phi^1_S : H^q(X,S) \to \hat{H}^q(X,S)$ for a sheaf S. For $q \geq 2$ uniqueness follows inductively from the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^{q-1}(X,\bar{S}) & \xrightarrow{\partial^0_S} & H^q(X,S) & \longrightarrow & 0 \\
 & & \downarrow \phi^{q-1}_{\bar{S}} & & \downarrow \phi^q_S & & \\
0 & \longrightarrow & \hat{H}^{q-1}(X,\bar{S}) & \xrightarrow{\hat{\partial}^0_S} & \hat{H}^q(X,S) & \longrightarrow & 0
\end{array}
$$

COROLLARY 2.5. By the theorem there exists unique homomorphisms $\phi: \mathcal{H} \to \hat{\mathcal{H}}$ and $\theta: \hat{\mathcal{H}} \to \mathcal{H}$.

$\theta \circ \phi$ by uniqueness is the identity homomorphism on $\hat{\mathcal{H}}$

$\phi \circ \theta$ by uniqueness is the identity homomorphism on $\hat{\mathcal{H}}$. It follows ϕ is an isomorphism. Hence, any two cohomology theories on X with coefficients in sheaves of K-modules over X are uniquely isomorphic.

THEOREM 2.6. Let $\mathcal{H}$ be a cohomology theory for X with coefficients in sheaves of K-modules over X. Let

$$\text{(i)} \qquad 0 \longrightarrow S \xrightarrow{\alpha} C_0 \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{\alpha_2} C_3 \xrightarrow{\alpha_3} \cdots$$

be a resolution of the sheaf S. There exists a canonical K-module homomorphism

$$\theta^q : H^q(\Gamma(C^*)) \longrightarrow H^q(X,S) \quad \text{for } q \text{ integer} \quad .$$

If $H^p(X,C_q) = 0$ for $p > 0$ and q integer then θ^q is a K-module isomorphism.

Proof. Since (i) is exact

$$\text{(ii)} \qquad 0 \longrightarrow \Gamma(S) \xrightarrow{\Gamma(\alpha)} \Gamma(C_0) \xrightarrow{\Gamma(\alpha_0)} \Gamma(C_1)$$

is an exact sequence of K-modules. By (a) of (1.1) $H^0(X,S) \simeq \Gamma(S)$. Since (ii) is exact, $\Gamma(\alpha)$ is a K-module monomorphism and $\text{Image}(\Gamma(\alpha)) = \text{Kernel}(\Gamma(\alpha_0))$. So $H^0(\Gamma(C^*)) = \text{Kernel}(\Gamma(\alpha_0))$ $= \text{Image}(\Gamma(\alpha)) \simeq \Gamma(S) \simeq H^0(X,S)$. Let θ^0 be the above K-module isomorphism.

Let $K_q = \text{Kernel}(\alpha_q)$ for $q \geq 0$. K_q is a subsheaf of C_q. The sequence

$$\text{(iii)} \qquad 0 \longrightarrow K_{q-1} \xrightarrow{\iota_{q-1}} C_{q-1} \xrightarrow{\beta_{q-1}} K_q \longrightarrow 0$$

where ι_{q-1} is the inclusion map and $\beta_q(a) = \alpha_q(a)$ for $a \in C_q$ is an exact sequence of sheaves for $q \geq 1$.

Since (iii) is a short exact sequence of sheaves by property (c) of (1.1), there exists a K-module homomorphism

$$\partial_q^p : H^p(X, K_q) \longrightarrow H^{p+1}(X, K_{q-1})$$

for p integer such that

$$\text{(v)} \quad \cdots \to H^p(X, K_{q-1}) \xrightarrow{\iota_{q-1*}^p} H^p(X, C_{q-1}) \xrightarrow{\beta_{q-1*}^p} H^p(X, K_q) \xrightarrow{\partial_q^p} H^{p+1}(X, K_{q-1}) \xrightarrow{\iota_{q-1*}^{p+1}} \cdots$$

is an exact sequence of K-modules.

Now by (a) of (1.1) there exists a K-module isomorphism $\eta_q : \Gamma(K_q) \to H^0(X, K_q)$. Let $\tau_q = \partial_q^0 \circ \eta_q : \Gamma(K_q) \to H^1(X, K_{q-1})$. Then $\text{Kernel}(\tau_q) \approx \text{Kernel}(\partial_q^0)$ and $\text{Image}(\tau_q) = \text{Image}(\partial_q^0)$. We know

$$0 \longrightarrow \Gamma(K_{q-1}) \xrightarrow{\Gamma(\iota_{q-1})} \Gamma(C_{q-1}) \xrightarrow{\Gamma(\beta_{q-1})} \Gamma(K_q)$$

is exact and so by (v) the sequence

$$\text{(vi)} \quad 0 \to \Gamma(K_{q-1}) \xrightarrow{\Gamma(\iota_{q-1})} \Gamma(C_{q-1}) \xrightarrow{\Gamma(\beta_{q-1})} \Gamma(K_q) \xrightarrow{\tau_q} H^1(X, K_{q-1}) \xrightarrow{\iota_{q-1*}^1} \cdots$$

is exact.

It is easily proved that $\text{Kernel}(\Gamma(\alpha_q)) \simeq \Gamma(K_q)$.

We know also that $\text{Image}(\Gamma(\alpha_{q-1})) = \text{Image}(\Gamma(\beta_{q-1}))$.

So

$$H^q(\Gamma(C^*)) = \frac{\text{Kernel}(\Gamma(\alpha_q))}{\text{Image}(\Gamma(\alpha_{q-1}))} = \frac{\text{Kernel}(\Gamma(\alpha_q))}{\text{Image}(\Gamma(\beta_{q-1}))} \simeq \frac{\Gamma(K_q)}{\text{Image}(\Gamma(\beta_{q-1}))} .$$

Since (vi) is exact there exists a K-module homomorphism

$$\gamma_q : \frac{\Gamma(K_q)}{\text{Image}(\Gamma(\beta_{q-1}))} \longrightarrow H^1(X,K_{q-1}) \quad .$$

It follows there exists a K-module homomorphism

$\theta_1^q : H^q(\Gamma(C^*)) \to H^1(X,K_{q-1})$. If $H^p(X,C_q) = 0$ for $p > 0$ then γ_q is a K-module isomorphism and so θ_1^q is a K-module isomorphism.

Let $q \geq r \geq 2$. The sequence of sheaves

$$0 \longrightarrow K_{q-r} \xrightarrow{\iota_{q-r}} C_{q-r} \xrightarrow{\beta_{q-r}} K_{q-r+1} \longrightarrow 0$$

where ι_{q-r} is the inclusion map and $\beta_{q-r}(b) = \alpha_{q-r}(b)$ for $b \in C_{q-r}$.

By property (c) of (1.1) for each $p \geq 0$ there exists a K-module homomorphism

$$\partial^p_{q-r+1} : H^p(X,K_{q-r+1}) \longrightarrow H^{p+1}(X,K_{q-r})$$

such that the sequence

$$\ldots \to H^p(X,C_{q-r}) \xrightarrow{\beta^p_{q-r*}} H^p(X,K_{q-r+1}) \xrightarrow{\partial^p_{q-r+1}}$$
$$H^{p+1}(X,K_{q-r}) \xrightarrow{\iota^{p+1}_{q-r*}} H^{p+1}(X,C_{q-r}) \xrightarrow{\beta^{p+1}_{q-r*}} \ldots$$

is exact.

Now if $H^p(X,C_{q-r}) = 0$ for $p > 0$ then it follows that ∂^p_{q-r+1} is a K-module isomorphism for $p > 0$.

Let $\overline{\theta}^q : H^q(\Gamma(C^*)) \to H^{q-1}(X,K_1)$ such that

$$\overline{\theta}^q = \partial_2^{q-2} \circ \partial_3^{q-3} \circ \ldots \circ \partial^2_{q-2} \circ \partial^1_{q-1} \circ \theta_1^q \quad .$$

If $H^p(X,C_q) = 0$ for all $p > 0$ and $q \geq 0$ then it is easily seen that $\overline{\theta}^q$ is a K-module isomorphism.

Since (i) is exact, we know α is a sheaf monomorphism. Let $\beta_0 : C_0 \to K_1$ such that $\beta_0(f) = \alpha_0(f)$ for $f \in C_0$. Now $K_1 = \text{Image}(\alpha_0)$ so β_0 is a sheaf epimorphism. Also $\text{Image}(\alpha) = \text{Kernel}(\alpha_0) = \text{Kernel}(\beta_0)$. Hence

$$\text{(viii)} \qquad 0 \longrightarrow S \xrightarrow{\alpha} C_0 \xrightarrow{\beta_0} K_1 \longrightarrow 0$$

is a short exact sequence of sheaves. It follows that there exists a K-module homomorphism $\partial^p : H^p(X,K_1) \to H^{p+1}(X,S)$ for $p \geqslant 0$ such that

$$\cdots \to H^p(X,S) \xrightarrow{\alpha^p_*} H^p(X,C_0) \xrightarrow{\beta^p_{0*}} H^p(X,K_1) \xrightarrow{\partial^p} H^{p+1}(X,S) \xrightarrow{\alpha^{p+1}_*} \cdots$$

is an exact sequence of K-modules. Now if $H^p(X,C_0) = 0$ for $p > 0$ then ∂^p is a K-module isomorphism for $p > 0$.

Let $\theta^q : H^q(\Gamma(C^*)) \to H^q(X,S)$ such that $\theta^q = \partial^{q-1} \circ \overline{\theta}^q$. θ^q is a K-module homomorphism. If $H^p(X,C_q) = 0$ for $p > 0$ and $q \geqslant 0$ then it easily follows that θ^q is a K-module isomorphism.

COROLLARY 2.7. It follows immediately that if (i) is a fine resolution then θ^q is a K-module isomorphism.

CHAPTER IV

SINGULAR COHOMOLOGY

In this chapter, it will be shown that if X is a homologically locally contractible (HLC) paracompact space, the sheaves of singular cochains with coefficients in K provide a torsion-free fine resolution of the constant sheaf $\mathcal{K}$.

