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BHARAT SARATH

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled

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submitted by Bharat Sarath in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

K Varadarajan

K. Varadarajan, Chairman
Department of Mathematics & Statistics

H. K. Farahat.

H.K. Farahat
Department of Mathematics & Statistics

W. K. Nicholson

W.K. Nicholson
Department of Mathematics & Statistics

V. H. Dyson

V.H. Dyson
Department of Philosophy

S K Sehgal

S.K. Sehgal, External Examiner
University of Alberta

26 Feb 76

date

ABSTRACT

This thesis deals with four broadly related topics, namely divisible modules, V-rings, M-injectivity and Krull-dimension. The first main theorem gives a condition for direct sums of modules divisible with respect to a given torsion theory to be divisible. Next, connections between V-rings and strongly regular rings are established and an example of a regular ring which is not a V-ring is given. Using an earlier theorem, noetherian V-rings or DSI-rings in our terminology are characterized in terms of semi-simple modules as well as through minimal generating sets. A definition of M-essential extension is given and M-injective hulls are characterized as minimal M-essential extensions. Certain standard results about injectivity are generalized to the M-injective case. Finally, the notion of Krull-dimension is introduced and rings over which all modules with Krull-dimension are noetherian are characterized. Calling such rings tall rings, it is shown that finite monoid rings over tall rings are tall though polynomial rings are never tall.

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INTRODUCTION

All rings in this thesis are assumed to possess an identity and all modules to be left unitary. Any property not explicitly stated to be otherwise is assumed to be a 'left' property, for example, 'ideal' will mean left ideal. Notation is generally explained as it is introduced but a list is also included at the end of the introduction.

This thesis divides up into roughly four fairly self-contained sections. The first consists of Chapters I and II which deal with divisibility of direct sums, the next of Chapters III and IV which deal with V-rings, the third of Chapters V and VI which deal with M-injectivity and the last of Chapter VII which deals with Krull-dimension and noetherian-ness. Though the motivation for the different sections are linked, the sections themselves are by and large independent of each other.

The main result of Chapter I, theorem 2.4, gives a criterion for the divisibility of direct sums of modules with respect to some torsion theory. Since divisibility with respect to the torsion theory consisting of all modules over a ring R is precisely the same as being injective in R-mod, we recover the theorem about injectivity of direct sums given in [25]. This is done in Chapter II mainly because the result is needed later on.

Chapter III introduces the notion of a V-ring (named after O. Villamayor), and discusses the connection between this notion and regularity (in the sense of Von Neumann). I. Kaplansky first observed

that for commutative rings, being a V-ring and being Von Neumann regular were equivalent. It is proved in Chapter III (theorem 2.8) that the equivalence holds for all rings whose maximal ideals are two-sided. Von Neumann regular rings whose maximal ideals are two-sided are called strongly regular rings and were introduced in [1]. For the sake of completeness we include some results from [1] and a basic theorem about V-rings from [19]. Chapter III also contains an example of a Von Neumann regular ring which is not a V-ring, namely the ring of endomorphisms of an infinite dimensional vector space. The first such example was given by C. Faith but the proof was somewhat more complicated.

In Chapter IV, attention is focused on noetherian V-rings. These are characterized as rings over which all semi-simple modules are injective, using results from Chapter II. The same characterization was obtained by Byrd [7] independently and using different methods. Next, we show in theorem 2.8 that noetherian V-rings can be alternatively characterized as rings over which every module has a minimal generating set and every minimal generating set of any submodule can be extended to a minimal generating set of the whole module. If we restrict this to fields we recover the classical theorem on the existence of bases. The final part of Chapter IV is the incorporation of an example of Cozzens [8] of a noetherian V-ring which is not regular. This example is given in full detail.

Chapters V and VI deal with the notion of M-injectivity introduced by Azumaya [2] and Sandomiersky [23]. A definition is given for

the M-injective hull and this is characterized as the maximal M-essential extension. The existence of M-injective hulls was given by M. Ryar [22] but the characterization of this as an M-essential extension is, to our knowledge, new. The rest of Chapter V generalizes a result of Bumby [6]. Chapter VI is devoted to generalizing a result of B. Osofsky which states that a ring R is semi-simple artinian if and only if every cyclic R-module is injective. We prove (lemma 3.7) that for a module M, if the endomorphism ring of M is regular and if every cyclic module is M-injective, then M is semi-simple artinian. Our interest in this stems from the following question posed by Faith in [12]. Here, he defines a class of rings called *PCI*-rings (proper cyclics injective) and asks whether all such rings are noetherian. A *PCI*-ring R is non-noetherian if and only if there exists a cyclic R-module M such that every cyclic R-module is M-injective but M is not semi-simple.

Hence, if we could show that the endomorphism ring of such an M was regular, then we could show that all *PCI*-rings are noetherian.

Chapter VII is somewhat different from the other chapters in that it has little connection with the notion of injectivity. In this chapter we introduce the notion of Krull-dimension as outlined in [15] and study rings over which all modules with Krull-dimension are noetherian. This question arose out of the observation that V-rings have this property. We characterize these rings (theorem 1.12) as rings over which every non-noetherian module has a non-noetherian submodule and call them 'tall' rings. The method of proof also yields

somewhat simpler proofs of theorem 4.1 and corollary 4.4 of [15] which are corollary 1.14 and theorem 1.11 in Chapter VII. In the second half of Chapter VII we prove theorem 3.3 that if R is a subring of S and if S is generated as an R-module by a finite set of elements in the centraliser of R in S, then R is a tall ring $\Leftrightarrow S$ is a tall ring. In particular, this means that every finite monoid ring over a tall ring is tall. Also, we give a proof that no polynomial ring is a tall ring.

Notations and Convention

(1) For any ring R, R-mod denotes the category of left R-modules.

Let R be a ring, M an R-module, $N \subset M$ a submodule and $x \in M$. Then

(2) $\mathcal{l}(x) = \{\lambda \in R | \lambda x = 0\}$ is the annihilator of x.

(3) We say N is 'large' or 'essential' in M if for all nonzero x in M, $Rx \cap N \neq 0$.

(4) By $[x : N]$ we denote the ideal $\{\lambda \in R | \lambda x \in N\}$.

(5) Given a subset $\{x_1,\ldots,x_n\} \subset M$ we denote the submodule generated by $x_1,\ldots,x_n$ by $\langle x_1,\ldots,x_n\rangle$.

(6) The word 'regular' is used in the sense of Von Neumann.

(7) $JR(M)$ is the Jacobson radical of M.

(8) Z_p denotes the integers modulo p.

CHAPTER I DIVISIBILITY OF DIRECT SUMS

§1. Torsion theories

In this chapter we determine necessary and sufficient conditions for the direct sum $\bigoplus_{\alpha\in J} M_\alpha$ of R-modules to be divisible with respect to a given torsion theory $(\mathcal{A},\mathcal{B})$ for the category R-mod of left R-modules. Throughout this thesis R will denote an associative ring with $1 \neq 0$ and all the modules we consider are unitary left modules. Before proving the main result of this chapter we briefly recall some basic definitions concerning torsion theories and indicate proofs of some known results that we need. We follow the terminology of S.E. Dickson [9].

DEFINITION 1.1 A torsion theory for R-mod consists of a pair $(\mathcal{A},\mathcal{B})$ of classes of R-modules satisfying the following conditions:

(i) $\mathcal{A} \cap \mathcal{B} = \{0\}$, the set consisting only of the module 0

(ii) $A \to A'' \to 0$ exact, $A \in \mathcal{A} \Rightarrow A'' \in \mathcal{A}$

(iii) $0 \to B' \to B$ exact, $B \in \mathcal{B} \Rightarrow B' \in \mathcal{B}$

(iv) Given any $M \in R$-mod, there exists an exact sequence $0 \to A \to M \to B \to 0$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$.

The modules in $\mathcal{A}$ are called the torsion modules for the torsion theory $(\mathcal{A},\mathcal{B})$ and the modules in $\mathcal{B}$ are termed torsion free.

LEMMA 1.2 Let $M \in R$-mod. Then

(i) $M \in \mathcal{A} \Leftrightarrow \mathrm{Hom}(M,B) = 0$ for all $B \in \mathcal{B}$

(ii) $M \in \mathcal{B} \Leftrightarrow \mathrm{Hom}(A,M) = 0$ for all $A \in \mathcal{A}$.

Proof For any $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $f \in \mathrm{Hom}(A,B)$ we have $A \to f(A) \to 0$ and $0 \to f(A) \to B$ exact. From 1.1 (ii) and (iii) we get $f(A) \in \mathcal{A} \cap \mathcal{B}$. Now, 1.1 (i) implies $f(A) = 0$. Hence $f = 0$.

Given any $M \in R$-mod, by 1.1 (iv) there exists an exact sequence $0 \to A \to M \to B \to 0$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$. If $\mathrm{Hom}(M,B) = 0$ for all $B \in \mathcal{B}$ we see that $A \to M \to 0$ is exact. From 1.1 (ii) we get $M \in \mathcal{A}$. If $\mathrm{Hom}(A,M) = 0$ for all $A \in \mathcal{A}$ we see that $0 \to M \to B$ is exact. Hence $M \in \mathcal{B}$ by 1.1 (iii). □

PROPOSITION 1.3 Let $(\mathcal{A},\mathcal{B})$ be any torsion theory for R-mod. Then $\mathcal{A}$ and $\mathcal{B}$ are closed under isomorphic images and extensions. $\mathcal{A}$ is closed under factor modules and arbitrary direct sums while $\mathcal{B}$ is closed under submodules and arbitrary direct products.

Proof The closure of $\mathcal{A}$ and $\mathcal{B}$ under isomorphic images, of $\mathcal{A}$ under factor modules and of $\mathcal{B}$ under submodules is immediate from 1.1 (ii) and (iii). If $0 \to A' \to A \to A'' \to 0$ is exact with A', A'' in $\mathcal{A}$, then for any $B \in \mathcal{B}$ the exactness of $0 \to \mathrm{Hom}(A'',B) \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A',B)$ together with 1.2 (i) implies $\mathrm{Hom}(A,B) = 0$. Hence by 1.2 (i) again we see that $A \in \mathcal{A}$. If $\{A_\alpha\}_{\alpha \in J}$ is any family with $A_\alpha \in \mathcal{A}$ for all $\alpha \in J$, then $\mathrm{Hom}(\bigoplus_{\alpha \in J} A_\alpha, B) = \prod_{\alpha \in J} \mathrm{Hom}(A_\alpha,B) = 0$ for any $B \in \mathcal{B}$. Hence $\bigoplus_{\alpha \in J} A_\alpha \in \mathcal{A}$.

Similarly, if $0 \to B' \to B \to B'' \to 0$ is any exact sequence with B', B'' in $\mathcal{B}$, then the exactness of $0 \to \mathrm{Hom}(A,B') \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B'')$ together with 1.2 (ii) yields $\mathrm{Hom}(A,B) = 0$ for all $A \in \mathcal{A}$. Hence $B \in \mathcal{B}$. Finally if $\{B_\alpha\}_{\alpha \in J}$ is any family with $B_\alpha \in \mathcal{B}$, then $\mathrm{Hom}(A, \prod_{\alpha \in J} B_\alpha) = \prod_{\alpha \in J} \mathrm{Hom}(A,B_\alpha) = 0$ for all $A \in \mathcal{A}$. Hence $\prod_{\alpha \in J} B_\alpha \in \mathcal{B}$. □

DEFINITION 1.4 An R-module M is said to be divisible with respect to a torsion theory $(\mathcal{A},\mathcal{B})$ if given any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{f} & N \\ & & {\scriptstyle g}\downarrow & & \\ & & M & & \end{array}$$

with $0 \to L \xrightarrow{f} N$ exact, and coker $f \in \mathcal{A}$ there exists an $h : N \to M$ such that $h \circ f = g$.

REMARK 1.5 It is clear that $(\mathcal{A},\mathcal{B})$ given by $\mathcal{A} = R$-mod and $\mathcal{B} = \{0\}$ is a torsion theory for R-mod. With respect to this torsion theory the notion of a divisible module is the same as that of an injective module.

In what follows immediately $(\mathcal{A},\mathcal{B})$ denotes a fixed torsion theory for R-mod and all the statements refer to the torsion theory $(\mathcal{A},\mathcal{B})$.

LEMMA 1.6 (i) The direct product of any family of divisible modules is divisible;

(ii) Any direct summand of a divisible module is divisible.

Proof Exactly analogous to the case of injective modules and hence omitted. □

THEOREM 1.7 The following conditions are equivalent:

(1) $\mathcal{A}$ is closed under submodules

(2) $\mathcal{B}$ is closed under injective hulls.

Proof (1) ⇒ (2). Let $B \in \mathcal{B}$ and E the injective hull of B. Let $A \in \mathcal{A}$ and $f \in \mathrm{Hom}(A,E)$. If $f(A) \neq 0$, since $B \subsetneq E$ is essential we get

$f(A) \cap B \neq 0$. Let $A' = f^{-1}(f(A) \cap B)$. Then (1) implies $A' \in \mathcal{A}$. From $f(A') = f(A) \cap B \neq 0$ we see that $f' = f|A'$ is a non-zero element of $\mathrm{Hom}(A',B)$ contradicting the fact that $B \in \mathcal{B}$. Hence $\mathrm{Hom}(A,E) = 0$ for all $A \in \mathcal{A}$. Thus $E \in \mathcal{B}$.

(2) $\Rightarrow$ (1). Let $A' \subset A$ with $A \in \mathcal{A}$. Let $B \in \mathcal{B}$ be arbitrary and $f \in \mathrm{Hom}(A',B)$. The injective hull E of B is in $\mathcal{B}$ by (2). Since E is injective, there exists $g : A \to E$ extending f. $A \in \mathcal{A}$, $E \in \mathcal{B} \Rightarrow g = 0$. Hence $f = 0$. Thus $\mathrm{Hom}(A',B) = 0$ for all $B \in \mathcal{B}$. Hence $A' \in \mathcal{A}$. □

DEFINITION 1.8 A torsion theory $(\mathcal{A},\mathcal{B})$ is called hereditary if it satisfies any of the equivalent conditions (1), (2) of Theorem 1.7.

Given a torsion theory $(\mathcal{A},\mathcal{B})$ for R-mod we will be considering the family $\mathcal{F} = \{I \mid I \subset R,\ R/I \in \mathcal{A}\}$ of left ideals of R.

PROPOSITION 1.9 When $(\mathcal{A},\mathcal{B})$ is a hereditary torsion theory $\mathcal{F}$ satisfies the following conditions:

(i) $I \in \mathcal{F}$, $I \subset J \subset R \Rightarrow J \in \mathcal{F}$

(ii) I,J in $\mathcal{F} \Rightarrow I \cap J \in \mathcal{F}$ and

(iii) $I \in \mathcal{F}$, $x \in R \Rightarrow [x : I] \in \mathcal{F}$ where $[x : I] = \{\lambda \in R \mid \lambda x \in I\}$.

Proof If $I \subset J \subset R$ then $R/I \to R/J \to 0$ is exact. Hence $R/J \in \mathcal{A}$. This proves (i).

If I,J are in $\mathcal{F}$, then from the exactness of $0 \to R/I \cap J \to R/I \oplus R/J$ and the hereditary nature of $(\mathcal{A},\mathcal{B})$ we get $R/I \cap J \in \mathcal{A}$. This proves (ii).

Let $\eta : R \to R/I$ denote the canonical quotient map. Then the left annihilator of $\eta(x) = \{\lambda \in R \mid \lambda x \in I\} = [x : I]$. Hence $R/[x:I] \simeq R\eta(x) \subset R/I$. Hence $R/[x:I] \in \mathcal{A}$. This proves (iii). □

LEMMA 1.10 Let (A,B) be any torsion theory for R-mod. Then any module M has a largest torsion submodule $T(M)$.

Proof Let $H = \{N | N \subset M, N \in A\}$. Let $C = \sum_{N \in H} N$, the sum in M and $D = \bigoplus_{N \in H} N$ the direct sum. Since there exists an exact sequence $D \to C \to 0$ and since $D \in A$ we get $C \in A$. Then, clearly $C = T(M)$ is the largest torsion submodule of M. □

§2. Divisibility of direct sums

This section contains the new results obtained in this chapter. Throughout this section (A,B) is a hereditary torsion theory for R-mod, F the associated "filter" of ideals, namely $F = \{I \subset R | R/I \in A\}$. For any $M \in R$-mod, $T(M)$ will denote the torsion submodule of M. We need the following well-known lemma. A proof of this can be found in [13].

LEMMA 2.1 A module M is divisible $\Leftrightarrow$ given any $f : I \to M$ with $I \in F$, there exists an extension $h : R \to M$ of f, or equivalently there exists an element $m \in M$ satisfying $f(\lambda) = \lambda m$ for all $\lambda \in I$.

Let $\{M_\alpha\}_{\alpha \in J}$ be a family of modules. We write $\underline{x}$ for the element $(x_\alpha)_{\alpha \in J}$ of $\prod_{\alpha \in J} M_\alpha$. For any $\underline{x} \in \prod_{\alpha \in J} M_\alpha$ we set $I_{\underline{x}} = \{\lambda \in R | \lambda \underline{x} \in \bigoplus_{\alpha \in J} M_\alpha\}$.

DEFINITION 2.2 An element $\underline{x} \in \prod_{\alpha \in J} M_\alpha$ is called "special" if there exists a finite subset F of J such that $\lambda x_\alpha = 0$ for all $\alpha \in J - F$ and $\lambda \in I_{\underline{x}}$.

Given any module M and any $x \in M$ we denote the annihilator $\{\lambda \in R | \lambda x = 0\}$ of x by $L(x)$.

DEFINITION 2.3 An element $x \in M$ is called a torsion element if and only if $x \in T(M)$.

It is well-known that $x \in T(M) \Leftrightarrow L(x) \in \mathsf{F}$ (Lemma 2.1 of [5]).

Let $\{M_\alpha\}_{\alpha \in J}$ be a given family of modules, N the factor module $\prod_{\alpha \in J} M_\alpha / \bigoplus_{\alpha \in J} M_\alpha$ and $p : \prod_{\alpha \in J} M_\alpha \to N$ the canonical quotient map.

THEOREM 2.4 $\bigoplus_{\alpha \in J} M_\alpha$ is divisible $\Leftrightarrow$ each M_α is divisible and every element $\underline{x} \in p^{-1}(T(N))$ is a special element.

Proof Assume each M_α divisible and every $\underline{x} \in p^{-1}(T(N))$ special. Let $f: I \to \bigoplus_{\alpha \in J} M_\alpha$ be any map with $I \in \mathsf{F}$. Since $\prod_{\alpha \in J} M_\alpha$ is divisible there exists an element $\underline{u} \in \prod_{\alpha \in J} M_\alpha$ such that $f(\lambda) = \lambda \underline{u}$ for all $\lambda \in I$. In particular $I\underline{u} \subset \bigoplus_{\alpha \in J} M_\alpha$. Hence $I \subset I_{\underline{u}}$. The map $\varphi: R \to N$ defined by $\varphi(\lambda) = \lambda p(\underline{u}) = p(\lambda \underline{u})$ for all $\lambda \in R$ satisfies $\varphi(I) = 0$. Let $\bar{\varphi}: R/I \to N$ denote the map induced by φ. Then $\bar{\varphi}(R/I) = Rp(\underline{u})$. Since $R/I \in \mathsf{A}$ we see that $Rp(\underline{u}) \in \mathsf{A}$. Hence $p(\underline{u}) \in T(N)$ or $\underline{u} \in p^{-1}(T(N))$. By assumption every element in $p^{-1}(T(N))$ is special. Hence $I_{\underline{u}}\, u_\alpha = 0$ for almost all $\alpha \in J$. From $I \subset I_{\underline{u}}$, we get $Iu_\alpha = 0$ for almost all $\alpha \in J$. Let $\underline{v} \in \bigoplus_{\alpha \in J} M_\alpha$ be the element whose α-component is u_α or 0 according as $Iu_\alpha \neq 0$ or $Iu_\alpha = 0$. Then it is clear that $\lambda \underline{v} = \lambda \underline{u}$ for all $\lambda \in I$. The map $h : R \to \bigoplus_{\alpha \in J} M_\alpha$ defined by $h(\lambda) = \lambda \underline{v}$ $(\lambda \in R)$ extends f. From Lemma 2.1 it follows that $\bigoplus_{\alpha \in J} M_\alpha$ is divisible. This proves the implication $\Leftarrow$.

Conversely, assume $\bigoplus_{\alpha \in J} M_\alpha$ divisible. M_α being a direct summand of $\bigoplus_{\alpha \in J} M_\alpha$ it follows that each M_α is divisible. Let $\underline{x} \in p^{-1}(T(N))$. Then clearly $L(p(\underline{x})) = I_{\underline{x}}$. Since $p(\underline{x}) \in T(N)$ it follows that $I_{\underline{x}} \in \mathsf{F}$. Consider the map $f : I_{\underline{x}} \to \bigoplus_{\alpha \in J} M_\alpha$ given by $f(\lambda) = \lambda \underline{x}$. The divisibility

of $\bigoplus_{\alpha\in J} M_\alpha$ now implies that there exists an extension $h: R \to \bigoplus_{\alpha\in J} M_\alpha$ of f. If $\underline{y} = h(1) \in \bigoplus_{\alpha\in J} M_\alpha$, then $y_\alpha = 0$ for almost all α and $I_{\underline{x}} x_\alpha = I_{\underline{x}} y_\alpha = 0$ for almost all $\alpha \in J$. Thus $\underline{x}$ is a special element. □

THEOREM 2.5 Suppose $\{M_\alpha\}_{\alpha\in J}$ is a family of R-modules such that for every countable subset K of J, $\bigoplus_{\alpha\in K} M_\alpha$ is divisible. Then $\bigoplus_{\alpha\in J} M_\alpha$ is itself divisible.

