TECHNICAL CONTRIBUTIONS

Note on Calculi for a Three-valued Logic for Logic Programming

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1 Introduction

From the time many-valued logic was first formulated by Lukasiewicz in the 1920ies, it has been associated with problems dealing with undefinedness, incompleteness and non-termination. In 1938, Kleene introduced many-valued logics to describe termination properties of recursive functions. Thus, it seems only natural that semantics of programming languages, especially of logic-oriented programming languages, be studied using suitable many-valued logics.

One of the main problems in the area of semantics for logic programming is the meaning of negation. Recently, Delahaye and Thibau [2] introduced a new three-valued logic to describe the semantics of logic programs with negation. In this note we present a sequent calculus and a natural deduction formulation of this logic.

This logic is Kleene's three-valued logic $\mathbf{K}_3$ together with a second implication connective $\rightarrow$, which is equal to Bochvar's external implication [7]. The set of truth values is $V = \{f, u, t\}$. Kleene's material implication $\Rightarrow$ is retained for use in the body of a clause in a logic program, while $\rightarrow$ is used between head and body. The fixpoint semantics developped using this logic enjoys several nice properties analogous to the fixpoint semantics of logic programs without negation or definite clause programs.

The truth tables for the logic **DT** are as follows:

$\wedge$	f	u	t
f	f	f	f
u	f	u	u
t	f	u	t

$\vee$	f	u	t
f	f	u	t
u	u	u	t
t	t	t	t

$\neg$	
f	t
u	u
t	f

These connectives are the standard operations in $\mathbf{L}_3$ and correspond to greatest lower bound, least upper bound, and complementation in the linear order

$f \leq u \leq t$ on V.

The following are the truth tables for the two implication connectives:

$\Rightarrow$	f	u	t
f	t	t	t
u	u	u	t
t	f	u	t

$\rightarrow$	f	u	t
f	t	t	t
u	t	t	t
t	f	f	t

As has been pointed out by Chestakov (cf. [5, p. 169]), the resulting propositional system is equivalent to the Lukasiewiczian system $\mathbf{L}_3$: The Lukasiewicz implication $\rightarrow_{\mathrm{L}}$ can be defined by

$$A \rightarrow_{\mathrm{L}} B := (A \Rightarrow B) \vee \left(\mathrm{J}_u(A) \wedge \mathrm{J}_u(B)\right),$$

where J_u is defined by $\mathrm{J}_u(A) := (A \Rightarrow \neg A) \wedge (\neg A \Rightarrow A)$, i.e., $\mathrm{J}_u(A)$ equals t if A is u, and equals f otherwise.

Semantics of the quantifiers $\forall$ (for all) and $\exists$ (there exists) are, analogously to $\wedge$ and $\vee$, the minimum and maximum of the set of truth values of all ground instances of the quantified formulas; they are the "generalized" $\wedge$ and $\vee$ in the terminology of [1]:

$\forall$	
$\{f\}, \{f,u\}, \{f,t\}, \{f,u,t\}$	f
$\{u\}, \{u,t\}$	u
$\{t\}$	t

$\exists$	
$\{f\}$	f
$\{f,u\}, \{u\}$	u
$\{f,t\}, \{u,t\}, \{f,u,t\}, \{t\}$	t

2 A Sequent Calculus for DT

Sequent calculus for classical logic was introduced by Gentzen [4] and has since been very important to proof theoretic studies. Many important theorems of first order logic and its extensions have natural formulations in sequent calculus and more elegant proofs than in, e.g., a framework based on Hilbert-style logical calculi. Sequent calculi have also been used for dealing with computational logic in the textbook of Gallier [3] and elsewhere. A useful feature of sequent calculi is the availability of a standard method for proving completeness due to Schütte—the method of reduction trees—which furthermore gives a non-algorithmic cut-elimination theorem for the logic being discussed. Furthermore, standard cut-elimination as done by Gentzen, as well as several other proof theoretic results obtained via sequent formulations carry over, *mutatis mutandis*.

We give in the following a three-valued sequent calculus for the logic **DT**. In the definitions of terms, formulas etc., we follow Takeuti [8]: free (a, b, ...) and bound variables (x, y, ...) are syntactically distinct. A three-valued sequent in first order three-valued logic is an expression of the form $\Pi \mid \Gamma \mid \Delta$, where Π, Γ, Δ are finite sequences of formulas. The calculus is as follows:

1. Axioms: all sequents of the form

$$A \mid A \mid A,$$

where A is any formula.