§1. Singular p-simplices

DEFINITION 1.1. The standard Euclidean 0-simplex $\Delta^0 = \{0\}$. Let p be an integer ≥ 1. The standard Euclidean p-simplex

$$\Delta^p = \{(a_1,\ldots,a_p) \in R^p \mid \sum_{i=1}^{p} a_i \leq 1 \text{ and } a_i \geq 0\} .$$

DEFINITION 1.2. A singular p-simplex σ in X is a continuous map $\Delta^p \to X$. Let $S_p(X)$ be the set of singular p-simplices in X. Let $S_p(X,K)$ be the free K-module generated by $S_p(X)$. A singular p-chain of X with coefficients in K is an element of $S_p(X,K)$.

Let Y be a subspace of X. Then it is clear that any continuous map $\sigma:\Delta^p \to Y$ can be regarded as a continuous map into X. Also $S_p(Y) = \{\sigma \in S_p(X) \mid \sigma(\Delta^p) \subset Y\}$.

DEFINITION 1.3. Let $p \geq 0$ be an integer. Let $K_i^p:\Delta^p \to \Delta^{p+1}$ for $0 \leq i \leq p+1$ such that $K_0^0(0) = 1$, $K_1^0(0) = 0$,

$$K_0^p(a_1,\ldots,a_p) = \left(1 - \sum_{i=1}^{p} a_i, a_1,\ldots,a_p\right)$$

for $p \geqslant 1$ and

$$K_i^p(a_1,\ldots,a_p) = (a_1,\ldots,a_{i-1},0,a_i,\ldots,a_p)$$

for $(1 \leqslant i \leqslant p+1)$.

Let $p \geqslant 1$ be an integer and $\sigma \in S_p(X)$. Let $i \geqslant 0$ be an integer satisfying $i \leqslant p$. Let $\sigma^i = \sigma \circ K_i^{p-1}$. The boundary of σ, $\partial_p \sigma \in S_{p-1}(X,K)$ is defined to be equal to $\sum_{i=0}^{p} (-1)^i \sigma^i$. Let $\tau = \sum_{j=1}^{k} a_j \sigma_j \in S_p(X,K)$ where $a_j \in K$ and $\sigma_j \in S_p(X)$. The boundary of τ, $\partial_p \tau \in S_{p-1}(X,K)$ is defined to be equal to $\sum_{i=0}^{p} \sum_{j=1}^{k} (-1)^i a_j \sigma_j^i$. Hence $\partial_p : S_p(X,K) \to S_{p-1}(X,K)$.

Now if Y is a subspace of X and $\sigma \in S_p(Y)$ then $\partial_p \sigma \in S_{p-1}(Y,K)$. Thus $\partial_p(S_p(Y,K)) \subset S_{p-1}(Y,K)$. Let

$$\partial_p^Y = \partial_p \big|_{S_p(Y,K)} .$$

LEMMA 1.4. If $i \leqslant j$ then $K_i^{p+1} \circ K_j^p = K_{j+1}^p \circ K_i^p$.

Proof. Let $p = 0$. Now $(K_0^1 \circ K_0^0)(0) = (0,1) = (K_1^1 \circ K_0^0)(0)$, $(K_0^1 \circ K_1^0)(0) = (1,0) = (K_2^1 \circ K_0^0)(0)$ and $(K_1^1 \circ K_1^0)(0) = (0,0) = (K_2^1 \circ K_1^0)(0)$.

Let $p \geqslant 1$. If $1 \leqslant i < j \leqslant p+1$ then

$$\begin{aligned} K_i^{p+1} \circ K_j^p(a_1,\ldots,a_p) &= (a_1,\ldots,a_{i-1},0,a_i,\ldots,a_{j-1},0,a_j,\ldots,a_p) \\ &= K_{j+1}^{p+1} \circ K_i^p(a_1,\ldots,a_p) . \end{aligned}$$

If $1 \leqslant i = j \leqslant p+1$ then

$$\begin{aligned} K_i^{p+1} \circ K_j^p(a_1,\ldots,a_p) &= (a_1,\ldots,a_{i-1},0,0,a_i,\ldots,a_p) \\ &= K_j^{p+1} \circ K_i^p(a_1,\ldots,a_p) . \end{aligned}$$

If $0 = i < j \leq p+1$ then

$$K_i^{p+1} \circ K_j^p(a_1,\ldots,a_p) = \left[1 - \sum_{i=1}^{p} a_i, a_1,\ldots,a_{j-1},0,a_j,\ldots,a_p\right]$$

$$= K_{j+1}^p \circ K_i^p(a_1,\ldots,a_n) \quad .$$

If $0 = i = j$ then

$$K_i^{p+1} \circ K_j^p(a_1,\ldots,a_p) = \left[0,\ 1 - \sum_{i=1}^{p} a_i, a_1,\ldots,a_p\right]$$

$$= K_{j+1}^{p+1} \circ K_i^p(a_1,\ldots,a_p) \quad .$$

Hence $K_i^{p+1} \circ K_j^p = K_{j+1}^{p+1} \circ K_i^p$.

LEMMA 1.5. $\partial_{p-1} \circ \partial_p : S_p(X,K) \to S_{p-2}(X,K)$ is the zero K-module homomorphism.

Proof. Let $\sigma \in S_p(X,K)$. Then

$$\partial_{p-1}\, \partial_p(\sigma) = \partial_{p-1} \left(\sum_{j=0}^{p} (-1)^j\, \sigma \circ K_j^{p-1}\right)$$

$$= \sum_{j=0}^{p} (-1)^j\, \partial_{p-1}(\sigma \circ K_j^{p-1})$$

$$= \sum_{j=0}^{p} \sum_{\ell=0}^{p-1} (-1)^{j+\ell}\, \sigma \circ K_j^{p-1} \circ K_\ell^{p-2} \quad . \qquad \text{(i)}$$

If $j \leq \ell$ then by lemma (1.5), $K_j^{p-1} \circ K_\ell^{p-2} = K_{\ell+1}^{p-1} \circ K_j^{p-2}$.

So for $j \leq \ell$

$$(-1)^{j+\ell}\, K_j^{p-1} \circ K_\ell^{p-2} = (-1)(-1)^{j+\ell+1}\, K_{\ell+1}^{p-1} \circ K_j^{p-2} \quad .$$

If $j > \ell$ then $j \geq \ell+1$. It follows from lemma (1.4) that for

$j > \ell$, $K^{p-1}_{\ell+1} \circ K^{p-2}_{j} = K^{p-1}_{j+1} \circ K^{p-2}_{\ell+1}$. So for $j > \ell$

$$(-1)^{j+\ell+1} K^{p-1}_{\ell+1} \circ K^{p-2}_{j} = (-1)(-1)^{j+\ell+2} K^{p-1}_{j+1} \circ K^{p-2}_{\ell+1} .$$

Hence every term in (i) cancels with another term and $\partial_{p-1} \partial_p(\sigma) = 0$. Thus we can conclude $\partial_{p-1} \circ \partial_p$ is the zero K-module homomorphism.

It follows that $\{S_p(X,K), \partial_p\}$ is a chain complex over K. It is called the singular chain complex of X over K. Also for Y any subspace of X, $\{S_p(Y,K), \partial^Y_p\}$ is a subcomplex of the chain complex $\{S_p(X,K), \partial_p\}$.

DEFINITION 1.6. Let $S^p(X,K)$ be the set of K-module homomorphisms: $S_p(X,K) \to K$. Let $f \in S^p(X,K)$ and $\sigma \in S_{p+1}(X,K)$. Define $d_p : S^p(X,K) \to S^{p+1}(X,K)$ by $\{d_p f\}(\sigma) = f(\partial_{p+1} \sigma)$.

Let Y be a subspace of X. Let $S^p(Y,K)$ be the set of K-module homomorphisms: $S_p(Y,K) \to K$. Let $g \in S^p(Y,K)$ then $d_p g \in S^{p+1}(Y,K)$. Let $d^Y_p = d_p\big|_{S^p(Y,K)}$.

DEFINITION 1.7. Since $\partial_{p-1} \circ \partial_p$ is the zero K-module homomorphism from lemma (1.5), it follows $d_{p+1} \circ d_p$ is the zero K-module homomorphism. Hence $\{S^p(X,K), d_p\}$ is a cochain complex of K-modules. Also for Y a subspace of X, $\{S^p(Y,K), d^Y_p\}$ is a subcomplex of $\{S^p(X,K), d_p\}$.

DEFINITION 1.8. The q th singular cohomology K-module of X is defined by

$$H^q(X,K) = \frac{\text{Kernel}(d_q)}{\text{Image}(d_{q-1})} .$$

§2. Presheaf of Singular p-Cochains

DEFINITION 2.1. Let $U \in T(X)$. Let $V \in T(X)$ and $V \subset U$. Let $\rho_{V,U}: S^p(U,K) \to S^p(V,K)$ be defined by $\{\rho_{V,U}(f)\}(\sigma) = f(\sigma)$ for $f \in S^p(U,K)$ and $\sigma \in S_p(V,K) \subset S_p(U,K)$. Now it is easily seen that $\{S^p(U,K), \rho_{V,U}\}$ is a presheaf called the presheaf of singular p-cochains over X. The associated sheaf $S^p_S(X,K)$ is called the sheaf of germs of singular p-cochains over X.

The presheaf $\{S^p(U,K), \rho_{V,U}\}$ satisfies condition (2) but not condition (1) of the definition of a complete presheaf.

DEFINITION 2.2. Let $U = \{U_\alpha\}_{\alpha \in J}$ be an open covering of X. Let $f \in S^p(X,K)$. f is called U-small if $f(\sigma) = 0$ whenever $\sigma \in S_p(U_\alpha,K)$ for some $\alpha \in J$. Let $S^p_0(X,K) = \{f \in S^p(X,K) \mid$ there exists an open covering $U = \{U_\alpha\}_{\alpha \in J}$ of X such that f is U-small$\}$.