Proof Assume if possible that $\bigoplus_{\alpha\in J} M_\alpha$ is not divisible. From theorem 2.4, there exists an element $\underline{x} \in p^{-1}(T(N))$ which is not special. Then $I_{\underline{x}} x_\alpha \neq 0$ for infinitely many $\alpha \in J$. Let K be an infinite countable subset of $\{\alpha \in J \mid I_{\underline{x}} x_\alpha \neq 0\}$. Let $N_K = \left(\prod_{\alpha\in K} M_\alpha\right) \Big/ \left(\bigoplus_{\alpha\in K} M_\alpha\right)$ and $p_K : \prod_{\alpha\in K} M_\alpha \to N_K$ the canonical quotient map. Let $\underline{y} \in \prod_{\alpha\in K} M_\alpha$ be the element given by $y_\alpha = x_\alpha$ for all $\alpha \in K$. Then clearly $I_{\underline{x}} \subset I_{\underline{y}}$. From $I_{\underline{x}} = L(p(\underline{x})) \in F$ we see that $I_{\underline{y}} \in F$. But $I_{\underline{y}} = L(p_K(\underline{y}))$. It follows that $p_K(\underline{y}) \in T(N_K)$. Clearly $I_{\underline{y}} y_\alpha \supset I_{\underline{x}} x_\alpha \neq 0$ for every $\alpha \in K$. Theorem 2.4 now implies that $\bigoplus_{\alpha\in K} M_\alpha$ is not divisible, a contradiction. This completes the proof of Theorem 2.5. □

THEOREM 2.6 Arbitrary direct sums of divisible modules divisible $\Rightarrow F$ satisfies the ascending chain condition.

Proof Suppose F does not satisfy the ascending chain condition. Then there exists an infinite sequence $I_1 \subset I_2 \subset I_3 \subset \ldots$ of ideals in F with $I_j \neq I_{j+1}$ for every $j \geq 1$. Let $A_j = R/I_j$, $\eta_j : R \to A_j$ the canonical projection and $x_j = \eta_j(1) \in A_j$. Let M_j be the divisible hull of A_j ([18], Proposition 0.7, page 10). Consider the element

$\underline{x} = (x_j)_{j\geq 1}$ of $\prod_{j\geq 1} M_j$. Let $N = (\prod_{j\geq 1} M_j)\big/(\oplus_{j\geq 1} M_j)$ and $p : \prod_{j\geq 1} M_j \to N$ the canonical quotient map. For any $\lambda \in I_j$ we have $\lambda x_k = 0$ whenever $k \geq j$. Hence $I_j \subset I_{\underline{x}}$ for all $j \geq 1$. In particular we get $I_{\underline{x}} \in F$. Hence $p(\underline{x}) \in T(N)$. Let λ_j be any element of I_{j+1} which is not in I_j. Then $\lambda_j x_j \neq 0$. Since $\lambda_j \in I_{\underline{x}}$ we see that $I_{\underline{x}} x_j \neq 0$ for each $j \geq 1$. Hence $\underline{x}$ is not a special element in $\prod_{j\geq 1} M_j$. By theorem 2.4, $\oplus_{j\geq 1} M_j$ is not divisible. This proves theorem 2.6.

2.7 REMARKS (1) When $A = R$-mod, $B = \{0\}$ theorem 2.4 gives necessary and sufficient conditions for a direct sum $\oplus_{\alpha\in J} M_\alpha$ to be injective. It then asserts that $\oplus_{\alpha\in J} M_\alpha$ is injective if and only if each M_α is injective and every element $\underline{x} \in \prod_{\alpha\in J} M_\alpha$ is special.

(2) Also when $A = R$-mod, $B = \{0\}$, theorem 2.5 asserts that if $\{M_\alpha\}_{\alpha\in J}$ is a family of R-modules with the property that $\oplus_{\alpha\in J} M_\alpha$ is injective for every countable subset K of J, then $\oplus_{\alpha\in J} M_\alpha$ is itself injective.

CHAPTER II SOME RESULTS ON DIRECT PRODUCTS

In this chapter we briefly list some results obtained by B. Sarath and K. Varadarajan [25] and later on improved by F.F. Mbuntum and K. Varadarajan. They are included here mainly for the sake of completeness. We use these results in our later chapters.

Throughout this chapter J denotes an infinite set and $\{M_\alpha\}_{\alpha\in J}$ a family of R-modules indexed by J. We choose a well-ordering on J and fix it throughout the following discussions. Unless otherwise stated by an ideal in R we mean a left ideal in R.

LEMMA 1.1 Suppose there exists an element $\underline{x} \in \prod_{\alpha\in J} M_\alpha$ and elements $a_\alpha \in R$ for all $\alpha \in J$ such that

(i) $a_\alpha x_\alpha \neq 0$ for each $\alpha \in J$ and

(ii) $a_\alpha x_\beta = 0$ whenever $\alpha < \beta$.

Then $\oplus_{\alpha\in J} M_\alpha$ is not a direct summand of $\prod_{\alpha\in J} M_\alpha$.
This is proposition 1 in [25]. □

For any $\underline{x} \in \prod_{\alpha\in J} M_\alpha$ let $P_\alpha(\underline{x}) = \{\lambda \in R \mid \lambda x_\beta = 0 \text{ for } \beta \geq \alpha\}$. Then clearly for any $\alpha < \beta$ in J we have $P_\alpha(\underline{x}) \subset P_\beta(\underline{x})$. For any ideal I in R let $P_\alpha(\underline{x},I) = P_\alpha(\underline{x}) \cap I$ and

$$\ell(\underline{x},I) = \begin{cases} \text{the number of distinct ideals in the collection } \{P_\alpha(\underline{x},I)\}_{\alpha\in J} \text{ if this number is finite;} \\ \infty \text{ otherwise.} \end{cases}$$

For any $\gamma \in J$ the truncated element $\underline{x}^\gamma \in \prod_{\alpha\in J} M_\alpha$ is defined by

$$x_\alpha^\gamma = \begin{cases} x_\alpha & \text{for all } \alpha < \gamma \\ 0 & \text{for all } \alpha \geq \gamma \end{cases}$$

LEMMA 1.2 Let I be any ideal in R. Suppose $\underline{x},\underline{u}$ are any two elements of $\prod_{\alpha\in J} M_\alpha$ satisfying $\lambda\underline{x} = \lambda\underline{u}$ for all $\lambda \in I$. Then

(i) $\lambda\underline{x}^\gamma = \lambda\underline{u}^\gamma$ for every $\gamma \in J$ and $\lambda \in I$

(ii) $P_\alpha(\underline{x},I) = P_\alpha(\underline{u},I)$ and $P_\alpha(\underline{x}^\gamma,I) = P_\alpha(\underline{u}^\gamma,I)$ for any α,γ in J.

In particular

$$l(\underline{x},I) = l(\underline{u},I) \text{ and } l(\underline{x}^\gamma,I) = l(\underline{u}^\gamma,I) \text{ for all } \gamma \in J.$$

Proof Trivial. □

LEMMA 1.3 Suppose $\underline{u} \in \bigoplus_{\alpha\in J} M_\alpha$ then for any ideal I of R we have

$$l(\underline{u},I) < \infty \text{ and } l(\underline{u}^\gamma,I) < \infty \text{ for all } \gamma \in J.$$

Proof Trivial. □

Suppose $\underline{x}$ and $\underline{u}$ are elements of $\prod_{\alpha\in J} M_\alpha$ satisfying $\lambda\underline{x} = \lambda\underline{u}$ for all $\lambda \in I$ and suppose $\underline{u} \in \bigoplus_{\alpha\in J} M_\alpha$. Then clearly $I \subset I_{\underline{x}}$. Combining lemmas 1.2 and 1.3 we get $l(\underline{x},I) < \infty$ and $l(\underline{x}^\gamma,I) < \infty$ for every $\gamma \in J$. The main result of chapter II asserts that the converse is also true. In fact a result which is slightly stronger than the converse is true.

§2. Ascent sequence

Let $\underline{x} \in \prod_{\alpha\in J} M_\alpha$ and $I \subset I_{\underline{x}}$. Suppose $l(\underline{x},I) < \infty$. Let $T_0 \subset T_1 \subset \ldots \subset T_r$ be the distinct ideals in the collection $\{P_\alpha(\underline{x},I)\}_{\alpha\in J}$ and δ_i the least ordinal in J with the property $P_{\delta_i}(\underline{x},I) = T_i$ $(0 \le i \le r)$. Clearly $\delta_0 = 0$. The $(r+1)$ tuple of ordinals $\{0,\delta_1,\ldots,\delta_r\}$ will be referred to as the ascent sequence of $\underline{x}$ w.r.t. I for the chosen well-ordering on J.

LEMMA 2.1 Suppose $I \subset I_{\underline{x}}$, $l(\underline{x},I) < \infty$ and $\{0,\delta_1,\ldots,\delta_r\}$ the ascent sequence of $\underline{x}$ w.r.t. I. Then each δ_i is a non-limiting ordinal.

Proof Suppose on the contrary a certain δ_i is a limiting ordinal. Since $P_{\delta_{i-1}}(\underline{x},I) \neq P_{\delta_i}(\underline{x},I)$ there exists a $\lambda \in P_{\delta_i}(\underline{x},I)$ with $\lambda \notin P_{\delta_{i-1}}(\underline{x},I)$. Then there exists an $\alpha_1 \in J$ with $\delta_{i-1} \leq \alpha_1 < \delta_i$ and $\lambda x_{\alpha_1} \neq 0$. We know that $P_{\delta_{i-1}}(\underline{x},I) = P_{\alpha}(\underline{x},I)$ for all $\delta_{i-1} \leq \alpha < \delta_i$. Assume k to be an integer ≥ 1 and that we have chosen elements $\alpha_1,\ldots,\alpha_k$ in J with $\delta_{i-1} \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k < \delta_i$ satisfying $\lambda x_{\alpha_j} \neq 0$ for $1 \leq j \leq k$. Since δ_i is a limiting ordinal there exists $\alpha' \in J$ with $\alpha_k < \alpha' < \delta_i$. From $\lambda \in P_{\delta_i}(\underline{x},I)$ and $\lambda \notin P_{\alpha'}(\underline{x},I) = P_{\delta_{i-1}}(\underline{x},I)$ we get an element α_{k+1} in J such that $\alpha' \leq \alpha_{k+1} < \delta_i$ and $\lambda x_{\alpha_{k+1}} \neq 0$. Iteration of this argument yields an infinite sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ of elements of J with $\lambda x_{\alpha_n} \neq 0$ for all $n \geq 1$. This contradicts $\lambda \in I \subset I_{\underline{x}}$. □

Recall that a well-ordering on J was called "admissible" in [25] if under the well-ordering J corresponds to the set of ordinals $< \Omega$ with Ω a limiting ordinal. (Equivalently J has no maximal element).

LEMMA 2.2 Suppose the well-ordering on J is admissible. Let $I \subset I_{\underline{x}}$, $l(\underline{x},I) < \infty$ and $\{0,\delta_1,\ldots,\delta_r\}$ the ascent sequence of $\underline{x}$ w.r.t. I. Then $I = P_{\delta_r}(\underline{x},I)$ and hence $\lambda\underline{x} = \lambda\underline{x}^{\delta_r}$.

Proof Let $\lambda \in I$. Since $I \subset I_{\underline{x}}$ we have $\lambda x_{\alpha} = 0$ except for a finite number of elements $\alpha_1 < \ldots < \alpha_s$ in J. (The elements $\alpha_1,\ldots,\alpha_s$ in general depend on λ). Since J has no maximal element there exists an

$\alpha \in J$ with $\alpha_s < \alpha$. Then $\lambda \in P_\alpha(\underline{x},I)$. But $P_\alpha(\underline{x},I)$ is one of the ideals T_i and hence $\lambda \in P_\alpha(\underline{x},I) \subset T_{\delta_r} = P_{\delta_r}(\underline{x},I)$. Since $\lambda \in I$ is arbitrary we get $I = P_{\delta_r}(\underline{x},I)$. □

PROPOSITION 2.3 Let the well-ordering on J be admissible. Let $\underline{x} \in \prod_{\alpha\in J} M_\alpha$ and $I \subset I_{\underline{x}}$. Suppose $l(\underline{x},I) < \infty$ and $l(\underline{x}^\gamma,I) < \infty$ for every limiting ordinal $\gamma \in J$. Then there exists an element $\underline{u} \in \bigoplus_{\alpha\in J} M_\alpha$ satisfying $\lambda\underline{x} = \lambda\underline{u}$ for all $\lambda \in I$.

Proof The proof of this is an exact replica of the proof of Theorem 2 of [25]. We have only to replace Proposition 6 of [25] by Lemma 2.2.

THEOREM 2.4 Let J be well-ordered in an arbitrary way (i.e. the well-ordering on J need not be "admissible"). Let $\underline{x} \in \prod_{\alpha\in J} M_\alpha$, $I \subset I_{\underline{x}}$. Then there exists an element $\underline{u} \in \bigoplus_{\alpha\in J} M_\alpha$ satisfying $\lambda\underline{x} = \lambda\underline{u}$ for all $\lambda \in I$ if and only if $l(\underline{x},I) < \infty$ and $l(\underline{x}^\gamma,I) < \infty$ for all limiting ordinals γ in J.

Proof In view of lemma 1.3 and proposition 2.3, to complete the proof of theorem 2.4 we have only to prove the existence of $\underline{u}$ in the case when J has a maximal element. Let $c \in J$ be the maximal element. Let J' = the set of ordinals $< c + \chi_0$, where χ_0 is the first countable ordinal. Let $N_\alpha = M_\alpha$ for all $\alpha \le c$ and $N_\alpha = 0$ for any $\alpha \in J'$ satisfying $\alpha > c$. Let $\underline{y} \in \prod_{\alpha\in J'} N_\alpha$ be given by $y_\alpha = x_\alpha$ for all $\alpha \in J$ and $y_\alpha = 0$ for $\alpha \in J' \setminus J$. Then it is trivial to see that $I_{\underline{y}} = I_{\underline{x}}$. Moreover every limiting ordinal in J' is already in J. It is trivial to see that $l(\underline{y},I) \le 1 + l(\underline{x},I)$ and that $l(\underline{y}^\gamma,I) = l(\underline{x}^\gamma,I)$ for every limiting ordinal γ in J. Actually

$$\ell(\underline{y},I) = \begin{cases} \ell(\underline{x},I) \text{ if } P_{\mathcal{C}}(\underline{x},I) = I \\ 1 + \ell(\underline{x},I) \text{ otherwise.} \end{cases}$$

Since J' is an admissible well-ordered set, proposition 2.3 is applicable to $\underline{y} \in \prod_{\alpha\in J'} N_\alpha$. We thus get an element $\underline{w} \in \bigoplus_{\alpha\in J'} N_\alpha$ such that $\lambda\underline{y} = \lambda\underline{w}$ for all $\lambda \in I$. If $\underline{u} \in \bigoplus_{\alpha\in J} M_\alpha$ is given by $u_\alpha = w_\alpha$ for all $\alpha \in J$ then clearly $\lambda\underline{x} = \lambda\underline{u}$ for all $\lambda \in I$. □

COROLLARY 2.5 Let J be well-ordered. An element $\underline{x} \in \prod_{\alpha\in J} M_\alpha$ is special in the sense of definition 2.2 of chapter I $\Leftrightarrow \ell(\underline{x},I_{\underline{x}}) < \infty$ and $\ell(\underline{x}^\gamma,I_{\underline{x}}) < \infty$ for every limiting ordinal $\gamma \in J$.

For any $\underline{x} \in \prod_{\alpha\in J} M_\alpha$ we denote $\ell(\underline{x},R)$ by $\ell(\underline{x})$. In other words

$$\ell(\underline{x}) = \begin{cases} \text{number of distinct } P_\alpha(\underline{x}) \\ \text{if this number is finite;} \\ \infty \text{ otherwise.} \end{cases}$$

PROPOSITION 2.6 Suppose $\bigoplus_{\alpha\in J} M_\alpha$ is a direct summand of $\prod_{\alpha\in J} M_\alpha$. Then $\ell(\underline{x}) < \infty$ for all $\underline{x} \in \prod_{\alpha\in J} M_\alpha$.

This is Theorem 1 in [25].

THEOREM 2.7 $\bigoplus_{\alpha\in J} M_\alpha$ is injective $\Leftrightarrow$ each M_α is injective and $\ell(\underline{x}) < \infty$ for any $\underline{x} \in \prod_{\alpha\in J} M_\alpha$.

Proof This differs from Theorem 3 of [25] in only one respect. The well-ordering chosen on J could be arbitrary. It need not be "admissible". For any ideal I of R it is easy to see that $\ell(\underline{x},I) \leq \ell(\underline{x})$. Theorem 2.7 is an immediate consequence of remark 2.7 (i) of chapter I and corollary 2.5 in the present chapter. □

CHAPTER III STRONGLY REGULAR RINGS

§1. *V*-Rings

In this chapter, we recall the definition of *V*-rings (after O.E. Villamayor). These rings, in a sense, generalize artinian semisimple rings, since *V*-rings can be characterized by the fact that all modules over them are Jacobson semisimple. Kaplansky first observed that, for commutative rings, the class of *V*-rings coincides with the class of regular rings. In the non-commutative case, however, it is known that neither class includes the other [8],[11]. An example of a regular ring which is not a *V*-ring was first given by C. Faith [11]. He actually states that the subring generated by the socle and the centre of the ring of endomorphisms of an infinite dimensional vector space over any field K is regular but not a (left) *V*-ring. We prove by more direct means that the complete ring of endomorphisms of an infinite dimensional vector space is itself not a *V*-ring. The example due to J.H. Cozzens [8] of a *V*-ring which is not regular is postponed to the next chapter on noetherian *V*-rings.

DEFINITION 1.1 A ring R is called a (left) *V*-ring if every simple (left) R-module is injective.

DEFINITION 1.2 A ring R is called a regular ring if for every $a \in R$ there exists an $a' \in R$ such that $aa'a = a$.

Regular rings were originally introduced by Von Neumann [30]. Clearly regularity is a left-right symmetric property, whereas that of

a V-ring is not. In fact, the example of a non-regular V-ring given by C. Faith which is mentioned in the introduction is also an example of a left V-ring which is not a right V-ring.

We now indicate the proof of a basic proposition about V-rings which is essentially due to O.E. Villamayor [19].

PROPOSITION 1.3. The following conditions on a ring R are equivalent:

(i) R is a V-ring.

(ii) Given an R-module M, a submodule $N \subset M$ and $x \in M$, $x \notin N$, there exists a maximal submodule T of M such that $T \supset N$ and $x \notin T$.

(iii) Given a module $M \neq 0$ and an element $x \neq 0$ in M, there exists a maximal submodule T of M with $x \notin T$.

(iv) The Jacobson radical $JR(M) = 0$ for all $M \in R$-mod.

(v) $JR(M) = 0$ for all cyclic modules $M \in R$-mod.

(vi) Given a left ideal $I \subset R$ and $x \in R$, $x \notin I$, there exists a maximal left ideal J of R satisfying $x \notin J \supset I$.

(vii) $JR(E(M)) = 0$ for all simple modules M, where $E(M)$ is the injective hull of M.

Proof (i) $\Rightarrow$ (ii). Let x, R, M, N be as in the hypothesis of (ii). Let $\mathcal{T}$ denote the collection of submodules S of M such that $N \subset S \subset M$ and $x \notin M$. Then $\mathcal{T}$ is non-empty and inductive and hence by Zorn's lemma contains a maximal element T. We show that T is actually a maximal submodule of M. Let $\eta : M \to M/T$ be the canonical quotient map. Then $R\eta(x)$ is a simple submodule of M/T from the choice of T. Hence

$M/T = R\eta(x) \oplus T'$, since R is a V-ring. If $0 \neq \bar{t} \in T'$, then pick $0 \neq t \in \eta^{-1}(\bar{t})$. Clearly $x \notin \langle T, t \rangle$, a contradiction of the maximality of T in Γ. Hence $T' = 0$ and so $M/T = R\eta(x)$ which is simple. Hence T is maximal.

(ii) $\Rightarrow$ (iii). Set $N = 0$ in (ii).

(iii) $\Rightarrow$ (iv). Obvious since $JR(M)$ is the intersection of all maximal submodules of M.

(iv) $\Rightarrow$ (v). Trivial.

(v) $\Rightarrow$ (vi). Under the hypothesis of (v) setting $\eta : R \to R/I$ to be the canonical homomorphism, $\eta(x) \neq 0$. Since $JR(R/I) = 0$, there exists a maximal submodule J of R/I with $\eta(x) \notin \bar{J}$. Then $J = \eta^{-1}(\bar{J})$ is a maximal ideal in R, with $I \subset J$, $x \notin J$.

(vi) $\Rightarrow$ (vii). We will prove that (vi) $\Rightarrow E(M) = M$ for every simple module M. In particular this gives $JR(E(M)) = JR(M) = 0$. Suppose $E(M) \neq M$. Let $a \in E(M)$ with $a \notin M$. Then $a \neq 0$ and hence $Ra \neq 0$. Since M is essential in $E(M)$, we see that $Ra \cap M \neq 0$. Since M is simple, we see that $Ra \cap M = M$ or that $Ra \supset M$. Let $f : R \to Ra$ denote the map $f(\lambda) = \lambda a$. Choose an $m \neq 0$ in M. Let $x \in R$ be such that $f(x) = m$. Then $x \notin l(a)$. By (vi) there exists a maximal left ideal J in R with $x \notin J$ and $J \supset l(a)$. Then $m = f(x) \notin f(J)$. Since M is simple it follows that $f(J) \cap M = 0$. But M is essential in Ra. Hence $f(J) = 0$. This yields $J = l(a)$ and hence $l(a)$ is itself a maximal ideal in R. Thus Ra is simple. In particular $Ra = M$, contradicting the choice of a. Thus $E(M) = M$.

