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Generalized Theories of Thermoelasticity

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "Generalized Theories of Thermoelasticity," submitted by Hong Li in partial fulfillment of the requirements for the degree of Master of Science .

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Abstract

This thesis is concerned with the generalized theories of thermoelasticity. Historical developments of the various theories of thermoelasticity are given in the introduction. Part 1 covers the basic laws and the generalized theories of thermoelasticity with one relaxation time as well as with two relaxation times. Part 2 of this thesis gives the theories of thermoelasticity derived recently be Green and Naghdi. In Part 3, we have formulated a one dimensional problem and then obtained a solution for the thermal shock problem for the stress-free as well as the fixed boundary. Numerical results are given in the form of tables and displayed graphically.

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Contents

List of Tables

List of Figures

List of Symbols

c	specific heat per unit mass
C_{ijkl}	elasticity tensor
e_{ij}	Green's strain tensor
$erf(\,)$	error function
F	Helmoholtz's free energy
F_i	external force per unit mass
$H(\,)$	Heaviside function
k_{ij}	thermal conductivity tensor
q_i	heat flux
R	strength of the internal heat source per unit mass
S	entropy per unit mass
s_{ij}	(second) Piola-Kirchoff stress tensor
t	time
T	absolute temperature
T_0	constant temperature
U	internal energy per unit mass
u_i	displacement vector
$x_i,\ x$	coordinate
α	coefficient of linear thermal expansion of the material
β_{ij}	thermoelasticity tensor
ϵ	thermoelastic coupling constant

ϵ_{ij}	Cauchy's strain tensor
$\lambda,\ \mu$	Lame's constants
ρ	mass density in the initial undeformed state
ρ_d	mass density in the deformed state
τ	relaxation time
σ	stress
σ_{ij}	Cauchy's stress tensor
θ	temperature change
θ_0	initial uniform temperature

Introduction

Thermoelasticity describes the behavior of elastic bodies under the influence of nonuniform temperature fields. It represents, therefore, a generalization of the theory of elasticity.

Heat phenomena in elasticity were first discussed by Duhamel [1] in 1837. In 1885, Neumann [2] rederived the equations obtained by Duhamel earlier for the classical theory of thermoelasticity. In this theory, the equations of motion or equilibrium contain the temperature term, but the equation of heat conduction is independent of the strain field, which contradicts the physical experiments.

Further development came in 1956, when Biot [3] introduced the coupled theory of thermoelasticity. This theory consists of two coupled partial differential equations in the displacement vector and temperature field one of which is hyperbolic, and the other is parabolic. Due to the nature of the parabolic-type equation, this theory predicts an infinite speed for heat propagation, that is, if a material conducting heat is subjected to a thermal disturbance, the effects of the disturbance will be felt instantaneously at distances infinitely far from its source. This prediction is unrealistic from a physical point of view, particularly in problems like those concerned with sudden heat inputs.

During the last three decades, a great deal of attention has been given to the generalized theories which are free from this drawback. These theories make use of modified versions of the classical Fourier's law of heat conduction and consequently involve hyperbolic-type heat transport equation admitting finite speed for heat propagation. According to these theories, heat propagation is to be viewed as a wave phenomenon rather than a diffusion phenomenon. A wave-like thermal disturbance is referred to as second sound—the first sound being the usual sound (wave)—and generalized theories predicting the occurrence of such disturbances are known as theories with finite wave speed or theories with second sound. These theories are motivated by experiments exhibiting the actual occurrence of second sound at low temperatures and for small interval of time.

One theory of the generalized thermoelasticity was introduced by Lord and Shulman [4] in 1967 for the isotropic case and extended by Dhaliwal and Sherief [5] in 1980 to the anisotropic case. By incorporating a heat flux-rate into the Fourier's law, this theory involves a hyperbolic-type heat transport equation admitting finite speed for heat propagation. Sherief and Dhaliwal [6] and Dhaliwal and Sherief [5] have established the uniqueness of solutions in the isotropic as well as anisotropic case for prescribed temperature on the entire boundary. Dhaliwal and Sherief [7] have employed a reciprocity theorem to derive an integral representation of solutions for the case of vibrations varying harmonically in time. Chester [8] has explained a clear physical meaning for the relaxation time and estimated the value. Ignaczak [9] has established the uniqueness of the solution for the heat-flux formulation of the theory in the isotropic case. A generalized one-dimensional thermal shock problem named Danilovskaya problem has been considered by many authors, e.g., Popov [10], Norwood and Warren [11], Kotenko and Lenyuk [12] , Rama Murthy [13], Sherief and Dhaliwal [14]. These authors have obtained different expressions by employing different notation. Other than this, some other one-dimensional half space problems have been studied by several authors under various other boundary conditions. Lord and Shulman [4], Achenbach [15] , Norwood and Warren [11], Lord and Lopez [16], Mengi and Turhan [17] and Rama Murthy [13] have investigated the cases of unit step function type sudden stress/strain or temperature input on the boundary. Rama Murthy ([13], [18]) has considered the unit step function type sudden velocity or temperature change, and constant velocity impact on the boundary. Chandrasekharaiah [19] has studies the case of thermal impulse on the strain free boundary. And Gawinecki [20] has studied the existence and uniqueness of solutions of thermoelastic equations in generalized as well as classical cases.

Another theory of the generalized thermoelasticity was developed by Green and Lindsay [21] in 1972. By including the temperature-rate among the constitutive variables, this theory also gives a hyperbolic-type heat transport equation admitting finite speed for heat propagation. A remarkable feature of this theory is that it does not violate the classical Fourier's law, if the material has a center of symmetry at each point. Moreover, even in the general anisotropic case, the heat conduction equation of this theory does not include the heat flux-rate term. This theory is based on an entropy production inequality proposed by Green and Laws [22]. Suhubi [23] has formulated this theory independently. Chandrasekharaiah ([19], [24]), Chandrasekharaiah and Srikantiah [25], Dhaliwal and Rokne [26] have considered the half-space problem in the cases of thermal impulse, unit step function type sudden changes in strain, displacement, or heat flux on the boundary. And Sherief [27] has solved a thermo-mechanical shock problem for thermoelasticity with two relaxation times. Wang and Dhaliwal [28] have found the fundamental solutions of the generalized thermoelastic equations of this theory. The detailed references regarding the developments in the generalized theory of thermoelasticity can be found in a review paper by Chandrasekharaiah [29].

In recent years, Green and Naghdi ([30], [31], [32]) put forth a new theory of thermoelasticity, which provides sufficient basic modifications in the constitutive equations to permit treatment of a much wider class of heat flow problems. The characterization of material response for the thermal phenomena in [30] and [31] is based on three types of constitutive response functions . The nature of these three types of constitutive equations is such that when the respective theories are linearized, type I is the same as the classical heat conduction theory (based on Fourier's law), type II predicts a finite speed and involves no energy dissipation, and type III permits the propagation of thermal signals at both infinite and finite speeds.

Some work concerning this theory has been done recently by Dhaliwal, Majumdar and Wang [33], in which they have considered the problem of thermoelastic waves in an infinite solid caused by a line heat source.

This thesis contains three parts.

Part 1: Thermoelasticity with second sound (or the generalized theories of thermomechanics). Detailed formulation of *Kinematic Relations, Law of Motion, Law of Conservation of Mass, Law of Conservation of Energy* and *Second Law of Thermodynamics* is given in Chapter 1. The generalized thermoelasticity with one relaxation time in both isotropic and anisotropic cases is discussed in Chapter 2. And as a special case, the constitutive equations and governing equations are obtained for one-dimensional problem for the isotropic case. In Chapter 3, the governing equations for the thermoelasticity with two relaxation times are obtained. And as a special case, the governing equations for the isotropic case are derived.

Part 2: Re-examination of the basic postulates of Thermomechanics. The new theory is outlined in Chapter 4, which contains a useful analogy between the concepts and equations of the purely thermal and the purely mechanical theories and three types of constitutive equations and their linear forms.

Part 3: One-dimensional thermal shock problems. To analyze the new theory, we have formulated a one-dimensional problem in Chapter 5. We have solved two one-dimensional thermal shock problems, one with stress-free boundary in Chapter 6, and another with fixed boundary in Chapter 7. Numerical results are given in the form of tables and displayed graphically.

Part I

Thermoelasticity With Second Sound

Chapter 1

Basic Laws

Like other branches of thermomechanics of deformable bodies, the thermoelasticity theory is based on the following fundamental equations ([34]):

1.1 Kinematic Relations

Let the position of a general point P of an elastic body in its initial state at time $t = 0$ be given by coordinates x_1, x_2, x_3 in a rectangular Cartesian coordinate system fixed in space. Let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ be the corresponding system of base vector (Figure 1.1). We then have

$$d\vec{r} = \vec{e}_1 dx_1 + \vec{e}_2 dx_2 + \vec{e}_3 dx_3 \equiv \vec{e}_i dx_i, \tag{1.1}$$

for the line element in the undeformed body.

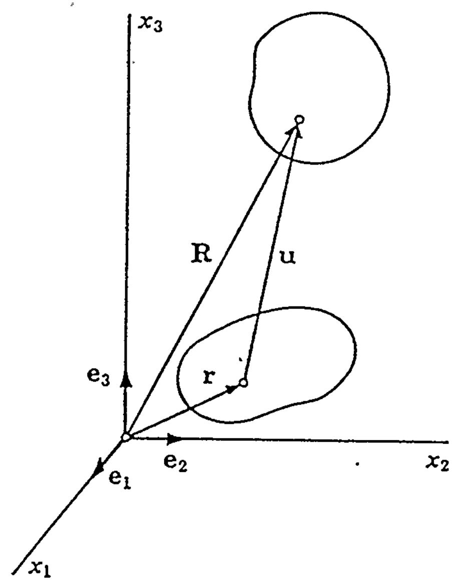
After the body has been deformed, the position vector of the point P will have changed from its initial value $\vec{r}$ to

$$\vec{R} = \vec{r} + \vec{u}, \tag{1.2}$$

where $\vec{u} = u_m \vec{e}_m$ represents the displacement vector. Analogous to equation (1.1), we write

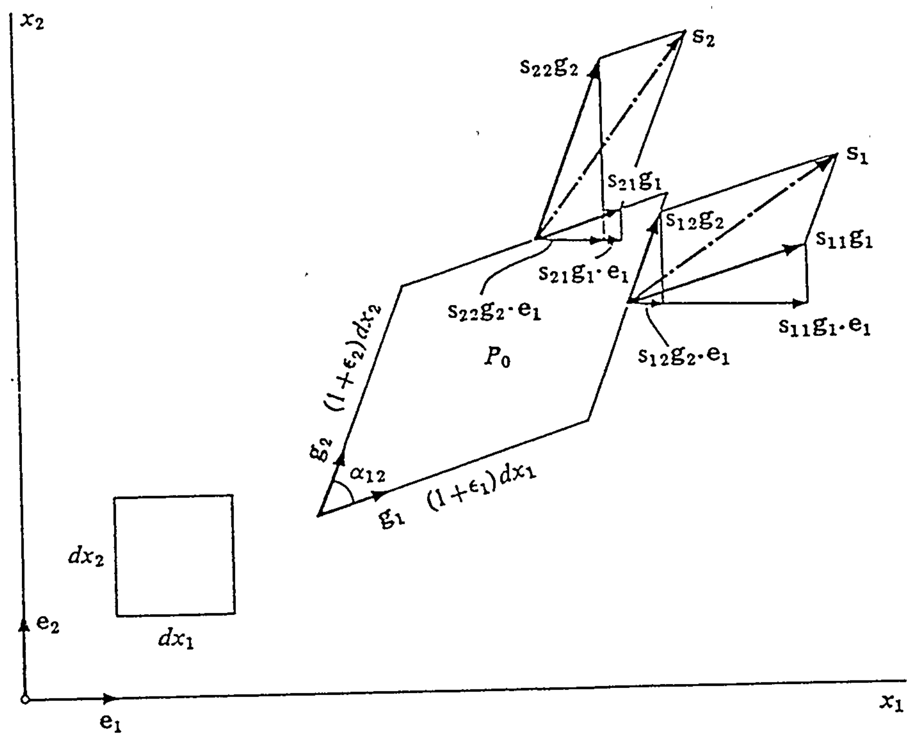
$$d\vec{R} = \vec{g}_i dx_i, \tag{1.3}$$

for the element in the deformed body.



x_3
R
u
r
e_3
e_2
e_1
x_2
x_1

Fig.1.1



x_2
s_2
$s_{22}g_2$
s_1
$s_{21}g_1$
$s_{12}g_2$
$s_{11}g_1$
$s_{21}g_1 \cdot e_1$
$s_{22}g_2 \cdot e_1$
$s_{11}g_1 \cdot e_1$
$s_{12}g_2 \cdot e_1$
$(1+\epsilon_2)dx_2$
P_0
g_2
α_{12}
g_1
$(1+\epsilon_1)dx_1$
dx_2
dx_1
e_2
e_1
x_1

Fig.1.2

Base vector $\vec{g}_i$ (Figure 1.2), forms a nonorthogonal triad varying from point to point. This triad represents the deformed, originally orthogonal, triad of the base vectors $\vec{e}_i$.

We consider all quantities (displacements and temperature) associated with the point as functions of these coordinates and of time t.

Differentiating equation (1.2) with respect to x_i, we obtain

$$\vec{g}_i = \vec{e}_i + \vec{u}_{,i} = \vec{e}_i + u_{m,i}\vec{e}_m, \tag{1.4}$$

where $u_{,i}$ represents the partial derivative of u with respect to x_i.

Using equations (1.1), (1.3) and (1.4), we find the expressions for the squared line elements $(dr)^2$ and $(dR)^2$ in the undeformed and deformed states of the body, respectively, as

$$(dr)^2 = \delta_{ij} dx_i dx_j, \qquad (dR)^2 = g_{ij} dx_i dx_j, \tag{1.5}$$

where

$$\delta_{ij} = \vec{e}_i \cdot \vec{e}_j = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j, \end{cases}$$

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j = \delta_{ij} + 2e_{ij}, \tag{1.6}$$

$$2e_{ij} = u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}. \tag{1.7}$$

The nine quantities $e_{ij} = e_{ji}$ represent the components of the Green's strain tensor defined as half the difference between the two metric tensors g_{ij} and δ_{ij} in the deformed and undeformed states of the elastic body, respectively.

The equations (1.7) are called *strain-displacement relations*, or *kinematic relations.*

1.2 Law of Motion

Consider a surface element dA whose normal was initially in the direction of $\vec{e}_1$ (Figure 1.2). The stress vector $\vec{s}_1$ acting on this element is then defined as the corresponding force divided by the initial area $dA = dx_2 dx_3$. Resolving $\vec{s}_1$ into three components in the directions of the three base vectors $\vec{g}_1$, $\vec{g}_2$, $\vec{g}_3$, one has

$$\vec{s}_1 = s_{11}\vec{g}_1 + s_{12}\,\vec{g}_2 + s_{13}\,\vec{g}_3.$$

In general, on a surface with normal initially in the direction of $\vec{e}_i$, we have

$$\vec{s}_i = s_{ij}\vec{g}_j. \tag{1.8}$$

The nine quantities s_{ij} constitute the (second) Piola-Kirchhoff stress tensor.

It is now a simple matter to formulate the equations of motion for an element of the body with initial volume $dV = dx_1 dx_2 dx_3$ and mass $dm = \rho dV = \rho_d dV_d$, where ρ and ρ_d are the mass density, and V and V_d are the volume occupied by the body in its reference configuration in the undeformed and deformed states, respectively. Taking components in the x_1-direction of all force vectors $s_{11}\,\vec{g}_1\,dx_2 dx_3$, ect., acting on the element (see Figure 1.2), multiplying with unit vector $\vec{e}_1$, and remembering that the stress vector $-s_{ij}\vec{g}_j$ on the left-hand surface of the element changes to the $s_{ij}\vec{g}_j + (s_{ij}\vec{g}_j)_{,1}dx_1 + \cdots$ on the right- hand surface, with similar relations for the other surfaces, one finds from Cauchy's law that

$$[(s_{1j}\vec{g}_j)_{,1} + (s_{2j}\vec{g}_j)_{,2} + (s_{3j}\vec{g}_j)_{,3}] \cdot \vec{e}_1 + \rho F_1 = \rho \ddot{u}_1,$$

where F_1 is the component in the x_1-direction of the external force vector F per unit mass and dot over the quantity represents the partial derivative with respect to time t. Two analogous relations are obtained for the x_2- and x_3-directions. From equation (1.4), one has

$$\vec{g}_j \cdot \vec{e}_k = \vec{e}_j \cdot \vec{e}_k + u_{m,j}\vec{e}_m \cdot \vec{e}_k = \delta_{kj} + u_{k,j}. \tag{1.9}$$

Hence, the three equations of motion read as

$$[s_{ij}(\delta_{kj} + u_{k,j})]_{,i} + \rho F_k = \rho \ddot{u}_k. \tag{1.10}$$

By taking moments of the force couples $s_{12}\ \vec{g}_2\ dx_2dx_3$ and $s_{21}\ \vec{g}_1\ dx_1dx_3$, acting on the element of Figure 1.2, one obtains

$$\begin{aligned} s_{12}\ \vec{g}_2\ dx_2dx_3 \times \vec{g}_1dx_1 + s_{21}\ \vec{g}_1\ dx_1dx_3 \times \vec{g}_2dx_2 \\ = (s_{12} - s_{21})\vec{g}_2 \times \vec{g}_1\ dx_1dx_2dx_3. \end{aligned}$$

And hence for a body in equilibrium, one must have

$$s_{12} = s_{21},$$

which can be generalized to

$$s_{ij} = s_{ji},\ \ i,j = 1,2,3. \tag{1.11}$$

We assume that this symmetry of the stress tensor holds also for a body in motion.

The equations (1.10) are called *the equations of motion.*

1.3 Law of Conservation of Mass

The Law of Conversation of Mass expresses the fact that the total mass of the body remains constant, that is,

$$\int_{V_d} \rho_d dV_d = \int_V \rho dV, \tag{1.12}$$

where ρ and ρ_d are respectively, the mass density in the undeformed and deformed states of the elastic body. Due to deformation, the volume of the element changes from its initial value $dV = dx_1dx_2dx_3$ to

$$dV_d = \vec{g}_1 \cdot (\vec{g}_2 \times d\vec{g}_3)dx_1dx_2dx_3 = \sqrt{g}dV, \tag{1.13}$$

where

$$g = \det |\, g_{ij} \,| = \det |\delta_{ij} + 2e_{ij}|.$$

From equations (1.12) and (1.13), one finds that

$$\rho_d = \frac{\rho}{\sqrt{g}}. \tag{1.14}$$

Equation (1.14) is called *the law of conservation of mass.*

1.4 Law of Conservation of Energy

The law of conservation of energy or *the first law of thermodynamics* is given by

$$\begin{aligned} &\frac{d}{dt}\int_m \frac{1}{2}\dot{u}_i\dot{u}_i dm + \frac{d}{dt}\int_m U dm \\ = &\int_m F_i\dot{u}_i dm + \oint_A f_i\dot{u}_i dA + \int_m R dm - \oint_A Q_i n_i dA, \end{aligned} \tag{1.15}$$

where m is the mass occupied by the body, A is the closed surface of m, U denotes internal energy per unit mass, F_i is the body force vector per unit mass, f_i is the applied surface stress, R is the heat produced per unit time and unit mass by heat sources distributed within the body and Q_i is the heat-flux through the surface of the body taken positive outwards. Both f_i and Q_i are referred to the unit area of the surface of the deformed body.

The first and second terms on the left-hand side of equation (1.15) represent the rate of change of kinetic energy and internal energy, respectively. They are equal to the rate of work done by all external forces, and to the amount of heat produced per unit time within the body plus the heat transported into the body from the outside.

Equation (1.15) may be transformed into a more convenient form with the aid of *the principle of rate of work.* This principle states that the rate of change of kinetic energy equals the rate of work of all forces, external and internal. Per unit of initial volume, the latter equals $-\vec{s}_i \cdot \dot{\vec{u}}_i$ or, using $\dot{\vec{u}} = \dot{u}_k \vec{e}_k$ and equations (1.8), (1.9) and (1.7), one obtains

$$-\vec{s}_i \cdot \dot{\vec{u}}_i = -s_{ij}\dot{u}_{k,i}\vec{g}_j \cdot \vec{e}_k = -s_{ij}\dot{u}_{k,i}(\delta_{kj} + u_{k,j}) = -s_{ij}\dot{e}_{ij}.$$

The principle of rate of work thus reads

$$\frac{d}{dt}\int_m \frac{1}{2}\dot{u}_i\dot{u}_i dm = \int_m F_i\dot{u}_i dm + \oint_A f_i\dot{u}_i dA - \int_{V_d} s_{ij}\dot{e}_{ij} dV_d.$$

Combining the above result with equation (1.15), gives

$$\int_m (\dot{U} - R)dm = \int_{V_d} s_{ij}\dot{e}_{ij} dV_d - \oint_A Q_i n_i dA.$$

Putting $dm = \rho dV$ and applying Gauss' theorem ($\vec{a}$ being an arbitrary vector)

$$\oint_A a_i n_i dA = \int_{V_d} \frac{1}{\sqrt{g}}(a_i\sqrt{g})dV_d,$$

with $dV_d = \sqrt{g}dV$ to the surface integral, one obtains, finally

$$\rho(\dot{U} - R) = s_{ij}\dot{e}_{ij} - q_{i,i}, \tag{1.16}$$

where $q_i = Q_i\sqrt{g}$ represents the heat-flux vector, referred to the unit area of the undeformed body.

Equation (1.16) is called *the law of conservation of energy.*

1.5 Second Law of Thermodynamics

The second law of Thermodynamics demanding positive production of entropy, in the form of the Clausius-Duhem inequality, states that

$$\frac{d}{dt}\int_m S dm - \int_m \frac{R}{T} dm + \oint_A \frac{Q_i n_i}{T} dA \geq 0, \tag{1.17}$$

where T is absolute temperature and S is entropy per unit mass. Using Gauss' theorem, we are led to

$$\rho(T\dot{S} - R) \geq -q_{i,i} + \frac{q_i}{T}T_{,i} \tag{1.18}$$

and upon elimination of R between equations (1.16) and (1.18), we obtain

$$\rho(\dot{U} - T\dot{S}) \leq S_{ij}\dot{e}_{ij} - \frac{q_i}{T}T_{,i}. \tag{1.19}$$

We now introduce the so-called Helmholtz's free energy function F, defined by

$$F = U - TS. \tag{1.20}$$

Let us assume that

$$\begin{aligned} F &= F(e_{ij}, T; T_{,i}), \\ S &= S(e_{ij}, T, T_{,i}), \\ q_i &= q_i(e_{ij}, T, T_{,i}), \\ s_{ij} &= s_{ij}(e_{ij}, T, T_{,i}), \end{aligned}$$

then

$$\dot{F} = \frac{\partial F}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial F}{\partial T}\dot{T} + \frac{\partial F}{\partial T_{,i}}\dot{T}_{,i}.$$

Substituting this into equation (1.16) and inequality (1.19), we find that

$$(\rho\frac{\partial F}{\partial e_{ij}} - s_{ij})\dot{e}_{ij} + \rho(\frac{\partial F}{\partial T} + S)\dot{T} + \rho(\frac{\partial F}{\partial T_{,i}})\dot{T}_{,i} + \rho(T\dot{S} - R) + q_{i,i} = 0,$$

$$(\rho\frac{\partial F}{\partial e_{ij}} - s_{ij})\dot{e}_{ij} + \rho(\frac{\partial F}{\partial T} + S)\dot{T} + \rho(\frac{\partial F}{\partial T_{,i}})\dot{T}_{,i} + \frac{q_i}{T}T_{,i} \leq 0.$$

Since $\dot{e}_{ij}$, $\dot{T}$ and $\dot{T}_{,i}$ are arbitrary, and q_i and the expressions within the parentheses in the inequality do not depend on these quantities, we conclude that

$$s_{ij} = \rho\frac{\partial F}{\partial e_{ij}}, \tag{1.21}$$

$$S = -\frac{\partial F}{\partial T}, \tag{1.22}$$

$$\frac{\partial F}{\partial T_{,i}} = 0, \tag{1.23}$$

$$q_{i,i} = \rho(R - T\dot{S}), \tag{1.24}$$

$$q_i T_{,i} \leq 0. \tag{1.25}$$

The above five equations describe the constitution of the thermoelastic material. Equation (1.21) represents the stress-strain law, while equation (1.22) defines entropy. Substitution of equation (1.22) into equation (1.24), together with equation (1.23), leads to the equation of heat conduction

$$q_{i,i} = \rho T(\frac{\partial^2 F}{\partial e_{ij}\partial T}\dot{e}_{ij} + \frac{\partial^2 F}{\partial T^2}\dot{T}) + \rho R. \tag{1.26}$$

We note that the temperature T and strain components e_{ij} are coupled in the above heat conduction equation.