LEMMA 2.3. $S^p_0(X,K)$ is a submodule of $S^p(X,K)$.

Proof. Let $f,g \in S^p_0(X,K)$ and $\lambda \in K$. Since $f \in S^p_0(X,K)$, there exists $U = \{U_\alpha\}_{\alpha \in I}$, an open cover of X, such that $f(\sigma) = 0$ for $\sigma \in S_p(U_\alpha,K)$ for some $\alpha \in I$. Since $g \in S^p_0(X,K)$, there exists $V = \{V_\beta\}_{\beta \in J}$, an open cover of X, such that $g(\tau) = 0$ for $\tau \in S_p(V_\beta,K)$ for some $\beta \in J$.

Let $W = \{U_\alpha \cap V_\beta\}_{(\alpha,\beta) \in I \times J}$. Now W is an open cover of X since U and V are open covers of X. Let $\theta \in S_p(U_\alpha \cap V_\beta,K)$. Then $(f+g)(\theta) = f(\theta) + g(\theta) = 0 + 0 = 0$. Thus $(f+g) \in S^p_0(X,K)$. Also $\lambda f \in S^p_0(X,K)$, since if $\sigma \in S_p(U_\alpha,K)$ then $(\lambda f)(\sigma) = \lambda(f(\sigma)) = \lambda 0 = 0$. It follows that $S^p_0(X,K)$ is a submodule of $S^p(X,K)$.

LEMMA 2.4. If $f \in S_0^p(X,K)$ then $d_p(f) \in S_0^{p+1}(X,K)$.

Proof. Let $f \in S_0^p(X,K)$. There exists $U = \{U_\alpha\}_{\alpha \in I}$, an open cover of X, such that $f(\sigma) = 0$ for $\sigma \in S_p(U_\alpha,K)$ for a $\alpha \in I$. Let $\tau \in S_{p+1}(U_\alpha,K)$. Then $\{d_p(f)\}(\tau) = f\left(\partial_{p+1}^{U_\alpha}(\tau)\right)$. We know $\partial_{p+1}^{U_\alpha}(\tau) \in S_p(U_\alpha,K)$ so $f\left(\partial_{p+1}^{U_\alpha}(\tau)\right) = 0$. Thus $\{d_p(f)\}(\tau) = 0$ and $d_p(f) \in S_0^{p+1}(X,K)$.

DEFINITION 2.5. By lemmas (2.3) and (2.4)

$$\cdots \longrightarrow 0 \longrightarrow S_0^0(X,K) \xrightarrow{\tilde{d}_0} S_0^1(X,K) \xrightarrow{\tilde{d}_1} S_0^2(X,K) \xrightarrow{\tilde{d}_2} \cdots$$

where $\tilde{d}_p(e) = d_p(e)$ for $e \in S_0^p(X,K)$ is a cochain complex of K-modules of $S^*(X,K)$. This cochain subcomplex is denoted $S_0^*(X,K)$. Let $N: S^*(X,K) \to \dfrac{S^*(X,K)}{S_0^*(X,K)}$ be the quotient map. The following theorem is well known in singular cohomology and will be assumed.

THEOREM 2.6. The map $N: S^*(X,K) \to \dfrac{S^*(X,K)}{S_0^*(X,K)}$ induces isomorphisms in cohomology.

LEMMA 2.7. $\Gamma(S_S^*(X,K)) \simeq \dfrac{S^*(X,K)}{S_0^*(X,K)}$.

Proof. Let $\gamma_p: S^p(X,K) \to \Gamma(S_S^p(X,K))$ for a $p \geq 0$ be defined by $\gamma_p(f) = \lambda$ for $f \in S^p(X,K)$ where $\lambda(x) = \rho_{x,X}\, f$. Let $i_p: S_0^p(X,K) \to S^p(X,K)$ be the inclusion map. Now if

$$0 \longrightarrow S_0^p(X,K) \xrightarrow{i_p} S^p(X,K) \xrightarrow{\gamma_p} \Gamma(S_S^p(X,K)) \longrightarrow 0$$

is an exact sequence for all $p \geq 0$ then $\Gamma(S_S^*(X,K)) \simeq \dfrac{S^*(X,K)}{S_0^*(X,K)}$.

The presheaf $\{S^p(U,K), \rho_{V,U}\}$ satisfies condition (2) of a complete presheaf. By Proposition Appendix 1, γ is an onto mapping.

Since i_p is an inclusion map, it is clear i_p is a one-to-one map.

Let $g \in S_0^p(X,K)$. There exists an open covering of X, $U = \{U_\alpha\}_{\alpha \in J}$, such that if $\sigma \in S_p(U_\alpha,K)$ for an $\alpha \in J$ then $g(\sigma) = 0$. Let $x \in X$. Now there exists a $\beta \in J$ such that $x \in U_\beta$. So $\{\gamma(g)\}(x) = \rho_{x,X}\, g = \rho_{x,U_\beta}\, \rho_{U_\beta,X}\, g = \rho_{x,U_\beta}(0) = 0$. Thus Image$(i) \subset$ Kernel(γ).

Let $h \in$ Kernel(γ). It follows that for $x \in X$, $\{\gamma(h)\}(x) = 0 = \rho_{x,X}\, h$. Hence there exists an open set $V_x \ni x$ satisfying $\rho_{V_x,X}\, h = 0$. So if $\sigma \in S_p(V_x,K)$ then $\rho_{V_x,X}\, h(\sigma) = 0$. Now $\{V_x\}_{x \in X}$ is an open cover of X and so $h \in S_0^p(X,K)$. Thus Kernel$(\gamma) \subset$ Image(i). We can conclude

$$\Gamma(S_S^*(X,K)) \simeq \frac{S^*(X,K)}{S_0^*(X,K)} .$$

§3. Homologically Locally Contractible Spaces

LEMMA 3.1. Let

$$\text{(i)} \qquad \cdots \longrightarrow A_{q+1} \xrightarrow{\partial_{q+1}} A_q \xrightarrow{\partial_q} A_{q-1} \xrightarrow{\partial_{q-1}} \cdots$$

be a chain complex of free K-modules. The chain complex (i) is exact if and only if there exists a K-module homomorphism $g_n : A_n \to A_{n+1}$ for n integer satisfying $\partial_{n+1} \circ g_n + g_{n-1} \circ \partial_n = Id_{A_n}$.

Proof. Let (i) be an exact sequence of K-modules. Let $\alpha_n : A_n \to \text{Image}(\partial_n)$ be defined by $\alpha_n(a) = \partial_n(a)$ for $a \in A_n$. Now $\text{Image}(\partial_n)$ is a submodule of the free K-module A_{n-1} so $\text{Image}(\partial_n)$ is a free K-module. Also α_n is a K-module epimorphism so there exists a K-module homomorphism $\beta_{n-1} : \text{Image}(\partial_n) \to A_n$ satisfying $\alpha_n \circ \beta_{n-1} = Id_{\text{Image}(\partial_n)}$. Let $f_n = Id - \beta_{n-1} \circ \partial_n$. Since $f_n(a) = a - \beta_{n-1}\,\partial_n(a)$ and $\partial_n f_n(a) = \partial_n(a) - \partial_n(a) = 0$, it follows $f_n(a) \in \text{Kernel}(\partial_n) = \text{Image}(\partial_{n+1})$. So $f_n : A_n \to \text{Image}(\partial_{n+1})$.

Let $g_n = \beta_n \circ f_n : A_n \to A_{n+1}$. Now g_n is a K-module homomorphism. Also

$$\begin{aligned}
&\partial_{n+1} \circ g_n + g_{n-1} \circ \partial_n \\
&= \partial_{n+1} \circ (\beta_n \circ (Id - \beta_{n-1} \circ \partial_n)) + \beta_{n-1} \circ (Id - \beta_{n-2} \circ \partial_{n-1}) \circ \partial_n \\
&= Id \circ (Id - \beta_{n-1} \circ \partial_n) + \beta_{n-1} \circ Id - \partial_n \\
&= Id_A \quad .
\end{aligned}$$

Suppose there exists a K-module homomorphism $g_n : A_n \to A_{n+1}$ for every n satisfying $\partial_{n+1} \circ g_n + g_{n-1} \circ \partial_n = Id_{A_n}$. Let $b \in \text{Kernel}(\partial_n)$ then $b = \partial_{n+1}(g_n(b)) \in \text{Image}(\partial_{n+1})$. It follows the sequence (i) of free K-modules is exact.

LEMMA 3.2. Let U be a subspace of X. Let $\ell_U : S_0(U,K) \to K$ be defined by $\ell_U(\tau) = \sum_{i=1}^{n} a_i$, where $\tau = \sum_{i=1}^{n} a_i\,\sigma_i$ with $a_i \in K$ and $\sigma_i \in S_0(U)$. Let $j_U : K \to S^0(U,K)$ be defined by $j_U(\lambda) = f$ for any $\lambda \in K$ where $f(\sigma) = \lambda$ for every $\sigma \in S_0(U)$. If $\tau = \sum_{i=1}^{n} a_i \sigma_i \in S_0(U,K)$ then it follows that $\{j_U(\lambda)\}(\tau) = \lambda\left(\sum_{i=1}^{n} a_i\right)$.

If the sequence of K-modules

(i) $$\cdots \longrightarrow S_1(U,K) \xrightarrow{\partial_1^U} S_0(U,K) \xrightarrow{\ell_U} K \longrightarrow 0$$

is exact then the sequence of K-modules

(ii) $$\cdots \longrightarrow 0 \longrightarrow K \xrightarrow{j_U} S^0(U,K) \xrightarrow{d_0^U} S^1(U,K) \xrightarrow{d_1^U} \cdots$$

is exact.