2. Structural rules:

Weakening:

$$\frac{\Pi \mid \Gamma \mid \Delta}{\Pi, A \mid \Gamma \mid \Delta}\ wf \qquad \frac{\Pi \mid \Gamma \mid \Delta}{\Pi \mid \Gamma, A \mid \Delta}\ wu \qquad \frac{\Pi \mid \Gamma \mid \Delta}{\Pi \mid \Gamma \mid \Delta, A}\ wt$$

Contraction:

$$\frac{\Pi, A, A \mid \Gamma \mid \Delta}{\Pi, A \mid \Gamma \mid \Delta}\ cf \qquad \frac{\Pi \mid \Gamma, A, A \mid \Delta}{\Pi \mid \Gamma, A \mid \Delta}\ cu \qquad \frac{\Pi \mid \Gamma \mid \Delta, A, A}{\Pi \mid \Gamma \mid \Delta, A}\ ct$$

Exchange:

$$\frac{\Pi, A, B, \Pi' \mid \Gamma \mid \Delta}{\Pi, B, A, \Pi' \mid \Gamma \mid \Delta}\ ef \qquad \frac{\Pi \mid \Gamma, A, B, \Gamma' \mid \Delta}{\Pi \mid \Gamma, B, A, \Gamma' \mid \Delta}\ eu \qquad \frac{\Pi \mid \Gamma \mid \Delta, A, B, \Delta'}{\Pi \mid \Gamma \mid \Delta, B, A, \Delta'}\ et$$

Cut:

$$\frac{\Pi, A \mid \Gamma \mid \Delta \quad \Pi \mid \Gamma, A \mid \Delta}{\Pi \mid \Gamma \mid \Delta}\ \text{cut}, fu \qquad \frac{\Pi, A \mid \Gamma \mid \Delta \quad \Pi \mid \Gamma \mid \Delta, A}{\Pi \mid \Gamma \mid \Delta}\ \text{cut}, ft$$

$$\frac{\Pi \mid \Gamma, A \mid \Delta \quad \Pi \mid \Gamma \mid \Delta, A}{\Pi \mid \Gamma \mid \Delta}\ \text{cut}, ut$$

3. Logical rules:

Rules for not:

$$\frac{\Pi \mid \Gamma \mid \Delta, A}{\Pi, (\neg A) \mid \Gamma \mid \Delta}\ \neg f \qquad \frac{\Pi \mid \Gamma, A \mid \Delta}{\Pi \mid \Gamma, (\neg A) \mid \Delta}\ \neg u \qquad \frac{\Pi, A \mid \Gamma \mid \Delta}{\Pi \mid \Gamma \mid \Delta, (\neg A)}\ \neg t$$

Rules for and:

$$\frac{\Pi, A, B \mid \Gamma \mid \Delta}{\Pi, (A \wedge B) \mid \Gamma \mid \Delta} \wedge f \qquad \frac{\Pi \mid \Gamma \mid \Delta, A \quad \Pi \mid \Gamma \mid \Delta, B}{\Pi \mid \Gamma \mid \Delta, (A \wedge B)} \wedge t$$

$$\frac{\Pi \mid \Gamma, A, B \mid \Delta \quad \Pi \mid \Gamma, A \mid \Delta, A \quad \Pi \mid \Gamma, B \mid \Delta, B}{\Pi \mid \Gamma, (A \wedge B) \mid \Delta} \wedge u$$

Rules for or:

$$\frac{\Pi, A \mid \Gamma \mid \Delta \quad \Pi, B \mid \Gamma \mid \Delta}{\Pi, (A \vee B) \mid \Gamma \mid \Delta} \vee f \qquad \frac{\Pi \mid \Gamma \mid \Delta, A, B}{\Pi \mid \Gamma \mid \Delta, (A \vee B)} \vee t$$

$$\frac{\Pi \mid \Gamma, A, B \mid \Delta \quad \Pi, A \mid \Gamma, A \mid \Delta \quad \Pi, B \mid \Gamma, B \mid \Delta}{\Pi \mid \Gamma, (A \vee B) \mid \Delta} \vee u$$

