UNIVERSITY OF CALGARY

What is a Differential Partial Combinatory Algebra?

by

Jonathan Gallagher

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF COMPUTER SCIENCE

CALGARY, ALBERTA

JANUARY, 2012

Abstract

In this thesis we combine Turing categories with Cartesian left additive restriction categories and again with differential restriction categories. The result of the first combination is a new structure which models nondeterministic computation. The result of the second combination is a structure which models the notion of linear resource consumption. We also study the structural background required to understand what new features Turing structure should have in light of addition and differentiation – most crucial to this development is the way in which idempotents split. For the combination of Turing categories with Cartesian left additive restriction categories we will also provide a model.

Acknowledgements

For this thesis, my deepest thanks and gratitude goes to Professor Robin Cockett. From finding opportunities for me to speak at conferences and workshops to helping me make my ideas into coherent thoughts to giving me a push back on track when I had failed – prof. Cockett has played a very important role in my development as a graduate student.

Next, I want to thank Dr. Geoff Cruttwell "getting me going" in my studies, and helping me with writing.

I am grateful to Professor Clifton Cunningham and Professor Michael Jacobson for serving on my thesis committee.

Two special "thank you's" to Professor Pieter Hofstra and Professor Peter Selinger. Professor Hofstra supported me to attend and speak at the Semigroups and Categories workshop at the University of Ottawa, and Professor Selinger supported me to attend and speak at the Octoberfest category theory workshop in Dalhousie University.

Table of Contents

List of Tables

Chapter 1

Introduction

The purpose of this thesis is to propose and justify the definition of a *differential Turing category*, and to pave the way to understanding what a differential partial combinatory algebra should be. Partial combinatory algebras (PCA)s were introduced by Feferman [24] to provide an algebraic formulation of computability. Turing categories were introduced by Cockett and Hofstra [13] to provide a categorical formulation of computability. Turing categories are closely related to PCAs, and a major objective of this thesis is to define differential structure for Turing categories and PCAs which maintains these relationships. To start, we will review the differential λ-calculus and the resource λ-calculus to build a motivation for adding differential structure to Turing categories and PCAs.

The idea of adding a derivative to a notion of computability was pioneered by Ehrhard and Regnier [23]. They investigated an extension of the λ-calculus in which terms have a formal derivative they called differential substitution. Adding formal differentiation to the λ-calculus, in this manner, produces a system in which "resource sensitive" computations, in the sense of [5], can be modeled.

The relationship between the differential λ-calculus and resource sensitivity arose from a relationship which was observed between certain models of Girard's linear logic [26], and Milner's Π-calculus (for communicating processes) [38] and how the latter can simulate the λ-calculus.

Girard's linear logic treats propositions as resources, and thus it is intuitively connected to the semantics of resource and concurrency. Girard, in attempting to a find a model of linear logic based on linear algebra, proposed a model which used Banach spaces [25]. However, as Ehrhard pointed out, this model did not completely describe the exponentials of linear logic [22]. Ehrhard continued Girard's investigation, and discovered that a complete model could be obtained using *Kothe spaces.*

The maps of the coKleisli category of the exponential comonad of Kothe spaces are infinitely differential maps. As Cartesian closed categories are models of the simply typed λ-calculus, Ehrhard was able to interpret λ-terms as differentiable maps, and compute their derivatives. In [23], Ehrhard and Regnier took a syntactic approach to studying differentiable λ-terms by adding to the λ-calculus a commutative monoid structure and a formal derivative operator: they called the result the *differential λ-calculus.* Ehrhard and Regnier immediately realized that there was a connection to Boudol's resource λ-calculus (see [5]). They noted in [23]:

> Intuitions behind the differential λ-calculus and the λ-calculus with resources are very similar.

Boudol's *resource λ-calculus* began with Milner's Π-calculus. To show that the Π calculus is a complete programming language, Milner defined a simulation of the λ-calculus within the Π-calculus [38]. Milner's simulation has the property that it soundly models β-reduction, and that if two terms are not equal in the λ-calculus, then they are not equal under translation. However, the Π-calculus distinguishes some terms which, in all λ-contexts, are indistinguishable [5].

In [5], Boudol defined the resource λ-calculus, and proved that this calculus is precisely the target of Milner's simulation of the λ-calculus. In [6], Boudol *et al* refined the calculus, and began to consider its rewriting theory. The resource λ-calculus was then further refined by Pagani and Tranquilli to show that the rewriting system is confluent modulo some equations [42].

In the resource λ-calculus, the second argument of application is a bag of terms, and terms in this bag are either linear (available for exactly one use), or infinite $(\_)^!$ (can be used many times or discarded). Here is a simple example.

$$(\lambda x.x[x^!])[M, N^!] \xrightarrow{\beta} M[N^!] + N[M, N!].$$

The β-reduction will linearly substitute the term $[M, N^!]$ for x in $x[x^!]$. Since the first x being substituted into is linear, and there are two options, a choice must be made: either M is substituted or N is substituted. Since M is linear, when M is substituted, it is depleted from the bag, but since N is infinite it remains in the bag: at this point we have $(M[x^!])[[N^!]/x] + (N[x^!])[[M, N^!]/x]$. In this case, the x being substituted into is infinite, so the substitution is a normal substitution, and we arrive at the result $M[N^!]+N[M, N^!]$.

Ehrhard and Regnier had observed that the differential substitution of the differential λ-calculus acts exactly like the linear substitution of the resource λ-calculus. This correspondence, however, was not made precise until a categorical semantics became available to unify the syntaxes.

Blute *et al* [3] initiated the categorical semantics of the differential λ-calculus using *differential categories.* A differential category is a symmetric monoidal category with a comonad and a differential combinator; however, only maps of a certain type can be differentiated. A standard example of a differential category is Ehrhard's category of Kothe spaces.

The differential λ-calculus actually arose from the coKleisli category of the exponential comonad on the category of Kothe spaces. Blute *et al* [2] thus axoimatized the coKleisli category of a differential category to continue the investigation into categorical models of the differential λ-calculus. These *Cartesian differential categories* are categories in which every map has a derivative. However, they are not necessarily closed, so these categories still did not model the full differential λ-calculus.

In [7], Bucciarelli *et al* defined Cartesian closed differential categories which are the closed extension of Blute *et al*'s Cartesian differential categories. This allowed them to make a precise the connection between Ehrhard and Regnier's differential λ-calculus and Boudol's resource λ-calculus. More precisely, Bucciarelli *et al* showed that the simply typed differential λ-calculus can be interpreted in a Cartesian closed differential category. Further they showed that the differential λ-calculus is a retract of the resource λ-calculus; in particular, this means that every model of the differential λ-calculus is a model of the resource λ-calculus.

Manzonetto [36] was the first to explicitly consider the untyped differential λ-calculus. Using the fact that Cartesian closed categories with a reflexive object provide models of the untyped λ-calculus, Manzonetto proved that Cartesian closed differential categories with a *linear* reflexive object provide *sound* models of the untyped differential λ-calculus. Manzonetto did not provide a *completeness* theorem, but he did conjecture that when the additive structure is idempotent then one can obtain a complete model of the differential λ-calculus [37]. Manzonetto's results further led to speculation that all models will have idempotent addition.

The syntax of the differential λ-calculus forces application to be linear in the first argument. To obtain models, one needs a pairing combinator which not only provides a pair type, but also correctly lifts differential structure to the model. One of the main contributions of this thesis, contrary to what was widely believed by the community, is to show that this does not require the idempotent induced by pairing to be linear. Instead, it is shown that a more sophisticated condition must be satisfied. This fact allows for the possibility of models in which addition is not idempotent.

It should be pointed that a first step in exploring applicative systems that are linear in the first argument was taken by Carraro *et al* [8] who introduced resource combinatory algebras as an algebraic formulation of the purely linear fragment of resource λ-calculus. They assume application is linear instead of linear in the first argument; further, their system does not provide a model of computability. Thus their approach is incompatible with the content of this thesis.

1.1 Organization

Cockett and Hofstra [13] showed that studying PCAs is essentially the same as studying Turing categories, and it is the interplay between PCAs and Turing structure which provides the driving force behind this thesis. The thesis displays this relationship as first additive and then differential structure is added. The organization of this thesis, then, reflects a program used to ensure that Turing structure interacts with the additive and differential structure in a consistent and seamless manner.

This thesis is broken into three chapters corresponding to the levels at which Turing structure is introduced: Cartesian restriction structure, Cartesian left additive restriction structure, and differential restriction structure. Each of these chapters displays the same five stages: the introduction of the underlying categorical structure, investigation of simple slice structure, idempotent splitting, the development of a sound and complete term logic, and the introduction of a Turing structure and PCA.

The simple slice category is part of a crucial story involving closed restriction structure. The complete development of closed restriction structure is not included in this thesis in order to keep its length manageable. However, the way in which the structure of a category transfers to the simple slice category suffices, in conjunction with the results for total closed structure, to understand the properties that a structurally appropriate applicative system should have.

The main results of this thesis are directly tied to the idempotent completions in the various settings. For Cartesian left additive restriction categories, one must split the idempotents such that in the idempotent completion, the retraction preserves addition. For differential restriction categories, one must split the idempotents such that in the idempotent completion, the retraction is strongly linear. These observations have important implications for appropriate Turing structure in these settings.

The development of a sound and complete term logic makes it easier to see and reason about PCAs. The author used the term logic in developing many of the results in the thesis: particularly concerning PCAs and the pairing combinator. However, the main body of the thesis sticks to categorical notation. The term logic is included because, once one gets used to it, it is an essential tool in understanding these settings.

Finally, we describe Turing structure and PCAs for these categorical structures. The first standard result about Turing categories is that they arise from splitting certain idempotents. For the Turing structure to fit with the underlying categorical structure, the way Turing structure splits idempotents must match the way the free idempotent completion splits the idempotents. This requirement forces certain properties of application, and in particular, explains why one needs, for differential Turing categories, application to be linear in its first argument.. The second standard result about Turing structure is that it is equivalent to the computable maps of a partial combinatory algebra. This in turn requires the combinatory completeness of an appropriate applicative system. Developing the definition of Turing structure for Cartesian left additive and differential restriction categories in a way that all of these equivalences are maintained accounts for the subtleties involved in these structures.

1.2 Contributions

This thesis introduces and explores left-addditive Turing categories; thus, the main new contributions start in that chapter. Both theorem 3.3.1 which concernts the idempotent splitting for Cartesian left additive restriction categories, and theorem 3.5.1 which provides a recognition theorem for left additive Turing categories, are new to this thesis.

This thesis also provides a characterization of left additive PCAs via proposition 3.5.2, and provides a model in which an idempotent sum is not required: proposition 3.5.3 and corollary 3.5.1.

For differential restriction structure, the main contribution is theorem 4.3.1 which concerns the idempotent splitting for differential restriction categories. From this, the thesis also provides the definition of a differential Turing category 4.6.1.

Chapter 2

Background

2.1 Restriction Categories

Partial maps play an imporant role in computability theory where one studies functions that are computations – computations which need not terminate. An explicit structure that allows one to reason about partial maps is crucial. This thesis will use *restriction categories* [16] which were introduced by Cockett and Lack to provide a categorical structure for handling partiality.

A widely known structure for describing partial maps are the *partial map categories* of Robinson and Rosolini [41] which are constructed from a class of monics. One property that these monics must have is that the pullback along such a monic must exist and again be in the class of monics, and this is because composition is defined by pullback. This makes reasoning in partial map categories quite complex.

Cockett and Lack's restriction categories capture partiality using an idempotent which acts like the domain of definition of a map. Unlike the partial map category approach, restriction categories are equationally defined, and thus, simplify calculations. However, the two approaches are closely related, as every partial map category is a restriction category, while every restriction category is a full subcategory of a partial map category [16].

For a historical introduction to partial structures, see [39].

Definition 2.1.1. *A* **restriction category** *is a category where each map f has an assignment:*

$$\frac{f : A \longrightarrow B}{\overline{f} : A \longrightarrow A}$$

which satisfies the following axioms:

[**R.1**] $\overline{f}\,f = f$;

[**R.2**] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$;

[**R.3**] $\overline{f}\,\overline{g} = \overline{\overline{f}\,g}$;

[**R.4**] $f\overline{h} = \overline{fh}\,f$.

Note, that in this thesis, we will write composition diagrammatically.

The standard example of a restriction category is Par [16], the category of sets and partial functions. The $\overline{(\_)}$ combinator is defined as

$$\overline{f}\,(x) = \begin{cases} x & f(x)\downarrow \\ \uparrow & \text{else} \end{cases},$$

where $\downarrow$ means is defined and $\uparrow$ means not defined. Another example is the category of topological spaces and maps which are continuous on an open susbset.

The following lemma shows some basic manipulations involving the restriction category axioms.

Lemma 2.1.1. *The following hold in a restriction category* $\mathbb{X}$*:*

(i) $\overline{f}\,\overline{f} = \overline{f}$;

(ii) $\overline{\overline{f}} = \overline{f}$;

(iii) $\overline{fg} = \overline{f}\,\overline{fg}$;

(iv) $\overline{fg} = \overline{f\overline{g}}$.

Proof. Consider the following manipulations.

(i) $\overline{f}\,\overline{f} = \overline{\overline{f}\,f} = \overline{f}$ by **R.3** then **R.1**.

(ii) $\overline{f} = \overline{\overline{f}}\,\overline{f} = \overline{f}\,\overline{\overline{f}} = \overline{\overline{f}\,\overline{f}} = \overline{\overline{f}}$ by **R.1**,**R.2**,**R.3** and point 1.

(iii) $\overline{fg} = \overline{\overline{f}fg} = \overline{f}\,\overline{fg}$ by **R.1**,**R.3**.

(iv) $\overline{fg} = \overline{f}\,\overline{fg} = \overline{fg}\,\overline{f} = \overline{\overline{fg}f} = \overline{f\overline{g}}$ by point 3 then **R.2**,**R.3**,**R.4**.

□

By lemma 2.1.1.1 we see that the maps $e = \overline{e}$ are idempotent, and we will refer to them as **restriction idempotents**.

A map is **total** when its domain of definition is its entire domain; that is, when $\overline{f} = 1$. The total maps in a restriction category form a subcategory which contains all the monic maps:

Lemma 2.1.2. *Let $\mathbb{X}$ be a restriction category. Then,*

(i) All monic maps (hence all identities and isomorphisms) are total;

(ii) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are both total then $fg : A \longrightarrow C$ is total too.

(iii) If fg is total then f is total.

Proof. Consider the following.

(i) Assume f is monic. Then since $\overline{f}f = f = 1f$ we have $\overline{f} = 1$.

(ii) Assume $\overline{f} = 1_A$ and $\overline{g} = 1_A$. Then applying lemma 2.1.1.4,

$$\overline{fg} = \overline{f\overline{g}} = \overline{f} = 1_A.$$

(iii) Assume $\overline{fg} = 1$. Then applying lemma 2.1.1.3,

$$\overline{f} = \overline{f}1 = \overline{f}\,\overline{fg} = \overline{fg} = 1.$$

□

The following proposition is now clear.

Proposition 2.1.1 (Cockett-Lack [16]). *The total maps of a restriction category $\mathbb{X}$ are a subcategory which contains all the monic maps.*

We will use $\mathsf{Total}(\mathbb{X})$ to denote the subcategory of total maps of a restriction category $\mathbb{X}$.

A natural relation on parallel, partial maps is that a map f is less defined than g when g restricted to the domain of f is f. In a restriction category we may define such a relation on maps.

$$f \leq g := \overline{f}\, g = f.$$

We prove that the above relation is a partial order which is preserved by composition.

Lemma 2.1.3. *Let $\mathbb{X}$ be a restriction category with $\leq$ defined as above. Then*

(i) $f \leq f$;

(ii) $f \leq g$ *and* $g \leq f$ *implies* $f = g$;

(iii) $f \leq g$ *and* $g \leq h$ *implies* $f \leq h$;

(iv) $f \leq g$ *implies* $hfk \leq hgk$;

(v) $f = g$ *iff* $\overline{f} = \overline{g}$ *and there is a* k *such that* $f, g \leq k$.

Proof. Consider:

(i) $\overline{f}\, f = f$ by **R.1**.

(ii) Assume $\overline{f}\, g = \overline{g}\, f = g$. Then, use this assumption, followed by **R.3**,**R.2**,**R.1**, and then the assumption again.

$$f = \overline{f}\, f = \overline{\overline{f}\, g}\, f = \overline{f}\, \overline{g}\, f = \overline{g}\, \overline{f}\, f = \overline{g}\, f = g.$$

(iii) Assume $\overline{f}\, g = f$ and $\overline{g}\, h = g$. Then use the assumption, then **R.3** followed by the assumption twice.

$$\overline{f}\, h = \overline{\overline{f}\, g}\, h = \overline{f}\, \overline{g}\, h = \overline{f}\, g = f.$$

(iv) Assume $\overline{f}\,g = f$. Then, use **R.4**, then lemma (2.1.1, **R.2**,the assumption, **R.4**, and then **R.1**.

$$\overline{hfk}\,hgk = h\overline{fk}\,gk = h\overline{f}\,\overline{fk}\,gk = h\overline{fk}\,\overline{f}\,gk = h\overline{fk}\,fk = \overline{hfk}\,hfk = hfk$$

(v) The "$\Rightarrow$" direction is obvious. Suppose $\overline{f} = \overline{g}$ and there is a k such that $f, g \leq k$; that is,

$$\overline{f}\,k = f \qquad \overline{g}\,k = g.$$

Thus,

$$f = \overline{f}\,k = \overline{g}\,k = g.$$

$\square$

The above also establishes that if $e = \overline{e}$ then $e \leq 1$ and that $\overline{g}\,f, f\overline{h} \leq f$.

Another standard relation on parallel, partial maps is that whenever both maps are defined, then they are equal. The compatibility relation is studied more extensively by Cockett and Hofstra in [11].

Definition 2.1.2. *$f, g : A \longrightarrow B$ in a restriction category are* **compatible** *when $\overline{f}\,g = \overline{g}\,f$, and denoted $f \smile g$.*

The compatibility relation is stable with respect to composition; it is also reflexive and symmetric. However, one must take care in reasoning about compatible maps because compatibility is not transitive.

Lemma 2.1.4. *In a restriction category we have,*

(i) $f \smile f$;

(ii) $f \smile g$ implies $g \smile f$;

(iii) $f \smile g$ implies $hfk \smile hgk$;

(iv) $f \smile g$ *iff* $\overline{f}\, g \leq f$ *iff* $\overline{g}\, f \leq g$;

(v) $f \leq g$ *iff* $f \smile g$ *and* $\overline{f} \leq \overline{g}$.

Proof. (1) and (2) are obvious. For (3), assume $\overline{f}\, g = \overline{g}\, f$ then consider

$$
\begin{aligned}
\overline{hfk}\, hgk = h\overline{fk}\, gk = h\overline{f}\, \overline{fk}\, gk & \qquad \mathbf{R.4}, \text{lemma } 2.1.1.3 \\
= h\overline{fk}\, \overline{f}\, gk = h\overline{fk}\, \overline{g}\, fk = h\overline{g}\, \overline{fk}\, fk & \qquad \mathbf{R.2}, \text{assumption}, \mathbf{R.2} \\
= h\overline{\overline{g}\, fk}\, fk = h\overline{\overline{f}\, gk}\, fk = h\overline{f}\, \overline{gk}\, fk & \qquad \mathbf{R.3}, \text{assumption}, \mathbf{R.3} \\
= h\overline{gk}\, \overline{f}\, fk = h\overline{gk}\, fk = \overline{hgk}\, hfk & \qquad \mathbf{R.2}, \mathbf{R.1}, \mathbf{R.4}.
\end{aligned}
$$

For (4), if $f \smile g$ then $\overline{f}\, g = \overline{g}\, f \leq f$. Conversely, if $\overline{f}\, g \leq f$ then $\overline{f}\, g = \overline{\overline{f}\, g}\, \overline{f} = \overline{f}\, \overline{g}\, f = \overline{g}\, f$. Thus $f \smile g$ iff $\overline{f}\, g \leq f$. Similarly $f \smile g$ iff $\overline{g}\, f \leq g$. For (5), "$\Rightarrow$" is obvious. Assume $f \smile g$ and $\overline{f} \leq \overline{g}$. Then $\overline{f}\, g = \overline{f}\, \overline{f}\, g = \overline{f}\, \overline{g}\, f = \overline{f}\, f = f$. □

Definition 2.1.3. *Let* $\mathbb{X}, \mathbb{Y}$ *be restriction categories. A* **restriction functor** $F : \mathbb{X} \longrightarrow \mathbb{Y}$ *is a functor where for all* f, $F(\overline{f}) = \overline{F(f)}$.

Lemma 2.1.5. *Restriction functors preserve total maps.*

Proof. Let F be a restriction functor, and f a total map. Then

$$\overline{F(f)} = F(\overline{f}) = F(1) = 1.$$

□

Definition 2.1.4. *Let* F, G *be restriction functors. A* **restriction natural transformation** $\alpha : F \Longrightarrow G$ *is a natural transformation where each component,* α_A *is total.*

Proposition 2.1.2 (Cockett-Lack [16]). *Restriction categories,functors, and natural transformations form a 2-category.*

2.2 Cartesian Restriction Categories

Products in a restriction category are investigated by Cockett and Lack in [14]. They considered the 2-category of restriction categories, restriction functors, and lax natural transformations, and defined a restriction category to have restriction products when the functors $\Delta : \mathbb{X} \longrightarrow \mathbb{X} \times \mathbb{X}$ and $! : \mathbb{X} \longrightarrow \mathbb{1}$ have right adjoints. Such categories can be directly axiomatized, and this axiomatization is provided below.

Definition 2.2.1. *A* **restriction terminal object,** $\top$ *is an object in a restriction category where for any A there is a unique, total map $!_A : A \longrightarrow \top$ such that $!_\top = 1_\top$ and if $f : A \longrightarrow B$ then*

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[dr, "f"'] & \top \arrow[d, phantom, "\leq" near start] \\ & B. \arrow[u, "!_B"'] \end{tikzcd}$$

A **restriction binary product** *of A, B is an object $A \times B$ with total projections $\pi_0 : A \times B \longrightarrow A$ and $\pi_1 : A \times B \longrightarrow B$ s.t. if $f : C \longrightarrow A$ and $g : C \longrightarrow B$ then there is a unique map $\langle f, g\rangle : C \longrightarrow A \times B$ with the property that $\langle f, g\rangle\pi_0 = \overline{g}\, f$ and $\langle f, g\rangle\pi_1 = \overline{f}\, g$.*

A **Cartesian restriction category** *is a restriction category with all restriction binary products and a restriction terminal object.*

Next, to set some notation, let $\mathbb{X}$ be a Cartesian restriction category, then set

$$A^0 := 1 \qquad A^{n+1} := A^n \times A.$$

In other words, $A^n = (\cdots(A \times A) \times \cdots A)$.

Next, we have a lemma regarding Cartesian restriction categories.

Lemma 2.2.1. *The following all hold in a Cartesian restriction category.*

(i) $\langle f, g\rangle\pi_0 = \overline{g}\, f$ *and* $\langle f, g\rangle\pi_1 = \overline{f}\, g$ *if and only if* $\langle f, g\rangle\pi_0 \leq f$, $\langle f, g\rangle\pi_1 \leq g$, $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$.

(ii) $(f \times g)\pi_0 = \overline{\pi_1 g}\,\pi_0 f$ *and* $(f \times g)\pi_1 = \overline{\pi_0 f}\,\pi_1 g$.

(iii) *If* $e = \overline{e}$ *then* $e\langle f, g\rangle = \langle ef, g\rangle = \langle f, eg\rangle$.

(iv) $f\langle g, h\rangle = \langle fg, fh\rangle$.

(v) $\langle f, g\rangle(h \times k) = \langle fh, hk\rangle$.

(vi) $(f \times g)(h \times k) = (fh \times gk)$.

(vii) *If* $f \leq f'$ *then* $\langle f, g\rangle \leq \langle f', g\rangle$*; if* $g \leq g'$ *then* $\langle f, g\rangle \leq \langle f, g'\rangle$.

(viii) π_0, π_1 *are lax natural.*

(ix) $\langle \pi_0, \pi_1\rangle = 1$

(x) $A \times B \cong B \times A$

(xi) *If* g *is total then* $\pi_0 f = (f \times g)\pi_0$*; if* f *is total then* $\pi_g = (f \times g)\pi_1$.

(xii) $\overline{f \times g} = \overline{f} \times \overline{g}$*;*

(xiii) $1 \times \overline{f} = \overline{\pi_f}$ *and* $\overline{f} \times 1 = \overline{\pi_0 f}$*;*

(xiv) $A \cong A \times \top \cong \top \times A$.

Proof.

(i) $\Rightarrow$: Assume $\langle f, g\rangle\pi_0 \leq f, \langle f, g\rangle\pi_1 \leq g, \overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$. Then,

$$\begin{aligned}
\langle f, g\rangle\pi_0 &= \overline{\langle f, g\rangle\pi_0}\, f && \text{assumption} \\
&= \overline{\langle f, g\rangle\overline{\pi_0}}\, f && \text{lemma (2.1.1.4} \\
&= \overline{f}\,\overline{g}\, f = \overline{g}\,\overline{f}\, f && \text{assumption and R.4} \\
&= \overline{g}\, f
\end{aligned}$$

A symmetric argument shows $\langle f, g\rangle\pi_1 = \overline{f}\, g$.

$\Leftarrow$: Assume that $\langle f, g\rangle\pi_0 = \overline{g}\, f$ and $\langle f, g\rangle\pi_1 = \overline{f}\, g$. The restriction ordering shows that $\langle f, g\rangle\pi_0 \leq f$ and $\langle f, g\rangle\pi_1 \leq g$. Further, consider,

$$\overline{\langle f, g\rangle} = \overline{\langle f, g\rangle\overline{\pi_1}} = \overline{\langle f, g\rangle\pi_1} = \overline{\overline{f}\, g} = \overline{f}\,\overline{g}\,.$$

(ii) Consider

$$(f \times g)\,\pi_0 = \langle \pi_0 f, \pi_1 g\rangle\pi_0 = \overline{\pi_1 g}\,\pi_0 f,$$

and similarly $(f \times g)\,\pi_1 = \overline{\pi_0 f}\,\pi_1 g$.

(iii) Assume $\overline{e} = e$. Then note,

$$\begin{aligned}
\langle f, eg\rangle\pi_0 &= \overline{eg}\, f \\
&= \overline{\overline{e} g}\, f && \text{assumption} \\
&= \overline{e}\,\overline{g}\, f && \text{R.3} \\
&= \overline{e}\,\langle f, g\rangle\pi_0 \\
&= e\langle f, g\rangle\pi_0. && \text{assumption}
\end{aligned}$$

Similarly $\langle f, eg\rangle\pi = e\langle f, g\rangle\pi_1$, but $\langle f, eg\rangle$ is the unique map such that

$$\begin{tikzcd}
 & Q \arrow[dl, "f"'] \arrow[d, dotted, "{\langle f,g\rangle}"] \arrow[dr, "eg"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] \arrow[ul, phantom, "\geq"] \arrow[ur, phantom, "\leq"] & B.
\end{tikzcd}$$

Thus, $e\langle f, g\rangle = \langle f, eg\rangle$. A similar argument shows $e\langle f, g\rangle = \langle ef, g\rangle$.