The sixteen scalar relations of equations (1.7), (1.10), (1.21) and (1.26) form the basic equations of themoelasticity. They contain u_i, e_{ij}, s_{ij} and T as sixteen unknown functions of space x_i and time t. They form the governing equations of the conventional coupled nonlinear theory of thermoelasticity.

1.6 Linear Approximation

In the linear approximation of the theory, we suppose that the field variables are small enough such that the second and higher degree terms in these variables may be neglected in the governing equations.

Then s_{ij} and e_{ij} reduce, respectively, to Cauchy's stress and strain tensors σ_{ij} and ϵ_{ij}. Equations (1.7), (1.10) and (1.24), with $\rho_d \approx \rho$, $T \approx \theta_0$, reduce to

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.27}$$

$$\sigma_{ij,j} + \rho F_i = \rho \ddot{u}_i, \tag{1.28}$$

$$q_{i,i} = \rho(R - \dot{S}\theta_0). \tag{1.29}$$

Here θ_0 is the initial uniform temperature (assumed to be positive).

Let us assume that

$$q_i = k_i - k_{ij}T_{,j} + k_{ijl}\epsilon_{jl},$$

where the coefficients k_i, k_{ij} and k_{ijl} are functions of x_i and T.

Due to the inequality (1.25), we find that

$$-q_i = k_i + k_{ij}T_{,j} + k_{ijl}\epsilon_{jl}.$$

By combining the above two equations, one gets

$$q_i = -k_{ij}T_{,j}. \tag{1.30}$$

For $V \equiv \rho F$, its linear expansion (retaining only up to quadratic terms) gives the following

$$V \equiv \rho F = V_0 + c_{ij}\epsilon_{ij} + \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl} - \beta_{ij}\epsilon_{ij}\theta + d\theta^2, \tag{1.31}$$

where V_0 is the energy at rest in the natural state, $\theta = T - \theta_0$ and the coefficients c_{ij}, C_{ijkl}, β_{ij}, and d are, generally, functions of x_i and T. And $V_0 \equiv 0$, if we assume that $V = 0$, when $\theta = 0$ and $\epsilon_{ij} = 0$.

Substituting from equation (1.31) into equation (1.21), we obtain

$$\sigma_{ij} = \frac{1}{2}(C_{ijkl} + C_{klij})\epsilon_{kl}, \tag{1.32}$$

since $c_{ij} \equiv 0$, if we assume that $\sigma_{ij} = 0$, when $\theta = 0$ and $\epsilon_{ij} = 0$.

The coefficients C_{ijkl} are the elastic moduli.

Considering the symmetry of the stress and strain tensors, from equations (1.32) and (1.31), we find that

$$C_{ijkl} = C_{jikl} = C_{ijlk}, \qquad \beta_{ij} = \beta_{ji}. \tag{1.33}$$

We notice from equation (1.21), that

$$\frac{\partial s_{ij}}{\partial e_{kl}} = \frac{\partial s_{kl}}{\partial e_{ij}},$$

and hence, we find that

$$C_{ijkl} = C_{klij}. \tag{1.34}$$

Let

$$c = \frac{\partial U}{\partial T},$$

then using equation (1.20), we find that

$$\begin{aligned} c &= \frac{\partial F}{\partial T} + S + T\frac{\partial S}{\partial T} \\ &= T\frac{\partial S}{\partial T} = -T\frac{\partial^2 F}{\partial T^2} \end{aligned}$$

$$\begin{aligned} &= -T\frac{\partial^2 F}{\partial \theta^2} = -\frac{2d}{\rho}T \\ &\approx -\frac{2d}{\rho}\theta_0, \end{aligned}$$

or

$$d \approx -\frac{\rho c}{2\theta_0}$$

.

Linear expansion (1.31) becomes

$$\rho F = \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl} - \beta_{ij}\epsilon_{ij}\theta - \frac{1}{2}(\frac{\rho c}{\theta_0})\theta^2.$$

From equations (1.21), (1.22), (1.23) together with equation (1.30), the following linear constitutive equations are obtained

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}\theta, \tag{1.35}$$

$$\rho S = \frac{\rho c}{\theta_0}\theta + \beta_{ij}\epsilon_{ij}, \tag{1.36}$$

$$q_i = -k_{ij}\theta_{,j}. \tag{1.37}$$

With the aid of equation (1.30), inequality (1.25) yields

$$k_{ij}\theta_{,i}\theta_{,j} \geq 0.$$

Thus the conductivity tensor is positive-definite.

For a homogeneous body, C_{ijkl}, β_{ij} and k_{ij} are constants.

Elimination S and q_i from equations (1.29), (1.36) and (1.37), σ_{ij} from equations (1.28) and (1.35) and using equations (1.27), (1.33) and (1.34), we obtain

$$k_{ij}\theta_{,ij} + \rho R = \rho c\dot{\theta} + \theta_0\beta_{ij}\dot{u}_{i,j}, \tag{1.38}$$

$$C_{ijkl}\ u_{k,lj} - \beta_{ij}\theta_{,j} + \rho F_i = \rho\ddot{u}_i. \tag{1.39}$$

Evidently, equation (1.38) is the heat transport equation and equation (1.39) is the (vector) equation of motion. These equations, which are coupled together, form a complete system of fields equations in the context of the linear conventional coupled thermoelasticity theory for homogeneous anisotropic solids.

1.7 Isotropic Case

For the isotropic case, we have

$$\begin{aligned} C_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mu, \\ k_{ij} &= k\delta_{ij}, \\ \beta_{ij} &= \beta\delta_{ij}, \end{aligned}$$

and the constitutive equations (1.35)-(1.37) reduce to the following equations

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} - \beta\delta_{ij}\theta, \tag{1.40}$$

$$\rho S = \frac{\rho c}{\theta_0}\theta + \beta\epsilon_{kk}, \tag{1.41}$$

$$q_i = -k\theta_{,i}, \tag{1.42}$$

where λ and μ are the isothermal Lamé constants, $\beta = (3\lambda + 2\mu)\alpha$ and α is the coefficient of linear thermal expansion of the material. The field equations (1.38) and (1.39) now take the form

$$k\nabla^2\theta + \rho R = \rho c\dot{\theta} + \theta_0\beta\dot{u}_{k,k}, \tag{1.43}$$

$$\mu\nabla^2 u_i + (\lambda + \mu)u_{k,k} - \beta\theta_{,i} + \rho F_i = \rho\ddot{u}_i. \tag{1.44}$$

We see that the equation (1.43) is of parabolic-type, and the equation (1.44) is of hyperbolic-type. Hence we say that the classical thermoelasticity predicts an infinite speed for heat propagation, that is, if an isotropic homogenous elastic continuum is subjected to a mechanical or thermal disturbance, the effect of the disturbance will be felt instantaneously at a distance infinitely far from its source. Moreover, this effect will be felt in both temperature and displacement fields, since the governing equations are coupled.

This theory is called the theory of coupled thermoelasticity.

Chapter 2

Generalized Theory of Thermoelasticity with One Relaxation Time

As mentioned before, the coupled theory of thermoelasticity predicts an infinite speed for heat propagation, which contradicts the physical experiments. This shortcoming of the theory comes from the fact that the equation governing the temperature distribution (heat transport equation), on which the theory is based, is a parabolic-type partial differential equation, which arises from the classical law of heat conduction.

In the derivation of the coupled theory of thermoelasticity, the heat conduction law is taken to be linear, having the general form

$$q_i = bT_{,i} + B_{ij}T_{,j}. \tag{2.1}$$

For an isotropic elastic solid, this reduces to the well known Fourier's law

$$q_i = -kT_{,i}. \tag{2.2}$$

For the generalized thermoelasticity in this chapter, equation (2.1) is replaced by a more general equation of the form:

$$q_i + a\dot{q}_i + A_{ij}\dot{q}_j = bT_{,i} + B_{ij}T_{,j}, \tag{2.3}$$

where a, A_{ij}, b, B_{ij} are material properties of the medium.

2.1 Isotropic Case

For an isotropic elastic solid, equation (2.3) reduces to

$$q_i + \tau \dot{q}_i = -kT_{,i}, \tag{2.4}$$

where τ, the relaxation time, represents the time-lag needed to establish steady-state heat conduction in an element of volume, when a temperature gradient is suddenly imposed on that element.

We introduce the Helmoholtz's free energy function in the form

$$F(e_{ij}, T) = U(e_{ij}, T) - TS(e_{ij}, T), \tag{2.5}$$

and the first law of thermodynamics

$$s_{ij}\dot{e}_{ij} + \rho T\dot{S} = \rho \dot{U}, \tag{2.6}$$

where

$$\rho \dot{S} = -q_{i,i} + \rho R. \tag{2.7}$$

It follows from equations (2.5) and (2.6) and the relation

$$\dot{F} = (\frac{\partial F}{\partial e_{ij}})\dot{e}_{ij} + (\frac{\partial F}{\partial T})\dot{T},$$

that s_{ij} and S can be expressed in terms of F as

$$s_{ij} = \rho \frac{\partial F}{\partial e_{ij}}, \tag{2.8}$$

$$S = -\frac{\partial F}{\partial T}. \tag{2.9}$$

Substituting from equation (2.9) into equation (2.7), we find that

$$q_{i,i} = \rho(\frac{\partial^2 F}{\partial T^2}\dot{T} + \frac{\partial^2 F}{\partial e_{ij}\partial T}\dot{e}_{ij}) + \rho R. \tag{2.10}$$

Combining equations (2.4) and (2.10), the energy equation in terms of F is obtained in the following form:

$$\begin{aligned} kT_{,ii} &= -\rho T[\frac{\partial^2 F}{\partial T^2}(\dot{T}+\tau\ddot{T}) + \frac{\partial^2 F}{\partial e_{ij}\partial T}(\dot{e}_{ij}+\ddot{e}_{ij})] \\ & -\rho\tau[\dot{T}^2(\frac{\partial^2 F}{\partial T^2} + T\frac{\partial^3 F}{\partial T^3}) + \dot{e}_{ij}\dot{T}(\frac{\partial^2 F}{\partial e_{ij}\partial T} + 2T\frac{\partial^3 F}{\partial e_{ij}\partial T^2}) \\ & +\dot{e}_{ij}^2 T\frac{\partial^3 F}{\partial e_{ij}^2\partial T}] - \rho(R+\tau\dot{R}). \end{aligned} \quad (2.11)$$

The middle bracketed set of terms on the right-hand side can be neglected within the framework of the usual assumptions of the linear theory, and thus the energy equation becomes

$$kT_{,ii} = -\rho T[\frac{\partial^2 F}{\partial T^2}(\dot{T}+\tau\ddot{T}) + \frac{\partial^2 F}{\partial e_{ij}\partial T}(\dot{e}_{ij}+\ddot{e}_{ij})] - \rho(R+\tau\dot{R}). \quad (2.12)$$

Now as usual in the isotropic case, the scalar function F can be expanded in the power series of the three strain invariants I_e, II_e, III_e and the temperature difference $\theta = T - \theta_0$, that is

$$F(e_{ij}, T) = F(I_e, II_e, III_e, \theta),$$

where

$$\begin{aligned} I_e &= e_{ij} = \frac{1}{1!}e_{ij}\delta_{ij}, \\ II_e &= \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} + \begin{vmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{vmatrix} + \begin{vmatrix} e_{11} & e_{13} \\ e_{31} & e_{33} \end{vmatrix} = \frac{1}{2!}\delta^{ij}_{lm}e_{il}e_{jm}, \quad (2.13) \\ III_e &= \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix} = \frac{1}{3!}\delta^{ijk}_{lmn}e_{il}e_{jm}e_{kn}, \end{aligned}$$

and

$$\delta_{ij} = \begin{cases} +1, & \text{when } i,\ j \text{ are the same integer,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\delta^{ij}_{lm} = \begin{cases} +1(-1), & \text{when } l,\ m \text{ are distinct integers from 1 to 3 and} \\ & i,\ j \text{ are an even(odd) permutation of } l,\ m, \\ 0, & \text{otherwise,} \end{cases}$$

$$i,\ j,\ l,\ m = 1,\ 2,\ 3,$$

with a similar definition for δ^{ijk}_{lmn}.

From equations (2.13), it is clear that

$$\begin{aligned} \frac{\partial I_e}{\partial e_{ij}} &= \delta_{ij}, \\ \frac{\partial II_e}{\partial e_{ij}} &= \delta_{ij} I_e - e_{ij}, \\ \frac{\partial III_e}{\partial e_{ij}} &= e_{ij} e_{jk} - e_{ij} I_e + \delta_{ij} II_e. \end{aligned} \tag{2.14}$$

We take F in the following form:

$$\begin{aligned} F(I_e, II_e, III_e, \theta) &= \rho(a_0 + a_1 I_e + a_2 II_e + a_3 III_e + a_4 \theta + a_5 I_e^2 + a_6 II_e^2 \\ &\quad + a_7 III_e^2 + a_8 \theta^2 + a_9 \theta I_e + a_{10} \theta II_e + \cdots), \end{aligned} \tag{2.15}$$

where $a_0,\ a_1,\ \cdots$ are constants. To linearize this theory, we keep terms of second order or less only in equation (2.15). From equations (2.15), (2.14) and (2.8), we arrive at

$$s_{ij} = a_1 \delta_{ij} + (a_2 + 2a_5)\delta_{ij} e_{kk} - a_2 e_{ij} + a_9 \theta \delta_{ij}, \tag{2.16}$$

where $a_1 = 0$, since we assume that $s_{ij} = 0$ when $e_{ij} = 0$, $\theta = 0$.

Taking

$$\begin{aligned} a_2 &= -2\mu, \\ a_5 &= \frac{1}{2}(\lambda + 2\mu), \\ a_9 &= -(3\lambda + 2\mu)\alpha, \end{aligned}$$

in equation (2.16), we get the familiar linear thermoelastic stress-strain relations

$$s_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}\theta, \tag{2.17}$$

where λ, μ are Lamé's constants and α is the coefficient of the linear thermal expansion of the material.

Defining the specific heat at constant deformation by

$$c = -T\frac{\partial^2 F}{\partial T^2}, \tag{2.18}$$

assuming c to be a constant and noticing that

$$\frac{\partial^2 F}{\partial e_{ij}\partial T} = \frac{\partial}{\partial T}\frac{\partial F}{\partial e_{ij}} = \frac{1}{\rho}\frac{\partial s_{ij}}{\partial T} = -\frac{(3\lambda + 2\mu)\alpha\delta_{ij}}{\rho}, \tag{2.19}$$

the linearized energy equation (2.12) may be written as

$$kT_{,ii} = \rho c(\dot{T} + \tau\ddot{T}) + (3\lambda + 2\mu)\alpha\theta_0(\dot{e}_{ij} + \ddot{e}_{ij}) - \rho(R + \tau\dot{R}), \tag{2.20}$$

where T has been replaced by θ_0 by assuming θ to be small.

This generalized heat conduction equation (2.20), together with the equation of motion (1.44) given by

$$\rho\ddot{u}_i = (\lambda + \mu)u_{j,ij} + \mu u_{i,jj} - (3\lambda + 2\mu)\alpha T_{,i} + \rho F_i, \tag{2.21}$$

form the governing equations of the generalized theory as thermoelasticity with one relaxation time.

It is easy to see that equations (2.20) and (2.21) are the counterparts of equations (1.43) and (1.44) in classical thermoelasticity.

Due to the hyperbolicity of the governing equations, this theory predicts a finite speed for heat propagation. And when $\tau = 0$, equation (2.20) reduces to equation (1.43), that is, this generalized theory reduces to the coupled theory of thermoelasticity.

2.2 One-dimensional Problem

For one-dimensional problem

stress	$\sigma = \sigma(x,t)$,
displacement	$u = u(x,t)$,
temperature	$\theta = \theta(x,t)$,

with $F_i = 0$, $R = 0$, equations (2.21), (2.20) and (1.44) reduce to

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} - (3\lambda+2\mu)\alpha\frac{\partial \theta}{\partial x}, \tag{2.22}$$

$$k\frac{\partial^2 \theta}{\partial x^2} = \rho c(\frac{\partial \theta}{\partial t} + \tau\frac{\partial \theta}{\partial t}) + (3\lambda+2\mu)\alpha\theta_0(\frac{\partial^2 u}{\partial x\partial t} + \tau\frac{\partial^3 u}{\partial x\partial t^2}), \tag{2.23}$$

$$\sigma = (\lambda+2\mu)\frac{\partial u}{\partial x} - (3\lambda+2\mu)\alpha\theta. \tag{2.24}$$

Using the following nondimensional variables

$$x' = (\frac{\lambda+2\mu}{\rho})^{\frac{1}{2}}\frac{\rho c}{k}x,$$

$$
\begin{aligned}
t' &= (\frac{\lambda+2\mu}{\rho})\frac{\rho c}{k},\\
\theta' &= \frac{\theta}{\theta_0},\\
\sigma' &= \frac{\sigma}{(3\lambda+2\mu)\alpha\theta_0},\\
u' &= [\rho(\frac{\lambda+2\mu}{\rho})^{\frac{3}{2}}\frac{1}{(3\lambda+2\mu)\alpha\theta_0}\frac{\rho c}{k}]u,\\
\tau' &= (\frac{\lambda+2\mu}{\rho})\frac{\rho c}{k}\tau,
\end{aligned}
$$

and dropping primes for convenience, we obtain the following

Equation of motion
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \tag{2.25}$$

Energy equation
$$\frac{\partial^2 \theta}{\partial x^2} = (1+\tau\frac{\partial}{\partial t})(\frac{\partial \theta}{\partial t} + \epsilon\frac{\partial^2 u}{\partial x \partial t}), \tag{2.26}$$

Constitutive equation
$$\sigma = \frac{\partial u}{\partial x} - \theta, \tag{2.27}$$

where

$$\epsilon = \frac{(3\lambda+2\mu)^2\alpha^2\theta_0}{(\lambda+2\mu)\rho c}.$$

Here, ϵ is the well known thermoelastic coupling constant, τ is the dimensionless relaxation time.

2.3 Anisotropic Case

In the most general homogeneous anisotropic medium, the second law of thermodynamics has the form of

$$s_{ij}\dot{e}_{ij} - q_{i,i} = \rho(\dot{U} - R), \tag{2.28}$$

where

$$\rho T\dot{S} = -q_{i,i} + \rho R. \tag{2.29}$$

Eliminating $q_{i,i}$ in the above equations, leads us to

$$\rho\dot{S} = \frac{\rho}{T}\dot{U} - \frac{1}{T}s_{ij}\dot{e}_{i,j}. \tag{2.30}$$

Noticing that

$$\dot{U} = \frac{\partial U}{\partial T}\dot{T} + \frac{\partial U}{\partial e_{ij}}\dot{e}_{ij},$$

we can write equation (2.30) as

$$\rho\dot{S} = \frac{\rho}{T}\frac{\partial U}{\partial T}\dot{T} + \frac{1}{T}(\rho\frac{\partial U}{\partial e_{ij}} - s_{ij})\dot{e}_{ij}. \tag{2.31}$$

The second law of thermoelasticity requires that $\dot{S}$ be an exact differential in T and e_{ij}, therefore

$$\begin{aligned} \rho\frac{\partial S}{\partial T} &= \frac{\rho}{T}\frac{\partial U}{\partial T}, \\ \rho\frac{\partial S}{\partial e_{ij}} &= \frac{1}{T}(\rho\frac{\partial U}{\partial e_{ij}} - s_{ij}). \end{aligned}$$

Using these relations and the identities

$$\begin{aligned} \frac{\partial^2 S}{\partial T\partial e_{ij}} &= \frac{\partial^2 S}{\partial e_{ij}\partial T}, \\ \frac{\partial^2 U}{\partial T\partial e_{ij}} &= \frac{\partial^2 U}{\partial e_{ij}\partial T}, \end{aligned}$$

together with equation (1.35), we get

$$\beta_{ij} = \frac{1}{T}(\rho\frac{\partial U}{\partial e_{ij}} - s_{ij}). \tag{2.32}$$

Substituting from equation (2.32) into equation (2.31), we get

$$\rho\dot{S} = \frac{\rho}{T}\frac{\partial U}{\partial T}\dot{T} + \beta_{ij}\dot{e}_{ij}. \tag{2.33}$$

Let

$$c = \frac{\partial U}{\partial T}$$

be the specific heat per unit mass in the absence of deformation (assumed independent of T in the neighborhood of the equilibrium state $T = \theta_0$).

Substituting c into equation (2.33) and integrating it with respect to t, we obtain

$$\rho S = \rho c \log T + \beta_{ij} e_{ij} + \text{ constant.} \tag{2.34}$$

If in equation (2.34), we choose the constant in such a way that $S = 0$, when $T = \theta_0$ and $e_{ij} = 0$, then equation (2.34), with this choice, takes the form

$$\rho S = \rho c \log(1 + \frac{\theta}{\theta_0}) + \beta_{ij} e_{ij}. \tag{2.35}$$

Approximating $\log(1 + \theta/\theta_0)$ by θ/θ_0, it further reduces to

$$\rho \theta_0 S = \rho c \theta + \theta_0 \beta_{ij} e_{ij}. \tag{2.36}$$

The linearized form of equation (2.29) is

$$q_{i,i} = -\rho \theta_0 \dot{S} + \rho R. \tag{2.37}$$

By using equation (2.36), equation (2.37) reduces to

$$q_{i,i} = -\rho c \dot{\theta} - \theta_0 \beta_{ij} \dot{e}_{ij} + \rho R. \tag{2.38}$$

We assume a generalized heat conduction equation of the form

$$q_i + \tau \dot{q}_i = -k_{ij} \theta_{,j}. \tag{2.39}$$

Now, taking divergence of both sides of equation (2.39) and using equation (2.38) and its time derivative, we arrive at

$$\rho c(\dot{\theta} + \tau \ddot{\theta}) + \theta_0 \beta_{ij}(\dot{e}_{ij} + \tau \ddot{e}_{ij}) - \rho(R + \tau \dot{R}) = \frac{\partial}{\partial x_i}(k_{ij} \theta_{,j}). \tag{2.40}$$

To get an equation satisfied by the displacements u_i, we substitute from equation (1.35) into equation (1.28), using the linearized strain-displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

and the symmetry condition

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \quad \beta_{ij} = \beta_{ji},$$

we find

$$\frac{\partial}{\partial x_j}(C_{ijkl}u_{k,l}) - \frac{\partial}{\partial x_j}(\beta_{ij}\theta) + \rho F = \rho \ddot{u}_i. \tag{2.41}$$

It is worth noting that for the case of isotropic case

$$\begin{aligned} C_{ijkl} &= \lambda\delta_{ij}\lambda_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mu, \\ k_{ij} &= k\delta_{ij}, \\ \beta_{ij} &= \beta\delta_{ij}, \end{aligned}$$

and equations (2.40) and (2.41) reduce to

$$kT_{,ii} = \rho c(\dot{T} + \tau\ddot{T}) + \beta\theta_0(\dot{e}_{ij} + \ddot{e}_{ij}) - \rho(R + \tau\dot{R}), \tag{2.42}$$

$$\rho\ddot{u}_i = (\lambda + \mu)u_{j,ij} + \mu u_{i,jj} - \beta T_{,i} + \rho F_i, \tag{2.43}$$

where

$$\beta = (3\lambda + 2\mu)\alpha.$$

It is clear that equations (2.42) and (2.43) are the same as equations (2.20) and (2.21) derived earlier for the isotropic case.

Chapter 3

Generalized Theory of Thermoelasticity with Two Relaxation Times

In the previous chapter, we considered the thermoelasticity theory with thermal relaxation formulated on the basis of the modification of the classical Fourier's law. And we observe that this theory admits second sound only because of the presence of the flux-rate term in the heat conduction equation.

In the present chapter, we consider a thermoelasticity theory with second sound which is not based on any predetermined form of the heat conduction law. This theory was developed by Green and Lindsay [21] in 1972.

3.1 Governing Equations

Like other thermodynamical theories of continua, the generalized theory of thermoelasticity with two relaxation times is also formulated on the basis of equations (1.7), (1.10), (1.14) and (1.16). But the entropy production inequality (1.18) for the homogeneous materials is now replaced by the following more general inequality:

$$\rho(T^*\dot{S} - R) + q_{i,i} - \frac{q_i}{T^*}T^*_{,i} \geq 0. \tag{3.1}$$

Here T^* is a constitutive function postulated to be positive.

It may readily be seen that inequality (1.18), on which the classical thermoelasticity is based, is a special case of inequality (3.1), for which $T^* = T$. The functions

T^* and its reciprocal $(T^*)^{-1}$ are known as thermodynamic temperature and coldness functions, respectively.

