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Mechanics of a Pseudo-rigid Disc Rolling in a Plane on a Line

by

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Abstract

This thesis presents the theory of reducing a non-holonomically constrained Hamiltonian system using the "non-holonomic bracket" technique, and explores the dynamics of a pseudo-rigid disc rolling in a plane on a line. We reduce the system through two symmetries, and find a relative equilibrium. We study the stability of the relative equilibrium in the reduced space using local eigenvalue techniques. Finally, we compare our results to other authors.

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Chapter 1

Introduction

In this thesis we shall explore, analytically, the mechanics of a pseudo-rigid disc rolling in the vertical plane along a straight line. Our goal is to gain further insight into the mechanics of rolling devices. We choose a pseudo-rigid disc for two reasons. First, to simplify our analysis, as a continuously de-formable disc would be nearly intractable analytically. Second, to ensure that we remain within the framework of modern Hamiltonian Mechanics.

The realm of pseudo-rigid mechanics lies between rigid Classical Mechanics and modern Continuum Mechanics. Bodies may deform, but only according to a finite number of parameters. In our case, the pseudo-rigid disc may only deform according to an affine linear transformation. This rescues us from the partial differential equations and infinite dimensions of Continuum Mechanics. Instead, we are able to exploit the rich theory of Hamiltonian Mechanics in finite dimensions. Following modern treatments of Hamiltonian Mechanics, we describe the configuration of the disc as a differentiable manifold. The governing Hamiltonian equations of motion are expressed in terms of the canonical symplectic form on the tangent bundle of the configuration space. In this way, we are able to exploit the geometry of the differential manifold instead of naively looking for solutions to a system of ordinary differential equations. As we shall see, this framework allows us to gain more qualitative knowledge than would be apparent from merely a set of equations of motion.

We impose two constraints on our rolling disc: that it remain in contact with the horizontal surface that it rolls upon (the contact constraint), and that it rolls without slipping (the no-slip constraint). As we shall demonstrate, the contact condition is a *holonomic constraint* - it is a restriction on the position of the system. The no-slip condition is a *linear non-holonomic constraint* - it is a restriction on the position and velocity of the system, and is non-integrable.

There are several different categories of symmetry, each yielding slightly different reduction techniques of varying complexity. Our pseudo-rigid rolling disc exhibits a symmetry with a *free* and *proper* action. The reduction technique employed in Chapter 3 is that of *non-holonomic brackets.* We will reduce the system and find a relative equilibrium of the reduced system corresponding to the "conveyor belt" motion.

We begin by presenting a historical motivation and the essential elements of the theory in Chapter 2. None of the results presented in Chapter 2 are new. In Chapter 3 we apply the theoretical results to our pseudo-rigid rolling disc. Most of the ground work for the reduction in Chapter 3 was done by hand; some tedious calculations were done using a computer algebra system.

Chapter 2

Theory

Historically, *Newton's laws of motion* formed the foundations of mechanics. The *Lagrangian* formulation is based upon the variational *Hamilton's principle*, which states:

$$\delta \int_a^b L(q^i, \dot{q}^i, t)dt = 0, \quad i = 1, \ldots, n, \tag{2.1}$$

where $L(q^i, \dot{q}^i, t)$ is the *Lagrangian* of the system (usually kinetic minus potential energy), expressed in terms of position coordinates q^i and their velocities $\dot{q}^i$. From Hamilton's principle we can derive the familiar *Euler-Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, n. \tag{2.2}$$

Equations of motion derived from either Newton's laws of motion or the Euler-Lagrange equations are second-order differential equations. Through the *Legendre transformation*, we can reformulate these second-order equations as a system of $2n$ first-order equations called *Hamilton's equations*. Hamilton's equations are:

$$\dot{q}^i = \frac{\partial H}{\partial p^i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n \tag{2.3}$$

where the Hamiltonian H is the total energy (usually kinetic plus potential energy) expressed in terms of position coordinates q^i and their conjugate momenta p^i.

In modern mathematics, a *Hamiltonian system* (M, ω, h) consists of a symplectic manifold (M, ω) and a smooth function $h : M \to \mathbb{R}$, called the *Hamiltonian* of the system. The vector field X_h determined by

$$X_h \lrcorner\, \omega = \mathrm{d}\, h \tag{2.4}$$

where $\lrcorner$ denotes the left-interior product, is called the *Hamiltonian vector field* corresponding to the Hamiltonian h. Non-degeneracy of ω guarantees that X_h exists [7, p187]. In local coordinates, $\omega = \mathrm{dq}^i \wedge \mathrm{dp}_i$ is the canonical symplectic form. Letting $X_h = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i}$, (2.4) becomes:

$$\dot{q}^i\, \mathrm{dp}_i - \dot{p}_i\, \mathrm{dq}^i \;=\; \frac{\partial h}{\partial p_i}\, \mathrm{dp}_i + \frac{\partial h}{\partial q_i}\, \mathrm{dq}^i\,. \tag{2.5}$$

Equating the dq^i and dp_i terms on both sides, we obtain:

$$\dot{q}^i = \frac{\partial h}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial h}{\partial q_i}, \quad i = 1, \ldots, n \tag{2.6}$$

which are exactly Hamilton's equations of motion (2.3). In this way, Hamilton's equations of classical mechanics and Hamiltonian systems of modern mathematics are related: integral curves of X_h satisfy Hamilton's equations (2.3) and are, therefore, classical equations of motion. The modern mathematical approach is powerful because the geometry of the problem is made explicit through the symplectic form. Instead of simply solving (2.3) to obtain equations of motion, we can appeal to the rich set of geometrical tools of differential geometry and gain more qualitative knowledge of the system.

2.1 Mechanical Systems with Constraints

A mechanical system is described by a smooth manifold, usually denoted Q, called the *configuration space* of the system. The coordinates of a point in the system's configuration space uniquely specify the location of each physical point of the mechanism in space. The number of coordinates is called the number of *degrees of freedom* of the system. The evolution of the mechanism in physical space is described by the evolution of a point in the system's configuration space [1, p2]. *Constraints* are restrictions on the position and/or velocity of the system. To describe a constraint we derive (usually from physical arguments) a set of *constraint functions*:

$$c_i : TQ \to \mathbb{R}, \qquad i = 1, \ldots, l, \tag{2.7}$$

where the i'th constraint is satisfied when $c_i(q, v) = 0$. A constraint is *linear* if the corresponding constraint function is a linear homogeneous function of v. We restrict ourselves to considering linear constraints only. We then define the *constraint forms* ϕ^i through:

$$c_i : v \mapsto \langle \phi^i(\tau_Q(v)) \,|\, v \rangle, \qquad i = 1, \ldots, l, \tag{2.8}$$

and the *constraint distribution* D by:

$$D = \bigcap_{i=1}^{l} \ker \phi^i \subset TQ. \tag{2.9}$$

Intuitively, given a configuration q, $D_q \subset T_qQ$ is the set of admissible velocities at q. We assume that the constraint distribution D is a sub-manifold of TQ. *Holonomic* constraints are integrable constraints; *non-holonomic* constraints are non-integrable constraints. A theorem of Frobenius tells us when a constraint is integrable or not. [1, p11]

Theorem 2.1 (Frobenius). If a distribution $D \subseteq TQ$ is involutive, then D is integrable. A distribution D is involutive if, for every pair X, Y of vector fields on Q with values in D, $[X, Y]$ has values in D as well. [3, p152]

If D is integrable we can effectively reduce the dimension of Q, and hence the number of degrees of freedom of the system, by imposing the constraint. Usually this is accomplished by eliminating a coordinate of Q through the constraint function. The evolution of the reduced system is governed by Hamilton's principle.

If D is not integrable, Hamilton's principle *can not* be applied. However, all is not lost – we can generalise Hamilton's principle to handle non-holonomic constraints. The generalisation of Hamilton's principle is called the *Hamilton-d'Alembert principle*. We begin with Newton's equations and arrive at the *d'Alembert principle* after imposing the non-holonomic constraints and making two fundamental hypotheses. We then reformulate the d'Alembert principle as the Hamilton-d'Alembert principle in a manner reminiscent of the reformulation of the Euler-Lagrange equations into Hamilton's equations. For the remainder of the chapter, we follow J.J. Duistermaat, R. Cushman, and J. Śniatycki work in *Geometry of Nonholonomic Systems* [4].

2.2 Newton's equations and the d'Alembert principle

The kinetic energy of a system can be expressed as $\frac{1}{2}\,\mathrm{k}(v,v)$ for some Riemannian metric k on Q. We call k the *kinetic energy metric* of the system. The metric k gives rise to two vector bundle isomorphisms

$$\mathrm{k}^\flat : TQ \to T^*Q. \quad \text{and} \quad \mathrm{k}^\sharp : T^*Q \to TQ \tag{2.10}$$

such that

$$\langle \mathrm{k}^\flat(v) | u \rangle = \mathrm{k}(v,u) \quad \forall\, u, v \in T_qQ \tag{2.11}$$

and $\mathrm{k}^\sharp = (\mathrm{k}^\flat)^{-1}$.

Let $t \mapsto q(t)$ be a smooth curve in Q describing a motion of our system. Then, $t \mapsto \dot{q}(t)$ denotes its lift to TQ, and $t \mapsto \ddot{q}(t)$ denotes the covariant derivative of $\dot{q}(t)$ with respect to the Levi-Civita connection of the kinetic energy metric. That is, $\dot{q}(t)$ is the velocity, and $\ddot{q}(t)$ is the acceleration of the system at time t. The *external force* acting on the system in configuration q is a covector in T_q^*Q. We can express the force as a map $\phi : \mathbb{R} \times TQ \longrightarrow T^*Q$. That is, $\phi(t, v)$ is the force acting on the system at time t, position q, and velocity v. In this way, we can express Newton's equations governing the motion $t \mapsto q(t)$, as:

$$\mathrm{k}^\flat\left(\ddot{q}(t)\right) = \phi\left(t, \dot{q}(t)\right). \tag{2.12}$$

Non-holonomically constrained systems are acted on by two forces: an external force $\phi_{ext} \in T^*Q$ and a reaction force $\phi_{constr} \in T^*Q$ of the constraints. The reaction force is exerted by the constraints to ensure that the constraints are satisfied. Before proceeding, we make three fundamental hypothesis:

Hypothesis 2.1. That the external and reaction forces are *additive.* That is, $\phi_{total} = \phi_{ext} + \phi_{constr}$.

Hypothesis 2.2. That the external force is *conservative.* That is, $\phi_{ext} = -\mathrm{d}\, f$ where $f \in C^\infty(Q)$.

Hypothesis 2.3. That the constraint D is *perfect.* That is, ϕ_{constr} does no work on virtual displacements in D. In other words:

$$\phi_{constr}(v) \in D^0_{\tau_Q(v)} \quad \forall\, v \in D, \tag{2.13}$$

where $D_q^0 = \left\{p \in T_q^*Q \,|\, \langle p \,|\, v\rangle = 0 \quad \forall\, v \in D_q\right\}$ is the annihilator of D_q.

With Hypothesis 2.1 and Hypothesis 2.2, Newton's equation (2.12) becomes:

$$\mathrm{k}^\flat\left(\ddot{q}(t)\right) = -\,\mathrm{d}\,f\left(q(t)\right) + \phi_{constr}\left(t, \dot{q}(t)\right) \tag{2.14}$$

A motion given by $t \mapsto q(t) \in Q$ with $\dot{q}(t) \in D$ which satisfies Newton's equations with constraint force $\phi_{constr} \in D^0$ is called a *dynamically admissible* motion. With Hypothesis 2.3, we arrive at:

Theorem 2.2 (d'Alembert principle). A smooth curve $t \mapsto q(t)$ is a dynamically admissible motion of a mechanical system with kinetic energy $k(v) = \frac{1}{2}\,\mathrm{k}(v, v)$ and potential energy $f \in C^\infty(Q)$ subject to the linear non-holonomic constraint $D \subseteq TQ$ if and only if

$$\mathrm{k}^\flat\left(\ddot{q}(t)\right) + \mathrm{d}\,f\left(q(t)\right) \in D^0_{q(t)} \quad \text{and} \quad \dot{q}(t) \in D_{q(t)}. \tag{2.15}$$

We note that, similar to the Euler-Lagrange equations, equation (2.15) is second-order.

2.3 Hamilton's equations and the Hamilton-d'Alembert principle

We now develop the Hamilton-d'Alembert principle for non-holonomically constrained systems. While equivalent to the d'Alembert principle, it is first-order: the Levi-Civita connection is absent.

The canonical 1-form on T^*Q is defined by:

$$\langle \theta_Q | u \rangle = \langle p \,|\, T\pi_Q(u) \rangle \quad \forall\, u \in T_p(T^*Q). \tag{2.16}$$

The canonical symplectic form on T^*Q is $\omega_Q = -\,\mathrm{d}\,\theta_Q$. Since $\mathrm{k}^\flat$ is a diffeomorphism, $\omega = (\mathrm{k}^\flat)^*\omega_Q$ is a symplectic form on TQ (where $(\mathrm{k}^\flat)^*\omega_Q$ is the pull-back of the two-form ω_Q). In other words, ω is a nondegenerate and closed two-form on TQ. It is interesting to note that the projection from TQ to Q under τ_Q of an integral curve of the Hamiltonian vector field X_k of the kinetic energy $k = \frac{1}{2}\,\mathrm{k}(v, v)$ is a geodesic of the Levi-Civita connection of k. In the theory of general relativity, geodesics play a central role – in travelling through curved space-time, particles follow geodesics.

We define

$$F = \{w \in T(TQ)\,|\,T\tau_Q(w) \in D\} \subset T(TQ), \tag{2.17}$$

and note that F is a distribution on TQ. In local coordinates, $T\tau_Q(q, v, \dot{q}, \dot{v}) = (q, \dot{q})$, and therefore F consists of all vectors $(q, v, \dot{q}, \dot{v}) \in T(TQ)$ such that $(q, \dot{q}) \in D$. We denote by F^0 the annihilator of F. That is, for $u \in TQ$

$$F_u^0 = \{p \in T^*(TQ)\,|\,\langle p|w\rangle = 0 \quad \forall w \in F_u\} \subset T^*(TQ) \tag{2.18}$$

Consistent with Hypothesis 2.3, forces in F^0 do no work on virtual displacements in F.

We now restate the d'Alembert principle in terms of the position q and its lift $\dot{q} \in TQ$, the velocity $u = \dot{q}$ and its lift $\dot{u} \in T(TQ)$, the Hamiltonian h, and the symplectic form ω.