(iv) Consider,

$$\begin{aligned}
\langle fg, fh\rangle\pi_0 &= \overline{fh}\, fg \\
&= f\overline{h}\, g && \text{R.4} \\
&= f\langle g, h\rangle\pi_0.
\end{aligned}$$

Similarly, $\langle fg, fh\rangle\pi_1 = f\langle g, h\rangle\pi_1$. Thus, $f\langle g, h\rangle$ will satisfy the same universal property as $\langle fg, fh\rangle$; therefore, $f\langle g, h\rangle = \langle fg, fh\rangle$.

(v) Consider,

$$\begin{aligned}
\langle f,g\rangle(h\times k) &= \langle f,g\rangle\langle\pi_0 h,\pi_1 k\rangle \\
&= \langle\langle f,g\rangle\pi_0 h,\langle f,g\rangle\pi_1 k\rangle && \text{part 4}\\
&= \langle\overline{g}\,fh,\overline{f}\,gk\rangle \\
&= \overline{g}\,\overline{f}\,\langle fh,gk\rangle && \text{part 3}\\
&= \overline{g}\,\langle\overline{f}\,fh,gk\rangle && \text{part 3}\\
&= \overline{g}\,\langle fh,gk\rangle && \text{R.1}\\
&= \langle fh,gk\rangle && \text{part 3,R.1}
\end{aligned}$$

(vi) Assume $f\leq f'$. Then,

$$\begin{aligned}
\overline{\langle f,g\rangle}\,\langle f',g\rangle &= \overline{f}\,\overline{g}\,\langle f',g\rangle \\
&= \overline{f}\,\langle f',g\rangle && \text{part 3,R.1}\\
&= \langle\overline{f}\,f',g\rangle && \text{part 3}\\
&= \langle f,g\rangle. && f\leq f'
\end{aligned}$$

Thus $\langle f,g\rangle\leq\langle f',g\rangle$. A symmetric argument shows that if $g\leq g'$ then $\langle f,g\rangle\leq\langle f,g'\rangle$.

(vii) Consider,

$$(f\times g)\,(h\times k)=\langle\pi_0 f,\pi_1 g\rangle\,(h\times k)=\langle\pi_0 fh,\pi_1 gk\rangle=(fh\times gk)\,.$$

(viii) We will show π_0 is lax natural; π_1 follows follows by a symmetric argument. Consider we must show,

$$\begin{tikzcd}
A\times B \arrow[r, "\pi_0"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\
C\times D \arrow[r, "\pi_0"'] & C.
\end{tikzcd}$$

Now,

$$
\begin{aligned}
\overline{(f \times g)\,\pi_0}\,\pi_0 f &= \overline{\overline{\pi_1 g}\,\pi_0 f}\,\pi_0 f \\
&= \overline{\pi_1 g}\,\overline{\pi_0 f}\,\pi_0 f \\
&= \overline{\pi_1 g}\,\pi_0 f \\
&= (f \times g)\,\pi_0,
\end{aligned}
$$

as required.

(ix) Consider that $\langle \pi_0, \pi_1 \rangle \pi_0 = \pi_0$ and $\langle \pi_0, \pi_1 \rangle \pi_1 = \pi_1$. Thus $\langle \pi_0, \pi_1 \rangle$ will satisfy the same universal property as 1; therefore, $\langle \pi_0, \pi_1 \rangle = 1$.

(x) We show $\langle \pi_1, \pi_0 \rangle : A \times B \longrightarrow B \times A$ is an involution thus an isomorphism. Consider,

$$
\langle \pi_1, \pi_0 \rangle \langle \pi_1, \pi_0 \rangle = \langle \langle \pi_1, \pi_0 \rangle \pi_1, \langle \pi_1, \pi_0 \rangle \pi_0 \rangle = \langle \pi_0, \pi_1 \rangle = 1.
$$

(xi) Assume g is total. Then,

$$
\begin{aligned}
(f \times g)\,\pi_0 &= \overline{\pi_1 g}\,\pi_0 f \\
&= \overline{\pi_1 \overline{g}}\,\pi_0 f \\
&= \overline{\pi_1}\,\pi_0 f && \text{assumption} \\
&= \pi_0 f.
\end{aligned}
$$

Similarly, if f is total, then $(f \times g)\,\pi_1 = \pi_1 g$.

(xii) Consider,

$$\begin{aligned}
\overline{f} \times \overline{g} &= \langle \pi_0 \overline{f}\,, \pi_1 \overline{g}\,\rangle \\
&= \langle \overline{\pi_0 f}\,\pi_0, \overline{\pi_1 g}\,\pi_1 \rangle && \text{R.4} \\
&= \overline{\pi_0 f}\,\overline{\pi_1 g}\,\langle \pi_0, \pi_1 \rangle \\
&= \overline{\pi_0 f}\,\overline{\pi_1 g} \\
&= \overline{\langle \pi_0 f, \pi_1 g \rangle} \\
&= \overline{f \times g}
\end{aligned}$$

(xiii) We will show that $1 \times \overline{f} = \overline{\pi_1 f}$, and $\overline{f} \times 1 = \overline{\pi_0 f}$ follows by a symmetric argument. Consider that

$$\begin{aligned}
1 \times \overline{f} &= \overline{1} \times \overline{f} && \text{1 is total} \\
&= \overline{1 \times f} \\
&= \overline{\pi_1 f}\,.
\end{aligned}$$

(xiv) We will show that $\langle 1_A, !_A \rangle : A \longrightarrow A \times \top$ is an isomorphism. Since ! is total,

$$\langle 1, ! \rangle \pi_0 = \overline{!}\,1 = 1.$$

Also, there is a unique total map $A \times \top \longrightarrow \top$. Thus, $!_{A \times \top} = \pi_1 = \pi_0 !_A$; therefore,

$$\pi_0 \langle 1_A, !_A \rangle = \langle \pi_0, \pi_0 !_A \rangle = \langle \pi_0, \pi_1 \rangle = 1.$$

$\square$

In a Cartesian restriction category the restriction structure is determined by the product structure:

Lemma 2.2.2. *Let $\mathbb{X}$ be a Cartesian restriction category. Then, the restriction structure may be interpreted as*

$$\overline{f} := \langle 1, f \rangle \pi_0,$$

or

$$\overline{f} := \langle f, 1\rangle \pi_1.$$

Proof. Consider,

$$\langle 1, f\rangle \pi_0 = \overline{f} \qquad \langle f, 1\rangle \pi_1 = \overline{f}.$$

□

Lemma 2.2.3. *Let $\mathbb{X}$ be a Cartesian restriction category. Then Cartesian restriction structure is unique up to isomorphism.*

Proof.

- Let $\top, \top'$ be restriction terminal objects. Let $!_A$ denote the unique total map $A \longrightarrow \top$ and $!'_A$ denote the unique total map $A \longrightarrow \top'$. Now the following commutes since there is a unique total map $\top \longrightarrow \top$ and $!_\top = 1_\top$.

$$\begin{tikzcd} \top \arrow[r, "!'_\top"] \arrow[rr, bend right, "1_\top"'] & \top' \arrow[r, "!_{\top'}"] & \top \end{tikzcd}$$

- Let $(\times, \pi_0, \pi_1, \langle \_, \_ \rangle)$ and $(\times', \pi'_0, \pi'_1, \langle \_, \_ \rangle')$ be two restriction product structures. It suffices to show that $\langle \pi'_0, \pi'_1 \rangle : A \times' B \longrightarrow A \times B$ is an isomorphism. Now,

$$\begin{aligned} \langle \pi'_0, \pi'_1 \rangle \langle \pi_0, \pi_1 \rangle' &= \langle \langle \pi'_0, \pi'_1 \rangle \pi_0, \langle \pi'_0, \pi'_1 \rangle \pi_1 \rangle' && \text{lemma (2.2.1.4)} \\ &= \langle \overline{\pi'_1}\, \pi'_0, \overline{\pi'_0}\, \pi'_1 \rangle \\ &= \langle \pi'_0, \pi'_1 \rangle \\ &= 1. \end{aligned}$$

Similary calculations show that $\langle \pi_0, \pi_1 \rangle' \langle \pi'_0, \pi'_1 \rangle = 1$; thus, $\langle \pi'_0, \pi'_1 \rangle$ is an isomorphism.

□

Definition 2.2.2. *Let $F : \mathbb{X} \longrightarrow \mathbb{Y}$ be a restriction functor. F is a* **Cartesian restriction functor** *in case F preserves all restriction product and restriction terminal object diagrams.*

Lemma 2.2.4. *The map $\sigma_\times = \langle F(\pi_0), F(\pi_1)\rangle : (\_\times\_)F \longrightarrow F(\_\times\_)$ is restriction natural.*

Proof. We must show that for all A, B the following diagram commutes.

$$\begin{tikzcd}
F(A\times B) \arrow[r, "\sigma_\times"] \arrow[d, "F(a\times b)"'] & F(A)\times F(B) \arrow[d, "F(a)\times F(b)"] \\
F(A'\times B') \arrow[r, "\sigma_\times"'] & F(A')\times F(B')
\end{tikzcd}$$

It does since,

$$\begin{aligned}
F(a\times b)\sigma_\times &= F(a\times b)\langle F(\pi_0), F(\pi_1)\rangle \\
&= \langle F(a\times b)F(\pi_0), F(a\times b)F(\pi_1)\rangle \\
&= \langle F((a\times b)\pi_0), F((a\times b)\pi_1)\rangle \\
&= \langle F(\overline{\pi_1 b}\,\pi_0 a), F(\overline{\pi_0 a}\,\pi_1 b)\rangle \\
&= \langle \overline{F(\pi_1 b)}\,F(\pi_0 a), \overline{F(\pi_0 a)}\,F(\pi_1 b)\rangle && \text{restriction functor} \\
&= \langle F(\pi_0 a), F(\pi_1 b)\rangle && \text{lemma (2.2.1.3),R.2,R.1} \\
&= \langle F(\pi_0)F(a), F(\pi_1)F(b)\rangle \\
&= \sigma_\times(F(a)\times F(b)) && \text{lemma (2.2.1.5)}
\end{aligned}$$

Since restriction functors preserve total maps, $\sigma_\times$ is a tuple of total maps hence $\sigma_\times$ itself is total. Therefore, $\sigma_\times$ is a restriction natural transformation. □

We end this subsection with a characterization of Cartesian restriction functors.

Proposition 2.2.1 (Cockett-Lack [14]). *Let $\mathbb{X}, \mathbb{Y}$ be Cartesian restriction categories, and let $F : \mathbb{X} \longrightarrow \mathbb{Y}$ be a restriction functor. F is a Cartesian restriciton functor iff $\sigma_\times$ and $!_{F(\top)}$ are isomorphisms.*

Proof. If F is a Cartesian restriction functor then lemma (2.2.3) shows that $\sigma_\times : F(A\times B) \longrightarrow F(A)\times F(B)$ is an isomorphism and that $!_F(\top)$ is an isomorphism. Conversely, if $\sigma_\times$

and $!_{F(\mathrm{T})}$ are isomorphisms then F preserves all restriction product and terminal object diagrams. $\square$

2.3 Simple Slices

When $\mathbb{X}$ is total (so that restriction products are categorical products) and A is an object of $\mathbb{X}$, the coKleisli category for the comonad $(\_ \times A)$ is called the simple slice category $\mathbb{X}$ over A and denoted $\mathbb{X}[A]$ [29] [27]. The coKleisli category is a well known categorical construction; see for example [34]. In [29], Jacobs shows that when a category $\mathbb{X}$ has finite products that the simple slice category is equivalent to the category of computations in context. In [31], such categories are used to prove the functional completeness of Cartesian closed categories. Thus, simple slice categories have an important role in the semantics of programming languages.

In a Cartesian restriction category, we will define $\mathbb{X}[A]$ in the same way. This section will prove that any comonad $\mathcal{S} = (S, \epsilon, \delta)$, where S is a restriction functor and ϵ is total, will lift Cartesian restriction structure to the coKleisli category. In particular, $(\_ \times A)$ is an example of such a comonad thus Cartesian restriction structure lifts to $\mathbb{X}[A]$.

2.3.1 CoKleisli Category

For notation: in calculations, we will use **boldface** for maps in the coKleisli category, and plain font for maps in the base category.

Note that for a comonad on a restriction category, $\overline{\delta_x \epsilon_{S(X)}} = 1$; thus, by lemma (2.1.2) δ_X is total at each component (i.e. δ is a restriction natural transformation); however, ϵ need not be. The following is a straightforward generalization of the fact that, for total categories, $\mathbb{X}_{\mathcal{S}}$ is a category (see for example [34]).

Proposition 2.3.1. *If $\mathbb{X}$ is a restriction category, and $\mathcal{S} = (S, \epsilon, \delta)$ is a comonad with*

S a restriction functor, then $\mathbb{X}_S$ is a restriction category where $\overline{\mathbf{f}} = \overline{f}\,\epsilon_X$. Further, the inclusion $\mathbb{X} \xrightarrow{J} \mathbb{X}_S$ is a restriction functor.

Proof. We must show the restriction identities. First note that from the comonad laws we have $\delta_X S(\epsilon_X) = \delta_X \epsilon_{S(X)}$ and for all g, $\delta_X S(g)\epsilon_Y = g$. Combining this with the fact that S is a restriction functor we have the following very useful identity,

$$(\overline{\mathbf{f}})^{\sharp} = \delta_X S(\overline{f}\,\epsilon_X) = \overline{f}\,.$$

The calculation for the above is

$$\delta_X S(\overline{f}\,\epsilon_X) = \delta_X S(\overline{f}\,)S(\epsilon_X) = \delta_X \overline{S(f)}\,S(\epsilon_X) = \overline{\delta_X S(F)}\,\delta_X S(\epsilon_X)$$
$$= \overline{\delta_X S(F)}\,\delta_X \epsilon_{S(X)} = \delta_X \overline{S(f)}\,\epsilon_{S(X)} = \delta_X S(\overline{f}\,)\epsilon_{S(X)} = \overline{f}$$

R.1 Using the above identity we have

$$\overline{\mathbf{f}}\,\mathbf{f} = (\overline{f}\,\epsilon_X)^{\sharp} f = \overline{f}\,f = f = \mathbf{f}.$$

R.2 Again using the above identity we have

$$\overline{\mathbf{f}}\,\overline{\mathbf{g}} = (\overline{f}\,\epsilon_X)^{\sharp}\overline{g}\,\epsilon_X = \overline{f}\,\overline{g}\,\epsilon_X = \overline{g}\,\overline{f}\,\epsilon_X = (\overline{g}\,\epsilon_X)^{\sharp}\overline{f}\,\epsilon_X = \overline{\mathbf{g}}\,\overline{\mathbf{f}}.$$

R.3 From the above identity,

$$\overline{\mathbf{f}}\,\overline{\mathbf{g}} = (\overline{f}\,\epsilon_X)^{\sharp}\overline{g}\,\epsilon_X = \overline{f}\,\overline{g}\,\epsilon_X = \overline{\overline{f}\,g}\,\epsilon_X = \overline{(\overline{f}\,\epsilon_X)^{\sharp} g}\,\epsilon_X = \overline{\overline{\mathbf{f}}\,\mathbf{g}}\,.$$

R.4 Again, use the above identity, and also that $f = \delta_X S(f)\epsilon_Y$.

$$\overline{\mathbf{f}\mathbf{h}}\,\mathbf{f} = \delta_X S(\overline{\delta_X S(f)h}\,\epsilon_X) = (\overline{\delta_X S(f)h}\,\epsilon_X)^{\sharp} f = \overline{\delta_X S(f)h}\,f$$
$$= \overline{\delta_X S(f)h}\,\delta_X S(f)\epsilon_Y = \delta_X S(f)\overline{h}\,\epsilon_Y = f^{\sharp}\overline{h}\,\epsilon_Y = \mathbf{f}\overline{\mathbf{h}}\,.$$

Therefore, $\mathbb{X}_S$ is a restriction category.

That the inclusion $\mathbb{X} \xrightarrow{J} \mathbb{X}_{\mathcal{S}}$ is a restriction functor: Consider

$$J(\overline{f}) = \epsilon_X \overline{f} = S(\overline{f})\epsilon_X = \overline{S(f)}\,\epsilon_X = \overline{S(f)\epsilon_X}\,\epsilon_X = \overline{\epsilon_X f}\,\epsilon_X = \overline{J(f)}\,\epsilon_X,$$

so that J preserves the restriction as required. $\square$

The above proof relies on the fact that S is a restriction functor. We did not need ϵ to be total, but for the next step we will. The following proposition is a generalization of the fact that for total categories, the coKleisli category of a Cartesian category is again a Cartesian category.

Proposition 2.3.2. *If $\mathbb{X}$ is a Cartesian restriction category, and $\mathcal{S} = (S, \epsilon, \delta)$ is a comonad with S a restriction functor and where ϵ is total then $\mathbb{X}_{\mathcal{S}}$ is a Cartesian restriction category. Further, the inclusion $\mathbb{X} \xrightarrow{J} \mathbb{X}_{\mathcal{S}}$ is a Cartesian restriction functor.*

Define $!_A : A \longrightarrow \top = !_{S(A)} : S(A) \longrightarrow \top$, and define the projections and pairings:

$$\frac{S(A \times B) \xrightarrow{\epsilon\pi_0} A}{\mathbf{A \times B \xrightarrow{\pi_0} A}} \qquad \frac{S(A \times B) \xrightarrow{\epsilon\pi_1} B}{\mathbf{A \times B \xrightarrow{\pi_1} B}} \qquad \frac{S(Z) \xrightarrow{\langle f, g\rangle} A \times B}{\mathbf{Z \xrightarrow{\langle f, g\rangle} A \times B}}$$

Proof. That $\top$ is a restriction terminal object in $\mathbb{X}_{\mathcal{S}}$ follows from the fact that it is a restriction terminal object in $\mathbb{X}$. The projections are total as

$$\overline{\epsilon\pi_i} = \overline{\epsilon\pi_i}\,\epsilon = \overline{\epsilon}\,\epsilon = \epsilon = \mathbf{1}.$$

Next we must show that $\mathbb{X}_{\mathcal{S}}$ has binary restriction product structure. Recall that $(\overline{f}\,\epsilon_A)^\sharp = \overline{f}$. Consider for π_0,

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle\boldsymbol{\pi}_0 = (\langle f, g\rangle)^\sharp \epsilon_{A\times B}\pi_0 = \langle f, g\rangle\pi_0 = \overline{g}\,f = (\overline{g}\,\epsilon_B)^\sharp f = \overline{\boldsymbol{g}}\,\boldsymbol{f}$$

Similarly for $\boldsymbol{\pi}_1$, $\langle \boldsymbol{f}, \boldsymbol{g}\rangle\boldsymbol{\pi}_1 = \overline{\boldsymbol{f}}\,\boldsymbol{g}$. Thus the restriction product laws hold, and now we must show that $\langle \boldsymbol{f}, \boldsymbol{g}\rangle$ is the unique map that makes these laws hold. Suppose $\boldsymbol{v}$ also satisfies $\boldsymbol{v}\boldsymbol{\pi}_0 = \overline{\boldsymbol{g}}\,\boldsymbol{f}$ and $\boldsymbol{v}\boldsymbol{\pi}_1 = \overline{\boldsymbol{f}}\,\boldsymbol{g}$. Then,

$$v\pi_0 = v^\sharp \epsilon_{A\times B}\pi_0 = \boldsymbol{v}\boldsymbol{\pi}_0 = \overline{\boldsymbol{g}}\,\boldsymbol{f} = (\overline{g}\,\epsilon_A)^\sharp f = \overline{g}\,f.$$

Similarly $v\pi_1 = \overline{f}g$. Thus, $v = \langle f, g\rangle$ by the uniqueness of $\langle f, g\rangle$ in $\mathbb{X}$. Thus $\mathbb{X}_{\mathcal{S}}$ is a Cartesian restriction category.

That the inclusion $\mathbb{X} \xrightarrow{J} \mathbb{X}_{\mathcal{S}}$ is a Cartesian restriction functor, use proposition (2.2.1). That $!_{J(\mathrm{T})}$ is an isomorphism follows because it is an isomorphism in $\mathbb{X}$. That $\sigma_\times$ is an isomorphism,

$$\langle J(\pi_0), J(\pi_1)\rangle = \langle \epsilon\pi_0, \epsilon\pi_1\rangle = \epsilon\langle \pi_0, \pi_1\rangle = \epsilon = 1.$$

Thus, by proposition (2.2.1), J is a Cartesian restriction functor. $\square$

In $\mathbb{X}_{\mathcal{S}}$, $\boldsymbol{f \times g} = \langle (\epsilon\pi_0)^\sharp f, (\epsilon\pi_1)^\sharp g\rangle = \sigma_\times(f \times g)$ which implies that J preserves products *strictly*, because

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{f} \times \boldsymbol{g}) &= \epsilon(f \times g) = \langle \epsilon\pi_0 f, \epsilon\pi_1 g\rangle = \langle S(\pi_0)\epsilon f, S(\pi_1)\epsilon g\rangle \\ &= \sigma_\times(\epsilon f \times \epsilon g) = \sigma_\times(J(f) \times J(g)) = \boldsymbol{J}(\boldsymbol{f}) \times \boldsymbol{J}(\boldsymbol{g}) \end{aligned}$$

Also note that the association map $\boldsymbol{a}_\times = J(a_\times)$.

2.3.2 The Simple Slice Category

To show that $(\_ \times A)$ is a comonad, we will use the coKleisli triple presentation of a comonad[1].

The map $(\_ \times A) : \mathbb{X} \longrightarrow \mathbb{X}$ is defined as $X \mapsto (X \times A)$ on objects. Next, define

$$\epsilon_X := (X \times A) \xrightarrow{\pi_0} X \text{ and } f^\sharp := (X \times A) \xrightarrow{\langle f, \pi_1\rangle} (Y \times A)$$

Note that

$$f^\sharp\epsilon = \langle f, \pi_1\rangle\pi_0 = f \qquad \pi_0^\sharp = \langle \pi_0, \pi_1\rangle = 1$$

and

$$f^\sharp g^\sharp = \langle f, \pi_1\rangle\langle g, \pi_1\rangle = \langle\langle f, \pi_1\rangle g, \pi_1\rangle = (f^\sharp g)^\sharp.$$

[1]The (co)Kleisli triple presentation of a (co)Monad was provided by Manes in [35]. Recall, its data consists of an object map, a family of morphisms ϵ, and a combinator $(\_)^\sharp$

Thus, using the coKleisli triple presentation of a comonad, the following proposition is obvious.

Proposition 2.3.3. *Let $\mathbb{X}$ be a Cartesian restriction category. Then, $(\_ \times A)$ is a comonad on $\mathbb{X}$.*

Note that $\overline{f} \times 1 = \overline{f} \times \overline{1} = \overline{f \times 1}$, so that $\_ \times A$ is a restriction functor. Also note that $\epsilon = \pi_0$ is total, thus by proposition (2.3.2),

Proposition 2.3.4. *$\mathbb{X}[A]$ is a Cartesian restriction category.*

To unwind this comonad further, given a map $f : X \longrightarrow Y$ then $S(f) = (f \times 1) : X \times A \longrightarrow Y \times A$ and that $\delta_X = \langle 1, \pi_1 \rangle = \Delta(1 \times \pi_1)$. Also, $\sigma_\times = \langle (\pi_0 \times 1), (\pi_1 \times 1) \rangle$.

2.4 Idempotent Splitting

Definition 2.4.1. *Let $e : A \longrightarrow A$ be an idempotent; e is a* **split idempotent** *when e factors,*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "r"'] & A \\ E \arrow[ur, "s"'] & \end{tikzcd}$$

where $sr = 1_E$.

Further, given a split idempotent like the one above, we call E a **retract** of A, and write $E \lhd A$. Obviously, an idempotent e splits into rs if and only if $sr = 1_E$, and also r is a retraction and s is a section thus a monic.

The following construction, called the Cauchy completion or idempotent splitting of a category, splits a class of idempotents into a retraction followed by a section. This construction can be found in Karoubi [30].

Definition 2.4.2. *Let $\mathbb{X}$ be a category, and $\mathcal{E}$ a class of idempotents in $\mathbb{X}$. $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ is the structure with*

Obj: (A, e), *such that A is an object in $\mathbb{X}$ and $e \in \mathcal{E}$ such that $e : A \longrightarrow A$.*

Arr: $(A, e) \xrightarrow{f} (B, e')$ *is a map $A \xrightarrow{f} B$ in $\mathbb{X}$ such that $efe' = f$.*

Id: $(A, e) \xrightarrow{e} (A, e)$.

Composition: $(A, e) \xrightarrow{f} (B, e') \xrightarrow{f'} (C, e'') = (A, e) \xrightarrow{ff'} (C, e'')$.

First, note that composition is well defined since

$$eff'e'' = eefe'e'f'e''e'' = efe'e'f'e'' = ff'.$$

Further note that a map f satisfies $efe' = f$ if and only if $ef = f = fe'$.

The following lemma will be quite useful in many proofs.

Lemma 2.4.1. *Let $e = ee$ be an idempotent in a restriction category. Then*

$$\overline{e}\,e = e = e\overline{e}.$$

Proof. Suppose $e = ee$. First, it is always the case that $e = \overline{e}\,e$. To show that $e = e\overline{e}$ consider

$$e\overline{e} = \overline{e\overline{e}}\,e = \overline{e}\,e = e.$$

□

The following, from [30], is straightforward to check.

Proposition 2.4.1 (Karoubi). *For every $\mathbb{X}$ and class $\mathcal{E}$ of idempotents,* $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ *is a category.*

In this section we show that a class of idempotents that is closed to taking products may be split to guarantee that the resultant category is a Cartesian restriction category. First, we show that one can split any class of maps in a restriction category and again obtain a restriction category. The following is a generalization of theorem 1 in [4] to restriction categories.

Proposition 2.4.2. *Let $\mathbb{X}$ be a restriction category. Then*

(i) $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ *has a restriction structure given by* $\overline{(A,e) \xrightarrow{f} (B,e')} = (A,e) \xrightarrow{e\overline{f}} (A,e)$.

(ii) If $\mathcal{E}$ *contains all the identities of* $\mathbb{X}$*, then there is a restriction preserving embedding* $\mathbb{X} \hookrightarrow \mathsf{Split}_{\mathcal{E}}(\mathbb{X})$.

(iii) All maps in $\mathcal{E}$ *split in* $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$.

Proof. (i) Note the restriction is well defined since

$$ee\overline{f}\, e = e\overline{ef}\, ee = e\overline{ef}\, e = ee\overline{f} = e\overline{f}\,.$$

Now, we must show that this gives a restriction structure.

[R.1] $e\overline{f}\, f = ef = f$.