(vii) $\Rightarrow$ (i). Let M be any simple module. Let T be a maximal submodule of $E(M)$. Since M is simple either $T \cap M = 0$ or $T \cap M = M$.

If $T \cap M = 0$, then $T = 0$ since M is essential in $E(M)$. In this case $E(M)$ is simple and hence $E(M) = M$. If $T \cap M = M$ for every maximal submodule T of $E(M)$, we will have $JR(E(M)) \supset M \neq 0$, contradicting (vii). Thus $JR(E(M)) = 0 \Rightarrow M = E(M)$. Thus every simple module is injective. □

Let W be an infinite dimensional vector space over a field K and $A = \mathrm{Hom}_K(W,W)$. It is customary to regard W as an A-module by defining $\theta \cdot x = \theta(x)$ for any $\theta \in A$ and $x \in W$. It is well-known [17], [21] that A is a regular ring and that W is a simple A-module. We will prove that W is not injective over A. Actually we will restrict to the case where W has a countable infinite base $\{v_n\}_{n\geq 1}$ over K. The proof we give can easily be modified to take care of the uncountable case as well.

PROPOSITION 1.4 With the notations as above, W is not injective over A.

Proof On the contrary assume W injective over A. For each integer $n \geq 1$ let $W_n = W$ and $g_n : W_n \to W$ the identity map of W. The maps g_n give rise to a unique map $g : \bigoplus_{n\geq 1} W_n \to W$ where $\bigoplus_{n\geq 1} W_n$ denotes the direct sum of the A-modules W_n. If W is injective $\exists$ some extension $h : \prod_{n\geq 1} W_n \to W$ of g. Let $q : A \to \prod_{n\geq 1} W_n$ be defined by $q(\theta) = (\theta(v_n))$. Clearly q is a homomorphism of A-modules. Let $w = f(1)$ where $f : A \to W$ is the map $f = h \circ q$. Then for any $\theta \in A$ we have $f(\theta) = f(\theta \cdot 1) = \theta \cdot f(1) = \theta \cdot w = \theta(w)$. For any integer $n \geq 1$ let

$$F_{n+1} = \text{vector subspace of } W \text{ spanned by } v_k \text{ for } k \geq n+1$$

and $I_n = \{\theta \in A \mid \theta(v_k) = 0 \text{ for } k \geq n+1\}$. Then for any $\theta \in I_n$ the element $q(\theta)$ is in the submodule $W_1 \oplus \ldots \oplus W_n$ of $\prod_{n\geq 1} W_n$. Since $h \mid \bigoplus_{n\geq 1} W_n = g$ we get

$$f(\theta) = h \circ q(\theta) = g(q(\theta)) = g(\theta(v_1),\ldots,\theta(v_n),0,0,\ldots)$$
$$= \theta(v_1 + \ldots + v_n) \quad \text{for every } \theta \in I_n.$$

Here $v_1 + \ldots + v_n$ denotes the sum in W. Let $\varphi_n : W \to W$ be the K-linear map determined by $\varphi_n(v_i) = v_i$ for $1 \leq i \leq n$ and $\varphi_n(v_k) = 0$ for $k \geq n+1$. Then clearly $\varphi_n \in I_n$. Hence

$$f(\varphi_n) = \varphi_n(v_1 + \ldots + v_n). \quad \text{But } f(\varphi_n) = \varphi_n(w).$$

It follows that $\varphi_n(w - \sum_{i=1}^{n} v_i) = 0$. Since $\operatorname{Ker} \varphi_n = F_{n+1}$ we get

$$w - \sum_{i=1}^{n} v_i \in F_{n+1} \qquad \ldots \quad \ldots \quad \ldots \quad (*)$$

Now, (*) should be valid for every $n \geq 1$. Since $\{v_n\}_{n\geq 1}$ is a base for W and F_{n+1} is generated by v_k for $k \geq n+1$ it is clear that $\exists$ no element $w \in W$ satisfying (*) for all $n \geq 1$. This contradiction proves that W is not an injective A-module. □

§2. Strongly regular rings

Even in the case of non-commutative rings, it is possible to find relationships between V-rings and regular rings. We prove here that for a ring whose maximal ideals are two-sided being a V-ring is equivalent to being a special kind of a regular ring. [14] and [19] are good sources for other results about the connections between V-rings and regular rings in the non-commutative case.

DEFINITION 2.1 A ring R is called a strongly regular ring if given $a \in R$, there exists an $x \in R$ such that $xa^2 = a$.

This definition is taken from [1]. In fact there, the definition was slightly different namely that given $a \in R$, there exists an

$x \in R$ such that $a^2x = a$ but the two are known to be equivalent [1]. For the sake of completeness we reproduce some of the results in [1].

LEMMA 2.2 Let R be a strongly regular ring. Then $JR(R) = 0$.

Proof Suppose $a \in JR(R)$ and $xa^2 = a$. Since $(1 - xa)$ is invertible, there exists $\mu \in R$ such that $\mu(1 - xa) = 1$. Then $\mu(1 - xa)a = a$. But $(1 - xa)a = 0$. Hence $0 = a$. Hence $JR(R) = 0$. □

LEMMA 2.3 Let R be a primitive strongly regular ring. Then R is a division ring.

Proof Since R is primitive, R is a dense subring of the ring of linear transformations of a vector space V over a division ring D. Suppose $\{e_1, e_2\}$ are linearly independent elements of V over D. We can assume that $\exists$ an $a \in R \subset \mathrm{Hom}_D(V,V)$ such that $ae_1 = 0$, $ae_2 = e_1$ from the density of R in $\mathrm{Hom}_D(V,V)$. Since $a^2(e_2) = 0$ it is impossible that $xa^2(e_2) = e_1$ for any $x \in R$. Hence any two elements in V are linearly dependent over D and hence $R = \mathrm{Hom}_D(V,V) \simeq D^0$ is a division ring. □

LEMMA 2.4 Let R be a ring and M a maximal left ideal of R. Suppose that given $a \in R$, $\exists\ b \in R$ such that ba is a central element (i.e. $bax = xba\ \forall\ x \in R$) and $ba^2 = a$. Then M is a two-sided ideal.

Proof Suppose $y \in R$, $y \notin M$ is given. We show first that $[y : M] = M$. Suppose $ay \in M$. Then let $b \in R$ be such that ba is central and $ba^2 = a$. Then $yba = bay \in M$. Since ba is central, $[ba : M] \supset M$ and since $y \notin M$ and M is maximal, $[ba : M] = R$. Hence $ba \in M$ and so $a = ba^2 = ba \cdot a = aba \in M$. Hence $[y : M] \subset M$ and since $[y : M]$ is maximal

$[y : M] = M$. Let $z \in M$ and $a \in R$. If $za \notin M$ then $a \notin M$. By what we have shown $[a : M] = M$ and hence $za \in M$, a contradiction. It follows that for any $z \in M$ and $a \in R$, $za \in M$. Hence M is a two-sided ideal. □

THEOREM 2.5 (Arens & Kaplansky [1]). Let R be a strongly regular ring. Then

(i) Given $a \in R$ there exists an $x \in R$ such that $a = a^2x = xa^2$ and $ax = xa$ is a central idempotent.

(ii) R is a regular ring.

(iii) Every maximal ideal is two-sided.

Proof Let M be a primitive (two-sided) ideal of R. Then R/M is a primitive strongly regular ring and hence a division ring by 2.3. So M is maximal. Let $\{M_\alpha\}_{\alpha \in J}$ be the set of all primitive ideals of R. Since $JR(R) = 0$ by 2.2, $\bigcap_{\alpha \in J} M_\alpha = 0$. Let $A = \prod_{\alpha \in J} R/M_\alpha$ and let $p_\alpha : A \to R/M_\alpha$ be the canonical projection maps. Since $\bigcap_{\alpha \in J} M_\alpha = 0$, there exists a (1-1) ring homomorphism $\mu : R \to A$. Given $a \in R$, there is an $x \in R$ such that $a = xa^2$. Now suppose $p_\alpha\mu(a) \neq 0$, for some α. Since $p_\alpha\mu(xa^2 - a) = p_\alpha(\mu(xa - 1)) \cdot p_\alpha(\mu(a)) = 0$, it follows that $p_\alpha\mu(xa - 1) = 0$. Hence $p_\alpha\mu(xax - x) = p_\alpha(\mu(xa - 1)) \cdot p_\alpha\mu(x) = 0$. If $p_\alpha \cdot \mu(x) = 0$, then $p_\alpha\mu(xa) = 0$, hence $p_\alpha\mu(xa - 1) = 1_\alpha \neq 0$, a contradiction. So $p_\alpha\mu(x) \neq 0$. Also $p_\alpha\mu(xax - x) = p_\alpha\mu(x) \cdot p_\alpha\mu(ax - 1)$. It follows that $p_\alpha\mu(ax - 1) = 0$. Hence $p_\alpha\mu(x) \cdot p_\alpha\mu(a) = 1_\alpha = p_\alpha\mu(a) \cdot p_\alpha\mu(x)$ {where 1_α = the unit element of R/M_α}. On the other hand if $p_\alpha\mu(a) = 0$, then $p_\alpha\mu(ax) = p_\alpha\mu(xa) = 0$. So $p_\alpha(ax) = p_\alpha(xa)$ $\forall$ $\alpha \in J$ and further $p_\alpha\mu(xa) =$ either 1_α or 0. Hence $p_\alpha\mu((xa)^2) = p_\alpha\mu(xa)$ $\forall$ $\alpha \in J$. Further, $\forall$ $b \in R$

$p_\alpha\mu(xab) = p_\alpha(\mu(xa)\mu(b)) = p_\alpha\mu(xa)p_\alpha\mu(b) = p_\alpha\mu(b)p_\alpha\mu(xa) = p_\alpha\mu(bxa)$ $\forall\ \alpha \in J$. From this we have $xa = ax$ is central and hence that $xa^2 = a^2x = a$ and $axa = a^2x = a$. This proves (i) and (ii). (iii) follows from (i) and 2.4. □

LEMMA 2.6 Let M be a maximal left ideal of R, $V = R/M$. If M is two-sided, for any $0 \neq v \in V$, we have $l(v) = M$.

Proof $l(v) = [v : M]$ is maximal since $Rv = V$ is simple. If M is two-sided and $v \neq 0$, $R \neq [v : M] \supset M$ and hence $l(v) = M$. □

PROPOSITION 2.7 Let R be a regular ring, $M \subset R$ a maximal left ideal. Suppose M is two-sided. Then $V = R/M$ is injective as an R-module.

Proof Let $I \subset R$ be any left ideal and $f : I \to R/M = V$ any homomorphism. Let $m \in I \cap M$. Since R is regular, there exists an $x \in R$ such that $mxm = m$. Since $xm \in I \cap M$, we have $f(m) = f(mxm) = mf(xm)$. But by 2.6, $mV = 0\ \forall\ m \in M$. Hence $f(m) = mf(xm) = 0$. Hence if $I \cap M = I$, we have $f = 0$. If $I \cap M \neq I$ then $I \cap M$ is maximal in I. Suppose $a \in I$, $a \notin I \cap M$ with $aya = a$ for $y \in R$. Define $g : R \to V$ by $g(r) = rf(ya)$ for any $r \in R$. Given any $i \in I$, $i = \lambda a + m$ for some $\lambda \in R$ and $m \in I \cap M$ (since $I \cap M$ is maximal in I). So $f(i) = \lambda f(a) + f(m) = \lambda f(a)$. Also $g(i) = g(\lambda a + m) = (\lambda a + m) \cdot f(ya)$ $= \lambda af(ya) = \lambda f(aya) = \lambda f(a)$. Thus $g|I = f$. □

THEOREM 2.8 Let R be a ring such that all maximal left ideals M of R are two-sided. Then the following are equivalent.

(i) R is regular.

(ii) R is a V-ring.

(iii) R is strongly regular.

Proof (i) $\Rightarrow$ (ii). From proposition 2.7, every simple R-module is injective hence R is a V-ring.

(ii) $\Rightarrow$ (iii). It follows immediately from 1.3 (ii) that given any $N \subset M$, $N = \cap \{N' | N' \supset N$ and N' maximal submodule of $M\}$. Now suppose $a \in R$ is given. If M is a maximal ideal of R and $a^2 \in M$, then $a \in [a : M] = M$ from 2.6. So $M \supset Ra^2 \Rightarrow M \supset Ra$. Since $Ra^2 = \cap \{M | M \supset Ra^2$ and M maximal in $R\}$ it follows that $Ra^2 \supset Ra$ and hence $\exists\ x \in R$ such that $xa^2 = a$.

(iii) $\Rightarrow$ (i). Immediate from (ii) of theorem 2.5. □

REMARK 2.9 This theorem implies the result of Kaplansky in the commutative case.

REMARK 2.10 Since every left ideal in a V-ring is the intersection of maximal left ideals containing it, we obtain that every left ideal in a strongly regular ring is two-sided. This can also be deduced directly from theorem 2.5.

REMARK 2.11 Strong regularity is a left-right symmetric concept so a strongly regular ring is also a right V-ring.

REMARK 2.12 In their paper [19] G.O. Michler and O.E. Villamayor show that the property of being a left (or right) V-ring is a Morita invariant property. It is easy to see that strong regularity is not a Morita invariant property. In fact if K is a field and $A = M_n(K)$ the ring of $n \times n$ matrices over K, then K and A are Morita equivalent.

K is strongly regular but for $n \geq 2$ A is not strongly regular since there are elements satisfying $a \neq 0$, $a^2 = 0$ for instance the matrix (a_{ij}) where $a_{12} = 1$, $a_{ij} = 0$ $\forall$ other i,j $1 \leq i, j \leq n$.

We conclude this chapter with two propositions which we require in the next chapter.

PROPOSITION 2.13 Let $\{R_\alpha\}_{\alpha \in I}$ be a family of strongly regular rings. Then $\prod_{\alpha \in J} R_\alpha = R$ is strongly regular.

Proof Obvious since multiplication in R is point-wise. □

PROPOSITION 2.14 Let R be a self-injective ring with $JR(R) = 0$ (R is self-injective if R is injective as a left R-module). Then R is a regular ring.

Proof This is an immediate consequence of Proposition 1, Chapter 4 §4 of [17]. In [17] everything is stated for right modules. We appeal to the analogous result for left modules. □

COROLLARY 2.15 Every self-injective V-ring is regular.

Proof Immediate from 1.3 (iv) and 2.14. □

The ring $\mathrm{Hom}_K(W,W)$ where W is an infinite dimensional vector space is right self-injective, and has a simple left module which is not injective. If it could be shown to be either left self-injective or having a simple non-injective right module we could prove that the converse to 2.15 is not necessarily true. (To our knowledge this question has not been settled).

CHAPTER IV *DSI*-RINGS

§1. *DSI*-Rings

DEFINITION 1.1 A ring R is called a *DSI*-ring if arbitrary direct sums of simple R-modules are injective. (Clearly every *DSI*-ring is a V-ring).

We shall first show that every *DSI*-ring is noetherian. This theorem was also derived by Byrd [7] using different techniques.

THEOREM 1.2 The following are equivalent for a ring R:

(i) R is a *DSI*-ring.

(ii) R is a noetherian V-ring.

Proof The implication R noetherian V-ring $\Rightarrow$ R a *DSI*-ring is an immediate consequence of the well-known result that direct sums of injective modules are injective over a noetherian ring [4].

To prove the implication R a *DSI*-ring $\Rightarrow$ R noetherian V-ring we have only to show that a *DSI*-ring is noetherian. Suppose on the contrary that R is not noetherian. Then there exists an infinite ascending sequence of ideals $I_1 \subset I_2 \subset I_3 \subset \ldots$ with $I_n \neq I_{n+1}$ for every n. Let $a_n \in I_{n+1}$ be such that $a_n \notin I_n$. Any *DSI*-ring is a priori a V-ring. By III 1.3 (vi) we can find a maximal left ideal J_n of R with $J_n \supset I_n$ and $a_n \notin J_n$. Let $M_n = {}^{R}/_{J_n}$. Then M_n is a simple R-module. Let $x_n = \eta_n(1)$ where $\eta_n : R \to {}^{R}/_{J_n} = M_n$ is the canonical quotient map. Since $a_n \notin J_n$ we have $a_n x_n \neq 0$. For $k \geq n+1$ we have $a_n \in I_{n+1} \subset I_k \subset J_k$.

Hence $a_n x_k = 0$ for $k \geq n+1$. Applying II, 1.1 we see that $\oplus_{n\geq 1} M_n$ is not a direct summand of $\Pi_{n\geq 1} M_n$. Since R is a *DSI*-ring by assumption, $\oplus_{n\geq 1} M_n$ has to be injective because each M_n is simple. When $\oplus_{n\geq 1} M_n$ is injective it is automatically a direct summand of $\Pi_{n\geq 1} M_n$. This contradiction shows that R has to be noetherian. □

COROLLARY 1.3 The following are equivalent for a ring R:

(i) R is a self-injective *DSI*-ring.

(ii) R is semisimple artinian.

Proof Let R be a self-injective *DSI*-ring. Then R is a self-injective *V*-ring in particular. Hence by III, 2.15 it is regular. Also from theorem 1.2, R is noetherian. It is well-known that a noetherian regular ring is semisimple Artin [17],[21].

Conversely, if R is semisimple artinian then every module over R is injective. In particular R is self-injective and direct sums of simple modules over R are injective. □

The condition that all direct sums of simple modules be injective is therefore quite a restrictive one. However, if we take a single injective simple module S, then arbitrary direct sums of copies of S are injective under a weaker hypothesis.

THEOREM 1.4 Let R be a ring in which every maximal left ideal is two-sided. Suppose M is a simple injective module over R. Let J be any indexing set and $M_\alpha = M$ for each $\alpha \in J$. Then $\oplus_{\alpha\in J} M_\alpha$ is an injective R-module.

Proof If J is finite there is nothing to prove. Let J be infinite. Let L denote the annihilator of M, namely $L = \{\lambda \in R | \lambda u = 0 \ \forall \ u \in M\}$. For any $v \in M$ let $\ell(v) = \{\lambda \in R | \lambda v = 0\}$. From III, 2.6, we get $\ell(u) = \ell(v)$ for any two non-zero elements u,v of M. Also the annihilator $L(0)$ of the zero element of M is the whole of R. It follows immediately from these, that $L(u)=L$ for any $u \neq 0$ in M. Once and for all choose an admissible well-ordering on J. Let $\underline{x} = (x_\alpha)_{\alpha \in J}$ be any element of $\prod_{\alpha \in J} M_\alpha$. We will show that $\ell(\underline{x}) \leq 2$. For this we consider two cases.

Case (1). Given any $\alpha \in J$ $\exists$ a $\beta \in J$ satisfying $\alpha \leq \beta$ and $x_\beta \neq 0$ in M. In this case it is clear that

$$P_\alpha(\underline{x}) = L \text{ for every } \alpha \in J.$$

Case (2). $\exists$ an $\alpha \in J$ such that $x_\beta = 0$ for $\beta \geq \alpha$. Then let α_0 be the least ordinal such that $x_\beta = 0$ for $\beta \geq \alpha_0$. Then it is clear that

$$P_\alpha(\underline{x}) = L \text{ for } \alpha < \alpha_0$$
$$P_\alpha(\underline{x}) = R \text{ for } \alpha \geq \alpha_0.$$

In any case $\ell(\underline{x}) \leq 2$ and hence by II, 2.7, it follows that $\oplus_{\alpha \in J} M_\alpha$ is injective. □

Let $\{K_i\}_{i \in I}$ be an infinite family of skew fields. From III, 2.13, it follows that $R = \prod_{i \in I} K_i$ is strongly regular. (Any skew-field K is strongly regular since given $a \in K$, $a^{-1}a^2 = a$). Since R is not noetherian, R is not *DSI*, but direct sums of copies of a single injective module are injective. In fact, each K_i has a simple module structure over R and it is shown in [26] that $\oplus_{i \in I} K_i$ is not an injective R-module. We now study the implications of the *DSI* condition on strongly regular rings and commutative rings.

THEOREM 1.5 A strongly regular ring R is a *DSI*-ring $\Leftrightarrow$ R is a direct product of finitely many skew fields.

Proof Assume R to be a strongly regular *DSI*-ring. Then from theorem 1.2 we see that R is noetherian. Since any strongly regular ring is a priori regular and any noetherian regular ring is semi-simple Artin, it follows that R is semi-simple Artin. Hence R is a direct sum of finitely many minimal left ideals of R. Since R is strongly regular all left ideals in R are two-sided. We will prove that each minimal ideal of R is a skew field. Let K be a minimal ideal of R and J any complementary ideal, so that $R = J \oplus K$. Then J is a maximal ideal of R and $R/J \simeq K$. Since every left ideal of R is two-sided, J is also a maximal left ideal of R. Hence in R/J there are no left ideals except R/J and 0. Hence $R/J \simeq K$ is a skew field. It now follows that R is a direct sum of a finite number of skew fields.

Conversely, assume that R is a direct product of finitely many skew fields. Each skew field is trivially seen to be strongly regular. By III 2.13, R itself is strongly regular. A finite direct product of skew fields is Artinian semi-simple. Hence all the modules over R are injective, in particular so are direct sums of simple modules over R. This completes the proof of theorem 1.5. □

COROLLARY 1.6 A commutative ring R is a *DSI*-ring if and only if it is a direct sum of finitely many fields.