Theorem 2.3 (Hamilton-d'Alembert principle). A curve $(t_0, t_1) \to Q : t \mapsto q(t)$ with lift $t \mapsto u(t) = \dot{q}(t) \in D$ is a dynamically admissible motion corresponding to the Hamiltonian $h = k + \tau_Q^* f$ subject to the non-holonomic constraint D if and only if

$$\dot{u}(t) \lrcorner\, \omega - \mathrm{d}\, h(\dot{q}(t))) \in F^0_{\dot{q}(t)} \tag{2.19}$$

for every $t \in (t_0, t_1)$.

2.4 Distributional Hamiltonian formulation

The Hamilton-d'Alembert principle does not give us an explicit equation for the motion of the system. In this section we derive such an equation.

We define $H = F \cap TD \subset T(TQ)$. H is a symplectic distribution on D. A point $(q, v, \dot{q}, \dot{v}) \in H$ is such that $(q, \dot{q}) \in D$, and $(\dot{q}, \dot{v}) \in TD$. The restriction of ω to vectors in H is non-degenerate, and denoted by ϖ. In this way, H is a symplectic distribution on D.

We define the *distributional Hamiltonian vector field* of $h \in C^\infty(D)$ with respect to the symplectic distribution (H, ϖ) on D to be the unique vector field Y_h on D with values in H such that:

$$Y_h \lrcorner\, \varpi = \partial_H h. \tag{2.20}$$

where $\partial_H h$ is the restriction of $\mathrm{d}\, h$ to H.

Theorem 2.4. A curve $t \mapsto q(t)$ in Q describes a dynamically admissible motion of a Hamiltonian system (TQ, h, ω), where $h = k + \tau_Q^* f$ subject to the linear non-holonomic constraint D if and only if it is the projection to Q under τ_Q of an integral curve $t \mapsto u(t)$ of the distributional Hamiltonian vector field Y_h of h.

Proof. Suppose that $t \mapsto q(t)$ describes a dynamically admissible motion of the system. Let $u(t) = \dot{q}(t)$. Then $\dot{u}(t) \in TD$ since $\dot{q} \in D$. Since $\dot{q}(t) = T\tau_Q(\dot{u}(t))$, if follows that $\dot{u}(t) \in F$. Therefore, $\dot{u}(t) \in H$.

Each $w \in F_{\dot{q}(t)}$ can be decomposed into $w = w_H + w_{H^\omega}$ with $w_H \in H$ and $w_{H^\omega} \in H^\omega$, where $H^\omega = \{w \in T(TQ) \,|\, \omega(w, v) = 0 \,\forall v \in H\}$ is the symplectic annihilator of H. Then

$$\langle \dot{u}(t) \lrcorner\, \omega - \mathrm{d}\, h \,|\, w \rangle = \langle \dot{u}(t) \lrcorner\, \omega - \mathrm{d}\, h \,|\, w_H + w_{H^\omega} \rangle \tag{2.21}$$

$$= \langle \dot{u}(t) \lrcorner\, \varpi - \partial_H h \,|\, w_H \rangle - \langle \partial_{H^\omega} h \,|\, w_{H^\omega} \rangle, \tag{2.22}$$

since $\dot{u}(t) \in H$. By the Hamilton-d'Alembert principle $\dot{u}(t) \lrcorner\, \omega - \mathrm{d}\, h(\dot{q}(t))) \in F^0_{\dot{q}(t)}$. Therefore, since $H \subseteq F$, we obtain:

$$\dot{u}(t) \lrcorner\, \varpi = \partial_H h, \tag{2.23}$$

and

$$\langle \mathrm{d}\, h | u \rangle = 0 \quad \forall u \in F \cup H^\omega. \tag{2.24}$$

Conversely, if $t \mapsto u(t)$ is an integral curve of the distributional Hamiltonian vector field Y_h of h, then equation (2.23) is trivially satisfied. Since Y_h has values in H and $F \cup H^\omega = F$ (proved in [4]), equation (2.24) is satisfied. Therefore equation (2.21) implies that $\dot{u} \lrcorner\, \omega - \mathrm{d}\, h \in F^0_{u(t)}$. Furthermore, if $t \mapsto q(t) = \tau_Q(u(t))$ is the projection of $t \mapsto u(t)$, then $\dot{q}(t) = u(t)$ (proved in [4]). □

Therefore, the governing equations for integral curves $t \mapsto u(t)$ of the distributional Hamiltonian vector field Y_h are:

$$\dot{u}(t) \lrcorner\, \varpi = \partial_H h(u(t)). \tag{2.25}$$

This is called the *distributional form of Hamilton's equations of motion.*

2.5 Non-holonomic brackets

The distributional form of Hamilton's equations can be reformulated in terms of a *non-holonomic bracket*, which is straight-forward to compute, and hence easier to work with compared to equation (2.25). In fact, once we have calculated the Poisson brackets between all the coordinate functions of D, we can use a computer algebra system to compute the non-holonomic brackets.

For every $f_1, f_2 \in C^\infty(D)$, we define the bracket $\{\ ,\ \}$ by:

$$\{f_1, f_2\} = \langle \mathrm{d}\, f_2 | Y_{f_1} \rangle \tag{2.26}$$

where Y_{f_1} is the distributional Hamiltonian vector field of f_1. The bracket defined above is the negative of the usual one; ie, the order of h and f in (2.28) is reversed. The bracket defined above is bilinear, anti-symmetric, and satisfies Leibniz' rule:

$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2 \{f_1, f_3\} \tag{2.27}$$

for all $f_1, f_2, f_3 \in C^\infty(D)$. We refer to this bracket as the *non-holonomic bracket.* It satisfies all the axioms of a Poisson algebra except the Jacobi identity (which is: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for all X, Y, Z.) – hence D is an almost Poisson algebra [10, p236]. Using this bracket, we can write the distributional form of Hamilton's equations of motion as:

$$\dot{f}(u(t)) = \{h, f\}(u(t)). \tag{2.28}$$

Consider the Poisson bracket between two functions, restricted to D:

$$
\begin{aligned}
[f_1, f_2]_D &= \langle \mathrm{d}\, f_2 | X_{f_1} \rangle_D \\
&= \langle \partial_H f_2 + \partial_{H^\omega} f_2 | Y_{f_1} + Z_{f_1} \rangle \\
&= \{f_1, f_2\} + \langle \mathrm{d}\, f_2 | Z_{f_1} \rangle, \quad \text{then} \qquad (2.29) \\
\{f_1, f_2\} &= [f_1, f_2]_D - \langle \mathrm{d}\, f_2 | Z_{f_1} \rangle. \qquad (2.30)
\end{aligned}
$$

Therefore, to compute non-holonomic brackets we need to evaluate the usual Poisson bracket and $\langle \mathrm{d}\, f_2 | Z_{f_1} \rangle$. This can be reduced to a matrix equation, as done in [4]. Let

$$
A = \begin{pmatrix} ([c_j, c_i]) & ([c_j, \tau_Q^* \phi_i]) \\ ([\tau_Q^* \phi_j, c_i]) & ([\tau_Q^* \phi_j, \tau_Q^* \phi_i]) \end{pmatrix} \qquad (2.31)
$$

where

$$
\begin{aligned}
[f_1, f_2] &= \langle \mathrm{d}\, f_2 | X_{f_1} \rangle, \quad f_1, f_2 \in C^\infty(D) \qquad (2.32) \\
&= \omega(\omega^\sharp(\mathrm{d}\, f_1), \omega^\sharp(\mathrm{d}\, f_2)) \\
[f_1, \tau_Q^* \phi_i] &= \langle \tau_Q^* \phi_i | X_{f_1} \rangle, \quad \phi_i \in T^*Q \qquad (2.33) \\
&= \omega(\omega^\sharp(\mathrm{d}\, f_1), \omega^\sharp(\tau_Q^* \phi_i)) \\
[\tau_Q^* \phi_i, \tau_Q^* \phi_j] &= \omega(\omega^\sharp(\tau_Q^* \phi_i), \omega^\sharp(\tau_Q^* \phi_j)). \qquad (2.34)
\end{aligned}
$$

Let $A^{-1} = (A^{ij})$ be the inverse of A. Then

$$
\{f_1, f_2\} = [f_1, f_2] - \sum_{i,j} ([f_2, c_i], [f_2, \tau_Q^* \phi_i])\, A^{ij} \begin{pmatrix} [f_1, c_j] \\ [f_1, \tau_Q^* \phi_j] \end{pmatrix}. \qquad (2.35)
$$

The above definition of the non-holonomic bracket may remind us of the "new Poisson bracket" as defined by P.A.M. Dirac in [5, p138]. He defines the new Poisson bracket $[\xi,\eta]^*$ between two functions ξ and η by:

$$[\xi,\eta]^* = [\xi,\eta] + [\xi,\theta_s]c_{ss'}[\theta_{s'},\eta]$$

where the θ_s are functions of positions q and their conjugate momenta p, and $c_{ss'}$ are related to the co-factors of $[\theta_s,\theta_{ss'}]$. The θ_s serve to transform the form of the Hamiltonian system. If the $\partial L/\partial\dot{q}$'s, where L is the Lagrangian, are not independent functions of the velocities, then there exist extra relations between positions and momenta. Dirac shows that these relations reduce the degrees of freedom of the system, and that they may be supplemented by extra relations between positions and momenta (ie, constraints) to further reduce the system. Taking the θ_s to be these relations, Dirac shows that the new Poisson brackets give rise to Hamilton's equations: $\dot{f} = [f,h]^*$. Furthermore, the resulting set of equations are simpler because the number of degrees of freedom are reduced.

2.6 Symmetry and Reduction

A *symmetry group* of a system is a group whose action preserves the distribution H, the two-form ϖ on H, and the Hamiltonian h.

Lemma 2.1. The action of a symmetry group preserves the non-holonomic bracket.

Proof. From (2.35) we note that the non-holonomic bracket can be defined entirely in terms of the Poisson bracket, which in turn is defined through the symplectic form ϖ (see (2.32), (2.33), and (2.34)). Therefore, since the action of a symmetry group preserves ϖ, then it also preserves the non-holonomic bracket. □

In the holonomic case, Noether's theorem states that a function $f \in C^\infty(TQ)$ is a constant of motion if and only if its Hamiltonian vector field X_f preserves the Hamiltonian h. Therefore, in the unconstrained case each one-parameter symmetry group has an associated conserved quantity, usually called a momentum. In general, this is not the case for constrained systems. The result that the distributional Hamiltonian vector field Y_f preserve the Hamiltonian h does not imply that it conserves either H or ϖ. That is, Y_f is not necessarily an infinitesimal symmetry of the constrained system, whereas X_f is an infinitesimal symmetry of the unconstrained system.

When a symmetry is present, we can *reduce* the phase space. That is, we can reduce the number of degrees of freedom of the system. Solutions of the reduced system are then reconstructed in the original system..

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let

$$\Phi : G \times D \to D : (g, u) \mapsto \Phi(g, u) = gu \tag{2.36}$$

be the smooth action of G on D. The group G is a *symmetry group* of the nonholomically constrained Hamiltonian system (D, H, ϖ, h) if the action Φ preserves the distribution H, the two-form ϖ on H, and the Hamiltonian h. [4]

The action Φ defines an equivalence relation on D by the relation of belonging to the same orbit. That is, for $u, v \in D$, we write $u \sim v$ if there exists a $g \in G$ such that $\Phi_g(u) = gu = v$. We denote by D/G the set of all such equivalence classes (orbits). This is often called the *orbit space.*

An action Φ is said to be *free* if it has no fixed points. That is, if $\Phi_g(u) = u$ implies that $g = e$, then Φ is free. An action is said to be *proper* if the mapping $\tilde{\Phi} : G \times D \to D \times D$ defined by $\tilde{\Phi}(g, u) = (u, \Phi_g(u))$ is proper. That is, if for every sequence (g_n, u_n) in $G \times D$ such that $g_n u_n$ converges to $v \in D$, there exists a subsequence (g_{n_k}, u_{n_k}) converging to (g, u) such that $v = gu$, then Φ is proper. If G is compact, this condition is automatically satisfied. [8] [4]

If the action Φ is free and proper, then $\bar{D} = D/G$ is a smooth manifold and $\rho : D \rightarrow \bar{D} : u \mapsto Gu$ is a smooth, locally trivial projection. All intrinsically defined vector fields, distributions, and brackets on D can be pushed forward onto the orbit space $\bar{D}$ via ρ. In particular, the vector field Y_h that describes the motion of the system, the distribution H, and the non-holonomic bracket $\{\, ,\}$ push down to $\bar{D}$. The non-holonomic bracket on D induces a non-holonomic bracket $\{\, ,\}_{\bar{D}}$ on $\bar{D}$ such that

$$\rho^* \{\bar{f}_1, \bar{f}_2\}_{\bar{D}} = \{\rho^* \bar{f}_1, \rho^* \bar{f}_2\} \tag{2.37}$$

for all $\bar{f}_1, \bar{f}_2 \in C^\infty(\bar{D})$, where $\rho^* : C^\infty(\bar{D}) \rightarrow C^\infty(D)$ is the pull-back by the orbit map ρ.

Let $t \mapsto \bar{u}(t)$ be a curve in the orbit space $\bar{D}$. It is an integral curve of the *inner derivation* corresponding to $\bar{h} \in C^\infty(\bar{D})$ if

$$\frac{d}{dt} \bar{f}(\bar{u}(t)) = \{\bar{h}, \bar{f}\}(\bar{u}(t)) \tag{2.38}$$

for all $\bar{f} \in C^\infty(\bar{D})$.

Theorem 2.5. For every G-invariant function h on D, the projections to $\bar{D}$ of integral curves of Y_h are integral curves of the inner derivation corresponding to $\bar{h}$, where $h = \rho^* \bar{h}$. [10]

By assumption, the Hamiltonian h is G-invariant. Therefore, solutions to Hamilton's equations of motion (2.28) push down to solutions of (2.38) on $\bar{D}$.

This leads us to the following reduction technique: first, using the algebra of G-invariant functions on D, we construct a set of basis functions for $\bar{D}$; second, using (2.37) and (2.38) we compute the equations of motion for the basis functions of $\bar{D}$; and finally, we reconstruct the equations of motion for $\bar{D}$ in D.

Chapter 3

Application

3.1 Statement of the Problem

The reference object of the rolling pseudo-rigid disc is a planar, solid disc of radius R, uniform mass distribution ρ, total mass M, and elastic shear modulus equal to μM. The disc rolls, without slipping, in the vertical plane on a straight line, under the influence of a constant gravitational force G. The disc may deform according to an affine linear transformation.