Proof. If the sequence of K-modules (i) is exact then by lemma (3.1) there exist K-module homomorphisms $g_p^U : S_p(U,K) \to S_{p+1}(U,K)$ for $p \geq 0$ and $m_U : K \to S_0(U,K)$ such that for $p \geq 1$

$$g_{p-1}^U \circ \partial_p^U + \partial_{p+1}^U \circ g_p^U = Id_{S_p(U,K)} ,$$

$$m_U \circ \ell_U + \partial_1^U \circ g_0^U = Id_{S_0(U,K)}$$

and

$$\ell_U \circ m_U = Id_K .$$

Now let $p \geq 1$. Let $h_p^U : S^p(U,K) \to S^{p-1}(U,K)$ be defined by $\{h_p^U(f)\}(\sigma) = \{f\}(g_p^U(\sigma))$ for $f \in S^p(U,K)$ and $\sigma \in S_{p-1}(U,K)$. It is clear that h_p^U is a K-module homomorphism.

Let $\theta \in S_p(U,K)$. Then

$$\begin{aligned}
&\{(d_{p-1}^U \circ h_p^U + h_{p+1}^U \circ d_p^U)(f)\}(\theta) \\
&= \{d_{p-1}^U(h_p^U(f))\}(\theta) + \{h_{p+1}^U(d_p^U(f))\}(\theta) \\
&= \{h_p^U(f)\}(\partial_p^U(\theta)) + \{d_p^U(f)\}(g_p^U(\theta)) \\
&= f(g_{p-1}^U(\partial_p^U(\theta))) + f(\partial_{p+1}^U\, g_p^U(\theta))
\end{aligned}$$

$$= f(g^U_{p-1} \circ \partial^U_p + \partial^U_{p+1} \circ g^U_p)(\theta)$$

$$= f(\theta) \quad .$$

Hence for $p \geq 1$,

$$d^U_{p-1} \circ h^U_p + h^U_{p+1} \circ d^U_p = Id_{S^p(U,K)} \quad .$$

Let $\alpha \in S^p(U,K)$ where $p \geq 1$ such that $d^U_p(\alpha) = 0$. Therefore

$$\alpha = (d^U_{p-1} \circ h^U_p + h^U_{p+1} \circ d^U_p)(\alpha)$$

$$= d^U_{p-1}(h^U_p(\alpha)) \quad .$$

Thus $\alpha \in \text{Image}(d^U_{p-1})$ and so $\text{Kernel}(d^U_p) \subset \text{Image}(d^U_{p-1})$. It follows from $d_p \circ d_{p-1} = 0$ that $\text{Kernel}(d^U_p) = \text{Image}(d^U_{p-1})$.

We will now prove j_U is a K-module monomorphism. If $j_U(\lambda) = 0$ where $\lambda \in K$ then $\{j_U(\lambda)\}(\theta) = \lambda\, \ell_U(\theta) = 0$ for $\theta \in S_0(U,K)$. Now ℓ_U is a K-module epimorphism so $\lambda = 0$ and j_U is a K-module monomorphism.

The next step is proving $\text{Kernel}(d^U_0) = \text{Image}(j_U)$. Let $\lambda \in K$ and $\tau \in S_1(U,K)$, then

$$\{d^U_0(j_U(\lambda)\}(\tau) = \{j_U(\lambda)\}(\partial^U_1(\tau))$$

$$= \{j_U(\lambda)\}(\tau \circ K^0_0) - \{j_U(\lambda)\}(\tau \circ K^0_1)$$

$$= \lambda - \lambda$$

$$= 0 \quad .$$

Hence $\text{Image}(j_U) \subset \text{Kernel}(d^U_0)$.

Let $n_U : S^0(U,K) \to K$ be defined by $n_U(f) \cdot \lambda' = f(m_U(\lambda'))$ for $f \in S^0(U,K)$ and $\lambda' \in K$. Let $\theta \in S_0(U,K)$ and $\theta = \sum_{i=1}^{m} b_i\, \theta_i$ where $b_i \in K$

and $\theta_i \in S_0(U)$. Thus

$$\begin{aligned}\{j_U(n_U(f)) + h_1^U(d_0^U(f))\}(\theta) &= \{j_U(n_U(f))\}(\theta) + \{h_1^U(d_0^U(f))\}(\theta) \\ &= (n_U(f))\left[\sum_{i=1}^{m} b_i\right] + \{d_0^U(f)\}(g_0^U(\theta)) \\ &= f\left\{m_U\left[\sum_{i=1}^{m} b_i\right]\right\} + f(\partial_1^U\, g_0^U(\theta)) \\ &= f(m_U\, \ell_U(\theta)) + f(\partial_1^U\, g_0^U(\theta)) \\ &= f(m_U\, \ell_U + \partial_1^U\, g_0^U)(\theta) \\ &= f(\theta)\ .\end{aligned}$$

So $j_U \circ n_U + h_1^U \circ d_0^U = Id_{S^0(U,K)}$. Let $g \in \text{Kernel}(d_0^U)$. It follows $g = j_U\, n_U(g)$ and $g \in \text{Image}(j_U)$. Thus $\text{Image}(j_U) \supset \text{Kernel}(d_0^U)$.

We can now conclude that sequence (ii) is exact.

LEMMA 3.3. $S_S^p(X,K)$ for $p \geq 0$ is a torsion-free fine sheaf of K-modules.

Proof. Let $\lambda \in K$, $\lambda \neq 0$ and $f \in S_S^p(X,K)_x$ for $x \in X$ such that $\lambda f = 0$. Since $f \in S_S^p(X,K)_x$, there exists $U \ni x$ open and $g \in S^p(U,K)$ such that $f = \rho_{x,U}\, g$. Thus $0 = \lambda f = \rho_{x,U}(\lambda g)$. Now there exists $V \ni x$ open such that $0 = \rho_{x,V}(0)$. Hence there exists $W \subset U \cap V$ open such that $\rho_{W,U}(\lambda g) = 0$. It follows $\lambda\rho_{W,U}g = 0$. So $\lambda\rho_{W,U}g(\sigma) = 0$ for $\sigma \in S_p(W,K)$. K is an integral domain so $\rho_{W,U}g(\sigma) = 0$. Hence $\rho_{W,U}g = 0$ and $f = \rho_{x,W}\,\rho_{W,U}g = 0$. Thus we can conclude $S_S^p(X,K)$ for $p \geq 0$ is a torsion-free sheaf.

Let $\{U_\alpha\}_{\alpha\in I}$ be a locally finite open cover of X. There exists a locally finite refinement $\{V_\alpha\}_{\alpha\in I}$ of $\{U_\alpha\}_{\alpha\in J}$ such that $\overline{V}_\alpha \subset U_\alpha$.

Let $\Gamma:X \to I$ be a set theoretic map such that $x \in V_{\Gamma(x)}$. Now $\Gamma^{-1}(\alpha) \subset V_\alpha$ so $\overline{\Gamma^{-1}(\alpha)} \subset \overline{V}_\alpha \subset U_\alpha$. Let $\phi_\alpha:X \to K$ such that

$$\phi_\alpha(x) = \begin{cases} 0 & \text{if } \Gamma(x) \neq \alpha \\ 1 & \text{if } \Gamma(x) = \alpha \quad \text{for } x \in X \end{cases} .$$

Hence $\sum_{\alpha \in I} \phi_\alpha = 1$. Also $\operatorname{supp}(\phi_\alpha) = \overline{\Gamma^{-1}(\alpha)} \subset U_\alpha$. Let $\ell^p_{\alpha,U}:S^p(U,K) \to S^p(U,K)$ be defined by $\{\ell^p_{\alpha,U}(f)\}(\sigma) = \phi_\alpha(\sigma(0))\cdot f(\sigma)$ where $f \in S^p(U,K)$, $\sigma \in S_p(U,K)$ and 0 is the origin in R^p. The K-module endomorphisms commute with restriction mappings, so there exists a presheaf homomorphism $\ell^p_\alpha = \{\ell^p_{\alpha,U}\}_{U \text{ open}} : \{S^p(U,K), \rho_{V,U}\} \to \{S^p(U,K), \rho_{V,U}\}$. There exists an associated sheaf homomorphism $\theta_\alpha : S^p_S(X,K) \to S^p_S(X,K)$.

The next step is proving $\sum_{\alpha \in I} \theta_\alpha = Id_{S^p_S(X,K)}$. Let $b \in S^p_S(X,K)_x$ where $x \in X$. There exists $U \ni x$ open and $a \in S^p(U,K)$ such that $b = \rho_{x,U}a$. Hence

$$\theta_\alpha(b) = \theta_\alpha(\rho_{x,U}(a)) = \rho_{x,U}(\ell_{\alpha,U}(a)) .$$

From this it follows

$$\sum_{\alpha \in I} \theta_\alpha(b) = \sum_{\alpha \in J} \rho_{x,U}(\ell_{\alpha,U}(a)) = \rho_{x,U} \sum_{\alpha \in I} \ell_{\alpha,U}(a) .$$

Now let $\sigma \in S_p(U,K)$ then

$$\begin{aligned} \sum_{\alpha \in I} \{\ell^p_{\alpha,U}(a)\}(\sigma) &= \sum_{\alpha \in I} \phi_\alpha(\sigma(0))\, a(\sigma) \\ &= \left(\sum_{\alpha \in I} \phi_\alpha(\sigma(0))\right) a(\sigma) \\ &= a(\sigma) . \end{aligned}$$

Hence $\sum_{\alpha\in I} \ell^p_{\alpha,U}(a) = a$ and $\sum_{\alpha\in I} \theta_\alpha(b) = \rho_{x,U}(a) = b$. Thus

$$\sum_{\alpha\in I} \theta_\alpha = Id_{S^p_S(X,K)} .$$

The last step is proving $\mathrm{supp}(\theta_\alpha) \subset U_\alpha$. Let $x \in X$ and $\theta_\alpha|_{S^p_S(X,K)} \neq 0$ for $\alpha \in I$. Let $f \in S^p_S(X,K)_x$. There exists $U \ni x$ open and $g \in S^p(U,K)$ satisfying $f = \rho_{x,U}g$. Now

$$\theta_\alpha(f) = \theta_\alpha(\rho_{x,U}(g)) = \rho_{x,U}(\ell^p_{\alpha,U}(g)) .$$

Suppose $x \notin V_\alpha$ then since X is normal there exists an open set $W \ni x$ such that $W \cap \overline{V}_\alpha = \phi$, Now $f = \rho_{x,W}\rho_{W,U}g$ so $\theta_\alpha(f) = \rho_{x,W}\ell^p_{\alpha,W}\rho_{W,U}g$. Let $\sigma \in S_p(W,K)$ then

$$\{\ell^p_{\alpha,W}(\rho_{W,U}(g))\}(\sigma) = \phi_\alpha(\sigma(0))\ \rho_{W,U}g(\sigma) = 0 .$$

Hence $\theta_\alpha(f) = 0$ and $\mathrm{supp}(\theta_\alpha) \subset \overline{V}_\alpha \subset U_\alpha$.