Proof Any *DSI*-ring is a priori a *V*-ring. Since any commutative *V*-ring is regular it follows that R is regular. But in the commutative case regularity and strong regularity are equivalent. Hence R is strongly regular. Theorem 1.5 now gives the desired result. □

The results in the next two sections illustrate the relationship between artinian semi-simple rings and *DSI*-rings. Since over an artinian semi-simple ring R, every R-module M is semi-simple, we can pick a set of minimal generators for M. (Of course these generators satisfy an independence relationship as well, namely that if $\{e_\alpha\}_{\alpha\in J}$ are minimal generators of M, then $\sum_{i=1}^{n} \lambda_{\alpha_i} e_{\alpha_i} = 0 \Rightarrow \lambda_{\alpha_i} e_{\alpha_i} = 0 \ \forall\ \lambda_{\alpha_i} \in R,\ \alpha_i \in J$). We show that every module M over a *DSI*-ring R possesses a minimal generating set, but first we introduce some notations and terminology. When C is a subset of a set B we write $B \sim C$ for the complement of C in B.

§2. Irredundant and redundant subsets of a module

Given a subset A of a module M, we write $\langle A\rangle$ for the submodule of M generated by A.

DEFINITION 2.1 A subset B of M is said to be irredundant if $A \subset B,\ \langle A\rangle = \langle B\rangle \Rightarrow A = B$.

In case B is not irredundant, we will refer to it as redundant.

DEFINITION 2.2 A subset A of M is said to be proper in M if $\langle A\rangle \neq M$.

LEMMA 2.3 Let B be an irredundant subset of M, $C \subset B$ then C is also irredundant.

Proof Suppose the contrary, that is $\langle D\rangle = \langle C\rangle$ for some $D \subsetneq C$. Then $A = D \cup (B \sim C)$ satisfies $A \subsetneq B, \langle A\rangle = \langle B\rangle$ contradicting the irredundancy of B. □

LEMMA 2.4 Let B be a subset of M. Then B is redundant if and only if $b \in \langle B \sim \{b\} \rangle$ for some $b \in B$.

Proof If $b \in \langle B \sim \{b\} \rangle$ then $\langle B \rangle = \langle B \sim \{b\} \rangle$ but clearly $B \sim \{b\} \neq B$ so B is redundant. If B is redundant, there exists a $C \subsetneq B$ with $\langle C \rangle = \langle B \rangle$, hence any $b \in B \sim C$ satisfies $b \in \langle B \sim \{b\} \rangle$. □

LEMMA 2.5 Let B be any irredundant subset of M. For each $b \in B$ let M_b be a maximal submodule of M satisfying

(i) $b \notin M_b$.

(ii) $M_b \supset \langle B \sim \{b\} \rangle$.

Let $I = \langle B \rangle$, $N = \bigcap_{b \in B} M_b$ and $j : {}^{M}/_{N} \to \prod_{b \in B} {}^{M}/_{M_b}$ the natural imbedding, that is $j(x + N) = (x + M_b)_{b \in B}$. Then j carries ${}^{I+N}/_{N}$ isomorphically onto $\bigoplus_{b \in B} {}^{M}/_{M_b}$.

Proof It suffices to check that j carries ${}^{I+N}/_{N}$ onto $\bigoplus_{b \in B} {}^{M}/_{M_b}$. Let $p_b : \prod_{b \in B} {}^{M}/_{M_b} \to {}^{M}/_{M_b}$ be the projection map for each $b \in B$. Since $\langle b \rangle + M_b = M$, $p_b \cdot j : {}^{I+N}/_{N} \to {}^{M}/_{M_b}$ is onto. If $b, b' \in B$ and $b \neq b'$ then from (ii) in the hypothesis, $p_b \cdot j(b') = 0$. The lemma follows from this and the fact that $\langle B \rangle = I$. □

LEMMA 2.6 Let $\{B_\alpha\}_{\alpha \in J}$ be a family of irredundant sets totally ordered by inclusion. Then $B = \bigcup_{\alpha \in J} B_\alpha$ is irredundant.

Proof Suppose B is redundant. From lemma 2.4 there exists a $b \in B$ with $b \in \langle B \sim \{b\} \rangle$. Then we can find $b_1, \ldots, b_r$ in $B \sim \{b\}$ with b in $\langle b_1, \ldots, b_r \rangle$. Since $\{B_\alpha\}_{\alpha \in J}$ is totally ordered by inclusion, there exists an $\alpha \in J$ such that $b, b_1, \ldots, b_r$ all lie in B_α. Then

$b \in \langle b_1,\ldots,b_r\rangle \subseteq \langle B_\alpha \sim \{b\}\rangle$ and by lemma 2.4, B_α itself is redundant, contradicting our assumption. □

DEFINITION 2.7 A subset B of M is called a minimal generating set for M if $\langle B\rangle = M$ and $A \subsetneq B \Rightarrow \langle A\rangle \neq M$.

If B is a minimal generating set it is clear that B is irredundant, and that conversely, if B is an irredundant subset of M with $\langle B\rangle = M$, then B is a minimal generating set for M.

THEOREM 2.8 The following conditions on a ring R are equivalent:

(i) R is a noetherian V-ring.

(ii) Given any minimal generating set of a submodule N of any module M, it can be extended to a minimal generating set for M. In particular for every R-module M there exists a minimal generating set.

Proof (i) ⇒ (ii). Let C be a minimal generating set for N. Let $E = \{B \mid B \text{ an irredundant subset of } M, B \supseteq C\}$. Clearly $C \in E$. Partially ordering E by inclusion, it follows from lemma 2.6 that every totally ordered subset has an upper bound. Hence, by Zorn's lemma there exists a maximal irredundant subset $B \supseteq C$ in M. Suppose $\langle B\rangle \neq M$. Since B is irredundant, from lemma 2.4 we get $b \notin \langle B \sim \{b\}\rangle \ \forall\ b \in B$. From III 1.3 (ii), there exist maximal submodules M_b of M with $\langle B \sim \{b\}\rangle \subseteq M_b \not\ni b \ \forall\ b \in B$. Let I, N, j be as in lemma 2.5. The proof now breaks into two cases.

Case (a) $I \not\supset N$. Then we can pick $u \in N$ with $u \notin I$. Then consider the set $B' = B \cup \{u\}$. From $u \notin I = \langle B \rangle$ we see that $B' \neq B$. Also for any $b \in B$, we have $\langle B' \sim \{b\} \rangle \subset \langle B \sim \{b\} \rangle + N \subset M_b$ and $b \notin M_b$. Thus $b \notin \langle B' \sim \{b\} \rangle$. Moreover, $u \notin I = \langle B \rangle = \langle B' \sim \{u\} \rangle$ and so by lemma 2.4, B' is irredundant, contradicting the maximality of B in E.

Case (b) $N \subset I$. In this case $^{I+N}/_{N} = {}^{I}/_{N} \neq {}^{M}/_{N}$ since by assumption $I \neq M$. Since j is a monomorphism, $j({}^{I}/_{N}) \subseteq j({}^{M}/_{N})$. By lemma 2.5, $j({}^{I}/_{N}) = \bigoplus_{b \in B} {}^{M}/_{M_b}$. Each of the modules ${}^{M}/_{M_b}$ is simple, and since R is a V-ring ${}^{M}/_{M_b}$ is injective ($\forall\, b \in B$), and since R is noetherian, $\bigoplus_{b \in B} {}^{M}/_{M_b}$ is injective. Since ${}^{I}/_{N} \simeq \bigoplus_{b \in B} {}^{M}/_{M_b}$ it follows that ${}^{I}/_{N}$ is an injective submodule of ${}^{M}/_{N}$. Hence ${}^{M}/_{N} = {}^{I}/_{N} \oplus {}^{I'}/_{N}$ for some submodule I' of M with $I \cap I' = N$. Since ${}^{I}/_{N} \neq {}^{M}/_{N}$, ${}^{I'}/_{N} \neq 0$; hence $I' \neq N$. Let $u \in I'$, $u \notin N$. Since $I' \cap I = N$ it follows that $u \notin I$.

Consider $B' = B \cup \{u\}$. For any $b \in B$ we claim that $b \notin \langle B' \sim \{b\} \rangle$. In fact $b \in \langle B' \sim \{b\} \rangle \Rightarrow b = v + \lambda u$ for some $v \in \langle B \sim \{b\} \rangle$ and $\lambda \in R$. Then $\lambda u = b - v \in \langle B \rangle \cap I' = I \cap I' = N$. Hence $b = v + \lambda u \in \langle B \sim \{b\} \rangle + N \subset M_b$ contradicting the fact that $b \notin M_b$. Thus $b \notin \langle B \sim \{b\} \rangle$. As already seen, $u \notin I = \langle B \rangle$ and hence by lemma 1.4, B' is irredundant contradicting the maximality of B in E. Hence $B = M$ and this completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). It suffices to show that direct sums of countable families of simple modules are injective, by I, 2.7 (ii). Let $\{S_k\}_{k \geq 1}$ be a countable family of simple modules and $S = \bigoplus_{k \geq 1} S_k$. Let $E(S)$ be the injective hull of S. Let $0 \neq x_k \in S_k$. Then $C = \{x_k \mid k \geq 1\}$ is a minimal generating set for S. From (ii) we get a minimal generating

set B of $E(S)$ with $B \supseteq C$. Suppose $B \neq C$. Let $b \in B \sim C$. By lemma 2.3, $\{b\}$ is irredundant, hence $b \neq 0$. (If $b = 0$, $b \in \langle\Phi\rangle = 0$). Since S is large in $E(S)$, $Rb \cap S \neq 0$. Let $0 \neq \mu b = \sum_{k=1}^{n} \lambda_k x_k$ with $\lambda_n x_n \neq 0$. Since S_n is simple, there exists a $\lambda \in R$ with $\lambda\lambda_n x_n = x_n$. Then $x_n = \lambda\lambda_n x_n = \lambda(\mu b - \sum_{k=1}^{n-1} \lambda_k x_k) \in \langle x_1,\ldots,x_{n-1},b\rangle$, but this means $x_1,\ldots,x_n$ b is redundant and hence by lemma 2.3 that B itself is redundant, a contradiction. So $B = C$ and hence $E(S) = S$. □

§3. Cozzens' Example

We conclude this chapter with an example of a *DSI*-ring which is not regular. This example was given by Cozzens [8]. First we introduce some terminology. Let R be a ring, $\varphi : R \to R$ a ring monomorphism.

DEFINITION 3.1 A map $D : R \to R$ is called a φ-derivation if

(i) $D(a + b) = D(a) + D(b)$

(ii) $D(ab) = \varphi(a)D(b) + D(a)b$.

If φ is the identity of R we call D an ordinary derivation.

Let D be a φ-derivation of R, $R[X]$ the polynomial ring over R.

DEFINITION 3.2 The ring of differential polynomials over R with φ-derivation D, $R[X,\varphi,D]$ is defined as follows:

(i) $R[X,\varphi,D]$ has the same elements and addition as $R[X]$.

Multiplication in R is the natural extension of:

(ii) aX for any $a \in R$ is the same as aX of $R[X]$;

(iii) $Xa = D(a) + \varphi(a)X$ for any $a \in R$.

Again, if φ is the identity we write $R[X,D]$ for $R[X,Id_R,D]$.

We now take R to be a field k, D to be an ordinary derivation, and look at the ring of differential polynomials $k[X,D]$.

PROPOSITION 3.3 $R = k[X,D]$ is a left principal ideal domain.

Proof It is sufficient to show that given $f,g \in R$, $\exists\ q,r \in R$ such that $\deg r < \deg g$ and $f = qg + r$. ($\deg f$ = degree of f is exactly the same as in the case of polynomial rings). The proof is almost exactly analogous to the case of the usual polynomial ring over a field.

If $\deg f < \deg g$ we can set $q = 0$, $r = f$. If $\deg f \geq \deg g$ we proceed by induction on the degree of f. Let $\deg f = n$. If $n = 1$ then $f = aX + b$, $g = cX + d$ and $f = ac^{-1}g + b - ac^{-1}d$ where $a,b,c,d \in k$, $c \neq 0$, or $f = ad^{-1}Xg + b - ad^{-1}D(d)$ if $c = 0$ and the proposition holds. Suppose the proposition has been proved for all integers $< n$. Let $f = \sum_{i=0}^{n} \lambda_i X^i$ and $g = \sum_{j=0}^{m} \mu_j X^j$ with $\lambda_i,\mu_j \in k$, $0 \leq i \leq n$, $0 \leq j \leq m$, $m \leq n$, $\mu_m \neq 0$. Then

$$\begin{aligned}\lambda_n\mu_m^{-1}X^{n-m}g &= (\lambda_n\mu_m^{-1}X^{n-m})\sum_{j=0}^{m}\mu_j X^j \\ &= \lambda_n\mu_m^{-1}\sum_{j=0}^{m}\sum_{r=0}^{n-m}\binom{n-m}{r}D^r(\mu_j)X^{n-m-r+j} \\ &= \lambda_n X^n + \sum_{r=0}^{n-1}\nu_r X^r \qquad \nu_r \in k,\ 0 \leq r \leq n-1.\end{aligned}$$

Hence $f - \lambda_n\mu_m^{-1}X^{n-m}g = \sum_{r\ 0}^{n-1}\nu_r' X^r = f'$ (say) where $\nu_r' \in k$ for $0 \leq r \leq n-1$. By the induction hypothesis, $f' = q'g + r$ where $\deg r < \deg g$ for some $q' \in R$ and hence $f = qg + r$ where $q = q' + \lambda_n\mu_{m-1}X^{n-m}$ and $\deg r < \deg g$. □

We define a left R-module structure on k as follows: Given $a \in k$, $f = \sum_{i=0}^{n}\lambda_i X^i \in R$ we set $fa = \sum_{i=0}^{n}\lambda_i D^i(a)$. This is a well

defined operation and the only bit of checking which is not absolutely obvious is that $(fg)a = f(ga)$ $\forall$ $f,g \in R$, $a \in k$. To check this we notice that if $f = \sum_{i=0}^{n} \lambda_i X^i$, $g = \sum_{j=0}^{m} \mu_j X^j$ with λ_j, μ_j in k, then $fg = \sum_{i=0}^{n} \sum_{j=0}^{m} \lambda_i \sum_{r=0}^{i} \binom{i}{r} D^r(\mu_j) X^{i+j-r}$.

Hence $$(fg)a = \sum_{j=0}^{m} \sum_{i=0}^{n} \lambda_i \binom{i}{r} D^r(\mu_j) D^{i+j-r}(a).$$

Now $$ga = \sum_{j=0}^{m} \mu_j D^j(a).$$

Hence $$f(ga) = \sum_{i=0}^{n} \lambda_i D^i \left(\sum_{j=0}^{m} \mu_j D^j(a) \right)$$

$$= \sum_{j=0}^{m} \sum_{i=0}^{n} \lambda_i D^i \left(\mu_j D^j(a) \right)$$

$$= \sum_{j=0}^{m} \sum_{i=0}^{n} \lambda_i \sum_{r=0}^{i} D^r(\mu_j) D^{i+j-r}(a).$$

So $(fg)a = f(ga)$ and k is an R-module.

LEMMA 3.4 Under the above left R-module structure, k is a simple R-module.

Proof Trivial.

Let D be an ordinary derivation over k and $R = k[X,D]$.

DEFINITION 3.5 An equation of the kind $\sum_{i=0}^{n} a_i D^i(y) = b$ where $b, a_i \in k$, $0 \leq i \leq n$, is known as a linear differential equation in D over k. (y is assumed to be a free variable). If $b = 0$ the equation is called homogeneous. In this case n is assumed to be > 0 so that we can meaningfully talk of solutions.

PROPOSITION 3.6 Let k be a field, D a derivation of k. Assume that every homogeneous linear differential equation over k has a non-zero solution in k. Then if $f \in k[X,D] = R$ (the ring of differential polynomials) and $\deg f = n \geq 1$, then there exists $g \in R$ such that

(i) $\deg g \leq n-1$

(ii) $af = (X + 1)g$ where $0 \neq a \in k$.

Proof Let $f = \sum_{i=0}^{m} a_i X^i$. We set $g = \sum_{j=0}^{m-1} b_j X^j$ and solve for $a, b_j \in k$; $0 \leq j \leq m-1$. For the lemma to hold we require that $af - (x + 1)g = 0$. Expanding and equating coefficients we get a series of equations:

$$\left.\begin{aligned} aa_0 - D(b_0) - b_0 &= 0 \\ aa_1 - b_0 - D(b_1) - b_1 &= 0 \\ &\vdots \\ aa_{m-1} - b_{m-2} - D(b_{m-1}) - b_{m-1} &= 0 \\ aa_m + b_{m-1} &= 0 \end{aligned}\right\}$$

We can solve for b_{m-1} from the last equation substitute in the one above and solve for b_{m-2} in terms of $a, a_m, a_{m-1}, D(a), D(a_{m-1})$ and proceeding on like this, we end up with an equation of the type $\sum_{i=0}^{m} c_i D^i(a) = 0$ where the c_i, $0 \leq i \leq m$ are dependent only on the a_i and their derivatives. Since by hypothesis $\sum_{i=0}^{m} c_i D^i(y) = 0$ has a non-trivial solution, we can find an $a \in K$ satisfying $\sum_{i=0}^{m} c_i D^i(a) = 0$ and substituting back in the equations (1) obtain b_i, $0 \leq i \leq m-1$. □

COROLLARY 3.7 The only irreducible elements of $R = k[X,D]$ are of the type $X - \alpha$ where $\alpha \in k$. Hence every simple module V over R is of type $R/R(X-\alpha)$.

Proof From lemma 3.6, it is clear that irreducible elements are precisely of this form. Since in a left principal ideal domain all maximal left ideals are generated by irreducible elements it follows that every simple module is of type $R/R(X-\alpha)$. □

LEMMA 3.8 Every simple $R = k[X,D]$-module V is divisible.

Proof Let $V = R/R(X-\alpha)$. We show V is divisible by induction on the degree of elements in R. Let $f \in R$ be given. If $\deg f = 0$, f is a unit and there is nothing to prove.

(i) $\deg f = 1$. Let $f = \gamma X + \delta$. Then $\gamma^{-1}f = X + \gamma^{-1}\delta$ and since divisibility of V by f is equivalent to divisibility by $\gamma^{-1}f$ we assume that $f = X - \beta$ without loss of generality. Let $\eta : R \to V$ be the canonical quotient map, and suppose $v \in V$ is given. We pick $h \in k \subset R$ such that $\eta(h) = v$. (This is possible since $f \in \eta^{-1}(v) \Rightarrow f = g(X - \alpha) + h$ where $\deg h < 1 \Rightarrow \deg h = 0$). By hypothesis there exists an $a \in k$ such that $D(a) - a(\beta - \alpha) = h$. Then $(X - \beta)a = a(X - \alpha) + h$ and so $(X - \beta)\eta(a) = \eta(h) = v$. Since v and f were arbitrary this proves that V is divisible by all elements of order 1.

(ii) $\deg f = m \geq 2$. Inductively we assume V is divisible by all g where $\deg g \leq m-1$. Now let $af = (X + 1)g$ where $\deg g \geq m-1$. Then $fV = a^{-1}(X + 1)gV = a^{-1}(X + 1)V$ since V is divisible by g and since a^{-1} is a unit $fV = V$, that is f divides V. Hence V is divisible. □

We can now prove the theorem of Cozzens about when $k[X,D]$ is a V-ring.

THEOREM 3.9 [Cozzens]. Let k be a field, $R = k[X,D]$ the ring of differential polynomials in D. Then the following are equivalent:

(i) R is a V-ring with a single simple module upto isomorphism.

(ii) Every linear differential equation in D over k has a solution and every homogeneous linear differential equation has a non-trivial solution.

Proof (i) $\Rightarrow$ (ii). k is a simple R-module by lemma 3, and hence injective. Let $\sum_{i=0}^{n} a_i D^i(y) = b$ be any linear differential equation where $b, a_i \in k$, $0 \leq i \leq n$. Let $\sum_{i=0}^{n} a_i X^i = f \in R$. Then since R is a domain and k is injective, k is divisible by f and hence $\exists\ c \in k$ such that $fc = b$. But $fc = \sum_{i=0}^{n} a_i D^i(c)$ by the module structure on k, (see lemma 3) and hence $\sum_{i=0}^{n} a_i D^i(y) = b$ has a solution. If $b = 0$, we have to show we can choose $c \neq 0$ in the above argument. We can assume if $b = 0$ that $\deg f \geq 1$ and hence that f is not a unit. Let $R \supset M \supset Rf$ where M is a maximal ideal of R. Let $\varphi : R/M \to k$ be an isomorphism (exists by hypothesis) and $\eta : R \to R/M$ the canonical quotient map. Then $\varphi\eta : R \to k$ is an epimorphism. Let $c = \varphi\eta(1)$. Then $c \neq 0$, but since $\eta f = 0$, $0 = \varphi\eta(f) = fc = \sum_{i=0}^{n} a_i D^i(c)$.

(ii) $\Rightarrow$ (i). Under the hypothesis we have shown that every simple R-module V is divisible and since R is a principal ideal domain divisibility is equivalent to injectivity. It remains to show that any two simple R-modules V_1 and V_2 are isomorphic. To prove this it is sufficient to show that there exists a non-zero homomorphism $\varphi : V_1 \to V_2$.