We use X and Y as coordinates for the reference space, and $\bar{X}$ and $\bar{Y}$ as coordinates for physical space. A reference point on the disc can be described with polar coordinates (r, θ):

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \tag{3.1}$$

where $r \in [0, R]$, and $\theta \in [0, 2\pi]$. Polar coordinates are used to derive the kinetic energy and constraint functions.

3.2 Configuration

The configuration of the flexible disc is described by a pair (B, b), where $B \in GL(2, \mathbb{R})$ is a linear deformation, and $b \in \mathbb{R}^2$ is a translation. The collection of all such pairs (B, b) forms a Lie group Q. As a manifold $Q = GL(2, \mathbb{R}) \times \mathbb{R}^2$. As a group, its multiplication is given by:

$$(A,a)(B,b) = (AB, Ab + a). \tag{3.2}$$

Coordinates for Q are x, y, e, f, g, and h, where

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}; \quad b = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.3}$$

Alternatively, the pair (B, b) can also be written as a 3×3 matrix:

$$\boldsymbol{B} = (B, b) = \begin{pmatrix} e & f & x \\ g & h & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

With this representation, group multiplication is simply given by matrix multiplication. In configuration q, the reference point (X, Y) is physically located at:

$$\begin{aligned} \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} &= B \begin{pmatrix} X \\ Y \end{pmatrix} + b \\ &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} er\cos\theta + fr\sin\theta + x \\ gr\cos\theta + hr\sin\theta + y \end{pmatrix}. \end{aligned} \tag{3.5}$$

In the rigid case $B \in SO(2, \mathbb{R})$, and therefore (B, b) would reduce to:

$$B = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}; \quad b = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.6}$$

where $\phi \in \mathbb{R}$.

3.3 Hamiltonian

3.3.1 Kinetic Energy

The kinetic energy of the flexible disc is given by:

$$
\begin{aligned}
T &= \frac{1}{2}\iint \rho\left(\dot{X}^2+\dot{Y}^2\right) dA \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^R \rho\left((\dot{e}r\cos\theta+\dot{f}r\sin\theta+\dot{x})^2+(\dot{g}r\cos\theta+\dot{h}r\sin\theta+\dot{y})^2\right) r dr d\theta \\
&= \frac{M}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{MR^2}{8}\left(\dot{e}^2+\dot{f}^2+\dot{g}^2+\dot{h}^2\right) \\
&= \frac{M}{2}\dot{b}^2+\frac{MR^2}{8}\operatorname{tr}\left(\dot{B}^T\dot{B}\right).
\end{aligned} \tag{3.7}
$$

3.3.2 Potential Energy

The potential energy is given by:

$$
V = MGy + W(e^2+f^2+g^2+h^2, eh-fg) \tag{3.8}
$$

where W is an arbitrary function of the two strain invariants, representing the stored elastic energy. We choose a Neo-hookean material (one which obeys Hooke's law – ie, the elastic restoring force depends linearly upon displacement), for which

$$
W(e^2+f^2+g^2+h^2, eh-fg) = \frac{\mu M}{2}\left(e^2+f^2+g^2+h^2+\frac{1}{(eh-fg)^2}-3\right).
$$

The -3 ensures that there is no stored elastic energy when the disc is not deformed. When the disc is not deformed the trace would be 2, and the determinant would be 1. The resulting potential energy is:

$$
\begin{aligned}
V &= MGy+\frac{\mu M}{2}\left(e^2+f^2+g^2+h^2+\frac{1}{(eh-fg)^2}-3\right) \\
&= MG\begin{pmatrix}0\\1\end{pmatrix}\cdot b+\frac{\mu M}{2}\left(\operatorname{tr}\left(B^TB\right)+\frac{1}{(\det B)^2}-3\right)
\end{aligned} \tag{3.9}
$$

3.3.3 Hamiltonian

The Hamiltonian (total energy function), denoted by E, is:

$$\begin{aligned} E &= T+V \\ &= \frac{M}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{MR^2}{8}\left(\dot{e}^2+\dot{f}^2+\dot{g}^2+\dot{h}^2\right) \\ &\quad +MGy+\frac{\mu M}{2}\left(e^2+f^2+g^2+h^2+\frac{1}{(eh-fg)^2}-3\right) \end{aligned} \tag{3.10}$$

3.3.4 Symplectic Form

The canonical symplectic form on T^*Q is

$$\omega_Q = \mathrm{dx}\wedge\mathrm{dp_x}+\mathrm{dy}\wedge\mathrm{dp_y}+\mathrm{de}\wedge\mathrm{dp_e}+\mathrm{df}\wedge\mathrm{dp_f}+\mathrm{dg}\wedge\mathrm{dp_g}+\mathrm{dh}\wedge\mathrm{dp_h}\ , \tag{3.11}$$

where p_k is the momentum in the direction k. Using the kinetic energy metric, we pull ω_Q back to TQ:

$$\begin{aligned} \omega &= M\left(\mathrm{dx}\wedge\mathrm{d\dot{x}}+\mathrm{dy}\wedge\mathrm{d\dot{y}}\right) \\ &\quad +\frac{MR^2}{4}\left(\mathrm{de}\wedge\mathrm{d\dot{e}}+\mathrm{df}\wedge\mathrm{d\dot{f}}+\mathrm{dg}\wedge\mathrm{d\dot{g}}+\mathrm{dh}\wedge\mathrm{d\dot{h}}\right). \end{aligned} \tag{3.12}$$

3.4 Constraints

We impose two constraints on the disc: first, that it remains in contact with the surface upon which it is rolling; second, that it rolls without slipping.

In order to derive the corresponding constraint functions, we must find the point of contact. The boundary of the disc is parameterised by θ:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} eR\cos\theta+fR\sin\theta+x \\ gR\cos\theta+hR\sin\theta+y \end{pmatrix}. \tag{3.13}$$

The tangent vector to the boundary is given by differentiating (3.13) with respect to θ:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}' = \begin{pmatrix} -eR\sin\theta + fR\cos\theta \\ -gR\sin\theta + hR\cos\theta \end{pmatrix}. \tag{3.14}$$

At the point of contact, denoted by $\theta = \theta_c$, the vertical component of (3.14) must be zero. That is, the tangent vector to the boundary is horizontal at the point of contact. Therefore:

$$gR\sin\theta_c = hR\cos\theta_c \tag{3.15}$$

$$\tan\theta_c = \frac{h}{g}. \tag{3.16}$$

Of the two solutions for θ_c, we choose the one corresponding to the lower of the two boundary points. Therefore:

$$\theta_c = \pi + \arctan\frac{g}{h}. \tag{3.17}$$

3.4.1 Contact

In order for the flexible disc to remain in contact with the surface that it is rolling upon, the vertical component of the position of the point of contact must be always zero. That is, $\bar{Y} = 0$. Therefore, substituting $\theta = \theta_c$ into (3.13), we have:

$$\begin{aligned} y &= -gR\cos\theta_c - hR\sin\theta_c \\ &= -gR\cos(\pi + \arctan\frac{h}{g}) - hR\sin(\pi + \arctan\frac{h}{g}) \\ &= gR\frac{g}{\sqrt{g^2+h^2}} + hR\frac{h}{\sqrt{g^2+h^2}} \\ &= R\sqrt{g^2+h^2} \end{aligned} \tag{3.18}$$

Note that $g^2+h^2 \neq 0$ since $B \in GL(2,\mathbb{R})$. The contact constraint yields a holonomic constraint:

$$y = R\sqrt{g^2+h^2}. \tag{3.19}$$

In the rigid case this would reduce to $y = R$ (see (3.6)).

3.4.2 No-slip Rolling

In order for the flexible disc to roll without slipping, the horizontal velocity of the point of contact must be always zero. That is, $\dot{X} = 0$. Therefore, differentiating (3.13) with respect to time and substituting $\theta = \theta_c$, we have:

$$\begin{aligned}
\dot{x} &= -\dot{e}R\cos\theta_c - \dot{f}R\sin\theta_c \\
&= -\dot{e}R\cos(\pi + \arctan\frac{h}{g}) - \dot{f}R\sin(\pi + \arctan\frac{h}{g}) \\
&= \dot{e}R\frac{g}{\sqrt{g^2+h^2}} + \dot{f}R\frac{h}{\sqrt{g^2+h^2}} \\
&= R\frac{\dot{e}g+\dot{f}h}{\sqrt{g^2+h^2}}
\end{aligned} \tag{3.20}$$

The no-slip condition yields a linear non-holonomic constraint:

$$\dot{x} = R\frac{\dot{e}g+\dot{f}h}{\sqrt{g^2+h^2}}. \tag{3.21}$$

In the rigid case, this would reduce to $\dot{x} = -R\dot{\phi}$ (see (3.6)).

3.5 Holonomic Reduction

The contact constraint (3.19) is clearly holonomic – it does not depend on the velocity of the system. Therefore, we can reduce the system by eliminating y. Let Q' be an embedding of Q given by (3.19). Its inclusion into Q is given by:

$$i : Q' \to Q : (x, e, f, g, h) \mapsto (x, R\sqrt{g^2 + h^2}, e, f, g, h) \tag{3.22}$$

Then

$$\begin{aligned} Ti : TQ' \to TQ \quad : \quad & \dot{x}\frac{\partial}{\partial x} + \dot{e}\frac{\partial}{\partial e} + \dot{f}\frac{\partial}{\partial f} + \dot{g}\frac{\partial}{\partial g} + \dot{h}\frac{\partial}{\partial h} \\ & \mapsto \dot{x}\frac{\partial}{\partial x} + R\frac{g\dot{g} + h\dot{h}}{\sqrt{g^2 + h^2}}\frac{\partial}{\partial y} + \dot{e}\frac{\partial}{\partial e} + \dot{f}\frac{\partial}{\partial f} + \dot{g}\frac{\partial}{\partial g} + \dot{h}\frac{\partial}{\partial h}. \end{aligned} \tag{3.23}$$

That is,

$$\dot{y} = R\frac{g\dot{g} + h\dot{h}}{\sqrt{g^2 + h^2}}. \tag{3.24}$$

3.6 Constraint Distribution

The no-slip constraint (3.21) is a non-holonomic constraint. We would like to construct the constraint distribution D and the distribution $H = F \cap TD$. From (3.21) we see that the constraint function is:

$$c(q, v) = \frac{\sqrt{g^2 + h^2}}{R}\dot{x} - g\dot{e} - h\dot{f}, \quad \text{and therefore} \tag{3.25}$$

$$\phi(q) = \frac{\sqrt{g^2 + h^2}}{R}\,\mathrm{dx} - g\,\mathrm{de} - h\,\mathrm{df} \in T_q^*Q. \tag{3.26}$$

Recall that the constraint distribution $D \subset TQ'$ is defined by (2.9), which is equivalent to:

$$D = \{u \in TQ' \mid \langle \phi \mid u \rangle = 0\}. \tag{3.27}$$

Claim 3.1. If $u \in D$, then

$$u = R\frac{g\dot{e} + h\dot{f}}{\sqrt{g^2 + h^2}}\frac{\partial}{\partial x} + \dot{e}\frac{\partial}{\partial e} + \dot{f}\frac{\partial}{\partial f} + \dot{g}\frac{\partial}{\partial g} + \dot{h}\frac{\partial}{\partial h} \tag{3.28}$$

Proof. Let $u = \dot{x}\frac{\partial}{\partial x} + \dot{e}\frac{\partial}{\partial e} + \dot{f}\frac{\partial}{\partial f} + \dot{g}\frac{\partial}{\partial g} + \dot{h}\frac{\partial}{\partial h} \in D$. Then

$$\langle \phi \mid u \rangle = \dot{x}\frac{\sqrt{g^2+h^2}}{R} - g\dot{e} - h\dot{f} = 0 \tag{3.29}$$

and therefore

$$\dot{x} = R\frac{g\dot{e} + h\dot{f}}{\sqrt{g^2+h^2}} \tag{3.30}$$

□

Note that, as a manifold, D has coordinates: x, e, f, g, h, $\dot{e}$, $\dot{f}$, $\dot{g}$, $\dot{h}$. Next, we find F; recall that

$$F = \{w \in T(TQ') \mid T\tau_{Q'}(w) \in D\}. \tag{3.31}$$

Claim 3.2. If $w \in F$, then

$$\begin{aligned} w \;=\; & R\frac{gw_e + hw_f}{\sqrt{g^2+h^2}}\frac{\partial}{\partial x} + w_e\frac{\partial}{\partial e} + w_f\frac{\partial}{\partial f} + w_g\frac{\partial}{\partial g} + w_h\frac{\partial}{\partial h} \\ & + w_{\dot{x}}\frac{\partial}{\partial \dot{x}} + w_{\dot{e}}\frac{\partial}{\partial \dot{e}} + w_{\dot{f}}\frac{\partial}{\partial \dot{f}} + w_{\dot{g}}\frac{\partial}{\partial \dot{g}} + w_{\dot{h}}\frac{\partial}{\partial \dot{h}} \end{aligned} \tag{3.32}$$

Proof. Let $w = w_x\frac{\partial}{\partial x} + w_e\frac{\partial}{\partial e} + \ldots + w_h\frac{\partial}{\partial h} + w_{\dot{x}}\frac{\partial}{\partial \dot{x}} + w_{\dot{e}}\frac{\partial}{\partial \dot{e}} + \ldots + w_{\dot{h}}\frac{\partial}{\partial \dot{h}} \in F$. Then

$$T\tau_{Q'}(w) = w_x\frac{\partial}{\partial x} + w_y\frac{\partial}{\partial y} + w_e\frac{\partial}{\partial e} + w_f\frac{\partial}{\partial f} + w_g\frac{\partial}{\partial g} + w_h\frac{\partial}{\partial h} \in D \tag{3.33}$$

and therefore, by comparison with (3.30), we must have:

$$w_x = R\frac{gw_e + hw_f}{\sqrt{g^2+h^2}}. \tag{3.34}$$

□.

Next, we find TD.