LEMMA 3.4. If for each $x \in X$ a base for open sets at x say $\mathcal{B}_x$ such that the sequence of K-modules

(i) $$\cdots \longrightarrow S_2(U,K) \xrightarrow{\partial_2^U} S_1(U,K) \xrightarrow{\partial_1^U} S_0(U,K) \xrightarrow{\ell_U} K \longrightarrow 0$$

is exact for all $U \in \mathcal{B}_x$ then the sequence of K-modules

(ii) $$0 \longrightarrow \mathcal{K} \xrightarrow{\Delta} S^0_S(X,K) \xrightarrow{D^0} S^1_S(X,K) \xrightarrow{D^1} \cdots$$

is exact where $\Delta : \mathcal{K} \to S^0_S(X,K)$ is defined by $\Delta_x(\lambda) = \rho_{x,U}\, j_U(\lambda)$ with U open in X and $x \in U$.

Proof. Let $x \in X$. Let $U \ni x$ be open and the sequence of K-modules

$$\cdots \longrightarrow S_2(U,K) \xrightarrow{\partial_2^U} S_1(U,K) \xrightarrow{\partial_1^U} S_0(U,K) \xrightarrow{\ell_U} K \longrightarrow 0$$

be exact. By lemma (3.2) the sequence of K-modules

$$\cdots \longrightarrow 0 \longrightarrow K \xrightarrow{j_U} S^0(U,K) \xrightarrow{d_0^U} S^1(U,K) \xrightarrow{d_1^U} \cdots$$

is exact. It follows from theorem (II 1.4) that the sequence of sheaves of K-modules

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{K} \xrightarrow{\Delta} S_S^0(X,K) \xrightarrow{D^0} S_S^1(X,K) \xrightarrow{D^1} \cdots$$

is exact, observing that the stalk of $S_S^p(X,K)$ at x is $\varinjlim_{u \in \mathcal{B}_x} S^p(U,K)$.

DEFINITION 3.5. X is called homologically locally contractible (H.L.C.) if for each $x \in X$ there exists a fundamental system of neighbourhoods $\mathcal{U}$ such that if $U \in \mathcal{U}$ then

$$\cdots \longrightarrow S_2(U,K) \xrightarrow{\partial_2^U} S_1(U,K) \xrightarrow{\partial_1^U} S_0(U,K) \xrightarrow{\ell_U} K \longrightarrow 0$$

is an exact sequence of K-modules.

DEFINITION 3.6. If X is a paracompact H.L.C. space, then by lemmas (3.3) and (3.4)

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{K} \xrightarrow{\Delta} S_S^0(X,K) \xrightarrow{D^0} S_S^1(X,K) \xrightarrow{D^1} \cdots$$

is a torsion-free fine resolution of $\mathcal{K}$. The qth singular cohomology K-module $H^q(X,T)$ over the sheaf T of K-modules is $H^q(\Gamma(S_S^*(X,K) \otimes T))$.

CHAPTER V

DE RHAM COHOMOLOGY

§1. De Rham Cohomology Real Vector Spaces.

DEFINITION 1.1. Let X be a C^∞ differentiable manifold of dimension n. Let U be open in X and $E^p(U)$ the set of C^∞ differential p-forms on U for $p \geq 0$. $E^p(U)$ is a vector space over the field of real numbers. There exists a linear transformation $d_U^p : E^p(U) \to E^{p+1}(U)$ for $p \geq 0$ called exterior differentiation such that $d_U^{p+1} \circ d_U^p$ is the zero linear transformation. It follows that there exists a sequence of real vector spaces

$$0 \longrightarrow E^0(U) \xrightarrow{d_U^0} E^1(U) \xrightarrow{d_U^1} E^2(U) \longrightarrow \cdots$$

with $\mathrm{Image}(d_U^{p-1}) \subset \mathrm{Kernel}(d_U^p)$ for $p \geq 1$.

In particular, there exists the sequence

$$0 \longrightarrow E^0(X) \xrightarrow{d_X^0} E^1(X) \xrightarrow{d_X^1} E^2(X) \longrightarrow \cdots$$

with $\mathrm{Image}(d_X^{p-1}) \subset \mathrm{Kernel}(d_X^p)$.

Let

$$H^p(E^*(X)) = \frac{\mathrm{Kernel}(d_X^p)}{\mathrm{Image}(d_X^{p-1})}$$

The real vector space $H^p(E^*(X))$ is called the pth de Rham cohomology real vector space of the C^∞ differentiable manifold X.

DEFINITION 1.2. Let $U \in T(X)$. Let $V \in T(X)$ and $V \subset U$. Let

$\rho_{V,U}: E^p(U) \to E^p(V)$ be defined by $\rho_{V,U}(\omega) = \omega|_V$ where $\omega \in E^p(U)$. Now trivially $d_V^p \circ \rho_{V,U} = \rho_{V,U} \circ d_U^p$ and the family $E^p = \{E^p(U), \rho_{V,U}\}_{U \text{ open}}$ is a presheaf of real vector spaces. E^p is easily seen to be a complete presheaf. The associated sheaf Ω^p is called the sheaf of germs of p-forms on X. Now $d^p = \{d_U^p\}_{U \text{ open}}$ is a presheaf homomorphism so there exists a sheaf homomorphism $D^p: \Omega^p \to \Omega^{p+1}$ associated to d^p.

Let R be the field of real numbers. $\{R, \rho_{V,U}\}$ where $\rho_{V,U}: R \to R$ is the identity mapping is a constant presheaf. Let $\mathcal{R} = X \times R$ be the associated sheaf.

For any open set U of X let $j_U: R \to E^0(U)$ be defined by $\{j_U(r)\}(x) = r$ for any $x \in U$ and $r \in R$. Now $j_V \circ \rho_{V,U} = \rho_{V,U} \circ j_U$ for V,U in $T(X)$ and $V \subset U$. So there exists an induced sheaf homomorphism $J: \mathcal{R} \to \Omega^0$.

We thus have an augmented sequence of real vector spaces

$$0 \longrightarrow R \xrightarrow{j_U} E^0(U) \xrightarrow{d_U^0} E^1(U) \xrightarrow{d_U^1} \cdots$$

for U open. Now $d_U^p \circ d_U^{p-1}$ and $d_U^0 \circ j_U$ are zero linear transformations. Hence by corollary (II 1.5) $D^p \circ D^{p-1}$ for $p \geq 1$ and $D^0 \circ J$ are zero linear transformations in the sequence

$$0 \longrightarrow \mathcal{R} \xrightarrow{J} \Omega^0 \xrightarrow{D^0} \Omega^1 \xrightarrow{D^1} \cdots .$$

§2. Existence of a Torsion-Free Fine Resolution.

POINCARE LEMMA 2.1. The sequence

$$0 \longrightarrow \mathcal{R} \xrightarrow{J} \Omega^0 \xrightarrow{D^0} \Omega^1 \xrightarrow{D^1} \Omega^2 \xrightarrow{D^2} \cdots$$

is an exact sequence of sheaves of real vector spaces.

Proof. The sequence

$$0 \longrightarrow \mathcal{R} \xrightarrow{J} \Omega^0 \xrightarrow{D^0} \Omega^1 \xrightarrow{D^1} \Omega^2 \xrightarrow{D^2} \cdots$$

is an exact sequence of sheaves if and only if the sequence

$$0 \longrightarrow \mathcal{R}_x \xrightarrow{J_x} \Omega^0_x \xrightarrow{D^0_x} \Omega^1_x \xrightarrow{D^1_x} \Omega^2_x \xrightarrow{D^2_x} \cdots$$

is an exact sequence of vector spaces for all $x \in X$. Let $x \in X$. Since X is a differentiable manifold, X is a locally Euclidean topological space. Hence there exists a coordinate neighbourhood W of x with coordinates $(x_1, \ldots, x_n)$ with $x = (0,0,\ldots,0)$. Let $\mathcal{U}$ be a family of spherical neighbourhoods $x_1^2 + x_2^2 + \ldots + x_n^2 < r^2$ of x which are contained in W. The family $\mathcal{U}$ is a fundamental system of the open sets of x.

Let $U \in \mathcal{U}$. We will now prove the sequence of real vector spaces

$$0 \longrightarrow R \xrightarrow{j_U} E^0(U) \xrightarrow{d^0_U} E^1(U) \xrightarrow{d^1_U} \cdots$$

is exact. Let $h^0_U : E^0(U) \to R$ be defined by $h^0_U(f) = f(0,0,\ldots,0) = f(x)$ for $f \in E^0(U)$. Let $h^p_U : E^p(U) \to E^{p-1}(U)$ be defined for $p \geq 1$ by

$$h^p_U(g(x_1, x_2, \ldots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p})$$

$$= \left\{ \int_0^1 g(u_1, \ldots, u_n) t^{p-1}\, dt \right\} \cdot \sum_{j=1}^{p} (-1)^{j-1} x_{i_j}\, dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p}$$

where $u_i = tx_i$ for $i = 1,\ldots,n$ and

$$g(x_1,\ldots,x_n)dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} \quad \in \quad E^p(U) \quad .$$

We will now verify that

$$j_U \circ h_U^0 + h_U^1 \circ d_U^0 \quad = \quad Id_{E^0(U)} \quad ,$$

and
$$d_U^{p-1} \circ h_U^p + h_U^{p+1} \circ d_U^p \quad = \quad Id_{E^p(U)}$$

for $p \geq 1$.