[R.2] We need that $e\overline{f}\, e\overline{g} = e\overline{g}\, e\overline{f}$. Consider,

$$e\overline{f}\, e\overline{g} = \overline{ef}\, e\overline{eg}\, e = \overline{ef}\, \overline{eeg}\, ee = \overline{ef}\, \overline{eg}\, e = \overline{eg}\, \overline{ef}\, e = \overline{eg}\, e\overline{f} = \overline{eg}\, ee\overline{f} = e\overline{g}\, e\overline{f}\,.$$

[R.3] We must show $e\overline{f}\, e\overline{g} = e\overline{e\overline{f}\, g}$. Now (using $e\overline{f} = e\overline{f}\, e$),

$$e\overline{e\overline{f}\, g} = e\overline{e\overline{f}\, eg} = e\overline{ef}\, \overline{eg} = e\overline{f}\, \overline{eg} = e\overline{f}\, \overline{g} = e\overline{f}\, e\overline{g}\,.$$

[R.4] Note, the burden is a bit trickier this time. We have $(A,e) \xrightarrow{f} (B,e') \xrightarrow{h} (C,e'')$, so that the restriction of h is $e'\overline{h}$ and the restriction of fh is $e\overline{fh}$. Thus, we must show that $fe'\overline{h} = e\overline{fh}\, f$. Now,

$$fe'\overline{h} = ef\overline{e'h}\, e' = ef\overline{h}\, e' = e\overline{fh}\, fe' = e\overline{fh}\, f.$$

Thus $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ has the restriction structure as described.

(ii) Define the inclusion on objects $A \mapsto (A, 1_A)$ and on arrows $A \xrightarrow{f} B \mapsto (A, 1_1) \xrightarrow{f} (B, 1_B)$. The inclusion obviously preserves the restriction. Note the inclusion is well defined since $1_A f 1_B = f$. This inclusion is clearly a functor, and is full and faithful since

$$\mathbb{X}(A,B) \cong \mathsf{Split}_{\mathcal{E}}(\mathbb{X})((A,1_A),(B,1_B)).$$

(iii) To see that $\mathcal{E}$ idempotents split in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ suppose $d \in \mathcal{E}$, and $(A,e) \xrightarrow{d} (A,e)$ is a map in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ (this map is necessarily idempotent). Note that $(A,d) \xrightarrow{d} (A,d)$ is also a map in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$; also $r = (A,e) \xrightarrow{d} (A,d)$ and $m = (A,d) \xrightarrow{d} (A,e)$ are maps in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ since $edd = dd = d$ and $dde = dd = d$ respectively. It is straightforward to show that $mr = (A,d) \xrightarrow{d} (A,d)$ and $rm = (A,e) \xrightarrow{d} (A,e)$, which completes the proof.

□

Now, we can consider when a map in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ is total. Suppose $(A,e) \xrightarrow{f} (B,e')$ is total. Then we have $e\overline{f} = e$.

Lemma 2.4.2. *Let $\mathbb{X}$ be a restriction category. Then the following are true.*

(i) A map $(A,e) \xrightarrow{f} (B,e')$ is total in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ iff $\overline{e} \leq \overline{f}$ in $\mathbb{X}$.

(ii) If $\mathcal{E}$ is exactly the class of restriction idempotents, then $(A,e) \xrightarrow{f} (B,e')$ is total iff $e \leq \overline{f}$ in $\mathbb{X}$.

Proof. Consider the following:

(i) Suppose $(A,e) \xrightarrow{f} (B,e')$ is total; i.e. $e\overline{f} = e$. Then $\overline{e}\,\overline{f} = \overline{e\overline{f}} = \overline{e}$. Conversely, suppose $\overline{e}\,\overline{f} = \overline{e}$. Then,

$$e\overline{f} = \overline{e}\,e\overline{f} = \overline{e}\,\overline{ef}\,e = \overline{e}\,\overline{f}\,e = \overline{e}\,e = e.$$

(ii) This follows immediately from (1) since $e = \overline{e}$.

□

Proposition 2.4.3. *Let $\mathbb{X}$ be a Cartesian restriction category, and $\mathcal{E}$ a class of idempotents which is closed under taking products (i.e. if $e_1, e_2 \in \mathcal{E}$ then $e_1 \times e_2 \in \mathcal{E}$). Then* $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ *can be given Cartesian restriction structure.*

Take the restriction terminal object to be $(\top, 1_\top)$, and the total maps into the terminal object to be

$$(A, e) \xrightarrow{e!_A} (\top, 1_\top).$$

Define the product of (B, e') and (C, e'') to be $(B \times C, e' \times e'')$. Take projections out of $(B \times C, e' \times e'')$ to be $(e' \times e'')\pi_0$ and $(e' \times e'')\pi_1$. Take the pairing of $(A, e) \xrightarrow{f} (B, e')$ and $(A, e) \xrightarrow{g} (C, e'')$ to be just $\langle f, g \rangle$.

Proof. First, note that the maps into the restriction terminal object are well defined as $ee!_A 1_\top = e!_A$. We must show that the restriction of $e!_A$ is e;

$$e\overline{e!_A} = e\overline{e\overline{!_A}} = e\overline{e} = \overline{e}\,e = e.$$

Next, we must show that if $(A, e) \xrightarrow{f} (B, e')$ then $fe!_B \leq e!_A$; by using the fact $\mathbb{X}$ has Cartesian restriction structure we have,

$$e\overline{fe!_B}\,e!_A = e\overline{f}\,e!_A = e\overline{f}\,!_A = ef!_B = fe!_B.$$

Now we must show that the structure defined gives a binary restriction product structure. First, note that $(e' \times e'')$ is always an idempotent when e' and e'' are, and thus $\mathcal{E}$ being closed under products is well defined. Next, consider that the pairing of maps is well defined as $e\langle f, g\rangle(e' \times e'') = \langle ef, eg\rangle(e' \times e'') = \langle efe', ege''\rangle = \langle f, g\rangle$. Also, note that projections are total. We will show the argument for π_0:

$$(e' \times e'')\overline{(e' \times e'')\pi_0} = (e' \times e'')\overline{(e' \times e')} = (e' \times e'').$$

Next, consider that the pairing of maps satisfies the correct laws for restriction product structure:

$$\langle f, g\rangle(e' \times e'')\pi_0 = \langle fe', ge''\rangle\pi_0 = \langle f, g\rangle\pi_0 = \overline{g}\,f,$$

and

$$\langle f, g\rangle(e' \times e'')\pi_1 = \langle fe', ge''\rangle\pi_1 = \langle f, g\rangle\pi_1 = \overline{f}\, g.$$

Finally, we show that $\langle f, g\rangle$ is the unique map with this property. Suppose $(A, e) \xrightarrow{v} (B \times C, e' \times e'')$ satisfies $v(e' \times e'')\pi_0 = \overline{g}\, f$ and $v(e' \times e'')\pi_1 = \overline{f}\, g$. Then note that $v(e' \times e'') = v$, so that $v\pi_0 = \overline{g}\, f$ and $v\pi_1 = \overline{f}\, g$. Then by the universal property in $\mathbb{X}$ we have that $v = \langle f, g\rangle$. This completes the proof. □

2.5 Term Logic

A functional type theory or term logic for a category permits equational reasoning about the maps. A term logic should be sound and complete with respect to the translation of the logic into the categorical setting. A soundness and completeness theorem for a translation means that theorems in the logic correspond precisely to equations in the category. The development of term logic we use here follows [18]; these results can be found in Cockett and Hofstra's notes [9].

Also worth noting is that a term logic can often be used as a programming language. This allows the development of categorical abstract machines, and often these machines allow one to discover new optimizations [17].

A **functional type theory** is given by three pieces: syntax, type inference judgments, and equations. The syntax of a functional type theory is generated inductively from a set of atomic terms and atomic types. Type inference judgments ensure that terms in the theory are well typed. Let $\mathbb{T}$ be a set of atomic types and V a set of atomic terms, Ω a set of function symbols, and let there be a mapping $\sigma : \Omega \longrightarrow \mathbb{T}^* \times \mathbb{T}$, called the **sorting** [2]. Note that the **arity**, $\omega : \Omega \longrightarrow \mathbb{N}$, can be defined in terms of the sorting as $\omega := \sigma; \pi_0; \mathsf{length}$.

[2] M^* is the free monoid of the set M

Recall some basic notation.

Definition 2.5.1. *A* **context** *is a finite bag of (term,type) pairs.*

Definition 2.5.2. *A* **term-in-context** *is a sequent*

$$\Gamma \vdash m : A$$

where Γ is a context, m is a term, and A is a type.

Term formation judgments

$$\frac{t_1 \quad \cdots \quad t_n}{t}$$

assert that if $t_1, \cdots, t_n$ are terms in the logic, then t a term in the logic. An **equation** is a judgment of the form;

$$\frac{\Gamma \vdash t_1 : A \quad \Gamma \vdash t_2 : A}{\Gamma \vdash_e t_1 = t_2 : A.}$$

However, the rule above will often be abbreviated by $t_1 =_\Gamma t_2 : A$, and when the context and type are clear as just $t_1 = t_2$. A **proved term** is a term which can be derived only from judgments.

The syntax of the Cartesian restriction terms and types are respectively generated by the following inductively defined sets. First let V be a set of variables. Then a pattern p is a tuple of variables with the property that each variable occurs exactly once in the tuple.

$$T := V \mid () \mid (T, \ldots, T) \mid \{p.T\}T \mid T_{|T}$$

$$Ty := \mathbb{T} \mid \mathbf{1} \mid Ty \times \cdots \times Ty$$

T is often called the **carrier set** of the term logic [20].

Basic terms:

$$\frac{}{\Gamma, x : A \vdash x : A}\ \textsc{Proj} \qquad \frac{\Gamma \vdash t : A \quad \Gamma, p : A \vdash s : B}{\Gamma \vdash \{p.s\}T : B}\ \textsc{Cut}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \cdots \quad \Gamma \vdash t_n : A_n \quad \sigma(f) = ([A_1, \ldots, A_n], B)}{\Gamma \vdash f(t_1, \ldots, t_n) : B}\ \textsc{Fun}$$

Cartesian terms:

$$\frac{}{\Gamma \vdash () : \mathbf{1}}\ \textsc{Unit} \qquad \frac{\Gamma \vdash t_1 : A_1 \quad \cdots \Gamma \vdash t_n : A_n}{\Gamma \vdash (t_1, \ldots, t_n) : A_1 \times \cdots \times A_n}\ \textsc{Tuple}$$

$$\frac{\Gamma, t_1 : A_1, \ldots, t_n : A_n \vdash t : A}{\Gamma, (t_1, \ldots, t_n) : A_1 \times \cdots A_n \vdash t : A}\ \textsc{Pat}$$

Partial Terms:

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash s : B}{\Gamma \vdash t_{|s} : A}\ \textsc{Rest}$$

Table 2.1: Cartesian Restriction Term Formation Rules

The term formation rules for Cartesian restriction term logic are given in table (2.5.3). Note that for the cut rule, variables are removed from context and put into the proved term. Since the proof of s depends on these variables, they are said to be **bound** in s. Also, for convenience we will write $t_{|s,s'}$ to denote $(t_{|s})_{|s'}$.

There are essentially six classes of equations placed on terms.

α-Renaming The terms introduced by the cut rule contain bound variables, and these variables can be renamed with fresh variables by a process called α-renaming. Two terms t_1, t_2 are α-equivalent if they can be made syntactically equal after a finite number of bound variable changes. α-equivalence is indeed an equivalence relation, and the rule is written,

$$\frac{\Gamma \vdash t_1 : A \quad \Gamma \vdash_e t_2 : A \quad t_1 \equiv_\alpha t_2}{\Gamma \vdash_e t_1 = t_2 : A}\ \alpha$$

Axioms Axioms specify basic equalities on terms. Axioms do not depend on any generated equivalence; we assume $\Gamma \vdash t_1 : A$ and $\Gamma \vdash t_2 : A$, and we write

$$\frac{}{\Gamma \vdash_e t_1 = t_2 : A.}\ \textsc{Axiom}_\varepsilon$$

ESB.1 $\{x.x\}t \Rrightarrow t$
ESB.2 $\{x.y\}t \Rrightarrow y_{\|t}$ when $y \neq x$
ESB.3 $\{x.f(t_1, \ldots, t_n)\}t \Rrightarrow f(\{x.t_1\}t, \ldots, \{x.t_n\}t)$
ESB.4 $\{().t\}k \Rrightarrow t_{\|k}$
ESB.5 $\{().t\}() \Rrightarrow t$
ESB.6 $\{x.()\}t \Rrightarrow ()_{\|t}$
ESB.7 $\{x.(t_1, \ldots, t_n)\}t \Rrightarrow (\{x.t_1\}t, \ldots, \{x.t_n\}t)$
ESB.8 $\{(x_1, \ldots, x_n).s\}(t_1, \ldots, t_n) \Rrightarrow \{x_1.\cdots(\{x_n.s\}t_n)\cdots\}t_1$
ESB.9 $\{x.s_{\|v}\}t \Rrightarrow \{x.s\}t_{\|\{x.v\}t}$

Table 2.2: Cut Elimination for Cartesian restriction terms

Equational We ensure that the equations generate an equivalence relation using the following rules.

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash_e t = t : A}\ \textsc{Refl} \qquad \frac{\Gamma \vdash_e t_2 = t_1 : A}{\Gamma \vdash_e t_1 = t_2 : A}\ \textsc{Symm}$$

$$\frac{\Gamma \vdash_e t_1 = t_2 : A \quad \Gamma \vdash_e t_2 = t_3 : A}{\Gamma \vdash_e t_1 = t_3 : A}\ \textsc{Tran}$$

Subterm We also have rules that ensure that terms with equal subterms are equal. There is one subterm equality rule for each term formation rule; for example, for the Fun rule,

$$\frac{\Gamma \vdash_e t_1 = t_1' : A_1 \quad \cdots \quad \Gamma \vdash_e t_n = t_n' : A_n \quad \sigma(f) = ([A_1, \ldots, A_n], B)}{\Gamma \vdash_e f(t_1, \ldots, t_n) = f(t_1', \ldots, t_n') : B}\ \textsc{SubFun}$$

Cut-elimination The equations for cut elimination of Cartesian restriction terms are in table 2.2. These equations are written directed because they can be viewed as a rewriting system on terms.

CRL.1 $t = t_{\mid t}$
CRL.2 $t_{\mid s,s'} = t_{\mid s',s}$
CRL.3 $t_{\mid s,s'} = t_{\mid s_{\mid s'}}$
CRL.4 $t = t_{\mid x}$
CRL.5 $f(t_1, \ldots, (t_k)_{\mid s}, \ldots, t_n) = f(t_1, \ldots, t_n)_{\mid s}$
CRL.6 $t_{\mid (s_1,\ldots,s_n)} = t_{\mid s_1,\ldots,s_n}$
CRL.7 $(t_1, \ldots, (t_k)_{\mid s}, \ldots, t_n) = (t_1, \ldots, t_n)_{\mid s}$

Table 2.3: Equations for Cartesian restriction terms

Equations The equations for Cartesian restriction term logic are in table 2.3.

A **theory** of Cartesian restriction term logic is $\mathcal{T} = (Ty, \Omega, \sigma, \xi)$ where ξ is the set of axioms that are imposed on Cartesian restriction term logic.

2.5.1 Cut Elimination

In this subsection, we will show that any term in Cartesian restriction term logic is equivalent to a term with no cuts. We will use the bag ordering technique to establish cut elimination.

Definition 2.5.3. *Let $(W, <)$ be a well founded set. Let $B_1, B_2 \in Bag(W)$, and define $<<$ on $Bag(W)$ by $B_1 << B_2$ if*

- $B_1 = M_1 \cup C$
- $B_2 = M_2 \cup C$
- *for all $x \in M_1$ there exists $y \in M_2$ such that $x < y$*

The bag ordering allows us to remove some common subbag of B_1, B_2, and then compare what is left pointwise.

Proposition 2.5.1 (Dershowitz). *If $(W,<)$ is a well founded set, then $(Bag(W),<<)$ is also a well founded set.*

Proof. See [21]. $\square$

Theorem 2.5.1 (Dershowitz). *Let $(W,<)$ be a well founded set, T be a set of terms, and $\Rrightarrow \subseteq T \times T$ a rewrite relation. Let $\nu : T \longrightarrow (W,<)$. If for every $a \Rrightarrow b$, $\nu(b) < \nu(a)$ then the rewrite rule terminates.*

Proof. Again see [21]. $\square$

Proposition 2.5.2. *Every Cartesian restriction term is equal to one with no cuts.*

Proof. (Sketch):

Consider the well founded set $(\mathbb{N} \times \mathbb{N}, \leq_{\text{lex}})$, and the map $|\_| : Terms \longrightarrow \mathbb{N}$ which is the size of the term. Define

$$\nu : Terms \longrightarrow Bag(\mathbb{N} \times \mathbb{N}, <<)$$

by

$$\nu(t) = \{| \ (|t'|, |s|) \mid \{p.s\}t' \text{ is a subterm of } t \ |\}$$

Now we must compare $\nu(t_1)$ and $\nu(t_2)$ for each $t_1 \Rrightarrow t_2$. However, using the bag ordering, we can eliminate $\nu(t_1) \cap \nu(t_2)$, so we only need to compare singleton bags. For example, take (ESB.4). Now, for all i $|t_i| \leq_{\text{lex}} |(t_1, \ldots, t_n)|$. Thus $(|t_i|, \_) \leq_{\text{lex}} (|(t_1, \ldots, t_n)|, |s|)$ as required. Or, for example, take (ESB.6). It is clear that $(|t|, |t_i|) \leq_{\text{lex}} (|t|, |f(t_1, \ldots, t_n)|)$ as required. The rest of the rules are similar. $\square$

2.5.2 Basic Equalities

This subsection presents some equalities between terms that are useful in establishing the completeness theorem for this term logic.

First, a lemma regarding some basic manipulations of restriction terms.

Lemma 2.5.1. *(i)* $t_{|s,s} = t_{|s}$

(ii) $t_{|()} = t$

(iii) If $x \notin s$ *then* $\{x.s\}t = s_{|t}$

(iv) If no variables of p *are in* s *then* $\{p.s\}t = s_{|t}$

(v) $(\{p.t'\}t)_{|s} = \{p.t'_{|s}\}t$

(vi) $(\{p.t'\}t)_{|s} = \{p.t'\}(t_{|s})$

Points *(iii)* – *(vi)* of the above lemma are all proved by induction. We will prove point *(v)* explicitly and sketch proofs of the rest.

Proof. Consider the following calculations:

(i) Using **CRL.3**, followed by **CRL.1** on $s_{|s}$,

$$t_{|s,s} = t_{|s_{|s}} = t_{|s}$$

(ii) Using **ESB.4** followed by **ESB.5**,

$$t_{|()} = \{().t\}() = t$$

(iii) The proof is by induction on the structure of s.

(iv) The proof is by induction on the length of the pattern p. For the case when $p = ()$ use **ESB.4**. When $p = x$ is a variable, use part 3 of this lemma. When $p = (p_1, \ldots, p_n)$,

$$\begin{aligned}
\{(p_1, \ldots, p_n).t\}(t_1, \ldots, t_n) &= \{p_1.(\cdots\{p_n.t\}t_n\cdots)\}t_1 \\
&= \{p_1.(\cdots t_{|t_n}\cdots)\}t_1 \\
&= \cdots \\
&= t_{|t_n,\ldots,t_1} = t_{|t_1,\ldots,t_n} \\
&= t_{|(t_1,\ldots,t_n)}
\end{aligned}$$

(v) Assume that t, t' do not contain any cuts. Then the proof proceeds by induction on the structure of the pattern p. When $p = ()$ consider that

$$(\{().t'\}t)_{|s} = t'_{|t,s} = t'_{|s,t} = \{().t'_{|s}\}t$$

When $p = x$, is a variable, induct on the structure of t'. The case for tuples is omitted because it is almost the same as the case for function application. Note, that parts 3 and 4 of this lemma are used extensively below.

$$\begin{aligned}
(\{x.()\}t)_{|s} &= ()_{|t,s} = (\{x.()\}t)_{|s,t} \\
&= (\{x.()\}t)_{|s_{|t}} = \{x.()\}t_{|\{x.s\}t} = \{x.()_{|s}\}t \\
(\{x.x\}t)_{|s} &= t_{|s} = (\{x.x\}t)_{|s,t} \\
&= \{x.x\}t_{|\{x.s\}t} = \{x.x_{|s}\}t \\
(\{x.f(t_1,\ldots,t_n)\}t)_{|s} &= f(\{x.t_1\}t_{|s},\ldots,\{x.t_n\}t_{|s}) = f(\{x.t_{1|s}\}t,\ldots,\{x.t_{n|s}\}t) \\
&= \{x.f(t_{1|s},\ldots,t_{n|s})\}t = \{x.f(t_1,\ldots,t_n)_{|s}\}t \\
(\{x.t_{|t'}\}s')_{|s} &= \{x.t\}s'_{|\{x.t'\}s',s} = \{x.t\}s'_{|\{x.t'\}s'_{|s}} \\
&= \{x.t\}s'_{|\{x.t'_{|s}\}s'} = \{x.t_{|t'_{|s}}\}s' = \{x.t_{|t',s}\}s'
\end{aligned}$$

Which completes the induction when p is a variable.

Finally, assume $p = (p_1,\ldots,p_n)$, and consider

$$\begin{aligned}
\{(p_1,\ldots,p_n).t\}(t_1,\ldots,t_n)_{|s} &= \{p_1.\cdots\{p_n.t\}t_n\cdots\}t_{1|s} \\
&= \{p_1.(\cdots\{p_n.t\}t_n\cdots)_{|s}\}t_1 \\
&= \cdots \\
&= \{p_1.\cdots\{p_n.t_{|s}\}t_n\cdots\}t_n \\
&= \{(p_1,\ldots,p_n).t_{|s}\}(t_1,\ldots,t_n)
\end{aligned}$$

This completes the induction and the proof of point (v).

(vi) For this, use induction on the length of the pattern p. The calculations are direct when $p = ()$ or when $p = (p_1, \ldots, p_n)$. Induct on the structure of t' when $p = x$, is a variable.

$\square$

Next, to make a few lemmas easier, we can define normal substitution $[t/p]\, s$ which does not update the restrictions appropriately.

Lemma 2.5.2. $\{p.t\}t' = ([t'/p]\, t)_{|t'}$

Proof. The proof is by induction on the pattern. First, if $p = ()$, then $\{().t\}t' = t_{|t'} = ([t'/()]\, t)_{|t'}$. Next assume that $p = x$, is a variable. Then proceed by induction on t. The case for tuples is omitted due to similarity with the case for function application. Consider,

$$
\begin{aligned}
\{x.()\}t' &= ()_{|t'} = ([t'/x]\, ())_{|t'} \\
\{x.x\}t' &= t' = t'_{|t'} = ([t'/x]\, x)_{|t'} \\
\{x.y\}t' &= y_{|t'} = ([t'/x]\, y)_{|t'} \\
\{x.f(t_1, \ldots, t_n)\}t' &= f(\{x.t_1\}t', \ldots, \{x.t_n\}t') \\
&= f(([t'/x]\, t_1)_{|t'}, \ldots, ([t'/x]\, t_n)_{|t'}) \\
&= f([t'/x]\, t_1, \ldots, [t'/x]\, t_n)_{|t', \ldots, t'} = f([t'/x]\, t_1, \ldots, [t'/x]\, t_n)_{|t'} \\
&= ([t'/x]\, f(t_1, \ldots, t_n))_{|t'} \\
\{x.t_{1|t_2}\}t' &= \{x.t_1\}t'_{|\{x.t_2\}t'} = [t'/x]\, t_{1|t',([t'/x]t_2)_{|t'}} \\
&= [t'/x]\, t_{1|[t'/x]t_2,t',t'} = [t'/x]\, t_{1|[t'/x]t_2,t'} \\
&= \left([t'/x]\, t_{1|[t'/x]t_2}\right)_{|t'} = \left([t'/x]\, (t_{1|t_2})\right)_{|t'}
\end{aligned}
$$

$\square$

This has a few consequences.

TCR.1 $[\![x : A \vdash x]\!] = 1_A$
TCR.2 $[\![(p_1, \ldots, p_n) \vdash x]\!] = \pi_i [\![p_i \vdash x]\!]$ (for $x \in p_i$)
TCR.3 $[\![\Gamma \vdash ()]\!] = !_{M(\Gamma)}$
TCR.4 $[\![\Gamma \vdash (t_1, \ldots, t_n)]\!] = \langle [\![\Gamma \vdash t_1]\!], \ldots, [\![\Gamma \vdash t_n]\!] \rangle$
TCR.5 $[\![\Gamma \vdash f(t_1, \ldots, t_n)]\!] = \langle [\![\Gamma \vdash t_1]\!], \ldots, [\![\Gamma \vdash t_n]\!] \rangle M(f)$
TCR.6 $[\![\Gamma \vdash \{p.s\}t]\!] = \langle [\![\Gamma \vdash t]\!], 1 \rangle [\![(p, \Gamma) \vdash s]\!]$
TCR.7 $[\![\Gamma \vdash t_{\vert s}]\!] = \overline{[\![\Gamma \vdash s]\!]}\, [\![\Gamma \vdash t]\!]$

Table 2.4: Translation of Cartesian restriction terms

Corollary 2.5.1.

(i) if $t = t'$ *and* $s = s'$ *then* $([t/p]\, s)_{\vert t} = ([t'/p]\, s')_{\vert t'}$

(ii) $\{p.s\}(\{p'.s'\}t) = \{p'.\{p.s\}s'\}t$

(iii) $\{p.\{p'.s'\}s\}t = \{p'.\{p.s'\}t\}(\{p.s\}t)$

2.5.3 Soundness

The interpretation of a logic into a category is **sound** if, whenever the axioms of the logic are satisfied, then all the equations are satisfied. In this section we will define a translation of Cartesian restriction terms into a Cartesian restriction category, and then show this translation is sound.

The translation of terms $[\![\_]\!]$ into a Cartesian restriction category is given in table 2.4.

Proposition 2.5.3 (Cockett and Hofstra [9]: proposition 1.10). *The translation defined in table 2.4 is sound.*

Proof. **CRL.1** This translates to $[\![\Gamma \vdash t]\!] = \overline{[\![\Gamma \vdash t]\!]}\, [\![\Gamma \vdash t]\!]$ which is **R.1**.

CRL.2 This translates to $\overline{[\![\Gamma \vdash s]\!]}\,\overline{[\![\Gamma \vdash s']\!]}\,[\![\Gamma \vdash t]\!] = \overline{[\![\Gamma \vdash s']\!]}\,\overline{[\![\Gamma \vdash s]\!]}\,[\![\Gamma \vdash t]\!]$ which is true because of **R.2**.