Claim 3.3. If $w \in TD$, then

$$\begin{aligned} w &= w_x \frac{\partial}{\partial x} + w_e \frac{\partial}{\partial e} + w_f \frac{\partial}{\partial f} + w_g \frac{\partial}{\partial g} + w_h \frac{\partial}{\partial h} \\ &\quad + R \frac{\left(g w_{\dot{e}} + w_g \dot{e} + h w_{\dot{f}} + w_h \dot{f}\right)(g^2 + h^2) - \left(g\dot{e} + h\dot{f}\right)(g w_g + h w_h)}{\sqrt{g^2 + h^2}^3} \frac{\partial}{\partial \dot{x}} \\ &\quad + w_{\dot{e}} \frac{\partial}{\partial \dot{e}} + w_{\dot{f}} \frac{\partial}{\partial \dot{f}} + w_{\dot{g}} \frac{\partial}{\partial \dot{g}} + w_{\dot{h}} \frac{\partial}{\partial \dot{h}} \end{aligned} \tag{3.35}$$

Proof. Let $t \mapsto v(t) = R\frac{g\dot{e}+h\dot{f}}{\sqrt{g^2+h^2}}\frac{\partial}{\partial x} + \dot{e}\frac{\partial}{\partial e} + \dot{f}\frac{\partial}{\partial f} + \dot{g}\frac{\partial}{\partial g} + \dot{h}\frac{\partial}{\partial h}$ be a curve in D, where x, e, f, g, h, $\dot{e}, \dot{f}, \dot{g}$, and $\dot{h}$ are all functions of time. Then, the $\frac{\partial}{\partial \dot{x}}$ component of the lift of $v(t)$ to TD is:

$$R \frac{\left(g w_{\dot{e}} + w_g \dot{e} + h w_{\dot{f}} + w_h \dot{f}\right)(g^2 + h^2) - \left(g\dot{e} + h\dot{f}\right)(g w_g + h w_h)}{\sqrt{g^2 + h^2}^3} \tag{3.36}$$

□

Now we are ready to find $H = F \cap TD$.

Claim 3.4. If $w \in H$, then

$$\begin{aligned} w &= R\frac{g w_e + h w_f}{\sqrt{g^2 + h^2}} \frac{\partial}{\partial x} + w_e \frac{\partial}{\partial e} + w_f \frac{\partial}{\partial f} + w_g \frac{\partial}{\partial g} + w_h \frac{\partial}{\partial h} \\ &\quad + R \frac{\left(g w_{\dot{e}} + w_g \dot{e} + h w_{\dot{f}} + w_h \dot{f}\right)(g^2 + h^2) - \left(g\dot{e} + h\dot{f}\right)(g w_g + h w_h)}{\sqrt{g^2 + h^2}^3} \frac{\partial}{\partial \dot{x}} \\ &\quad + w_{\dot{e}} \frac{\partial}{\partial \dot{e}} + w_{\dot{f}} \frac{\partial}{\partial \dot{f}} + w_{\dot{g}} \frac{\partial}{\partial \dot{g}} + w_{\dot{h}} \frac{\partial}{\partial \dot{h}} \end{aligned} \tag{3.37}$$

Proof. Simply take the intersection of F and TD above. □

Note that $T(TQ')$ has 20 dimensions, and the distribution H has 17 parameters: x, e, f, g, h, $\dot{e}$, $\dot{f}$, $\dot{g}$, $\dot{h}$, w_e, w_f, w_g, w_h, $w_{\dot{e}}$, $w_{\dot{f}}$, $w_{\dot{g}}$, $w_{\dot{h}}$.

3.7 Non-holonomic Brackets

We would like to derive the non-holonomic bracket for our system. This will enable us to compute the distributional equations of motion in a straight-forward manner. We follow the algorithm in section 2.5. Recall our non-holonomic constraint:

$$c = \frac{\sqrt{g^2+h^2}}{R}\dot{x} - g\dot{e} - h\dot{f}, \tag{3.38}$$

$$\phi = \frac{\sqrt{g^2+h^2}}{R}\,\mathrm{dx} - g\,\mathrm{de} - h\,\mathrm{df}. \tag{3.39}$$

Then

$$\tau_{Q'}^*\phi = \frac{\sqrt{g^2+h^2}}{R}\,\mathrm{dx} - g\,\mathrm{de} - h\,\mathrm{df}, \tag{3.40}$$

$$\begin{aligned} \mathrm{dc} &= \frac{\dot{x}}{R\sqrt{g^2+h^2}}(d\,\mathrm{dg} + h\,\mathrm{dh}) - \dot{e}\,\mathrm{dg} - \dot{f}\,\mathrm{dh} + \frac{\sqrt{g^2+h^2}}{R}\,\mathrm{d\dot{x}} - g\,\mathrm{d\dot{e}} - h\,\mathrm{d\dot{f}} \\ &= \left(\frac{g\dot{x}}{R\sqrt{g^2+h^2}} - \dot{e}\right)\mathrm{dg} + \left(\frac{h\dot{x}}{R\sqrt{g^2+h^2}} - \dot{f}\right)\mathrm{dh} + \frac{\sqrt{g^2+h^2}}{R}\,\mathrm{d\dot{x}} - g\,\mathrm{d\dot{e}} - h\,\mathrm{d\dot{f}}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} X_c &= \frac{1}{M}\left(\frac{\sqrt{g^2+h^2}}{R}\frac{\partial}{\partial x}\right) \\ &+ \frac{4}{MR^2}\left(\left(\dot{e} - \frac{g\dot{x}}{R\sqrt{g^2+h^2}}\right)\frac{\partial}{\partial \dot{g}} + \left(\dot{f} - \frac{h\dot{x}}{R\sqrt{g^2+h^2}}\right)\frac{\partial}{\partial \dot{h}} - g\frac{\partial}{\partial e} - h\frac{\partial}{\partial f}\right). \end{aligned} \tag{3.42}$$

Thus

$$\begin{aligned} [c, \tau_Q^*\phi] &= \langle \tau_Q^*\phi \mid X_c \rangle \\ &= \frac{1}{MR^2}\left(g^2+h^2\right) + \frac{4}{MR^2}\left(g^2+h^2\right) \\ &= \frac{5}{MR^2}\left(g^2+h^2\right). \end{aligned} \tag{3.43}$$

Then

$$A^{-1} = \frac{1}{[c,c][\tau_Q^*\phi,\tau_Q^*\phi] - [\tau_Q^*\phi,c][c,\tau_Q^*\phi]} \begin{pmatrix} [\tau_Q^*\phi,\tau_Q^*\phi] & -[c,\tau_Q^*\phi] \\ -[\tau_Q^*\phi,c] & [c,c] \end{pmatrix}$$
$$= \frac{MR^2}{5\,(g^2+h^2)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.44}$$

Therefore, the non-holonomic bracket of two functions, restricted to D, is given by:

$$\{f_1,f_2\} = [f_1,f_2] + \frac{[f_2,c][f_1,\tau_Q^*\phi] - [f_2,\tau_Q^*\phi][f_1,c]}{\frac{5}{MR^2}\,(g^2+h^2)}. \tag{3.45}$$

The Poisson brackets between the coordinate functions of TQ', c, and $\tau_Q^*\phi$ are:

$$[x,c] = \langle \mathrm{dc} \mid X_x\rangle = \left\langle \mathrm{dc} \mid \frac{-1}{M}\frac{\partial}{\partial \dot{x}} \right\rangle = \frac{-\sqrt{g^2+h^2}}{MR} \tag{3.46}$$
$$[\dot{x},c] = \langle \mathrm{dc} \mid X_{\dot{x}}\rangle = \left\langle \mathrm{dc} \mid \frac{1}{M}\frac{\partial}{\partial x} \right\rangle = 0 \tag{3.47}$$
$$[x,\tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_x\rangle = 0 \tag{3.48}$$
$$[\dot{x},\tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_{\dot{x}}\rangle = \frac{\sqrt{g^2+h^2}}{MR} \tag{3.49}$$

$$[e,c] = \langle \mathrm{dc} \mid X_e\rangle = \left\langle \mathrm{dc} \mid \frac{-4}{MR^2}\frac{\partial}{\partial \dot{e}} \right\rangle = \frac{4g}{MR^2} \tag{3.50}$$
$$[\dot{e},c] = \langle \mathrm{dc} \mid X_{\dot{e}}\rangle = \left\langle \mathrm{dc} \mid \frac{4}{MR^2}\frac{\partial}{\partial e} \right\rangle = 0 \tag{3.51}$$
$$[e,\tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_e\rangle = 0 \tag{3.52}$$
$$[\dot{e},\tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_{\dot{e}}\rangle = \frac{-4g}{MR^2} \tag{3.53}$$

$$[f,c] = \langle \mathrm{dc} \mid X_f\rangle = \frac{4h}{MR^2} \tag{3.54}$$
$$[\dot{f},c] = \langle \mathrm{dc} \mid X_{\dot{f}}\rangle = 0 \tag{3.55}$$
$$[f,\tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_f\rangle = 0 \tag{3.56}$$
$$[\dot{f},\tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_{\dot{f}}\rangle = \frac{-4h}{MR^2} \tag{3.57}$$

$$[g, c] = \langle \mathrm{dc} \mid X_g \rangle = 0 \tag{3.58}$$

$$[\dot{g}, c] = \langle \mathrm{dc} \mid X_{\dot{g}} \rangle = \frac{4}{MR^2}\left(\frac{g\dot{x}}{R\sqrt{g^2+h^2}} - \dot{e}\right) \tag{3.59}$$

$$[g, \tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_g \rangle = 0 \tag{3.60}$$

$$[\dot{g}, \tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_{\dot{g}} \rangle = 0 \tag{3.61}$$

$$[h, c] = \langle \mathrm{dc} \mid X_h \rangle = 0 \tag{3.62}$$

$$[\dot{h}, c] = \langle \mathrm{dc} \mid X_{\dot{h}} \rangle = \frac{4}{MR^2}\left(\frac{h\dot{x}}{R\sqrt{g^2+h^2}} - \dot{f}\right) \tag{3.63}$$

$$[h, \tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_h \rangle = 0 \tag{3.64}$$

$$[\dot{h}, \tau_Q^*\phi] = \langle \tau_Q^*\phi \mid X_{\dot{h}} \rangle = 0. \tag{3.65}$$

The non-zero Poisson brackets among the coordinate functions are:

$$[x, \dot{x}] = \langle \mathrm{d\dot{x}} \mid X_x \rangle = \frac{-1}{M} \tag{3.66}$$

$$[e, \dot{e}] = \frac{-1}{MR^2} \tag{3.67}$$

$$[f, \dot{f}] = \frac{-1}{MR^2} \tag{3.68}$$

$$[g, \dot{g}] = \frac{-1}{MR^2} \tag{3.69}$$

$$[h, \dot{h}] = \frac{-1}{MR^2}. \tag{3.70}$$

Therefore, the non-zero non-holonomic brackets among the coordinate functions of TQ' are:

$$\begin{aligned}
\{x, \dot{x}\} &= -\frac{4}{5M} && (3.71)\\
\{x, \dot{e}\} &= -\frac{4}{5MR}\frac{g}{\sqrt{g^2+h^2}} && (3.72)\\
\{x, \dot{f}\} &= -\frac{4}{5MR}\frac{h}{\sqrt{g^2+h^2}} && (3.73)\\
\{\dot{x}, e\} &= \frac{4}{5MR}\frac{g}{\sqrt{g^2+h^2}} && (3.74)\\
\{\dot{x}, f\} &= \frac{4}{5MR}\frac{h}{\sqrt{g^2+h^2}} && (3.75)\\
\{\dot{x}, \dot{g}\} &= \frac{4}{5MR^2}\frac{g\dot{x} - \dot{e}R\sqrt{g^2+h^2}}{g^2+h^2} && (3.76)\\
\{\dot{x}, \dot{h}\} &= \frac{4}{5MR^2}\frac{h\dot{x} - \dot{f}R\sqrt{g^2+h^2}}{g^2+h^2} && (3.77)
\end{aligned}$$

$$\begin{aligned}
\{e, \dot{e}\} &= -\frac{4}{MR^2} + 4\frac{4}{5MR^2}\frac{g^2}{g^2+h^2} && (3.78)\\
\{e, \dot{f}\} &= 4\frac{4}{5MR^2}\frac{gh}{g^2+h^2} && (3.79)\\
\{\dot{e}, f\} &= -4\frac{4}{5MR^2}\frac{gh}{g^2+h^2} && (3.80)\\
\{\dot{e}, \dot{g}\} &= -4\frac{4}{5MR^2}\frac{g}{g^2+h^2}\left(\frac{g\dot{x}}{R\sqrt{g^2+h^2}} - \dot{e}\right) && (3.81)\\
\{\dot{e}, \dot{h}\} &= -4\frac{4}{5MR^2}\frac{g}{g^2+h^2}\left(\frac{h\dot{x}}{R\sqrt{g^2+h^2}} - \dot{f}\right) && (3.82)
\end{aligned}$$

$$\{f, \dot{f}\} = -\frac{4}{MR^2} + 4\frac{4}{5MR^2}\frac{h^2}{g^2+h^2} \tag{3.83}$$

$$\{\dot{f}, \dot{g}\} = -4\frac{4}{5MR^2}\frac{h}{g^2+h^2}\left(\frac{g\dot{x}}{R\sqrt{g^2+h^2}} - \dot{e}\right) \tag{3.84}$$

$$\{\dot{f}, \dot{h}\} = -4\frac{4}{5MR^2}\frac{h}{g^2+h^2}\left(\frac{h\dot{x}}{R\sqrt{g^2+h^2}} - \dot{f}\right) \tag{3.85}$$

$$\{g, \dot{g}\} = -\frac{4}{MR^2} \tag{3.86}$$

$$\{h, \dot{h}\} = -\frac{4}{MR^2}. \tag{3.87}$$

3.8 Non-holonomic Equations of Motion

Using the non-holonomic brackets computed in the last section, we can easily compute the equations of motion for the distribution form of Hamilton's equations. Recall that:

$$\begin{aligned} E &= \frac{M}{2}\left(\dot{x}^2 + R^2\frac{\left(g\dot{g} + h\dot{h}\right)^2}{g^2+h^2}\right) + \frac{MR^2}{8}\left(\dot{e}^2 + \dot{f}^2 + \dot{g}^2 + \dot{h}^2\right) \\ &+ MGR\sqrt{g^2+h^2} + \frac{\mu M}{2}\left(e^2 + f^2 + g^2 + h^2 + \frac{1}{(eh-fg)^2} - 3\right). \end{aligned} \tag{3.88}$$

For an arbitrary coordinate function z, the non-holonomic Hamilton's equation for the evolution of z is:

$$\begin{aligned}
\frac{d}{dt}z = \{E, z\} &= M\{\dot{x}, z\}\dot{x} \\
&+\frac{MR^2}{g^2+h^2}(g\dot{g}+h\dot{h})\Big(\{g,z\}\dot{g}+\{\dot{g},z\}g+\{h,z\}\dot{h}+\{\dot{h},z\}h\Big) \\
&-\frac{MR^2}{(g^2+h^2)^2}(g\dot{g}+h\dot{h})^2\Big(\{g,z\}g+\{h,z\}h\Big) \\
&+\frac{MR^2}{4}\Big(\{\dot{e},z\}\dot{e}+\{\dot{f},z\}\dot{f}+\{\dot{g},z\}\dot{g}+\{\dot{h},z\}\dot{h}\Big) \\
&+\frac{MGR}{\sqrt{g^2+h^2}}\Big(\{g,z\}g+\{h,z\}h\Big) \\
&+\mu M\Big(\{e,z\}e+\{f,z\}f+\{g,z\}g+\{h,z\}h\Big) \\
&-\frac{\mu M}{(eh-fg)^3}\Big(\{e,z\}h+\{h,z\}e-\{f,z\}g-\{g,z\}\Big). \qquad (3.89)
\end{aligned}$$