In order to obtain the constitutive equations, we introduce an energy function F^* through the equation

$$F^* = U - T^*S, \tag{3.2}$$

and in general

$$F^* = F^*(T, \dot{T}, T_{,i}, e_{ij}),$$
$$T^* = T^*(T, \dot{T}, T_{,i}, e_{ij}).$$

Note that unlike in the classical thermoelasticity, $\dot{T}$ is now included among the constitutive variables. If we set $T^* = T$, then F^* reduces to the Helmholtz's free energy F defined by equation (1.20).

Noticing that

$$\begin{aligned}\dot{F}^* &= \frac{\partial F^*}{\partial T}\dot{T} + \frac{\partial F^*}{\partial \dot{T}}\ddot{T} + \frac{\partial F^*}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial F^*}{\partial e_{ij}}\dot{e}_{ij},\\ \dot{T}^* &= \frac{\partial T^*}{\partial T}\dot{T} + \frac{\partial T^*}{\partial \dot{T}}\ddot{T} + \frac{\partial T^*}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial T^*}{\partial e_{ij}}\dot{e}_{ij},\\ \dot{U} &= \dot{F}^* + \dot{T}^*S + T^*\dot{S}\end{aligned}$$

and substituting these results into equation (1.16) and inequality (1.19), one finds

$$\begin{aligned}&\rho(\frac{\partial F^*}{\partial T} + S\frac{\partial T^*}{\partial T})\dot{T} + \rho(\frac{\partial F^*}{\partial \dot{T}} + S\frac{\partial T^*}{\partial \dot{T}})\ddot{T} + \rho(\frac{\partial F^*}{\partial T_{,i}} + S\frac{\partial T^*}{\partial T_{,i}})\dot{T}_{,i}\\ &\quad+\{\rho(\frac{\partial F^*}{\partial e_{ij}} + S\frac{\partial T^*}{\partial e_{ij}}) - s_{ij}\}\dot{e}_{ij} + \rho(T^*\dot{S} - R) + q_{i,i} = 0,\end{aligned}$$

$$\rho(\frac{\partial F^*}{\partial T} + S\frac{\partial T^*}{\partial T})\dot{T} + \rho(\frac{\partial F^*}{\partial \dot{T}} + S\frac{\partial T^*}{\partial \dot{T}})\ddot{T} + \rho(\frac{\partial F^*}{\partial T_{,i}} + S\frac{\partial T^*}{\partial T_{,i}})\dot{T}_{,i}$$

$$+\{\rho(\frac{\partial F^*}{\partial e_{ij}}\dot{e}_{ij} + S\frac{\partial T^*}{\partial e_{ij}}) - s_{ij}\}\dot{e}_{ij}$$
$$+\frac{q_i}{T^*}(\frac{\partial T^*}{\partial T}T_{,i} + \frac{\partial T^*}{\partial T_{,j}}T_{,ji} + \frac{\partial T^*}{\partial \dot{T}}\dot{T}_{,i} + \frac{\partial T^*}{\partial e_{kl}}\dot{e}_{kl,i}) \leq 0.$$

The above inequality should be valid for all temperature and displacement fields, and for all reference bodies, it being assumed that the energy and momentum equations balance by suitable choice of specific heat supply Q and externally applied body force F_i. Hence, we conclude ([35]) that

$$\frac{\partial F^*}{\partial \dot{T}} + S\frac{\partial T^*}{\partial \dot{T}} = 0,$$
$$\rho(\frac{\partial F^*}{\partial e_{ij}} + S\frac{\partial T^*}{\partial e_{ij}}) - s_{ij} = 0,$$
$$\rho(\frac{\partial F^*}{\partial T} + S\frac{\partial T^*}{\partial T})\dot{T} + \rho(\frac{\partial F^*}{\partial T_{,i}} + S\frac{\partial T^*}{\partial T_{,i}})\dot{T}_{,i} + \rho(T^*\dot{S} - R) + q_{i,i} = 0,$$
$$\frac{\partial T^*}{\partial e_{ij}} = 0,$$
$$\frac{\partial T^*}{\partial T_{,i}} = 0,$$
$$\rho(\frac{\partial F^*}{\partial T_{,i}} + S\frac{\partial T^*}{\partial T_{,i}}) + \frac{q_i}{T^*}\frac{\partial T^*}{\partial \dot{T}} = 0,$$
$$\rho(\frac{\partial F^*}{\partial T} + S\frac{\partial T^*}{\partial T})\dot{T} + \frac{q_i}{T^*}\frac{\partial T^*}{\partial T}T_{,i} \leq 0,$$

which lead us to

$$s_{ij} = \rho\frac{\partial F^*}{\partial e_{ij}}, \quad (3.3)$$
$$\frac{\partial F^*}{\partial \dot{T}} + S\frac{\partial T^*}{\partial \dot{T}} = 0, \quad (3.4)$$
$$\frac{q_i}{\rho T^*}\frac{\partial T^*}{\partial \dot{T}} + \frac{\partial F^*}{\partial T_{,i}} = 0, \quad (3.5)$$
$$(\frac{\partial F^*}{\partial T} + S\frac{\partial T^*}{\partial T})\dot{T} + \frac{\partial F^*}{\partial T_{,i}}\dot{T}_{,i} + \dot{S}T^* + \frac{1}{\rho}q_{i,i} = R, \quad (3.6)$$
$$(\frac{\partial F^*}{\partial T} + S\frac{\partial T^*}{\partial T})\dot{T} + \frac{q_i}{\rho T^*}\frac{\partial T^*}{\partial T}T_{,i} \leq 0. \quad (3.7)$$

It is seen that

$$T^* = T^*(T, \dot{T}).$$

The above expression and equations (3.3)-(3.5) are the constitutive equations for s_{ij}, S and q_i, respectively, and equation (3.6) is the energy equation. These equations along with the fundamental equations (1.7), (1.10), (1.14), (1.16) and inequality (3.7) constitute the governing equations of nonlinear thermoelasticity with two relaxation times.

If we drop $\dot{T}$ from the list of constitutive variables, equations (3.3)-(3.6) and inequality (3.7) reduce to the corresponding equations (1.21)-(1.24) and inequality (1.25) of classical thermoelasticity.

If $\partial T^*/\partial \dot{T} \neq 0$ and F^* depends on $T_{,i}$, equation (3.5) gives

$$\frac{\partial q_i}{\partial T_{,j}} = \frac{\partial q_j}{\partial T_{,i}}.$$

When q_i is a linear function of $T_{,i}$, this relation gives

$$k_{ij} = k_{ji},$$

which means that the conductivity tensor k_{ij} is symmetric.

3.2 Linear Approximation

Here we consider the usual kind of linear theory in which the changes of temperature, displacement components, and their space and time derivatives are small. Then s_{ij} and e_{ij} reduce to Cauchy's stress and strain tensors σ_{ij} and ϵ_{ij}, respectively. And we assume that

$$T^*(T, 0) = T = T_0 + \theta.$$

In the general anisotropic case, for small strain and small temperature increment theory, we assume the following expansions (with zero initial stress and zero initial heat flux):

$$
\begin{aligned}
V^* \equiv \rho F^* &= V_0 - a\theta - b\dot{\theta} - \frac{1}{2}d\theta^2 - e\theta\dot{\theta} - \frac{1}{2}f\dot{\theta}^2 + a_i\theta\theta_{,i} \\
&\quad + \alpha\frac{c_i}{\theta_0}\dot{\theta}\theta_{,i} + a_{ijk}\epsilon_{ij}\theta_{,k} - \beta_{ik}\epsilon_{ik}\theta + b_{ik}\epsilon_{ik}\dot{\theta} \\
&\quad + \frac{1}{2}\alpha\frac{k_{rs}}{\theta_0}\theta_{,r}\theta_{,k} + \frac{1}{2}C_{ikrs}\epsilon_{ik}\epsilon_{rs}, \\
T^* &= T_0 + \theta + \theta_0 + \alpha\theta + \beta\theta\dot{\theta} + \frac{1}{2}\gamma\dot{\theta}^2.
\end{aligned}
$$

All the coefficients in the above expressions are constants for a homogeneous body.

Substituting the above expressions into equations (3.3)-(3.5) and inequality (3.7), with $T^* \approx \theta_0$, we obtain

$$\sigma_{ik} = C_{ikrs}\epsilon_{rs} - \beta_{ik}\theta + b_{ik}\dot{\theta} + a_{ikr}\theta_{,r}, \tag{3.8}$$

$$q_i = -\frac{\theta_0}{\alpha}(a_i\theta + \alpha\frac{c_i}{\theta_0}\dot{\theta} + a_{rsi}\epsilon_{rs} + \alpha\frac{k_{ij}}{\theta_0}\theta_{,j}), \tag{3.9}$$

$$\rho S = \frac{1}{\alpha}\{b + (e - \frac{b\beta}{\alpha})\theta + (f - \frac{b\gamma}{\alpha})\dot{\theta} - \frac{\alpha}{\theta_0}c_i\theta_{,i} - b_{ik}\epsilon_{ik}\}, \tag{3.10}$$

$$
\begin{aligned}
&(\frac{b}{\alpha} - a)\dot{\theta} + \{\frac{1}{\alpha}(e - \frac{b\beta}{\alpha}) - d\}\theta\dot{\theta} + a_i\theta_{,i}\dot{\theta} - \frac{1}{\alpha}a_i\theta\theta_{,i} \\
&+(-\beta_{ij} - \frac{b_{ij}}{\alpha})\epsilon_{ij}\dot{\theta} - \frac{a_{rsi}}{\alpha}\epsilon_{rs}\theta_{,i} + \{\frac{1}{\alpha}(f - \frac{b\gamma}{\alpha}) - e\}\dot{\theta}^2 \\
&\qquad -2\frac{c_i}{\theta_0}\dot{\theta}\theta_{,i} - \frac{k_{ij}}{\theta_0}\theta_{,i}\theta_{,j} \leq 0.
\end{aligned} \tag{3.11}
$$

The above inequality then yields the following restriction on the coefficients

$$b = a\alpha,$$

$$
\begin{aligned}
d\alpha &= e - \frac{b\beta}{\alpha} = e - a\beta, \\
a_i &= 0, \\
b_{ij} &= -\alpha\beta_{ij}, \\
a_{ijk} &= 0, \\
\theta_0(d\alpha - h)\dot{\theta}^2 &+ 2c_i\dot{\theta}\theta_{,i} + k_{ij}\theta_{,i}\theta_{,j} \geq 0,
\end{aligned}
$$

where

$$h\alpha = f - \frac{b\gamma}{\alpha}.$$

Let

$$c = -T^* \frac{\partial^2 F^*}{\partial T^2},$$

then with $T^* \approx \theta_0$ and from the expansion for V^*, we find that

$$c = \frac{dT^*}{\rho} \approx \frac{d\theta_0}{\rho},$$

or

$$d \approx \frac{\rho c}{\theta_0}.$$

Let us define α_0 by

$$\alpha_0 = \frac{h\theta_0}{\rho c}.$$

Now, we can reduce equations (3.8)-(3.10) and inequality (3.11) to the following

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}(\theta + \alpha\dot{\theta}), \tag{3.12}$$

$$q_i = -(c_i\dot{\theta} + k_{ij}\theta_{,j}), \tag{3.13}$$

$$\rho S = \frac{\rho c}{\theta_0}(\theta + \alpha_0\dot{\theta}) - \frac{c_i}{\theta_0}\theta_{,i} + \beta_{ij}\epsilon_{ij}, \tag{3.14}$$

$$\rho c(\alpha - \alpha_0)\dot{\theta}^2 + 2c_i\dot{\theta}\theta_{,i} + k_{ij}\theta_{,i}\theta_{,j} \geq 0, \tag{3.15}$$

where $b = 0$, since we assume that $\rho S = 0$, when $\theta = \dot{\theta} = \theta_{,i} = e_{ij} = 0$.

For a homogeneous body, we obtain the governing equations

$$k_{ij}\theta_{,ij} + \rho R = \rho c(\dot{\theta} + \alpha_0\ddot{\theta}) - 2c_i\dot{\theta}_{,i} + \theta_0\beta_{ij}\dot{u}_{i,j}, \tag{3.16}$$

$$C_{ijkl}u_{k,lj} - \beta_{ij}(\theta + \alpha\dot{\theta})_{,j} + F_i = \rho\ddot{u}_i. \tag{3.17}$$

It may be seen that equations (3.16) and (3.17) are the counterparts of equations (2.40) and (2.41) of the previous chapter.

For inequality (3.15) to hold for all arbitrary $\dot{\theta}$ and $\theta_{,i}$, it is necessary that

$$\alpha \geq \alpha_0,$$

and

$$2c_i\dot{\theta}\theta_{,i} + k_{ij}\theta_{,i}\theta_{,j} \geq 0.$$

If a body has a center of symmetry at each point, then since the sign of $2c_i\dot{\theta}\theta_{,i}$ can be changed, this implies that

$$c_i = 0,$$

and hence

$$k_{ij}\theta_{,i}\theta_{,j} \geq 0,$$

which means that the conductivity tensor k_{ij} is positive-definite.

For this case ($c_i = 0$), equations (3.12)-(3.14) and inequality (3.15) reduce to

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}(\theta + \alpha\dot{\theta}), \tag{3.18}$$

$$q_i = -k_{ij}\theta_{,j}, \tag{3.19}$$

$$\rho S = \frac{\rho c}{\theta_0}(\theta + \alpha_0\dot{\theta}) + \beta_{ij}\epsilon_{ij}, \tag{3.20}$$

$$\rho c(\alpha - \alpha_0)\dot{\theta}^2 + k_{ij}\theta\theta_{,i}\theta_{,j} \geq 0. \tag{3.21}$$

3.3 Isotropic Case

For an isotropic medium, we have

$$\begin{aligned} C_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mu, \\ k_{ij} &= k\delta_{ij}, \\ \beta_{ij} &= \beta\delta_{ij}. \end{aligned}$$

The fundamental equations (3.18)-(3.20) in this case take the form

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}e_{kk} - \beta\delta_{ij}(\theta + \alpha\dot{\theta}), \tag{3.22}$$

$$q_i = -k\theta_{,i}, \tag{3.23}$$

$$\rho S = \frac{\rho c}{\theta_0}(\theta + \alpha_0\dot{\theta}) + \beta\epsilon_{kk}. \tag{3.24}$$

And the governing equations (3.16) and (3.17), with $c_i = 0$, become

$$k\theta_{,ii} + \rho R = \rho c(\dot{\theta} + \alpha_0\ddot{\theta}) + \theta_0\beta\epsilon_{kk}, \tag{3.25}$$

$$\rho\ddot{u}_i = (\lambda + \mu)u_{j,ij} + \mu u_{i,j} - \beta\theta_{,i} + \rho F_i. \tag{3.26}$$

It may be seen that in equation (3.23), Fourier's law of heat conduction is not violated, but equation (3.25) is still of hyperbolic type, with $\alpha_0 > 0$.

It may be seen that equations (3.25) and (3.26) are the counterparts of equations (2.20) and (2.21) of the previous chapter.

Part II

Re-examination of the Basic Postulates of Thermomechanics

Chapter 4

Re-examination of the Basic Postulates of Thermomechanics

As mentioned earlier, Green and Naghdi ([30], [31]) re-examined the basic postulates of thermomechanics. Their analysis contains a useful analogy between the concepts and equations of the purely thermal and purely mechanical theories and three types of constitutive equations, labeled as type I, II and III.

Consider a finite elastic body B with material points X and identify the material point (particle) X with its position $\mathbf{X}$ in a fixed reference configuration $\boldsymbol{k}_0$. In the present configuration $\boldsymbol{k}$ at time t, the body occupies a region of space R bounded by a closed surface ∂R. Similarly, in the present configuration, an arbitrary material volume of B occupies a part of the region of space, which we denote by $P(\subseteq R)$, bounded by a closed surface ∂P. The place occupied by the material point X in the current configuration $\boldsymbol{k}$ is $\mathbf{x}$.

For purely mechanical theories, we use the following notation:

(a) displacement: $\mathbf{x} = \chi(\mathbf{X}, t)$, χ a sufficiently smooth vector function

(b) particle velocity $\mathbf{v}$ at $\mathbf{x}$: $\mathbf{v} = \dot{\mathbf{x}}$

(c) deformation gradient tensor: $\mathbf{F}=\partial\chi/\partial\mathbf{X}$

(d) velocity gradient tensor: $\mathbf{L}=\partial\mathbf{v}/\partial\mathbf{x}$

(e) externally applied body forces per unit mass: $\mathbf{b}=\mathbf{b}(\mathbf{x}, t)$

(f) external rate of work per unit mass: $\mathbf{bv}$

(g) internal surface force per unit area over ∂P: $\mathbf{t}$

(h) rate of internal surface work per unit area: $\mathbf{tv}$

(i) internal body force: f

For purely thermal theory, we need the following notation:

(a_1) thermal displacement: $\alpha = \alpha(\mathbf{X}, t)$

(b_1) empirical temperature: $T = \dot{\alpha}$

(c_1) temperature: θ which depends on T and the properties of the material such that

$$\theta > 0, \; \frac{\partial \theta}{\partial T} > 0$$

(d_1) thermal displacement gradient: $\boldsymbol{\beta} = \partial \alpha / \partial \mathbf{X}$

(e_1) temperature gradient: $\gamma = \partial T / \partial \mathbf{X}$ which relates to $\boldsymbol{\beta}$ by $\dot{\boldsymbol{\beta}} = \mathbf{F}^T \gamma$

(f_1) external rate of supply of entropy per unit mass: s

(g_1) external rate of supply of heat per unit mass: $r = \theta s$

(h_1) internal rate of production of entropy per unit mass: ξ

(i_1) internal rate of production of heat per unit mass: $\theta \xi$

(j_1) entropy density per unit mass: η

(k_1) heat density per unit mass: $\theta \eta$

(l_1) internal flux of entropy per unit mass: $-k$

(m_1) internal flux of heat per unit mass: $-h = -\theta k$

where a superposed dot denotes material time derivative, keeping $\mathbf{X}$ fixed.

In Table 4.1, we give the correspondence between the mechanical and thermal variables.

Table 4.1: Correspondence between mechanical and thermal variables

mechanical variable	thermal variables
$\mathbf{x}$	α
$\mathbf{v}$	T
$\mathbf{F}$	β
$\mathbf{L}$	γ
$\mathbf{b}$	s
$\mathbf{b} \cdot \mathbf{v}$	$r = \theta s$
$\mathbf{t}$	k
$\mathbf{t} \cdot \mathbf{v}$	$h = \theta k$
f	ξ
$\dot{\mathbf{v}}$	$\dot{\eta}$

The local field equation for the balance of entropy is

$$\rho\dot{\eta} = \rho(s + \xi) - \text{div } \mathbf{p}, \tag{4.1}$$

where

$$k = \mathbf{p} \cdot \mathbf{n}, \ \mathbf{q} = \theta\mathbf{p}$$

and $\mathbf{p}$ is the entropy flux vector and $\mathbf{q}$ is the heat flux vector. We also record here the *reduced* energy equation

$$\mathbf{T} \cdot \mathbf{L} - \mathbf{p} \cdot \mathbf{g} - \rho(\dot{\psi} + \eta\dot{\theta}) - \rho\theta\xi = 0, \tag{4.2}$$

which has been obtained from the local field equation for the energy balance after elimination of the external body force and the external supply of entropy. In equation (4.2), ψ is the specific Helmholtz's free energy and $\mathbf{g}$ is the temperature gradient defined by

$$\mathbf{g} = \text{grad } \theta,$$

where the grad operator stands for $\partial(\ )/\partial\mathbf{x}$.

4.1 Classical Thermoelasticity (Type I)

For the theory of thermoelasticity of type I, we require the constitutive equations for

$$\mathbf{T},\ \mathbf{p},\ \psi,\ \eta,\ \theta,\ \xi \tag{4.3}$$

and assume that these are functions of the variables:

$$(T, \gamma, \mathbf{F}; \mathbf{X}). \tag{4.4}$$

However, for simplicity, in what follows we suppress explicit dependence on $\mathbf{X}$ and regard the material to be homogeneous. Introduction of constitutive assumptions of the forms

$$\psi = \hat{\psi}(T, \gamma, \mathbf{F}),\ \theta = \hat{\theta}(T, \gamma, \mathbf{F}),\ \mathbf{T} = \hat{\mathbf{T}}(T, \gamma, \mathbf{F}),\ \text{etc.} \tag{4.5}$$

into the reduced energy equation (4.2), after some rearrangements of terms, yields

$$\begin{aligned}\hat{\mathbf{p}} \cdot \frac{\partial \hat{\theta}}{\partial T}\gamma + \rho\hat{\theta}\hat{\xi} + \rho(\frac{\partial \hat{\psi}}{\partial T} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial T})\dot{T} + \rho(\frac{\partial \hat{\psi}}{\partial \gamma} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \gamma}) \cdot \dot{\gamma} \\ +[-\hat{\mathbf{T}}(\mathbf{F}^T)^{-1} + \rho(\frac{\partial \hat{\psi}}{\partial \mathbf{F}} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \mathbf{F}})] \cdot \dot{\mathbf{F}} \\ +(\frac{\partial \hat{\theta}}{\partial \gamma} \otimes \hat{\mathbf{p}}) \cdot \frac{\partial \gamma}{\partial \mathbf{x}} + (\frac{\partial \hat{\theta}}{\partial \mathbf{F}} \otimes \hat{\mathbf{p}}) \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = 0,\end{aligned} \tag{4.6}$$

where the symbol $\otimes$ denotes tensor product and for clarity, we have temporarily used the symbols such as $\hat{\theta}$ and $\hat{\eta}$ in order to distinguish between the response functions and their values.

Equation (4.6) has the form (A.1) of Appendix A with $N = 5$, namely

$$a + a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4 + a_5 y_5 = 0$$

with

$$\begin{aligned}
a &= \hat{\mathbf{p}} \cdot \frac{\partial \hat{\theta}}{\partial T}\gamma + \rho \hat{\theta} \hat{\xi}, \\
a_1 &= \rho \left(\frac{\partial \hat{\psi}}{\partial T} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial T} \right), \\
a_2 &= \rho \left(\frac{\partial \hat{\psi}}{\partial \gamma} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial \gamma} \right), \\
a_3 &= -\hat{\mathbf{T}}(\mathbf{F}^T)^{-1} + \rho \left(\frac{\partial \hat{\psi}}{\partial \mathbf{F}} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial \mathbf{F}} \right), \\
a_4 &= \frac{1}{2} \left(\frac{\partial \hat{\theta}}{\partial \gamma} \otimes \hat{\mathbf{p}} + \hat{\mathbf{p}} \otimes \frac{\partial \hat{\theta}}{\partial \gamma} \right), \\
a_5 &= \left(\frac{\partial \hat{\theta}}{\partial \mathbf{F}} \otimes \hat{\mathbf{p}} \right),
\end{aligned} \tag{4.7}$$

and

$$y_1 = \dot{T},\ y_2 = \dot{\gamma},\ y_3 = \dot{\mathbf{F}},\ y_4 = \frac{\partial \gamma}{\partial \mathbf{x}},\ y_5 = \frac{\partial \mathbf{F}}{\partial \mathbf{x}},$$

and hence equation (4.6) holds only if

$$a = a_1 = a_2 = a_3 = a_4 = a_5 = 0. \tag{4.8}$$

From equations (4.7) and (4.8), using $a_4 = a_5 = a_2 = 0$, we obtain

$$\frac{\partial \hat{\theta}}{\partial \gamma} = \mathbf{0},\ \frac{\partial \hat{\theta}}{\partial \mathbf{F}} = \mathbf{0},\ \frac{\partial \hat{\psi}}{\partial \gamma} = \mathbf{0}. \tag{4.9}$$

From equations (4.5) and (4.9), we conclude that

$$\theta = \theta(T),\ \psi = \psi(T, \mathbf{F}).$$

With the choice of

$$T = \theta + \theta_0\ (\theta_0 > 0),\ \frac{\partial \theta}{\partial T} = 1 \tag{4.10}$$

and without loss of generality, we may replace T by θ in all constitutive developments of this section. We may use the same symbols for a function and its value. From equations (4.7)-(4.10), we obtain

$$\eta = -\frac{\partial \psi}{\partial \theta}, \ \mathbf{T} = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \ \mathbf{p} \cdot \boldsymbol{\gamma} + \rho \theta \xi = 0, \tag{4.11}$$

with $\mathbf{p} = \mathbf{p}(\theta, \mathbf{F}, \mathbf{g})$.