CRL.3 Using **R.3**,

$$[\![\Gamma \vdash t_{|s_{|s'}}]\!] = \overline{[\![\Gamma \vdash s_{|s'}]\!]}\,[\![\Gamma \vdash t]\!] = \overline{\overline{[\![\Gamma \vdash s']\!]}\,[\![\Gamma \vdash s]\!]}\,[\![\Gamma \vdash t]\!]$$
$$= \overline{[\![\Gamma \vdash s']\!]}\,\overline{[\![\Gamma \vdash s]\!]}\,[\![\Gamma \vdash t]\!] = [\![\Gamma \vdash t_{|s,s'}]\!]$$

CRL.4 Note that when x is variable then $[\![\Gamma \vdash x]\!]$ is a composition of projections, and so it is total. Then

$$[\![\Gamma \vdash t_{|x}]\!] = \overline{[\![\Gamma \vdash x]\!]}\,[\![\Gamma \vdash t]\!] = [\![\Gamma \vdash t]\!]$$

CRL.5 By lemma (2.2.1), when $e = \overline{e}$ then $e\langle f_1, \ldots, f_n\rangle = \langle f_1, \ldots, ef_j, \ldots, f_n\rangle$. Then

$$[\![\Gamma \vdash f(t_1, \ldots, (t_k)_{|s}, \ldots, t_n)]\!] = \langle [\![\Gamma \vdash t_1]\!], \ldots, \overline{[\![\Gamma \vdash s]\!]}\,[\![\Gamma \vdash t_k]\!], \ldots, [\![\Gamma \vdash t_n]\!]\rangle M(f)$$
$$= \overline{[\![\Gamma \vdash s]\!]}\,\langle t_1, \ldots, t_n\rangle M(f) = [\![\Gamma \vdash f(t_1, \ldots, t_n)_{|s}]\!]$$

CRL.6 Using lemma (2.2.1), $\overline{\langle f_1, \ldots, f_n\rangle} = \overline{f_1} \cdots \overline{f_n}$:

$$[\![\Gamma \vdash t_{|(s_1, \ldots, s_n)}]\!] = \overline{\langle [\![\Gamma \vdash s_1]\!], \ldots, [\![\Gamma \vdash s_n]\!]\rangle}\,[\![\Gamma \vdash t]\!]$$
$$= \overline{[\![\Gamma \vdash s_1]\!]} \cdots \overline{[\![\Gamma \vdash s_n]\!]}\,[\![\Gamma \vdash t]\!] = [\![\Gamma \vdash t_{|s_1, \ldots, s_n}]\!]$$

CRL.7

$$[\![\Gamma \vdash (t_1, \ldots, (t_k)_{|s}, \ldots, t_n)]\!] = \langle [\![\Gamma \vdash t_1]\!], \ldots, \overline{[\![\Gamma \vdash s]\!]}\,[\![\Gamma \vdash t_k]\!], \ldots, [\![\Gamma \vdash t_n]\!]\rangle$$
$$= \overline{[\![\Gamma \vdash s]\!]}\,\langle t_1, \ldots, t_n\rangle = [\![\Gamma \vdash (t_1, \ldots, t_n)_{|s}]\!]$$

Therefore the interpretation is sound. □

2.5.4 Completeness

To show the completeness of the interpretation 2.4, we will show that for each theory there is a Cartesian restriction category generated freely from the logic. This means that there is an interpretation or model of each theory in a Cartesian restriction category.

For each Cartesian restriction theory $\mathcal{T}$ define the classifying category, $\mathbb{C}[\mathcal{T}]$ to be the structure with

Obj: Products of atomic types.

Arr: A map $f : A \longrightarrow B$ is a term judgment $\Gamma : A \vdash t : B$, under the equivalence relation given by provable equality in the logic. We will often use the following notation for such an f: $A \longrightarrow B; \Gamma \mapsto t$, and if the types are clear, just $\Gamma \mapsto t$.

Id: The identity map $1_A = A \longrightarrow A; \Gamma \mapsto \Gamma$.

Comp: Composition is given by substitution.

$$\frac{(\Gamma \mapsto t)(\Gamma' \mapsto t')}{\Gamma \mapsto \{p'.t'\}t}$$

Rest: The restriction of a map is given by:

$$\frac{f = \Gamma \mapsto t}{\overline{f} = \Gamma \mapsto \Gamma_{|t}}$$

Products: For the total map into the terminal object use $! = \Gamma \mapsto ()$. The projections are $\pi_0 = (\Gamma, \Gamma') \mapsto \Gamma$ and $\pi_1 = (\Gamma, \Gamma') \mapsto \Gamma'$. Define the pairing as

$$\frac{f = \Gamma \mapsto t_1 \quad g = \Gamma \mapsto t_2}{\langle f, g \rangle = \Gamma \mapsto (t_1, t_2).}$$

We must first show:

Lemma 2.5.3. *For any Cartesian restriction theory $\mathcal{T}$, $\mathbb{C}[\mathcal{T}]$ is a category.*

Proof. That the identities act as identities, we must show that for any patterns p, p' we have $(p \mapsto p)(p \mapsto t) = p \mapsto t$ and $(p \mapsto t)(p' \mapsto p') = p \mapsto t$. The proof proceeds by induction on the pattern. For the base case, the pattern is a variable. Recall that the restriction of a variable disappears.

$$\begin{aligned}(x \mapsto x)(y \mapsto t) &= x \mapsto \{y.t\}x = x \mapsto [x/y]\, t_{|x} \\ &= x \mapsto [x/y]\, t = y \mapsto t\end{aligned}$$

where the last equation is α-equivalence. For the right law,

$$(p \mapsto t)(x \mapsto x) = p \mapsto \{x.x\}t = p \mapsto t.$$

For the inductive case, assume the patterns are nonempty, and for all subpatterns, p_i, $p_i \mapsto p_i$ acts as an identity. For the left law,

$$\begin{aligned}&((p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n))((q_1, \ldots, q_n) \mapsto t) \\ &= (p_1, \ldots, p_n) \mapsto \{(q_1, \ldots, q_n).t\}(p_1, \ldots, p_n) \\ &= (p_1, \ldots, p_n) \mapsto \{q_1. \cdots \{q_n.t\}p_n \cdots\}p_1 \\ &= (p_1, \ldots, p_n) \mapsto [p_1/q_1] \cdots [p_n/q_n]\, t \\ &= (q_1, \ldots, q_n) \mapsto t\end{aligned}$$

The last equation is α-equivalence. Note also that the restriction of a pattern disappears, so the fourth step is sound. For the right law,

$$\begin{aligned}&(p \mapsto (t_1, \ldots, t_n))((p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n)) \\ &= p \mapsto \{(p_1, \ldots, p_n).(p_1, \ldots, p_n)\}(t_1, \ldots, t_n) \\ &= p \mapsto (\{(p_1, \ldots, p_n).p_1\}(t_1, \ldots, t_n), \ldots, \{(p_1, \ldots, p_n).p_n\}(t_1, \ldots, t_n)) \\ &= p \mapsto ((\{p_1.p_1\}t_1)_{|t_2, \ldots, t_n}, \ldots, (\{p_n.p_n\}t_n)_{|t_1, \ldots, t_{n-1}}) \\ &= p \mapsto (\{p_1.p_1\}t_1, \ldots, \{p_n.p_n\}t_n) \\ &= p \mapsto (t_1, \ldots, t_n)\end{aligned}$$

where the last step is the inductive hypothesis.

That composition is associative, use corollary (2.5.1).

$$
\begin{aligned}
((p_1 \mapsto t_1)(p_2 \mapsto t_2))(p_3 \mapsto t_3) &= (p_1 \mapsto \{p_2.t_2\}t_1)(p_3 \mapsto t_3) \\
&= p_1 \mapsto \{p_3.t_3\}(\{p_2.t_2\}t_1) \\
&= p_1 \mapsto \{p_2.\{p_3.t_3\}t_2\}t_1 \\
&= (p_1 \mapsto t_1)(p_2 \mapsto \{p_3.t_3\}t_2) \\
&= (p_1 \mapsto t_1)((p_2 \mapsto t_2)(p_3 \mapsto t_3))
\end{aligned}
$$

□

Next, we check that $\mathbb{C}[\mathcal{T}]$ is a restriction category.

Lemma 2.5.4. *For every Cartesian restriction theory, $\mathbb{C}[\mathcal{T}]$ is a restriction category given by the above restriction data.*

Proof. **R.1**

$$
\begin{aligned}
\overline{f}\, f &= (p \mapsto p_{|t})(p \mapsto t) = p \mapsto \{p.t\}p_{|t} \\
&= p \mapsto (\{p.t\}p)_{|t} = p \mapsto ([p/p]\, t)_{|t} \\
&= p \mapsto t_{|p,t} = p \mapsto t_{|t} = p \mapsto t = f
\end{aligned}
$$

R.2

$$
\begin{aligned}
\overline{f}\,\overline{g} &= (p \mapsto p_{|t})(p \mapsto p_{|s}) = p \mapsto \{p.p_{|s}\}p_{|t} \\
&= p \mapsto (\{p.p\}p)_{|t,s} = p \mapsto (\{p.p\}p)_{|s,t} = (p \mapsto p_{|s})(p \mapsto p_{|t}) = \overline{g}.\overline{f}
\end{aligned}
$$

R.3

$$
\begin{aligned}
\overline{f}\,\overline{g} &= (p \mapsto p_{|t})(p \mapsto p_{|s}) = p \mapsto p_{|t,s} = p \mapsto p_{|t_{|s}} \\
&= \overline{p \mapsto t_{|s}} = \overline{p \mapsto \{p.s\}p_{|t}} = \overline{(p \mapsto p_{|t})(p \mapsto s)} = \overline{\overline{f}\, g}
\end{aligned}
$$

R.4

$$\begin{aligned}
f\overline{h} &= (p \mapsto t)(q \mapsto q_{|s}) = p \mapsto \{q.q_{|s}\}t \\
&= p \mapsto \{q.q\}t_{|\{q.s\}t} = p \mapsto t_{|\{q.s\}t} \\
&= p \mapsto \{p.t\}p_{|\{q.s\}t} = (p \mapsto p_{|\{q.s\}t})(p \mapsto t) \\
&= \overline{(p \mapsto \{q.s\}t)}\,(p \mapsto t) = \overline{(p \mapsto t)(q \mapsto s)}\,(p \mapsto t) = \overline{fh}\,f
\end{aligned}$$

Thus, $\mathbb{C}[\mathcal{T}]$ is a restriction category. ◻

Next, we show that $\mathbb{C}[\mathcal{T}]$ has the Cartesian restriction structure proposed.

Proposition 2.5.4 (Cockett and Hofstra [9]: theorem 1.12). *For every Cartesian restriction theory $\mathcal{T}$, $\mathbb{C}[\mathcal{T}]$ is a Cartesian restriction category.*

Proof. First, we show that $\mathbb{C}[\mathcal{T}]$ has a restriction terminal object. First note that $\overline{!} = \overline{p \mapsto ()} = p \mapsto p_{|()} = p \mapsto p$, so that ! is total, and $() \mapsto ()$ is the identity. Further,

$$f! = (p \mapsto t)(q \mapsto ()) = p \mapsto \{q.()\}t = p \mapsto ()_{|t} = p \mapsto \{q.()\}p_{|t} = \overline{f}\,!$$

Where the equality $\{q.()\}t = ()_{|t}$ can be proved by induction with the base case being **ESB.6**.

Next, we show that the data given defines partial products. First, that the projections are total is given by straightforward calculations, for example, for π_0,

$$\overline{\pi_0} = \overline{(p,q) \mapsto p} = (p,q) \mapsto (p,q)_{|p} = (p,q) \mapsto (p_{|p},q) = (p,q) \mapsto (p,q).$$

For the pairing law for π_0 note that no variable of t_1 occurs in t_0, then

$$\begin{aligned}
\langle f,g\rangle\pi_0 &= (p \mapsto (t_0,t_1))((p,q) \mapsto p) = p \mapsto \{q.\{p.p\}t_0\}t_1 \\
&= p \mapsto \{q.t_0\}t_1 = p \mapsto t_{0|t_1} = \overline{p \mapsto t_1}\,(p \mapsto t_0) = \overline{g}\,f
\end{aligned}$$

Similarly, $\langle f,g\rangle\pi_1 = \overline{f}\, g$. To see that $\langle f,g\rangle$ is unique with respect to this property, assume $p \mapsto (v_1,v_2)$ also satisfies, $(p \mapsto (v_1,v_2))\pi_0 = \overline{p \mapsto v_2}\,(p \mapsto v_1)$ and $(p \mapsto (v_1,v_2))\pi_1 = \overline{p \mapsto v_1}\,(p \mapsto v_2)$. Then consider that

$$(p \mapsto (v_1,v_2))\pi_0 = p \mapsto ([v_2/q]\, v_1)_{|v_2}$$

Since q and v_1 do not share variables, this implies that $t_0 = [v_2/q]\, v_1 = v_1$. Similarly, $v_2 = t_1$. This completes the proof that $\mathbb{C}[\mathcal{T}]$ is a Cartesian restriction category. $\square$

2.6 Turing Categories

Turing categories were introduced by Cockett and Hofstra [13] to give a categorical characterization of computability theory.

Let $\mathbb{X}$ be a Cartesian restriction category, and let $f : A \times B \longrightarrow C$ $t : T \times B \longrightarrow C$. f has a t **index** $h : A \longrightarrow T$ if h is total and the following diagram commutes

$$\begin{tikzcd} T \times B \arrow[r, "t"] & C \\ A \times B \arrow[u, "h\times 1"] \arrow[ur, "f"'] & \end{tikzcd}$$

A map $\tau_{BC} : T \times B \longrightarrow C$ is **universal** if for every A and $f : A \times B \longrightarrow C$ f has τ_{BC} index. Essentially, we are saying that τ is universal when it acts as a universal Turing machine, for every f, we get a "code" h which is used by τ to compute f. This leads to the following definition:

Definition 2.6.1. *A Cartesian restriction category* $\mathbb{X}$ *is a* **Turing category** *when there is an object* T*, called a* **Turing object**, *such that for any two objects* B, C *there is a map* τ_{BC} *which is universal.*

An object T for which every other object is a retract is a **universal object**.

The standard example of a Turing category is given by the computable functions over the natural numbers. For some fixed programming language, one may encode each program as a natural number. There is a universal function $\phi(n,m)$ which applies to m the program whose number is n. We may look at the category of functions that are computable in this manner; $\mathbb{N}$ is a Turing object and the universal map is ϕ.

From the definition we have an immediate observation

Lemma 2.6.1. *The Turing object T of a Turing category is universal.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
T \arrow[r, bend left, "{\langle 1,! \rangle}"] & T \times T \arrow[l, bend left, "\pi_0"] \arrow[r, "\tau"] & A \arrow[dl, bend left, "{\langle 1,! \rangle}"] \\
& A \times T \arrow[u, "h \times 1"] \arrow[ur, "\pi_0"] &
\end{tikzcd}$$

Take $m = \langle 1, ! \rangle (h \times 1)\pi_0 = h$ and $r = \langle 1, ! \rangle \tau$. Then it is obvious that $mr = 1_A$. $\square$

Theorem 2.6.1. *[Recognition of Turing categories(Cockett-Hofstra [13]: theorem 3.4)] Let $\mathbb{X}$ be a Cartesian restriction category. Then $\mathbb{X}$ is a Turing category if and only if there is an object T for which every other object is a retract and there is a universal map $T \times T \xrightarrow{\bullet} T$.*

Proof. Lemma (2.6.1) shows the implication "$\implies$." Thus assume $\mathbb{X}$ is a Cartesian restriction category with an object T for which every object is a retract and there is a universal $T \times T \xrightarrow{\bullet} T$. Our goal is to show $\mathbb{X}$ is in fact a Turing category. Now any two objects B, C are retracts of T; thus we have

$$\begin{tikzcd}
B \arrow[r, bend left, "m_B"] & T \arrow[l, bend left, "r_B"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, bend left, "m_C"] & T \arrow[l, bend left, "r_C"]
\end{tikzcd}$$

Define

$$\tau_{BC} := T \times B \xrightarrow{1 \times m_B} T \times T \xrightarrow{\bullet} T \xrightarrow{r_C} C$$

Our goal is to show τ_{BC} is universal. Suppose $f : A \times B \longrightarrow C$. We must show there is a total $h : A \longrightarrow T$ such that

$$\begin{tikzcd} T \times B \arrow[r, "\tau_{BC}"] & C \\ A \times B \arrow[u, "h\times 1"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. However, note that $(1 \times m_B)(1 \times r_B) = 1_{A\times B}$, and that there is a total h such that

$$\begin{tikzcd} T \times T \arrow[r, "\bullet"] & T \\ A \times T \arrow[u, "h\times 1"] \arrow[ur, "(1\times r_B)fm_C"'] & \end{tikzcd}$$

commutes because $\bullet$ is universal. Then consider the following diagram

$$\begin{tikzcd} T \times B \arrow[r, "1\times m_B"] & T \times T \arrow[r, "\bullet"] & T \arrow[r, "r_C"] & C \\ A \times B \arrow[u, "h\times 1"] \arrow[r, "1\times m_B"'] & A \times T \arrow[u, "h\times 1"] \arrow[r, "1\times r_B"'] & A \times B \arrow[u, "fm_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

which proves that τ_{BC} is universal, and thus that $\mathbb{X}$ is a Turing category. $\square$

In a Turing category, the Turing object plays the role of a weak exponential object, and the index maps play the role of exponential transposes. There is also a way in which Turing categories arise from models of the untyped lambda calculus. We can consider an object A in a Cartesian closed category $\mathbb{X}$ for which $(A \Rightarrow A) \lhd A$; these are called **reflexive objects**. Note this makes every power of A a retract of A. Thus we can look at the full subcategory of $\mathbb{X}$ whose objects are the finite powers of A, called $\mathsf{Comp}_{\Rightarrow}(A)$. Define $A \times A \xrightarrow{\bullet} A = A \times A \xrightarrow{r \times 1} (A \Rightarrow A) \times A \xrightarrow{\mathsf{eval}} A$. Let $f : A^n \times A \longrightarrow A$. Then $\lambda(f)m$ is total, and the following commutes.

$$\begin{tikzcd} A \times A \arrow[r, "(r\times 1)\mathsf{eval}"] & A \\ A^n \times A \arrow[u, "\lambda(f)m\times 1"] \arrow[ur, "f"'] & \end{tikzcd}$$

Thus by theorem (2.6.1) $\mathsf{Comp}_{\Rightarrow}(A)$ is a Turing category.

Another perspective on Turing categories comes from PCAs.

In a Cartesian restriction category, an **applicative system** is $\mathbb{A} = (A, \bullet)$ where $A \times A \xrightarrow{\bullet} A$. Also define

$$\bullet^{(0)} := A \xrightarrow{\Delta} A \times A \xrightarrow{\bullet} A \qquad \bullet^{(1)} := \bullet \qquad \bullet^{(n+1)} := (\bullet \times 1)\bullet^{(n)} : A^2 \times A^n \longrightarrow A$$

A map $f : A^n \longrightarrow A$ is **$\mathbb{A}$-computable** if there is a total point $h : \top \longrightarrow A$ such that

$$\begin{tikzcd} A \times A^n \arrow[r, "\bullet^{(n)}"] & A \\ A^n \arrow[u, "\langle !h, 1\rangle"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. h is called a **code** for f. The notion $\mathbb{A}$-computable may also be extended to maps $A^n \longrightarrow A^m$. When $n = 0$, $f : \top \longrightarrow A^m$ is $\mathbb{A}$-computable if $!f : A \longrightarrow A$ is $\mathbb{A}$-computable. For $m = 0$, $f : A^n \longrightarrow \top$ is $\mathbb{A}$-computable when $\overline{f} : A^n \longrightarrow A^n$ is $\mathbb{A}$-computable. For $n, m > 0$ $f : A^n \longrightarrow A^m$ is $\mathbb{A}$-computable if each $f\pi_j$ is $\mathbb{A}$-computable.

To denote the term $\langle a, b\rangle\bullet$ we will write $a \bullet b$, or if the context is clear just $a\,b$. Also for notation, it will be assumed that application binds to the left, so the term $(a\,b)\,c$ can be written as just $a\,b\,c$.

For an applicative system, $\mathbb{A} = (A, \bullet)$, call the following structure $\mathsf{Comp}(\mathbb{A})$:

Obj: Powers of A

Arr: A map $A^n \xrightarrow{f} A^m$ is a map which $\mathbb{A}$-computable in the underlying category.

For an arbitrary applicative system $\mathbb{A}$, the structure $\mathsf{Comp}(\mathbb{A})$ is not a category; for example, there is no reason that the identity map is $\mathbb{A}$-computable.

If presented in the term logic, a map in $\mathsf{Comp}(\mathbb{A})$ is of the form,

$$\Gamma \longrightarrow t$$

where t is a term whose nodes are $\bullet$ and whose leaves are variables in Γ. An applicative system is **combinatory complete** if every such t can be represented by a single element

$\underline{t}$ such that

$$\underline{t}\, M_1 \cdots M_n = \{x_n.\cdots\{x_1.t\}M_1\cdots\}M_n$$

where $x_1, \ldots, x_n$ are the variables in Γ. It is a classic result that combinatory completeness devolves into having two particular elements s, k such that

$$s\, m_1\, m_2\, m_3 = m_1\, m_3\, (m_2\, m_3) \qquad k\, m_1\, m_2 = m_{1|m_2}.$$

In a Cartesian restriction category, s, k arise as codes for the following maps respectively

$$A \times A \times A \xrightarrow{\langle\langle\pi_0,\pi_2\rangle,\langle\pi_1,\pi_2\rangle\rangle} A^4 \xrightarrow{\bullet\times\bullet} A \times A \xrightarrow{\bullet} A \qquad A \times A \xrightarrow{\pi_0} A$$

An applicative system with s, k is a **partial combinatory algebra** (PCA).

Proposition 2.6.1 (Curry-Schonfinkel). *An applicative system is combinatory complete if and only if it is a PCA.*

Proof. The proof of the combinatory completeness theorem proceeds by constructing points using only s, k that compute any map. The construction of points is given by simulating the lambda calculus

$$\lambda^* x.x = s\, k\, k$$

$$\lambda^* x.z = k\, z$$

$$\lambda^* x.(t_1\, t_2) = s\, (\lambda^* x.t_1)\, (\lambda^* x.t_2).$$

Then, by induction,

$$(\lambda^* x.t)\, m = \{x.t\}m$$

as desired. □

To lift these classical results up to results about applicative systems in Cartesian restriction categories, codes must be total. Thus, we must insist that $s\, x\, y$ and $k\, x$ are total for all x, y.

If $\mathbb{A}$ is a PCA in a Cartesian restriction category, then $\mathsf{Comp}(\mathbb{A})$ may be given the structure of a Cartesian restriction category. The identity is coded by $s\,k\,k$. Composition is given by the combinator $c := \lambda^* x\,y\,z.x\,(y\,z)$, so that if f has code $\underline{f}$ and g has code $\underline{g}$, then gf has name $c\,\underline{f}\,\underline{g}$. Pairing is given by the combinator $\lambda^* a\,b\,p.p\,a\,b$. π_0 is given by k and π_1 is given $\lambda^* x\,y.y$. For restriction, use that $\overline{f} = \langle 1, f\rangle\pi_0$. The converse is also true; if $\mathsf{Comp}(\mathbb{A})$ is a Cartesian restriction category, then s, k may be defined.

Moreover, this construction of combinators makes A into a Turing object and $\bullet^{(n)}$ into a universal map for each n. Thus, $\mathsf{Comp}(\mathbb{A})$ may be given the structure of a Turing category when $\mathbb{A}$ is a PCA.

A summary of the above results is:

Theorem 2.6.2 (Cockett and Hofstra [13]: theorems 4.5 and 4.6). *Let $\mathbb{X}$ be a Cartesian restriction category, and $\mathbb{A}$ an applicative system. Then:*

(i) $\mathbb{A}$ is a PCA if and only if $\mathsf{Comp}(\mathbb{A})$ *is a Cartesian restriction category.*

(ii) If $\mathbb{A}$ is a PCA then $\mathsf{Comp}(\mathbb{A})$ *is a Turing category.*

Every PCA generates a Turing category, and the converse is also true.

Theorem 2.6.3. *Let $\mathbb{X}$ be a Turing category with Turing object T and universal map $T \times T \xrightarrow{\bullet} T$. Then $(T, \bullet)$ is a PCA.*

Thus, we have given another view on Turing categories using the strong connection between Turing categories and PCAs. When moving Turing structure into left additive and differential restriction structure, these connections should be preserved.

Chapter 3

Cartesian Left Additive Restriction Categories

3.1 Cartesian Left Additive Restriction Categories

Cartesian left additive structure is first introduced in [2] as a prerequisite to Cartesian differential structure. The idea behind Cartesian left additive restriction categories is that they should axiomatize arbitrary partial maps between commutative monoids. Much of this material is in Cockett *et al* [15]. Such maps have an obvious addition; if $f, g : A \longrightarrow B$

$$(f + g)(a) = f(a) + g(a)$$

hence if $h : C \longrightarrow A$,

$$(f + g)(h(a)) = f(h(a)) + g(h(a))$$

then it is obvious that the sum is preserved with composition coming from the left. However, these maps are not homomorphisms, so in general

$$h((f + g)(a)) = h(f(a) + g(a)) \neq h(f(a)) + h(g(a)).$$

Since the maps in a restriction setting are partial, the sum of maps should be defined precisely where both maps are. The above observations are captured in the following definition:

Definition 3.1.1. *Let $\mathbb{X}$ be a restriction category. $\mathbb{X}$ is a* **left additive restriction category** *in case each $\mathbb{X}(A, B)$ is a commutative monoid such that $\overline{f + g} = \overline{f}\,\overline{g}$ and $\overline{0} = 1$, and it is left additive; that is, $f(g + h) = fg + fg$ and $f0 = \overline{f}\,0$.*

Note in the definition above that $f0 = \overline{f}\,0 \neq 0$. This is because 0 is total, but $f0$ need not be total.

Some results about left additive structure:

Proposition 3.1.1 (Cockett *et al* [15]: Proposition 3.2). *In any left additive restriction category:*

(i) $f + g = \overline{g}\, f + \overline{f}\, g$;

(ii) *if* $e = \overline{e}$, *then* $e(f+g) = ef + g = f + eg$;

(iii) *if* $f \leq f', g \leq g'$, *then* $f + g \leq f' + g'$;

(iv) *if* $f \smile f', g \smile g'$, *then* $(f+g) \smile (f'+g')$.

Proof. (i)

$$f + g = \overline{f+g}\,(f+g) = \overline{f}\,\overline{g}\,(f+g) = \overline{g}\,\overline{f}\,f + \overline{f}\,\overline{g}\,g = \overline{g}\,f + \overline{f}\,g$$

(ii)

$$\begin{aligned} & f + eg \\ = \;& \overline{eg}\,f + \overline{f}\,eg \text{ by (i)} \\ = \;& \overline{e}\,\overline{g}\,f + \overline{e}\,\overline{f}\,g \\ = \;& \overline{e}\,(\overline{g}\,f + \overline{f}\,g) \\ = \;& e(f+g) \text{ by (i)} \end{aligned}$$

(iii) Suppose $f \leq f'$, $g \leq g'$. Then:

$$\begin{aligned} & \overline{f+g}\,(f'+g') \\ = \;& \overline{f}\,\overline{g}\,(f'+g') \\ = \;& \overline{g}\,\overline{f}\,f' + \overline{f}\,\overline{g}\,g' \\ = \;& \overline{g}\,f + \overline{f}\,g \text{ since } f \leq f', g \leq g' \\ = \;& f + g \text{ by (i).} \end{aligned}$$

so $(f+g) \leq (f'+g')$.

(iv) Suppose $f \smile f'$, $g \smile g'$. This means that $\overline{f}\, f' = \overline{f'}\, f$ and $\overline{g}\, g' = \overline{g'}\, g$. Then we use part (3) of this lemma in the following calculation.

$$\begin{aligned}
\overline{f'+g'}\,(f+g) &= \overline{f'}\,\overline{g'}\,(f+g) \\
&= (\overline{f'}\, f + \overline{g'}\, g) \\
&= (\overline{f}\, f' + \overline{g}\, g') \\
&= \overline{f}\,\overline{g}\,(f'+g') \\
&= \overline{f+g}\,(f'+g')
\end{aligned}$$

Thus, $f+g \smile f'+g'$.