We note again that the non-holonomic bracket defined in this thesis $\{E, z\}$ is the negative of the usual one. Therefore, we obtain the following equations of motion:

$$\frac{d}{dt}x = \{E, x\} = \frac{4}{5}\dot{x} + \frac{R}{5}\frac{g\dot{e}+h\dot{f}}{\sqrt{g^2+h^2}} \qquad (3.90)$$

$$\frac{d}{dt}e = \{E, e\} = \frac{1}{5}\dot{e} + \frac{4}{5}h\frac{\dot{e}h-\dot{f}g}{g^2+h^2} + \frac{4}{5R}\frac{g\dot{x}}{\sqrt{g^2+h^2}} \qquad (3.91)$$

$$\frac{d}{dt}f = \{E, f\} = \frac{1}{5}\dot{f} + \frac{4}{5}g\frac{\dot{f}g-\dot{e}h}{g^2+h^2} + \frac{4}{5R}\frac{h\dot{x}}{\sqrt{g^2+h^2}} \qquad (3.92)$$

$$\frac{d}{dt}g = \{E, g\} = \dot{g} + 4g\frac{g\dot{g}+h\dot{h}}{g^2+h^2} \qquad (3.93)$$

$$\frac{d}{dt}h = \{E, h\} = \dot{h} + 4h\frac{g\dot{g}+h\dot{h}}{g^2+h^2} \qquad (3.94)$$

$$\begin{aligned}
\frac{d}{dt}\dot{x} = \{E, \dot{x}\} &= -\frac{4\mu}{5R}\frac{eg+fh}{\sqrt{g^2+h^2}} + \frac{R}{5}\frac{\dot{e}\dot{g}+\dot{f}\dot{h}}{\sqrt{g^2+h^2}} + \frac{4R}{5}\frac{\dot{e}g^2\dot{g}+\dot{g}g\dot{g}h+\dot{e}gh\dot{h}+\dot{f}h^2\dot{h}}{\sqrt{g^2+h^2}^3} \\
&-\frac{1}{5}\dot{x}\frac{g\dot{g}+h\dot{h}}{g^2+h^2} - \frac{4}{5}\dot{x}\frac{g^3\dot{g}+g\dot{g}h^2+g^2h\dot{h}+h^3\dot{h}}{(g^2+h^2)^2} \qquad (3.95)
\end{aligned}$$

$$\frac{d}{dt}\dot{e} = \{E, \dot{e}\} = -\frac{16}{5}\frac{\dot{e}g^3\dot{g} + \dot{f}g^2\dot{g}h + \dot{e}g^2h\dot{h} + \dot{f}gh^2\dot{h}}{(g^2+h^2)^2} - \frac{4}{5}\frac{\dot{e}g\dot{g} + \dot{f}g\dot{h}}{(g^2+h^2)} - \frac{4\mu}{R^2}e$$
$$+\frac{4\mu}{R^2}\frac{h}{(eh-fg)^3} + \frac{16\mu}{5R^2}\frac{eg^2+fgh}{(g^2+h^2)} + \frac{16}{5R}\dot{x}\frac{g^4\dot{g} + g^3h\dot{h} + gh^3\dot{h}}{\sqrt{g^2+h^2}^5}$$
$$+\frac{4}{5R}\dot{x}\frac{g^2\dot{g} + gh\dot{h}}{\sqrt{g^2+h^2}^3} \tag{3.96}$$

$$\frac{d}{dt}\dot{f} = \{E, \dot{f}\} = -\frac{16}{5}\frac{\dot{e}g^2\dot{g}h + \dot{f}g\dot{g}h^2 + \dot{e}gh^2\dot{h} + \dot{f}h^3\dot{h}}{(g^2+h^2)^2} - \frac{4}{5}\frac{\dot{e}\dot{e}h + \dot{f}h\dot{h}}{(g^2+h^2)}$$
$$-\frac{4\mu}{R^2}f - \frac{4\mu}{R^2}\frac{g}{(eh-fd)^3} + \frac{16\mu}{5R^2}\frac{egh+fh^2}{(g^2+h^2)}$$
$$+\frac{16}{5R}\dot{x}\frac{g^3\dot{h}h + g\dot{g}h^3 + g^2h^2\dot{h} + h^4\dot{h}}{\sqrt{g^2+h^2}^5} + \frac{4}{5R}\dot{x}\frac{g\dot{g}h + h^2\dot{h}}{\sqrt{g^2+h^2}^3} \tag{3.97}$$

$$\frac{d}{dt}\dot{g} = \{E, \dot{g}\} = 4\frac{g^3\dot{g}^2 + 2g^2\dot{g}h\dot{h} + gh^2\dot{h}^2}{(g^2+h^2)^2} + \frac{4}{5}\frac{\dot{e}^2g - 5g\dot{g}^2 + \dot{e}\dot{f}h - 5\dot{g}h\dot{h}}{(g^2+h^2)}$$
$$-\frac{4\mu}{R^2}g - \frac{4\mu}{R^2}\frac{f}{(eh-fd)^3} - \frac{4G}{R}\frac{g}{\sqrt{g^2+h^2}} - \frac{4}{5R}\dot{x}\frac{\dot{e}g^2+\dot{f}gh}{\sqrt{g^2+h^2}^3}$$
$$-\frac{4}{5R}\frac{\dot{e}\dot{x}}{\sqrt{g^2+h^2}} + \frac{4}{5R^2}\frac{g\dot{x}^2}{(g^2+h^2)} \tag{3.98}$$

$$\frac{d}{dt}\dot{h} = \{E, \dot{h}\} = 4\frac{g^2\dot{g}^2h + 2g\dot{g}h^2\dot{h} + h^3\dot{h}^2}{(g^2+h^2)^2} + \frac{4}{5}\frac{\dot{e}\dot{f}g + \dot{f}^2h - 5g\dot{g}\dot{h} - 5h\dot{h}^2}{(g^2+h^2)}$$
$$-\frac{4\mu}{R^2}h + \frac{4\mu}{R^2}\frac{e}{(eh-fg)^3} - \frac{4G}{R}\frac{h}{\sqrt{g^2+h^2}} - \frac{4}{5R}\dot{x}\frac{\dot{e}gh+\dot{f}h^2}{\sqrt{g^2+h^2}^3}$$
$$-\frac{4}{5R}\frac{\dot{f}\dot{x}}{\sqrt{g^2+h^2}} + \frac{4}{5R^2}\frac{h\dot{x}^2}{(g^2+h^2)}. \tag{3.99}$$

3.9 $\mathbb{R}$-Symmetry; x-translation

There are two obvious symmetries present in the system. First, translation in the x-direction. Second, rotation of the body in reference space. In order to prove that a given symmetry preserves the constraint distribution D, the distribution H, and the Hamiltonian E, it suffices to show that it preserves the constraint functions, and the potential and kinetic energies.

Claim 3.5. The pair (A, a), where

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad a = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \tag{3.100}$$

is a physical $\mathbb{R}$-symmetry, parameterised by λ. This corresponds to translation by λ in the x-direction.

Proof. Consider the composition $(A, a)(B, b)$:

$$\left(I, \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \right) (B, b) = \left(B, b + \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \right). \tag{3.101}$$

Therefore, under the action of this symmetry, only x is acted upon. As x doesn't appear in T or V, or the contact and no-slip constraints, they are all preserved by the symmetry. □

3.10 $\mathbb{S}^1$-Symmetry; body rotation

A *body symmetry* is a symmetry (C, c), with composition given by:

$$(B, b)(C, c) = (BC, Bc + b). \tag{3.102}$$

Claim 3.6. The pair (C, c), where

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2, \mathbb{R}); \quad c = 0 \tag{3.103}$$

is a body symmetry, parameterised by θ. This corresponds to rotating the body through the angle θ.

Proof. The proof of this claim is deferred to section 3.10.2. □

3.10.1 Action of $\mathbb{S}^1$-Symmetry

We would like to understand the action of the $\mathbb{S}^1$ symmetry. For the remainder of this section, it is understood that $y = R\sqrt{g^2+h^2}$.

Let $\Phi_\theta^{Q'}$ be the action of the $\mathbb{S}^1$-symmetry group at θ. Let $B' = \Phi_\theta^{Q}(B)$. Then

$$\begin{aligned}
B' &= \Phi_\theta^{Q'}(B) \\
&= \begin{pmatrix} e & f & x \\ g & h & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e\cos\theta + f\sin\theta & -e\sin\theta + f\cos\theta & x \\ g\cos\theta + h\sin\theta & -g\sin\theta + h\cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{3.104}$$

That is

$$e' = e\cos\theta + f\sin\theta \tag{3.105}$$

$$f' = -e\sin\theta + f\cos\theta \tag{3.106}$$

$$g' = g\cos\theta + h\sin\theta \tag{3.107}$$

$$h' = -g\sin\theta + h\cos\theta \tag{3.108}$$

$$x' = x \tag{3.109}$$

$$y' = y. \tag{3.110}$$

The action $\Phi_\theta^{Q'}$ can be expressed as a left-action $A = BCB^{-1}$. The matrix elements of $(eh-fg)A$ are:

$$A_{11} = (eh-fg)\cos\theta + (eg+fh)\sin\theta \quad (3.111)$$

$$A_{12} = -(e^2+f^2)\sin\theta \quad (3.112)$$

$$A_{13} = (eh-fg)x(1-\cos\theta) - ((eg+fh)x + (e^2+f^2)y)\sin\theta \quad (3.113)$$

$$A_{21} = (g^2+h^2)\cos\theta \quad (3.114)$$

$$A_{22} = (eh-fg)\cos\theta - (eg+fh)\sin\theta \quad (3.115)$$

$$A_{23} = (eh-fg)y(1-\cos\theta) + (-(g^2+h^2)x + (eg+fh)y)\sin\theta \quad (3.116)$$

$$A_{31} = 0 \quad (3.117)$$

$$A_{32} = 0 \quad (3.118)$$

$$A_{33} = eh-fg. \quad (3.119)$$

The action $\Phi^{Q'}$ can be lifted to TQ'. Let

$$\dot{B} = \operatorname{diag}\left[\begin{pmatrix} e & f & x \\ g & h & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \dot{e} & \dot{f} & \dot{x} \\ \dot{g} & \dot{h} & \dot{y} \\ 0 & 0 & 0 \end{pmatrix}\right] \quad (3.120)$$

be the matrix representation of TQ'.

Claim 3.7. The lift $\Phi_\theta^{TQ'} : TQ' \to TQ'$ of $\Phi_\theta^{Q'}$ acts diagonally on TQ' and is given in matrix form by:

$$\dot{C} = \operatorname{diag}\left[\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]. \quad (3.121)$$

Proof. Let

$$q(t) = (x(t), e(t), f(t), g(t), h(t)) \in Q' \tag{3.122}$$

be a curve in Q', with tangent vector

$$u(t) = \frac{dx(t)}{dt}\frac{\partial}{\partial x} + \frac{de(t)}{dt}\frac{\partial}{\partial e} + \ldots + \frac{dh(t)}{dt}\frac{\partial}{\partial h} \in T_{q(t)}Q'. \tag{3.123}$$

Then $\Phi_\theta^{Q'}(q(t))$ is another curve in Q' and its tangent vector $v(t)$ is given by:

$$\begin{aligned} v(t) \;=\; & \frac{dx(t)}{dt}\frac{\partial}{\partial x} \\ & + \left(\frac{de(t)}{dt}\cos\theta + \frac{df(t)}{dt}\sin\theta\right)\frac{\partial}{\partial e} \\ & + \left(-\frac{de(t)}{dt}\sin\theta + \frac{df(t)}{dt}\cos\theta\right)\frac{\partial}{\partial f} \\ & + \left(\frac{dg(t)}{dt}\cos\theta + \frac{dh(t)}{dt}\sin\theta\right)\frac{\partial}{\partial g} \\ & + \left(-\frac{dg(t)}{dt}\sin\theta + \frac{dh(t)}{dt}\cos\theta\right)\frac{\partial}{\partial h}. \end{aligned} \tag{3.124}$$

Therefore, $\Phi_\theta^{TQ'}$ maps the vector $u \in T_qQ'$ to the vector

$$\begin{aligned} v \;=\; & u_x\frac{\partial}{\partial x} \\ & +(u_e\cos\theta + u_f\sin\theta)\frac{\partial}{\partial e} + (-u_e\sin\theta + u_f\cos\theta)\frac{\partial}{\partial f} \\ & +(u_g\cos\theta + u_h\sin\theta)\frac{\partial}{\partial g} + (-u_g\sin\theta + u_h\cos\theta)\frac{\partial}{\partial h} \in T_{\Phi_\theta^{Q'}(q)}Q'. \end{aligned} \tag{3.125}$$

Clearly $\Phi_\theta^{TQ'}$ is diagonal. □

We now consider the restriction of $\Phi^{TQ'}$ to D.

Claim 3.8. The action $\Phi_\theta^D = \Phi_\theta^{TQ'}\big|_D : D \to TQ'$ preserves D. That is $\Phi_\theta^D : D \to D$.