In the rest of this section, we consider the linear theory of thermoelasticity type I and linearize the foregoing constitutive results. Thus we assume that the temperature θ represents departure from an equilibrium temperature θ_0 and $\mathbf{u} = \chi - \mathbf{X}$ is the displacement vector from an equilibrium state with zero stress such that both θ and $\mathbf{u}$ are small of $0(\varepsilon)$ and further we restrict our attention to an isotropic material. For such a linearized theory, it will suffice to assume that the specific Helmholtz's free energy is a quadratic function of infinitesimal temperature and infinitesimal strain

$$\mathbf{E} = \frac{1}{2}[\text{Grad } \mathbf{u} + (\text{Grad } \mathbf{u})^T], \tag{4.12}$$

so that the reduced energy equation becomes

$$\rho_0 \psi = \frac{1}{2}\lambda(\text{tr}\mathbf{E})^2 + \mu \text{tr}\mathbf{E}^2 - \frac{1}{2}\frac{c\theta^2}{\theta_0} - \frac{E\beta^*}{3(1-2\nu)}\theta \text{tr}\mathbf{E}, \tag{4.13}$$

where β^* is the coefficient of volume expansion.

After simple substitution and rearrangement, we obtain the following coupled system of partial differential equations:

$$(\lambda + \mu)\text{grad div } \mathbf{u} + \mu \nabla^2 \mathbf{u} - \frac{E\beta^*}{3(1-2\nu)}\text{grad } \theta + \rho \mathbf{b} = \rho \ddot{\mathbf{u}}, \tag{4.14}$$

$$c\dot{\theta} + \frac{E\beta^*\theta_0}{3(1-2\nu)}\text{div } \dot{\mathbf{u}} = \rho r + k\nabla^2 \theta, \tag{4.15}$$

where the notation "grad" and "div" stand respectively for the gradient operator and divergence operator with respect to **X**.

The system of equations (4.14) and (4.15) predicts propagation of waves with damping, which is due to the thermal part of the equations.

4.2 Thermoelasticity (Type II)

Using the similar assumptions as in section 4.1, we regard

$$T,\ \boldsymbol{\beta},\ \mathbf{F} \tag{4.16}$$

as the independent variables.

Introduction of constitutive assumptions of the forms

$$\psi = \hat{\psi}(T, \boldsymbol{\beta}, \mathbf{F}),\ \theta = \hat{\theta}(T, \boldsymbol{\beta}, \mathbf{F}),\ \mathbf{T} = \hat{\mathbf{T}}(T, \boldsymbol{\beta}, \mathbf{F}) \tag{4.17}$$

and similar assumptions for $\mathbf{p}$, η, and ξ into the reduced energy equation (4.2), after some rearrangement, results in

$$\rho\hat{\theta}\hat{\xi} + \rho\left(\frac{\partial\hat{\psi}}{\partial T} + \hat{\eta}\frac{\partial\hat{\theta}}{\partial T}\right)\dot{T} + \left[\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}} + \hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\right)\cdot\mathbf{F}^T\gamma + \hat{\mathbf{p}}\cdot\gamma\frac{\partial\hat{\theta}}{\partial T}\right]$$
$$+[-\hat{\mathbf{T}} + \rho\left(\frac{\partial\hat{\psi}}{\partial\mathbf{F}} + \hat{\eta}\frac{\partial\hat{\theta}}{\partial\mathbf{F}}\right)\mathbf{F}^T]\cdot\mathbf{L} + \left(\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\otimes\hat{\mathbf{p}}\right)\cdot\frac{\partial\boldsymbol{\beta}}{\partial\mathbf{x}} + \left(\frac{\partial\hat{\theta}}{\partial\mathbf{F}}\otimes\hat{\mathbf{p}}\right)\cdot\frac{\partial\mathbf{F}}{\partial\mathbf{x}} = 0. \tag{4.18}$$

Equation (4.18) has the form (A.1) of Appendix A with $N = 5$, but now with

$$\begin{aligned}
a &= \rho\hat{\theta}\hat{\xi}, \\
a_1 &= \rho\left(\frac{\partial\hat{\psi}}{\partial T} + \hat{\eta}\frac{\partial\hat{\theta}}{\partial T}\right), \\
a_2 &= \rho\mathbf{F}\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}} + \hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\right) + \hat{\mathbf{p}}\frac{\partial\hat{\theta}}{\partial T},
\end{aligned}$$

$$a_3 = \left[-\hat{\mathbf{T}} + \rho\left(\frac{\partial\hat{\psi}}{\partial\mathbf{F}} + \hat{\eta}\frac{\partial\hat{\theta}}{\partial\mathbf{F}}\right)\mathbf{F}^T\right],$$
$$a_4 = \frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}} \otimes \hat{\mathbf{p}},$$
$$a_5 = \frac{\partial\hat{\theta}}{\partial\mathbf{F}} \otimes \hat{\mathbf{p}}$$

and with variables y_1, y_2, ..., y_5 taken to be

$$y_1 = \dot{T},\ y_2 = \boldsymbol{\gamma},\ y_3 = \mathbf{L},\ y_4 = \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{x}},\ y_5 = \frac{\partial\mathbf{F}}{\partial\mathbf{x}}.$$

And hence we find that equation (4.18) holds only if

$$a = a_1 = a_2 = a_3 = a_4 = a_5 = 0. \tag{4.19}$$

Now $a_4 = a_5 = 0$ leads us to

$$\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}} = \mathbf{0},\ \frac{\partial\hat{\theta}}{\partial\mathbf{F}} = \mathbf{0}. \tag{4.20}$$

In view of the above results, we find

$$\theta = \theta(T),\ \psi = \psi(T, \boldsymbol{\beta}, \mathbf{F}), \tag{4.21}$$

where without ambiguity in equation (4.21), we have used the same symbols for the functions θ, ψ and their values. With the choice of

$$T = \theta - \theta_0\ (T > -\theta_0),\ \frac{\partial\theta}{\partial T} = 1 \tag{4.22}$$

and without loss of generality, we may replace T by θ in all constitutive developments of this section.

From equations (4.18)-(4.22), we obtain

$$\eta = -\frac{\partial\psi}{\partial\theta},\ \mathbf{T} = \rho\frac{\partial\psi}{\partial\mathbf{F}}\mathbf{F}^T,\ \mathbf{p} = -\rho\mathbf{F}\frac{\partial\psi}{\partial\boldsymbol{\beta}},\ \xi = 0. \tag{4.23}$$

It is important to note that in this development there is an absence of internal rate of production of entropy ξ so that there is no dissipation of energy.

In the rest of this section, we consider the linearized version of the foregoing constitutive developments and then discuss the complete linear theory of thermoelasticity of type II. As in Section 4.1, again we assume that the temperature θ, the thermal displacement α and the relative displacement $\mathbf{u} = \chi - \mathbf{X}$ which respectively represent departures from an equilibrium temperature θ_0, an equilibrium mean thermal displacement α_0 and an equilibrium position of the state of the body with zero stress–are all small of $0(\varepsilon)$ and we restrict our attention to an isotropic material. In the development of such a linearized theory, it will suffice to assume that the specific Helmholtz's free energy is a quadratic function of the infinitesimal temperature, the infinitesimal thermal displacement gradient $\boldsymbol{\beta} = \text{Grad}\ \alpha$ and the infinitesimal strain E defined in equation (4.12), so that

$$\rho_0\psi = \frac{1}{2}\lambda(\text{tr}\ \mathbf{E})^2 + \mu\text{tr}\mathbf{E}^2 - \frac{c\theta^2}{2\theta_0} - \frac{E\beta^*}{3(1-2\nu)}\theta\text{tr}\mathbf{E} + \frac{k^*}{2\theta_0}\boldsymbol{\beta}\cdot\boldsymbol{\beta}, \tag{4.24}$$

where k^* is a constant.

After some simple substitution and rearrangement, the following coupled system of partial differential equations are obtained

$$(\lambda+\mu)\text{grad div}\ \mathbf{u} + \mu\nabla^2\mathbf{u} - \frac{E\beta^*}{3(1-2\nu)}\text{grad}\ \theta + \rho_0\mathbf{b} = \rho\ddot{\mathbf{u}}, \tag{4.25}$$

$$c\ddot{\theta} + \frac{E\beta^*\theta_0}{3(1-2\nu)}\text{div}\ \ddot{\mathbf{u}} = \rho_0 r + k^*\nabla^2\theta. \tag{4.26}$$

The system of equations (4.25) and (4.26) permits propagation of harmonic waves without damping.

4.3 Thermoelasticity (Type III)

For the theory of thermoelasticity of type III, we require constitutive equations for

$$\mathbf{T},\ \mathbf{p},\ \psi,\ \eta,\ \theta,\ \xi$$

and we assume these are functions of the independent variables

$$T,\ \gamma,\ \boldsymbol{\beta},\ \mathbf{F};\mathbf{X}. \tag{4.27}$$

However, for simplicity, in what follows we suppress explicit dependence on $\mathbf{X}$ and regard the material to be homogeneous. Introduction of constitutive assumptions of the forms

$$\psi = \hat{\psi}(T,\gamma,\boldsymbol{\beta},\mathbf{F}),\ \theta = \hat{\theta}(T,\gamma,\boldsymbol{\beta},\mathbf{F}),\ \mathbf{T} = \hat{\mathbf{T}}(T,\gamma,\boldsymbol{\beta},\mathbf{F}),\ \text{ect.}, \tag{4.28}$$

into the reduced energy equation (4.2), after some rearrangements of terms, yields

$$\begin{aligned}
&\hat{\mathbf{p}}\cdot\frac{\partial\hat{\theta}}{\partial T}\gamma+\rho\hat{\theta}\hat{\xi}\\
&+\rho\left(\frac{\partial\hat{\psi}}{\partial T}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial T}\right)\dot{T}+\rho\left(\frac{\partial\hat{\psi}}{\partial\gamma}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\gamma}\right)\cdot\dot{\gamma}+\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\right)\cdot\mathbf{F}^{T}\gamma\\
&+\left[-\hat{\mathbf{T}}\cdot\mathbf{L}+\rho\left(\frac{\partial\hat{\psi}}{\partial\mathbf{F}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\mathbf{F}}\right)\cdot\dot{\mathbf{F}}\right]\\
&+\left(\frac{\partial\hat{\theta}}{\partial\gamma}\otimes\hat{\mathbf{p}}\right)\cdot\frac{\partial\gamma}{\partial\mathbf{x}}+\left(\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\otimes\hat{\mathbf{p}}\right)\cdot\frac{\partial\boldsymbol{\beta}}{\partial\mathbf{x}}+\left(\frac{\partial\hat{\theta}}{\partial\mathbf{F}}\otimes\hat{\mathbf{p}}\right)\cdot\frac{\partial\mathbf{F}}{\partial\mathbf{x}}=0.
\end{aligned} \tag{4.29}$$

Equation (4.29) has the form (A.1) of Appendix A with $N=6$, namely

$$a+a_1y_1+a_2y_2+a_3y_3+a_4y_4+a_5y_5+a_6y_6=0, \tag{4.30}$$

with

$$a \;=\; \hat{\mathbf{p}}\cdot\frac{\partial\hat{\theta}}{\partial T}\gamma+\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\right)\cdot\mathbf{F}^{T}\gamma+\rho\hat{\theta}\hat{\xi},$$

$$
\begin{aligned}
a_1 &= \rho\left(\frac{\partial\hat{\psi}}{\partial T}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial T}\right),\\
a_2 &= \rho\left(\frac{\partial\hat{\psi}}{\partial \gamma}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial \gamma}\right),\\
a_3 &= -\hat{\mathbf{T}}(\mathbf{F}^T)^{-1}+\rho\left(\frac{\partial\hat{\psi}}{\partial \mathbf{F}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial \mathbf{F}}\right),\\
a_4 &= \frac{1}{2}\left(\frac{\partial\hat{\theta}}{\partial \gamma}\otimes\hat{\mathbf{p}}+\hat{\mathbf{p}}\otimes\frac{\partial\hat{\theta}}{\partial \gamma}\right),\\
a_5 &= \frac{\partial\hat{\theta}}{\partial \mathbf{F}}\otimes\hat{\mathbf{p}},\\
a_6 &= \frac{\partial\hat{\theta}}{\partial \beta}\otimes\hat{\mathbf{p}}
\end{aligned}
\tag{4.31}
$$

and

$$
y_1=\dot{T},\ y_2=\dot{\gamma},\ y_3=\dot{\mathbf{F}},\ y_4=\frac{\partial\gamma}{\partial\mathbf{x}},\ y_5=\frac{\partial\mathbf{F}}{\partial\mathbf{x}},\ y_6=\frac{\partial\beta}{\partial\mathbf{x}}.
$$

The equation (4.29) will hold only if

$$
a=a_1=a_2=a_3=a_4=a_5=a_6=0. \tag{4.32}
$$

Now $a_4=a_5=a_2=a_6=0$ leads us to

$$
\frac{\partial\hat{\theta}}{\partial\gamma}=\mathbf{0},\ \frac{\partial\hat{\theta}}{\partial\mathbf{F}}=\mathbf{0},\ \frac{\partial\hat{\psi}}{\partial\gamma}=\mathbf{0},\ \frac{\partial\hat{\theta}}{\partial\beta}=\mathbf{0}. \tag{4.33}
$$

From equation (4.33), we conclude that

$$
\theta=\theta(T),\ \psi=\psi(T,\mathbf{F},\beta), \tag{4.34}
$$

where without ambiguity in equation (4.34), we have used the same symbols for the functions θ, ψ and their values. Hence with the choice

$$
T=\theta+\theta_0\ (\theta_0>0),\frac{\partial\theta}{\partial T}=1 \tag{4.35}
$$

and without loss in generality, we may replace T by θ in the constitutive developments of this section.

From equations (4.31)-(4.35), we obtain

$$\eta = -\frac{\partial \psi}{\partial \theta}, \ \mathbf{T} = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \ \mathbf{p} \cdot \gamma + \rho \frac{\partial \psi}{\partial \boldsymbol{\beta}} \cdot \mathbf{F}^T \gamma + \rho \theta \xi = 0. \tag{4.36}$$

In the rest of this section, we consider the linear theory of thermoelasticity of type III and linearize the foregoing constitutive results. Thus we assume that the temperature θ represents departure from an equilibrium temperature θ_0 and $\mathbf{u} = \chi - \mathbf{X}$ is the displacement vector from a state with zero stress such that both θ and $\mathbf{u}$ are small of $0(\varepsilon)$. We also assume that both time and space derivatives of θ and $\mathbf{u}$ are of small of $0(\varepsilon)$. For such a linearized theory, it will suffice to assume that the specific Helmholtz's free energy is a quadratic function of infinitesimal temperature, infinitesimal strain $\mathbf{E}$ and $\boldsymbol{\beta}$. In the context of the linearized theory and for an isotropic material, the specific Helmholtz's free energy is

$$\rho_0 \psi = \frac{1}{2} \lambda (\text{tr } \mathbf{E})^2 + \mu \text{tr } \mathbf{E}^2 - \frac{c\theta^2}{2\theta_0} - \frac{E\beta^*}{3(1-2\nu)} \theta \text{tr } \mathbf{E} + \frac{k^*}{2\theta_0} \boldsymbol{\beta} \cdot \boldsymbol{\beta}. \tag{4.37}$$

Substitution yields the following coupled system of partial differential equations:

$$(\lambda + \mu) \text{grad div } \mathbf{u} + \mu \nabla^2 \mathbf{u} - \frac{E\beta^*}{3(1-2\nu)} \text{grad } \theta + \rho_0 \mathbf{b} = \rho_0 \ddot{\mathbf{u}}, \tag{4.38}$$

$$\rho_0 c \ddot{\theta} + \frac{E\beta^* \theta_0}{3(1-2\nu)} \text{dvi } \ddot{\mathbf{u}} = \rho_0 \dot{r} + k \nabla^2 \dot{\theta} + k^* \nabla^2 \theta. \tag{4.39}$$

Part III

One-dimensional Thermal Shock Problems

Chapter 5

One-dimensional Problem

In this part, we will use the theory developed in Section 4.3 to solve some one-dimensional problems. The one-dimensional problem due to its relative simplicity, has had a broad treatment in the literature. The particular problems to be treated here are that of an isotropic homogeneous thermoelastic half-space, and the boundary conditions considered are the same as those considered by Boley and Tolin [36] to solve the corresponding coupled thermoelasticity problems.

For a homogeneous, isotropic elastic body, the basic equations for the linear generalized theory of thermoelasticity of type III developed in [32] are

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}-\gamma\theta_{,i}+\rho f_i=\rho\ddot{u}_i, \tag{5.1}$$

$$\rho c\ddot{\theta}+\gamma\theta_0\ddot{u}_{i,i}=\rho\dot{Q}+k\dot{\theta}_{,ii}+k^*\theta_{,ii}, \tag{5.2}$$

$$\sigma_{ij}=\lambda u_{i,i}\delta_{ij}+\mu(u_{i,j}+u_{j,i})-\gamma\theta, \tag{5.3}$$

where

$$\begin{aligned}\gamma &= \frac{1}{3}E\beta^*/(1-2\nu),\\ k^* &= \text{a constant},\\ \beta^* &= \text{coefficient of volume expassion},\end{aligned}$$

a comma followed by a suffix denotes material derivative and a superposed dot denotes the derivative with respect to time.

To transform the above equations to nondimensional form, we define the following nondimensional variables

$$
\begin{aligned}
x_i' &= x_i/l, \\
t' &= ta_0/l, \\
\theta' &= \theta/\theta_0, \\
u_i' &= u_i/l, \\
\sigma_{ij}' &= \sigma_{ij}/\mu, \\
\rho' &= \rho/\rho_0, \\
Q' &= Ql/a_0^3, \\
f_i' &= f_i l/a_0^2,
\end{aligned}
$$

where

$$
\begin{aligned}
l &= \text{a standard length}, \\
a_0 &= \text{a standard speed}, \\
\rho_0 &= \text{a standard mass density}.
\end{aligned}
$$

The basic equations (5.1)-(5.3), dropping primes for convenience, reduce to the following

$$\rho\alpha_1\ddot{u}_i = \alpha_2 u_{j,ij} + u_{i,jj} - \alpha_3\theta_{,i} + \rho\alpha_1 f_i, \tag{5.4}$$

$$\theta_{,ii} + \alpha_4\dot{\theta}_{,ii} + \rho\alpha_5\dot{Q} = \rho\alpha_6\ddot{\theta} + \alpha_7\ddot{u}_{i,i}, \tag{5.5}$$

$$\sigma_{ij} = \frac{\lambda}{\mu}u_{i,i}\delta_{ij} + (u_{i,j} + u_{j,i}) - \alpha_3\theta, \tag{5.6}$$

where

$$\alpha_1 = \rho_0 a_0^2/\mu,$$

$$\alpha_2 = (\lambda+\mu)/\mu,$$
$$\alpha_3 = \gamma\theta_0/\mu,$$
$$\alpha_4 = ka_0/k^*l,$$
$$\alpha_5 = \rho_0 a_0^4/(k^*\theta_0),$$
$$\alpha_6 = \rho_0 c a_0^2/k^*,$$
$$\alpha_7 = \gamma a_0^2/k^*.$$

For a one-dimensional problem, all quantities depend only on one space coordinate x and time t, such that:

stress	$\sigma = \sigma(x,t)$,
displacement	$u = u(x,t)$,
temperature	$\theta = \theta(x,t)$.

For this case, equations (5.4)-(5.6), with $Q = 0$, $f_i = 0$, reduce to

$$\rho\alpha_1\ddot{u} = (\alpha_2+1)u'' - \alpha_3\theta', \tag{5.7}$$

$$\theta'' + \alpha_4\dot{\theta}'' = \rho\alpha_6\ddot{\theta} + \alpha_7\ddot{u}', \tag{5.8}$$

$$\sigma = (\alpha_2+1)u' - \alpha_3\theta, \tag{5.9}$$

where prime and dot denote derivatives with respect to x and t, respectively, and σ denotes the normal stress.

Introducing the thermoelastic potential function φ defined by

$$u = \frac{\partial\varphi}{\partial x}, \tag{5.10}$$

equations (5.7)-(5.9) reduce to

$$\rho\alpha_1\ddot{\varphi} = (\alpha_2+1)\varphi'' - \alpha_3\theta, \tag{5.11}$$

$$\theta'' + \alpha_4\dot{\theta}'' = \rho\alpha_6\ddot{\theta} + \alpha_7\ddot{\varphi}'', \tag{5.12}$$

$$\sigma = (\alpha_2+1)\varphi'' - \alpha_3\theta \quad = \rho\alpha_1\ddot{\varphi}. \tag{5.13}$$

Applying the Laplace transform defined by

$$\bar{g}(x,p) = \int_0^\infty g(x,t)\exp(-pt)dt, \qquad Re(p) > 0, \tag{5.14}$$

to equations (5.10)-(5.13), we arrive at

$$\bar{u} = \frac{d}{dx}\bar{\varphi}, \tag{5.15}$$

$$\bar{\theta} = \frac{1}{\alpha_3}\{(\alpha_2+1)\frac{d^2}{dx^2} - \rho\alpha_1 p^2\}\bar{\varphi}, \tag{5.16}$$

$$\alpha_7 p^2\frac{d^2}{dx^2}\bar{\varphi} = \{(1+\alpha_4 p)\frac{d^2}{dx^2} - \rho\alpha_6 p^2\}\bar{\theta}, \tag{5.17}$$

$$\bar{\sigma} = \rho\alpha_1 p^2\bar{\varphi}, \tag{5.18}$$

where we have used the following initial conditions

$$u(x,t) = \dot{u}(x,t) = \theta(x,t) = \dot{\theta}(x,t) = 0, \quad \text{at} \quad t = 0. \tag{5.19}$$

Now eliminating $\bar{\theta}$ between equations (5.16) and (5.17), we obtain the following differential equation for $\bar{\varphi}$:

$$\{(1+\alpha_4 p)\frac{d^4}{dx^4} - (b_1 p + b_2)p^2\frac{d^2}{dx^2} + b_3 p^4\}\bar{\varphi} = 0, \tag{5.20}$$

where

$$b_1 = \frac{\rho\alpha_1\alpha_4}{(\alpha_2+1)},$$

$$b_2 = \rho\alpha_6 + \frac{(\rho\alpha_1 + \alpha_3\alpha_7)}{(\alpha_2+1)},$$

$$b_3 = \frac{\rho^2\alpha_1\alpha_6}{(\alpha_2+1)}.$$

Using the regularity condition

$$\bar{\varphi} \longrightarrow 0, \qquad \text{as} \qquad x \longrightarrow \infty, \tag{5.21}$$

the solution of equation (5.20) for $\bar{\varphi}$ is given by

$$\bar{\varphi} = A_1 \exp(-\lambda_1 x) + A_2 \exp(-\lambda_2 x), \tag{5.22}$$

where A_1, A_2 are functions of p, and λ_1 and λ_2 are the positive roots of the equation

$$(1+\alpha_4 p)\lambda^4 - (b_1 p + b_2)p^2\lambda^2 + b_3 p^4 = 0, \tag{5.23}$$

given by

$$\lambda_i = p\left\{\frac{(b_1 p + b_2) + (-1)^{i+1}\sqrt{(b_1 p + b_2)^2 - 4b_3(1+\alpha_4 p)}}{2(1+\alpha_4 p)}\right\}^{\frac{1}{2}}, \qquad i = 1, 2. \tag{5.24}$$

By simple substitution from equation (5.22) into equations (5.15), (5.16) and (5.18), we arrive at the following expressions for $\bar{u}$, $\bar{\theta}$ and $\bar{\sigma}$:

$$\bar{u} = -\lambda_1 A_1 \exp(-\lambda_1 x) - \lambda_2 A_2 \exp(-\lambda_2 x), \tag{5.25}$$

$$\bar{\theta} = B_1 A_1 \exp(-\lambda_1 x) + B_2 A_2 \exp(-\lambda_2 x), \tag{5.26}$$

$$\bar{\sigma} = \rho\alpha_1 p^2 \left\{A_1 \exp(-\lambda_1 x) + A_2 \exp(-\lambda_2 x)\right\}, \tag{5.27}$$

in terms of the two unknown functions A_1 and A_2, which are to be determined by the associated boundary conditions at $x = 0$ and

$$\begin{aligned} B_1 &= C_1{\lambda_1}^2 - C_2 p^2, \\ B_2 &= C_1{\lambda_2}^2 - C_2 p^2. \end{aligned} \tag{5.28}$$

$$\begin{aligned} C_1 &= \frac{\alpha_2 + 1}{\alpha_3}, \\ C_2 &= \rho\frac{\alpha_1}{\alpha_3}. \end{aligned} \tag{5.29}$$

Chapter 6

One-dimensional Thermal Shock Problem –Stress-free Boundary

We consider now a thermal shock problem for a homogeneous isotropic elastic half-space $x \geq 0$ with stress-free boundary $x = 0$. At time $t = 0$, the stress-free boundary is suddenly heated to a uniform temperature and left in that state. The problem is to determine the distribution of stress and temperature for $x > 0$ and $t > 0$.

The problem is usually named after the Russian lady scientist V. I. Danilovskaya, who had first studied it in the context of classical thermoelasticity by neglecting the coupling term in the heat transport equation.