□

Blute *et al* noticed that the **additive maps**, these are the maps h which satisfy $(f+g)h = fh + gh$ and $0h = 0$, form a subcategory. However, additivity does not translate directly into the partial world, as there is no reason for $(f+g)h$ and $fg+gh$ to agree on where they are defined. Thus to define an additive map in a left additive restriction category requires a weaker relation than equality.

Definition 3.1.2. *A map h in a left additive restriction category is* **additive** *when for all f, g, $(f+g)h \smile fh + gh$ and $0h \smile 0$*

We also have the following alternate characterizations of additivity:

Lemma 3.1.1. *A map f is additive if and only if for any x, y,*

$$\overline{xf}\,\overline{yf}\,(x+y)f \leq xf + yf \text{ and } 0f \leq 0$$

or

$$(x\overline{f} + y\overline{f})f \leq xf + yf \text{ and } 0f \leq 0.$$

Proof. The first part is an instance of lemma (2.1.4). Then note that by **[R.4]**

$$\overline{xf}\,\overline{yf}\,(x+y) = \overline{xf}\,x + \overline{yf}\,y = \overline{f}\,x + \overline{f}\,y,$$

so that we get the second part. □

Proposition 3.1.2 (Cockett *et al* [15]: proposition 3.6). *In any left additive restriction category,*

(i) total maps are additive if and only if $\overline{(x+y)f} = \overline{xf+yf}$*;*

(ii) restriction idempotents are additive;

(iii) additive maps are closed under composition;

(iv) if $g \leq f$ *and* f *is additive, then* g *is additive;*

(v) 0 maps are additive, and additive maps are closed under addition.

Proof. In each case, the 0 axiom is straightforward, so we only show the addition axiom.

(i) It suffices to show that if f is total, then $\overline{(x+y)f} = \overline{xf+yf}$. Indeed, if f is total,

$$\overline{(x+y)f} = \overline{x+y} = \overline{x}\,\overline{y} = \overline{xf}\,\overline{yf} = \overline{xf+yf}\,..$$

(ii) Suppose $e = \overline{e}$. Then by **[R.4]**,

$$(xe+ye)\overline{e} = \overline{\overline{xe+ye}\,\overline{e}}\,(xe+ye) \leq xe+ye$$

so that e is additive.

(iii) Suppose f and g are additive. Then

$$\begin{aligned}
& \overline{xfg}\,\overline{yfg}\,(x+y)fg \\
= \ & \overline{xfg}\,\overline{yfg}\,\overline{xf}\,\overline{yf}\,(x+y)fg \\
\leq \ & \overline{xfg}\,\overline{yfg}\,(xf+yf)g \text{ since } f \text{ is additive,} \\
\leq \ & xfg+yfg \text{ since } g \text{ is additive,}
\end{aligned}$$

as required.

(iv) If $g \leq f$, then $g = \overline{g} f$, and since restriction idempotents are additive, and the composites of additive maps are additive, g is additive.

(v) For any 0 map, $(x+y)0 = 0 = 0+0 = x0 + y0$, so it is additive. For addition, suppose f and g are additive. Then we have

$$(x+y)f \smile xf + yf \text{ and } (x+y)g \smile xg + yg.$$

Since adding preserves compatibility, this gives

$$(x+y)f + (x+y)g \smile xf + yf + xg + yg.$$

Then using *left* additivity of x, y, and $x+y$, we get

$$(x+y)(f+g) \smile x(f+g) + y(f+g)$$

so that $f+g$ is additive.

□

The above notion of additive map may be strengthened to assert that the domain is additively closed. This means that whenever fh and gh are defined then $(f+g)h$ is defined, and $0h$ is total.

Definition 3.1.3. *A map h in a left additive restriction category is* **strongly additive** *when for all f, g, $fh + gh \leq (f+g)h$ and $0h = 0$.*

An alternate description, which can be useful for some proofs, is the following:

Lemma 3.1.2. *f is strongly additive if and only if $(x\overline{f} + y\overline{f})f = xf + yf$ and $0f = 0$.*

Proof.

$$
\begin{aligned}
& xf + yf \leq (x+y)f \\
\Leftrightarrow\ & \overline{xf + yf}\,(x+y)f = xf + yf \\
\Leftrightarrow\ & \overline{xf}\,\overline{yf}\,(x+y)f = xf + yf \\
\Leftrightarrow\ & (\overline{xf}\,x + \overline{yf}\,y)f = xf + yf \\
\Leftrightarrow\ & (x\overline{f} + y\overline{f}\,)f = xf + yf \text{ by } [\mathbf{R.4}].
\end{aligned}
$$

□

Proposition 3.1.3 (Cockett *et al* [15]: proposition 3.9). *In a left additive restriction category,*

(i) strongly additive maps are additive, and if f is total, then f is additive if and only if it is strongly additive;

(ii) f is strongly additive if and only if $\overline{f}$ is strongly additive and f is additive;

(iii) identities are strongly additive, and if f and g are strongly additive, then so is fg;

(iv) 0 maps are strongly additive, and if f and g are strongly additive, then so is $f+g$;

(v) if f is strongly additive and has a partial inverse g, then g is also strongly additive.

Proof. In most of the following proofs, we omit the proof of the 0 axiom, as it is straightforward.

(i) Since $\leq$ implies $\smile$, strongly additive maps are additive, and by previous discussion, if f is total, the restrictions of $xf + yf$ and $(x+y)f$ are equal, so $\smile$ implies $\leq$.

(ii) When f is strongly additive then f is additive. To show that $\overline{f}$ is strongly additive we have:

$$
\begin{aligned}
& (x\overline{f} + y\overline{f})\overline{f} \\
= \;& \overline{(x\overline{f} + y\overline{f})f}\,(x\overline{f} + y\overline{f}) \text{ by } [\mathbf{R.4}], \\
= \;& \overline{xf + yf}\,(x\overline{f} + y\overline{f})\text{by 3.1.2 as } f \text{ is strongly additive}, \\
= \;& \overline{x\overline{f}}\;\overline{y\overline{f}}\,(x\overline{f} + y\overline{f}) \\
= \;& x\overline{f} + y\overline{f}
\end{aligned}
$$

Together with $0\overline{f} = \overline{0f}\,0 = \overline{0}\,0 = 0$, this implies, using Lemma 3.1.2, that $\overline{f}$ is strongly additive.

Conversely, suppose $\overline{f}$ is strongly additive and f is additive. First, observe:

$$
\begin{aligned}
\overline{xf + yf} &= \overline{xf}\,\overline{yf} \\
&= \overline{x\overline{f}}\;\overline{y\overline{f}} \\
&= \overline{x\overline{f} + y\overline{f}} \\
&= \overline{(x\overline{f} + y\overline{f})\overline{f}} \text{ by 3.1.2 as } \overline{f} \text{ is strongly additive} \\
&= \overline{(x\overline{f} + y\overline{f})f}
\end{aligned}
$$

This can be used to show:

$$
\begin{aligned}
xf + yf &= \overline{xf + yf}\,(xf + yf) \\
&= \overline{(x\overline{f} + y\overline{f})f}\,(xf + yf) \text{ by the above} \\
&= \overline{xf + yf}\,(x\overline{f} + y\overline{f})f \text{as } f \text{ is additive} \\
&= \overline{(x\overline{f} + y\overline{f})f}\,(x\overline{f} + y\overline{f})f \text{ by the above} \\
&= (x\overline{f} + y\overline{f})f,
\end{aligned}
$$

For the zero case we have:

$$\begin{aligned} 0f &= \overline{0f}\,0 \text{ since } f \text{ is additive} \\ &= \overline{0\overline{f}}\,0 \\ &= \overline{0}\,0 \text{ since } \overline{f} \text{ is strongly additive} \\ &= 0 \end{aligned}$$

Thus, by lemma 3.1.2, f is strongly additive.

(iii) Identities are total and additive, so are strongly additive. Suppose f and g are strongly additive. Then

$$\begin{aligned} & xfg + yfg \\ \leq\ & (xf + yf)g \text{ since } g \text{ strongly additive,} \\ \leq\ & (x + y)fg \text{ since } f \text{ strongly additive,} \end{aligned}$$

so fg is strongly additive.

(iv) Since any 0 is total and additive, 0's are strongly additive. Suppose f and g are strongly additive. Then

$$\begin{aligned} & x(f + g) + y(f + g) \\ =\ & xf + xg + yf + yf \text{ by left additivity,} \\ \leq\ & (x + y)f + (x + y)g \text{ since } f \text{ and } g \text{ are strongly additive,} \\ =\ & (x + y)(f + g) \text{ by left additivity,} \end{aligned}$$

so $f + g$ is strongly additive.

(v) Suppose f is strongly additive and has a partial inverse g. Using the alternate form of strongly additive,

$$
\begin{aligned}
& (x\overline{g} + y\overline{g})g \\
= \; & (xgf + ygf)g \\
= \; & (xg\overline{f} + yg\overline{f})fg \text{ since } f \text{ is strongly additive,} \\
= \; & (xg\overline{f} + yg\overline{f})\overline{f} \\
= \; & xg\overline{f} + yg\overline{f} \text{ since } \overline{f} \text{ strongly additive,} \\
= \; & xg + yg
\end{aligned}
$$

and $0g = 0fg = 0\overline{f} = 0$, so g is strongly additive.

□

In the standard example of commutative monoids and partial maps, addition on a pairing of maps is defined componentwise, and has the effect that for all f, g, h, k, $(f \times g) + (h \times k) = (f + h) \times (g + k)$. In combining Cartesian restriction and left additive restriction structure, this interaction between the product functor and the addition on homsets will be the same.

Definition 3.1.4. *Let $\mathbb{X}$ be a Cartesian restriction category and a left additive restriction category. $\mathbb{X}$ is a* **Cartesian left additive restriction category** *in case $(f + g) \times (h + k) = (f \times h) + (g \times k)$, $0 = 0 \times 0$, and π_0, π_1, Δ are additive.*

For total Cartesian left additive categories, there is a structure theorem which states that Cartesian left additive structure is determined by each object, A, being a commutative monoid with addition canonically defined as $\pi_0 + \pi_1 : A \times A \longrightarrow A$ [2]. This theorem also holds in Cartesian left additive restriction categories.

Theorem 3.1.1 (Cockett *et al* [15]: theorem 3.11). *Let $\mathbb{X}$ be a Cartesian restriction category. Then $\mathbb{X}$ is a Cartesian left additive restriction category iff each object is a total* *commutative monoid*

$$+_A : A \times A \longrightarrow A \qquad\qquad 0_A : 1 \longrightarrow A$$

such that the following exchange [1] *axiom holds:*

$$+_{A\times B} = (A\times B)\times(A\times B) \xrightarrow{ex} (A\times A)\times(B\times B) \xrightarrow{+_A\times +_B} A\times B.$$

Proof.

("$\Rightarrow$"): Suppose $\mathbb{X}$ is a Cartesian left additive restriction category. Then define

$$+_A := \pi_0 + \pi_1 \quad \text{and} \quad 0_A := 0.$$

The identity laws hold since $(0\times 1)(\pi_0+\pi_1) = \pi_0 0 + \pi_1 = \pi_1$, and $(1\times 0)(\pi_0+\pi_1) = \pi_0 + \pi_1 0 = \pi_0$. Further, the following diagram commutes.

$$\begin{tikzcd}
(A\times A)\times A \arrow[r, "\langle \pi_0\pi_0, \pi_1\times 1\rangle"] \arrow[d, "+_A\times 1"'] & A\times(A\times A) \arrow[r, "1\times +_A"] & A\times A \arrow[d, "+_A"] \\
A\times A \arrow[rr, "+_A"'] & & A
\end{tikzcd}$$

To prove the diagram commutes, consider that

$$\begin{aligned}
\langle \pi_0\pi_0, \pi_1\times 1\rangle(1+_A)+_A &= \langle \pi_0\pi_0, (\pi_1\times 1)+_A\rangle(\pi_0+\pi_1) \\
&= \pi_0\pi_0 + (\pi_1\times 1)+_A \\
&= \pi_0\pi_0 + (\pi_1\times 1)(\pi_0+\pi_1) \\
&= \pi_0\pi_0 + (\pi_0\pi_1 + \pi_1) \\
&= (\pi_0\pi_0 + \pi_0\pi_1) + \pi_1 && \text{associativity} \\
&= \pi_0(\pi_0+\pi_1) + \pi_1 \\
&= \pi_0 +_A + \pi_1 \\
&= (+_A \times 1) +_A .
\end{aligned}$$

[1] *Recall that the exchange map is defined by* $ex := \langle \pi_0\times\pi_0, \pi_1\times\pi_1\rangle$ *and that it satisfies, for example,* $\langle\langle f,g\rangle,\langle h,k\rangle\rangle ex = \langle\langle f,h\rangle,\langle g,k\rangle\rangle$ *and* $(\Delta\times\Delta)ex = \Delta$.

Thus, the structure gives a monoid, and this monoid is total since $\overline{0_A} = 1$ and $\overline{\pi_0 + \pi_1} = \overline{\pi_0}\,\overline{\pi_1} = 1$. This monoid is commutative since $\pi_0 + \pi_1 = \pi_1 + \pi_0$. The exchange law holds because,

$$\begin{aligned}
+_{A\times B} &= \pi_0 + \pi_1 \\
&= (\pi_0 + \pi_1)\langle \pi_0, \pi_1 \rangle \\
&= \langle (\pi_0 + \pi_1)\pi_0, (\pi_0 + \pi_1)\pi_1 \rangle \\
&= \langle \pi_0\pi_0 + \pi_1\pi_0, \pi_0\pi_1 + \pi_1\pi_1 \rangle && \pi \text{ is additive} \\
&= \langle (\pi_0 \times \pi_0)+_A, (\pi_1 \times \pi_1)+_B \rangle \\
&= ex(+_A \times +_B).
\end{aligned}$$

("$\Leftarrow$"): Given a total commutative monoid structure on each object, the left additive structure on $\mathbb{X}$ is defined by:

$$\frac{A \xrightarrow{f} B \quad A \xrightarrow{g} B}{A \xrightarrow[f+g := \langle f, g\rangle +_B]{} B}\ \text{add} \qquad \frac{}{A \xrightarrow[0_{AB} := !_A 0_B]{} B}\ \text{zero}$$

That this gives a commutative monoid on each $\mathbb{X}(A, B)$ follows directly from the commutative monoid axioms on B and the cartesian structure.

For the right identity law we must show $f + 0 = \langle f, !_A 0_B \rangle +_B = f$. Consider the following diagram.

$$\begin{tikzcd}
A \arrow[rrr, "{\langle f, !_A 0_B \rangle}"] \arrow[dr, "{\langle 1, !_A \rangle}"] \arrow[dddrrr, bend right, "f"'] & & & B \arrow[ddd, "+_B"] \\
& A \times 1 \arrow[urr, "f \times 0_B"] \arrow[dr, "f \times 1"] & & \\
& & B \times 1 \arrow[uur, "1 \times 0"'] \arrow[dr, "\pi_0"] & \\
& & & B.
\end{tikzcd}$$

The right most shape commutes because of the laws for commutative monoids, and the other shapes commute because of the coherences of Cartesian restriction structure. The left identity holds similarly.

For associativity, precompose the monoid associativity diagram with $\langle\langle f,g\rangle,h\rangle$; around the bottom is $(f+g)+h$ and around the top is $f+(g+h)$.

For commutativity, use the symmetry $\langle\pi_1,\pi_0\rangle{+_B} = {+_B}$ of the commutative monoid laws,

$$\langle f,g\rangle{+_B} = \langle f,g\rangle\langle\pi_1,\pi_0\rangle{+_B} = \langle g,f\rangle\,{+_B}\,.$$

For the interaction with restriction,

$$\overline{0} = \overline{!_A 0_B} = \overline{!_A \overline{0_B}} = \overline{!_A} = 1,$$

and

$$\overline{f+g} = \overline{\langle f,g\rangle{+_B}} = \overline{\langle f,g\rangle\overline{+_B}} = \overline{\langle f,g\rangle} = \overline{f}\,\overline{g}\,.$$

For the interaction with composition,

$$f0 = f!_B 0_C = \overline{f}\,!_A 0_C = \overline{f}\,0,$$

and

$$f(g+h) = f(\langle g,h\rangle{+_C}) = \langle fg,fh\rangle{+_C} = fg+fh.$$

The requirement that $(f+g)\times(h+k) = (f\times h)+(g\times k)$ follows from the exchange axiom:

$$\begin{tikzcd}
A\times C \arrow[rr, "\langle f\times h,g\times k\rangle"] \arrow[dr, "\langle f,g\rangle\times\langle h,k\rangle"'] \arrow[ddrr, bend right, "\langle f,g\rangle+_B\times\langle h,k\rangle+_D"'] & & (B\times D)\times(B\times D) \arrow[dl, "\mathrm{ex}"'] \arrow[dd, "+_{B\times D}"] \\
& (B\times B)\times(D\times D) \arrow[dr, "+_B\times+_D"] & \\
& & B\times D
\end{tikzcd}$$

the right triangle is the exchange axiom, and the other two shapes commute by the cartesian coherences. Also

$$0 = 0\langle\pi_0,\pi_1\rangle = \langle 0,0\rangle = \langle\pi_0 0,\pi_1 0\rangle = 0\times 0.$$

It remains to prove that the diagonal map and projections are additive.

Since π_0 is total, π_0 is additive in case for all $f, g : A \longrightarrow B \times C$, $(f+g)\pi_0 = f\pi_0 + g\pi_0$, which is shown by the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\langle f,g\rangle"] \arrow[dr, "{\langle\langle f\pi_0,g\pi_0\rangle,\langle f\pi_1,g\pi_1\rangle\rangle}"'] \arrow[drr, bend right=40, "\langle f\pi_0,g\pi_0\rangle"'] & (B\times C)\times(B\times C) \arrow[r, "+_{(B\times C)}"] \arrow[d, "ex"] & B\times C \arrow[r, "\pi_0"] & B \\
 & (B\times B)\times(C\times C) \arrow[r, "\pi_0"'] \arrow[ur, "+_B\times +_C"'] & B\times B \arrow[ur, "+_B"'] &
\end{tikzcd}$$

A similar argument shows that π_1 is additive. Since Δ is total, Δ is additive when for all $f, g : A \longrightarrow B$, $f\Delta + g\Delta = (f+g)\Delta$. This is shown by the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\langle f\Delta,g\Delta\rangle"] \arrow[d, "\langle f,g\rangle"] \arrow[dd, bend right=60, "\langle f,g\rangle+_B"'] & (B\times B)\times(B\times B) \arrow[d, "ex"] \arrow[dd, bend left=60, "+_{B\times B}"] \\
B\times B \arrow[r, "\Delta"] \arrow[ur, "\Delta\times\Delta"] \arrow[d, "+_B"] \arrow[dr, "\langle +_B,+_B\rangle"'] & (B\times B)\times(B\times B) \arrow[d, "+_B\times +_B"] \\
B \arrow[r, "\Delta"'] & B\times B
\end{tikzcd}$$

This completes the proof that $\mathbb{X}$ is a Cartesian left additive restriction category. $\square$

Proposition 3.1.4 (Cockett *et al* [15]: proposition 3.1.4). *In a cartesian left additive restriction category:*

(i) $\langle f, g\rangle + \langle f', g'\rangle = \langle f + f', g + g'\rangle$ *and* $\langle 0, 0\rangle = 0$*;*

(ii) if f *and* g *are additive, then so is* $\langle f, g\rangle$*;*

(iii) the projections are strongly additive, and if f *and* g *are strongly additive, then so is* $\langle f, g\rangle$*,*

(iv) f *is additive if and only if*

$$(\pi_0 + \pi_1)f \smile \pi_0 f + \pi_1 f \text{ and } 0f \smile 0;$$

(that is, in terms of the monoid structure on objects, $(+)(f) \smile (f \times f)(+)$ *and* $0f \smile 0$*),*

(v) f *is strongly additive if only if*

$$(\pi_0 + \pi_1)f \geq \pi_0 f + \pi_1 f \text{ and } 0f = 0;$$

(that is, $(+)(f) \geq (f \times f)(+)$ *and* $0f \geq 0$*).*

Note that f being strongly additive only implies that $+$ and 0 are lax natural transformations.

Proof. (i) Since the second term is a pairing, it suffices to show they are equal when post-composed with projections. Post-composing with π_0, we get

$$\begin{aligned}
& (\langle f, g\rangle + \langle f', g'\rangle)\pi_0 \\
= \; & \langle f, g\rangle\pi_0 + \langle f', g'\rangle\pi_0 \text{ since } \pi_0 \text{ is additive,} \\
= \; & \overline{g}\, f + \overline{g'}\, f' \\
= \; & \overline{g}\,\overline{g'}\,(f + f') \\
= \; & \overline{g + g'}\,(f + f') \\
= \; & \langle f + f', g + g'\rangle\pi_0
\end{aligned}$$

as required. The 0 result is direct.

(ii) We need to show

$$(x + y)\langle f, g\rangle \smile x\langle f, g\rangle + y\langle f, g\rangle;$$

however, since the first term is a pairing, it suffices to show they are compatible when post-composed by the projections. Indeed,

$$(x + y)\langle f, g\rangle\pi_0 = (x + y)\overline{g}\, f \smile x\overline{g}\, f + y\overline{g}\, f$$

while since π_0 is additive,

$$(x\langle f,g\rangle + y\langle f,g\rangle)\pi_0 = x\langle f,g\rangle\pi_0 + y\langle f,g\rangle\pi_0 = x\overline{g}\,f + y\overline{g}\,f$$

so the two are compatible, as required. Post-composing with π_1 is similar.

(iii) Since projections are additive and total, they are strongly additive. If f and g are strongly additive,

$$\begin{aligned}
& x\langle f,g\rangle + y\langle f,g\rangle \\
= \; & \langle xf, xg\rangle + \langle yf, yg\rangle \\
= \; & \langle xf + yf, xg + yg\rangle \text{ by (i)} \\
\leq \; & \langle (x+y)f, (x+y)g\rangle \text{ since } f \text{ and } g \text{ are strongly additive,} \\
= \; & (x+y)\langle f,g\rangle
\end{aligned}$$

so $\langle f,g\rangle$ is strongly additive.

(iv) If f is additive, the condition obviously holds. Conversely, if we have the condition, then f is additive, since

$$(x+y)f = \langle x,y\rangle(\pi_0+\pi_1)f \smile \langle x,y\rangle(\pi_0 f + \pi_1 f) = xf + yf$$

as required.

(v) Similar to the previous proof.

□

3.2 Simple Slices

Suppose S is a comonad, and S is a restriction functor on a Cartesian left additive restriction category $\mathbb{X}$; note that we are *not* assuming that S preserves addition. We can define maps $\mathbf{+}_A = \epsilon +_A$ and $\mathbf{0}_A = \epsilon 0_A$ to be the components of a total commutative monoid structure on the objects of $\mathbb{X}_{\mathcal{S}}$. That is, we can define monoid structure on the objects of $\mathbb{X}_{\mathcal{S}}$ as the image, under the inclusion $\mathbb{X} \xrightarrow{J} \mathbb{X}_{\mathcal{S}}$, of the canonical monoid structure on $\mathbb{X}$. We will prove that this does give a total commutative monoid structure on $\mathbb{X}_{\mathcal{S}}$. This implies that $\mathbb{X}_{\mathcal{S}}$ is a Cartesian left additive restriction category, and that the inclusion $\mathbb{X} \xrightarrow{J} \mathbb{X}_{\mathcal{S}}$ is a Cartesian left additive restriction functor. In particular, Cartesian left additive restriction structure lifts to $\mathbb{X}[A]$.

The following proposition is a generalization of Blute *et al* [2], proposition 1.3.3 to the setting of Cartesian left additive restriction categories.

Proposition 3.2.1. *If $\mathbb{X}$ is a Cartesian left additive restriction category, and $\mathcal{S} = (S, \epsilon, \delta)$ is a comonad with S a restriction functor and where ϵ is total then $\mathbb{X}_{\mathcal{S}}$ is a Cartesian left additive restriction category. Further, the inclusion $\mathbb{X} \hookrightarrow \mathbb{X}_{\mathcal{S}}$ is a Cartesian left additive restriction functor.*

Proof. In proposition (2.3.2), we proved that the inclusion $J : \mathbb{X} \longrightarrow \mathbb{X}_{\mathcal{S}}$ strictly preserves products. This fact is quite useful for proving that $\mathbb{X}_{\mathcal{S}}$ has Cartesian left additive restriction structure. Note that since J is a functor, it preserves composition, and in $\mathbb{X}_{\mathcal{S}}$ this means that for all h, k, $J(hk) = \boldsymbol{J(hk)} = \boldsymbol{J(h)J(k)} = (J(h))^{\sharp}J(k)$.

Now, for the left unit law we must show $(\mathbf{0} \times \mathbf{1})\boldsymbol{+_A} = \boldsymbol{\pi_1}$. Consider

$$(\mathbf{0} \times \mathbf{1})\boldsymbol{+_A} = (J(0) \times J(1))^{\sharp}J(+_A) = (J(0 \times 1))^{\sharp}J(+_A)$$
$$= J((0 \times 1)+_A) = J(\pi_1) = \boldsymbol{\pi_1}.$$

Similarly, the right unit law holds. For associativity, we must show $\boldsymbol{a_\times}(\mathbf{1} \times \boldsymbol{+_A})\boldsymbol{+_A} = (\boldsymbol{+_A} \times \mathbf{1})\boldsymbol{+_A}$. Consider

$$\boldsymbol{a_\times}(\mathbf{1} \times \boldsymbol{+_A})\boldsymbol{+_A} = (J(a_\times))^{\sharp}(J(1) \times J(+_A))^{\sharp}+_A$$
$$= J(a_\times(1 \times +_A)+_A) = J((+_A \times 1)+_A) = (J(+_A) \times J(1))^{\sharp}J(+_A)$$
$$= (\boldsymbol{+_A} \times \mathbf{1})\,\boldsymbol{+_A}.$$

Finally, we must show the exchange law, $+_{A\times B} = ex(+_A \times +_B)$, holds. Note, that $ex = J(ex)$, so that

$$+_{A\times B} = J(+_{A\times B}) = J(ex(+_A \times +_B)) = (J(ex))^{\sharp}(J(+_A) \times J(+_B)) = ex(+_A \times +_B).$$

Thus, $\mathbb{X}_S$ is a Cartesian left additive restriction functor, and $\mathbb{X} \xrightarrow{J} \mathbb{X}_S$ is a Cartesian left additive restriction functor. $\square$

Note that the addition in $\mathbb{X}_S$ is

$$f + g = \langle f, g\rangle +_A = (\langle f, g\rangle)^{\sharp}\epsilon +_A = \langle f, g\rangle +_A,$$

in other words, the addition in $\mathbb{X}_S$ is simply inherited from $\mathbb{X}$.

The following is now obvious.