Proof. Let

$$u = \frac{g\dot{e} + h\dot{f}}{\sqrt{g^2+h^2}}\frac{\partial}{\partial x} + \dot{e}\frac{\partial}{\partial e} + \dot{f}\frac{\partial}{\partial f} + \dot{g}\frac{\partial}{\partial g} + \dot{h}\frac{\partial}{\partial h} \in D_q. \tag{3.126}$$

If Φ_θ^D preserves D, then there is some

$$v = \frac{g' v_e + h' v_f}{\sqrt{g'^2+h'^2}}\frac{\partial}{\partial x} + v_e\frac{\partial}{\partial e} + v_f\frac{\partial}{\partial f} + v_g\frac{\partial}{\partial g} + v_h\frac{\partial}{\partial h} \in D_q \tag{3.127}$$

such that

$$\begin{aligned} v &= \Phi_\theta^D(u) \\ &= \frac{g\dot{e} + h\dot{f}}{\sqrt{g^2+h^2}}\frac{\partial}{\partial x} \\ &\quad +(\dot{e}\cos\theta + \dot{f}\sin\theta)\frac{\partial}{\partial e} + (-\dot{e}\sin\theta + \dot{f}\cos\theta)\frac{\partial}{\partial f} \\ &\quad +(\dot{g}\cos\theta + \dot{h}\sin\theta)\frac{\partial}{\partial g} + (-\dot{g}\sin\theta + \dot{h}\cos\theta)\frac{\partial}{\partial h} \end{aligned} \tag{3.128}$$

Comparing components, we see that the following must hold:

$$\frac{g' v_e + h' v_f}{\sqrt{g'^2+h'^2}} = \frac{g\dot{e} + h\dot{f}}{\sqrt{g^2+h^2}} \tag{3.129}$$

$$v_e = \dot{e}\cos\theta + \dot{f}\sin\theta \tag{3.130}$$

$$v_f = -\dot{e}\sin\theta + \dot{f}\cos\theta \tag{3.131}$$

$$v_g = \dot{g}\cos\theta + \dot{h}\sin\theta \tag{3.132}$$

$$v_h = -\dot{g}\sin\theta + \dot{h}\cos\theta \tag{3.133}$$

The last four equations determine v_e, v_f, v_g and v_h. It remains to show that the first equation is satisfied. Consider the $\frac{\partial}{\partial x}$ component:

$$\begin{aligned}
\frac{g'v_e + h'v_f}{\sqrt{g'^2 + h'^2}} &= \frac{(g\cos\theta + h\sin\theta)\left(\dot{e}\cos\theta + \dot{f}\sin\theta\right) + (-g\sin\theta + h\cos\theta)\left(-\dot{e}\sin\theta + \dot{f}\cos\theta\right)}{\sqrt{(g\cos\theta + h\sin\theta)^2 + (-g\sin\theta + h\cos\theta)^2}} \\
&= \frac{g\dot{e}\left(\cos^2\theta + \sin^2\theta\right) + h\dot{f}\left(\cos^2\theta + \sin^2\theta\right)}{\sqrt{g^2 + h^2}} \\
&= \frac{g\dot{e} + h\dot{f}}{\sqrt{g^2 + h^2}} \qquad (3.134)
\end{aligned}$$

Thus, the first equation is satisfied. □

Infinitesimal Generator

The infinitesimal left-generator ξ of $\Phi_{\theta}^{Q'}$ is given by

$$\begin{aligned}
\xi &= \left.\frac{d}{d\theta}A\right|_{\theta=0} \qquad (3.135) \\
&= \frac{1}{eh - fg}\begin{pmatrix} eg + fh & -(e^2 + f^2) & -(eg + fh)x + (e^2 + f^2)y \\ g^2 + h^2 & -(eg + fh) & -(g^2 + h^2)x + (eg + fh)y \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

3.10.2 Proof that the $\mathbb{S}^1$-Symmetry is a Symmetry

Proof. In order to preserve T, our symmetry must satisfy:

$$\frac{M}{2}\left(\dot{B} + \dot{b}\right)^2 + \frac{M}{8}\operatorname{tr}\left(\left(\dot{B}C\right)^T\left(\dot{B}C\right)\right) = \frac{M}{2}\left(\dot{b}\right)^2 + \frac{M}{8}\operatorname{tr}\left(\dot{B}^T\dot{B}\right) \qquad (3.136)$$

But $c = 0$. We are left with:

$$\begin{aligned}
\operatorname{tr}\left(\left(\dot{B}C\right)^T\left(\dot{B}C\right)\right) &= \operatorname{tr}\left(\dot{B}^T\dot{B}\right) \qquad (3.137) \\
\operatorname{tr}\left(C^T\dot{B}^T\dot{B}C\right) &= \operatorname{tr}\left(\dot{B}^T\dot{B}\right) \qquad (3.138)
\end{aligned}$$

But $C \in SO(2,\mathbb{R})$ implies $C^TC = I$. We are left with:

$$\begin{aligned}
\operatorname{tr}\left(C^T\dot{B}^T\dot{B}C\right) &= \operatorname{tr}\left(\dot{B}^T\dot{B}\right) && (3.139)\\
\operatorname{tr}\left(C^TC\dot{B}^T\dot{B}\right) &= \operatorname{tr}\left(\dot{B}^T\dot{B}\right) && (3.140)\\
\operatorname{tr}\left(\dot{B}^T\dot{B}\right) &= \operatorname{tr}\left(\dot{B}^T\dot{B}\right) && (3.141)
\end{aligned}$$

Therefore T is preserved. In order to preserve V, our symmetry must satisfy:

$$\begin{aligned}
MG(Bc+b)\begin{pmatrix}0\\1\end{pmatrix} + \frac{\mu M}{2}\left(\operatorname{tr}\left((BC)^T(BC)\right) + \frac{1}{(\det BC)^2} - 3\right) &= \\
MGb\begin{pmatrix}0\\1\end{pmatrix} + \frac{\mu M}{2}\left(\operatorname{tr}\left(B^TB\right) + \frac{1}{(\det B)^2} - 3\right) & \quad (3.142)
\end{aligned}$$

But $c = 0$ implies:

$$MG(Bc+b)\begin{pmatrix}0\\1\end{pmatrix} = MGb\begin{pmatrix}0\\1\end{pmatrix} \qquad (3.143)$$

Furthermore, $C \in SO(2,\mathbb{R})$ implies:

$$\operatorname{tr}\left((BC)^T(BC)\right) = \operatorname{tr}\left(B^TB\right) \qquad (3.144)$$

and

$$\frac{1}{(\det BC)^2} = \frac{1}{(\det B)^2} \qquad (3.145)$$

Therefore V is preserved. Recall the holonomic contact constraint (3.19):

$$y^2 = R^2(g^2+h^2)$$

Then, with our body symmetry, we have:

$$\begin{aligned}
g'^2 + h'^2 &= (g\cos\theta - h\sin\theta)^2 + (g\sin\theta + h\cos\theta)^2 \\
&= g^2\left(\cos^2\theta + \sin^2\theta\right) + h^2\left(\cos^2\theta + \sin^2\theta\right) \\
&\quad -2gh\cos\theta\sin\theta + 2gh\sin\theta\cos\theta \\
&= g^2 + h^2 \\
&= y^2 \\
&= y'^2 \quad \text{from (3.110).}
\end{aligned} \tag{3.146}$$

Therefore the contact constraint is preserved. Similarly, with our body symmetry, (3.21) becomes:

$$\begin{aligned}
\dot{e}'g' + \dot{f}'h' &= \left(\dot{e}\cos\theta - \dot{f}\sin\theta\right)(g\cos\theta - h\sin\theta) \\
&\quad + \left(\dot{e}\sin\theta + \dot{f}\cos\theta\right)(g\sin\theta + h\cos\theta) \\
&= \dot{e}g\cos^2\theta - \dot{e}h\cos\theta\sin\theta - \dot{f}g\sin\theta\cos\theta + \dot{f}h\sin^2\theta \\
&\quad +\dot{e}g\sin^2\theta + \dot{e}h\sin\theta\cos\theta + \dot{f}g\cos\theta\sin\theta + \dot{f}h\cos^2\theta \\
&= \dot{e}g\left(\cos^2\theta + \sin^2\theta\right) + \dot{f}h\left(\cos^2\theta + \sin^2\theta\right) \\
&= \dot{e}g + \dot{f}h \\
&= \dot{x}y \\
&= \dot{x}'y' \quad \text{from (3.109) and (3.110).}
\end{aligned} \tag{3.147}$$

Therefore the no-slip constraint is preserved. □

3.10.3 Free and Proper

Claim 3.9. The action $\Phi_0^{Q'}$ is free and proper.

Proof. Properness is trivial, as $\mathbb{S}^1$ is compact. Furthermore, the infinitesimal left-generator ξ of $\Phi_\theta^{Q'}$ is nowhere zero. Towards a contradiction, suppose $\xi = 0$. Then $e^2 + f^2 = 0$ and $g^2 + h^2 = 0$, which implies that $B = 0 \notin GL(2, \mathbb{R})$; this is a contradiction. Finally, there are no discrete isotropies either, as the action of $\Phi_\theta^{Q'}$ is a body-rotation through the angle θ; and $\theta \in [0, 2\phi)$. Therefore $\Phi_\theta^{Q'}$ has no fixed points. □

3.11 Non-holonomic Reduction

The algebra of polynomials on $Q'/\mathbb{R}/\mathbb{S}^1$ which are invariant under the actions of the $\mathbb{R}$ and $\mathbb{S}^1$ symmetries is generated by:

$$\sigma_1 = e^2 + f^2 \tag{3.148}$$

$$\sigma_2 = eg + fh \tag{3.149}$$

$$\sigma_3 = e\dot{e} + \dot{f}f \tag{3.150}$$

$$\sigma_4 = e\dot{g} + \dot{h}f \tag{3.151}$$

$$\sigma_5 = eh - fg \tag{3.152}$$

$$\sigma_6 = e\dot{f} - \dot{e}f \tag{3.153}$$

$$\sigma_7 = e\dot{h} - \dot{g}f \tag{3.154}$$

$$\sigma_8 = g^2 + h^2 \tag{3.155}$$

$$\sigma_9 = g\dot{e} + h\dot{f} \tag{3.156}$$

$$\sigma_{10} = g\dot{g} + h\dot{h} \tag{3.157}$$

$$\sigma_{11} = g\dot{f} - \dot{e}h \tag{3.158}$$

$$\sigma_{12} = g\dot{h} - \dot{g}h \tag{3.159}$$

$$\sigma_{13} = \dot{e}^2 + \dot{f}^2 \tag{3.160}$$

$$\sigma_{14} = \dot{e}\dot{g} + \dot{f}\dot{h} \tag{3.161}$$

$$\sigma_{15} = \dot{e}\dot{h} - \dot{g}\dot{f} \tag{3.162}$$

$$\sigma_{16} = \dot{g}^2 + \dot{h}^2 \tag{3.163}$$

$$\sigma_{17} = \dot{x} \tag{3.164}$$

subject to the relations (cross products):

$$\sigma_5^2 = \sigma_1\sigma_8 - \sigma_2^2 \tag{3.165}$$

$$\sigma_6^2 = \sigma_1\sigma_{13} - \sigma_3^2 \tag{3.166}$$

$$\sigma_7^2 = \sigma_1\sigma_{16} - \sigma_4^2 \tag{3.167}$$

$$\sigma_{11}^2 = \sigma_{13}\sigma_8 - \sigma_9^2 \tag{3.168}$$

$$\sigma_{12}^2 = \sigma_8\sigma_{16} - \sigma_{10}^2 \tag{3.169}$$

$$\sigma_{15}^2 = \sigma_{13}\sigma_{16} - \sigma_{14}^2 \tag{3.170}$$

and (linear dependence):

$$\sigma_{13} = \alpha^2\sigma_1 + 2\alpha\beta\sigma_2 + \beta^2\sigma_8 \tag{3.171}$$

$$\sigma_{14} = \alpha\gamma\sigma_1 + (\alpha\delta + \gamma\beta)\sigma_2 + \beta\delta\sigma_8 \tag{3.172}$$

$$\sigma_{16} = \gamma^2\sigma_1 + 2\gamma\delta\sigma_2 + \delta^2\sigma_8, \tag{3.173}$$

where

$$\alpha = \frac{\sigma_3\sigma_8 - \sigma_9\sigma_2}{\sigma_1\sigma_8 - \sigma_2\sigma_2} \qquad \beta = \frac{\sigma_1\sigma_9 - \sigma_2\sigma_3}{\sigma_1\sigma_8 - \sigma_2\sigma_2} \tag{3.174}$$

$$\gamma = \frac{\sigma_4\sigma_8 - \sigma_{10}\sigma_2}{\sigma_1\sigma_8 - \sigma_2\sigma_2} \qquad \delta = \frac{\sigma_1\sigma_{10} - \sigma_2\sigma_4}{\sigma_1\sigma_8 - \sigma_2\sigma_2}. \tag{3.175}$$

Furthermore, we have the following inequalities

$$\sigma_1 > 0 \tag{3.176}$$

$$\sigma_5 > 0 \tag{3.177}$$

$$\sigma_8 > 0 \tag{3.178}$$

$$\sigma_{13} \geq 0 \tag{3.179}$$

$$\sigma_{16} \geq 0. \tag{3.180}$$

For convenience, we sometimes write $\sqrt{\sigma_8}$ as ρ_8. The non-holonomic constraint becomes the relation

$$\sigma_9 = \frac{\sigma_{17}\sqrt{\sigma_8}}{R} \tag{3.181}$$
$$= \frac{\sigma_{17}\rho_8}{R} \tag{3.182}$$

We have 17 invariants and 10 relations. We are left with 7 equations of motion.

3.11.1 Reduced Equations of Motion

The reduced non-holonomic Hamilton's equations of motion are (see Appendix A for a discussion on how these equations were found):

$$\frac{d}{dt}\sigma_1 = 2\sigma_3 - 2\frac{4}{5}\frac{\sigma_9\sigma_2}{\rho_8^2} + 2\frac{4}{5R}\frac{\sigma_2\sigma_{17}}{\rho_8} \tag{3.183}$$

$$\frac{d}{dt}\sigma_2 = \frac{1}{5}\sigma_9 + \sigma_4 + 4\frac{\sigma_2\sigma_{10}}{\rho_8^2} + \frac{4}{5R}\rho_8\sigma_{17} \tag{3.184}$$

$$\frac{d}{dt}\sigma_3 = \sigma_{13} - 4\frac{\mu}{R^2}\sigma_1 - 4\frac{4}{5}\frac{\sigma_2\sigma_9\sigma_{10}}{\rho_8^4} - \frac{4}{5}\frac{\sigma_9^2 + \sigma_{14}\sigma_2 + 4\frac{\mu}{R^2}\sigma_2^2}{\rho_8^2}$$
$$+\frac{4}{5R}\frac{\sigma_9\sigma_{17}}{\rho_8} + \frac{4}{R}\frac{\sigma_{10}\sigma_2\sigma_{17}}{\rho_8^3} + 4\frac{\mu}{R^2}\frac{1}{\sigma_5^2} \tag{3.185}$$

$$\frac{d}{dt}\sigma_4 = \sigma_{14} - 4\frac{\mu}{R^2}\sigma_2 - \frac{4}{5R}\frac{(\sigma_3 - \sigma_{10})\sigma_{17}}{\rho_8}$$
$$-4\frac{G}{R}\frac{\sigma_2}{\rho_8} - \frac{4}{5R}\frac{\sigma_9\sigma_2\sigma_{17}}{\rho_8^3} + 4\frac{\sigma_5\sigma_{12}\sigma_{10}}{\rho_8^4}$$
$$+\frac{4}{5}\frac{\sigma_9(\sigma_3 + \sigma_{10})}{\rho_8^2} + \frac{4}{5R^2}\sigma_2\sigma_{17}^2\rho_8^2 \tag{3.186}$$

$$\frac{d}{dt}\sigma_8 = 10\sigma_{10} \tag{3.187}$$

$$\frac{d}{dt}\sigma_{10} = \sigma_{16} - 4\frac{\mu}{R^2}\rho_8^2 + 4\frac{\mu}{R^2}\frac{1}{\sigma_5^2} - 4\frac{G}{R}\rho_8 - 2\frac{4}{5R}\frac{\sigma_{17}\sigma_9}{\rho_8}$$
$$\frac{4}{5}\frac{\sigma_9^2}{\rho_8^2} + \frac{4}{5R^2}\sigma_{17}^2 + 4\frac{\sigma_{10}^2}{\rho_8^2} \tag{3.188}$$

$$\frac{d}{dt}\sigma_{17} = \frac{4R}{5}\frac{\sigma_9\sigma_{10}}{\rho_8^3} - \frac{1}{5}\frac{4\mu\sigma_2/R + R\sigma_{14}}{\rho_8} - \frac{\sigma_{10}\sigma_{17}}{\rho_8^2}. \tag{3.189}$$

Note that σ_{12} appears in equation (3.186). The sign of σ_{12} is not known. However, using the relations in section 3.11 we can show that:

$$\sigma_{12}^2 = \frac{(\sigma_{10}\sigma_2 - \rho_8^2\sigma_4)^2}{\rho_8^2\sigma_1 - \sigma_2^2}, \quad \text{and therefore} \tag{3.190}$$

$$\sigma_{12} = \pm\frac{|\sigma_{10}\sigma_2 - \rho_8^2\sigma_4|}{\sqrt{\rho_8^2\sigma_1 - \sigma_2^2}}, \tag{3.191}$$

since $\rho_8^2\sigma_1 - \sigma_2^2 = \sigma_5^2$ and $\sigma_5 > 0$.