The data of the problem suggests that this is a one-dimensional problem, all the field variables depend on x and t only. Under the assumptions made, the initial and boundary conditions are given by

$$u(x,0) = \dot{u}(x,0) = \theta(x,0) = \dot{\theta}(x,0) = 0, \tag{6.1}$$

$$\sigma(0,t) = 0, \quad \theta(0,t) = T_0 H(t) \tag{6.2}$$

and the regularity conditions are

$$(\sigma(x,t),\ \theta(x,t)) \longrightarrow 0, \quad \text{as} \quad x \longrightarrow \infty, \quad t > 0, \tag{6.3}$$

where $T_0 \neq 0$ is the uniform temperature input applied to the boundary and $H(t)$ is the Heaviside unit step function, i.e.,

$$H(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ 1, & \text{for } t \geq 0^+ . \end{cases}$$

Taking the Laplace transform of the boundary and regularity conditions, we find that

$$\begin{aligned} \bar{\sigma}(x,t) &= 0, \quad \text{at} \quad x = 0, \\ \bar{\theta}(x,t) &= \frac{T_0}{p}, \quad \text{at} \quad x = 0 \end{aligned} \tag{6.4}$$

and

$$\{\bar{\sigma}(x,t), \bar{\theta}(x,t)\} \longrightarrow 0, \quad \text{as} \quad x \longrightarrow \infty, \quad t > 0. \tag{6.5}$$

The expressions for $\bar{\theta}$ and $\bar{\sigma}$ given by equations (5.26) and (5.27) satisfy the regularity conditions (6.5), and the boundary conditions (6.4) will also be satisfied, if

$$\begin{aligned} A_1 &= \frac{T_0}{p(B_1 - B_2)}, \\ A_2 &= -\frac{T_0}{p(B_1 - B_2)}. \end{aligned}$$

Substituting the above expressions for A_1 and A_2 in equations (5.25)-(5.27), we find that

$$\bar{u} = -\frac{T_0}{p(B_1 - B_2)}\{\lambda_1 \exp(-\lambda_1 x) - \lambda_2 \exp(-\lambda_2 x)\}, \tag{6.6}$$

$$\bar{\theta} = \frac{T_0}{p(B_1 - B_2)}\{B_1 \exp(-\lambda_1 x) - B_2 \exp(-\lambda_2 x)\}, \tag{6.7}$$

$$\bar{\sigma} = \frac{\rho \alpha_1 p T_0}{B_1 - B_2}\{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\}. \tag{6.8}$$

Theoretically, we can take the inverse Laplace transform of equations (6.6)-(6.8), and find the expressions for the quantities concerned, but it is difficult to find the inverse transforms of these equations in the present form. We shall try to find the inverse transforms for small values of time (large values of p) by expanding the above expressions in the inverse powers of p to a few terms.

6.1 General Case

To find the small time approximate solution, we let $z = 1/p$ in equation (5.24) and obtain

$$\begin{aligned}\lambda_i &= \frac{1}{z}\{\frac{(b_1+b_2z)+(-1)^{i+1}\sqrt{(b_1+b_2z)^2-4b_3(z^2+\alpha_4 z)}}{2(z+\alpha_4)}\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}\alpha_4^{\frac{1}{2}}z}\{(b_1+b_2z)+(-1)^{i+1}b_1(1+\frac{2b_1b_2-4b_3\alpha_4}{b_1^2}z)^{\frac{1}{2}}\}^{\frac{1}{2}}(1+\frac{z}{\alpha_4})^{-\frac{1}{2}}.\end{aligned}$$

Expanding the following expressions in powers of z and retaining only necessary terms, we find

$$\begin{aligned}(1+\frac{2b_1b_2-4b_3\alpha_4}{b_1^2}z)^{\frac{1}{2}} &\approx 1+\frac{b_1b_2-2b_3\alpha_4}{b_1^2}z, \\ (1+\frac{z}{\alpha_4})^{-1} &\approx 1-\frac{z}{\alpha_4}.\end{aligned}$$

Using a similar method of expansion, we obtain

$$\begin{aligned}\lambda_1 &\approx \frac{1}{\sqrt{2}\alpha_4^{\frac{1}{2}}z}\{(2b_1+\frac{2b_1b_2-2b_3\alpha_4}{b_1}z)(1-\frac{z}{\alpha_4})\}^{\frac{1}{2}} \\ &\approx \frac{1}{b_1^{\frac{1}{2}}\alpha_4^{\frac{1}{2}}z}(b_1^2+\frac{b_1b_2\alpha_4-b_3\alpha_4^2-b_1^2}{\alpha_4}z)^{\frac{1}{2}} \\ &= \frac{b_1^{\frac{1}{2}}}{\alpha_4^{\frac{1}{2}}z}(1+\frac{b_1b_2\alpha_4-b_3\alpha_4^2-b_1^2}{\alpha_4b_1^2}z)^{\frac{1}{2}} \\ &\approx \frac{b_1^{\frac{1}{2}}}{\alpha_4^{\frac{1}{2}}z}(1+\frac{1}{2}\frac{b_1b_2\alpha_4-b_3\alpha_4^2-b_1^2}{\alpha_4b_1^2}z)\end{aligned}$$

$$= (\frac{b_1}{\alpha_4})^{\frac{1}{2}}\frac{1}{z} + \frac{1}{2}\frac{b_1 b_2 \alpha_4 - b_3\alpha_4^2 - b_1^2}{(\alpha_4 b_1)^{\frac{3}{2}}},$$

$$\begin{aligned} \lambda_2 &\approx \frac{1}{\sqrt{2}\alpha_4^{\frac{1}{2}} z}\{2b_3\alpha_4 z(1-\frac{z}{\alpha_4})\}^{\frac{1}{2}} \\ &\approx \frac{1}{\sqrt{2}\alpha_4^{\frac{1}{2}} z}(2b_3\alpha_4 z)^{\frac{1}{2}} \\ &= b_3{}^{\frac{1}{2}} z^{-\frac{1}{2}}. \end{aligned}$$

Now we may write

$$\lambda_1 \approx b_{10}p + b_{11}, \qquad \lambda_2 \approx b_{20}p^{\frac{1}{2}}, \tag{6.9}$$

where

$$\begin{aligned} b_{10} &= (\frac{b_1}{\alpha_4})^{\frac{1}{2}} = b_5(\alpha_2+1)^{\frac{1}{2}}, \\ b_{20} &= b_3^{\frac{1}{2}}, \\ b_{11} &= \frac{1}{2}\frac{(b_1 b_2\alpha_4 - b_3\alpha_4^2 - b_1^2)}{(b_1\alpha_4)^{\frac{3}{2}}}. \end{aligned}$$

Now from equation (5.28), we find that

$$\begin{aligned} B_1 - B_2 &= C_1(\lambda_1^2 - \lambda_2^2) \\ &= C_1[b_{10}^2 p^2 + (2b_{10}b_{11} - b_{20}^2)p + b_{11}^2] \\ &= C_1 b_{10}^2 p^2 (1 + \frac{2b_{10}b_{11} - b_{20}^2}{b_{10}^2}p^{-1} + \frac{b_{11}^2}{b_{10}^2}p^{-2}), \end{aligned}$$

from which, we obtain

$$\begin{aligned} \frac{1}{B_1 - B_2} &\approx \frac{1}{C_1 b_{10}^2 p^2}[1 - \frac{2b_{10}b_{11} - b_{20}^2}{b_{10}^2}p^{-1} - \frac{b_{11}^2}{b_{10}^2}p^{-2} + (\frac{2b_{10}b_{11} - b_{20}^2}{b_{10}^2})^2 p^{-2}] \\ &= \frac{1}{C_1 b_{10}^2 p^2}(1 - \bar{D}_1 p^{-1} - \bar{D}_2 p^{-2}), \end{aligned}$$

where

$$\bar{D}_1 = \frac{2b_{10}b_{11} - b_{20}^2}{b_{10}^2},$$
$$\bar{D}_2 = \frac{4b_{10}b_{11}b_{20}^2 - 3b_{11}^2b_{10}^2 - b_{20}^4}{b_{10}^4}.$$

Using the same method of the expansion, we find that

$$\begin{aligned}
\frac{\lambda_1}{p(B_1 - B_2)} &\approx \frac{1}{C_1 b_{10}^2 p^3}(1 - \bar{D}_1 p^{-1} - \bar{D}_2 p^{-2})(b_{10}p + b_{11}) \\
&\approx \frac{1}{C_1 b_{10}^2}[b_{10}p^{-2} + (b_{11} - \bar{D}_1 b_{10})p^{-3} + (-\bar{D}_2 b_{10} - \bar{D}_1 b_{11})p^{-4}], \\
\frac{\lambda_2}{p(B_1 - B_2)} &\approx \frac{1}{C_1 b_{10}^2 p^3}(1 - \bar{D}_1 p^{-1} - \bar{D}_2 p^{-2}) b_{20} p^{\frac{1}{2}} \\
&\approx \frac{1}{C_1 b_{10}^2}(b_{20}p^{-\frac{5}{2}} - \bar{D}_1 b_{20} p^{-\frac{7}{2}} - \bar{D}_2 b_{20} p^{-\frac{9}{2}}), \\
\frac{B_1}{p(B_1 - B_2)} &\approx \frac{1}{C_1 b_{10}^2 p^3}(1 - \bar{D}_1 p^{-1} - \bar{D}_2 p^{-2})[(C_1 b_{10}^2 - C_2)p^2 + 2C_1 b_{10} b_{11} p + C_1 b_{11}^2] \\
&\approx \frac{1}{C_1 b_{10}^2}\{(C_1 b_{10}^2 - C_2)p^{-1} + [2C_1 b_{10} b_{11} - \bar{D}_1(C_1 b_{10}^2 - C_2)]p^{-2} \\
&+ [C_1 b_{11}^2 - 2C_1 \bar{D}_1 b_{10} b_{11} - \bar{D}_2(C_1 b_{10}^2 - C_2)]p^{-3}\}, \\
\frac{B_2}{p(B_1 - B_2)} &\approx \frac{1}{C_1 b_{10}^2 p^3}(1 - \bar{D}_1 p^{-1} - \bar{D}_2 p^{-2})(C_1 b_{20}^2 p - C_2 p^2) \\
&\approx \frac{1}{C_1 b_{10}^2}[-C_2 p^{-1} + (C_1 b_{20}^2 + C_2 \bar{D}_1)p^{-2} + (C_2 \bar{D}_2 - C_1 \bar{D}_1 b_{20}^2)p^{-3}].
\end{aligned}$$

Using the above expansions in equations (6.6)-(6.8), we get

$$\begin{aligned}
\bar{u} &\approx \frac{T_0}{C_1 b_{10}^2}\{(\bar{E}_1 p^{-2} + \bar{E}_2 p^{-3} + \bar{E}_3 p^{-4}) \exp(-b_{10}px - b_{11}x) \\
&\quad -(\bar{E}_4 p^{-\frac{5}{2}} + \bar{E}_5 p^{-\frac{7}{2}} + \bar{E}_6 p^{-\frac{9}{2}}) \exp(-b_{20}p^{\frac{1}{2}}x)\}, \qquad (6.10) \\
\bar{\theta} &\approx \frac{T_0}{C_1 b_{10}^2}\{\bar{F}_1 p^{-1} + \bar{F}_2 p^{-2} + \bar{F}_3 p^{-3}) \exp(-b_{10}px - b_{11}x) \\
&\quad - (\bar{F}_4 p^{-1} + \bar{F}_5 p^{-2} + \bar{F}_6 p^{-3}) \exp(-b_{20}p^{\frac{1}{2}}x)\}, \qquad (6.11) \\
\bar{\sigma} &\approx \frac{\rho\alpha_1 T_0}{C_1 b_{10}^2}(\bar{G}_1 p^{-1} + \bar{G}_2 p^{-2} + \bar{G}_3 p^{-3}) \\
&\quad \{\exp(-b_{10}px - b_{11}x) - \exp(-b_{20}p^{\frac{1}{2}}x)\}, \qquad (6.12)
\end{aligned}$$

where

$$
\begin{aligned}
\bar{E}_1 &= -b_{10} \quad , \\
\bar{E}_2 &= \bar{D}_1 b_{10} - b_{11}, \\
\bar{E}_3 &= b_{11}\bar{D}_1 + \bar{D}_2 b_{10}, \\
\bar{E}_4 &= -b_{20}, \\
\bar{E}_5 &= \bar{D}_1 b_{20}, \\
\bar{E}_6 &= \bar{D}_2 b_{20}, \\
\bar{F}_1 &= C_1 b_{10}^2 - C_2, \\
\bar{F}_2 &= 2C_1 b_{10} b_{11} - \bar{D}_1 (C_1 b_{10}^2 - C_2), \\
\bar{F}_3 &= C_1 b_{11}^2 - 2C_1 \bar{D}_1 b_{10} b_{11} - \bar{D}_2 (C_1 b_{10}^2 - C_2), \\
\bar{F}_4 &= -C_2, \\
\bar{F}_5 &= C_1 b_{20}^2 + C_2 \bar{D}_1, \\
\bar{F}_6 &= C_2 \bar{D}_2 - C_1 \bar{D}_1 b_{20}^2, \\
\bar{G}_1 &= 1, \\
\bar{G}_2 &= -\bar{D}_1, \\
\bar{G}_3 &= -\bar{D}_2.
\end{aligned}
$$

Now, to obtain the inverse Laplace transforms of equations (6.10)-(6.12), we will need the following results ([37], p.494)

$$
\begin{aligned}
L^{-1}[p^{-v-1}] &= \frac{t^v}{\Gamma(v+1)}, \qquad v > -1, \\
L^{-1}[\exp(-ap)] &= \delta(t-a), \qquad a > 0, \\
L^{-1}[p^{-\frac{2+n}{2}} \exp(-ap^{-\frac{1}{2}} x)] &= (4t)^{\frac{1}{2}n} i^n erfc(\frac{ax}{2\sqrt{t}}),
\end{aligned}
$$

$$n = 0, 1, 2, \dots ,$$

where $L^{-1}[\,]$ denotes the inverse Laplace transform, $\delta(\,)$ is the Dirac delta function, and $erf()$ is the usual error function, and also we have the following notation

$$\begin{aligned} erfc(x) &= 1 - erf(x), \\ ierfc(x) &= i^1 erfc(x) = \int_x^\infty erfc(\xi) d\xi, \\ i^n erfc(x) &= \int_x^\infty i^{n-1} erfc(\xi) d\xi, \\ & \qquad\qquad n = 2, 3, 4\dots . \end{aligned}$$

We note that

$$\begin{aligned} \exp(-\lambda_1 x) &= \exp(-b_{10} p x) \cdot \exp(-b_{11} x), \\ \exp(-\lambda_2 x) &= \exp(-b_{20} p^{\frac{1}{2}} x). \end{aligned}$$

Using the above results and the following convolution theorem

$$L^{-1}[g_1(p) \cdot g_2(p)] = \int_0^t f_1(t - z) f_2(z) dz,$$

where

$$L^{-1}[g_1(p)] = f_1(t), \quad L^{-1}[g_2(p)] = f_2(t)$$

in equations (6.10)-(6.12), we obtain

$$\begin{aligned} u(x,t) &\approx \frac{T_0}{C_1 b_{10}^2} \{ \exp(-b_{11} x) \sum_{j=1}^{3} \bar{E}_j \int_0^t \delta(t - b_{10} x - z) \frac{z^j}{\Gamma(j+1)} dz \\ & - \sum_{j=4}^{6} \bar{E}_j (4t)^{(j-\frac{5}{2})} i^{(2j-5)} erfc(\frac{b_{20} x}{2\sqrt{t}}) \}, \\ \theta(x,t) &\approx \frac{T_0}{C_1 b_{10}^2} \{ \exp(-b_{11} x) \sum_{j=1}^{3} \bar{F}_j \int_0^t \delta(t - b_{10} x - z) \frac{z^{j-1}}{\Gamma(j)} dz \end{aligned}$$

$$- \sum_{j=4}^{6} \bar{F}_j (4t)^{j-4} i^{2(j-4)} erfc(\frac{b_{20}x}{2\sqrt{t}})\},$$

$$\sigma(x,t) \approx \frac{\rho\alpha_1 T_0}{C_1 b_{10}^2} \sum_{j=1}^{3} \bar{G}_j \{\exp(-b_{11}x) \int_0^t \delta(t - b_{10}x - z) \frac{z^{(j-1)}}{\Gamma(j)} dz$$

$$- (4t)^{(j-1)} i^{2(j-1)} erfc(\frac{b_{20}x}{2\sqrt{t}})\},$$

which may be further simplified to:

$$u(x,t) \approx \frac{T_0}{C_1 b_{10}^2} \{\exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} \bar{E}_j \frac{(t - X_{10})^j}{(j+1)!}$$

$$- \sum_{j=4}^{6} \bar{E}_j (4t)^{(j-\frac{5}{2})} i^{(2j-5)} erfc(\frac{X_{20}}{2\sqrt{t}})\}, \quad (6.13)$$

$$\theta(x,t) \approx \frac{T_0}{C_1 b_{10}^2} \{\exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} \bar{F}_j \frac{(t - X_{10})^{(j-1)}}{j!}$$

$$- \sum_{j=4}^{6} \bar{F}_j (4t)^{(j-4)} i^{2(j-4)} erfc(\frac{X_{20}}{2\sqrt{t}})\}, \quad (6.14)$$

$$\sigma(x,t) \approx \frac{\rho\alpha_1 T_0}{C_1 b_{10}^2} \{\exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} \bar{G}_j \frac{(t - X_{10})^{(j-1)}}{j!}$$

$$- \sum_{j=1}^{3} \bar{G}_j (4t)^{(j-1)} i^{2(j-1)} erfc(\frac{X_{20}}{2\sqrt{t}})\}, \quad (6.15)$$

where

$$X_{11} = b_{11}x \quad X_{10} = b_{10}x \quad X_{20} = b_{20}x.$$

Due to the presence of the error function in equations (6.13)-(6.15), we conclude that this theory predicts an infinite speed for heat propagation. To analyze the results given above, we use the following values of the parameters involved in those equations:

$$b_1 = 2.8,\ b_2 = 5.2,\ b_3 = 3.35,\ b_4 = 2.25,$$

$$\alpha_1 = 1.25,\ \alpha_2 = 0.25,\ \alpha_3 = 0.25,\ \alpha_4 = 3.1,\ \rho_0 = 1.0. \quad (6.16)$$

The numerical values of temperature and stress distribution for $t = 0.02,\ 0.04,\ 0.06$ are given in Tables 6.1-6.3. And these values are displayed in Figures 6.1-6.6.

The graph of σ/T_0 in Figures 6.2 and 6.4 has a sharp corner at $x = 0.05$, since the value of σ/T_0 decreases from $x = 0$ to $x = 0.05$ and it starts increasing for $x > 0.05$ and the graph is a straight line from $x = 0$ to $x = 0.05$. This sharp turn of the graph at $x = 0.05$ may be avoided by obtaining values of σ/T_0 at large number of points in the interval [0,0.1] (e.g. x =0.0, 0.01, 0.02, 0.03, ..., 0.09, 0.1). A similar explanation applies for the graph of σ/T_0 in Figure 6.6 for $0 \le x \le 0.2$.

It is due to the loss of accuracy in the approximation that these graphs show that σ/T_0 is not equal to zero at $x = 0$, although it is prescribed to be zero at $x = 0$.

Table 6.1: Numerical values of temperature and stress for $t = 0.02$

x	θ/T_0	σ/T_0
.0000	.9983	.0099
.0500	.7181	-.1882
.1000	.3993	-.1033
.1500	.1882	-.0482
.2000	.0745	-.0190
.2500	.0245	-.0062
.3000	.0067	-.0017
.3500	.0015	-.0004
.4000	.0003	-.0001
.4500	.0000	.0000
.5000	.0000	.0000
.5500	.0000	.0000
.6000	.0000	.0000
.6500	.0000	.0000
.7000	.0000	.0000
.7500	.0000	.0000
.8000	.0000	.0000
.8500	.0000	.0000
.9000	.0000	.0000
.9500	.0000	.0000

Table 6.2: Numerical values of temperature and stress for $t = 0.04$

x	θ/T_0	σ/T_0
.0000	.9966	.0203
.0500	.8303	-.2303
.1000	.5752	-.1561
.1500	.3684	-.0983
.2000	.2171	-.0572
.2500	.1173	-.0306
.3000	.0579	-.0150
.3500	.0261	-.0067
.4000	.0107	-.0027
.4500	.0040	-.0010
.5000	.0013	-.0003
.5500	.0004	-.0001
.6000	.0001	.0000
.6500	.0000	.0000
.7000	.0000	.0000
.7500	.0000	.0000
.8000	.0000	.0000
.8500	.0000	.0000
.9000	.0000	.0000
.9500	.0000	.0000

Table 6.3: Numerical values of temperature and stress for $t = 0.06$

x	θ/T_0	σ/T_0
.0000	.9947	.0311
.0500	.7757	.0216
.1000	.6655	-.1902
.1500	.4764	-.1331
.2000	.3232	-.0886
.2500	.2073	-.0560
.3000	.1254	-.0335
.3500	.0715	-.0189
.4000	.0384	-.0101
.4500	.0193	-.0050
.5000	.0091	-.0024
.5500	.0041	-.0010
.6000	.0017	-.0004
.6500	.0007	-.0002
.7000	.0002	-.0001
.7500	.0001	.0000
.8000	.0000	.0000
.8500	.0000	.0000
.9000	.0000	.0000
.9500	.0000	.0000

6.2 A Special Case

Because the constitutive equations (5.4)-(5.6) include a diffusion type equation for heat conduction, generally, this theory predicts an infinite speed for the heat propagation. But for a special case, when $k^* \gg k$, that is, $\alpha_4 \approx 0, \quad b_1 \approx 0$, equation (5.24) becomes

$$\lambda_i = c_i p, \qquad i = 1, 2, \tag{6.17}$$

where

$$c_i = \sqrt{\frac{b_2 + (-1)^{i+1}\sqrt{b_2^2 - 4b_3}}{2}}.$$

Now from equations (5.27), (5.28) and (6.17), we have

$$\begin{aligned} \frac{1}{B_1 - B_2} &= \frac{1}{C_1(\lambda_1^2 - \lambda_2^2)} \\ &= \frac{1}{C_1(c_1^2 - c_2^2)p^2} = \frac{N}{p^2}, \end{aligned} \tag{6.18}$$

where

$$N = \frac{1}{C_1 c_1^2 - C_1 c_2^2}.$$

Substituting from equations (6.17) and (6.18) in equations (6.6)-(6.8), we obtain

$$\bar{u} = -\frac{T_0 N}{p^2}\{c_1 \exp(-\lambda_1 x) - c_2 \exp(-\lambda_2 x)\}, \tag{6.19}$$

$$\bar{\theta} = \frac{T_0 N}{p}\{N_1 \exp(-\lambda_1 x) - N_2 \exp(-\lambda_2 x)\}, \tag{6.20}$$

$$\bar{\sigma} = \frac{\rho \alpha_1 T_0 N}{p}\{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\}, \tag{6.21}$$

where

$$\begin{aligned} N_1 &= C_1 c_1^2 - C_2, \\ N_2 &= C_1 c_2^2 - C_2. \end{aligned}$$

Taking the inverse Laplace transforms of equations (6.19)-(6.21), we obtain

$$u(x,t) = -\frac{T_0 N}{2}\{c_1 H(t-c_1x)(t-c_1x) - c_2 H(t-c_2x)(t-c_2x)\}, \quad (6.22)$$

$$\theta(x,t) = T_0 N\{N_1 H(t-c_1x) - N_2 H(t-c_2x)\}, \quad (6.23)$$

$$\sigma(x,t) = \rho\alpha_1 T_0 N\{H(t-c_1x) - H(t-c_2x)\}. \quad (6.24)$$

For obtaining the numerical values of θ and σ, we have used the same numerical values of the parameters as given in equations (6.16). The numerical values of the temperature and stress for $t = 0.15,\ 0.25,\ 0.50$ for various of x are given in Tables 6.4-6.6. The jumps in temperature and stress fields occur at $x = x_1,\ x_2$ as given below:

t	0.15	0.25	0.50
x_1	.0711	.1186	.2371
x_2	.1728	.2882	.5760

The numerical values of temperature and stress are displayed in Figure 6.7-6.12.