Let $V_1 = {}^{R}/_{R(X-\alpha)}$, $V_2 = {}^{R}/_{R(X-\beta)}$. Again let $a \in k$ be a solution of $D(a) + a(\beta - \alpha) = 0$. Then $(X - \alpha)a = a(X - \beta)$. Let $\eta_1 : R \to V_1$, $\eta_2 : R \to V_2$ be the canonical quotient maps and define $\varphi : V_1 \to V_2$ by $\varphi(\eta_1(r)) = \varphi(\eta_2(ra))$ $\forall$ $r \in R$. It is easy to see that φ is a non-zero homomorphism. □

COROLLARY 3.10 If a field k and a derivation D satisfy the hypotheses of theorem 3.9 then $k[X,D]$ is a V-ring which is not regular.

Proof Any regular domain is a skew field. $k[X;D]$ is a domain but $f \in k[X;D]$ is not invertible if $\deg f \geq 1$. Hence $k[X;D]$ is not regular. □

We now only have to demonstrate a field k and a derivation D of k satisfying the condition (ii) in theorem 3.9. For this, we quote a result of Kolchin [16] which states: Given a field k of characteristic 0 and a derivation D of k, there exists a field $U \supset k$ and a derivation $\bar{D}$ of U extending D, such that the equation $p(y,\bar{D}(y),\ldots,\bar{D}^n(y)) = 0, n$ arbitrary has a solution in U for all polynomials $p(X_1,\ldots,X_{n+1}) \in U[X_1,\ldots,X_{n+1}] - U$. Further, every homogeneous linear differential equation in $\bar{D}$ over U has a non-zero solution over U.

Such a field U is called a universal differential field and $U,\bar{D}$ clearly satisfy (ii) of theorem 3.9, hence $U[X,\bar{D}]$ is a V-ring which is not regular. □

CHAPTER V *M*-ESSENTIAL EXTENSIONS AND A GENERALISATION OF BUMBY'S RESULT

§1. *M*-injectivity and *M*-injective hulls

The notions of *M*-injectivity and *M*-projectivity have been introduced and studied by F. Sandomiersky [23] and G. Azumaya [2]. These notions attempt to generalise the classical definitions of injectivity and projectivity in terms of an arbitrary module M in the category of modules over a ring R.

DEFINITION 1.1 Let R be a ring, M an R-module. Given any other R-module N, we say N is M-injective if for every diagram $0 \longrightarrow M' \xrightarrow{i} M$, with f from M' to N, with $0 \to M' \xrightarrow{i} M$ exact there exists $g : M \to N$ such that $g \circ i = f$.

M-projectivity is defined dually.

It is clear that if we replace M by the left R-module R, the definition of R-injectivity is exactly the same as injectivity. A module M which is itself M-injective coincides with the standard definition of a quasi-injective module and perhaps merits the name self-injective. The case of *M*-projective modules is not so clear. It is not certain whether it is possible to pick an R-module M such that the class of projective R-modules and the class of M-projective R-modules coincide. We refer the interested reader to [3],[29] for some results in this direction.

The following definition was introduced by G. Azumaya [2].

DEFINITION 1.2 Let A,B be R-modules and $f : A \to B$ a monomorphism. Then for an R-module M, f is said to be an M-monomorphism if there exists a homomorphism $\varphi : M \to B$ with $B = \varphi(M) + f(A)$.

We now embark on some definitions and results which will allow us to characterize M-injective hulls as maximal M-essential extensions for any given R-module M. We note here that M. Ryar [22] had shown the existence of 'minimal' M-injective hulls, but that the concept of M-essentialness and the generalization of the results of Eckmann and Schopf [10] are to our knowledge, new.

DEFINITION 1.3 A homomorphism $f : A \to B$ is said to be M-admissible if given $x \in B$ there exist finitely many submodules $f(A) = B_0 \subset B_1 \subset \ldots \ldots \subset B_k$ of B and $N_1,\ldots,N_k$ of M (all depending on x) such that

(i) $x \in B_k$ and

(ii) the inclusion $B_{i-1} \hookrightarrow B_i$ is an N_i-monomorphism for $1 \leq i \leq k$.

DEFINITION 1.4 Let $A \subset B$. We say that the inclusion $A \hookrightarrow B$ is M-essential (or that B is an M-essential extension of A) if

(i) the inclusion $A \hookrightarrow B$ is M-admissible and

(ii) $C \subset B, C \cap A = 0 \Rightarrow C = 0$.

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1.5 Every homomorphism is clearly R-admissible. Hence in case $M = R$, condition (i) in definition 1.4 is superfluous. Thus an R-essential

extension is nothing but an essential extension in the usual sense. An M-essential extension is a priori essential. But the converse is not true. For instance, when $R = Z$ the ring of integers, the inclusion $Z \subset Q$ into the rationals is essential, but not Z_p-essential for any prime p.

1.6 For any $N \subset M$, any N-monomorphism $f : A \to B$ is M-admissible.

1.7 Suppose $A \subset B$ is an M-monomorphism. Let $\varphi : M \to B$ be such that $B = A + \varphi(M)$. Then for $A \subset H \subset B$ we have $H = A + \varphi(N)$ where $N = \varphi^{-1}(H)$. Thus $A \subset H$ is an N-monomorphism. Clearly $H \subset B$ is an M-monomorphism.

As an immediate consequence of the above two observations we see that if $C \subset D$ is M-admissible and $C \subset H \subset D$, then $C \subset H$ and $H \subset D$ are both M-admissible. In particular if $C \subset D$ is M-essential, then $C \subset H$ and $H \subset D$ are M-essential.

LEMMA 1.8 If $f : A \to B$ and $g : B \to C$ are M-admissible maps so is $g \circ f : A \to C$.

Proof Let $y \in C$. Since $g : B \to C$ is M-admissible, there exist finitely many submodules $g(B) = C_0 \subset C_1 \subset \ldots \ldots \subset C_l$ of C with $y \in C_l$, submodules $M_1, \ldots, M_l$ of M and maps $\varphi_j : M_j \to C_j$ satisfying $C_j = C_{j-1} + \varphi_j(M_j)$ for $1 \leq j \leq l$. Then $C_l = g(B) + \varphi_1(M_1) + \ldots + \varphi_l(M_l)$. From $y \in C_l$ we get $y = g(b) + \sum_{j=1}^{l} \varphi_j(u_j)$ with $b \in B$ and $u_j \in M_j$. Since $A \xrightarrow{f} B$ is M-admissible there exist finitely many submodules $f(A) = B_0 \subset B_1 \subset \ldots \subset B_k$ with $b \in B_k$, submodules $N_1, \ldots, N_k$ of M and maps $\theta_i : N_i \to B_i$ satisfying $B_i = B_{i-1} + \theta_i(N_i)$ for $1 \leq i \leq k$. Define submodules H_i (for $0 \leq i \leq h + l$) of C as follows:

$$H_i = \begin{cases} g(B_i) & \text{for } 0 \le i \le k \\ g(B_k) + \sum_{j=1}^{i-k} \varphi_j(M_n) & \text{for } k+1 \le i \le k+l. \end{cases}$$

Writing $N_{k+j} = M_j$ for $1 \le j \le l$ let $\alpha_i : N_i \to H_i$ for $1 \le i \le k+l$ be

$$\alpha_i = \begin{cases} g \circ \theta_i & \text{for } 1 \le i \le k \\ \varphi_{i-k} & \text{for } k+1 \le i \le k+l. \end{cases}$$

Then clearly $H_0 = gf(A)$, $H_i = H_{i-1} + \alpha_i(N_i)$ for $1 \le i \le k+l$ and $y \in H_{k+l}$. This proves that $g \circ f : A \to C$ is M-admissible. □

COROLLARY 1.9 If $A \subset B \subset C$ and $A \hookrightarrow B$, $B \hookrightarrow C$ are M-essential then $A \hookrightarrow C$ is M-essential.

Given an M-monomorphism $f : A \to B$, and a map $g : A \to H$ where H is M-injective it is proved by Azumaya [2] that there exists an $h : B \to H$ satisfying $h \circ f = g$. The following lemma is a slight extension of this result of Azumaya.

LEMMA 1.10 Let H be M-injective and $f : A \to B$ an M-admissible monomorphism. Given any $g : A \to H$ there exists a map $h : B \to H$ satisfying $h \circ f = g$.

Proof Let $\mathcal{F}$ be the family of pairs (D,φ) where $f(A) \subset D \subset B$ and $\varphi : D \to H$ satisfies $\varphi \circ f = g$. Since $f : A \to B$ is a monomorphism there exists a unique map $\theta : f(A) \to H$ such that $\theta \circ f = g$. Thus $(f(A),\theta) \in \mathcal{F}$ showing that $\mathcal{F}$ is non-empty. Partially order $\mathcal{F}$ by $(D,\varphi) < (D',\varphi')$ whenever $D \subset D'$ and $\varphi' | D = \varphi$. By Zorn's lemma there exists a maximal

element (D,h) in $\mathcal{F}$. If $D \neq B$ there exists an $x \in B$ with $x \notin D$. Since $f: A \to B$ is M-admissible there exists finitely many submodules $f(A) = B_0 \subset B_1 \subset \ldots \subset B_k$ of B, $N_1,\ldots,N_k$ of M such that $x \in B_k$ and $B_{i-1} \hookrightarrow B_i$ is an N_i-monomorphism for $1 \leq i \leq k$. It follows that $D + B_{i-1} \hookrightarrow D + B_i$ is an N_i-monomorphism for $1 \leq i \leq k$.

Since H is M-injective it follows from proposition 2.5 (4) of [23] that H is N-injective for any submodule N of M. In particular H is N_i-injective for $1 \leq i \leq k$. Observing that $D + B_0 = D$ and using the fact that $D + B_{i-1} \hookrightarrow D + B_i$ is an N_i-monomorphism we extend h successively to maps $\varphi_i : D + B_i \to H$ $(i = 1,\ldots,k)$. Then $\varphi_k : D + B_k \to H$ extends h. In particular $\varphi_k \circ f = h \circ f = g$. Moreover $x \in D + B_k$ and $x \notin D$. Thus $(D,h) < (D + B_k, \varphi_k)$ in $\mathcal{F}$, contradicting the maximality of (D,h). This shows that $D = B$. □

LEMMA 1.11 If H is an M-injective module the only M-essential extension of H is H.

Proof Let $H \hookrightarrow D$ be any M-essential extension of H. Since the inclusion $j : H \to D$ is then M-admissible, by lemma 1.10 we get a map $h : D \to H$ such that $h \circ j = Id_H$. Thus $D = H \oplus C$ for some $C \subset D$. Then $C \cap H = 0$ and from condition (ii) in definition 1.4 we get $C = 0$. Hence $D = H$. □

DEFINITION 1.12 Let A be an R-module. An R-module B such that $A \subset B$, B is M-injective and $A \hookrightarrow B$ is M-essential is called an M-injective hull of A.

LEMMA 1.13 Let $A \subset D \subset B$ be such that $A \subsetneq D$ is M-admissible. Then for any $E \subset B$ the inclusion $E + A \subsetneq E + D$ is M-admissible.

Proof Any element of $E + D$ is of the form $x + y$ with $x \in E$, $y \in D$. Since $A \subsetneq D$ is M-admissible there exist finitely many submodules $A = D_0 \subset D_1 \subset \ldots \subset D_k$ of D, $N_1, \ldots, N_k$ of M such that $y \in D_k$ and $D_{i-1} \subsetneq D_i$ is an N_i-monomorphism for $1 \leq i \leq k$. Then $E + A = E + D_0 \subset E + D_1 \subset \ldots \subset E + D_k$ are such that $x + y \in E + D_k$ and $E + D_{i-1} \subsetneq E + D_i$ is an N_i-monomorphism for $1 \leq i \leq k$. □

COROLLARY 1.14 Let $A \subset D_i \subset B$ be such that $A \subsetneq D_i$ is M-admissible for $i = 1,2$. Then $A \subsetneq D_1 + D_2$ is M-admissible.

Proof By lemma 1.11, $D_1 + A \subsetneq D_1 + D_2$ is M-admissible. But $D_1 + A = D_1$. By lemma 1.8, the composite of $A \subsetneq D_1$ and $D_1 \subsetneq D_1 + D_2$ is M-admissible. □

§2. Existence and uniqueness up to isomorphism of M-injective hulls

We first recall the following test for M-injectivity of a module H [2]. For H to be M-injective it is necessary and sufficient that, given any $u \in M$, any left ideal I of R and any homomorphism $f : Iu \to H$ there exists an extension $h : Ru \to H$ of f.

THEOREM 2.1 For any module A there exists an M-injective hull.

Proof Let E be an injective hull of A. Let $F = \{D \mid A \subset D \subset E;$ $A \subsetneq D$ is M-admissible$\}$. Then F has a unique maximal element. In fact we will show that $H = \bigcup_{D \in F} D$ is a submodule of E and that $H \in F$.

Let x,y be in H. Then there exist D_1, D_2 in $\mathcal{F}$ such that $x \in D_1$, $y \in D_2$. By corollary 1.12, $A \hookrightarrow D_1 + D_2$ is M-admissible. Hence $D_1 + D_2 \in \mathcal{F}$. Hence $x + y \in D_1 + D_2 \subset H$. For any $x \in H$ and $\lambda \in R$ it is clear that $\lambda x \in H$. Hence H is a submodule of E. Since any $x \in H$ lies in a $D \in \mathcal{F}$, from the very definition of an M-admissible map it follows that $A \hookrightarrow H$ is M-admissible.

Since $A \hookrightarrow E$ is essential and $A \subset H \subset E$ it follows that $A \hookrightarrow H$ is essential. This combined with the fact that $A \hookrightarrow H$ is M-admissible shows that $A \hookrightarrow H$ is M-essential. We will now prove that H is M-injective. Consider any map $Iu \xrightarrow{f} H$ where $u \in M$, I an ideal in R. Since E is injective (in particular M-injective) there exists an extension $g : Ru \to E$ of $j \circ f$ where $j : H \to E$ is the inclusion. Let $x = g(u) \in E$. Then clearly $H \hookrightarrow H + Rx$ is an Ru-monomorphism and $Ru \subset M$. It follows that $A \hookrightarrow H + Rx$ is M-admissible. By the maximality of H in $\mathcal{F}$ we see that $H = H + Rx$. Hence $x \in H$ and hence $g(Ru) \subset H$. Thus $g : Ru \to H$ is an extension of f. This completes the proof that H is an M-injective hull of A. □

PROPOSITION 2.2 Let $f : A \to A'$ be any monomorphism, $A \hookrightarrow B$ an M-essential extension and E' any M-injective module containing A'. Then there exists a map $h : B \to E'$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow i & & \downarrow i' \\ B & \xrightarrow{h} & E \end{array}$$

is commutative, where i and i' denote the respective inclusions. Moreover any such h is a monomorphism.

Proof The existence of h is an immediate consequence of lemma 1.8. Let $C = \ker h$. Then $C \cap A = \ker f = 0$, since f is a monomorphism. Since $A \subsetneq B$ is essential we get $C = 0$. Thus h is a monomorphism.

THEOREM 2.3 Suppose $\varphi : A \to A'$ is an isomorphism, E and E' arbitrary M-injective hulls of A and A' respectively. Then there exists a $\psi : E \to E'$ such that

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ E & \xrightarrow{\psi} & E' \end{array}$$

is commutative where i and i' denote the respective inclusions. Moreover any such ψ is an isomorphism.

In particular, if E, E' are M-injective hulls of the same module A there exists an isomorphism $\psi : E \to E'$ such that $\psi|A = Id_A$.

Proof The existence of ψ, and that any such ψ is a monomorphism follow from proposition 2.2. Then $\psi(E)$ being isomorphic to E is M-injective. Moreover $A' \subset \psi(E) \subset E'$. Since $A' \subsetneq E'$ is M-essential it follows that $\psi(E) \subsetneq E'$ is M-essential. From lemma 1.11 we now have $\psi(E) = E'$. □

REMARK 2.4 If E is an M-injective hull of A and H any M-injective module containing A, from proposition 2.2 we see that there exists a monomorphism $h : E \to H$ such that $h|A = Id_A$. Then $A \subsetneq h(E)$ is M-essen-

tial, and since $h(E)$ is M-injective the only M-essential extension of $h(E)$ is $h(E)$ itself. Thus $h(E)$ is a maximal M-essential extension of A in H.

Conversely, let H be any M-injective module containing A and F any maximal M-essential extension of A in H. Let G be any M-injective hull of F. Again by proposition 2.2 there exists a monomorphism $h : G \to H$ such that $h|F = Id_F$. It follows that $h(G)$ is an M-essential extension of F. By corollary 1.9, it follows that $A \subset\!\!\to h(G)$ is M-essential. Since $A \subset\!\!\to F$ is a maximal M-essential extension in H it follows that $F = h(G)$ and hence $G = F$. This shows that F itself is M-injective.

We shall now prove the M-injective analogue of a result of Bumby [6] which states that given two injective modules M,N, if there exist monomorphisms $f : N \to M$ and $g : M \to N$ then M and N are isomorphic.

§3. Analogue of Bumby's result

DEFINITION 3.1 A module A will be referred to as M-admissible if $0 \to A$ is M-admissible.

THEOREM 3.2 Let A and B be M-admissible, M-injective modules. Suppose each of the modules A,B admits a monomorphism into the other. Then $A \simeq B$.

Proof Without loss of generality we can assume $A \subset B$. Let $f : B \to A$ be a monomorphism. From $0 \subset A \subset B$ and the M-admissibility of $0 \to B$ we see that the inclusion $A \subset\!\!\to B$ is M-admissible. An application of lemma 1.10 immediately yields $B = A \oplus K$ for some submodule K of B. From the M-injectivity of B we see that K is also M-injective.

Let D be the submodule $\sum_{k\geq 1} f^k(K)$ of A (the sum in A of the submodules $f^k(K)$). Then it is clear that $f : D \oplus K \to D$ is onto. However, since f is a monomorphism, it follows that $f : D \oplus K \to D$ is an isomorphism. Now $D \subset A$ and A is M-injective. Hence there exists an M-injective hull E of D in A. Then clearly $E \oplus K$ is an M-injective hull of $D \oplus K$ in $A \oplus K = B$. By theorem 2.3, the isomorphism $f : D \oplus K \to D$ extends to an isomorphism $\beta : E \oplus K \to E$. From the M-admissibility of $0 \to A$ it follows that $E \subsetneq A$ is M-admissible. Again an application of lemma 1.10 yields $A = E \oplus L$ for some $L \subset A$. The map $h : \beta \oplus Id_L : E \oplus K \oplus L \to E \oplus L$ is an isomorphism. But $E \oplus K \oplus L \simeq E \oplus L \oplus K = A \oplus K = B$. Thus h gives rise to an isomorphism of B onto A. □

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3.3 When R is considered as a left module over itself in the usual way the notion of an R-injective module in the sense of Azumaya is the same as that of an injective module over R. Moreover any homomorphism $f : A \to B$ is R-admissible in the sense of definition 1.1. Hence every module is R-admissible. Thus theorem 3.2 in the special case when $M = R$ gives Bumby's result.

3.4 For the validity of theorem 3.2 we cannot remove the restrictions that A and B be M-admissible. In fact when M is a simple (or a semi-simple) module every module over R is M-injective. Choosing R to be the ring of integers and $M = Z_p$ where p is a prime we see that any Z-module is Z_p-injective. Let $A = \bigoplus_{n\geq 1} Q$ a countable direct sum

of copies of Q (rationals) and $B = Z \oplus A$. Then each of A and B admits a monomorphism into the other. Clearly, $A \not\cong B$, since A is divisible and B is not.

CHAPTER VI *PCI*-RINGS AND AN EXTENSION OF OSOFSKY'S RESULT

This chapter is devoted to extending the following theorem due to B.L. Osofsky. A ring R is artinian semi-simple $\Leftrightarrow$ every cyclic module is (R-)injective. We prove that a cyclic R-module M is semi-simple artinian $\Leftrightarrow$ every cyclic R-module is M-injective and $\mathrm{End}_R(M)$ is regular. Since one of the first steps in Osofsky's proof is that $R = \mathrm{End}_R(R)$ is regular if every cyclic R-module is injective, the condition $\mathrm{End}_R(M)$ regular in our proof seems necessary. In main, we follow [20], using the results of the previous chapter to replace injective hulls by M-injective hulls. The only other main modification required is the replacement of a result of Johnson [see (20] by our proposition 2.5.

Before proceeding with the main results of this chapter, we explain our motivation by proving a hypothetical 'theorem' dealing with V-rings. C. Faith [12] has introduced the notion of a *PCI*-ring, that is, a ring R such that R/I is injective for all ideals $I \neq 0$ (proper cyclics injective). Clearly every *PCI*-ring is a V-ring. The example of a *DSI*-ring given by Cozzens [8] in Chapter II is also that of a *PCI*-ring. Faith proves in [12] that a *PCI*-ring is either artinian semi-simple or a simple ore domain and then poses the question whether these domains are noetherian, in other words, are all *PCI*-rings *DSI*? We prove the following implication. There exists a *PCI*-ring R which is not *DSI* $\Rightarrow$ there exists a cyclic R-module M such that every cyclic R-module is M-injective but M itself not semi-simple.

§1. *PCI*-Rings

LEMMA 1.1 Let R be a *PCI*-ring which is not *DSI*. Then there exists a cyclic module M such that $\mathrm{Soc}(M)$ is an infinite direct sum of simple modules and is essential in M.