Proposition 3.2.2. *For every Cartesian left additive restriction category $\mathbb{X}$, $\mathbb{X}[A]$ is a Cartesian left additive restriction category.*

Note, that if a map $f : X \times C \longrightarrow Y$ in $\mathbb{X}[A]$ is strongly additive then $(\pi_0 + \pi_1)f \geq \pi_0 f + \pi_1 f$ and $0f = 0$. This means

$$X \times C \times A \xrightarrow{\langle \pi_0 + \pi_1, \pi_2\rangle f} Y \geq X \times C \times A \xrightarrow{\langle \pi_0, \pi_2\rangle f + \langle \pi_1, \pi_2\rangle f} Y,$$

and $\langle 0, \pi_1\rangle f = f$. Thus:

Proposition 3.2.3. *A map f is strongly additive if and only if f is strongly additive in its first variable.*

3.3 Idempotent Splitting

As we shall see, idempotents that split as rs where r is strongly additive are important to the development of left additive Turing categories. In this section, we will show that any Cartesian left additive restriction category can be completed to one in which "retractively additive" idempotents split in the above manner.

Definition 3.3.1. *Let $\mathbb{X}$ be a Cartesian left additive restriction category. An idempotent e is* **retractively additive** *when,* $0\overline{e} = 0$, $\overline{(\pi_0 + \pi_1)e} \geq \overline{e \times e}$ *and* $(\pi_0 + \pi_1)e \geq (\pi_0 e + \pi_1 e)e$.

Note that the last conidition is equivalent to having, for all f, g, $(f+g)e \geq (fe+ge)e$. The following lemma will be useful in the next section.

Lemma 3.3.1. *If $\overline{e}$ is strongly additive then for all f, g*

$$\overline{fe+ge} = \overline{(fe+ge)e}\,.$$

Proof. Use the fact that $e\overline{e} = \overline{e}\overline{e}\,e = \overline{e}\,e = e$. Also use lemma (3.1.2)

$$\overline{(fe+ge)e} = \overline{(fe\overline{e}+ge\overline{e})\overline{e}} = \overline{f\overline{e}+g\overline{e}} = \overline{fe}\,\overline{ge} = \overline{fe+ge}$$

□

Retractively additive idempotents have the following property,

Lemma 3.3.2. *The following hold in any Cartesian left additive restriction category,*

(i) If $(f+g)e \geq (fe+ge)e$ then e is retractively additive if and only if $\overline{e}$ is strongly additive.

(ii) If e is strongly additive then e is retractively additive.

(iii) If e is retractively additive then so is $\overline{e}$.

(iv) If e is strongly additive, e' is retractively additive, and $ee' = e'e$ then ee' is retractively additive.

(v) If e, e' are retractively additive then so is $e \times e'$.

Part 2 in particular says that all identities are retractively additive.

Proof.

(i) Assume $(f+g)e \geq (fe+ge)e$.

Then assume e is retractively additive. Then it suffices to show that $(\pi_0+\pi_1)\overline{e} \geq \pi_0\overline{e} + \pi_1\overline{e}$. Note that $\overline{e\times e} = \overline{\pi_0 e}\,\overline{\pi_1 e} = \overline{\pi_0 e + \pi_1 e}$. Then,

$$
\begin{aligned}
(\pi_0+\pi_1)\overline{e} &= \overline{(\pi_0+\pi_1)e}\,(\pi_0+\pi_1) \geq \overline{\pi_0 e + \pi_1 e}\,(\pi_0+\pi_1) \\
&= \overline{\pi_0 e}\,\overline{\pi_1 e}\,(\pi_0+\pi_1) = \pi_0\overline{e} + \pi_1\overline{e}.
\end{aligned}
$$

Assume, $\overline{e}$ is strongly additive. Then it suffices to show $\overline{(\pi_0+\pi_1)e} \geq \overline{e\times e}$. Consider,

$$
\overline{(\pi_0+\pi_1)e} \geq \overline{\pi_0\overline{e} + \pi_1\overline{e}} = \overline{\pi_0 e}\,\overline{\pi_1 e} = \overline{\pi_0 e + \pi_1 e} = \overline{e\times e}
$$

(ii) Suppose e is strongly additive. Then $\overline{e}$ is also strongly additive. Also,

$$
(f+g)e = (f+g)ee \geq (fe+ge)e.
$$

Thus by part (i) of this lemma, e is retractively additive.

(iii) Suppose e is retractively additive. Then by part (i) of this lemma $\overline{e}$ is strongly additive. By part (ii) of this lemma $\overline{e}$ is retractively additive.

(iv) If e is strongly additive and e' is retractively additive. Then,

$$
\begin{aligned}
0\overline{ee'} &= \overline{0ee'}\,0 = 0ee'0 = 0e\overline{e'}\,0 \\
&= 0\overline{e'}\,0 = 00 = 0.
\end{aligned}
$$

Also,

$$
\begin{aligned}
(f+g)\overline{ee'} &= \overline{(f+g)ee'}\,(f+g) \geq \overline{(fe+ge)e'}\,(f+g) \\
&\geq \overline{(fee'+gee')e'}\,(f+g) = \overline{fee'+gee'}\,(f+g) \qquad \text{lemma (3.3.1)} \\
&= f\overline{ee'} + g\overline{ee'}
\end{aligned}
$$

Thus $\overline{ee'}$ is strongly additive.

Next,

$$(fee' + gee')ee' = (fee' + gee')e'e \leq (fe + ge)e'e$$
$$\leq (f + g)ee'e = (f + g)ee'$$

Thus, ee' is retractively additive.

(v) Since both maps are pairings, it suffices to show the inequality after either projection. For π_0

$$\begin{aligned}
&(f+g)(e \times e')\pi_0 \\
&= (f+g)\overline{\pi_1 e'}\,\pi_0 e \\
&= \overline{(f\pi_1 + g\pi_1)e'}\,(f\pi_0 + g\pi_0)e \\
&\geq \overline{(f\pi_1 e' + g\pi_1 e')e'}\,(f\pi_0 e + g\pi_0 e)e \\
&= \overline{(\overline{f\pi_0 e}\, f\pi_1 e' + \overline{g\pi_0 e}\, g\pi_1 e')e'}\,\overline{f\pi_1 e'}\,\overline{g\pi_1 e'}\,(f\pi_e + g\pi_0 e)e \\
&= (f(e \times e') + g(e \times e'))(e \times e')\pi_0.
\end{aligned}$$

Thus, $(f+g)(e\times e')\pi_0 \geq (f(e\times e')+g(e\times e'))(e\times e')\pi_0$. Similarly, $(f+g)(e\times e')\pi_1 \geq (f(e \times e') + g(e \times e'))(e \times e')\pi_1$. Thus,

$$(f+g)(e \times e') \geq (f(e \times e') + g(e \times e'))(e \times e').$$

□

On part (iv) of the above lemma: $ee' = e'e$ is needed to ensure that ee' is actually an idempotent. One should also note that generally, retractively additive idempotents that commute are not closed to composition. This is because if e, e' are retractively additive then $\overline{e}, \overline{e'}$ are strongly additive. However, it does not follow that $\overline{ee'}$ is strongly additive. For an explicit counter example, take $e, e' : \mathbb{N} \longrightarrow \mathbb{N}$ defined,

$$e(n) = 1 \qquad e'(n) = \begin{cases} \uparrow & n = 1 \\ n & \text{else} \end{cases}.$$

It is easy to show that e, e' are both retractively additive. However, $\overline{ee'}$ is not strongly additive because

$$(0\overline{ee'})(n) = \overline{ee'}(0) = \uparrow = \emptyset(0).$$

Where $\emptyset$ is the nowhere defined map rather than the 0 map. Thus ee' is not strongly additive.

Definition 3.3.2. *Let $e = rs$ and $sr = 1$. e* **left additively splits** *if r is strongly additive.*

Next, we have an observation linking left additively split idempotents and retractively additive idempotents.

Lemma 3.3.3. *If e left additively splits, then e is retractively additive.*

Proof. Suppose e left additively splits rs where r is strongly additive. Then $\overline{e} = \overline{rs} = \overline{r}$ is strongly additive. Next, we will show that $(fr + gr)s = (fe + ge)e$. First, $(fr + gr)s = (fer + ger)s \leq (fe + ge)e$. Now,

$$\begin{aligned}\overline{(fr + gr)s} &= \overline{fr + gr} = \overline{fr}\,\overline{gr} = \overline{f\overline{r}}\,\overline{g\overline{r}} \\ &= \overline{fe}\,\overline{ge} = \overline{fe + ge}.\end{aligned}$$

Thus $(fr + gr)s = (fe + ge)e$. This allows,

$$(f + g)e = (f + g)rs \geq (fr + gr)s = (fe + ge)e,$$

as required. □

Now, every left additive split idempotent is retractively additive, but retractively additive idempotents need not split. A Cartesian left additive restriction category is a **split Cartesian left additive restriction category** when every retractively additive idempotent splits with a left additive retraction (i.e. left additively).

In a Cartesian left additive restriction category, suppose $e = A \xrightarrow{r} E \xrightarrow{s} A$. Then consider the following diagram,

$$\begin{tikzcd}
& A \arrow[rr, "e"] \arrow[dr, "r"] & & A \\
X \arrow[rr, "{f,g}"] & & E \arrow[ur, "s"'] &
\end{tikzcd}$$

One could add f, g by just $f + g$. However one coud add $fs + gs$ then postcompose with r; this results in $(fs + gs)r$. Now,

Proposition 3.3.1. *If rs is a left additively split idempotent, then $f + g = (fs + gs)r$ and $0 = 0r$.*

Proof. The 0 case is by definition. Now,

$$(fs + gs)r \geq fsr + gsr = f + g.$$

Next, use lemma (3.3.1) in the following calculation,

$$\begin{aligned}
\overline{(fs+gs)r} &= \overline{(fs+gs)\overline{r}} = \overline{(fs+gs)e} \\
&= \overline{(fse+gse)e} = \overline{fse+gse} = \overline{fs+gs} \\
&= \overline{fs}\,\overline{gs} = \overline{f}\,\overline{g} = \overline{f+g}.
\end{aligned}$$

This proves that $(fs + gs)r = f + g$. ☐

The above proposition may be restated in terms of the monoid structure: if rs left additively splits, then

$$+_B = (s \times s) +_A r \qquad 0_B = 0_A r.$$

Also, left additive splittings are unique up to an additive isomorphism.

Proposition 3.3.2. *If*

$$\begin{tikzcd} & B \arrow[dr, "s"] & \\ A \arrow[ur, "r"] \arrow[rr, "e"] \arrow[dr, "r'"'] & & A \\ & B' \arrow[ur, "s'"'] & \end{tikzcd}$$

commutes with r, r' strongly additive and $sr = 1$, $s'r' = 1$ then there is a unique isomorphism $\alpha : B \longrightarrow B'$ such that $r\alpha = r'$ and $\alpha s' = s$. Moreover, α is strongly additive.

Proof. Define $\alpha := B \xrightarrow{s} A \xrightarrow{r'} B'$.

First $\alpha s'r = sr's'r = ser = sr = 1$. Thus α is an isomorphism.

Next,

$$r\alpha = rsr' = er' = r',$$

$$\alpha s' = sr's' = se = s.$$

Thus α satisfies the required identities. Suppose β satisfies the same identities. In particular $\alpha s' = \beta s'$ Since s' is monic, it follows that $\alpha = \beta$. Thus α is unique. Now,

$$0\alpha = 0r\alpha = 0r' = 0$$

Also,

$$(f+g)\alpha = (fs+gs)r\alpha = (fs+gs)r' \geq (fsr' + gsr') = f\alpha + g\alpha.$$

Thus α is strongly additive. □

Next, we give a construction that completes a Cartesian left additive restriction category to be a split Cartesian left additive restriction category. Let $\mathbb{X}$ be a Cartesian left additive restriction category. If $\mathcal{E}$ is a product closed collection of retractively additive idempotents that contains all the identities, then first each monoid of $\mathbb{X}$ lifts directly into $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$. Further, each $e \in \mathcal{E}$ splits

$$1_A \xrightarrow{e} e \xrightarrow{e} 1_A.$$

If e is retractively additive then for e to be left additively split, proposition (3.3.1) tells us that

$$+_e := (e \times e) +_A e \qquad 0_e = 0e.$$

Denote $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ with this additive structure by $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$. Then

Lemma 3.3.4. *Let $\mathbb{X}$ be a Cartesian left additive restriction category. If $\mathcal{E}$ is a product closed collection of retractively additive idempotents that contains all the identities, then $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$ is a Cartesian left additive restriction category.*

Proof. Since $\mathcal{E}$ is product closed, $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$ is a Cartesian restriction category. We will use theorem (3.1.1) to show that $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$ is a Cartesian left additive restriction category. Thus we must show that each e is a total commutative monoid.

First, we will show that the monoid structure is total.

$$\overline{0e} = \overline{0\overline{e}} = \overline{0} = ! : \top \to \top.$$

To show that addition is total we must show that $(e \times e)\overline{(e \times e) +_A e} = e \times e$. It suffices to show that $\overline{(e \times e) +_A e} = \overline{e \times e}$. Use lemma (3.3.1),

$$\overline{(e \times e) +_A e} = \overline{(\pi_0 e + \pi_1 e)e} = \overline{\pi_0 e + \pi_1 e} = \overline{e \times e}.$$

Thus the monoid structure is total.

To show show that the unit laws hold, we will show that the right unit law holds, as the left holds by symmetry. First,

$$(fe + 0e)e \leq (f + 0)e = fe.$$

Also,

$$\overline{(fe + 0e)e} = \overline{fe + 0e} = \overline{fe}\,\overline{0e} = \overline{fe}.$$

Thus the right unit law holds. The left law holds, again, by symmetry.

For associativity note,

$$
\begin{aligned}
((\pi_0 e+\pi_1 e)e+\pi_2 e)e &\geq ((\pi_0+\pi_1)e+\pi_2 e)e \\
&\geq ((\pi_0+\pi_1)+\pi_2)e = (\pi_0+(\pi_1+\pi_2))e,
\end{aligned}
$$

$$(\pi_0 e+(\pi_1 e+\pi_2 e)e)e \geq (\pi_0 e+(\pi_1+\pi_2)e)e \geq (\pi_0+(\pi_1+\pi_2))e,$$

and using lemma (3.3.1) multiple times,

$$
\begin{aligned}
\overline{((\pi_0 e+\pi_1 e)e+\pi_2 e)e} &= \overline{(\pi_0 e+\pi_1 e)e+\pi_2 e} = \overline{(\pi_0 e+\pi_1 e)e}\,\overline{\pi_2 e} \\
&= \overline{\pi_0 e}\,\overline{\pi_1 e}\,\overline{\pi_2 e} = \cdots \\
&= \overline{(\pi_0 e+(\pi_1 e+\pi_2 e)e)e}\,.
\end{aligned}
$$

Then by lemma (2.1.3), $((\pi_0 e+\pi_1 e)e+\pi_2 e)e = (\pi_0 e+(\pi_1 e+\pi_2 e)e)e$.

Commutativity is obvious.

For exchange use the naturality of ex:

$$
\begin{aligned}
+_{(e\times e')} &= ((e\times e')\times(e\times e'))\,+_{(A\times B)}\,(e\times e') = ((e\times e')\times(e\times e'))ex(+_A\times +_B)(e\times e') \\
&= ex((e\times e)\times(e'\times e'))(+_A\times +_B)(e\times e') = ex((e\times e)+_A e\times(e'\times e')+_B e').
\end{aligned}
$$

Thus by theorem (3.1.1), $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$ is a Cartesian left additive restriction category. $\square$

Theorem 3.3.1. *Let $\mathbb{X}$ be a Cartesian left additive restriction category, and $\mathcal{E}$ be the collection of retractively additive idempotents. Then $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$ is a split Cartesian left additive restriction category.*

Proof. Lemma 3.3.4 shows that $\mathsf{Split}_{\mathcal{E}'_+}(\mathbb{X})$ is a Cartesian left additive restriction category. Thus it remains to show that all retractively additive maps left additively split. Let $e \in I$, and suppose $d \xrightarrow{e} d$. Then e splits,

$$
\begin{tikzcd}
d \arrow[rr, "e"] \arrow[dr, "e"'] & & d \\
& e \arrow[ur, "e"'] &
\end{tikzcd}
$$

We must show that $d \xrightarrow{e} e$ is strongly additive, the 0 case is obvious. Thus it remains to show that if $f, g : d' \longrightarrow d$ then $(fd + gd)de \geq (fe + ge)e$. Now, since e is retractively additive,

$$(fd + gd)de = (f + g)e \geq (fe + ge)e,$$

as required. $\square$

Also, interestingly, if $e \in \mathcal{E}$ is strongly additive then e splits in $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$ into strongly additive pieces. This is proved by the following lemma.

Lemma 3.3.5. *Let $e = ee$ be left additively split as rs. Then e is strongly additive if and only if s is strongly additive.*

Proof. If s is strongly additive, then e is strongly additive as strongly additive maps are closed to composition. Thus it remains to show that if r and e are both strongly additive then s is strongly additive. Since s is total, we must show that $0s = 0$ and $(\pi_0 + \pi_1)s = \pi_0 s + \pi_1 s$. For the zero case,$0s = (0r)s = 0$ because r is strongly additive. For the additive case, use proposition (3.3.1), then use lemma (3.1.2):

$$\begin{aligned}(\pi_0 + \pi_1)s &= (\pi_0 s + \pi_1 s)rs = (\pi_0 se + \pi_1 se)e \\ &= \pi_0 se\overline{e} + \pi_1 se\overline{e})e = \pi_0 se + \pi_1 se \\ &= \pi_0 s + \pi_1 s\end{aligned}$$

$\square$

Then it is clear that if $e \in \mathcal{E}$ is strongly additive then it is also retractively additive; thus, e splits in $\mathsf{Split}_{\mathcal{E}_+}(\mathbb{X})$ into rs where r is strongly additive. The above lemma shows that now, s must also be strongly additive.

Basic terms:

$$\frac{}{\Gamma, x : A \vdash x : A}\ \textsc{Proj} \qquad \frac{\Gamma \vdash t : A \quad \Gamma, pat : A \vdash s : B}{\Gamma \vdash \{pat.s\}T : B}\ \textsc{Cut}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \cdots \quad \Gamma \vdash t_n : A_n \quad \sigma(f) = ([A_1, \ldots, A_n], B)}{\Gamma \vdash f(t_1, \ldots, t_n) : B}\ \textsc{Fun}$$

Cartesian terms:

$$\frac{}{\Gamma \vdash () : \mathbf{1}}\ \textsc{Unit} \qquad \frac{\Gamma \vdash t_1 : A_1 \quad \cdots \Gamma \vdash t_n : A_n}{\Gamma \vdash (t_1, \ldots, t_n) : A_1 \times \cdots \times A_n}\ \textsc{Tuple}$$

$$\frac{\Gamma, t_1 : A_1, \ldots, t_n : A_n \vdash t : A}{\Gamma, (t_1, \ldots, t_n) : A_1 \times \cdots A_n \vdash t : A}\ \textsc{Pat}$$

Additive Terms:

$$\frac{}{\Gamma \vdash 0 : A}\ \textsc{Zero} \qquad \frac{\Gamma \vdash t_1 : A \quad \Gamma \vdash t_2 : A}{\Gamma t_1 + t_2 : A}\ \textsc{Add}$$

Partial Terms:

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash s : B}{\Gamma \vdash t_{|s} : A}\ \textsc{Rest}$$

Table 3.1: Cartesian Left Additive Restriction Term Formation Rules

3.4 Term Logic

We extend the term logic for Cartesian restriction categories to Cartesian left additive restriction categories, and then prove this extension is sound and complete with respect to a translation into Cartesian left additive restriction categories.

The syntax is extended with a formal 0 term and formal sums of terms, which we assume forms a monoid on the terms.

$$T := V \mid () \mid (T, \ldots, T) \mid \{p.T\}T \mid 0 \mid T + T \mid T_{|T}$$

$$Ty := \mathbb{T} \mid \mathbf{1} \mid Ty \ \times \cdots \times \ Ty$$

The type judgments are extended in table (3.1).

Of the six classes of equalities we only need to update the cut elimination rules and equations for the logic.

Cut-elimination The equalities for cut elimination of Cartesian left additive restriction terms is presented in table 2.2.

Table 2.2, and
ESB.10 $\{x.0\}t \Rrightarrow 0_{\mid t}$
ESB.11 $\{0.t\}0 \Rrightarrow t$
ESB.12 $\{x.(s_1 + s_2)\}t \Rrightarrow \{x.s_1\}t + \{x.s_2\}t$

Table 3.2: Cut elimination for Cartesian left additive restriction terms

Table 2.3 and
CLARC.8 $t_{\mid 0} = t$
CLARC.9 $t_1 + \cdots + (t_k)_{\mid s} + \cdots + t_n = (\sum_i^n t_i)_{\mid s}$
CLARC.10 $t_{\mid \sum_i^n s_i} = t_{\mid s_1, \ldots, s_n}$

Table 3.3: Equations for Cartesian left additive restriction terms

Equations The equations for Cartesian left additive restriction term logic are presented in table 3.3.

We note that cut elimination holds in this logic as well. We also note that lemmas (2.5.1 and 2.5.2) and corollary (2.5.1) have analogous counterparts. The proofs by induction in lemma (2.5.1) simply require two extra cases – one for the zero and one for the sum – and these new cases are easy to show.

3.4.1 Soundness

First, extend the translation as in table 3.4.

Table 2.4 and
TCR.8 $[\![\Gamma \vdash 0]\!] = 0$
TCR.9 $[\![\Gamma \vdash t_1 + t_2]\!] = [\![\Gamma \vdash t_1]\!] + [\![\Gamma \vdash t_2]\!]$

Table 3.4: Translation of Cartesian left additive restriction terms

To prove the soundness of this translation, it suffices to prove that [**CLARC**.8-10] hold.

Theorem 3.4.1. *The translation defined in table (3.4) is sound.*

Proof. **CLARC.8** For this use the fact that 0 is total,

$$[\![\Gamma \vdash t_{|0}]\!] = \overline{[\![\Gamma \vdash 0]\!]}\,[\![\Gamma \vdash t]\!] = [\![\Gamma \vdash t]\!]$$

CLARC.9 Using proposition (3.1.1), if $e = \overline{e}$, then $e(f_1 + \cdots + f_n) = f_1 + \cdots + ef_i + \cdots + f_n$. Then,

$$[\![\Gamma \vdash t_1 + \cdots + (t_k)_{|s} + \cdots + t_n]\!] = [\![\Gamma \vdash t_1]\!] + \cdots + \overline{[\![\Gamma \vdash s]\!]}\,[\![\Gamma \vdash t_k]\!] + \cdots [\![\Gamma \vdash t_n]\!]$$
$$= \overline{[\![\Gamma \vdash s]\!]} \sum_i [\![\Gamma \vdash t_i]\!] = [\![\Gamma \vdash \left(\sum_i t_i\right)_{|s}]\!]$$

CLARC.10 Using $\overline{f+g} = \overline{f}\,\overline{g}$:

$$[\![\Gamma \vdash t_{|\sum_i s_i}]\!] = \overline{[\![\Gamma \vdash \sum_i s_i]\!]}\,[\![\Gamma \vdash t]\!] = \overline{[\![\Gamma \vdash s_1]\!]} \cdots \overline{[\![\Gamma \vdash s_n]\!]}\,[\![\Gamma \vdash t]\!] = [\![\Gamma \vdash t_{|s_1,\ldots,s_n}]\!]$$

□

3.4.2 Completeness

We form the classifying category, now of a Cartesian left additive restriction theory. To do so we must add a zero map $\Gamma \mapsto 0$, and define addithition,

$$\frac{f = \Gamma \mapsto t_1 \quad g = \Gamma \mapsto t_2}{f + g = \Gamma \mapsto t_1 + t_2} \quad .$$

Then it is straightforward to show that $\mathbb{C}[\mathcal{T}]$ is a Cartesian left additive restriction category. Note that 0 is total and $f0 = \overline{f}\,0$. Also,

$$(p \mapsto t)(q \mapsto s_1 + s_2) = p \mapsto \{q.s_1 + s_2\}t$$
$$= p \mapsto \{q.s_1\}t + \{q.s_2\}t = (p \mapsto t)(q \mapsto s_1) + (p \mapsto t)(q \mapsto s_2)$$

And $\overline{p \mapsto f + g} = p \mapsto p_{|f+g} = p \mapsto p_{|f,g} = \overline{p \mapsto f}\,\overline{p \mapsto g}$. Thus $\mathbb{C}[\mathcal{T}]$ is a left additive restriction category. The projections and $\Delta = p \mapsto (p, p)$ are easily shown to be additive. As the addition is defined componentwise, it now follows that

Proposition 3.4.1. *$\mathbb{C}[\mathcal{T}]$ is a Cartesian left additive restriction category.*

3.5 Turing Categories

Now, we are ready to define left additive Turing categories.

Definition 3.5.1. *A Turing category $\mathbb{X}$ that has left additive structure is a* **left additive Turing category** *when*

LT.1 *Each universal map $\tau_{BC} : T \times B \longrightarrow C$ is strongly additive in its first argument.*

If a universal map is strongly additive in its first argument, call it an **additive universal map**.

The above definition mimics the fact that in a Cartesian closed left additive category, the evaluation map is additive in its first variable. Intuitively, the sum in left additive Turing categories is meant to capture non-deterministic computation. For example, we have (in term logic notation)

$$(t_1 + t_2) \bullet m = t_1 \bullet m + t_2 \bullet m$$

If either t_1 or t_2 is applied to some argument m then the result is either $t_1 \bullet m$ or $t_2 \bullet m$. However, in general,

$$t\,(m_1 + m_2) \neq t\,m_1 + t\,m_2.$$

The justification for calling this non-determinism comes first from Plotkin [40]; Plotkin used Smyth's powerdomains [43] to model nondeterministic state transformations. de

Liguoro and Piperno then refined this approach in [19] where they used *Smyth nondeterministic algebras* to formally model nondeterminism. de Liguoro and Piperno defined a *semilinear applicative system* to be what is called here a left additive applicative system, with the additional constraint that the sum is idempotent. They then showed that models of semilinear applicative systems are Smyth nondeterministic algebras.

One expects a recognition theorem for left additive Turing categories similar to theorem (2.6.1). A first guess might be that every object is a retract of the Turing object where both the section and retraction are strongly additive. However, this does not work. Of course, one may add structure to the definition of left additive Turing categories to force this condition. However, doing so excludes the very natural example we will present shortly.