Table 6.4: Numerical values of temperature and stress for $t = 0.15$

x	θ/T_0	σ/T_0
.0000	1.0000	.0000
x_1^-	1.0000	.0000
x_1^+	.0668	-.0677
.1000	.0668	-.0677
x_2^-	.0668	-.0677
x_2^+	.0000	.0000
.2000	.0000	.0000
.3000	.0000	.0000

Table 6.5: Numerical values of temperature and stress for $t = 0.25$

x	θ/T_0	σ/T_0
.0000	1.0000	.0000
.1000	1.0000	.0000
x_1^-	1.0000	.0000
x_1^+	.0668	-.0677
.2000	.0668	-.0677
x_2^-	.0668	-.0677
x_2^+	.0000	.0000
.3000	.0000	.0000
.4000	.0000	.0000

Table 6.6: Numerical values of temperature and stress $t = 0.5$

x	θ/T_0	σ/T_0
.0000	1.0000	.0000
.1000	1.0000	.0000
.2000	1.0000	.0000
x_1^-	1.0000	.0000
x_1^+	.0668	-.0677
.3000	.0668	-.0677
.4000	.0668	-.0677
.5000	.0668	-.0677
x_2^-	.0668	-.0677
x_2^+	.0000	.0000
.6000	.0000	.0000

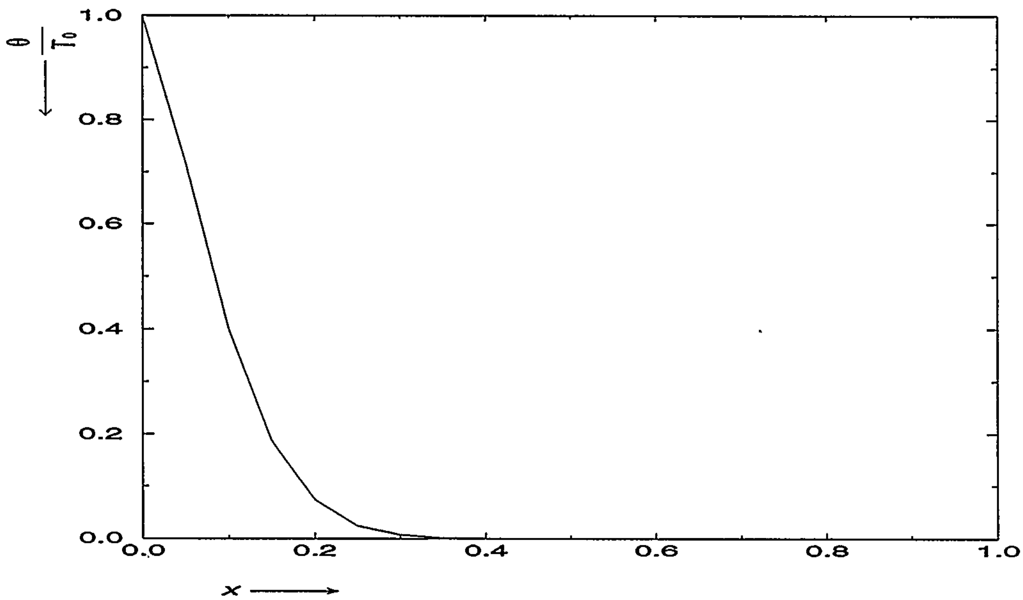


Fig.6.1 Numerical values of temperature θ/T_0 against x for a free boundary problem at $t = 0.02$

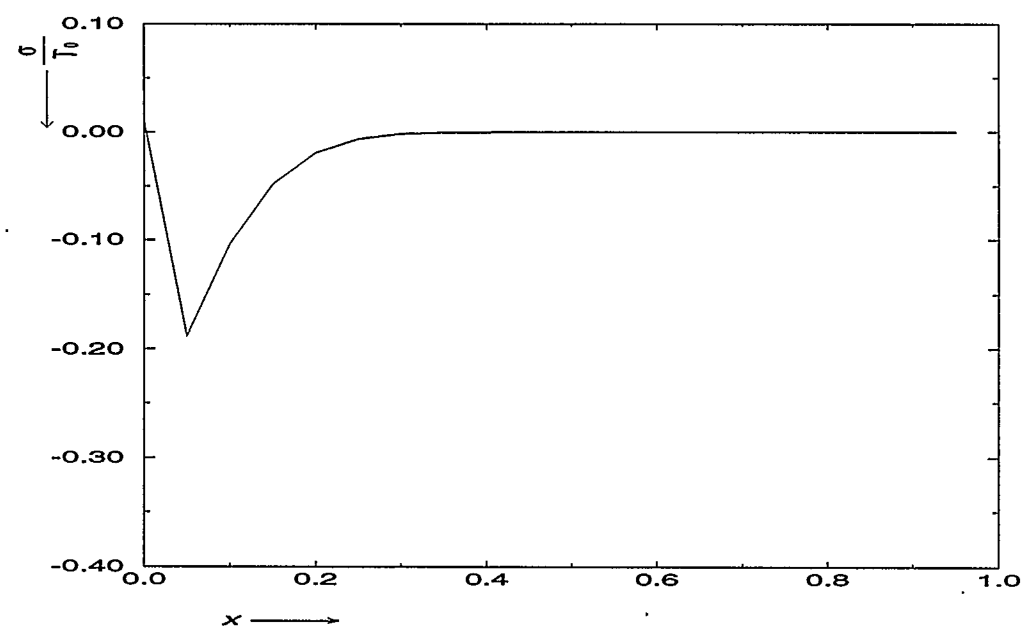


Fig.6.2 Numerical values of stress σ/T_0 against x for a free boundary problem at $t = 0.02$

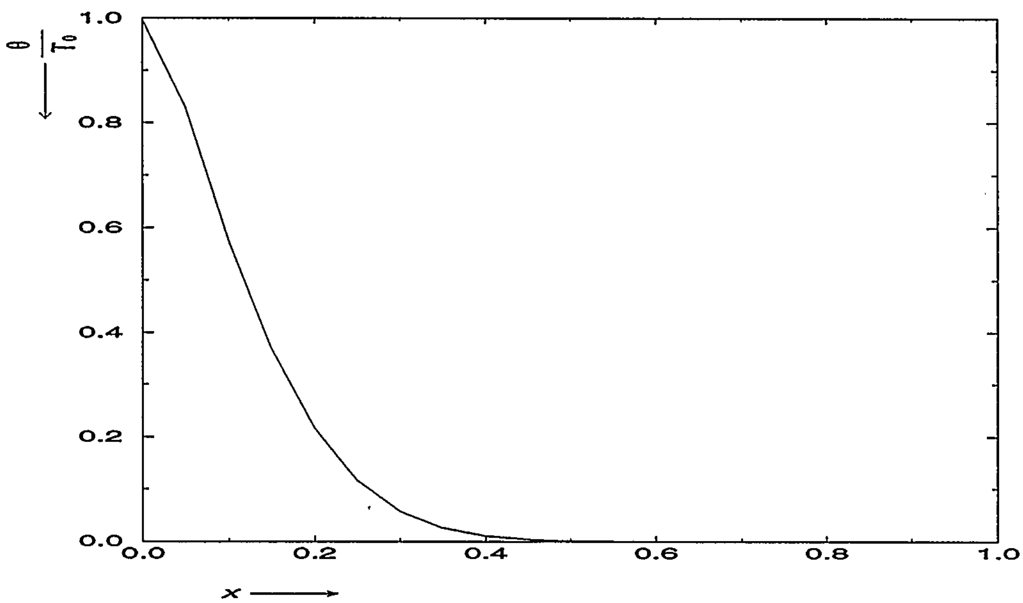


Fig.6.3 Numerical values of temperature θ/T_0 against x for a free boundary problem at $t = 0.04$

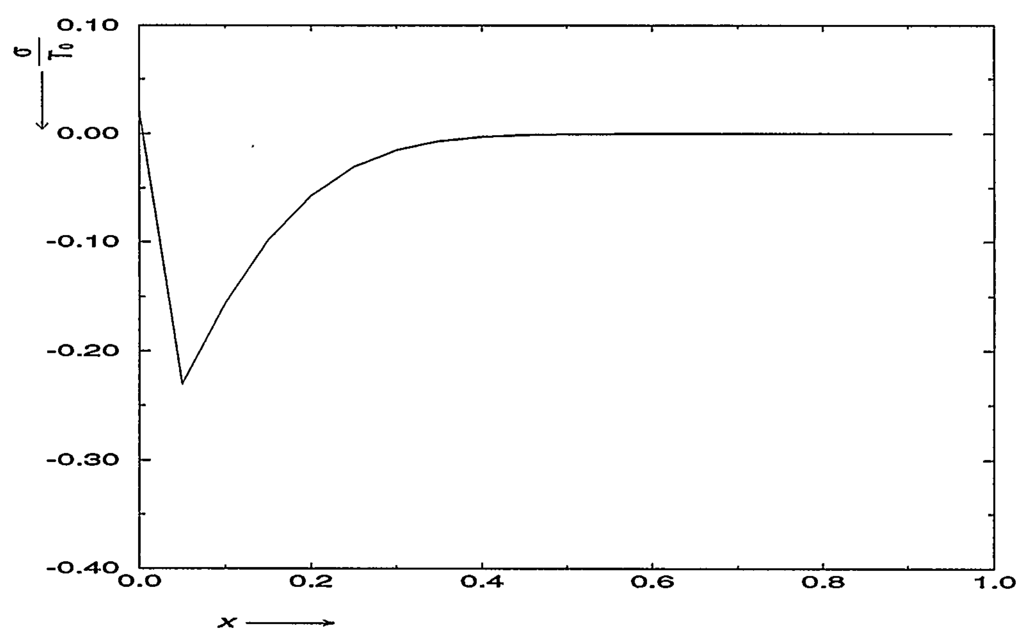


Fig.6.4 Numerical values of stress σ/T_0 against x for a free boundary problem at $t = 0.04$

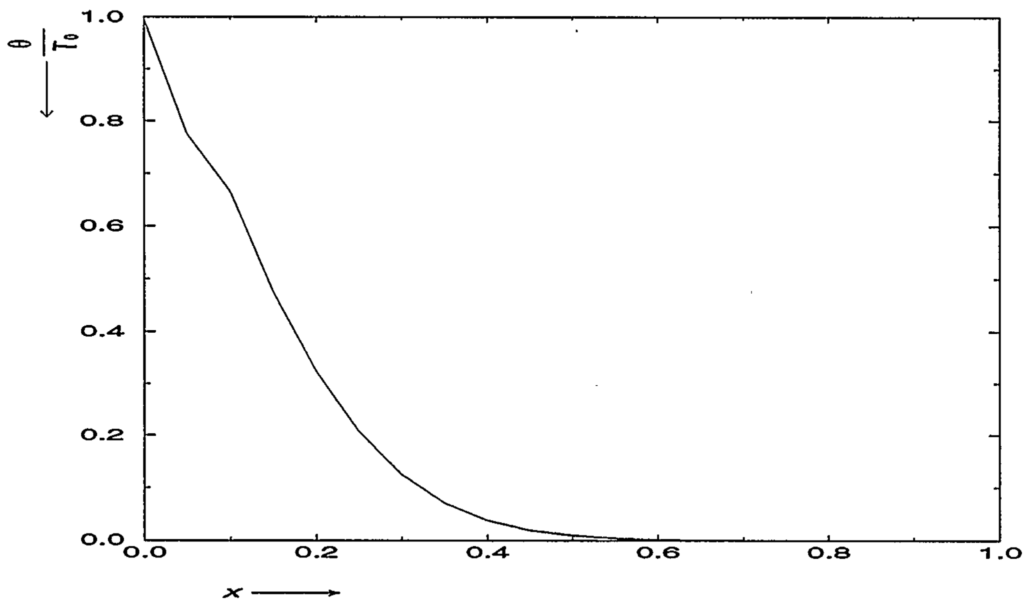


Fig.6.5 Numerical values of temperature θ/T_0 against x
for a free boundary problem at $t = 0.06$

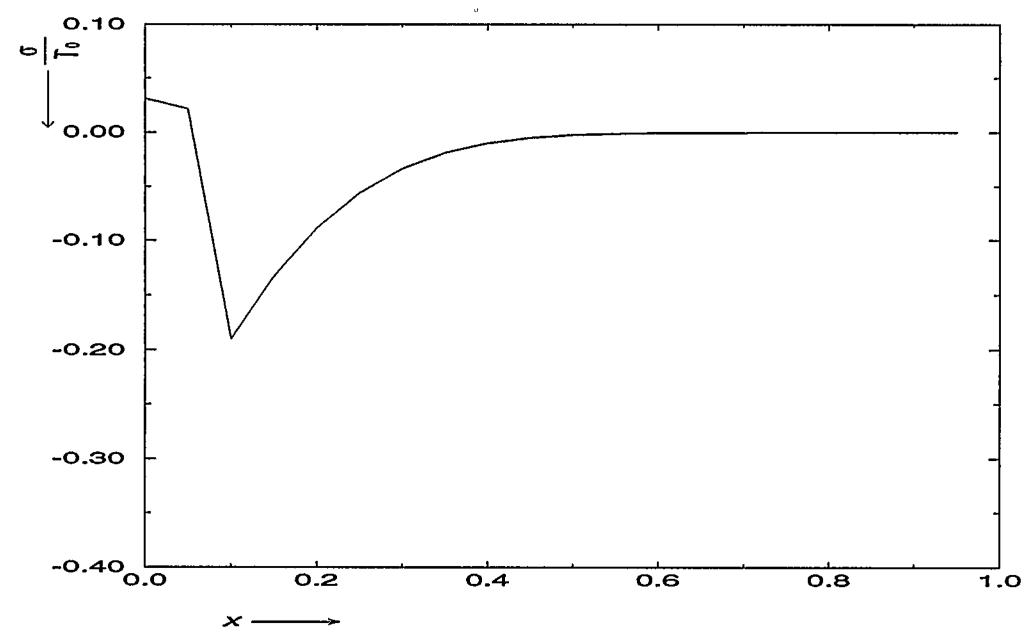


Fig.6.6 Numerical values of stress σ/T_0 against x
for a free boundary problem at $t = 0.06$

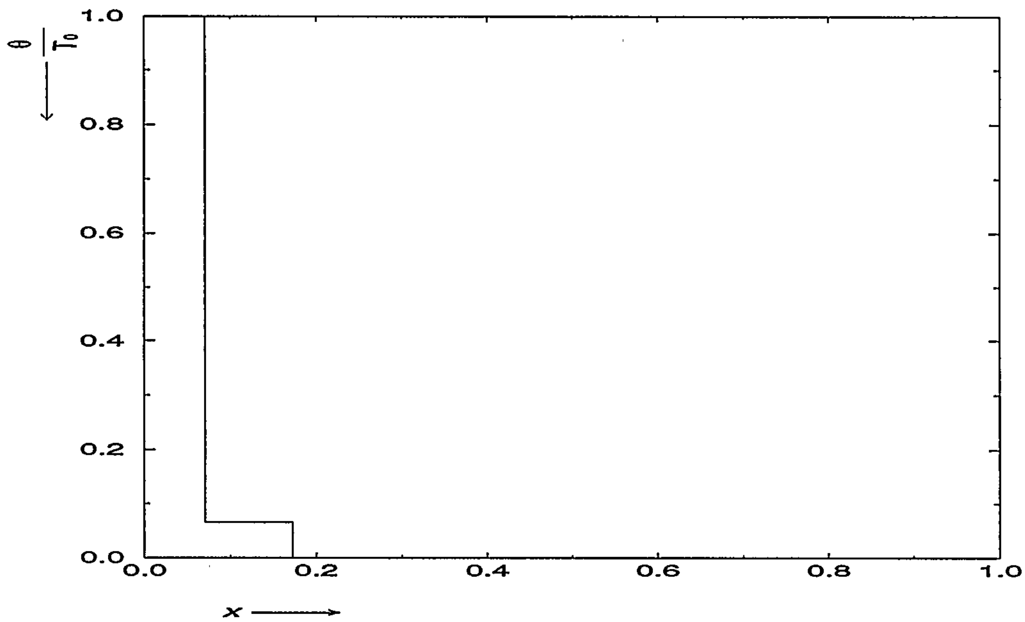


Fig.6.7 Numerical values of temperature θ/T_0 against x
for a free boundary problem (a special case) at $t = 0.15$

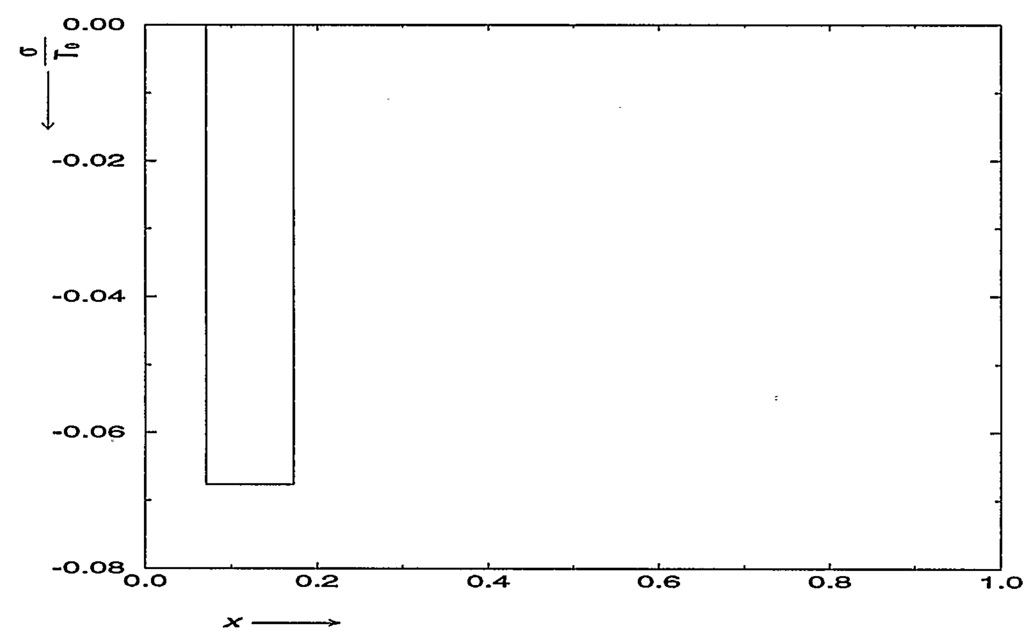


Fig.6.8 Numerical values of stress σ/T_0 against x
for a free boundary problem (a special case) at $t = 0.15$

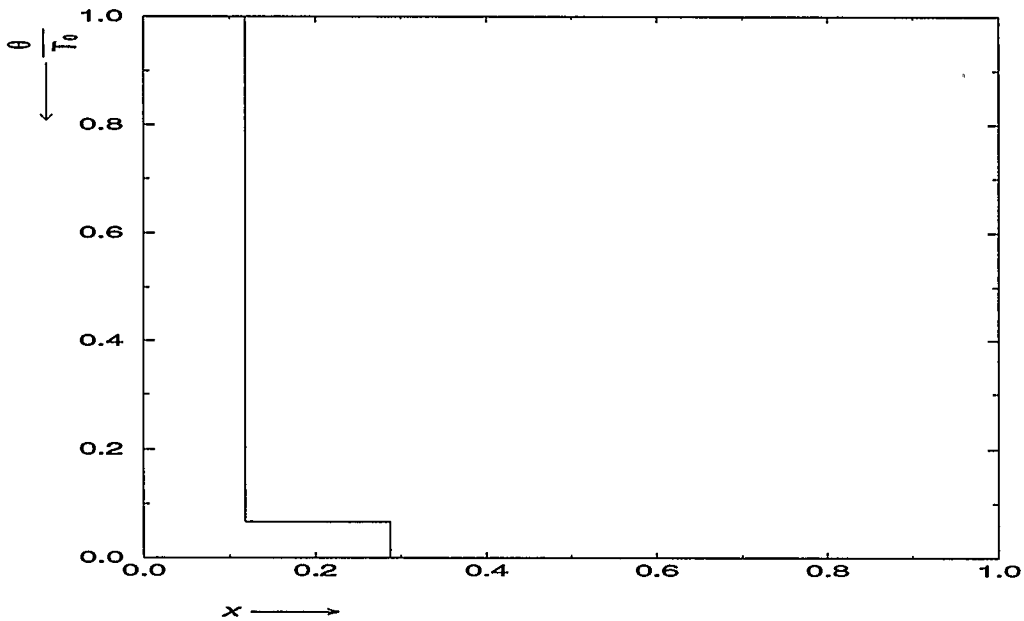


Fig.6.9 Numerical values of temperature θ/T_0 against x
for a free boundary problem (a special case) at $t = 0.25$

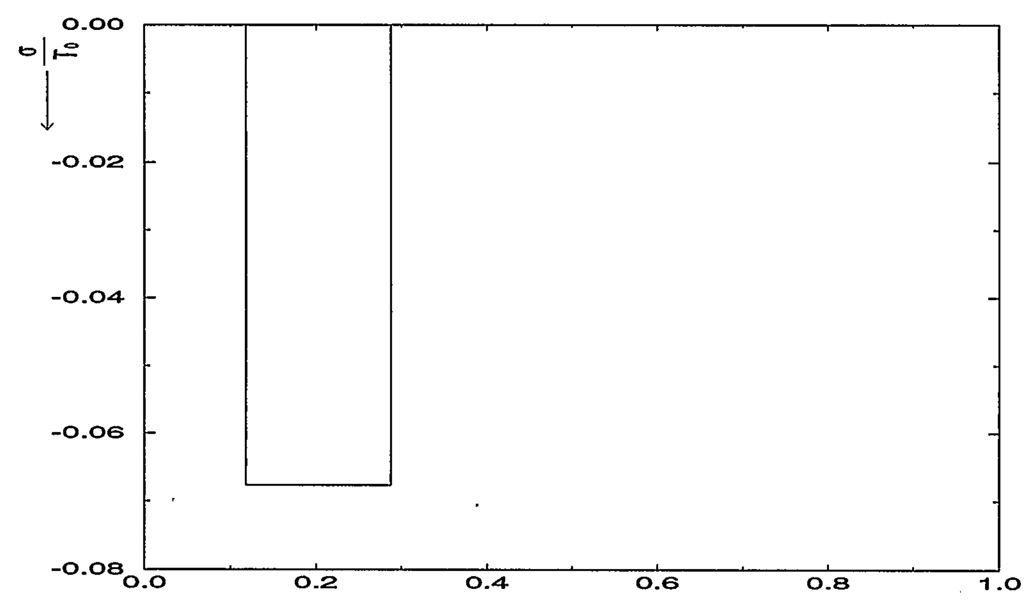


Fig.6.10 Numerical values of stress σ/T_0 against x
for a free boundary problem (a special case) at $t = 0.25$

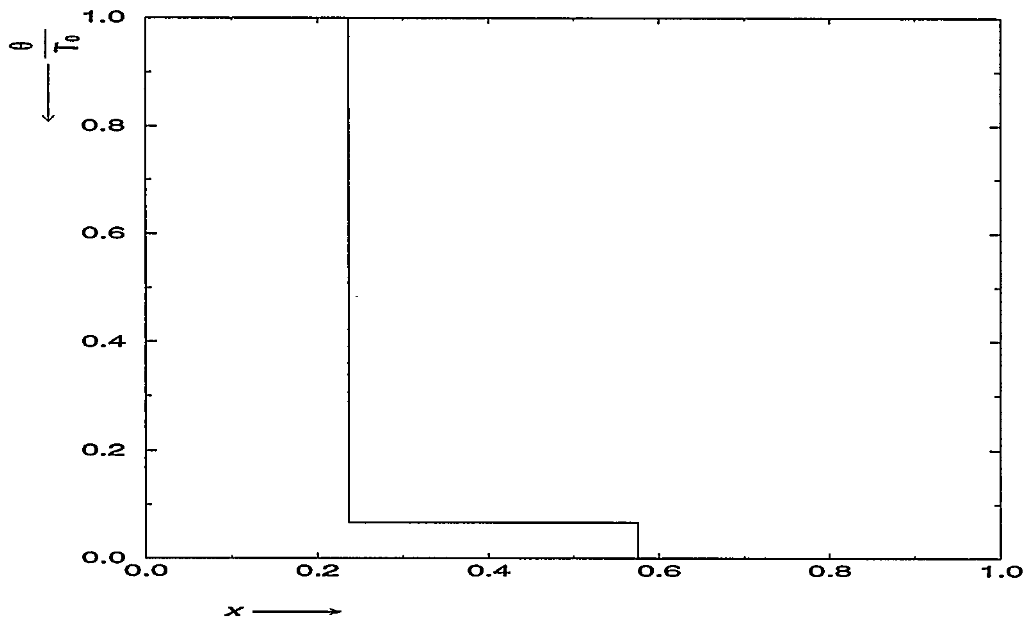


Fig.6.11 Numerical values of temperature θ/T_0 against x
for a free boundary problem (a special case) at $t = 0.5$

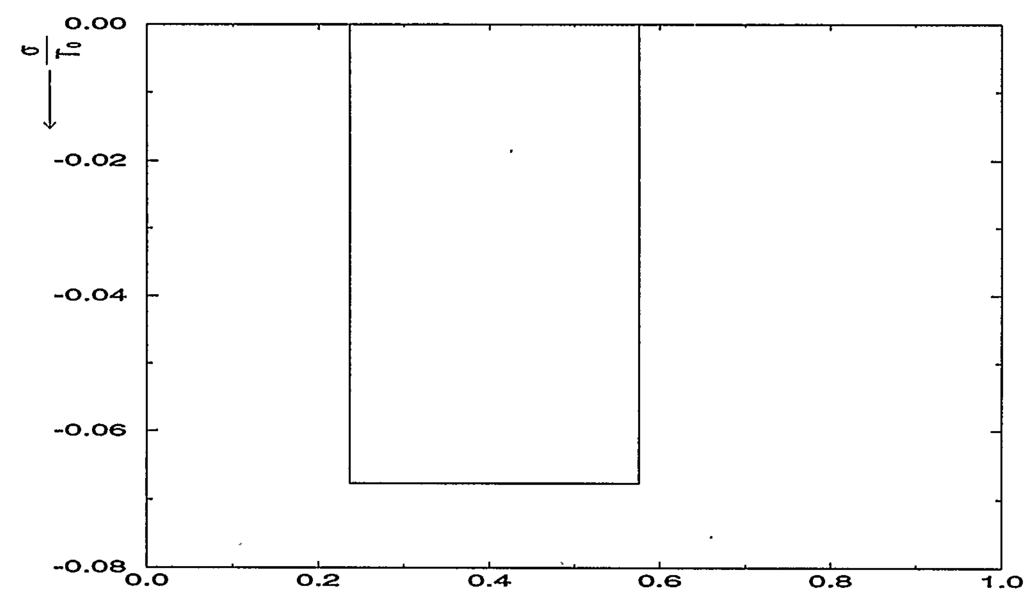


Fig.6.12 Numerical values of stress σ/T_0 against x
for a free boundary problem (a special case) at $t = 0.5$

Chapter 7

One-dimensional Thermal Shock Problem –Fixed Boundary

Now, we consider the problem of an elastic half-space $x \geq 0$ which is kept at a uniform reference temperature, with its plane boundary held rigidly fixed and subjected to sudden heating such that the initial and boundary conditions are

$$u(x,0) = \dot{u}(x,0) = \theta(x,0) = \dot{\theta}(x,0) = 0,$$
$$u(0,t) = 0, \quad \theta(0,t) = T_0 H(t),$$

and the regularity conditions are

$$(u(x,t),\ \theta(x,t)) \longrightarrow 0, \quad \text{as} \quad x \longrightarrow \infty, \quad t > 0, \tag{7.1}$$

where $T_0 \neq 0$ is the uniform temperature applied to the boundary and $H(t)$ is the Heaviside unit step function.