Let $f \in E^0(U)$. Now $j(h_U^0(f)) = j(f(0,0,\ldots,0)) = f(0,0,\ldots,0)$.

Also

$$d_U^0(f) \quad = \quad \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x_1,\ldots,x_n)\,dx_k \quad .$$

Hence

$$\begin{aligned} h_U^1(d_U^0(f)) \quad &= \quad \sum_{k=1}^{n} h_U^1\left(\frac{\partial f}{\partial x_k}(x_1,\ldots,x_n)dx_k\right) \\ &= \quad \sum_{k=1}^{n} \left[\int_0^1 \frac{\partial f}{\partial u_k}(u_1,\ldots,u_n)\,dt\right] x_k \\ &= \quad \sum_{k=1}^{n} \left[\int_0^1 \frac{\partial f}{\partial u_k}(u_1,\ldots,u_n)\,x_k\right] dt \quad . \end{aligned}$$

Now since $u_i = tx_i$ it follows that $\frac{du_i}{dt} = x_i$. Therefore

$$\sum_{k=1}^{n} \frac{\partial f}{\partial u_k}(u_1,\ldots,u_n)x_k \quad = \quad \frac{df(u_1,\ldots,u_n)}{dt} \quad .$$

Hence

$$\begin{aligned} h_U^1\, d_U^0(f) \quad &= \quad \int_0^1 \sum_{k=1}^{n} \frac{\partial f}{\partial u_k}(u_1,\ldots,u_n)\, x_k\, dt \\ &= \quad \int_0^1 \frac{df(u_1,\ldots,u_n)}{dt} \quad dt \end{aligned}$$

$$= f(u_1,\ldots,u_n)\Big|_{t=0}^{t=1}$$

$$= f(x_1,\ldots,x_n) - f(0,\ldots,0) \quad .$$

So

$$\begin{aligned} j_U(h_U^0(f)) + h_U^1(d_U^0(f)) &= f(0,\ldots,0) + f(x_1,\ldots,x_n) - f(0,\ldots,0) \\ &= f(x_1,\ldots,x_n) \quad . \end{aligned}$$

It follows immediately that $j_U \circ h_U^0 + h_U^1 \circ d_U^0 = Id_{E^0(U)}$.

Let

$$c = g(x_1,\ldots,x_n)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \in E^p(U)$$

where $p \geq 1$. Now

$$h_U^p(c) = \left\{ \int_0^1 g(u_1,\ldots,u_n)t^{p-1}\, dt \right\} \sum_{j=1}^{p} (-1)^{j-1} x_{i_j}\, dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p}$$

where $u_k = tx_k$ for $k = 1,\ldots,n$. Hence since d_U^{p-1} is an anti-derivation

$$\begin{aligned} d_U^{p-1}(h_U^p(c)) = &\left\{ d_U^{p-1} \left(\int_0^1 g(u_1,\ldots,u_n)t^{p-1}\, dt \right) \right\} \sum_{j=1}^{p} (-1)^{j-1} x_{i_j}\, dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p} \\ &+ \left(\int_0^1 g(u_1,\ldots,u_n)t^{p-1}\, dt \right) \left\{ d_U^{p-1} \left(\sum_{j=1}^{p} (-1)^{j-1} x_{i_j}\, dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p} \right) \right\} . \end{aligned}$$

Now

$$d_U^{p-1}\left[\int_0^1 g(u_1,\ldots,u_n)t^{p-1}\,dt\right]$$

$$= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left[\int_0^1 g(u_1,\ldots,u_n)t^{p-1}\,dt\right] dx_k$$

$$= \sum_{k=1}^{n} \left[\int_0^1 \frac{\partial g}{\partial x_k}(u_1,\ldots,u_n)t^{p-1}\,dt\right] dx_k \ .$$

By the chain rule and the fact $u_k = tx_k$ we know that $\frac{\partial g}{\partial x_k} = \frac{\partial g}{\partial u_k}\frac{\partial u_k}{\partial x_k} = \frac{\partial g}{\partial u_k} t$. So

$$d_U^{p-1}\left[\int_0^1 g(u_1,\ldots,u_n)t^{p-1}\,dt\right]$$

$$= \sum_{k=1}^{n} \left[\int_0^1 \frac{\partial g}{\partial u_k}(u_1,\ldots,u_n)t^{p}\,dt\right] dx_k \ .$$

Also

$$d_U^{p-1}\left[\sum_{j=1}^{p} (-1)^{j-1} x_{i_j}\, dx_{i_1} \wedge \ldots \wedge \widehat{dx}_{i_j} \wedge \ldots \wedge dx_{i_p}\right]$$

$$= \sum_{j=1}^{p} (-1)^{j-1}\, dx_{i_j} \wedge dx_{i_1} \wedge \ldots \wedge \widehat{dx}_{i_j} \wedge \ldots \wedge dx_{i_p}$$

$$= \sum_{j=1}^{p} (-1)^{2(j-1)1}\, dx_{i_1} \wedge \ldots \wedge dx_{i_j} \wedge \ldots \wedge dx_{i_p}$$

$$= p\, dx_{i_1} \wedge \ldots \wedge dx_{i_j} \wedge \ldots \wedge dx_{i_p} \ .$$

It follows that

$$d_U^{p-1}(h_U^p(c)) = \sum_{k=1}^{n} \left[\int_0^1 \frac{\partial g}{\partial u_k}(u_1,\ldots,u_n)t^{p}\,dt\right] \sum_{j=1}^{p} (-1)^{j-1} x_{i_j}\, dx_k \wedge dx_{i_1} \wedge \ldots \wedge \widehat{dx}_{i_j} \wedge \ldots \wedge dx_{i_p}$$

$$+ \left(\int_0^1 g(u_1,\ldots,u_n)t^{p-1}\, dt \right) p\; dx_{i_1} \wedge \ldots \wedge dx_{i_p} .$$

Also

$$h_U^{p+1}(d_U^p(c)) = h_U^{p+1}\left(\sum_{k=1}^{n} \frac{\partial g}{\partial x_k}(x_1,\ldots,x_n)\; dx_k \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} \right)$$

$$= \sum_{k=1}^{n} \left(h_U^{p+1}\left(\frac{\partial g}{\partial x_k}(x_1,\ldots,x_n) dx_k \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} \right) \right)$$

$$= \sum_{k=1}^{n} \left(\int_0^1 \frac{\partial g}{\partial u_k}(u_1,\ldots,u_n)t^p\, dt \right) \left\{ x_k dx_{i_1} \wedge \ldots \wedge dx_{i_p} \right.$$

$$\left. + \sum_{j=1}^{p} (-1)^{j-2}\, x_{i_j}\, dx_k \wedge dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p} \right\}$$

$$= \sum_{k=1}^{n} \left(\int_0^1 \frac{\partial g}{\partial u_k}(u_1,\ldots,u_n)t^p\, dt \right) \left\{ x_k dx_{i_1} \wedge \ldots \wedge dx_{i_p} \right.$$

$$\left. - \sum_{j=1}^{p} (-1)^{j-1}\, x_{i_j}\, dx_k \wedge dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p} \right\}.$$

Hence

$$d_U^{p-1}(h_U^p(c)) + h_U^{p+1}(d_U^p(c))$$

$$= \sum_{k=1}^{n} \left(\int_0^1 \frac{\partial g}{\partial u_k}(u_1,\ldots,u_n)t^p\, dt \right)$$

$$\left(\sum_{j=1}^{p} (-1)^{j-1}\, x_{i_j}\, dx_k \wedge dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p} \right)$$

$$+ \left(\int_0^1 g(u_1,\ldots,u_n)t^{p-1}\, dt \right) p\; dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

$$- \sum_{k=1}^{n} \left(\int_0^1 \frac{\partial g}{\partial u_k}(u_1,\ldots,u_n)t^p\, dt \right)$$

$$\left(\sum_{j=1}^{p} (-1)^{j-1}\, x_{i_j}\, dx_k \wedge dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_p} \right)$$

$$+ \sum_{k=1}^{n} \left(\int_0^1 \frac{\partial g}{\partial u_k} (u_1,\ldots,u_n) t^p \, dt \right) \left\{ x_k \, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \right\}$$

$$= \left(\int_0^1 g(u_1,\ldots,u_n) t^{p-1} \, dt \right) p \, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

$$+ \sum_{k=1}^{n} \left(\int_0^1 \frac{\partial g}{\partial u_k} (u_1,\ldots,u_n) t^p \, dt \right) \left\{ x_k \, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \right\} .$$

Integrating by parts we get

$$d_U^{p-1}(h_U^p(c)) + h_U^{p+1}(d_U^p(c))$$

$$= g(u_1,\ldots,u_n) t^p \Big|_{t=0}^{t=1} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

$$+ \left(\int_0^1 \sum_{k=1}^{n} \frac{\partial g}{\partial u_k} (u_1,\ldots,u_n) t^p \, dt - \sum_{k=1}^{n} \int_0^1 \frac{\partial g}{\partial u_k} (u_1,\ldots,u_n) t^p \, dt \right) [x_k \, dx_{i_1} \wedge \ldots \wedge dx_{i_p}]$$

$$= g(x_1,\ldots,x_n) \, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

$$= c .$$

We may thus conclude that

$$d_U^{p-1} \circ h_U^p + h_U^{p+1} \circ d_U^p = Id .$$

Let $a \in E^p(U)$ where $p \geq 1$ satisfy $d_U^p(a) = 0$ then

$$a = d_U^{p-1}(h_U^p(a)) + h_U^{p+1}(d_U^p(a)) = d_U^{p-1}(h_U^p(a)) .$$

Thus $a \in \text{Image}(d_U^{p-1})$. Let $y \in E^0(U)$ satisfy $d_U^0(y) = 0$. Then

$$y = j_U(h_U^0(y)) + h_U(d_U^0(y)) = j_U(h_U^0(y)) .$$

Hence $y \in \text{Image}(j_U)$.