Proof Since R is not *DSI* let $\{S_i\}_{i=1}^{\infty}$ be a collection of simple R-modules whose direct sum is not injective. Let $S = \bigoplus_{i=1}^{\infty} S_i$ and let $E(S)$ be the injective hull of S. Let $0 \neq x \in E(S)$, $x \notin S$. Then $0 \neq Rx \cap S$ is the socle of Rx. Suppose that the socle of Rx is a finite direct sum of simple modules. Then, since R is *PCI*, it follows that $\mathrm{Soc}Rx$ is injective. Hence $Rx = \mathrm{Soc}Rx \oplus T$ for some $T \subset Rx$. We have $T \cap S \subset Rx \cap S = \mathrm{Soc}Rx$. Thus $T \cap S \subset T \cap \mathrm{Soc}Rx = 0$. Since S is essential in $E(S)$, it follows that $T = 0$. Hence $Rx = \mathrm{Soc}Rx$. But then $Rx = \mathrm{Soc}Rx \subset \mathrm{Soc}E(S) = S$. This contradicts $x \notin S$. It follows that $\mathrm{Soc}Rx$ is an infinite direct sum of simple modules. Also $\mathrm{Soc}Rx = Rx \cap S$ is essential in $Rx \cap E(S) = Rx$. □

LEMMA 1.2 Let R be a *PCI*-ring which is not *DSI*. Let M be any R-module with $\mathrm{Soc}M$ essential in M. Then $\mathrm{Hom}_R(N,R) = 0$ for all submodules N of M.

Proof In the proof we will use the result due to C. Faith [12] which asserts that if R is a *PCI*-ring which is not *DSI* then R is a simple ore domain. Let $S = \mathrm{Soc}M$. Then for any submodule N of M, $\mathrm{Soc}N = S \cap N$ and $S \cap N$ is essential in N. Let $N \subset M$ and $f \in \mathrm{Hom}_R(N,R)$.

If $f \neq 0$, there exists an $x \in N$ with $f(x) \neq 0$. Since R is a domain we have $L(f(x)) = 0$. It follows that $L(x) = 0$ and that $R \simeq Rx$ as R-modules. But $Rx \subset M$. Hence SocRx is essential in Rx. It follows that Soc$R$ is essential in R. However Soc$R$ is a two-sided ideal of R. Since R is simple, either Soc$R$ = 0 or Soc$R$ = R. If Soc$R$ = 0, it is not essential in R. Hence Soc$R$ = R. Then R is a semi-simple artinian ring and hence *DSI*, contradicting the hypothesis of the lemma. Thus Hom(N,R) = 0 for all $N \subset M$. □

PROPOSITION 1.3 Let R be a *PCI*-ring which is not *DSI*. Then there exists a cyclic R-module M satisfying the following two conditions:

(i) Every cyclic R-module is M-injective and

(ii) M is not semi-simple.

Proof By 1.1, there exists a cyclic module M such that

(a) Soc$M$ is an infinite direct sum of simple modules and

(b) Soc$M$ is essential in M.

Since M is cyclic, and Soc$M$ an infinite direct sum we have Soc$M \neq M$. Thus M is not semi-simple.

Let N be any submodule of M. Let A be any cyclic R-module and $f : N \to A$ any homomorphism. If $A \simeq R$, from lemma 1.2, we get $f = 0$. Hence the zero homomorphism $0 : M \to A$ extends f. If $A \not\simeq R$, then A is a proper cyclic module and hence injective. Hence there exists an extension $g : M \to A$ of f. This proves that every cyclic R-module is M-injective. □

Thus to answer Faith's question positively we have only to show that there does not exist a non-semi-simple cyclic module M with all

cyclic modules M-injective. Though we have not completely succeeded in answering Faith's question, the motivation for the results obtained in this chapter comes from Faith's problem. Actually we prove the following result.

Let R be any ring and M an R-module such that $\mathrm{End}(M)$ is a regular ring. Then M is semi-simple $\Leftrightarrow$ every cyclic R-module is M-injective.

§2. Uniqueness of M-injective hulls

Let M be an R-module and $u \in M$.

DEFINITION 2.1 An element $e \in R$ will be referred to as a u-idempotent if $eu \neq 0$ and $Ru = Reu \oplus R(1 - e)u$.

When e is a u-idempotent and $(1 - e)u \neq 0$ then $(1 - e)$ is also u-idempotent. Moreover from $u = eu + (1 - e)u$ it follows that the projection p_e of Ru onto Reu is given by $p(ru) = reu$ for any $r \in R$. In particular $eu = e^2u$ and $l(u)e \subset l(u)$. Conversely, if $e \in R$ satisfies $0 \neq eu = e^2u$ and $l(u)e \subset l(u)$, then e is a u-idempotent. Suppose $\lambda \in R$ is any element with $\lambda u = eu$ where e is a u-idempotent. Then for any $r \in l(u)$ we have $r\lambda u = reu = 0$. Hence $l(u)\lambda \subset l(u)$. Moreover $\lambda u \in Reu$ and $p_e | Reu = Id_{Reu}$. Hence $p_e(\lambda u) = \lambda u$. In other words $\lambda eu = \lambda u$. But $\lambda(eu) = \lambda(\lambda u) = \lambda^2 u$. Thus $0 \neq \lambda u = \lambda^2 u$. This shows that λ is also a u-idempotent.

In this section we will be making repeated use of the fact, that when $A \subset B$ the inclusion $A \subsetneq B$ is essential (alternatively expressed as "A is large in B") if and only if, given $0 \neq b \in B$ there exists an $r \in R$ such that $0 \neq rb \in A$.

LEMMA 2.2 Let $e \in R$ be a u-idempotent and I an ideal in R satisfying $I \supset l(eu)$. If Ieu is large in Reu, then Iu is large in Ru.

Proof Suppose $0 \neq \lambda u \in Ru$. If $\lambda eu = 0$, then $0 \neq \lambda u = \lambda(1 - e)u \in l(eu)u \subset Iu$. If $\lambda eu \neq 0$, the largeness of Ieu in Reu yields an $r \in R$ with $0 \neq r\lambda eu \in Ieu$.

Hence $r\lambda eu = seu$ for some $s \in I$. Then $(r\lambda - s) \in l(eu) \subset I$. Hence $r\lambda = (r\lambda - s) + s \in I$. From $p_e(r\lambda u) = r\lambda eu \neq 0$ we get $r\lambda u \neq 0$. Thus $0 \neq r\lambda u \in Iu$. □

For the rest of this section and in section 3 completely, M denotes a *cyclic* module and x a fixed generator of M. For any x-idempotent $e \in R$ the projection $Rx \to Rex$ will be denoted by p_e. We write L for $l(x)$ and End(M) for the ring of endomorphisms of M. An element $\lambda \in R$ will be called "special" if $L\lambda \subset L$.

PROPOSITION 2.3 (i) If $N \neq 0$ is a direct summand of M, there exists an x-idempotent $e \in R$ such that $N = Rex$.

(ii) If every cyclic submodule of M is a direct summand of M, given any $\lambda \in R$ with $\lambda x \neq 0$, there exists a $\mu \in R$ such that $e = \mu\lambda$ is an x-idempotent and $\lambda x = \lambda ex$.

(iii) Suppose End(M) is regular (in the sense of Von-Neumann). Let $\lambda \in R$ be special and $\lambda x \neq 0$. Then there exists a special $\mu \in R$ with $e = \mu\lambda$ an x-idempotent and $\lambda x = \lambda ex$.

Proof (i) If $p : M \to N$ denotes the projection of M onto the direct summand N, then $p(x) = ex$ for some $e \in R$. It is easily checked that e is an x-idempotent and that $N = Rex$.

(ii) $R\lambda x \neq 0$ is then a direct summand of M. If $p : M \to R\lambda x$ denotes the projection of M onto $R\lambda x$ then $p(x) = \mu\lambda x$ for some $\mu \in R$. Then $e = \mu\lambda$ is an x-idempotent satisfying $\lambda x = \lambda e x$.

(iii) When $\lambda \in R$ is special, $\alpha_\lambda : M \to M$ given by $\alpha_\lambda(rx) = r\lambda x$ for any $r \in R$, is a well-defined endomorphism of M. The regularity of End(M) yields an element $\theta \in \text{End}(M)$ such that $\alpha_\lambda \circ \theta \circ \alpha_\lambda = \alpha_\lambda$. Then $\theta(x) = \mu x$ for a special $\mu \in R$. Now,

$\lambda x = \alpha_\lambda(x) = \alpha_\lambda \circ \theta \circ \alpha_\lambda(x) = \alpha_\lambda \circ \theta(\lambda x) = \alpha_\lambda(\lambda\mu x) = \lambda\mu\lambda x$. Hence $e = \mu\lambda$ satisfies $ex \neq 0$, $e^2 x = \mu\lambda\mu\lambda x = \mu\lambda x = ex$. Also $Le = L\mu\lambda \subset L\lambda \subset L$. This proves that e is an x-idempotent. □

PROPOSITION 2.4 Suppose every cyclic submodule of M is a direct summand of M and End(M) is regular. Let $0 \neq A \subset M$ and $A \subset E_i \subset M$ be such that A is large in E_i $(i = 1,2)$. Then A is large in $E_1 + E_2$.

Proof Since A is large in E_1 it suffices to prove that E_1 is large in $E_1 + E_2$. Let $0 \neq \lambda_1 x + \lambda_2 x \in E_1 + E_2$ with $\lambda_i x \in E_i$ $(i = 1,2)$. If $\lambda_2 x = 0$ then $0 \neq \lambda_1 x + \lambda_2 x \in E_1$. Let $\lambda_2 x \neq 0$. Then by (ii) of proposition 2.3, there exists a $\mu \in R$ such that $e_2 = \mu\lambda_2$ is an x-idempotent and $\lambda_2 x = \lambda_2 e_2 x$. We consider the element $\mu(\lambda_1 + \lambda_2)x$. For the proof of this proposition we have to deal with three different cases.

Case (1) $l(\mu\lambda_1 x) \neq l(\mu\lambda_2 x)$

Without loss of generality we can assume that there exists an element $r \in l(\mu\lambda_1 x)$ with $r \notin l(\mu\lambda_2 x)$. Then $r\mu(\lambda_1 + \lambda_2)x = r\mu\lambda_2 x \neq 0$. Since A is large in E_2 and $r\mu\lambda_2 x \in E_2$ we get an element $s \in R$ with $0 \neq sr\mu\lambda_2 x \in A$. Then $sr\mu(\lambda_1 + \lambda_2)x = sr\mu\lambda_2 x$ is a non-zero element of E_1.

Case (2) $l(\mu\lambda_1 x) = l(\mu\lambda_2 x)$ and $\mu(\lambda_1 + \lambda_2)x = 0$

Then $e_2 x = \mu\lambda_2 x = -\mu\lambda_1 x$. Hence $\lambda_2 x = \lambda_2 e_2 x = -\lambda_2\mu\lambda_1 x \in E_1$. It follows that $\lambda_1 x + \lambda_2 x \in E_1$ and $\lambda_1 x + \lambda_2 x$ is non zero.

Case (3) $l(\mu\lambda_1 x) = l(\mu\lambda_2 x)$ and $\mu(\lambda_1 + \lambda_2)x \neq 0$

Let $c = \mu(\lambda_1 + \lambda_2)$. Since $e_2 = \mu\lambda_2$ is an x-idempotent, we have $l(\mu\lambda_2 x) \supset L$. The equality $l(\mu\lambda_1 x) = l(\mu\lambda_2 x)$ gives $l(\mu\lambda_1 x) \supset L$. Hence for any $a \in L$, $acx = a\mu\lambda_1 x + a\mu\lambda_2 x = 0$. Thus $Lc \subset L$.

Proposition 2.3 (iii) now yields a special element $\nu \in R$ such that $e = \nu c$ is an x-idempotent and $cx = cex$.

Let $J = [\mu\lambda_2 x : E_1]$ and $J_2 = [\mu\lambda_2 x : A]$. From $A \subset E_1$ we get $J_2 \subset J$. Since $\mu\lambda_2 x \in E_2$ and A is large in E_2 it follows that $J_2\mu\lambda_2 x$ is large in $R\mu\lambda_2 x$. Since $J \supset J_2$, it follows that $J\mu\lambda_2 x$ is large in $R\mu\lambda_2 x$. But $e_2 = \mu\lambda_2$ is an x-idempotent and $J \supset [\mu\lambda_2 x : 0] = [e_2 x : 0] = l(e_2 x)$. By lemma 2.2, Jx is large in Rx.

We assert that $e\nu \notin L$. In fact, if $e\nu \in L$, from $e\nu c \in Lc \subset L$ we will have $0 = e\nu cx = eex = ex$, a contradiction. Thus $e\nu x \neq 0$. The largeness of Jx in Rx gives an element $r \in R$ with $0 \neq re\nu x \in Jx$. Since $J \supset l(e_2 x) \supset l(x) = L$, it follows that $re\nu cx = re\nu(\mu\lambda_1 x + \mu\lambda_2 x) \in E_1$. Also, $re\nu cx = reex = rex$. Moreover $\nu \in R$ being special we have $L\nu \subset L$. Since $re\nu \notin L$, it follows that $re \notin L$. Thus $0 \neq rex = re\nu cx = re\nu\mu(\lambda_1 x + \lambda_2 x) \in E_1$.

This completes the proof of proposition 2.4. □

PROPOSITION 2.5 Suppose every cyclic submodule of M is a direct summand of M, End(M) is regular and M is M-injective. Then any $A \subset M$ has a unique (actually unique) M-injective hull in M.

Proof We need to prove this proposition only when $A \neq 0$. By hypothesis M itself is M-injective. Hence by IV; 2.2, there exists an M-injective hull of A in M. Suppose E_1, E_2 are any two M-injective hulls of A in M. Then A is large in E_i $(i = 1,2)$. Every cyclic submodule of M being a direct summand of M, A is large in $E_1 + E_2$ (proposition 2.4). Since E_1 is M-injective we get $M = E_1 \oplus H$ for some $H \subset M$. If $D = (E_1 + E_2) \cap H$, then $E_1 + E_2 = E_1 \oplus D$. Since E_1 is large in $E_1 + E_2$ it follows that $D = 0$. Hence $E_2 \subset E_1$. Similarly $E_1 \subset E_2$. □

§3. Generalization of a result of Osofsky

DEFINITION 3.1 A family $\{e_\alpha\}_{\alpha \in J}$ of x-idempotents will be said to be x-orthogonal (or simply orthogonal when there is no possibility of confusion) if $e_\alpha e_\beta x = 0$ whenever $\alpha \neq \beta$ in J.

REMARK 3.2 It is easily seen that if $\{e_\alpha\}_{\alpha \in J}$ is an orthogonal family of x-idempotents in R, then the sum $\sum_{\alpha \in J} Re_\alpha x$ in Rx is direct. In fact if $re_\alpha x = \sum_{\beta \neq \alpha} \lambda_\beta e_\beta x$, then $(re_\alpha - \sum_{\beta \neq \alpha} \lambda_\beta e_\beta)x = 0$. Thus $(re_\alpha - \sum_{\beta \neq \alpha} \lambda_\beta e_\beta) \in L \subset l(e_\alpha x)$. Hence $(re_\alpha - \sum_{\beta \neq \alpha} \lambda_\beta e_\beta)e_\alpha x = 0$. In other words $re_\alpha x = 0$.

LEMMA 3.3 Suppose M has the property that every submodule of M has a unique M-injective hull in M. Let $\{e_\alpha\}_{\alpha \in J}$ be any set of orthogonal x-idempotents in R. Then for any $K \subset J$ there exists an x-idempotent $E_K \in R$ such that

$$e_\alpha E_K x = e_\alpha x \qquad \text{whenever } \alpha \in K$$
$$e_\beta E_K x = 0 = E_K e_\beta x \qquad \text{whenever } \beta \notin K$$
$$E_K x + E_{J-K} x = E_J x.$$

Proof This is the analogue of Lemma 3 of [20]. We briefly indicate the proof. For any $K \subset J$ let $(\sum_{\alpha\in K} Re_\alpha x)^*$ denote the unique M-injective hull of $\sum_{\alpha\in K} Re_\alpha x$ in M. Then $(\sum_{\alpha\in K} Re_\alpha x)^*$ is a direct summand of M. If $\varphi : M \to (\sum_{\alpha\in K} Re_\alpha x)^*$ is any map with $\varphi | (\sum_{\alpha\in K} Re_\alpha x)^* = Id_{(\sum_{\alpha\in K} Re_\alpha x)^*}$, then $\varphi(x) = e_K x$ for some x-idempotent e_K and $(\sum_{\alpha\in K} Re_\alpha x)^* = Re_K x$. Then for any $F \subset G \subset J$ we have $Re_F x \subset Re_G x$ and $Re_G x = Re_F x \oplus Re_{G-F} x$. In particular $Re_J x = Re_K x \oplus Re_{J-K} x$. If $\pi_K : Re_J x \to Re_K x$ denotes the projection and $E_K \in R$ any element with $\pi_K(e_J x) = E_K x$, then the elements $E_K \in R$ satisfy the requirements of the lemma. Actually, $Rx = Re_J x \oplus R(1 - e_J x) = Re_K x \oplus Re_{J-K} x \oplus R(1 - e_J x)$. If $p_K : Rx \to Re_K x$ denotes the projection, then $E_K x = p_K(x)$. □

We carry over the terminology from lemma 3.3 for the rest of this chapter.

LEMMA 3.4 If $bx \in RE_J x$, $bx \notin A = \sum_{\alpha\in J} Re_\alpha x$, then $Rbx \cap A \notin \sum_{i=1}^{n} Re_{\alpha_i} x$ for any finite subset $\{\alpha_i\}_{i=1}^{n} \subset J$.

Proof Suppose $Rbx \cap A \subset \sum_{i=1}^{n} Re_{\alpha_i} x$ for some $\alpha_i \in J$, $1 \le i \le n$. Rbx is the M-injective hull of $Rbx \cap A$. Since $\sum_{i=1}^{n} Re_{\alpha_i} x$ is M-injective and contains $Rbx \cap A$, from the uniqueness of M-injective hulls it follows that $Rbx \subset \sum_{i=1}^{n} Re_{\alpha_i} x \subset A$, a contradiction. □

LEMMA 3.5 Suppose $ax = \sum_{i=1}^{n} \nu_i e_{\alpha_i} x \in \sum_{j=1}^{m} RE_{A_j} x$ where $\alpha_i \in J$, $1 \le i \le n$, $A_j \subset J$, $1 \le j \le m$, $\nu_i e_{\alpha_i} x \neq 0 \ \forall\ 1 \le i \le n$. Then $\alpha_i \in \bigcup_{j=1}^{m} A_j \ \forall\ 1 \le i \le n$.

Proof First suppose $j = 1$. Let $ax = \lambda E_{A_1} x$. Then $ax = aE_{A_1} x = \sum_{\alpha_i \in A_1} \nu_{\alpha_i} e_{\alpha_i} x$ from lemma 3.3. Since the $\{Re_\alpha\}_{\alpha \in J}$ are independent submodules it follows that $\alpha_i \in A_1 \ \forall\ 1 \le i \le n$. Noting that $\sum_{j=1}^{m} RE_{A_j} = \sum_{j=1}^{m} RE_{\bigcup_{j=1}^{m} A_j}$ (from the uniqueness of M-injective hulls) our proof is complete. □

LEMMA 3.6 Suppose M has the property that every submodule of M has a unique M-injective hull in M. Let $\{e_\alpha\}_{\alpha \in J}$ be an infinite family of orthogonal x-idempotents in R. Let $A = \sum_{\alpha \in J} Re_\alpha x$ and $\eta : M \to M/A$ be the canonical quotient map. Let $P(J)$ denote the set of subsets of J and $U \subset P(J)$ satisfy the condition that any $K \in U$ is an infinite subset of J. Let $C_K = \eta(RE_K x)$. Then the sum $\sum_{K \in U} C_K$ in M/A is direct if and only if for any finite number of elements $K_1, \ldots, K_n$ in U, each of the sets $K_i \cap (\bigcup_{i \neq j} K_j)$ is finite $(1 \le i \le n)$.

Proof Suppose for some $K_i \in U$, $0 \le i \le n$ we have $K_0 \cap [\bigcup_{i=1}^{n} K_i] = L$ infinite. Then $RE_L x \subset RE_{K_0} x$, $RE_L x \subset \sum_{i=1}^{n} RE_{K_i} x$ and $E_L x \notin A$, since $RE_L x \supset \sum_{\alpha \in L} Re_\alpha x$. Hence $0 \neq \eta(E_L x) \in RC_{K_0} \cap \sum_{i=1}^{n} RC_{K_i}$.

Suppose $0 \neq c \in RC_{K_0} \cap \sum_{i=1}^{n} RC_{K_i}$ for some $K_i \in U$, $0 \le i \le n$. Let $bx \in RE_{K_0} x \cap \sum_{i=1}^{n} RE_{K_i} x$ be such that $\eta(bx) = c$. Since $c \neq 0$, $bx \notin A$. So by 3.4, $Rbx \not\subset \sum_{i=1}^{n} Re_{\alpha_i} x$ for any finite subset $\{\alpha_i\}_{i=1}^{n}$ of J. Suppose $L = K_0 \cap [\bigcup_{i=1}^{m} K_i]$ is finite. Pick $\lambda \in R$ such that $\lambda bx = \sum_{j=1}^{m} \mu_j e_{\beta_j} x \notin \sum_{\alpha \in L} Re_\alpha x$. Then $\mu_j e_{\beta_j} x \neq 0$ for some j, $m \le j \le 1$

with $\beta_j \notin L$. Since $\lambda bx \in RE_{K_0}x$, $\lambda bx \in \sum_{i=1}^{n} RE_{K_i}x$, from 3.5 it follows that $\beta_j \in K_0$, $\beta_j \in \bigcup_{i=1}^{n} K_i$, contradicting $\beta_j \notin L$. Hence L is infinite. □

LEMMA 3.7 Suppose M has the property that every submodule of M has a unique M-injective hull in M. Suppose there exists an infinite family $\{e_\alpha\}_{\alpha\in J}$ of orthogonal x-idempotents in R. Let $A = \sum_{\alpha\in J} Re_\alpha x$. Then the module M/A is not M-injective.