The boundary and regularity conditions may be transformed to

$$\bar{u}(x,p) = 0, \qquad \text{at} \qquad x = 0, \tag{7.2}$$
$$\bar{\theta}(x,p) = \frac{T_0}{p}, \qquad \text{at} \qquad x = 0, \tag{7.3}$$
$$\left\{\bar{u}(x,p), \bar{\theta}(x,p)\right\} \longrightarrow 0, \qquad \text{as} \qquad x \longrightarrow \infty. \tag{7.4}$$

The expressions for $\bar{u}$ and $\bar{\theta}$ given by equations (5.25) and (5.26) satisfy the regularity conditions (7.4), and the boundary conditions (7.2) and (7.3), will be satisfied, if

$$\begin{aligned} A_1 &= \frac{T_0}{p}\frac{\lambda_2}{B_1\lambda_2 - B_2\lambda_1}, \\ A_2 &= -\frac{T_0}{p}\frac{\lambda_1}{B_1\lambda_2 - B2\lambda_1}. \end{aligned}$$

Substituting for A_1 and A_2 into equations (5.25)-(5.27), we find that

$$\bar{u} = -\frac{T_0\lambda_1\lambda_2}{p(B_1\lambda_2 - B_2\lambda_1)}\{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\}, \quad (7.5)$$

$$\bar{\theta} = \frac{T_0}{p(B_1\lambda_2 - B_2\lambda_1)}\{B_1\lambda_2\exp(-\lambda_1 x) - B_2\lambda_1\exp(-\lambda_2 x)\}, \quad (7.6)$$

$$\bar{\sigma} = \frac{\rho\alpha_1 p T_0}{B_1\lambda_2 - B_2\lambda_1}\{\lambda_2\exp(-\lambda_1 x) - \lambda_1\exp(-\lambda_2 x)\}. \quad (7.7)$$

Here, we use the small time approximation discussed in Chapter 6, to find the approximate results for this problem.

7.1 General Case

Substituting for B_1 and B_2 from equation (5.28), we find that

$$B_1\lambda_2 - B_2\lambda_1 = (C_1\lambda_1^2 - C_2p^2)\lambda_2 - (C_1\lambda_2^2 - C_2p^2)\lambda_1.$$

Now expanding the above expression in powers of $1/p$, using equation (6.9) for values of λ_1 and λ_2, we obtain

$$\begin{aligned} B_1\lambda_2 - B_2\lambda_1 &\approx (C_1b_{10}^2 - C_2)b_{20}p^{\frac{5}{2}} + 2C_1b_{10}b_{11}b_{20}p^{\frac{3}{2}} + C_1b_{11}^2b_{20}p^{\frac{1}{2}} + C_2b_{10}p^3 \\ &\quad -(C_1b_{20}^2b_{10} - C_2b_{11})p^2 - C_1b_{11}b_{20}^2p \\ &\approx C_2b_{10}p^3(1 + \frac{(C_1b_{10}^2 - C_2)b_{20}}{C_2b_{10}}p^{-\frac{1}{2}} - \frac{C_1b_{20}^2b_{10} - C_2b_{11}}{C_2b_{10}}p^{-1}), \end{aligned}$$

and hence

$$
\begin{aligned}
\frac{1}{B_1\lambda_2 - B_2\lambda_1} &\approx \frac{1}{p^3 C_2 b_{10}}\Big(1 - \frac{(C_1 b_{10}^2 - C_2) b_{20}}{C_2 b_{10}} p^{-\frac{1}{2}} + \frac{C_1 b_{20}^2 b_{10} - C_2 b_{11}}{C_2 b_{10}} p^{-1} \\
&\quad + \frac{(C_1 b_{10}^2 - C_2)^2 b_{20}^2}{C_2^2 b_{10}^2} p^{-1}\Big) \\
&\approx \frac{1}{p^3 C_2 b_{10}}(1 - D_1 p^{-\frac{1}{2}} - D_2 p^{-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 &= \frac{(C_1 b_{10}^2 - C_2) b_{20}}{C_2 b_{10}}, \\
D_2 &= \frac{b_{11}}{b_{10}} + \frac{C_1}{C_2} {b_{20}}^2 - \frac{{b_{20}}^2}{{b_{10}}^2} - \frac{{C_1}^2}{{C_2}^2} {b_{10}}^2 {b_{20}}^2.
\end{aligned}
$$

Using the same method of expansion, we find that

$$
\begin{aligned}
\frac{\lambda_1\lambda_2}{p(B_1\lambda_2 - B_2\lambda_1)} &\approx \frac{1}{p^4 C_2 b_{10}}(1 - D_1 p^{-\frac{1}{2}} - D_2 p^{-1})(b_{10} b_{20} p^{\frac{3}{2}} + b_{11} b_{20} p^{\frac{1}{2}}) \\
&\approx \frac{1}{C_2 b_{10}}(b_{10} b_{20} p^{-\frac{5}{2}} - D_1 b_{10} b_{20} p^{-3} - (D_2 b_{10} b_{20} - b_{11} b_{20}) p^{-\frac{7}{2}}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{B_1\lambda_2}{p(B_1\lambda_2 - B_2\lambda_1)} &\approx \frac{1}{p^4 C_2 b_{10}}(1 - D_1 p^{-\frac{1}{2}} - D_2 p^{-1}) \cdot \\
&\quad [(C_1 b_{10}^2 - C_2) b_{20} p^{\frac{5}{2}} + 2 C_1 b_{10} b_{11} b_{20} p^{\frac{3}{2}} + C_1 b_{11}^2 b_{20} p^{\frac{1}{2}}] \\
&\approx \frac{1}{C_2 b_{10}}(b_{20}(C_1 b_{10}^2 - C_2) p^{-\frac{3}{2}} + (C_2 - C_1 b_{10}^2) D_1 b_{20} p^{-2} \\
&\quad + (2 C_1 b_{10} b_{11} - (C_1 b_{10}^2 - C_2) D_2) b_{20} p^{-\frac{5}{2}}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{B_2\lambda_1}{B_1\lambda_2 - B_2\lambda_1} &\approx \frac{1}{p^4 C_2 b_{10}}(1 - D_1 p^{-\frac{1}{2}} - D_2 p^{-1}) \cdot \\
&\quad [-C_2 b_{10} p^3 + (C_1 b_{20}^2 b_{10} - C_2 b_{11}) p^2 + C_1 b_{11} b_{20}^2 p] \\
&\approx \frac{1}{C_2 b_{10}}[-C_2 b_{10} p^{-1} + D_1 C_2 b_{10} p^{-\frac{3}{2}}
\end{aligned}
$$

$$+(b_{10}C_2D_2 + C_1b_{10}b_{20}^2 - C_2b_{11})p^{-2}],$$

$$\begin{aligned}
\frac{\lambda_2 p}{B_1\lambda_2 - B_2\lambda_1} &\approx \frac{1}{p^2C_2b_{10}}(1 - D_1p^{-\frac{1}{2}} - D_2p^{-1})b_{20}p^{\frac{1}{2}} \\
&\approx \frac{1}{C_2b_{10}}(b_{20}p^{-\frac{3}{2}} - D_1b_{20}p^{-2} - D_2b_{20}p^{-\frac{5}{2}}),
\end{aligned}$$

$$\begin{aligned}
\frac{\lambda_1 p}{B_1\lambda_2 - B_2\lambda_2} &\approx \frac{1}{p^2C_2b_{10}}(1 - D_1p^{-\frac{1}{2}} - D_2p^{-1})(b_{10}p + b_{11}) \\
&\approx \frac{1}{C_2b_{10}}(b_{10}p^{-1} - b_{10}D_1p^{-\frac{3}{2}} + (-b_{10}D_2 + b_{11})p^{-2}).
\end{aligned}$$

Substituting the results in equations (7.5)-(7.7), we find that

$$\begin{aligned}
\bar{u} &\approx \frac{T_0}{C_2b_{10}}\left(E_1p^{-\frac{5}{2}} + E_2p^{-3} + E_3p^{-\frac{7}{2}}\right) \\
&\quad \left\{\exp(-b_{10}px - b_{11}x) - \exp(-b_{20}p^{\frac{1}{2}}x)\right\}, && (7.8) \\
\bar{\theta} &\approx \frac{T_0}{C_2b_{10}}\{(F_1p^{-\frac{3}{2}} + F_2p^{-2} + F_3p^{-\frac{5}{2}})\exp(-b_{10}px - b_{11}x) && (7.9) \\
&\quad - (F_4p^{-1} + F_5p^{-\frac{3}{2}} + F_6p^{-2})\exp(-b_{20}p^{\frac{1}{2}}x)\}, \\
\bar{\sigma} &\approx \frac{\rho\alpha_1T_0}{C_2b_{10}}\{(G_1p^{-\frac{3}{2}} + G_2p^{-2} + G_3p^{-\frac{5}{2}})\exp(-b_{10}px - b_{11}x) && (7.10) \\
&\quad - (G_4p^{-1} + G_5p^{-\frac{3}{2}} + G_6p^{-2})\exp(-b_{20}p^{\frac{1}{2}}x)\},
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= -b_{10}b_{20}, \\
E_2 &= b_{10}b_{20}D_1, \\
E_3 &= D_2b_{10}b_{20} - b_{11}b_{20}, \\
F_1 &= b_{20}(C_1b_{10}^2 - C_2), \\
F_2 &= (C_2 - C_1b_{10}^2)D_1b_{20},
\end{aligned}$$

$$
\begin{aligned}
F_3 &= (2C_1b_{10}b_{11} - (C_1b_{10}^2 - C_2)D_2)b_{20}, \\
F_4 &= -C_2b_{10}, \\
F_5 &= D_1C_2b_{10}, \\
F_6 &= b_{10}C_2D_2 + C_1b_{10}b_{20}^2 - C_2b_{11}, \\
G_1 &= b_{20}, \\
G_2 &= -b_{20}D_1, \\
G_3 &= -b_{20}D_2, \\
G_4 &= b_{10}, \\
G_5 &= -b_{10}D_1, \\
G_6 &= -b_{10}D_2 + b_{11}.
\end{aligned}
$$

Taking the inverse Laplace transform of equations (7.8)-(7.10), we obtain

$$
\begin{aligned}
u(x,t) &\approx \frac{T_0}{C_2b_{10}}\sum_{j=1}^{3} E_j\{\exp(-b_{11}x)\int_0^t \delta(t - b_{10}x - z)\frac{z^{(1+\frac{j}{2})}}{\Gamma(2+\frac{j}{2})}dz \\
&\quad - (4t)^{(1+\frac{j}{2})} i^{(2+j)} erfc(\frac{b_{20}x}{2\sqrt{t}})\}, \qquad (7.11) \\
\theta(x,t) &\approx \frac{T_0}{C_2b_{10}}\{\exp(-b_{11}x)\sum_{j=1}^{3} F_j \int_0^t \delta(t - b_{10}x - z)\frac{z^{\frac{j}{2}}}{\Gamma(1+\frac{j}{2})}dz \\
&\quad - \sum_{j=4}^{6} F_j (4t)^{\frac{j-4}{2}} i^{(j-4)} erfc(\frac{b_{20}x}{2\sqrt{t}})\}, \qquad (7.12) \\
\sigma(x,t) &\approx \frac{\rho\alpha_1T_0}{C_2b_{10}}\{\exp(-b_{11}x)\sum_{j=1}^{3} G_j \int_0^t \delta(t - b_{10}x - z)\frac{z^{\frac{j}{2}}}{\Gamma(1+\frac{j}{2})}dz \\
&\quad - \sum_{j=4}^{6} G_j (4t)^{\frac{j-4}{2}} i^{(j-4)} erfc(\frac{b_{20}x}{2\sqrt{t}})\}. \qquad (7.13)
\end{aligned}
$$

Evaluation of the integrals in equations (7.11)-(7.13) leads us to

$$
u(x,t) \approx \frac{T_0}{C_2b_{10}}\sum_{j=1}^{3} E_j\{\exp(-X_{11})H(t - X_{10})\frac{(t - X_{10})^{(1+\frac{j}{2})}}{\Gamma(2+\frac{j}{2})}
$$

$$- (4t)^{(1+\frac{j}{2})} i^{(2+j)} erfc(\frac{X_{20}}{2\sqrt{t}})\}, \tag{7.14}$$

$$\theta(x,t) \approx \frac{T_0}{C_2 b_{10}} \{\exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} F_j \frac{(t - X_{10})^{\frac{j}{2}}}{\Gamma(1+\frac{j}{2})}$$

$$- \sum_{j=4}^{6} F_j (4t)^{\frac{j-4}{2}} i^{(j-4)} erfc(\frac{X_{20}}{2\sqrt{t}})\}, \tag{7.15}$$

$$\sigma(x,t) \approx \frac{\rho \alpha_1 T_0}{C_2 b_{10}} \{\exp(-X_{11}) H(t - X_{10}) \sum_{j=1}^{3} G_j \frac{(t - X_{10})^{\frac{j}{2}}}{\Gamma(1+\frac{j}{2})}$$

$$- \sum_{j=4}^{6} G_j (4t)^{\frac{j-4}{2}} i^{(j-4)} erfc(\frac{X_{20}}{2\sqrt{t}})\}, \tag{7.16}$$

where

$$X_{11} = b_{11} x \; X_{10} = b_{10} x \; X_{20} = b_{20} x.$$

For obtaining the numerical values of θ and σ, we have used the same numerical values of the parameters as given in equations (6.16).

The numerical values of the temperature and stress for $t = 0.02,\ 0.04,\ 0.06$ are given in Tables 7.1-7.3. And these values are displayed in Figures 7.1-7.6.

The explanation for sharp corners in the graph of σ/T_0 in Figures 7.2, 7.4 and 7.6 is the same as given on page 66.

Table 7.1: Numerical values of temperature and stress for $t = 0.02$

x	θ/T_0	σ/T_0
.0000	.9988	-.1755
.0500	.6633	-.1734
.1000	.3675	-.0949
.1500	.1728	-.0442
.2000	.0682	-.0174
.2500	.0224	-.0057
.3000	.0061	-.0015
.3500	.0014	-.0003
.4000	.0003	-.0001
.4500	.0000	.0000
.5000	.0000	.0000
.5500	.0000	.0000
.6000	.0000	.0000
.6500	.0000	.0000
.7000	.0000	.0000
.7500	.0000	.0000
.8000	.0000	.0000
.8500	.0000	.0000
.9000	.0000	.0000
.9500	.0000	.0000

Table 7.2: Numerical values of temperature and stress for $t = 0.04$

x	θ/T_0	σ/T_0
.0000	.9971	-.1388
.0500	.7743	-.2130
.1000	.5345	-.1442
.1500	.3412	-.0907
.2000	.2006	-.0527
.2500	.1081	-.0282
.3000	.0533	-.0138
.3500	.0240	-.0062
.4000	.0098	-.0025
.4500	.0036	-.0009
.5000	.0012	-.0003
.5500	.0004	-.0001
.6000	.0001	.0000
.6500	.0000	.0000
.7000	.0000	.0000
.7500	.0000	.0000
.8000	.0000	.0000
.8500	.0000	.0000
.9000	.0000	.0000
.9500	.0000	.0000

Table 7.3: Numerical values of temperature and stress for $t = 0.06$

x	θ/T_0	σ/T_0
.0000	.9952	-.1070
.0500	.8052	-.1765
.1000	.6225	-.1756
.1500	.4443	-.1229
.2000	.3006	-.0818
.2500	.1923	-.0517
.3000	.1161	-.0309
.3500	.0661	-.0174
.4000	.0353	-.0093
.4500	.0178	-.0046
.5000	.0084	-.0022
.5500	.0037	-.0009
.6000	.0015	-.0004
.6500	.0006	-.0001
.7000	.0002	-.0001
.7500	.0001	.0000
.8000	.0000	.0000
.8500	.0000	.0000
.9000	.0000	.0000
.9500	.0000	.0000

7.2 A Special Case

Because the constitutive equations (5.4)-(5.6) include a diffusion type of equation for heat conductivity, generally, this theory predicts a infinite speed for the heat propagation. But for a special case, when $k^* \gg k$, that is, $\alpha_4 \approx 0, \quad b_1 \approx 0$, expression (5.22) becomes

$$\lambda_i = c_i p, \qquad i = 1, 2, \tag{7.17}$$

where

$$c_i = \sqrt{\frac{b_2 + (-1)^{i+1}\sqrt{b_2^2 - 4b_3}}{2}}.$$

Now from equations (6.28) and (7.17), we have

$$\begin{aligned} \frac{1}{B_1\lambda_2 - B_2\lambda_1} &= \frac{1}{(C_2\lambda_1^2 - C_2p^2)\lambda_2 - (C_1\lambda_2^2 - C_2p^2)\lambda_1} \\ &= \frac{1}{(C_1c_1^2c_2 - C_2c_2 - C_1c_1c_2^2 + C_2c_1)p^3}, \\ &= \frac{M}{p^3}, \end{aligned} \tag{7.18}$$

where

$$M = \frac{1}{C_1c_1^2c_2 - C_2c_2 - C_1c_1c_2^2 + C_2c_1}.$$

Substituting from equations (7.17) and (7.18) in equations (7.5)-(7.7), we obtain

$$\bar{u} = -\frac{T_0Mc_1c_2}{p^2}\{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\}, \tag{7.19}$$

$$\bar{\theta} = \frac{T_0M}{p}\{M_1\exp(-\lambda_1 x) - M_2\exp(-\lambda_2 x)\}, \tag{7.20}$$

$$\bar{\sigma} = \frac{\rho\alpha_1T_0M}{p}\{c_1\exp(-\lambda_1 x) - c_2\exp(-\lambda_2 x)\}, \tag{7.21}$$

where

$$M_1 = (C_1c_1^2 - C_2)c_2,$$
$$M_2 = (C_1c_2^2 - C_2)c_1.$$

Taking the inverse Laplace transforms of equations (7.19)-(7.21), we find

$$u = -\frac{T_0c_1c_2M}{2}\{H(t-c_1x)(t-c_1x) - H(t-c_2x)(t-c_2x)\}, \quad (7.22)$$
$$\theta = T_0M\{M_1H(t-c_1x) - M_2H(t-c_2x)\}, \quad (7.23)$$
$$\sigma = \rho\alpha_1T_0M\{c_1H(t-c_1x) - c_2H(t-c_2x)\}. \quad (7.24)$$

For obtaining the numerical values of θ and σ, we have used the same numerical values of the parameters as given in equations (6.16). The numerical values of the temperature and stress for $t = 0.15,\ 0.25,\ 0.50$ for various of x are given in Tables 7.4-7.6. The jumps in temperature and stress fields occur at $x = x_1,\ x_2$ as given below:

t	0.15	0.25	0.50
x_1	.0711	.1186	.2371
x_2	.1728	.2882	.5760

The numerical values of temperature and stress are displayed in Figures 7.7-7.12.