Using the relations in section 3.11 the above equations of motion can be simplified:

$$\frac{d}{dt}\sigma_1 = 2\sigma_3 \tag{3.192}$$

$$\frac{d}{dt}\sigma_2 = \sigma_4 + \frac{1}{R}\rho_8\sigma_{17} + 4\frac{\sigma_2\sigma_{10}}{\rho_8^2} \tag{3.193}$$

$$\begin{aligned}\frac{d}{dt}\sigma_3 = {} & \frac{1}{\sigma_2^2 - \sigma_1\rho_8^2}\left[\frac{4\mu}{R^2}\left(\rho_8^2\sigma_1^2 - \frac{9}{5}\sigma_1\sigma_2^2 + \frac{4}{5}\frac{\sigma_2^4}{\rho_8^2} - 1\right)\right. \\ & -\frac{1}{R^2}\rho_8^2\sigma_1\sigma_{17}^2 + \frac{2}{R}\rho_8\sigma_2\sigma_3\sigma_{17} - \rho_8^2\sigma_3^2 - \frac{4}{5R}\frac{\sigma_2^2\sigma_4\sigma_{17}}{\rho_8} + \frac{4}{5}\sigma_2\sigma_3\sigma_4 \\ & \left. +\sigma_{10}\left(\frac{4}{5R}\frac{\sigma_2^3\sigma_{17}}{\rho_8^3} - \frac{4}{5}\frac{\sigma_2^2\sigma_3}{\rho_8^2}\right)\right]\end{aligned} \tag{3.194}$$

$$\begin{aligned}\frac{d}{dt}\sigma_4 = {} & -\frac{4\mu}{R^2}\sigma_2 - \frac{4G}{R}\frac{\sigma_2}{\rho_8} - \sigma_{10}\left(\frac{1}{\sigma_2^2 - \rho_8^2\sigma_1}\left(\frac{1}{R}\rho_8\sigma_1\sigma_{17} - \sigma_2\sigma_3\right)\right) \\ & -\frac{1}{\sigma_2^2 - \rho_8^2\sigma_1}\left(-\frac{1}{R}\rho_8\sigma_{17}\sigma_2\sigma_4 + \rho_8^2\sigma_3\sigma_4\right) \\ & \pm\sigma_{10}\frac{4|\sigma_{10}\sigma_2 - \rho_8^2\sigma_4|}{\rho_8^4}\end{aligned} \tag{3.195}$$

$$\frac{d}{dt}\sigma_8 = 10\sigma_{10} \tag{3.196}$$

$$\begin{aligned}\frac{d}{dt}\sigma_{10} = {} & -\frac{4\mu}{R^2}\rho_8^2 - \frac{4G}{R}\rho_8 + 4\frac{\sigma_{10}^2}{\rho_8^2} + \frac{1}{\sigma_2^2 - \rho_8^2\sigma_1}\left[-\frac{4\mu}{R^2}\right. \\ & \left.\sigma_{10}\left(-\sigma_1\sigma_{10} + 2\sigma_2\sigma_4\right) - \rho_8^2\sigma_4^2\right]\end{aligned} \tag{3.197}$$

$$\begin{aligned}\frac{d}{dt}\sigma_{17} = {} & \frac{1}{\sigma_2^2 - \rho_8^2\sigma_1}\left[\sigma_{10}\left(-\frac{1}{5}\sigma_2^2\sigma_{17} + \frac{R}{5}\frac{\sigma_2\sigma_3}{\rho_8}\right) + \frac{1}{5}\sigma_{17}\sigma_2\sigma_4 - \frac{R}{5}\rho_8\sigma_3\sigma_4\right. \\ & \left.\frac{4\mu}{5R}\rho_8\sigma_1\sigma_2 - \frac{4\mu}{5R}\frac{\sigma_2^3}{p8}\right]\end{aligned} \tag{3.198}$$

3.12 Fixed Points of the Reduced Motion

A point q_0 of a dynamical system $\dot{q} = f(q)$ is called a *fixed point* if $f(q_0) = 0$ [9, p102]. A fixed point of the reduced system is called a *relative equilibrium* of the original system.

We solve the reduced system for fixed points. Clearly, (3.192) and (3.196) imply that $\sigma_3 = 0$ and $\sigma_{10} = 0$. Then (3.193) becomes:

$$\frac{d}{dt}\sigma_2 = \sigma_4 + \frac{1}{R}\rho_8\sigma_{17}, \tag{3.199}$$

and therefore $\sigma_4 = \frac{1}{R}\rho_8\sigma_{17}$. Then (3.198) becomes:

$$\frac{d}{dt}\sigma_{17} = \frac{\sigma_2}{\sigma_2^2 - \rho_8^2\sigma_1}\left[\frac{1}{5}\sigma_{17}\sigma_4 + \frac{4\mu}{5R}\rho_8\sigma_1 - \frac{4\mu}{5R}\frac{\sigma_2^2}{\rho_8}\right] \tag{3.200}$$

and therefore, either $\sigma_2 = 0$ or

$$\frac{1}{5}\sigma_{17}\sigma_4 + \frac{4\mu}{5R}\rho_8\sigma_1 - \frac{4\mu}{5R}\frac{\sigma_2^2}{\rho_8} = 0 \tag{3.201}$$

We proceed along the $\sigma_2 = 0$ branch, as the other branch is empty. Then (3.194) becomes:

$$\frac{d}{dt}\sigma_3 = \frac{-1}{\sigma_1\rho_8^2}\left(\frac{4\mu}{R^2}\left(\rho_8^2\sigma_1^2 - 1\right) - \frac{1}{R^2}\rho_8^2\sigma_1\sigma_{17}^2\right) \tag{3.202}$$

and therefore:

$$\sigma_{17}^2 = 4\mu\left(\sigma_1 - \frac{1}{\rho_8^2\sigma_1}\right). \tag{3.203}$$

Then (3.197) becomes:

$$\frac{d}{dt}\sigma_{10} = -\frac{4\mu}{R^2}\rho_8^2 - \frac{4G}{R}\rho_8 + \frac{1}{\rho_8^2\sigma_1}\left[\frac{4\mu}{R^2} + \frac{1}{R^2}\rho_8^4\sigma_{17}^2\right] \tag{3.204}$$

and therefore:

$$\sigma_1 = \frac{\mu + \sqrt{\mu^2 - 4G\mu R\rho_8^5}}{2GR\rho_8^3}. \tag{3.205}$$

Finally, (3.195) becomes:

$$\frac{d}{dt}\sigma_4 = -4\frac{\sigma_4}{\rho_8^2} \pm 4\frac{|-\rho_8^2\sigma_4|}{\rho_8^4} \tag{3.206}$$

and therefore $\sigma_{12} \geq 0$.

3.12.1 Fixed Points

The above is summarised by:

$$\sigma_1 = \frac{\mu + \sqrt{\mu^2 - 4G\mu R\rho_8^5}}{2GR\rho_8^3} \tag{3.207}$$

$$\sigma_2 = 0 \tag{3.208}$$

$$\sigma_3 = 0 \tag{3.209}$$

$$\sigma_4 = \frac{\rho_8\sigma_{17}}{R} \tag{3.210}$$

$$\sigma_{10} = 0 \tag{3.211}$$

$$\sigma_{12} \geq 0 \tag{3.212}$$

$$\sigma_{17}^2 = 4\mu\left(\sigma_1 - \frac{1}{\rho_8^2\sigma_1}\right) \tag{3.213}$$

Six of the seven invariants are dependent; ρ_8 is free.

3.13 Stability

We would like to know if the relative equilibrium found above is stable or not. The answer to this question has interesting physical consequences. If the relative equilibrium is stable, the disc will continue rolling near the relative equilibrium even if it slightly perturbed.

Let ϕ_t denote the flow of the dynamical system $\dot{q} = f(q)$. A fixed point q_0 of $\dot{q} = f(q)$ is *stable* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $q \in N_\delta(q_0)$ and $t \geq 0$ we have:

$$\phi_t(q) \in N_\epsilon(q_0). \tag{3.214}$$

The fixed point q_0 is *unstable* if it is not stable. [9, p129]

Theorem 3.1. If q_0 is a stable fixed point of $\dot{q} = f(q)$, then no eigenvalue of $\mathrm{D}\, f(q_0)$ has positive real part. [9, p130]

If one eigenvalue of $\mathrm{D}\, f(q_0)$ has positive real part, then the fixed point q_0 is unstable (ie, not stable).

Let f denote the dynamical system defined by the seven reduced equations of motion, and q_0 denote the relative equilibrium found previously. Computing the eigenvalues of $\mathrm{D}\, f(q_0)$ by hand would be tedious. Instead, we set the constants G, M, μ, and R to unity and use a computer algebra system to compute the eigenvalues, as functions of ρ_8, for us. Figures 3.1 through 3.7 are plots of the real parts of the seven eigenvalues as functions of ρ_8.

From these plots, we make two observations. Fist, that for almost all ρ_8 the fixed point will be unstable, as at least one of the eigenvalues has positive real part. Second, that there is a region near $\rho_8 = 0.3$ where all the eigenvalues have zero real part. In this region, eigenvalue techniques are not robust enough to determine stability. When the disc is at rest (ie, $\sigma_{17} = 0$), we can solve the fixed point equations (3.207)

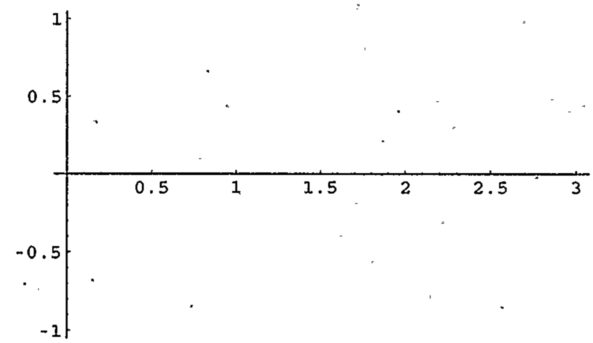

Figure 3.1: Real Part of the First Eigenvalue vs ρ_8.

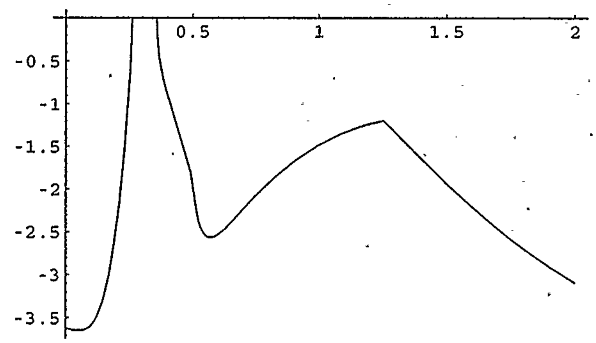

Figure 3.2: Real Part of the Second Eigenvalue vs ρ_8.

and (3.213) for ρ_8. We find that the relative equilibrium is numerically located at $\rho_8 = 0.3142\ldots$. This falls within the region where all the eigenvalues have zero real part. We may physically argue that the rest state should be stable, and our results here would be consistent with that argument.

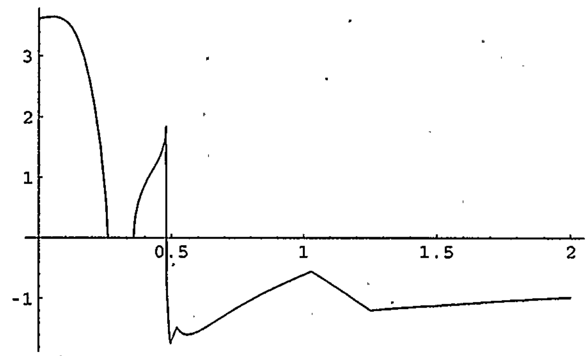

Figure 3.3: Real Part of the Third Eigenvalue vs ρ_8.

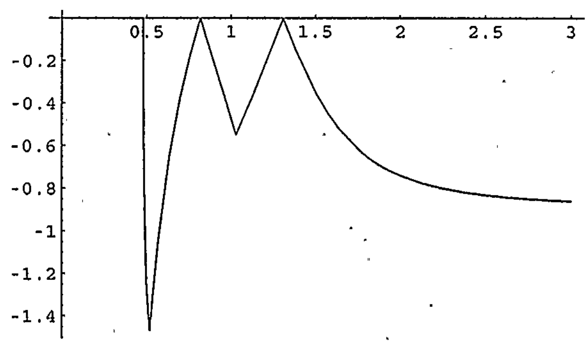

Figure 3.4: Real Part of the Fourth Eigenvalue vs ρ_8.