From the section on idempotent splitting, one might expect that the retraction to be strongly additive. This is indeed the case, and it is actually the following theorem that inspired the work on idempotent splitting:

Theorem 3.5.1. *Let $\mathbb{X}$ be a Cartesian left additive restriction category. Then $\mathbb{X}$ is a left additive Turing category with Turing object T if and only if each object is a retract of T where the retraction is strongly additive and there is an additive universal map $T \times T \xrightarrow{\bullet} T$.*

Proof. Assume $\mathbb{X}$ is a left additive Turing category. Then from theorem (2.6.1), every object is a retract of T. Thus it remains to show that the retraction is strongly additive. For this, reconsider the following diagram.

$$\begin{tikzcd}
T \arrow[r, bend left, "\langle 1,! \rangle"] & T \times T \arrow[l, bend left, "\pi_0"] \arrow[r, "\tau"] & A \arrow[dl, bend left, "\langle 1,! \rangle"] \\
& A \times T \arrow[u, "h \times 1"] \arrow[ur, "\pi_0"'] &
\end{tikzcd}$$

The retraction $r = \langle 1, !\rangle\tau$. Now, using that τ is additive in its first argument,

$$(f+g)r = (f+g)\langle 1, !\rangle\tau_{\mathrm{T}A} = \langle f+g, !\rangle\tau \geq \langle f, !\rangle\tau + \langle g, !\rangle\tau = f\langle 1, !\rangle\tau + g\langle 1, !\rangle\tau = fr + gr$$

and

$$0r = 0\langle 1, !\rangle\tau = \langle 0, !\rangle\tau = 0$$

Conversely, assume that $\mathbb{X}$ is a Cartesian left additive restriction category such where each object is a retract of T and there retraction is strongly additive. Further suppose $T \times T \xrightarrow{\bullet} T$ is an additive universal map. From theorem (2.6.1), $\mathbb{X}$ is a Turing category. Thus it remains to prove that the derived universal map $(1 \times m) \bullet r$ is strongly additive in the first variable. Note that $\bullet r$ is strongly additive in the first variable because $\bullet$ is strongly additive in the first variable, and r is assumed to be strongly additive. Consider,

$$\langle f+g, k\rangle(1 \times m) \bullet r = \langle f+g, km\rangle \bullet r = \langle f, km\rangle \bullet r + \langle g, km\rangle \bullet r$$
$$= \langle f, k\rangle(1 \times m) \bullet r + \langle g, k\rangle(1 \times m) \bullet r,$$

thus $\tau = (1 \times m) \bullet r$ is strongly additive in the first variable. □

3.5.1 Left additive PCAs

Next, we must discuss left additive PCAs. In a Cartesian restriction category, an **additive applicative system** is $\mathbb{A} = (A, \bullet, +, 0)$ such that such that $(A, +, 0)$ is a total commutative monoid, and $(A, \bullet)$ is an applicative system where further $(x_1+x_2)\,y \geq x_1\,y+x_2\,y$ and $0\,y = 0$. In a Cartesian left additive restriction category, an additive applicative system is just $\mathbb{A} = (A, \bullet)$ where $\bullet$ is strongly additive in its first argument.

If $\mathbb{A}$ is an additive applicative system, then $\mathbb{A}$ is a *combinatory complete* if every term in $\bullet, +$ can be represented by a point. A **left additive partial combinary algebra** (LPCA) is an additive applicative system with s, k just as before. Alternatively, an LPCA is a PCA in an additive applicative system. An analog of the Curry-Schonfinkel theorem holds for additive applicative systems as well.

Proposition 3.5.1. *An additive applicative system is combinatory complete if and only if it is an LPCA.*

Proof. Again, codes are constructed using a simulation of the lambda calculus, but now we may add codes:

$$\begin{aligned}
\lambda^* x.x &= s\,k\,k \\
\lambda^* x.z &= k\,z \\
\lambda^* x.(t_1\,t_2) &= s\,(\lambda^* x.t_1)\,(\lambda^* x.t_2) \\
\lambda^* x.0 &= 0 \\
\lambda^* x.(t_1 + t_2) &= (\lambda^* x.t_1) + (\lambda^* x.t_2).
\end{aligned}$$

By induction,

$$(\lambda^* x.t)m = \{x.t\}m.$$

□

Also, if $\mathbb{A}$ is an LPCA, then the commutative monoid structure is always computable.

Lemma 3.5.1. *If $\mathbb{A}$ is an LPCA, then the commutative monoid structure is $\mathbb{A}$-computable.*

Proof. First note that $0\,x = 0$, so 0 is $\mathbb{A}$-computable. To show that $+$ is computable, it suffices to show that $\pi_0 + \pi_1$ is $\mathbb{A}$-computable. We will show that $\lambda^* xy.x + y$ is a code for $\pi_0 + \pi_1$. Consider,

$$\begin{aligned}
(\lambda^* xy.x + y)\,1\,1 &= ((\lambda^* xy.x) + (\lambda^* xy.y))\,1\,1 \\
&= ((\lambda^* xy.x)\,1\,1) + ((\lambda^* xy.y)\,1\,1) \\
&= \pi_0 + \pi_1.
\end{aligned}$$

□

When $\mathbb{A}$ is an LPCA, $\mathsf{Comp}(\mathbb{A})$ has the structure of a Cartesian left additive restriction category.

Theorem 3.5.2. *For an additive applicative system $\mathbb{A}$, $\mathbb{A}$ is an LPCA if and only if* $\mathsf{Comp}(\mathbb{A})$ *is a Cartesian left additive restriction category.*

Proof. Assume that $\mathbb{A}$ is an additive applicative system. First, if $\mathsf{Comp}(\mathbb{A})$ is a Cartesian left additive restriction category, then $\mathbb{A}$ is a combinatory complete. Thus, $\mathbb{A}$ is an LPCA by proposition 3.5.1.

Conversely assume that $\mathbb{A}$ is an LPCA. Then $\mathsf{Comp}(\mathbb{A})$ is a Cartesian restriction category by theorem 2.6.2. We will show that $\mathsf{Comp}(\mathbb{A})$ is a Cartesian left additive restriction category by theorem 3.1.1. First, A^0 is obviously a total commutative monoid with $0 = 1$ and $+ =!$. Next, $(A, +, 0)$ is a total commutative monoid since, by lemma 3.5.1, $+, 0$ are $\mathbb{A}$-computable. Then define

$$+_{n+m} := ex((+_n) \times (+_m)) \qquad 0_{n+m} := 0_n \times 0_m.$$

Thus, $\mathsf{Comp}(\mathbb{A})$ is a Cartesian left additive restriction category by theorem 3.1.1. □

Combining theorem 3.5.2 with theorem 2.6.2, if $\mathbb{A}$ is an LPCA, then $\mathsf{Comp}(\mathbb{A})$ is a Cartesian left additive restriction category and a Turing category. Moreover,

Proposition 3.5.2. *For an LPCA $\mathbb{A}$,* $\mathsf{Comp}(\mathbb{A})$ *is a left additive Turing category.*

Proof. By theorem 3.5.1, it suffices to show for each n, A^n is a left additive retract of A. Doing so devolves into showing that there are left additive retractions $\top, A^2 \lhd A$. For $\top$ take $0 : \top \longrightarrow A$ and $! : A \longrightarrow \top$; ! is strongly additive. For $A \times A$ take the section to be

$$(a, b) \mapsto \lambda^* p.p\,a\,b.$$

The retraction is the pairing of the following two maps

$$P : a \mapsto a\,(\lambda^* xy.x) \qquad Q : a \mapsto a\,(\lambda^* xy.y)$$

Both P and Q are strongly additive, and this follows since $\bullet$ is strongly additive in the first variable. Consider, for example

$$(f+g)P = (f+g)\bullet(\lambda^* xy.x) \geq (f \bullet \lambda^* xy.x) + (g \bullet \lambda^* xy.x) = fP + gP.$$

Similarly, Q is strongly additive. Further, the pairing of strongly additive maps is again strongly additive, thus the retraction $A \times A \xrightarrow{\langle P,Q\rangle} A$ is strongly additive. This completes the proof that $\mathsf{Comp}(\mathbb{A})$ is a left additive Turing category. $\square$

3.5.2 Bag PCAs

Next, we will construct a model of an additive PCA. The intuition comes from the bag monad $B : \mathsf{Par} \longrightarrow \mathsf{Par}$; recall if A is a set, then $B(A)$ is the free commutative monoid generated by A. Moreover the bag monad is strong; there is a map

$$B(A) \times A \xrightarrow{\theta} B(A \times A); \left(\left(\sum_i x_i\right), y\right) \mapsto \sum_i (x_i, y).$$

Suppose that $\mathbb{A} = (A, \bullet)$ is a PCA. Then $B(A)$ can be viewed as a nondeterministic choice of programs. Moreover, using strength, the composite

$$B(A) \times A \xrightarrow{\theta} B(A \times A) \xrightarrow{B(\bullet)} B(A),$$

give the effect of nondeterministically applying a choice of programs to a single argument. We will generalize the needed structure of the bag monad to produce a class of examples with this nondeterministic flavor. It should be noted that such monads are the *algebra modalities* of Blute *et al* [3] with the added condition of strength on the monad. It should be noted that algebra modalities are related to the storage modalities of Blute *et al* [1] or equivalently the linear exponential monads of Hyland and Schalk [28]; storage modalities are used to study the semantics of linear logic with exponentials.

Definition 3.5.2. *Let $\mathbb{X}$ be a Cartesian restriction category. A* **bag monad** *is (B, η, μ, θ) where $B : \mathbb{X} \longrightarrow \mathbb{X}$ is a restriction functor, (B, η, μ) is a monad, (B, θ) is a strong functor, and further the following are satisfied:*

(i) *η is a strong restriction natural transformation; that is, $\overline{\eta_A} = 1_A$ and $(\eta \times 1)\theta = \eta$;*

(ii) *μ is a strong restriction natural transformation; that is, $\overline{\mu_A} = 1_{B^2(A)}$ and $(\mu \times 1)\theta = \theta B(\theta)\mu$;*

(iii) *For each A, there are restriction natural transformations $[] : \top \longrightarrow B(A)$ and $\oplus : B(A)^2 \longrightarrow B(A)$ which makes $B(A)$ into a total commutative monoid;*

(iv) *μ is a homomorphism of this monoid structure; that is,*

$$\begin{tikzcd}
\top \arrow[rr, "{[]}"] \arrow[dr, "{[]}"'] & & B(A) \\
& B^2(A) \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B^2(A) \times B^2(A) \arrow[r, "\mu\times\mu"] \arrow[d, "\oplus"'] & B(A) \times B(A) \arrow[d, "\oplus"] \\
B^2(A) \arrow[r, "\mu"'] & B(A)
\end{tikzcd}$$

(v) *the monoid structure is strong with respect to $\times$; that is,*

$$\begin{tikzcd}
\top \times A \arrow[r, "{[]\times k}"] \arrow[d, "!"'] & B(A) \times A \arrow[d, "\theta"] \\
\top \arrow[r, "{[]}"'] & B(A \times A)
\end{tikzcd}
\qquad
\begin{tikzcd}
B(A)^2 \times A \arrow[rr, "\oplus\times 1"] \arrow[d, "1\times\Delta"'] & & B(A) \times A \arrow[dd, "\theta"] \\
B(A)^2 \times A^2 \arrow[d, "ex"'] & & \\
(B(A) \times A)^2 \arrow[r, "\theta\times\theta"'] & B(A \times A)^2 \arrow[r, "\oplus"'] & B(A \times A).
\end{tikzcd}$$

Lemma 3.5.2. *Let $\mathbb{X}$ be a Cartesian restriction category, let B be a bag monad, and let A be an object for which there is a total isomorphism $A \xrightarrow{\alpha} B(A)$. Then, A can be given the structure of a total commutative monoid.*

Proof. Define the monoid structure as

$$\begin{tikzcd}
A \times A \arrow[r, "+"] \arrow[d, "\alpha\times\alpha"'] & A \\
B(A) \times B(A) \arrow[r, "\oplus"'] & B(A) \arrow[u, "\alpha^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "{[]}"'] & A \\
& B(A) \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

First, $0, +$ are clearly total.

For the right unit law, the identity is $([]\alpha^{-1} \times 1)(\alpha \times \alpha) \oplus \alpha^{-1} = \pi_1$.

$$([]\alpha^{-1}\times)(\alpha \times \alpha) \oplus \alpha^{-1} = (1 \times \alpha)([] \times 1) \oplus \alpha^{-1} = (1 \times \alpha)\pi_1\alpha^{-1}$$
$$= \pi_1\alpha\alpha^{-1} = \pi_1.$$

The left unit law is similar. For associativity, use that $a_\times$ is natural, and that $\oplus$ is associative.

$$a_\times(1 \times +)+ = a_\times(1 \times ((\alpha \times \alpha) \oplus \alpha^{-1}))(\alpha \times \alpha) \oplus \alpha^{-1}$$
$$= a_\times(\alpha \times (\alpha \times \alpha)\oplus) \oplus \alpha^{-1} = a_\times(\alpha \times (\alpha \times \alpha))(1 \times \oplus) \oplus \alpha^{-1}$$
$$= ((\alpha \times \alpha) \times \alpha)a_\times(1 \times \oplus) \oplus \alpha^{-1} = ((\alpha \times \alpha) \times \alpha)(\oplus \times 1) \oplus \alpha^{-1}$$
$$= ((\alpha \times \alpha) \oplus \alpha^{-1}\alpha \times \alpha) \oplus \alpha^{-1} = ((\alpha \times \alpha) \oplus \alpha^{-1} \times 1)(\alpha \times \alpha) \oplus \alpha^{-1},$$

For commutativity, use that $c_\times$ is natural and that $\oplus$ is commutative.

$$c_\times(\alpha \times \alpha) \oplus \alpha^{-1} = (\alpha \times \alpha)c_\times \oplus \alpha^{-1} = (\alpha \times \alpha) \oplus \alpha^{-1},$$

as required. □

Furthermore, α induces an algebra of B.

Lemma 3.5.3. *Let $\mathbb{X}$ be a Cartesian restriction cateogry, let B be a bag monad, and let A be an object for which there is a total isomorphism $A \xrightarrow{\alpha} B(A)$. Then the map*

$$\begin{tikzcd}
B(A) \arrow[r, "\nu"] \arrow[d, "B(\alpha)"'] & A \\
B^2(a) \arrow[r, "\mu"'] & B(A) \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

makes (A, ν) into an algebra of B.

Proof. For the unit law, use the naturality of η and that unit law for the monad,

$$\eta B(\alpha)\mu\alpha^{-1} = \alpha\eta\mu\alpha^{-1} = \alpha\alpha^{-1} = 1.$$

For the interaction with μ, use the multiplication law for the monad, followed by the naturality of μ:

$$B(B(\alpha)\mu\alpha^{-1})B(\alpha)\mu\alpha^{-1} = B^2(\alpha)B(\mu)B(\alpha^{-1})B(\alpha)\mu\alpha^{-1}$$
$$= B^2(\alpha)B(\mu)\mu\alpha^{-1} = B^2(\alpha)\mu\mu\alpha^{-1} = \mu B(\alpha)\mu\alpha^{-1}$$

Thus ν is an algebra of B. $\square$

Suppose $\mathbb{X}$ is a Cartesian restriction category, $(B, \eta, \mu, \theta, [], \oplus)$ is a bag monad, and let $\mathbb{A} = (A, \bullet, s, k)$ be a PCA with a total isomorphism $\alpha : A \longrightarrow B(A)$. Then one can define the following application on A.

$$\begin{tikzcd}
A \times A \arrow[rr, "\star"] \arrow[d, "\alpha\times 1"'] & & A \\
B(A) \times A \arrow[dr, "\theta"'] & & B(A) \arrow[u, "\nu"'] \\
& B(A \times A) \arrow[ur, "B(\bullet)"'] &
\end{tikzcd}$$

Think of B as the bag monad acting on $\mathbb{N}$; $\star$ interprets the program position as a bag of numbers; intuitively, $n, m \mapsto (\{\!| x_1, \ldots, x_j |\!\}, m)$. Each of these numbers is then regarded as a program and applied nondeterministically to some argument, and the result is a bag of values; intuitively,

$$\{\!| (x_1, m), \ldots, (x_j, m) |\!\} \mapsto \{\!| x_1 \bullet m, \ldots, x_j \bullet m |\!\}$$

Then each of these values is interpreted as a nondeterministic choice of answers, and the result is a bag of bags. μ is applied to flatten the result into one large bag. Finally, this bag is converted back to a number.

Given a bag monad B and a PCA $\mathbb{A}$, if there is an isomorphism $A \xrightarrow{\alpha} B(A)$ and codes for $\eta\alpha^{-1}$ and $\star$, then $\mathbb{A}$ is a **bag pca**. If $\mathbb{A}$ is a bag PCA, then denote the induced applicative system $(A, \star)$ by $\mathbb{A}_+$. We are going to prove that for every bag PCA $\mathbb{A}$, the induced $\mathbb{A}_+$ is an LPCA;the first step is:

Lemma 3.5.4. *Let $\mathbb{X}$ be a Cartesian restriction category, let B be a bag monad, and let $\mathbb{A} = (A, \bullet, s, k)$ be an object for which there is a total isomorphism $A \xrightarrow{\alpha} B(A)$. If the maps $\eta\alpha^{-1}$ and $\star$ are $\bullet$-computable, then there are $s_\star, k_\star$ that make $\mathbb{A}_+ = (A, \star, s_\star, k_\star)$ into a PCA.*

Proof. First define $k_1 : A \longrightarrow A$ such that $k_1 \star 1 = \pi_0$. Then define $k_\star : \top \longrightarrow A$ such that $k_\star \star 1 = k_1$. It will then follow that $k_\star$ is a code for π_0 becasue,

$$k_\star \star 1 \star 1 = k_1 \star 1 = \pi_0.$$

Define

$$k_1 := (k \bullet 1)\eta\alpha^{-1}.$$

Then,

$$\begin{aligned}
k_1 \star 1 &= ((k \bullet 1)\eta\alpha^{-1}) \star 1 \\
&= (((k \bullet 1)\eta\alpha^{-1}) \times 1)(\alpha \times 1)\theta B(\bullet)\nu \\
&= ((k \bullet 1)\eta \times 1)\theta B(\bullet)\nu \\
&= ((k \bullet 1) \times 1)\eta B(\bullet)\nu \quad \text{strength} \\
&= ((k \bullet 1) \times 1) \bullet \eta\nu \quad \text{naturality} \\
&= \pi_0\eta\nu \quad (k \bullet 1) \bullet 1 = \pi_0 \\
&= \pi_0 \quad \text{algebra.}
\end{aligned}$$

Now recall that if f has code $\underline{f}$ and g has code $\underline{g}$ then the code for fg is $\lambda^* x.\underline{g} \bullet (\underline{f} \bullet x)$. We have supposed that $\eta\alpha^{-1}$ has a code in $\bullet$ and $k \bullet 1$ has a code in $\bullet$ because $\mathbb{A}$ is a PCA. The code for $k_\star$ is essentially the code for the composite, $(k \bullet 1)\eta\alpha^{-1}$; explicitly first define

$$k_2 := (\lambda^* x.\underline{\eta\alpha^{-1}} \bullet (\underline{k \bullet 1} \bullet x)),$$

Then define,

$$k_\star := k_2\eta\alpha^{-1}.$$

Then,

$$\begin{aligned}
(k_2\eta\alpha^{-1} \times 1)\star &= ((k_2\eta\alpha^{-1}) \times 1)(\alpha \times 1)\theta B(\bullet)B(\alpha)\mu\alpha^{-1} \\
&= (k_2\eta \times 1)\theta B(\bullet)B(\alpha)\mu\alpha^{-1} \\
&= (k_2 \times 1)\eta B(\bullet)B(\alpha)\mu\alpha^{-1} \quad \text{strength} \\
&= (k_2 \times 1) \bullet \eta B(\alpha)\mu\alpha^{-1} \quad \text{naturality} \\
&= (k \bullet 1)\eta\alpha^{-1}\eta B(\alpha)\mu\alpha^{-1} \quad \text{assumption} \\
&= (k \bullet 1)\eta\alpha^{-1}\alpha\eta\mu\alpha^{-1} \quad \text{naturality} \\
&= (k \bullet 1)\eta\eta\mu\alpha^{-1} \\
&= (k \bullet 1)\eta\alpha^{-1} \quad \text{monad laws.}
\end{aligned}$$

Therefore there is a code $k_\star$ such that $k_\star \star x \star y = x$.

Next we must show there is a code for the map $(\pi_0 \star \pi_2) \star (\pi_1 \star \pi_2)$. Define,

$$s_\star := (\lambda^* x.\underline{\eta\alpha^{-1}}(\lambda^* y.\underline{\eta\alpha^{-1}}(\lambda z.\underline{\star}(\underline{\star}xz)(\underline{\star}yz))))\eta\alpha^{-1} : \top \longrightarrow A.$$

Note that $s_\star$ exists because of the assumption that $\star$ has a $\bullet$-code, and that $\eta\alpha^{-1}$ has a

$\bullet$-code. Now, consider,

$$
\begin{aligned}
&(s_\star \times 1 \times 1 \times 1)(\star \times 1 \times 1)(\star \times 1)\star \\
&= ((\lambda^* x.\underline{\eta\alpha^{-1}}(\lambda^* y.\underline{\eta\alpha^{-1}}(\lambda z.\underline{\star}(\underline{\star} xz)(\underline{\star} yz))))\eta\alpha^{-1} \times 1 \times 1 \times 1) \\
&\quad (\alpha \times 1 \times 1 \times 1)(\theta \times 1 \times 1)(B(\bullet) \times 1 \times 1)(\nu \times 1 \times 1)(\star \times 1)\star \\
&= ((\lambda^* x.\underline{\eta\alpha^{-1}}(\lambda^* y.\underline{\eta\alpha^{-1}}(\lambda z.\underline{\star}(\underline{\star} xz)(\underline{\star} yz))) \times 1)\eta \times 1 \times 1) \\
&\quad (B(\bullet) \times 1 \times 1)(\nu \times 1 \times 1)(\star \times 1)\star \quad \text{strength} \\
&= ((\lambda^* x.\underline{\eta\alpha^{-1}}(\lambda^* y.\underline{\eta\alpha^{-1}}(\lambda z.\underline{\star}(\underline{\star} xz)(\underline{\star} yz))) \times 1) \bullet \times 1 \times 1) \\
&\quad (\eta \times 1 \times 1)(\nu \times 1 \times 1)(\star \times 1)\star \quad \text{naturality} \\
&= ((\lambda^* x.\underline{\eta\alpha^{-1}}(\lambda^* y.\underline{\eta\alpha^{-1}}(\lambda z.\underline{\star}(\underline{\star} xz)(\underline{\star} yz))) \times 1) \bullet \times 1 \times 1) \\
&\quad (\star \times 1)\star \quad \text{algebra law} \\
&= ((\lambda^* y.\underline{\eta\alpha^{-1}}(\lambda^* z.\underline{\star}(\underline{\star}\pi_0 z)(\underline{\star} yz)))\eta\alpha^{-1} \times 1 \times 1)(\star \times 1)\star \\
&= \cdots \\
&= ((\lambda^* z.\underline{\star}(\underline{\star}\pi_0 z)(\underline{\star}\pi_1 z)) \times 1)\star \\
&= \cdots \\
&= \underline{\star}(\underline{\star}\pi_0\pi_2)(\underline{\star}\pi_1\pi_2) \\
&= (\pi_0 \star \pi_2) \star (\pi_1 \star \pi_2).
\end{aligned}
$$

This completes the proof that $(A, \star, k_\star, s_\star)$ is a PCA. $\square$

Next, we show that if $\mathbb{A}$ is a bag PCA, then $\mathbb{A}_+$ is an LPCA.

Proposition 3.5.3. *Let $\mathbb{A}$ be bag PCA. Then $\mathbb{A}_+$ is an LPCA.*

By lemma 3.5.2, A may be given the structure of a total commutative monoid $(A, +, 0)$. By lemma 3.5.4 $\mathbb{A}_+$ is a PCA. Thus, it remains to show that $\star$ is strongly additive in its first argument.

Proof. For the zero case:

$$
\begin{aligned}
0 \star k = \langle 0, k\rangle \star &= \langle ![]\alpha^{-1}, k\rangle \star \\
&= \langle ![]\alpha^{-1}, k\rangle(\alpha \times 1)\theta B(\bullet)\nu \\
&= \langle ![], k\rangle\theta B(\bullet)\nu \\
&= \langle !, 1\rangle([] \times k)\theta B(\bullet)\nu \\
&= \langle !, 1\rangle ![] B(\bullet)\nu \qquad \text{since } ([] \times k)\theta = ![] \\
&= \langle !, 1\rangle ![] B(\bullet\alpha)\mu\alpha^{-1} \\
&= \langle !, 1\rangle ![]\mu\alpha^{-1} \qquad \text{naturality} \\
&= ![]\mu\alpha^{-1} \\
&= ![]\alpha^{-1} \quad \text{since } []\mu = [] \\
&= 0
\end{aligned}
$$

For the sum case:

$$
\begin{aligned}
&(g + h) \star k = \langle g + h, k\rangle \star \\
&= \langle\langle g\alpha, h\alpha\rangle \oplus \alpha^{-1}, k\rangle(\alpha \times 1)\theta B(\bullet)\nu \\
&= \langle\langle g\alpha, h\alpha\rangle, k\rangle(\oplus \times 1)\theta B(\bullet)\nu \\
&= \langle\langle g\alpha, h\alpha\rangle, k\rangle(1 \times \Delta)ex(\theta \times \theta) \oplus B(\bullet)\nu \quad ;\text{since } (\oplus \times 1)\theta = (1 \times \Delta)ex(\theta \times \theta)\oplus \\
&= \langle\langle g\alpha, h\alpha\rangle, \langle k, k\rangle\rangle ex(\theta \times \theta) \oplus B(\bullet)B(\alpha)\mu\alpha^{-1} \\
&= \langle\langle g\alpha, k\rangle, \langle h\alpha, k\rangle\rangle(\theta \times \theta)(B(\bullet) \times B(\bullet))(B(\alpha) \times B(\alpha)) \oplus \mu\alpha^{-1} \qquad \text{naturality} \\
&= \langle\langle g\alpha, k\rangle, \langle h\alpha, k\rangle\rangle(\theta \times \theta)(B(\bullet)B(\alpha)\mu \times B(\bullet)B(\alpha)\mu) \oplus \alpha^{-1} \qquad \mu \text{ is a homomorphism} \\
&= \langle\langle g\alpha, k\rangle\theta B(\bullet)B(\alpha)\mu, \langle h\alpha, k\rangle\theta B(\bullet)B(\alpha)\mu\rangle \oplus \alpha^{-1} \\
&= \langle\langle g\alpha, k\rangle\theta B(\bullet)B(\alpha)\mu\alpha^{-1}, \langle h\alpha, k\rangle\theta B(\bullet)B(\alpha)\mu\alpha^{-1}\rangle(\alpha \times \alpha) \oplus \alpha^{-1} \\
&= (g \star k) + (h \star k)
\end{aligned}
$$

□

Corollary 3.5.1. *For a bag PCA* $\mathbb{A}$, $\mathsf{Comp}(\mathbb{A}_+)$ *is a left additive Turing category.*

Simulations as described by Cockett and Hofstra [13] [10] and Longley [32] [33] give a way to compare applicative structures. For PCAs $\mathbb{A}, \mathbb{B}$ in a Cartesian restriction category $\mathbb{X}$, Cockett and Hofstra [13] proved that the definition of simulation $\phi : \mathbb{A} \longrightarrow \mathbb{B}$ is equivalent to having a code $u : \top \longrightarrow B$ such that

$$(u \times \phi \times \phi)\bullet_B^2 = \bullet_A \phi.$$

Also important is when PCAs $\mathbb{A}, \mathbb{B}$ are **simulation equivalent**. In [13] Cockett and Hofstra prove that $\mathbb{A}, \mathbb{B}$ in $\mathbb{X}$ are simulation equivalent exactly when there are simulations $\phi : \mathbb{A} \longrightarrow \mathbb{B}$ and $\psi : \mathbb{B} \longrightarrow \mathbb{A}$ such that $\phi\psi$ is $\mathbb{A}$-computable and has an $\mathbb{A}$-computable retraction, and $\psi\phi$ is $\mathbb{B}$-computable and has a $\mathbb{B}$-computable retraction.