Proof Let $\{L_i\}_{i\geq 1}$ be a countable set of infinite disjoint subsets of J with $\bigcup_{i\geq 1} L_i = J$. Let $\mathcal{D}$ be the collection of subsets D of $P(J)$ (the set of all subsets of J) satisfying the following conditions: (i) $D \supset \{L_i\}_{i\geq 1}$, (ii) Given any finite collection K_i, $0 \leq i \leq n$ of elements of D, $K_0 \cap [\bigcup_{i=1}^{n} K_i]$ is finite. It is easy to see that $\mathcal{D}$ is non-empty and inductive when partially ordered by inclusion, and hence contains a maximal element, say D. We set $D' = D - \{L_i\}_{i\geq 1}$. Let $\eta : M \to M/A$ be the canonical quotient map and let $N = \sum_{B\in D} RE_B$. Set $\varphi : Nx \to M/A$ by $\varphi(E_{L_i}x) = \eta(E_{L_i}x)$ $\forall\ i \geq 1$, $\varphi(E_B x) = 0$ $\forall\ B \in D'$. (φ is well defined from lemma 3.6). Suppose M/A were M-injective. Then, since $\varphi(Nx) \subset \sum_{i\geq 1} R(\eta E_{L_i}x)$, there exists $c \in \left(\sum_{i\geq 1} R\eta(E_{L_i}x)\right)^* \subset M/A$ such that $\varphi(nx) = nc$. Let $\eta(bx) = c$. Then $\eta(E_{L_i}x) \in R\eta(bx)$ $\forall\ i \geq 1$. Since $R\eta(E_B x) \cap \sum_{i\geq 1} R\eta E_{L_i}x = 0$ $\forall\ B \in D'$, and $c \in \left(\sum_{i\geq 1} R\eta(E_{L_i}x)\right)^*$ it follows that $R\eta(bx) \cap R\eta(E_B x) = 0$ $\forall\ B \in D'$. Now from the argument of lemma 3.6, it follows that $\{\beta \in L_i \mid e_\beta x \in Rbx\}$ is infinite and $\{\beta \in B \mid e_\beta x \in Rbx\}$ is finite $\forall\ B \in D'$. Pick $j_i \in L_i$ such that

$e_{j_i} x \in Rbx \ \forall \ i \geq 1$ and set $T = \{j_i\}_{i \geq 1}$. From the disjoint nature of the L_i's, it follows that $j_n \in L_m$ if and only if $n = m$. From the maximality of D it follows that $T \cap B$ is infinite for some $B \in D$. As already pointed out, $T \cap L_i = \{j_i\} \ \forall \ i \geq 1$ so we can assume that $B \in D'$. But $T \cap B \subseteq \{\beta \in B | e_\beta x \in Rbx\}$ is finite $\forall \ B \in D'$, a contradiction. Hence M/A is not M-injective. □

LEMMA 3.8 Suppose End(M) is regular, M is M-injective and every cyclic submodule is a direct summand of M. Then there cannot exist any infinite family of orthogonal x-idempotents in R.

Proof Immediate consequence of proposition and lemma 3.7. □

LEMMA 3.9 Let M have the property that every cyclic submodule of M is a direct summand of M. If there exist no infinite families of orthogonal x-idempotents in M, then M is semi-simple.

Proof We first show that $\mathrm{Soc}M \neq 0$. If M itself is simple, there is nothing to prove. If not, there exists a cyclic submodule N_1 of M with $0 \neq N_1 \neq M$. Proposition 2.3 (i) yields an x-idempotent e_1 such that $N_1 = Re_1 x$. If $R(1 - e_1)x$ is simple there is nothing to prove. If not, there exists a cyclic submodule N_2 of $R(1 - e_1)x$ with $0 \neq N_2 \neq R(1 - e_1)x$. Then $M = N_2 \oplus N_2'$ for some $N_2' \subset M$. Since $R(1 - e_1)x \supset N_2$ we get $R(1 - e_1)x = N_2 \oplus H_2'$ where $H_2' = N_2' \cap R(1 - e_1)x$. Thus $M = N_1 \oplus N_2 \oplus H_2'$. If $e_2 x$ is the projection of x onto N_2 then e_1, e_2 are orthogonal x-idempotents, $N_2 = Re_2 x$ and $M = Re_1 x \oplus Re_2 x \oplus R(1 - e')x$ where $e' = e_1 + e_2$. If $R(1 - e')x$ is not simple we can repeat the above construction. Since there are no infinite families of orthogonal

x-idempotents in R, the above process has to stop after a finite number of steps. Thus we will get a finite number of orthogonal x-idempotents $e_1,\ldots,e_k$ with $R(1 - e)x$ simple where $e = e_1 + \ldots + e_k$. Also $(1 - e) = \nu$ is an x-idempotent.

Thus M has a simple submodule of the form $R\nu x$ with ν an x-idempotent. Let G be the set whose elements are families of orthogonal x-idempotents $\{e_\alpha\}_{\alpha\in J}$ satisfying the condition that each $Re_\alpha x$ is a simple submodule of M. If $g = (e_\alpha)_{\alpha\in J}$, $g' = (e_\beta')_{\beta\in J'}$ are elements of G we define $g \leq g'$ if $J \subset J'$ and $e_\alpha' = e_\alpha$ for all $\alpha \in J$. Then applying Zorn's lemma we get a maximal element $g \in G$. Since there are no infinite families of orthogonal x-idempotents in R, g will be a finite collection $(e_i)_{i=1}^{k}$. We claim that $M = Re_1x \oplus \ldots \oplus Re_kx$. This will prove the semi-simplicity of M, since each Re_ix is simple. If $M \neq Re_1x \oplus \ldots \oplus Re_kx$ we will have $R(1 - e)x \neq 0$. Then using the argument given in the earlier paragraph we can obtain a simple module $N = Re_{k+1}x$ with $\{e_i\}_{i=1}^{k+1}$ an orthogonal family of x-idempotents. This contradicts the maximality of $g = \{e_i\}_{i=1}^{k}$ in G.

PROPOSITION 3.10 Let M be a cyclic module with End(M) regular. If every cyclic module is M-injective then M is semi-simple.

Proof Immediate consequence of lemmas 3.9 and 3.10.

§4. Regularity of End(M) and M-injectivity of cyclic modules

In this section we relax the restriction that M be cyclic. We first observe that if M is semi-simple, then End(M) is regular. Explicitly we have not seen a proof of this fact in literature.* The

*See [31].

proof of this is exactly similar to the well-known fact that $\mathrm{End}_K(V)$ is regular when V is a vector space over a field K. For the sake of completeness we briefly indicate the proof.

LEMMA 4.1 The ring of endomorphisms $\mathrm{End}(M)$ of a semi-simple module M is regular.

Proof Let $f \in \mathrm{End}(M)$. Let $N = \ker f$, $j : N \to M$ and $i : f(M) \to M$ the respective inclusions. Since M is semi-simple the exact sequences

$$0 \longrightarrow N \xrightarrow{j} M \xrightarrow{f} f(M) \longrightarrow 0$$

$$0 \longrightarrow f(M) \xrightarrow{i} M \longrightarrow {}^{M}\!/_{f}(M) \longrightarrow 0$$

split. Let $\theta : f(M) \to M$, $\varphi : M \to f(M)$ be such that $f \circ \theta = Id_{f(M)}$ and $\varphi \circ i = Id_{f(M)}$. Then the element $g = \theta \circ \varphi$ of $\mathrm{End}(M)$ is seen to satisfy $f \circ g \circ f = f$. □

When $M = M_1 \oplus M_2$ with $\mu_i : M_i \to M$, $p_i : M \to M_i$ $(i = 1,2)$ respectively denoting the inclusion and projection maps, we can represent any $\alpha \in \mathrm{End}(M)$ as a matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ where $\alpha_{ij} = p_i \circ \alpha \circ \mu_j \in \mathrm{Hom}(M_j, M_i)$.

LEMMA 4.2 Suppose $\mathrm{End}(M)$ is regular and M_1 is a direct summand of M. Then $\mathrm{End}(M_1)$ is regular.

Proof Let $M = M_1 \oplus M_2$. Let $\varphi \in \mathrm{End}(M_1)$ and $\alpha = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(M)$. Since $\mathrm{End}(M)$ is regular, there exists a $\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \in \mathrm{End}(M)$ with $\alpha \circ \beta \circ \alpha = \alpha$. But $\alpha \circ \beta \circ \alpha = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varphi\beta_{11}\varphi & 0 \\ 0 & 0 \end{pmatrix}$. Hence $\theta = \beta_{11} \in \mathrm{End}(M_1)$ satisfies $\varphi \circ \theta \circ \varphi = \varphi$. □

REMARK 4.3 Suppose $M = M_1 \oplus M_2$ and $\mathrm{End}(M_i)$ regular ($i = 1,2$). Then in general it is not true that $\mathrm{End}(M)$ is regular. In fact if $\mathrm{End}(M)$ is regular, in addition to the regularity of $\mathrm{End}(M_i)$ ($i = 1,2$) we also get the following as a consequence:

Given $\alpha_{21} \in \mathrm{Hom}(M_1,M_2)$ (resp. $\alpha_{12} \in \mathrm{Hom}(M_2,M_1)$) there exists $\beta_{12} \in \mathrm{Hom}(M_2,M_1)$ (resp. $\beta_{21} \in \mathrm{Hom}(M_1,M_2)$) such that $\alpha_{21}\circ\beta_{12}\circ\alpha_{21} = \alpha_{21}$ (resp. $\alpha_{12} = \alpha_{12}\circ\beta_{21}\circ\alpha_{12}$). In fact the regularity of $\mathrm{End}(M)$ yields a $\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$ such that $\begin{pmatrix} 0 & 0 \\ \alpha_{21} & 0 \end{pmatrix}\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \alpha_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha_{21} & 0 \end{pmatrix}$ in $\mathrm{End}(M)$. This in turn gives $\alpha_{21}\circ\beta_{12}\circ\alpha_{21} = \alpha_{21}$.

EXAMPLE 4.4 Let R be a regular ring which is not semi-simple (for instance $R = \mathrm{End}_K(V)$ where V is an infinite dimensional vector space over a field K). Then $\mathrm{Soc}R \neq R$. Let I be a maximal left ideal of R with $I \supset \mathrm{Soc}R$. Then $M_1 = R/I$ is a simple R-module and hence $\mathrm{End}(M_1)$ is a regular ring (lemma 4.1). Also $\mathrm{End}(R) \simeq R$ is regular. However $\mathrm{End}(M_1 \oplus R)$ is not regular. In fact for any $\alpha \in \mathrm{Hom}(R,M_1) \simeq M_1$ we have $\alpha(\lambda) = \alpha(\lambda.1) = \lambda\alpha(1) = 0$ for any $\lambda \in \mathrm{Soc}R$. Since M_1 is simple for any $\beta \in \mathrm{Hom}(M_1,R)$ we have $\beta(M_1) \subset \mathrm{Soc}R$. Hence $\alpha\circ\beta = 0$ for any $\alpha \in \mathrm{Hom}(R,M_1)$, $\beta \in \mathrm{Hom}(M_1,R)$. Thus if we choose $\alpha \neq 0$, for every $\beta \in \mathrm{Hom}(M_1,R)$ we have $\alpha\circ\beta\circ\alpha \neq \alpha$. From remark 4.3, we see that $\mathrm{End}(M_1 \oplus R)$ is not regular.

THEOREM 4.5 Let M be a module with $\mathrm{End}(M)$ regular. Then the following conditions are equivalent:

(1) M is semi-simple.

(2) Every module is M-injective.

(3) Every finitely generated module is M-injective.

(4) Every cyclic module is M-injective.

Proof (1) $\Rightarrow$ (2). Given any module A and any $f: N \to A$ with $N \subset M$, the semi-simplicity of M implies that N is a direct summand of M and hence f can be extended to a $g: M \to A$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). It suffices to show that every cyclic submodule of M is semi-simple. Suppose N is any cyclic submodule of M. Since every cyclic module is M-injective it follows that N is M-injective and hence N is a direct summand of M. From lemma 4.2 it follows that End(N) is regular. Also from proposition (2.5 (4) of [23]) it follows that every cyclic module is N-injective. From proposition 3.10 it now follows that N is semi-simple. □

We remark once again that all the material in section 3 and the essential idea in theorem 4.5 are taken from [20]. The proofs here are adaptations of the proofs in [20].

CHAPTER VII TALL RINGS

§1. Krull dimension

The notion of Krull-dimension is becoming an important tool for the study of noetherian modules. We work with the definition given in [15], and examine some relationships between having Krull-dimension and being noetherian. The starting point of this investigation is the observation that over a V-ring the class of noetherian modules and the class of modules with Krull-dimension coincide (which is not true in general). M. Teply had originally proved (the proof is unpublished) that V-rings with Krull-dimension are noetherian. Apparently, the proof is very similar to the one first noted by us, essentially using 1.7 (ii). Our aim in this chapter is to study rings in general, over which the classes of modules with Krull-dimension and noetherian modules are equivalent.

DEFINITION 1.1 The Krull-dimension of a module M over a ring R, (written K-dim M) is defined by transfinite recursion as follows: (i) K-dim $M = -1$ if and only if $M = 0$. (ii) Given an ordinal α, assuming that the concept K-dim $N < \alpha$ has already been defined, we set K-dim $M = \alpha$ if and only if K-dim $M \not< \alpha$ and there exists no infinite descending chain $M = I_0 \supset I_1 \supset \ldots$ of submodules of M with $K\text{-dim}\left(I_i/I_{i+1}\right) \not< \alpha \ \forall\ i \geq 1$.

Given a family $\{I_i\}$ of submodules of M we will call them independent if their sum $\Sigma\ I_i$ in M is a direct sum. Given any ring R,

there always exist modules that do not have Krull-dimension over R, but noetherian modules always have Krull-dimension. This is the content of our next two lemmas.

LEMMA 1.2 [15] If an R-module M contains an infinite independent family of submodules then M does not have Krull-dimension.

Proof Suppose on the contrary there are modules M possessing an infinite independent family of submodules and having Krull-dimension. Among them pick a module, say M, with the least Krull-dimension. Let K-dim $M = \alpha$. Then all the R-modules N with K-dim $N = \beta < \alpha$, will not contain an infinite independent family of submodules. Let $\{T_i\}_{i\geq 1}$ be a disjoint partition of the integers ≥ 1, with each T_i infinite. Let $W_i = \bigcup_{j>i} T_j$ and $J_i = \bigoplus_{j\in W_i} I_j$. Then $M = J_0 \supset J_1 \supset J_2$ is a descending chain with J_i/J_{i+1} containing an infinite independent family of submodules. Hence K-dim $J_i/J_{i+1} \geq \alpha$ which contradicts K-dim $M = \alpha$. This proves the lemma. □

LEMMA 1.2' Every noetherian module has Krull-dimension.

Proof Let M be a noetherian module without Krull-dimension. Let $N = \{N | N$ is a submodule of M and M/N does not have Krull-dimension$\}$ N is non-empty since $0 \in N$ and hence has a maximal element, say N, since M is noetherian. Then M/N does not have Krull-dimension but every proper factor of M/N does. Let $A = \{A | N \subsetneq A \subset M, A$ a submodule$\}$. Let β = l.u.b. $\{K\text{-dim } M/A | A \in A\}$ (β exists since A is a set). Let $M/N = I_0/N \supset I_1/N \supset \ldots$ be a descending chain of submodules of M/N. From the definition it is evident that if K-dim $A = \alpha$, and B is a submodule

of A then $K\text{-dim } B \leq \alpha$. So either $I_j/N = 0 \ \forall\ j \geq n$ (for some integer n), or $K\text{-dim } \left(I_i/N \Big/ I_{i+1}/N\right) = K\text{-dim } I_i/I_{i+1} \leq K\text{-dim } M/I_{i+1} \leq \beta$ for all integers $i \geq 1$. In any case $K\text{-dim } M \leq \beta + 1$. □

We now set about trying to determine those rings over which the converse of 1.2' holds.

Let R be a ring and S a subcategory of R-mod.

DEFINITION 1.3 S is called a Serre subcategory of R-mod if:

(i) S is non-empty.

(ii) $M,N \in S \Rightarrow \text{Hom}_R(M,N) = \text{Hom}_S(M,N)$ (S is full).

(iii) If $0 \to L \to M \to N \to 0$ is exact in R-mod then $M \in S \Leftrightarrow L,N \in S$.

From (i) and (iii) it is clear that $0 \in S$. From property (iii) it follows easily that S is closed under finite products and hence finite limits. We also note that from [15] 1.1, it follows that the set of all modules with Krull-dimension over a ring R forms a Serre subcategory of R-mod.

Let R be a ring, S any subcategory of R-mod. We define two maps $G(S),H(S) : R\text{-mod} \to R\text{-mod}$ as follows:

(i) $\forall\ M \in S,\ G(S)(M) = H(S)(M) = 0$.

(ii) $\forall\ M \notin S \quad G(S)(M) = \cap\ \{N \mid N \subset M,\ M/N \in S\}$.

(iii) $\forall\ M \notin S \quad H(S)(M) = \cap\ \{N \mid N \subset M,\ N \notin S\}$.

If S is a Serre subcategory, then from 1.3 (iii) it follows that $H(S)(M) \subset G(S)M\ \forall\ M \in R\text{-mod}$.

Before stating the main results of this chapter we introduce a definition and fix some terminology.

DEFINITION 1.4 A module M has finite uniform dimension if no homomorphic image of M contains an infinite independent family of submodules. Otherwise we say M does not have finite uniform dimension.

For the rest of this section R denotes a ring, S a Serre subcategory of R-mod. N denotes the Serre subcategory of all noetherian R-modules K, K_α those of all modules with Krull-dimension and those of Krull dimension $\leq \alpha$ respectively. An immediate consequence of lemma 1.2 is the following.

LEMMA 1.5 $M \in K \Rightarrow M$ has finite uniform dimension.

LEMMA 1.6 Given any $0 \neq M \in R$-mod, then $H(K)(M) \neq M \neq H(K_\beta)(M)$ for all ordinals β.

Proof If $0 \neq M \in K_\beta$, then $H(K_\beta)(M) = 0 \neq M$. If $M \notin K_\beta$, then $H(K_\beta)(M) = M \Rightarrow K\text{-dim } N \leq \beta$ for all $N \subseteq M$. In this case, given any descending chain $M \supset N_1 \supset N_2 \supset \ldots$ there exists an integer i_0 such that $K\text{-dim } {}^{N_i}/_{N_{i+1}} < \beta$ for all $i \geq i_0$. Hence $K\text{-dim } M \leq \beta$, a contradiction. Finally, let $\beta = \text{l.u.b.}\{K\text{-dim } N \mid N \subset M, N \in K\}$. Then $H(K_\beta)(M) = H(K)(M)$. Hence $M \neq H(K)(M)$ for all $M \neq 0$. □

PROPOSITION 1.7 Let S be a Serre subcategory of R-mod. Let $M \notin S$ and $H(S)(N) \neq N$ for all $0 \neq N \in R$-mod. Suppose further that ${}^{M}/_{I} \notin S$ for all finitely generated $I \subset M$. Then either (i) there exists an exact sequence $0 \to N \to M \to P \to 0$ with both N and P not in S, or (ii) M does not have finite uniform dimension.

REMARK (i) and (ii) are not necessarily exclusive.

Proof Let $I \subset M$ be finitely generated and $\eta: M \to M/I$ be the canonical quotient map. Let $\bar{A} = G(S)(M/I)$, $A = \eta^{-1}(\bar{A})$. The proof splits into three cases.

(i) $M/A \in S$. Then since $M/I \notin S$ by hypothesis, it follows from 1 (iii) that $\bar{A} = A/I \notin S$. Hence $H(S)(\bar{A}) \neq \bar{A}$. Therefore there exists an $\bar{N} \supsetneq \bar{A}$ with $\bar{N} \notin S$. Then $M/I \big/ \bar{N} \notin S$, since $M/I \big/ \bar{N} \in S \Rightarrow G(S)(M/I) \subset \bar{N} \Rightarrow \bar{A} \subset \bar{N}$, a contradiction. Let $N = \eta^{-1}(\bar{N})$. From $\bar{N} \notin S$ we get $N \notin S$. Now $0 \to N \to M \to M/N \to 0$ is exact with N and $M/N \simeq M/I \big/ \bar{N}$ not in S.

(ii) $M/A \notin S$, and $\bar{A} \notin S$. Then $A \notin S$ and $0 \to A \to M \to M/A \to 0$ is exact.

(iii) Suppose for all finitely generated $I \subset M$ we have $G(S)(M/I) \in S$. Then $M/I \big/ G(S)(M/I) \notin S$. Since $\{0\}$ is finitely generated we get $G(S)(M) \in S$. Hence $M \neq G(S)(M)$. It follows that there exists $0 \neq a_1 \in M$, $N_1 \subset M$ with $a_1 \in N_1$ and $M/N_1 \in S$.