Rules for Kleene's implication $\Rightarrow$:

$$\frac{\Pi \mid \Gamma \mid \Delta, A \quad \Pi, B \mid \Gamma \mid \Delta}{\Pi, (A \Rightarrow B) \mid \Gamma \mid \Delta} \Rightarrow f \qquad \frac{\Pi, A \mid \Gamma \mid \Delta, B}{\Pi \mid \Gamma \mid \Delta, (A \Rightarrow B)} \Rightarrow t$$

$$\frac{\Pi \mid \Gamma, A, B \mid \Delta \quad \Pi \mid \Gamma, A \mid \Delta, A \quad \Pi, B \mid \Gamma, B \mid \Delta}{\Pi \mid \Gamma, (A \Rightarrow B) \mid \Delta} \Rightarrow u$$

Rules for Bochvar's external implication $\rightarrow$:

$$\frac{\Pi \mid \Gamma \mid \Delta, A \quad \Pi, B \mid \Gamma, B \mid \Delta}{\Pi, (A \rightarrow B) \mid \Gamma \mid \Delta} \rightarrow f \qquad \frac{\Pi, A \mid \Gamma, A \mid \Delta, B}{\Pi \mid \Gamma \mid \Delta, (A \rightarrow B)} \rightarrow t$$

Rules for the universal quantifier:

$$\frac{\Pi, A(t) \mid \Gamma \mid \Delta}{\Pi, (\forall x)A(x) \mid \Gamma \mid \Delta} \forall f \qquad \frac{\Pi \mid \Gamma \mid \Delta, A(a)}{\Pi \mid \Gamma\Delta, (\forall x)A(x)} \forall t$$

$$\frac{\Pi \mid \Gamma, A(a) \mid \Delta, A(a) \quad \Pi \mid \Gamma, A(t) \mid \Delta}{\Pi \mid \Gamma, (\forall x)A(x) \mid \Delta} \forall u$$

Rules for the existential quantifier:

$$\frac{\Pi, A(a) \mid \Gamma \mid \Delta}{\Pi, (\exists x)A(x) \mid \Gamma\Delta} \exists f \qquad \frac{\Pi \mid \Gamma \mid \Delta, A(t)}{\Pi \mid \Gamma \mid \Delta, (\exists x)A(x)} \exists t$$

$$\frac{\Pi, A(a) \mid \Gamma, A(a) \mid \Delta \quad \Pi \mid \Gamma, A(t) \mid \Delta}{\Pi \mid \Gamma, (\exists x)A(x) \mid \Delta} \exists u$$

In $(\forall u)$, $(\forall t)$, $(\exists f)$, and $(\exists u)$, a has to satisfy the *eigenvariable* condition: a must not occur in the lower sequent. Note that there is no rule $(\rightarrow u)$, since $A \rightarrow B$ does never take the truth vale u.

3 Natural Deduction for DT

In [4], Gentzen also introduced the calculus of natural deduction **NJ**. It differs from other concepts of calculi in that there are no axioms, but one deduces a formula from certain assumptions and some rules of inference allow certain assumptions to be *cancelled*. A formula is deducible in **NJ** iff it is deducible from a set of assumptions all of which are cancelled.

The original formulation of natural deduction by Gentzen is, of course, an intuitionistic system. In fact, there is a strong correspondence between natural deduction and the sequent calculus, **NJ**-deductions being, in a sense, **LJ** proof trees written sideways. From this point of view, the "intuitionism" of **NJ** lies in the fact that there is always only *one* formula being derived, just as there is at most one formula to the right of the sequent arrow in **LJ**, the intuitionistic sequent calculus. We thus immediately obtain a classical natural deduction system by allowing *sequences* of formulas to be derived. To generalize this system to a three-valued logic, one additionally has to allow for two different kinds of assumptions corresponding to the truth values f and u. Also note, that in natural deduction, we deal with *sets* of formulas, and the comma stands for set union.