We can now conclude that the sequence

$$0 \longrightarrow R \xrightarrow{j_U} E^0(U) \xrightarrow{d_U^0} E^1(U) \xrightarrow{d_U^1} \cdots$$

is exact for $U \in \mathcal{U}$. Since $\mathcal{U}$ is a fundamental system of neighbourhoods of x the sequence of real vector spaces

$$0 \longrightarrow \mathcal{R}_x \xrightarrow{J_x} \Omega_x^0 \xrightarrow{D_x^0} \Omega_x^1 \xrightarrow{D_x^1} \cdots$$

is exact for $x \in X$ because exactness commutes with direct limits. It follows that the sequence of sheaves

$$0 \longrightarrow \mathcal{R} \xrightarrow{J} \Omega^0 \xrightarrow{D^0} \Omega^1 \xrightarrow{D^1} \cdots$$

is exact.

LEMMA 2.2. Ω^p is a torsion-free fine sheaf.

Proof. Since for $x \in X$ Ω_x^p is a real vector space, Ω_x^p is torsion-free. Thus Ω^p is a torsion-free sheaf for $p \geq 0$.

Let $\{U_\alpha\}_{\alpha \in J}$ be a locally finite open covering of X. Since X is a C^∞ differentiable manifold there exists a C^∞ partition of unity $\{\phi_\alpha\}_{\alpha \in J}$ subordinate to the covering $\{U_\alpha\}_{\alpha \in J}$. For each open set U and integer $p \geq 0$ let $\gamma_{U,\alpha}: E^p(U) \to E^p(U)$ be defined by $\gamma_{U,\alpha}(h) = (\phi_\alpha|_U)\cdot h$ for $h \in E^p(U)$. Now

$$\gamma_\alpha = \{\gamma_{U,\alpha}\}_{U \text{ open}} : \{E^p(U), \rho_{V,U}\} \to \{E^p(U), \rho_{V,U}\}$$

is a presheaf homomorphism. Let $\theta_\alpha : \Omega^p \to \Omega^p$ be the associated sheaf homomorphism. We will now prove $\sum_{\alpha \in J} \theta_\alpha = Id_{\Omega^p}$.

Let $b \in \Omega^p_x$. There exists $W \ni x$ open and $c \in E^p(W)$ such that $b = \rho_{x,W}\, c$ is satisfied. Now

$$\sum_{\alpha\in J} \theta_\alpha(b) = \sum_{\alpha\in J} \theta_\alpha(\rho_{x,W}\, c) = \sum_{\alpha\in J} \rho_{x,W}(\gamma_{W,\alpha}(c))$$

$$= \rho_{x,W}\left(\sum_{\alpha\in J}(\phi_\alpha|_W)\cdot c\right) = \rho_{x,W}\left(c \sum_{\alpha\in J}(\phi_\alpha|_W)\right)$$

$$= \rho_{x,W}\, c = b .$$

Thus $\sum_{\alpha\in J} \theta_\alpha = Id_{\Omega^p}$. We know that $\mathrm{supp}(\phi_\alpha) \subset U_\alpha$ so $\mathrm{supp}(\theta_\alpha) \subset \mathrm{supp}(\phi_\alpha) \subset U_\alpha$. It follows that Ω^p for a $p \geqslant 0$ is a fine sheaf.

THEOREM 2.3. The sequence of sheaves

(i) $$0 \longrightarrow \mathcal{R} \xrightarrow{J} \Omega^0 \xrightarrow{D^0} \Omega^1 \xrightarrow{D^1} \cdots$$

is a torsion-free fine resolution of $\mathcal{R}$.

Proof. By lemma (2.1) the sequence (i) is exact and by lemma (2.2) Ω^p for $p \geqslant 0$ is a torsion-free fine sheaf. Hence (i) is a torsion-free fine resolution of $\mathcal{R}$.

CHAPTER VI

DE RHAM'S THEOREM

Let X be a C^∞ differentiable n-manifold.

§1. Presheaf of Differentiable Singular p-Cochains

DEFINITION 1.1. Let $\sigma \in S_p(X)$. If there exists an open set P in R^p with $\Delta_p \subset P$ such that σ has a C^∞ extension $\sigma^*: P \to X$ then σ is a differentiable singular p-simplex in X. Let $S_{p,\infty}(X)$ be the set of differentiable singular p-simplices in X. Let $S_{p,\infty}(X,K)$ be the free K-module generated by $S_{p,\infty}(X)$.

If U is an open set in X then U is a C^∞ manifold and so $S_{p,\infty}(U)$ and $S_{p,\infty}(U,K)$ are defined. Clearly $S_{p,\infty}(U) \subset S_{p,\infty}(X)$ and hence $S_{p,\infty}(U,K) \subset S_{p,\infty}(X,K)$.

Let $\tau \in S_{p,\infty}(X)$. Now there exists $V \supset \Delta_p$ open in R^p such that there is a C^∞ extension $\tau^*: V \to X$ for τ. Now $K_i^{p-1}: \Delta_{p-1} \to \Delta_p$ for $0 \leqslant i \leqslant p$ is a C^∞ map. Hence $\tau \circ K_i^{p-1} \in S_{p-1,\infty}(X)$ and therefore $\partial_p(\tau) \in S_{p-1,\infty}(X,K)$. Let $\partial_{p,\infty}: S_{p,\infty}(X,K) \to S_{p-1,\infty}(X,K)$ denote the restriction of ∂_p to $S_{p,\infty}(X,K)$.

DEFINITION 1.2. A differentiable singular p-cochain f in X is a K-module homomorphism $f: S_{p,\infty}(X,K) \to K$. Let $S_\infty^p(X,K)$ be the set of differentiable singular p-cochains in X. $S_\infty^p(X,K)$ with the usual addition and scalar multiplication is a K-module.

Let U be open in X. A differentiable singular p-cochain g in U

is a K-module homomorphism $g:S_{p,\infty}(U,K) \to K$. Let $S_{\infty}^{p}(U,K)$ be the set of differentiable singular p-cochains in U. $S_{\infty}^{p}(U,K)$ with the usual addition and scalar multiplication is a K-module. Let $V \subset U$ be open in X. Let $\rho_{V,U}:S_{\infty}^{p}(U,K) \to S_{\infty}^{p}(V,K)$ be defined by $\rho_{V,U}h = h|_{S_{p,\infty}(V,K)}$. Then $\{S_{\infty}^{p}(U,K),\rho_{V,U}\}$ is a presheaf of K-modules called the presheaf of differentiable singular p-cochains.

The associated sheaf $S_{S,\infty}^{p}(X,K)$ is called the sheaf of germs of differentiable singular p-cochains.

Let $d_p^{X*}:S_{\infty}^{p}(X,K) \to S_{\infty}^{p+1}(X,K)$ be defined by $\{d_p^{X*}(f)\}(\sigma) = f(\partial_{p+1,\infty}\sigma)$ for $f \in S_{\infty}^{p}(X,K)$ and $\sigma \in S_{p+1,\infty}(X,K)$. Since $\partial_{p+1,\infty} \circ \partial_{p+2,\infty}$ is the zero K-module homomorphism, $d_{p+2}^{X*} \circ d_{p+1}^{X*}$ is the zero K-module homomorphism.

Since for U open in X and $V \subset U$ open in X, $\rho_{V,U} \circ d_p^{U*} = d_p^{V*} \circ \rho_{V,U}$, there exists an induced sheaf homomorphism $D^{p*}:S_{S,\infty}^{p}(X,K) \to S_{S,\infty}^{p+1}(X,K)$ such that

$$D_x^{p*} \circ \rho_{x,U} = \rho_{x,U} \circ d_p^{U*} \quad \text{for } U \ni x .$$

It is easily seen that $D^{p+1*} \circ D^{p*}$ is the zero sheaf homomorphism for $p \geq 1$.

Let U be open in X. Let $j_U^*:K \to S_{\infty}^{0}(U,K)$ be defined by $\{j_U^*(\lambda)\}(\sigma) = \lambda \sum_{i=1}^{m} a_i$ where $\lambda \in K$, $\sigma \in S_{\infty}^{0}(U,K)$, $\sigma = \sum_{i=1}^{m} a_i \sigma_i$, $a_i \in K$ and $\sigma_i \in S_{\infty}^{0}(U)$. Now j_U^* is a K-module homomorphism and if $V \subset U$ open then $\rho_{V,U} \circ j_U^* = j_V^* \circ \rho_{V,U}$. So there exists an induced sheaf homomorphism satisfying $\Delta_x^* \circ \rho_{x,U} = \rho_{x,U} \circ j_U^*$ for $x \in U$.

LEMMA 1.3. $S_{S,\infty}^{p}(X,K)$ is a torsion-free fine sheaf of K-modules.

Proof. By a similar argument to lemma (IV 3.3).

§2. De Rham's Theorem

From here on $K = R$ (field of real numbers).

THEOREM 2.1. Let U be open in X. Let $\alpha_U^p : E^p(U) \to S_\infty^p(U,R)$ be defined by $\{\alpha_U^p(\tau)\}(\sigma) = \int_\sigma \tau$ for $\tau \in E^p(U)$ and $\sigma \in S_\infty^p(U,R)$. $\alpha_U^* : E^*(U) \to S_\infty^*(U,R)$ is a map of cochain complexes.

Proof. α_U^* is a map of cochain complexes if

$$\begin{array}{ccc} E^p(U) & \xrightarrow{d_p^U} & E^{p+1}(U) \\ \downarrow \alpha_U^p & & \downarrow \alpha_U^{p+1} \\ S_\infty^p(U,R) & \xrightarrow{d_p^{U*}} & S_\infty^{p+1}(U,R) \end{array}$$

commutes for every p. We know

$$\{\alpha_U^{p+1}(d_p^U(\tau))\}(\sigma) = \int_\sigma d_p^U(\tau)$$

for $\tau \in E^p(U)$ and $\sigma \in S_{p+1,\infty}(U,R)$.