Taking this as the start of an induction, suppose $a_1, \ldots, a_r$, $N_1, \ldots, N_r$ have been constructed with (1) $a_i \notin N_i$ $(1 \leq i \leq r)$ and (2) $a_i \in N_j$ for all $i \neq j$. Let $I = \langle a_1, \ldots, a_n \rangle$ and $N = \bigcap_{i=1}^{r} N_i$. Then M/N imbeds into $\prod_{i=1}^{r} M/N_i$, and S is closed under finite products and submodules. Hence $M/N \in S$. Again let $\eta: M \to M/I$ be the canonical quotient map and $\bar{A} = G(S)(M/I)$. Now $\bar{A} \in S$ by assumption and since $M/I \big/ \eta(N) \simeq M/N+I \in S$ and $M/I \notin S$, we get $\eta(N) \notin S$. Also by the definition of $G(S)$ we get $G(S)(M/I) \subset \eta(N)$. Thus $\bar{A} \subset \eta(N)$. From $\bar{A} \in S$, $\eta(N) \notin S$ we get $\bar{A} \neq \eta(N)$. Hence there exists $0 \neq \bar{a} \in \eta(N)$, $\bar{a} \notin \bar{A}$. This means there exists $\bar{L} \subset M/I$ with $M/I \big/ \bar{L} \in S$ and $\bar{a} \notin \bar{L}$. Take $a_{r+1} \in N \cap \eta^{-1}(\bar{a})$, $N_{r+1} = \eta^{-1}(\bar{L})$. Clearly $a_{r+1} \notin N_{r+1}$ and $a_{r+1} \in N_i$,

$N_{r+1} \ni a_i \ \forall\ 1 \le i \le r$, so we have constructed a_{r+1} and N_{r+1} satisfying the conditions (i) and (ii). Hence by induction there exists $\{a_i\}_{i\ge1} \subset M$ and $\{N_i\}_{i\ge1}$ submodules of M satisfying $a_i \notin N_i$, $a_i \in N_j$ $\forall\ i \ne j \ge 1$. Setting $N = \bigcap_{i\ge1} N_i$ and $\eta: M \to M/N$ to be the canonical quotient map, it follows that $\{R\eta(a_i)\}_{i\ge1}$ are independent in M/N. □

LEMMA 1.8 Let S be a Serre subcategory of R-mod satisfying $H(S)(N) \ne N$ for all $0 \ne N$. Suppose given $M \notin S$ there exists an $A \subset' M$ such that $A/I \notin S$ for every finitely generated $I \subset A$. Then $K \subseteq S$.

Proof Suppose there is an $M \in K$, $M \notin S$. Of all such choose one of least Krull-dimension, say α. Then $M \in K_\alpha$ and $K_\beta \subset S\ \forall\ \beta < \alpha$, where $\alpha > -1$ since $0 \in S$. Since $M \notin S$ there is an $A \subset M$ such that $A/I \notin S$ for any finitely generated $I \subset A$. Hence by proposition 1.7 and lemma 1.2 there is an exact sequence $0 \to N_1 \to A \to P \to 0$ with N_1 and P not in S. Now straightforward induction allows us to construct $M \supset N_1 \supset N_2 \supset \ldots$ where $N_i/N_{i+1} \notin S\ \forall\ i \ge 1$. This means $K\text{-dim } N_i/N_{i+1} \ge \alpha$ contradicting K-dim $M = \alpha$. □

THEOREM 1.9 The following are equivalent for a ring R:

(i) $H(N)(M) \ne M\ \forall\ 0 \ne M \in R$-mod where N is the subcategory of all noetherian R-modules.

(ii) $K = N$.

Proof That $N \subset K$ is well known [15]. Suppose $M \in R$-mod $M \notin N$. Then there is a submodule $N \subset M$ such that N is not finitely generated and hence $N/I \notin N$ for any finitely generated submodule I of N. Hence by lemma 1.8, $K \subset N$. Hence $K = N$. □

We now give a proof of a theorem first proved in [15]. The original proof involved the notion of a localizing subcategory, but it can also be deduced by more direct arguments, in a slightly generalised form.

LEMMA 1.10 Let R be a ring, S a Serre subcategory of R-mod containing all finitely generated modules. Let K be the subcategory of all R-modules with Krull-dimension. Then $H(S)M \neq M \ \forall \ 0 \neq M \in R\text{-mod} \Rightarrow S \supset K$.

Proof If $M \notin S$ then $M/I \notin S$ for any finitely generated $I \subset M$ since $I \in S$. Hence from lemma 1.8, $K \subset S$. □

THEOREM 1.11 (Gordon Robson [15]) If the ring R has K-dim α (as a left R-module) then every left R-module with K-dim has K-dim $\leq \alpha$.

Proof Let $K = \{M \in R\text{-mod} \mid M \text{ has } K\text{-dim}\}$, $K_\alpha = \{M \in R\text{-mod} \mid K\text{-dim } M \leq \alpha\}$. Then clearly K_α contains all finitely generated R-modules. Hence from lemmas 1.6 and 1.10 (with $S = K_\alpha$) we get $K \subset K_\alpha$. This proves the theorem. □

We shall now concentrate on rings R which satisfy the condition that all R-modules with Krull-dimension are noetherian.

THEOREM 1.12 The following are equivalent for a ring R:

(i) Every R-module with Krull-dimension is noetherian.

(ii) Every non-noetherian R-module M has a proper non-noetherian submodule.

(iii) Given $M \in R$-mod, M non-noetherian there is an exact sequence $0 \to N \to M \to P \to 0$ of R-modules with N,P non-noetherian.

Proof That (i) $\Leftrightarrow$ (ii) is the essence of Theorem 1.9. (iii) $\Rightarrow$ (ii) is obvious. To show (ii) $\Rightarrow$ (iii) we let N = {all noetherian R-modules} and notice that $M \in R\text{-mod}$ $M \notin N \Rightarrow$ there exists $N \subset M$ with N non-finitely generated. Hence $N/I \notin N$ for any finitely generated $I \subset N$. Applying proposition 1.7 to N with $N = S$ and noticing that in any module M without finite uniform dimension we have an exact sequence $0 \to N \to M \to P \to 0$ of R-modules with N,P non-noetherian, our proof is complete. □

PROPOSITION 1.13 Let S be a Serre category $M \notin S$. Suppose $I \subset M$, I finitely generated $\Rightarrow I \in S$. Suppose further that $H(S)N \neq N$ $\forall$ $N \notin S$. Then M does not have finite uniform dimension.

Proof Since $H(S)M \neq M$ we can pick $a_1 \in M$, $N_1 \subset M$ with $a_1 \notin N_1$ and $N_1 \notin S$. Suppose we have $N_1,\ldots,N_r$, $a_1,\ldots,a_r$ with $a_i \notin N_i$, $a_i \in N_j$ $\forall$ $i \neq j$, and $\bigcap_{i=1}^{r} N_i \notin S$. Let $a_1,\ldots,a_r = I$, $N_1 \cap \ldots \cap N_r = N$. Now $(N+I)/I \cong N/(N \cap I) \notin S$ since $N \cap I \in S$. Pick $N_{r+1}/N \subseteq (N+I)/I$ and $a_{r+1} \in N$ s.t. $a_{r+1} \notin N_{r+1}$, and $N_{r+1}/I \notin S$. Since $N_{r+1} = N \cap N_{r+1} + I$ we see that $N \cap N_{r+1} \notin S$. Finally $a_{r+1} \notin N_{r+1}$ and $a_{r+1} \in N_j$ $\forall$ $1 \leq j \leq r$. The rest of the proof is the same as in the proof of 1.7 (iii). □

COROLLARY 1.14 Let $M = \varinjlim T_i$ where $T_i \in K_\alpha$. If M has Krull-dimension then $M \in K_\alpha$.

Proof Set $S = K_\alpha$ in proposition 1.13. □

§2. Tall rings

DEFINITION 2.1 An R-module M will be called 'tall' if there is an exact sequence $0 \to N \to M \to P \to 0$ of R-modules with N,P non-noetherian. A ring R will be called 'tall' if every non-noetherian R-module is tall, that is R satisfies the equivalent conditions of theorem 1.12.

LEMMA 2.2 Let R be a tall ring, $\varphi : R \to S$ a ring epimorphism. Then S is a tall ring.

Proof Let M be a non-noetherian S-module. Make M an R-module through φ by setting $rm = \varphi(r)m \ \forall \ r \in R, \ m \in M$. Then N is an R-submodule of $M \Leftrightarrow N$ is an S-submodule of M. Hence M is a non-noetherian R-module, and since R is a tall ring, has a proper non-noetherian R-submodule N. Then N is also a proper non-noetherian S-submodule and hence by theorem 1.12 S is a tall ring. □

The classical example of an artinian but non-noetherian module is Z_{p^∞}. Z_{p^∞} is in fact the injective hull of $Z_p = Z/pZ$. We show that whether a ring R is a tall ring can be determined by examining the essential extensions of simple R-modules.

PROPOSITION 2.3 Let R be a ring which is not tall. Then there exists a simple R-module S and an R-module B such that $S \subset B$ and

(i) S is essential in B.

(ii) B is not noetherian but has no proper non-noetherian submodules.

Proof Since R is not a tall ring, there exists a non-noetherian R-module M which has no proper non-noetherian submodules. Let $a \in M$.

Let N be a maximal submodule of M not containing a. Since N is a proper submodule, N is noetherian and M/N is non-noetherian. Setting $B = M/N$ and $S = R\eta(a)$ where $\eta : M \to B$ is the canonical quotient map, the proposition follows. □

We now show that a polynomial ring is never a tall ring. To show this it is sufficient to prove that a polynomial ring in one variable over any ring is not tall.

PROPOSITION 2.4 Let R be any ring and $R[X]$ the polynomial ring in the commuting indeterminate X. Then $R[X]$ is not tall.

Proof Let $M \subset R$ be a maximal ideal of R, $S = R/M$. Let $S_i = S$ for all integers $i \geq 1$ and $T = \bigoplus_i S_i$. Let $u \in S$, and define $u_i \in T$ $(i \geq 1)$ to be the element with u in the i^{th} coordinate and zero elsewhere. Set $Xu_1 = 0$, $Xu_i = u_{i-1}$ and leave action by elements of R unchanged. This extends to an operation by $R[X]$ on T which makes T an R-module. A verification exactly analogous to the case of Z_{p^∞} shows that T is not noetherian but has no proper non-noetherian submodules. □

REMARK 2.5 This shows that subrings of tall rings are not tall, since we can take the field extension $k(X)$ of a field k by an indeterminate X. $k[X]$ is a subring of $k(X)$ which is not tall, while any field is obviously tall.

§3. Monoid rings over tall rings

In contrast to proposition 2.4 we will show that, given any finite monoid with identity G, and a ring R, then the monoid ring $R[G]$ is a tall ring $\Leftrightarrow$ R is a tall ring. In the course of the proof we need the following lemma which is easy to verify.

LEMMA 3.1 Let M be an R-module, A,B,C,D submodules of M with $A \supset B$, $C \supset D$. Then there exists an epimorphism $^{A+C}/_{B+D} \to {}^{A}/_{B+A\cap C} \oplus {}^{C}/_{D+A\cap C}$.

We now fix the following notation:

R is a subring of S containing the identity of S.

$G = \{1 = g_0, g_1, \ldots, g_n\}$ is a finite subset of the centraliser of R in S, and as an R-module, $S = \langle G \rangle$. If M is any S-module and A an R-submodule of M, then we define R-submodules A_i $(0 \leq i \leq n)$ of M by $A_i = \sum_{0\leq j\leq i} g_i A$. Clearly $A = A_0 \subset A_1 \subset \ldots \subset A_n$, and A_n is an S-submodule S. Setting $g_i^{-1}(A) = \{m \in M | g_i m \in A\}$, it is clear that $g_i^{-1}A$ is an R-submodule of M.

PROPOSITION 3.2 Let R, S, G be as above, and M an S-module. If M is non-noetherian when considered as an R-module in the natural way, then there exists an S-submodule N of M with both N and $^{M}/_{N}$ non-noetherian as R-modules.

Proof We will construct R-submodules $A^{(0)}, A^{(1)}, \ldots, A^{(n)}$ of M such that $A_i^{(i)}$ and $^{M}/_{A_i^{(i)}}$ are both non-noetherian as R-modules. (For each R-submodule C of M, $C_i = \sum_{0\leq j\leq i} g_j C$). Since M is non-noetherian and R is a tall ring, there exists an R-submodule $A^{(0)} = A_0^{(0)}$ of M with $A^{(0)}$ and $^{M}/_{A^{(0)}}$ non-noetherian. Suppose $A^{(0)}, \ldots, A^{(r)}$ have been constructed

with the required property. If $A_r^{(r)} + g_{r+1}A_r \;\; A_r^{(r)}$ is noetherian, set $A^{(r+1)} = A^{(r)}$. Then $A_{r+1}^{(r+1)} \supset A_r^{(r)}$ and is non-noetherian, and $M_{r+1}^{(r+1)} \;\; A_r^{(r)}$ is noetherian. Since $M/A_r^{(r)}$ is non-noetherian it follows that $M/A_{r+1}^{(r+1)}$ is non-noetherian. Now suppose $A_r^{(r)} + g_{r+1}A^{(r)} \;\; A_r^{(r)} \simeq g_{r+1}A^{(r)} \;\; A_r^{(r)} \cap g_{r+1}A^{(r)}$ is non-noetherian. Then the proof splits into two cases.

Case (1) $B = A_r^{(r)} \cap g_{r+1}A^r$ is not noetherian. Then there exists a non-noetherian submodule C of B with B/C also non-noetherian. We set $A^{(r+1)} = g_{r+1}^{-1}(C)$. Then $A^{(r+1)} \subset A^{(r)}$, and, as an R-module, C is a homomorphic image of $A^{(r+1)}$. So $A^{(r+1)}$ is non-noetherian as an R-module.

Case (2) $B = A_r^{(r)} \cap g_{r+1}A^r$ is noetherian. In this case, since $g_{r+1}A_r$ is not noetherian and R is a tall ring we can get a non-noetherian submodule C of $g_{r+1}A^{(r)}$ with $C \supset B$ and $g_{r+1}A^{(r)}/C$ non-noetherian. Again we set $A^{(r+1)} = g_{r+1}^{-1}(C)$. Then $A^{(r+1)}$ is non-noetherian. In both cases, by lemma 2.10 there exists an epimorphism

$$A_r^{(r)} + g_{r+1}A^{(r)} / A_r^{(r+1)} + g_{r+1}A^{(r+1)} \to A_r^{(r)} / A_r^{(r+1)} + B \oplus g_{r+1}A^{(r)} / g_{r+1}A^{(r+1)} + B.$$

In case (1), we have $g_{r+1}A^{(r+1)} = C \subset B$ and hence the second factor in the direct sum reduces to $g_{r+1}A^{(r)}/B$ which is non-noetherian by assumption. In case (2), the second factor reduces to $g_{r+1}A^{(r)}/C$ which is non-noetherian by the choice of C. It follows that $M/A_{r+1}^{(r+1)} = M/A_r^{(r+1)} + g_{r+1}A^{(r+1)}$ is non-noetherian. Setting $N = A_n^{(n)}$, N is an S-submodule satisfying the requirements of the proposition. □

THEOREM 3.3 Let R, G, S be as in proposition 3.2. Then if R is a tall ring then so is S.

Proof Let M be a non-noetherian S-module. By theorem 1.12 it is sufficient to show that M has a proper non-noetherian submodule to prove that S is a tall ring. Now clearly M is non-noetherian as an R-module and by proposition 3.2 has an S-submodule N such that N and M/N are non-noetherian as R-modules. In particular $M \neq N$. Again by applying proposition 3.2 to N, we get an S-submodule N_1 such that N_1 and N/N_1 are non-noetherian as R-modules and hence $N_1 \subseteq N$. Let k be an integer ≥ 1 and assume inductively that we have constructed S-modules $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_k \subset N$, with each N/N_j non-noetherian as an R-module. Then there exists an S-submodule I of N/N_k with $(N/N_k)/I$ and I both non-noetherian as R-modules. Let $\eta : N \to N_k$ be the canonical quotient map, and $N_{k+1} = \eta^{-1}(I)$. Then $N_k \subseteq N_{k+1} \subseteq N$, and N/N_{k+1} is not R-noetherian. Proceeding thus, we get an infinite ascending chain of S-submodules $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_k \subseteq \ldots$ of N. Thus N is a non-noetherian proper submodule of M and hence our proof is complete. □

COROLLARY 3.4 Let $G = \{e = g_0, g_1, \ldots, g_n\}$ be a finite monoid with identity e. Let R be a ring and $S = R[G]$ the monoid ring. Then S is a tall ring $\Leftrightarrow$ R is a tall ring.

Proof Since R is an epimorphic image of S, S a tall ring $\Rightarrow$ R is a tall ring by lemma 2.2. If R is a tall ring, the subring Re of S is a tall ring, S is generated by G as an Re-module, and since e is the identity of G, each $g_i \in G$ is in the centraliser of Re. Hence by theorem 3.3 S is a tall ring. □

We now give some examples of tall rings.

EXAMPLE 3.5 Every V-ring R is a tall ring, since for all R-modules M, $H(M) \subset JR(M) = 0$, so $H(M) \neq M$ if $M \neq 0$ and so by theorems 1.9, 1.12 R is a tall ring.

EXAMPLE 3.6 Every left perfect ring R is a tall ring, since again, for all modules M, $JR(M) = M \Leftrightarrow M = 0$ so $H(M) = M \Leftrightarrow M = 0$.

Before our final example of a tall ring which is neither a perfect ring nor a V-ring, we prove some simple lemmas.

LEMMA 3.7 Every perfect V-ring R is semi-simple artinian.

Proof If R is a perfect ring $R/JR(R)$ is semi-simple artinian and if R is a V-ring $JR(R) = 0$ so the lemma follows. □

LEMMA 3.8 Let R be a ring, G a group and $R(G)$ the group ring. Then $R[G]$ perfect $\Rightarrow R$ perfect.

Proof Perfect rings are characterised by the fact that all modules have projective covers. Let R be perfect $\varphi : R \to S$ an onto ring homomorphism. Let M be an S-module. Make M an R-module by setting $rm = \varphi(r)m \ \forall \ r \in R, m \in M$. Let $\ker\varphi = I$. Let P be an R-projective cover for M. Then P/IP is a projective cover for M as an S-module. This is well known and easy to check. Hence every homomorphic image of a perfect ring is perfect. Hence $R[G]$ perfect $\Rightarrow R$ perfect. □

LEMMA 3.9 [19] Let R be a V-ring, G a finite group. Then if $R[G]$ is a V-ring the order of G is invertible in R.

Proof Suppose $G = \{e = g_0, g_1, \ldots, g_{n-1}\}$ and n is not invertible in R. Let M be a maximal ideal of R containing Rn and let $S = R/M$. We make S an $R[G]$-module by setting $gs = s$ for all $s \in S$, $g \in G$. Then the element $\sum_{i=0}^{n-1} g_i = \lambda$ annihilates S but we can define $f: R[G]\lambda \to S$ by $f(a\lambda) = a\eta(1)$ for all $a \in R[G]$, where $\eta: R \to S$ is the canonical quotient map. In fact for any $a = r_0 + r_1 g_1 + \ldots + r_{n-1} g_{n-1}$ in $R[G]$ with $r_i \in R$ we have $a\lambda = (r_0 + r_1 + \ldots + r_{n-1})(e + g_1 + \ldots + g_{n-1}) = \varepsilon(a)\lambda$ where $\varepsilon: R[G] \to R$ is the augmentation map. Hence $a\lambda = 0 \Leftrightarrow \varepsilon(a) = 0$. For the action on S we have $a.s = \varepsilon(a)s$ for any $s \in S$. Hence $a\lambda = 0 \Rightarrow \varepsilon(a) = 0 \Rightarrow \varepsilon(a)\eta(1) = 0$. Thus f is well-defined. Moreover $f \neq 0$. Since $\lambda S = 0$, there exists no extension of f to a homomorphism $R[G] \to S$. Hence S is not $R[G]$-injective. Clearly S is a simple $R[G]$-module. Thus $R[G]$ is not a V-ring. □

EXAMPLE 3.10 Let R be an infinite direct product of copies of Z_2 (the integers modulo 2). Let G be the cyclic group of order 2. Then R is a V-ring and hence a tall ring (example 3.5). Since R is not semi-simple artinian R is not perfect by lemma 3.7. Hence $R[G]$ is not perfect by 3.8. Since 2 is not invertible in R, $R[G]$ is not a V-ring by 3.9, but $R[G]$ is a tall ring by 3.3. So $R[G]$ is a tall ring which is neither a perfect ring nor a V-ring.

We conclude this chapter with a proposition which may be of practical use in determining when certain rings are tall.

DEFINITION 3.11 Let R be a ring. We define a collection $F(R)$ of two sided ideals of R by $F(R) = \{I \mid R/I \text{ is a tall ring}\}$.

PROPOSITION 3.12 R is a tall ring $\Leftrightarrow$ for any non-noetherian R-module M, there exists an $I \in F(R)$ with $IM \neq M$. In particular if $F(R)$ contains a nilpotent ideal then R is a tall ring.

Proof If R is a tall ring then $\{0\} \in F(R)$ hence one implication is trivial. If a non-noetherian R-module M is given suppose $I \in F(R)$ and $IM \neq M$. If M/IM is noetherian then IM is non-noetherian and M has a proper non-noetherian submodule. If M/IM is non-noetherian, then M/IM is also a non-noetherian R/I-module. Let $\eta : M \to M/IM$ be the R-quotient map and $N \subset M/IM$ a proper non-noetherian R/I-submodule. The set $\eta^{-1}(N)$ has a natural R-module structure and is a proper non-noetherian submodule of M. Hence by 1.12, R is a tall ring. The last sentence follows from the fact that if I is a nilpotent ideal then $IM \neq M$ for any module $M \neq 0$. □

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