Table 7.4: Numerical values of temperature and stress for $t = 0.15$

x	u/T_0	σ/T_0
.0000	1.0000	-.0883
x_1^-	1.0000	-.0883
x_1^+	.1481	-.1501
.1000	.1481	-.1501
x_2^-	.1481	-.1501
x_2^+	.0000	.0000
.2000	.0000	.0000
.3000	.0000	.0000
.4000	.0000	.0000
.5000	.0000	.0000
.6000	.0000	.0000
.7000	.0000	.0000

Table 7.5: Numerical values of temperature and stress for $t = 0.25$

x	u/T_0	σ/T_0
.0000	1.0000	-.0883
.1000	1.0000	-.0883
x_1^-	1.0000	-.0883
x_1^+	.1481	-.1501
.2000	.1481	-.1501
x_2^-	.1481	-.1501
x_2^+	.0000	.0000
.3000	.0000	.0000
.4000	.0000	.0000
.5000	.0000	.0000
.6000	.0000	.0000
.7000	.0000	.0000

Table 7.6: Numerical values of temperature and stress for $t = 0.5$

x	u/T_0	σ/T_0
.0000	1.0000	-.0883
.1000	1.0000	- .0883
.2000	1.0000	- .0883
x_1^-	1.0000	-.0883
x_1^+.	1481	-.1501
.3000	.1481	-.1501
.4000	.1481	-.1501
.5000	.1481	-.1501
x_2^-	.1481	-.1501
x_2^+	.0000	.0000
.6000	.0000	.0000

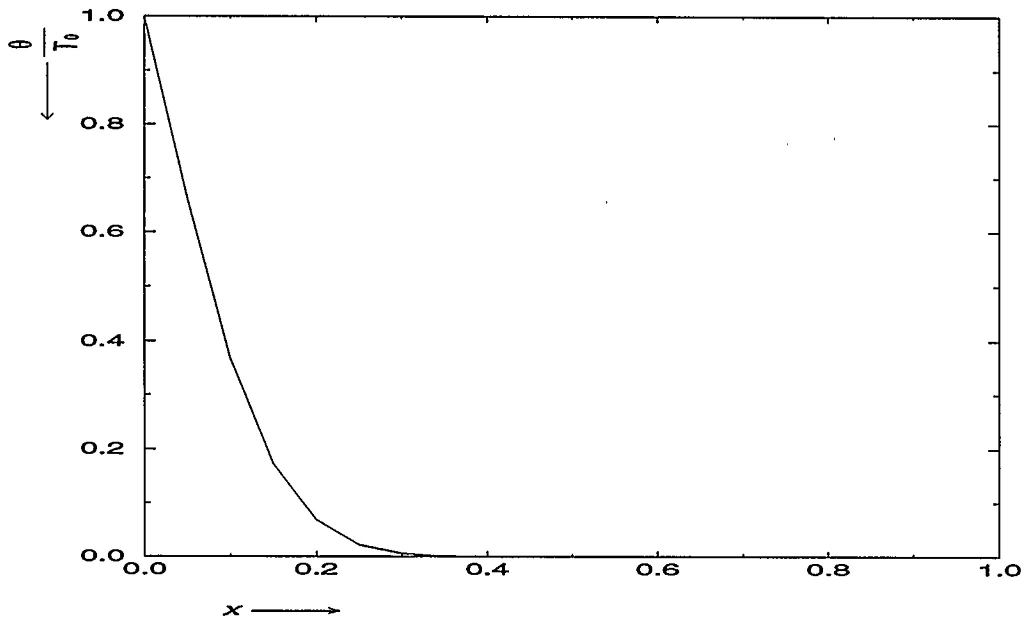


Fig.7.1 Numerical values of temperature θ/T_0 against x
for a fixed boundary problem at $t = 0.02$

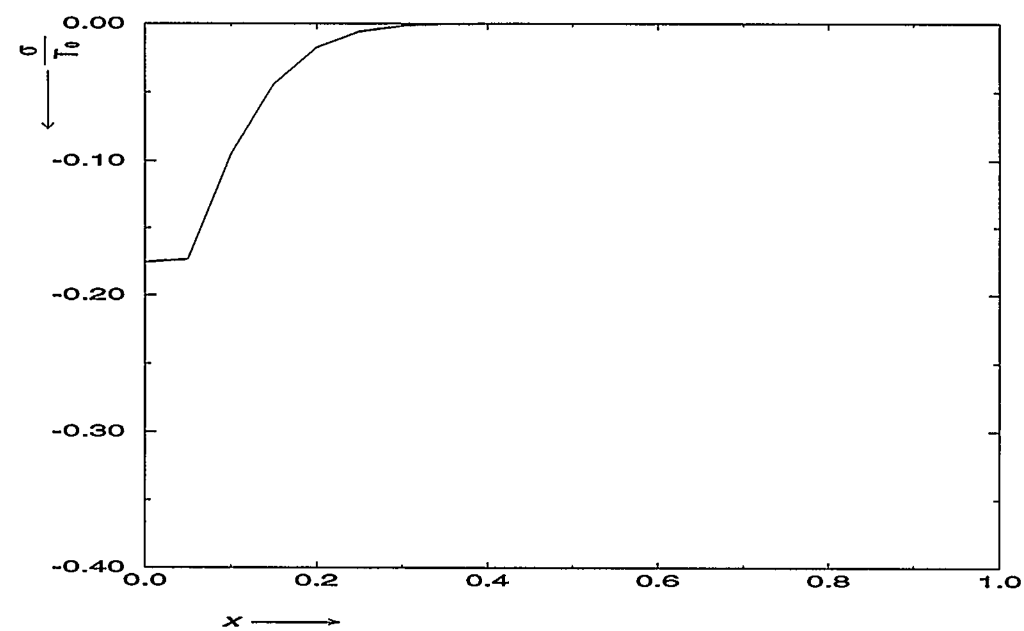


Fig.7.2 Numerical values of stress σ/T_0 against x
for a fixed boundary problem at $t = 0.02$

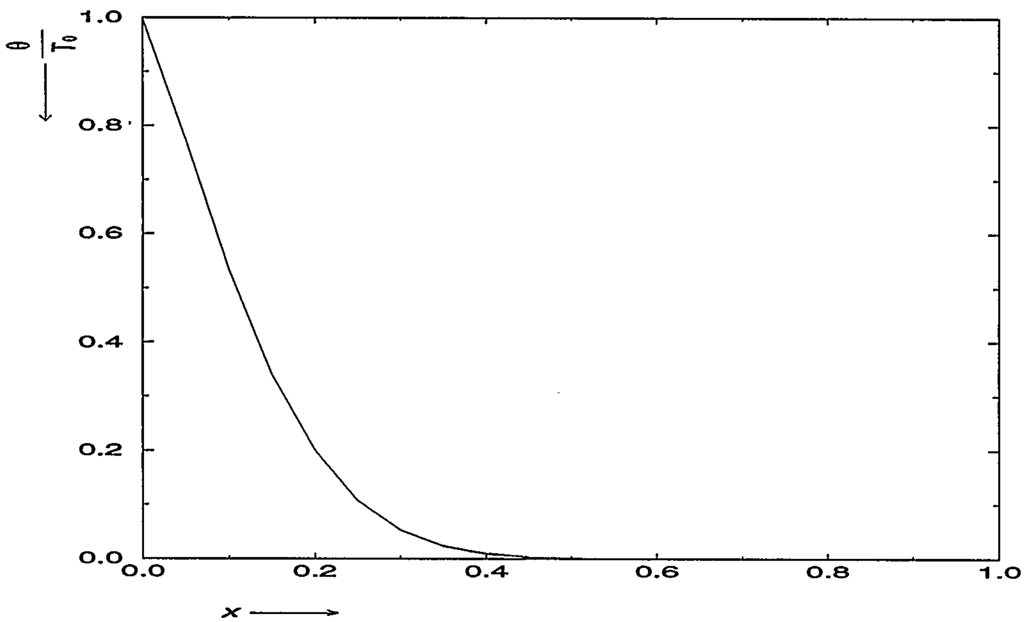


Fig.7.3 Numerical values of temperature θ/T_0 against x for a fixed boundary problem at $t = 0.04$

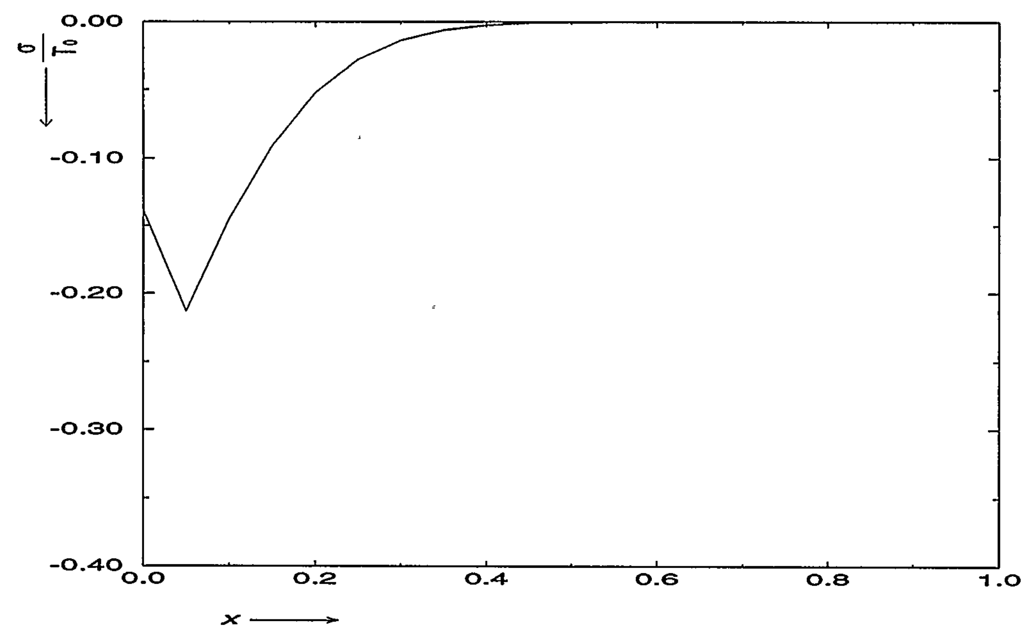


Fig.7.4 Numerical values of stress σ/T_0 against x for a fixed boundary problem at $t = 0.04$

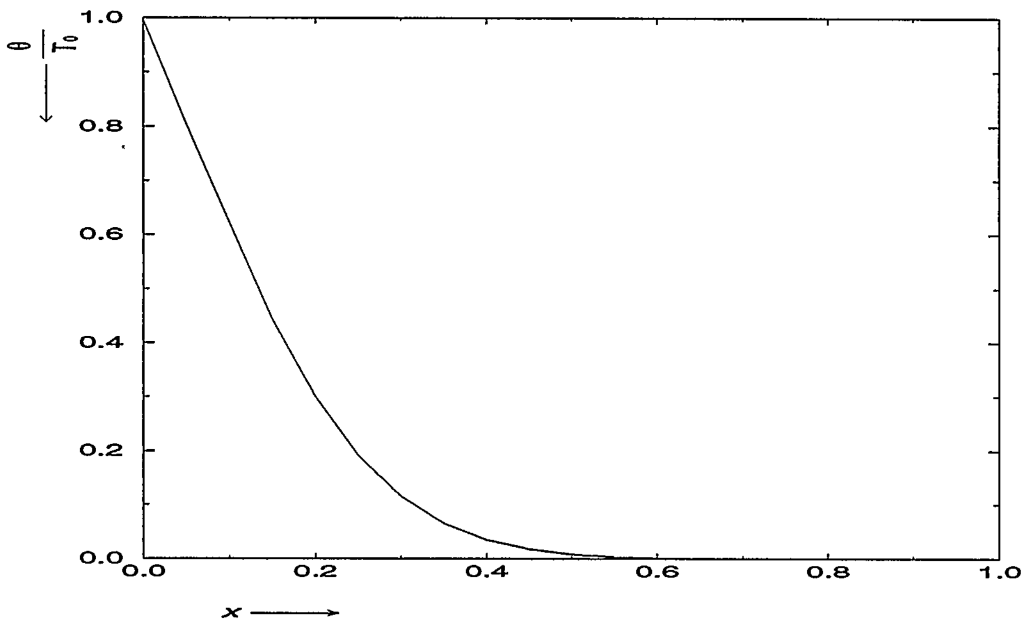


Fig.7.5 Numerical values of temperature θ/T_0 against x for a fixed boundary problem at $t = 0.06$

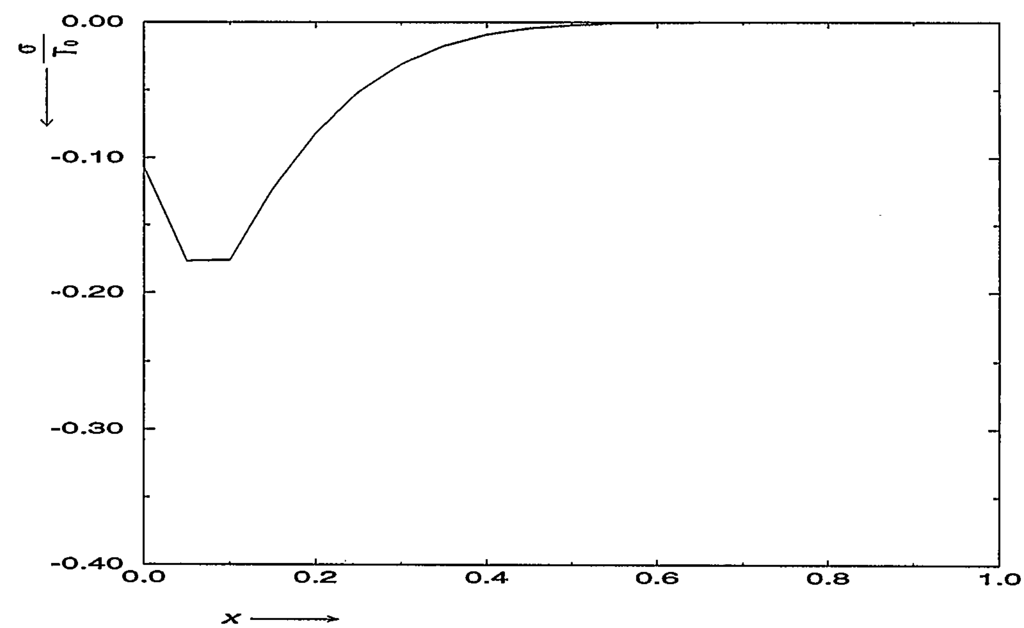


Fig.7.6 Numerical values of stress σ/T_0 against x for a fixed boundary problem at $t = 0.06$

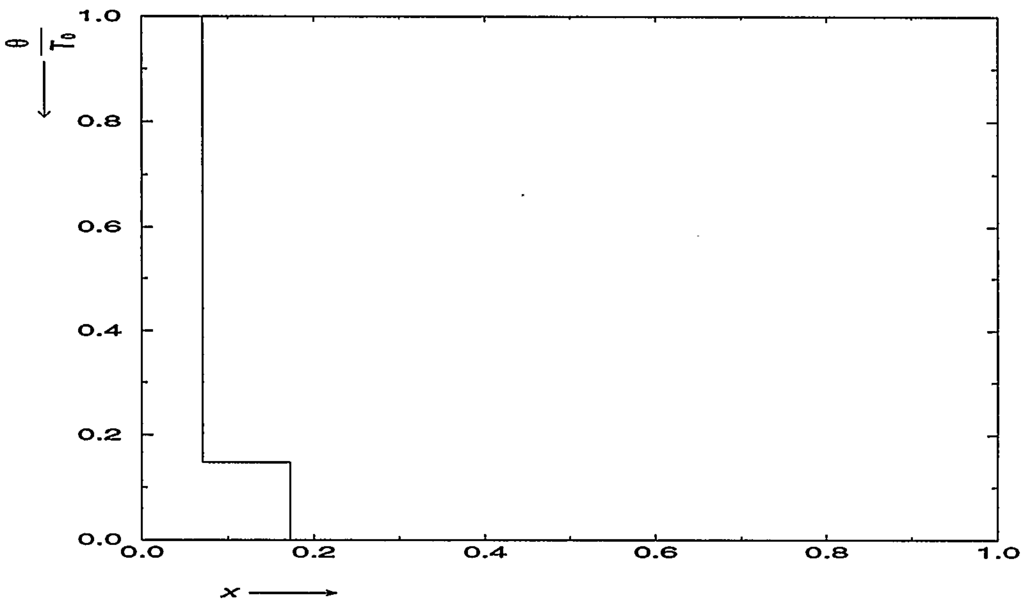


Fig.7.7 Numerical values of temperature θ/T_0 against x for a fixed boundary problem (a special case) at $t = 0.15$

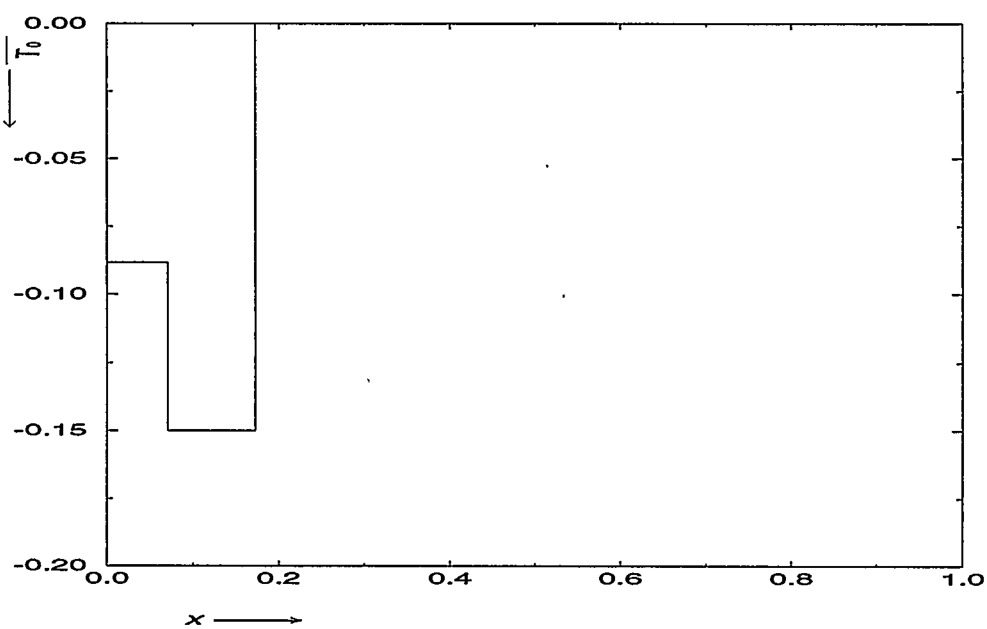


Fig.7.8 Numerical values of stress σ/T_0 against x for a fixed boundary problem (a special case) at $t = 0.15$

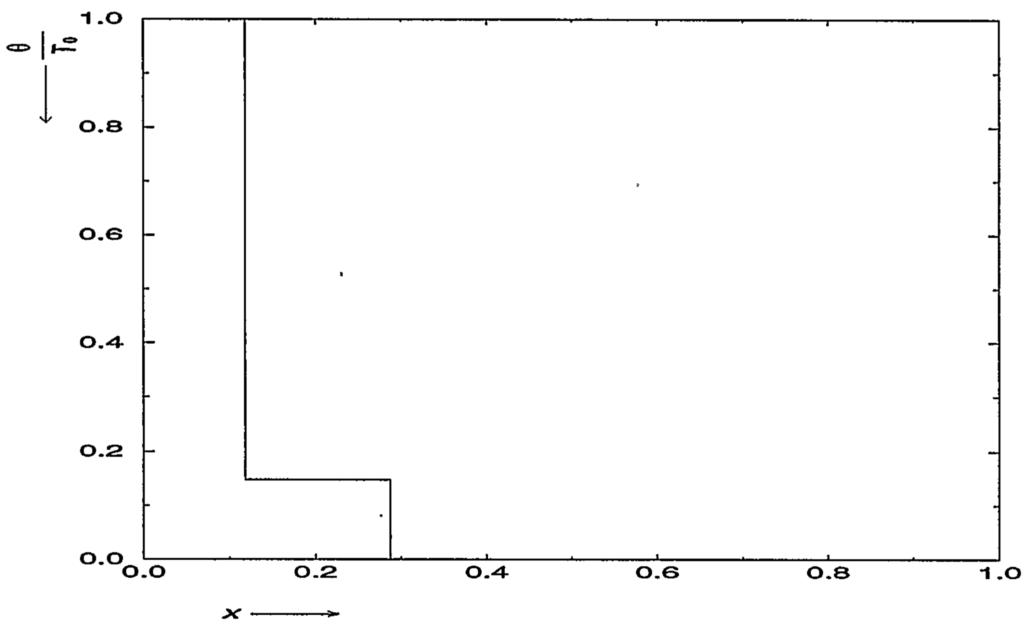


Fig.7.9 Numerical values of temperature θ/T_0 against x for a fixed boundary problem (a special case) at $t = 0.25$

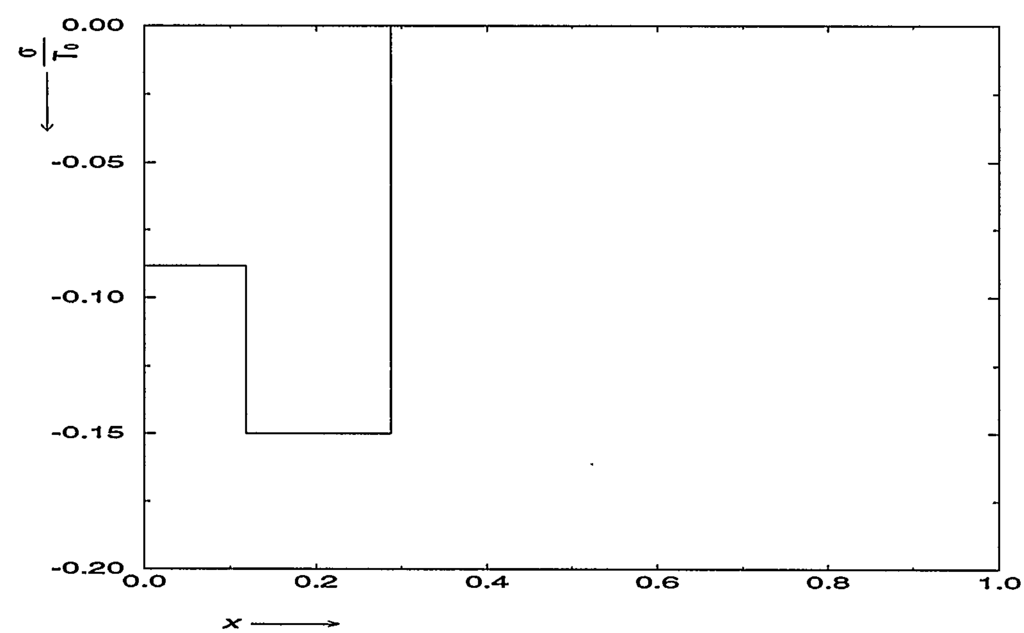


Fig.7.10 Numerical values of stress σ/T_0 against x for a fixed boundary problem (a special case) at $t = 0.25$

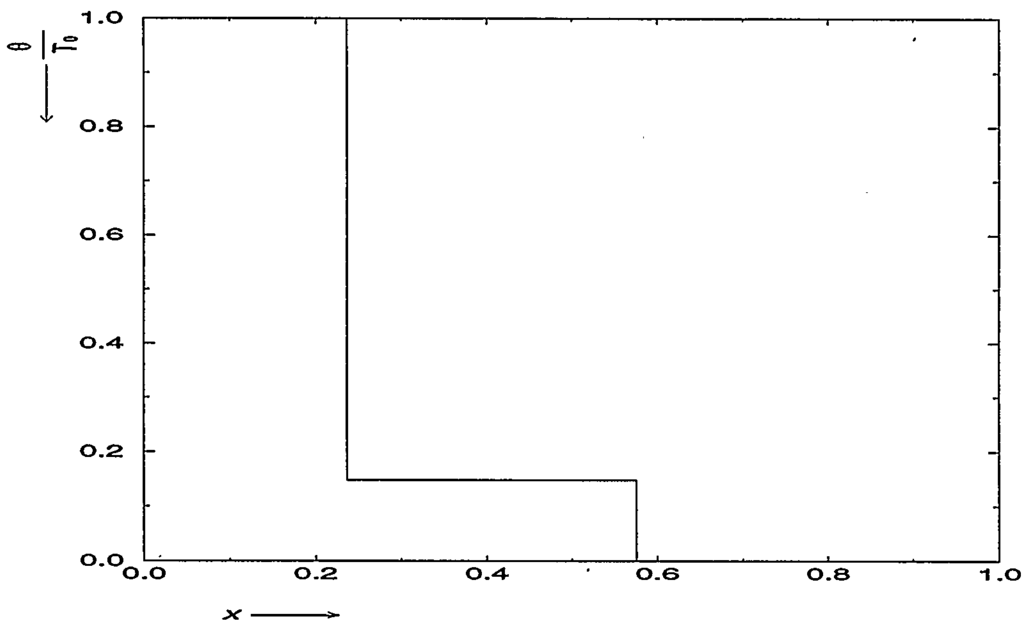


Fig.7.11 Numerical values of temperature θ/T_0 against x for a fixed boundary problem (a special case) at $t = 0.5$

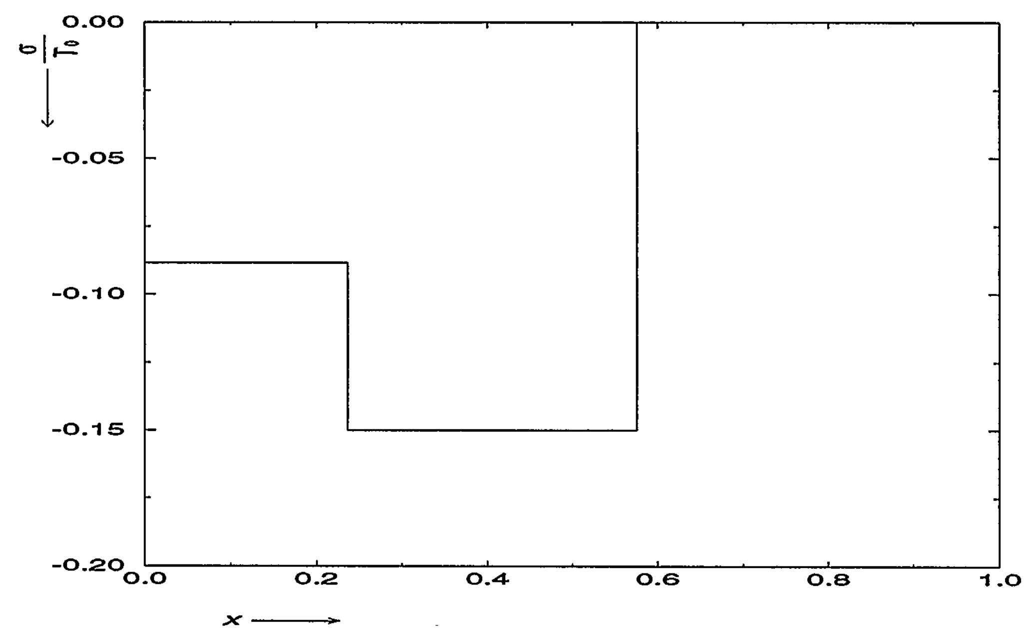


Fig.7.12 Numerical values of stress σ/T_0 against x for a fixed boundary problem (a special case) at $t = 0.5$

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Appendix A

The purpose of this appendix is to obtain the conditions which are both n ecessary and sufficient for the requirement that an equation of the form (4.6) be satisfied identically being independen t of these rate quantities. For this purpose, we first observe that an equation of the type (4.6) can be recast in the form

$$\sum_{n=i}^{N} a_n y_n + a = 0. \tag{A.1}$$

Suppose that (A.1) holds as an identity for all arbitrary values of y_n in some range of values which includes the values $y_n = 0$, where the coefficients $a_1, ..., a_n$ are functions of other variables independent of all y_n. Then, (A.1) is a linear identity in y_n and it follows that necessary and sufficient conditions for (A.1) to hold are

$$a = 0, \; a_n = 0 \; (n = 1, 2, ...N). \tag{A.2}$$

The variables in (A.1) may represent scalars, or a_n, y_n may be vectors with $a_n y_n$ as scalar product of the vectors.