By Stokes theorem

$$\int_\sigma d_p^U(\tau) = \int_{\partial_{p+1,\infty}^U \sigma} \tau .$$

Hence

$$\{\alpha_U^{p+1}(d_p^U(\tau))\}(\sigma) = \sum_{i=0}^{p+1} (-1)^i \int_{\sigma^i} \tau .$$

Also

$$\{d_p^* \alpha_U^p(\tau)\}(\sigma) = \{\alpha_U^p(\tau)\}(\partial_{p+1,\infty}^U \sigma)$$

$$= \{\alpha_U^p(\tau)\} \left(\sum_{i=0}^{p+1} (-1)^i \sigma^i \right)$$

$$= \sum_{i=0}^{p+1} (-1)^i \int_{\sigma^i} \tau \quad .$$

It follows that $\alpha_U^{p+1} \circ d_p^U = d_p^{U*} \circ \alpha_U^p$.

LEMMA 2.2. Since $\alpha_U = \{\alpha_U^p\}$ is a map of presheaves of cochain complexes, there exists an induced linear transformation

$$\alpha_*^p : H^p(E^*(X)) \rightarrow H^p(\Gamma(S^*_{S,\infty}(X,R))) \quad .$$

Proof. Let U be open and $V \subset U$ be also open. Then

$$\begin{array}{ccc} E^p(U) & \xrightarrow{\rho_{V,U}} & E^p(V) \\ \downarrow{\alpha_U^p} & & \downarrow{\alpha_V^p} \\ S_\infty^p(U,R) & \xrightarrow{\rho_{V,U}} & S_\infty^p(V,R) \end{array}$$

is commutative. Hence there exists an induced sheaf homomorphism $\beta^p : \Omega^p \rightarrow S^p_{S,\infty}(X,R)$, which in turn induces

$$\{\Gamma(\beta^p)\} : \Gamma(X,\Omega^p) \rightarrow \Gamma(X,S^p_{S,\infty}(X,R)) \quad .$$

Now $\{\Gamma(\beta^p)\}$ is a homomorphism of cochain complexes so there exists induced linear transformations $H^p(E^*(X)) \rightarrow H^p(\Gamma(S^*_{S,\infty}(X,R)))$ for all q.

THEOREM 2.3. $\{\Gamma(\beta^p)\}$ induces isomorphisms in cohomology.

Proof. Since in any differentiable manifold contractible open sets form a fundamental system of neighbourhoods of a point, it follows that

$$0 \longrightarrow \mathcal{R} \xrightarrow{\Delta_*} S^0_{S,\infty}(X,R) \xrightarrow{D^0_*} S^1_{S,\infty}(X,R) \xrightarrow{D^1_*} \cdots$$

is a torsion-free fine resolution of $\mathcal{R}$.

We thus have two torsion-free fine resolutions of $\mathcal{R}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{R} & \xrightarrow{J} & \Omega^0 & \xrightarrow{D^0} & \Omega^1 & \xrightarrow{D^1} & \Omega^2 & \longrightarrow & \cdots \\
 & & \downarrow I_D & & \downarrow \beta^0 & & \downarrow \beta^1 & & \downarrow \beta^2 & & \\
0 & \longrightarrow & \mathcal{R} & \xrightarrow{\Delta_*} & S^0_{S,\infty}(X,R) & \xrightarrow{D^0_*} & S^1_{S,\infty}(X,R) & \xrightarrow{D^1_*} & S^2_{S,\infty}(X,R) & \longrightarrow & \cdots
\end{array}$$

where each square is commutative. Now $\beta^q:\Omega^q \to S^p_{S,\infty}(X,R)$ is a sheaf homomorphism so by (III 2.5) $\alpha^q_*:H^q(\Gamma(X,\Omega^*)) \to H^q(\Gamma(X,S^*_{S,\infty}(X,R)))$ is a real vector space isomorphism and this is the unique isomorphism stated in the proof of uniqueness of cohomology theory over X with sheaf coefficients.

Now the presheaf $\{E^p(U),\rho_{V,U}\}$ is complete and thus $E^p(X)$ can be identified with $\Gamma(X,\Omega^p)$. Thus the cohomology of $\Gamma(X,\Omega^*)$ is the same as de Rham cohomology of X.

Combining this with theorem (2.3) we get

DE RHAM'S THEOREM 2.4. There exists a canonical isomorphism

$$\gamma^q:H^q_{\text{de Rham}}(X) \to H^q_{\text{Singular}}(X,R) \quad .$$

Proof. We know by (IV 2.7) that

$$\Gamma(S^*_{S,\infty}(X,R)) \simeq \frac{S^*_{\infty}(X,R)}{S^*_{0,\infty}(X,R)} \quad .$$

Since

$$N: S^*_{S,\infty}(X,R) \rightarrow \frac{S^*_\infty(X,R)}{S^*_{0,\infty}(X,R)}$$

induces isomorphism in cohomology by theorem (IV 2.5) it follows that

$$H^q(\Gamma(S^*_{s,\infty}(X,R)) \simeq H^q_{\text{Singular}}(X,R) \quad .$$

Now, by (2.3), we know that

$$H^q_{\text{de Rham}}(X) \simeq H^q(\Gamma(S^*_\infty(X,R))$$

so there exists a canonical isomorphism

$$\gamma^q : H^q_{\text{de Rham}}(X) \rightarrow H^q_{\text{Singular}}(X,R) \quad .$$

Here we assume the result that for a paracompact C^∞ manifold the inclusion $S^*_\infty(X,R) \rightarrow S^*(X,R)$ induces isomorphism in cohomology.

APPENDIX

PROPOSITION 1. Let $P = \{S_U, \rho_{V,U}\}$ be a presheaf satisfying condition (2) of a complete presheaf then $\gamma: S_X \to \Gamma(S(P))$ given by $\{\gamma(f)\}(x) = \rho_{x,X} f$ where $f \in S_X$ and $x \in X$, a paracompact space, is an onto mapping.

Proof. Let $\sigma \in \Gamma(S(P)$. Let $x \in X$. There exists an open set $U \ni x$ in X and an element $a_U \in S_U$ such that $\sigma(x) = \rho_{x,U} a_U$. Hence $\gamma(a_U) \in \Gamma(S(P), U)$ has the property that $\gamma(a_U)(x) = \sigma(x)$. Hence there exists U' open with $x \in U' \subset U$ such that $\gamma(a_U)|_{U'} = \sigma|_{U'}$. This means $\gamma(\rho_{U',U}(a_U)) = \sigma|_{U'}$. Let $b_{U'} = \rho_{U',U}(a_U) \in S_{U'}$ then $\gamma(b_{U'}) = \sigma|_{U'}$. Since X is paracompact, it follows that there exists a locally finite open covering $\{U'_\alpha\}_{\alpha \in J}$ of X and elements $b_\alpha \in P_{U'_\alpha}$ such that $\gamma(b_\alpha) = \sigma|_{U'_\alpha}$ for all $\alpha \in J$.

Let $\{V_\alpha\}_{\alpha \in J}$ be a refinement of $\{U_\alpha\}_{\alpha \in J}$ such that $\overline{V}_\alpha \subset U'_\alpha$ for all $\alpha \in J$.

Let $x \in X$ and $J_x = \{\alpha \in J | x \in \overline{V}_\alpha\}$. Now if α and $\beta \in J_x$ then $x \in U'_\alpha \cap U'_\beta$. Also $\rho_{x,U'_\alpha} b_\alpha = \rho_{x,U'_\beta} b_\beta$. Hence there exists $W_{\alpha\beta} \ni x$ open such that $U'_\alpha \cap U'_\beta \supset W_{\alpha\beta}$ and $\rho_{W_{\alpha\beta},U'_\alpha} b_\alpha = \rho_{W_{\alpha\beta},U'_\beta} b_\beta$. Let $W_x = \bigcap_{\substack{\alpha \in J_x \\ \beta \in J_x}} W_{\alpha\beta}$.

W_x is an open set since J_x is finite. Also $x \in W_x$ and $W_x \subset \bigcap_{\alpha \in J_x} U'_\alpha$. It follows that $\rho_{W_x,U'_\alpha} b_\alpha = \rho_{W_x,U'_\beta} b_\beta$ for α and $\beta \in J_x$. Now since $\{\overline{V}_\alpha\}_{\alpha \in J}$ is a locally finite closed cover, there exists $Y_x \ni x$ such that $L_x = \{\alpha \in J | Y_x \cap \overline{V}_\alpha \neq \phi\}$ is finite. Let

$$Z_x = Y_x \cap \left(\bigcap_{\alpha \in L_x - J_x} [\overline{V}_\alpha]' \right) \cap W_x .$$

Now $x \in Z_x$ and since L_x is finite, Z_x is an open set. It follows that

(i) $\quad Z_x \cap \overline{V}_\beta = \phi \quad$ if $\beta \notin J_x$,

(ii) $\quad Z_x \subset \bigcap_{\beta \in J_x} U'_\beta$, and

(iii) $\quad \rho_{Z_x,U'_\alpha}(b_\alpha) = \rho_{Z_x,U'_\beta}(b_\beta) \quad$ if α and $\beta \in J_x$.

Let $a_x = \rho_{Z_x,U'_\alpha} b_\alpha$ for $\alpha \in J_x$. Let $p \in Z_x \cap Z_y$. It follows from (i) that $J_p \subset J_x \cap J_y$. Let $\alpha \in J_p$. Then it follows that $a_x = \rho_{Z_x,U'_\alpha} b_\alpha$ and $a_y = \rho_{Z_y,U'_\alpha} b_\alpha$. So that

$$\rho_{Z_x \cap Z_y, Z_x}(a_x) = \rho_{Z_x \cap Z_y, Z_x} \rho_{Z_x,U'_\alpha} b_\alpha$$

$$= \rho_{Z_x \cap Z_y, U'_\alpha} b_\alpha = \rho_{Z_x \cap Z_y, Z_y} \rho_{Z_y,U'_\alpha} b_\alpha$$

$$= \rho_{Z_x \cap Z_y, Z_y} a_y .$$

It follows from property (2) of completeness that there exists an element $a \in S_X$ such that $\rho_{Z_x,X}(a) = a_x$. It follows immediately that $\gamma(a) = \sigma$ and γ is an onto mapping.

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