3.14 Reconstruction

We reconstruct the relative equilibrium found in the previous section. Let:

$$\begin{aligned} B &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} a\cos\theta - b\sin\theta & a\sin\theta + b\cos\theta \\ b\cos\theta - c\sin\theta & b\sin\theta + c\cos\theta \end{pmatrix}, \end{aligned} \tag{3.215}$$

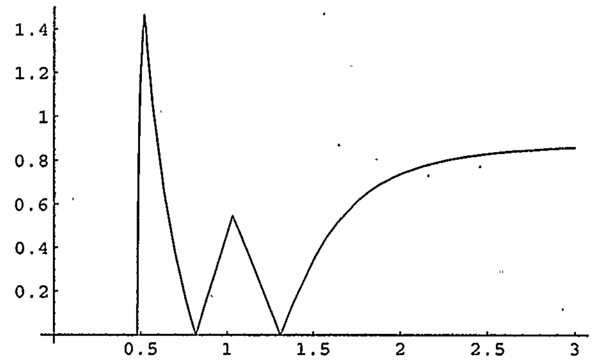

Figure 3.5: Real Part of the Fifth Eigenvalue vs ρ_8.

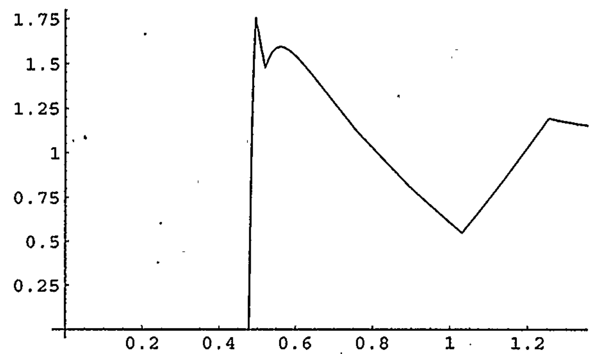

Figure 3.6: Real Part of the Sixth Eigenvalue vs ρ_8.

where a, b, c, and θ are functions of time. At the fixed point, $\sigma_2 = 0$, therefore:

$$\begin{aligned}
\sigma_2 &= eg + fh \\
&= (a\cos\theta - b\sin\theta)(b\cos\theta - c\sin\theta) + (a\sin\theta + b\cos\theta)(b\sin\theta + c\cos\theta) \\
&= b(a+c) = 0. \qquad (3.216)
\end{aligned}$$

Therefore $b = 0$ or $a = -c$. We proceed along the $b = 0$ branch, as the $a = -c$ branch is empty. We are left with:

$$B = \begin{pmatrix} a\cos\theta & a\sin\theta \\ -c\sin\theta & +c\cos\theta \end{pmatrix}. \qquad (3.217)$$

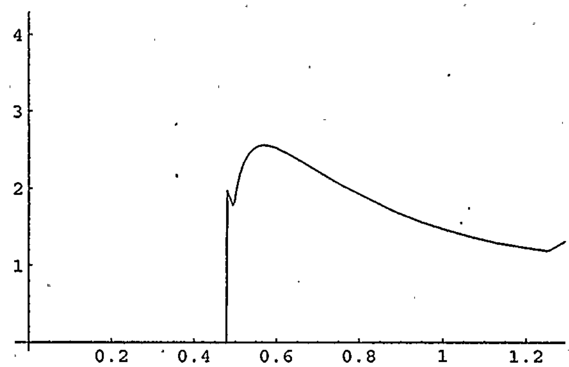

Figure 3.7: Real Part of the Seventh Eigenvalue vs ρ_8.

At the fixed point, $\sigma_3 = 0$, therefore:

$$\begin{aligned}\sigma_3 &= e\dot{e} + f\dot{f} = 0 \\ &= (a\cos\theta)(\dot{a}\cos\theta - a\dot{\theta}\sin\theta) + (a\sin\theta)(\dot{a}\sin\theta + a\dot{\theta}\cos\theta) \\ &= a\dot{a} = 0. \end{aligned} \tag{3.218}$$

Therefore $a = 0$ or $\dot{a} = 0$. The $a = 0$ branch is not interesting, as the disc would be a point. Therefore $\dot{a} = 0$, and hence a is constant. Similarly, at the fixed point $\sigma_{10} = 0$, therefore:

$$\begin{aligned}\sigma_{10} &= g\dot{g} + h\dot{h} \\ &= c\dot{c} = 0. \end{aligned} \tag{3.219}$$

Therefore, c is constant. At the fixed point, $\sigma_4 = \frac{\rho_8 \sigma_{17}}{R}$, therefore:

$$\begin{aligned}
\sigma_4 &= e\dot{g} + \dot{h}f \\
&= (a\cos\theta)(-c\dot{\theta}\cos\theta) + (-\dot{\theta}\sin\theta)(a\sin\theta) \\
&= -ac\dot{\theta} \\
&= \frac{\rho_8 \sigma_{17}}{R} \\
&= \frac{\dot{x}\sqrt{g^2 + h^2}}{R} \\
&= \frac{\pm c\dot{x}}{R}
\end{aligned} \tag{3.220}$$

Therefore $\pm c\dot{x} = Rac\dot{\theta}$ and hence $\dot{\theta} = -\frac{\dot{x}}{aR}$ (the $+\frac{\dot{x}}{aR}$ branch is not physical as the disc would be moving forward in the x direction, but rolling counter-clockwise). At the fixed point,

$$\sigma_{17}^2 = 4\mu\left(\sigma_1 - \frac{1}{\rho_8^2 \sigma_1}\right), \tag{3.221}$$

and therefore

$$\begin{aligned}
\dot{x}^2 &= 4\mu\left((e^2 + f^2) - \frac{1}{(e^2 + f^2)(g^2 + f^2)}\right) \\
&= 4\mu\left(a^2 - \frac{1}{a^2 c^2}\right).
\end{aligned} \tag{3.222}$$

Hence $\dot{x}$ and $\dot{\theta}$ are constant since a and c are constant. That is, the disc rolls with constant velocity in the x-direction. Finally, at the fixed point,

$$\sigma_1 = \frac{\mu + \sqrt{\mu^2 - 4G\mu R\rho_8^5}}{2GR\rho_8^3}, \tag{3.223}$$

and therefore

$$a^2 = \frac{\mu + \sqrt{\mu^2 - 4G\mu Rc^5}}{2GRc^3}. \tag{3.224}$$

The above is summarised by:

$$a^2 = \frac{\mu + \sqrt{\mu^2 - 4G\mu Rc^5}}{2GRc^3} \quad (3.225)$$

$$b = 0 \quad (3.226)$$

$$c = \text{constant} \quad (3.227)$$

$$\dot{\theta} = -\frac{\dot{x}}{aR} \quad (3.228)$$

$$\dot{x}^2 = 4\mu\left(a^2 - \frac{1}{a^2c^2}\right). \quad (3.229)$$

All the variables are determined, except for c, which is free but cannot depend on time.

3.15 Comparison of Results

In the section "Driven motions and the conveyor-belt problem" (Section 2.3) of [6], the authors present a solution of our problem in the case where the motion is *driven*. A driven motion is a motion induced by a prescribed motion of the centre of the disc (as an axle would). The horizontal translation and vertical position of the centre of the disc are prescribed:

$$x(t) = \alpha t, \quad \text{and} \quad (3.230)$$

$$y(t) = \beta \quad (3.231)$$

where α and β are constants. The deformation matrix B is decomposed in a manner similar to the last section:

$$\begin{aligned} B &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}, \end{aligned} \tag{3.232}$$

where ω is a constant angular speed. The authors arrive at the following solution for the driven motion:

$$a^2 = \frac{\alpha^2}{2kR^2} + \sqrt{\left(\frac{\alpha^2}{2kR^2}\right)^2 + \left(\frac{R}{\beta}\right)^2} \tag{3.233}$$

$$b = 0 \tag{3.234}$$

$$c = \frac{\beta}{R} \tag{3.235}$$

$$\omega = \frac{\alpha}{Ra} \tag{3.236}$$

$$4\lambda = (\omega^2 - k) + \frac{k}{a^2c^4} \tag{3.237}$$

$$\mu = 0 \tag{3.238}$$

where k is a constant, and λ and μ are Lagrange multipliers. The motion described above is one in which the shape of the disc is preserved, and it appears to glide horizontally (as opposed to rotating). If one were to move alongside the centre of the disc, it would appear to be rolling as on a conveyor belt. That is, the disc is experiencing constant (angular) velocity body rotation and horizontal translation, but its shape is otherwise fixed. This description corresponds exactly to our relative equilibrium.

In order to compare our solution with theirs, we make the following substitutions: $y = \beta$, $\dot{x} = \alpha$ and $\dot{\theta} = \omega$. The contact constraint $y = R\sqrt{g^2 + h^2}$ becomes $\beta = Rc$, which implies that $c = \frac{\beta}{R}$. Therefore, (3.235) is recovered and c is constant (which is consistent with our solution). Our solution becomes:

$$
\begin{aligned}
a^2 &= \frac{\mu + \sqrt{\mu^2 - 4G\mu R\left(\frac{\beta}{R}\right)^5}}{2GR\left(\frac{\beta}{R}\right)^3} && (3.239)\\
b &= 0 && (3.240)\\
c &= \frac{\beta}{R} && (3.241)\\
\omega &= \mp\frac{\alpha}{aR} && (3.242)\\
\alpha^2 &= 4\mu\left(a^2 - \frac{1}{a^2\left(\frac{\beta}{R}\right)^2}\right). && (3.243)
\end{aligned}
$$

Rearranging (3.243), we recover (3.233):

$$
\begin{aligned}
\alpha^2 &= 4\mu\left(a^2 - \frac{1}{a^2\left(\frac{\beta}{R}\right)^2}\right) && (3.244)\\
\alpha^2 a^2 &= 4\mu a^4 - 4\mu\frac{R^2}{\beta^2} && (3.245)\\
a^2 &= \frac{\alpha^2}{8\mu} + \sqrt{\left(\frac{\alpha^2}{8\mu}\right)^2 + \left(\frac{R}{\beta}\right)^2}. && (3.246)
\end{aligned}
$$

We have recovered the driven motion, but we have an extra relation (3.239). This arises because our solution is not driven. However, the authors note that if $\lambda = \frac{G}{\beta}$, the conveyor belt motion is no longer driven. That is, the motion happens on its own, as does our solution. Substituting $\lambda = \frac{G}{\beta}$ into (3.237), we have:

$$
\begin{aligned}
4\frac{G}{\beta} &= \left(\frac{\alpha^2}{a^2R^2} - k\right) + \frac{kR^4}{a^2\beta^4} && (3.247)\\
4\frac{G}{\beta} &= \frac{\alpha^2}{a^2R^2} - k + \frac{kR^4}{a^2\beta^4} && (3.248)\\
4\frac{G}{\beta} &= \frac{R^2k}{a^2R^2}\left(a^2 - \frac{R^2}{a^2\beta^2}\right) - k + \frac{kR^4}{a^2\beta^4} && (3.249)\\
4g\beta^3a^4 &= k\beta^4a^4 - kR^2\beta^2 - k\beta^4a^4 + kR^4a^2 && (3.250)\\
4g\beta^3a^4 &= kR^4a^2 - kR^2\beta^2 && (3.251)
\end{aligned}
$$

and therefore

$$a^2 = \frac{kR^4 + \sqrt{k^2R^8 - 16GkR^2\beta^5}}{8G\beta^3}. \tag{3.252}$$

Changing into our variables ($k = \frac{4\mu}{R^2}$), we have:

$$\begin{aligned} a^2 &= \frac{4\mu R^2 + \sqrt{(4\mu)^2R^4 - 64G\mu(cR)^5}}{8G(cR)^3} \\ &= \frac{\mu + \sqrt{\mu^2 - 4G\mu Rc^5}}{2GRc^3}, \end{aligned} \tag{3.253}$$

which is exactly (3.225). Therefore, the conveyor belt solutions found here and in [6] match exactly.

3.16 Conclusion

Using the reduction theory for linear non-holonomically constrained Hamiltonian systems, as developed by Jędrzej Śniatycki and Richard Cushman, we were able to reduce the pseudo-rigid rolling disc and find a relative equilibrium. We reconstructed the relative equilibrium in the original space and obtained the "conveyor belt" motion as in [6]. However, in [6], the authors looked for solutions of a certain type – postulating the form of the solution and verifying that it was indeed a solution. Our approach was constructive – we found the solution directly. Furthermore, in the reduced space, analysing the stability of the relative equilibrium could be done using local eigenvalue techniques.

Appendix A

Computer Algebra Techniques

The computer algebra system *Mathematica* was used to perform many of the tedious calculations done throughout this paper. The most tedious calculations appear when finding the evolution of a given function using (2.28) and (2.38). Ultimately, when computing the non-holonomic bracket between an arbitrary coordinate function z and the Hamiltonian E, $\{E, z\}$, the most basic computation is the evaluation of the non-holonomic bracket between the coordinate functions. Furthermore, the non-holonomic bracket is defined in terms of the Poisson bracket. As such, we first define the Poisson bracket, denoted by pb, as:

- bilinear:

```
pb[0, X_] := 0
pb[X_Plus, Y_] := Block[{pb}, Distribute[pb[X, Y]]];
pb[X_, Y_Plus] := Block[{pb}, Distribute[pb[X, Y]]];
pb[(a_)?NumericQ*(X_), Z_] := a*pb[X, Z];
pb[Z_, (a_)?NumericQ*(X_)] := a*pb[Z, X];
```

- anti-symmetric:

```
pb[X_Symbol, Y_Symbol] /; !OrderedQ[{X, Y}]
        := -pb[Sequence @@ Sort[{X, Y}]];
pb[X_, X_] := 0
```

- and satisfying the Leibnitz rule[1]:

```
pb[Z_, (X_)*(Y_)] := pb[Z, X]*Y + X*pb[Z, Y];
pb[(X_)*(Y_), Z_] := -pb[Z, X]*Y - X*pb[Z, Y];
```

Second, we define the Poisson bracket between all of the coordinate and constraint functions. For example:

```
pb[x, xd] = -1/M;
pb[c, x] = Sqrt[g^2 + h^2]/M/R;
```

Finally, we define the non-holonomic bracket, denoted by hb, by:

```
A = {{pb[c, c], pb[c, phi]}, {pb[phi, c], pb[phi, phi]}};
hb[F_, H_] := pb[F, H] -
      ({pb[H, c], pb[H, phi]}.Inverse[A].{{pb[F, c]}, {pb[F, phi]}})[[1]]
```

Once this is done, we can use Mathematica to compute the evolution of a coordinate function z by simple evaluating `hb[E,z]`.

[1]There were in fact many more definitions required to handle all the various rules of differentiation needed to compute $\{z, E\}$. For brevity they are not included here. Ideally we could instruct Mathematica to use the pre-defined differentiation rules when evaluating the Poisson bracket.

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