Rules for not:

$$\dfrac{\begin{array}{c}\Pi,[A]\mid\Gamma\\ \vdots\\ \Delta\end{array}}{\Delta,\neg A}\,I\neg \qquad \dfrac{\begin{array}{c}\Pi,[\neg A]\mid\Gamma\\ \vdots\\ \Delta,\neg A\end{array}\qquad\begin{array}{c}\Pi\mid\Gamma,[A]\\ \vdots\\ \Delta\end{array}}{\Delta}\,E_u\neg \qquad \dfrac{\begin{array}{c}\Pi\mid\Gamma,[\neg A]\\ \vdots\\ \Delta,\neg A\end{array}\qquad\begin{array}{c}\Pi\mid\Gamma\\ \vdots\\ \Delta,A\end{array}}{\Delta}\,E_f\neg$$

Rules for and:

$$\dfrac{\begin{array}{c}\Pi\mid\Gamma\\ \vdots\\ \Delta,A\end{array}\quad\begin{array}{c}\Pi\mid\Gamma\\ \vdots\\ \Delta,B\end{array}}{\Delta,(A\wedge B)}\,I\wedge \qquad \dfrac{\begin{array}{c}\Pi\mid\Gamma,[A\wedge B]\\ \vdots\\ \Delta,(A\wedge B)\end{array}\qquad\begin{array}{c}\Pi,[A],[B]\mid\Gamma\\ \vdots\\ \Delta\end{array}}{\Delta}\,E_f\wedge$$

$$\dfrac{\begin{array}{c}\Pi,[A\wedge B]\mid\Gamma\\ \vdots\\ \Delta,(A\wedge B)\end{array}\quad\begin{array}{c}\Pi\mid\Gamma,[A],[B]\\ \vdots\\ \Delta\end{array}\quad\begin{array}{c}\Pi\mid\Gamma,[A]\\ \vdots\\ \Delta,A\end{array}\quad\begin{array}{c}\Pi\mid\Gamma,[B]\\ \vdots\\ \Delta,B\end{array}}{\Delta}\,E_u\wedge$$

Rules for or:

$$\dfrac{\begin{array}{c}\Pi \mid \Gamma \\ \vdots \\ \Delta, A, B\end{array}}{\Delta, (A \vee B)}\, I\vee \qquad \dfrac{\begin{array}{c}\Pi \mid \Gamma, [A \vee B] \\ \vdots \\ \Delta, (A \vee B)\end{array} \quad \begin{array}{c}\Pi, [A] \mid \Gamma \\ \vdots \\ \Delta\end{array} \quad \begin{array}{c}\Pi, [B] \mid \Gamma \\ \vdots \\ \Delta\end{array}}{\Delta}\, E_f\vee$$

$$\dfrac{\begin{array}{c}\Pi, [A \vee B] \mid \Gamma \\ \vdots \\ \Delta, (A \vee B)\end{array} \quad \begin{array}{c}\Pi \mid \Gamma, [A], [B] \\ \vdots \\ \Delta\end{array} \quad \begin{array}{c}\Pi, [A] \mid \Gamma, [A] \\ \vdots \\ \Delta\end{array} \quad \begin{array}{c}\Pi, [B] \mid \Gamma, [B] \\ \vdots \\ \Delta\end{array}}{\Delta}\, E_u\vee$$

Rules for Kleene's implication $\Rightarrow$:

$$\dfrac{\begin{array}{c}\Pi, [A] \mid \Gamma \\ \vdots \\ \Delta, B\end{array}}{\Delta, (A \Rightarrow B)}\, I\Rightarrow \qquad \dfrac{\begin{array}{c}\Pi \mid \Gamma, [A \Rightarrow B] \\ \vdots \\ \Delta, (A \Rightarrow B)\end{array} \quad \begin{array}{c}\Pi \mid \Gamma \\ \vdots \\ \Delta, A\end{array} \quad \begin{array}{c}\Pi, [B] \mid \Gamma \\ \vdots \\ \Delta\end{array}}{\Delta}\, E_f\Rightarrow$$

$$\dfrac{\begin{array}{c}\Pi, [A \Rightarrow B] \mid \Gamma \\ \vdots \\ \Delta, (A \Rightarrow B)\end{array} \quad \begin{array}{c}\Pi \mid \Gamma, [A], [B] \\ \vdots \\ \Delta\end{array} \quad \begin{array}{c}\Pi \mid \Gamma, [A] \\ \vdots \\ \Delta, A\end{array} \quad \begin{array}{c}\Pi, [B] \mid \Gamma, [B] \\ \vdots \\ \Delta\end{array}}{\Delta}\, E_u\Rightarrow$$

Rules for Bochvar's external implication $\rightarrow$:

$$\dfrac{\begin{array}{c}\Pi, [A] \mid \Gamma, [A] \\ \vdots \\ \Delta, B\end{array}}{\Delta, (A \rightarrow B)}\, I\rightarrow \qquad \dfrac{\begin{array}{c}\Pi \mid \Gamma, [A \rightarrow B] \\ \vdots \\ \Delta, (A \rightarrow B)\end{array} \quad \begin{array}{c}\Pi \mid \Gamma \\ \vdots \\ \Delta, A\end{array}}{\Delta, B}\, E_f\rightarrow$$

Rules for the universal quantifier:

$$\dfrac{\begin{array}{c}\Pi \mid \Gamma \\ \vdots\, * \\ \Delta, A(a)\end{array}}{\Delta, (\forall x)A(x)}\, I\forall \qquad \dfrac{\begin{array}{c}\Pi \mid \Gamma, [(\forall x)A(x)] \\ \vdots \\ \Delta, (\forall x)A(x)\end{array} \quad \begin{array}{c}\Pi, [A(t)] \mid \Gamma \\ \vdots \\ \Delta\end{array}}{\Delta}\, E_f\forall$$

$$\dfrac{\begin{array}{c}\Pi, [(\forall x)A(x)] \mid \Gamma \\ \vdots \\ \Delta, (\forall x)A(x)\end{array} \quad \begin{array}{c}\Pi \mid \Gamma, [A(a)] \\ \vdots\, * \\ \Delta, A(a)\end{array} \quad \begin{array}{c}\Pi \mid \Gamma, [A(t)] \\ \vdots \\ \Delta\end{array}}{\Delta}\, E_u\forall$$

Rules for the existential quantifier:

$$\dfrac{\begin{array}{c}\Pi \mid \Gamma \\ \vdots \\ \Delta, A(t)\end{array}}{\Delta, (\exists x)A(x)}\ I\exists \qquad \dfrac{\begin{array}{c}\Pi \mid \Gamma, [(\exists x)A(x)] \\ \vdots \\ \Delta, (\exists x)A(x)\end{array} \quad \begin{array}{c}\Pi, [A(a)] \mid \Gamma \\ \vdots\ * \\ \Delta\end{array}}{\Delta}\ E_f\exists$$

$$\dfrac{\begin{array}{c}\Pi, [(\exists x)A(x)] \mid \Gamma \\ \vdots \\ \Delta, (\exists x)A(x)\end{array} \quad \begin{array}{c}\Pi, [A(a)] \mid \Gamma, [A(a)] \\ \vdots\ * \\ \Delta\end{array} \quad \begin{array}{c}\Pi \mid \Gamma, [A(t)] \\ \vdots \\ \Delta\end{array}}{\Delta}\ E_u\exists$$

In $(I\forall)$, $(E_u\forall)$, $(E_f\exists)$, and $(E_u\exists)$, the variable a again has to satisfy the eigenvariable condition: a must not occur in $\Pi \mid \Gamma$ and must not be the eigenvariable of a quantifier inference in the deduction $*$. Again, note that there is no rule $(E_u\rightarrow)$.

In ordinary natural deduction one usually works with a special symbol $\perp$ denoting falsehood, and replaces $\neg A$ by $A \supset \perp$. This comes from the fact that in **NJ**, it would be a bit awkward to allow an empty end formula (which would correspond to an empty right-hand side in **LJ**). Since our calculus is not intuitionistic, we allow any number (including none) of end formulas. Still, it would be possible to introduce a device analogous to $\perp$ (e.g., some propositional constant or a formula which does not assume the truth value t) to represent the empty set of formulas. In an intuitionistic system for **DT**, one could then represent $\neg A$ as $A \Rightarrow \perp$, where $\perp$ always takes f as its truth value. This works only because $\Rightarrow$ is defined by $\neg A \vee B$, and because f leaves disjunctions invariant; it would not work with $\rightarrow$ and/or a constant taking u as its value.

Another fact that should be pointed out is that the natural deduction calculus for **DT** also enjoys a certain "normal form" property, as does **NJ** (cf. [6]), but the resulting "normal forms" are not unique. The reason is that there may be more than one formula at the bottom of a deduction, to which reductions can be applied. One could, however, define the notion of *formula thread*, i.e., an end formula and its predecessors in the deduction. Normal forms of each formula thread would then be unique.

The essential feature of these calculi is the use of three places for formulas in the sequent corresponding to the three truth values, resp. the use of two different places for assumptions in the natural deduction formulation.

Finally, note that the choice of the truth value t as the designated truth value in the natural deduction axiomatization is not essential—one could just as well give natural deduction systems in each of the other two truth values.

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