If $\mathbb{A}$ is a bag PCA, then we claim that $\mathbb{A}_+$ is simulation equivalent to $\mathbb{A}$.

Proposition 3.5.4. *Every bag PCA* $\mathbb{A}$ *is simulation equivalent to* $\mathbb{A}_+$.

Take $\phi = 1_A : \mathbb{A} \longrightarrow \mathbb{A}_+$ and $\psi = 1_A : \mathbb{A}_+ \longrightarrow \mathbb{A}$. If ϕ and ψ can be proved to be simulations, then note that simulation equivalence devolves to the fact that 1_A is both $\mathbb{A}$-computable and $\mathbb{A}_+$-computable. By assumption, there is a code $\underline{\star}$ such that $\underline{\star} \bullet 1 \bullet 1 = \star$, thus by the above discussion, ψ is a simulation. Thus, it remains to show that ϕ is a simulation.

Proof. We must show there is a code $\underline{\bullet}$ such that $(\underline{\bullet} \times 1 \times 1)\star^2 = \bullet$. This can be accomplished by showing that there is a $\bullet' : A \longrightarrow A$ such that $(\bullet' \times 1)\star = \bullet$, and then showing there is a $\underline{\bullet} : \top \longrightarrow A$ such that $(\underline{\bullet} \times 1)\star = \bullet'$. Then

$$(\underline{\bullet} \times 1 \times 1)(\star \times 1)\star = ((\underline{\bullet} \times 1) \star \times 1)\star = (\bullet' \times 1)\star = \bullet,$$

which proves that $\underline{\bullet}$ is a $\star$-code for $\bullet$.

Define

$$\bullet' := \eta\alpha^{-1}.$$

Consider,

$$\begin{aligned}(\eta\alpha^{-1} \times 1)\star &= (\eta\alpha^{-1} \times 1)(\alpha \times 1)\theta B(\bullet)\nu \\ &= (\eta \times 1)\theta B(\bullet)\nu \\ &= \eta B(\bullet)\nu \quad \text{strength} \\ &= \bullet\eta\nu \quad \text{naturality} \\ &= \bullet \text{ . algebra law.}\end{aligned}$$

By assumption, $\underline{\eta\alpha^{-1}}$ is a code for $\eta\alpha^{-1}$, then define

$$\underline{\bullet} := \underline{\eta\alpha^{-1}}\eta\alpha^{-1}.$$

Consider,

$$\begin{aligned}(\underline{\bullet} \times 1))\star &= (\underline{\eta\alpha^{-1}} \times 1)(\eta\alpha^{-1} \times 1)(\alpha \times 1)\theta B(\bullet)B(\alpha)\mu\alpha^{-1} \\ &= (\underline{\eta\alpha^{-1}} \times 1)(\eta \times 1)\theta B(\bullet)B(\alpha)\mu\alpha^{-1} \\ &= (\underline{\eta\alpha^{-1}} \times 1)\eta B(\bullet)B(\alpha)\mu\alpha^{-1} \quad \text{strength} \\ &= (\underline{\eta\alpha^{-1}} \times 1) \bullet \eta B(\alpha)\mu\alpha^{-1} \quad \text{naturality} \\ &= \eta\alpha^{-1}\eta B(\alpha)\mu\alpha^{-1} \quad (\underline{\eta\alpha^{-1}} \times 1)\bullet = \eta\alpha^{-1} \\ &= \eta\alpha^{-1}\alpha\eta\mu\alpha^{-1} \quad \text{naturality} \\ &= \eta\eta\mu\alpha^{-1} \\ &= \eta\alpha^{-1} \quad \text{monad law.}\end{aligned}$$

Thus $\underline{\bullet}$ is a $\star$-code for $\bullet$ which completes the proof that $\mathbb{A}_+$ is simulation equivalent to $\mathbb{A}$. $\square$

3.6 Remarks

3.6.1 Simple Slices

There is always an additive structure on $\mathbb{X}[A]$. This additive structure is actually part of a deeper structural story. The coherence condition for Cartesian closed left additive categories, $\lambda(f+g) = \lambda(f) + \lambda(g)$, is determined by forcing the additive structure of $\mathbb{X}[A]$ to coincide with additive structure on the Kleisli category of $(A \Rightarrow \_)$ (see Blute *et al* [2]).

However, generalizing the above coherence to the restriction case is more subtle. One might expect that eval should be strongly additive in its first coordinate as a result of this coherence; i.e., $\langle h+g, k\rangle \mathsf{eval} \geq \langle h, k\rangle \mathsf{eval} + \langle g, k\rangle \mathsf{eval}$. However instead of an inequality, one can show equality, and this suggests that this coherence condition might be too strong.

3.6.2 Idempotent Splitting

It was widely believed, when this investigation began, that in order to obtain a left additive restriction category by splitting the idempotents of a left additive restriction category, one could only split strongly additive idempotents.

An indication that this was incorrect came from the difficulty of axiomatizing a left additive Turing category in which each object was the splitting of a strongly additive idempotent. In particular, one could not obtain a reasonable recognition theorem for these Turing categories.

3.6.3 Turing Categories

Of course, once one realizes that the idempotents need not be strongly additive, and in particular, that the section of a split idempotent need not be strongly additive, the necessity for Manzonetto's encoding of pairing combinator[37] is no longer needed. In fact, the standard pairing combinator works as long as one is in an additive applicative system.

Chapter 4

Differential Restriction Categories

4.1 Differential Restriction Categories

Cartesian left additive restriction categories provide the structure on which differential restriction categories rests. The total derivative of smooth partial functions between finite dimensional, real vector spaces provides a motivating example of differential restriction structure where the derivative is given by the Jacobian matrix. Given a smooth partial function, say $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the total derivative of f at a point in $\mathbb{R}^n$ (the Jacobian of f at a point), $D[f](\mathrm{x})$ is a linear map $\mathbb{R}^n \multimap \mathbb{R}^m$. However, the derivative may be generalized to settings without closed structure by using the uncurried form of the derivative, $D[f] : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$. The second argument is the point at which the derivative is being taken; the first is the direction in which the derivative is being evaluated. A key example of a differential restriction category is developed by Cockett *et al* in [15] which is based on the notion of taking the formal derivative of a rational polynomial with coefficients in a commutative ring.

Definition 4.1.1. *A* **differential restriction category** *is a Cartesian left additive restriction category with a differential combinator*

$$\frac{f : X \longrightarrow Y}{D[f] : X \times X \longrightarrow Y}$$

that satisfies

DR.1 $D[0] = 0$ *and* $D[f+g] = D[f] + D[g]$*;*

DR.2 $\langle 0, g\rangle D[f] = \overline{gf}\,0$ *and* $\langle g+h, k\rangle D[f] = \langle g, k\rangle D[f] + \langle h, k\rangle D[f]$*;*

DR.3 $D[\pi_0] = \pi_0\pi_0$ *and* $D[\pi_1] = \pi_0\pi_1$;

DR.4 $D[\langle f, g\rangle] = \langle D[f], D[g]\rangle$;

DR.5 $D[fg] = \langle D[f], \pi_1 f\rangle D[g]$;

DR.6 $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle D[D[f]] = \overline{h}\langle g, k\rangle D[f]$;

DR.7 $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle D[D[f]] = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle D[D[f]]$;

DR.8 $D[\overline{f}\,] = (1 \times \overline{f}\,)\pi_0$;

DR.9 $\overline{D[f]} = 1 \times \overline{f}$.

Note that $D[!_A] = !_{A\times A}$ since $D[!_A]$ is a total map $A \times A \longrightarrow \top$. A brief description of these axioms may help with their meaning. **DR.1** says that $D[\_]$ is a linear operator. **DR.2** says that the derivative of a map is additive in the first coordinate. **DR.3** says that projections are linear. **DR.4** says that the derivative respects the product structure correctly. **DR.5** is the chain rule. We shall see shortly that **DR.6** says that the derivative of a map is linear in its first variable. **DR.7** says that the order in which partial derivatives is taken does not matter. **DR.8** says that restriction idempotents are linear. **DR.9** says that the partiality of the derivative of a map is determined by the "point" of differentiation.

An important notion for Cartesian differential categories is that of a linear map; in that setting a map f is linear if $D[f] = \pi_0 f$. However, note in a differential restriction category, equal maps have equal restrictions; thus

$$(\overline{f} \times 1) = \overline{\pi_0 f} = \overline{D[f]} = (1 \times \overline{f}\,) = \overline{\pi_1 f} = (1 \times \overline{f}\,),$$

is true if and only if $\overline{f} = 1$; i.e. f is total. Hence a map in a differential restriction category, f, is **linear** when $D[f] \smile \pi_0 f$.

Proposition 4.1.1 (Cockett *et al*: proposition 3.20). *In a differential restriction category:*

(i) if f is total, f is linear if and only if $D[f] = \pi_0 f$;

(ii) if f is linear, then f is additive;

(iii) restriction idempotents are linear;

(iv) if f and g are linear, so is fg;

(v) if $g \leq f$ and f is linear, then g is linear;

(vi) 0 maps are linear, and if f and g are linear, so is $f + g$;

(vii) projections are linear, and if f and g are linear, so is $\langle f, g\rangle$;

(viii) if f is linear and has a partial inverse g, then g is also linear.

Proof. (i) It suffices to show that if f is total, $\overline{D[f]} = \overline{\pi_0 f}$. Indeed, if f is total,

$$\overline{D[f]} = 1 \times \overline{f} = \overline{f} \times 1 = \overline{\pi_0 f}.$$

(ii) For the 0 axiom use [**R.4**], that f is linear, and then [**DR.2**].

$$\begin{aligned} 0f &= \overline{0f}\,0f = \overline{\langle 0,0\rangle\pi_1 f}\,\langle 0,0\rangle\pi_0 f \\ &= \langle 0,0\rangle\overline{\pi_1 f}\,\pi_0 f \leq \langle 0,0\rangle D[f] \\ &= \overline{0f}\,0 \leq 0 \end{aligned}$$

and for the addition axiom use that f is linear and [**DR.2**],

$$\begin{aligned} &\overline{(x+y)f)}(xf+yf) = \overline{(x+y)f}(\overline{xf}xf + \overline{y}\,\overline{xf}\,\overline{x}yf) \\ &= \overline{(x+y)f}(\overline{xf}xf + \overline{\overline{y}xf}\,\overline{x}yf) = \overline{(x+y)f}(\overline{\langle x,x\rangle\pi_1 f}\langle x,x\rangle\pi_0 f + \overline{\langle y,x\rangle\pi_1 f}\langle y,x\rangle\pi_0 f) \\ &= \overline{(x+y)f}(\langle x,x\rangle\overline{\pi_1 f}\pi_0 f + \langle y,x\rangle\overline{\pi_1 f}\pi_0 f) \leq \overline{(x+y)f}(\langle x,x\rangle D[f] + \langle y,x\rangle D[f]) \\ &= \overline{\langle x+y,x\rangle\pi_0 f}\langle x+y,x\rangle D[f] = \langle x+y,x\rangle\overline{\pi_0 f}D[f] \\ &= \langle x+y,x\rangle\overline{\pi_1 f}\pi_0 f = \overline{x+y,x\rangle\pi_1 f}\langle x+y,x\rangle\pi_0 f \\ &= \overline{\overline{x+y}xf}\,\overline{x}(x+y)f \leq (x+y)f \end{aligned}$$

as required.

(iii) Suppose $e = \overline{e}$. Then consider

$$\overline{\pi_1 e}\,\pi_0 \overline{e} = \overline{\pi_1 e}\,\overline{\pi_0 e}\,\pi_0 \leq \overline{\pi_1 e}\,\pi_0 = \langle \pi_0 e, \pi_1 e\rangle \pi_0 = (1 \times e)\pi_0 = D[e],$$

so that e is linear

(iv) Suppose f and g are linear; then consider

$$\begin{aligned}
&D[fg] = \langle D[f], \pi_1 f\rangle D[g] \\
&\geq \langle \overline{\pi_1 f}\pi_0 f, \pi_1 f\rangle \overline{\pi_1 g}\pi_0 g = \overline{\langle \overline{\pi_1 f}\pi_0 f, \pi_1 f\rangle \pi_1 g}\langle \overline{\pi_1 f}\pi_0 f, \pi_1 f\rangle \pi_0 g \\
&= \overline{\overline{\pi_1 f \pi_0 f}\pi_1 f g}\,\overline{\pi_1 f \pi_1 f}\pi_0 f g = \overline{\pi_1 f \pi_1 f g \pi_0 f}\pi_0 f g \\
&= \overline{\pi_1 f g}\pi_0 f g.
\end{aligned}$$

(v) If $g \leq f$, then $g = \overline{g}\, f$; since restriction idempotents are linear and the composite of linear maps is linear, g is linear.

(vi) Since $D[0] = 0 = \pi_0 0$, 0 is linear. Suppose f and g are linear; then consider

$$\begin{aligned}
&\overline{\pi_0(f+g)}D[f+g] = \overline{\pi_0 f + \pi_0 g}(D[f] + D[g]) \\
&= \overline{\pi_0 f}\,\overline{\pi_0 g}(D[f] + D[g]) = \overline{\pi_0 f}D[f] + \overline{\pi_0 g}D[g] \\
&= \overline{\pi_1 f}\pi_0 f + \overline{\pi_1 g}\pi_0 g = \overline{\pi_1 f}\,\overline{\pi_1 g}\pi_0(f+g) \\
&\leq \pi_0(f+g)
\end{aligned}$$

as required.

(vii) By **[DR.3]**, projections are linear. Suppose f and g are linear; then consider

$$\begin{aligned}
&D[\langle f, g\rangle] = \langle D[f], D[g]\rangle \\
&\geq \langle \overline{\pi_1 f}\pi_0 f, \overline{\pi_1 g}\pi_0 g\rangle = \overline{\pi_1 f}\,\overline{\pi_1 g}\pi_0 \langle f, g\rangle \\
&= \overline{\overline{\pi_1 f}\pi_1 g}\pi_0\langle f, g\rangle = \overline{\overline{\pi_1 \overline{f}}\pi_1 \overline{g}}\pi_0 \langle f, g\rangle \\
&= \overline{\pi_1 \overline{f}\,\overline{g}}\pi_0\langle f, g\rangle = \overline{\pi_1 \langle f, g\rangle}\pi_0 \langle f, g\rangle
\end{aligned}$$

as required.

(viii) If g is the partial inverse of a linear map f, then

$$\begin{aligned}
& D[g] \geq (\overline{g} \times \overline{g})D[g] \\
& = (gf \times gf)D[g] = (g \times g)(f \times f)D[g] \\
& = (g \times g)\langle \pi_0 f, \pi_1 f \rangle D[g] = (g \times g)\langle \overline{\pi_1 f}\, \pi_0 f, \pi_1 f \rangle D[g] \\
& = (g \times g)\langle \overline{\pi_0 f}\, D[f], \pi_1 f \rangle D[g] = (g \times g)\overline{\pi_0 f}\, \langle D[f], \pi_1 f \rangle D[g] \\
& = (g \times g)\overline{\pi_0 f}\, D[fg] = (g \times g)\overline{\pi_0 f}\, D[\overline{f}\,] \\
& = (g \times g)\overline{\pi_0 f}\,(1 \times \overline{f}\,)\pi_0 = \overline{(g \times g)\pi_0 f}\,(g \times g)(1 \times \overline{f}\,)\pi_0 \\
& = \overline{\overline{\pi_1 g}\, \pi_0 g f}\,(g \times g)\pi_0 = \overline{\pi_1 g}\, \overline{\pi_0 \overline{g}}\, \overline{\pi_1 g}\, \pi_0 g \\
& = \overline{\pi_1 g}\, \pi_0 g
\end{aligned}$$

as required.

□

The above lemma establishes that the collection of linear maps in $\mathbb{X}$ is a partial inverse closed, differential restriction subcategory $\mathsf{Lin}(\mathbb{X})$ such that the inclusion $\mathfrak{L} : \mathsf{Lin}(\mathbb{X}) \longrightarrow \mathbb{X}$ preserves the deriviative, and has the property that if $\overline{e_1} = e_1 \neq e_2 = \overline{e_2}$ then $\mathfrak{L}(e_1) \neq \mathfrak{L}(e_2)$.

Partial differentials in a differential restriction category may be obtained by "zeroing out" the components on which the derivative is not being taken: the partial derivatives in the first and second variables (respectively) are:

$$\frac{A \times B \xrightarrow{f} C}{A \times (A \times B) \xrightarrow[(\langle 1,0 \rangle \times 1)D[f]]{} C}\ D_{\times,0}$$

$$\frac{A \times B \xrightarrow{f} C}{B \times (A \times B) \xrightarrow[(\langle 0,1 \rangle \times 1)D[f]]{} C}\ D_{\times,1}.$$

The following is a generalization of [2] lemma 4.5.1 to the restriction setting.

Lemma 4.1.1. *In any differential restriction category*

$$D[f] = (\pi_0 \times 1)D_{\times,0}[f] + (\pi_1 \times 1)D_{\times,1}[f].$$

Proof. Consider,

$$\begin{aligned}
&(\pi_0 \times 1)D_{\times,0}[f] + (\pi_1 \times 1)D_{\times,1}[f] \\
&= (\pi_0 \times 1)(\langle 1,0\rangle \times 1)D[f] + (\pi_1 \times 1)(\langle 0,1\rangle \times 1)D[f] \\
&= \langle\langle \pi_0\pi_0, 0\rangle, \pi_1\rangle D[f] + \langle\langle 0, \pi_0\pi_1\rangle, \pi_1\rangle D[f] \\
&= \langle\langle \pi_0\pi_0, 0\rangle + \langle 0, \pi_0\pi_1\rangle, \pi_1\rangle D[f] && \text{DR.2} \\
&= \langle\langle \pi_0\pi_0, \pi_0\pi_1\rangle, \pi_1\rangle D[f] \\
&= \langle \pi_0, \pi_1\rangle D[f] = D[f]
\end{aligned}$$

as required. ◻

Partial derivatives allow one to speak of being linear in one argument. Of interest is when a function is **linear in its first argument**. To explain what this means choose $A \times B \xrightarrow{f} C$. The partial derivative with respect to the first variable has the type $A \times (A \times B) \xrightarrow{D_{\times,0}[f]} C$, and to say $D_{\times,0}[f]$ is linear is the requirement that $D_{\times,0}[f] \smile (1 \times \pi_1)f = a_\times^{-1}(\pi_0 \times 1)f$.

If f is linear in the first variable then $\langle 1,0\rangle f$ is linear.

Lemma 4.1.2. *If a map in a differential restriction category is linear in its first variable then*

$$D[\langle 1,0\rangle f] \smile \pi_0\langle 1,0\rangle f.$$

Proof. Assume that f is linear in its first variable, and then consider the following:

$$
\begin{aligned}
& D[\langle 1,0\rangle f] = \langle D[\langle 1,0\rangle], \pi_1\langle 1,0\rangle\rangle D[f] \\
& = \langle\langle D[1], D[0]\rangle, \pi_1\langle 1,0\rangle\rangle D[f] = \langle\langle \pi_0, 0\rangle, \pi_1\langle 1,0\rangle\rangle D[f] \\
& = (\langle 1,0\rangle \times \langle 1,0\rangle)D[f] = (1 \times \langle 1,0\rangle)(\langle 1,0\rangle \times 1)D[f] \\
& \smile (1 \times \langle 1,0\rangle)(1 \times \pi_1)f = (1 \times 0)f \\
& = \langle \pi_0, \pi_1 0\rangle f = \langle \pi_0, 0\rangle f = \langle \pi_0, \pi_0 0\rangle f \\
& = \pi_0\langle 1,0\rangle f
\end{aligned}
$$

□

DR.6 obviously implies that the derivative of a map is linear in its first variable. Moreover,

Lemma 4.1.3. *DR.6 is equivalent to* $(\langle 1,0\rangle \times 1)D[D[f]] = (1 \times \pi_1)D[f]$.

Proof. Assume DR.6. Then,

$$
\begin{aligned}
(\langle 1,0\rangle \times 1)D[D[f]] &= \langle \pi_0\langle 1,0\rangle, \pi_1\langle \pi_0, \pi_1\rangle\rangle D[D[f]] \\
&= \langle\langle \pi_0, 0\rangle, \langle \pi_1\pi_0, \pi_1\pi_1\rangle\rangle D[D[f]] \\
&= \langle \pi_0, \pi_1\pi_1\rangle D[f] && \text{DR.6} \\
&= (1 \times \pi_1)D[f].
\end{aligned}
$$

Conversely assume that $(\langle 1,0\rangle \times 1)D[D[f]] = (1 \times \pi_1)D[f]$. Then

$$
\begin{aligned}
& \langle\langle g,0\rangle, \langle h,k\rangle\rangle D[D[f]] = \overline{h}\langle g,k\rangle D[f] = \langle g, \langle h,k\rangle\rangle(\langle 1,0\rangle \times 1)D[D[f]] \\
& = \langle g, \langle h,k\rangle\rangle(1 \times \pi_1)D[f] = \langle g, \langle h,k\rangle\pi_1\rangle D[f] = \overline{h}\langle g,k\rangle D[f]
\end{aligned}
$$

□

Corollary 4.1.1. *For all f in a differential restriction category,*

$$D[\langle 1,0\rangle D[f]] = \pi_0\langle 1,0\rangle D[f] = \langle \pi_0, 0\rangle D[f].$$

4.2 Simple Slices

Differential restriction structure does not lift to the coKleisli category like Cartesian left additive restriction structure does [1]. However, the simple slice category $\mathbb{X}[A]$ is always a differential restriction category, and in this section we prove it.

The following theorem generalizes Blute *et al* [2] corollary 4.5.2 to the restriction setting; however, here a direct proof is given instead of using the term logic appeal that Blute *et al* used.

Theorem 4.2.1. *Let $\mathbb{X}$ be a differential restriction category. Then for each A, $\mathbb{X}[A]$ is a differential restriction category where the derivative is defined as*

$$\boldsymbol{D}[\boldsymbol{f}] := a_\times D_{\times,0}[f].$$

Again, of great help is that

$$(\overline{\boldsymbol{f}})^\sharp = \overline{f}.$$

Also, for readability let $N := (\langle 1,0\rangle \times 1)$.

Proof. The proof is by calculation in $\mathbb{X}[A]$.

DR.1 For the zero case,

$$\boldsymbol{D}[0] = a_\times D_{\times,0}[0] = 0.$$

For the sum use left additivity,

$$\boldsymbol{D}[\boldsymbol{f}+\boldsymbol{g}] = a_\times N D[f+g] = a_\times N(D[f]+D[g]) = a_\times N D[f]+a_\times N D[g] = \boldsymbol{D}[\boldsymbol{f}]+\boldsymbol{D}[\boldsymbol{g}].$$

DR.2 For the zero case,

$$\begin{aligned}\langle 0,\boldsymbol{g}\rangle \boldsymbol{D}[\boldsymbol{f}] &= \langle\langle 0,g\rangle,\pi_1\rangle a_\times N D[f] = \langle 0,\langle g,\pi_1\rangle\rangle N D[f]\\ &= \langle 0,\langle g,\pi_1\rangle\rangle D[f] = \overline{\langle g,\pi_1\rangle f}\, 0 = \overline{\boldsymbol{g}\boldsymbol{f}}\, 0.\end{aligned}$$

[1] One can put a differential restriction structure on the coKleisli category for certain comonads; however, $(\_ \times A)$ is not an example of such a comonad.

For the sum, use the that $\langle 1, 0\rangle$ is total and additive.

$$\begin{aligned}
&\langle g+h, k\rangle D[f] \\
&= \langle\langle g+h, k\rangle, \pi_1\rangle a_\times D_{\times,0}[f] = \langle g+h, \langle k, \pi_1\rangle\rangle(\langle 1, 0\rangle \times 1)D[f] \\
&= (g\langle 1,0\rangle + h\langle 1,0\rangle, \langle k, \pi_1\rangle)D[f] = \langle g, \langle k,\pi_1\rangle\rangle D_{\times,0}[f] + \langle h, \langle k, \pi_1\rangle\rangle D_{\times,0}[f] \\
&= \langle g, k\rangle D[f] + \langle h, k\rangle D[f].
\end{aligned}$$

DR.3 For π_0,

$$\begin{aligned}
D[\pi_0] &= a_\times ND[\pi_0\pi_0] = a_\times(\langle 1, 0 \times 1)\pi_0\pi_0\pi_0 \\
&= a_\times\pi_0\pi_0 = \pi_0\pi_0\pi_0 = \langle\pi_0\pi_0, \pi_1\rangle\pi_0\pi_0 \\
&= \pi_0\pi_0.
\end{aligned}$$

Similarly, $D[\pi_1] = \pi_0\pi_1$.

DR.4 Consider,

$$D[\langle f, g\rangle] = a_\times ND[\langle f, g\rangle] = a_\times N\langle D[f], D[g]\rangle = \langle D[f], D[g]\rangle.$$

DR.5 Consider,

$$\begin{aligned}
D[fg] &= a_\times ND[fg] = a_\times N\langle D[\langle f, \pi_1\rangle], \pi_1\langle f, \pi_1\rangle\rangle D[g] \\
&= a_\times(\langle 1, 0\rangle \times 1)\langle\langle D[f], \pi_0\pi_1\rangle, \langle\pi_1 f\pi_1\pi_1\rangle\rangle D[g] \\
&= \langle\langle D[f], 0\rangle, \langle a_\times\pi_1 f, a_\times\pi_1\pi_1\rangle\rangle D[g] \\
&= \langle D[f], \langle\pi_1 f, \pi_1\rangle\rangle(\langle 1, 0\rangle \times 1)D[g] = \langle D[f], \pi_1 f\rangle D[g].
\end{aligned}$$

DR.6 Consider,

$$\langle\langle \boldsymbol{g}, \boldsymbol{0}\rangle, \langle \boldsymbol{h}, \boldsymbol{k}\rangle\rangle \boldsymbol{D}[\boldsymbol{f}] = \langle\langle\langle g,0\rangle,\langle h,k\rangle\rangle,\pi_1\rangle a_\times ND[a_\times ND[f]]$$

$$= \langle\langle\langle g,0\rangle,0\rangle,\langle\langle h,k\rangle,\pi_1\rangle\rangle D[a_\times ND[f]]$$

$$= \langle\langle\langle g,0\rangle,0\rangle,\langle\langle h,k\rangle,\pi_1\rangle\rangle (a_\times N \times a_\times N) D[D[f]]$$

$$= \langle\langle\langle g,0\rangle,0\rangle,\langle\langle h,0\rangle,\langle k,\pi_1\rangle\rangle D[D[f]]$$

$$= \overline{\langle h,0\rangle}\,\langle\langle g,0\rangle,\langle k,\pi_1\rangle\rangle D[f] = \overline{h}\,\langle g,\langle k,\pi_1\rangle\rangle(\langle 1,0\rangle \times 1)D[f]$$

$$= \overline{h}\,\langle\langle g,k\rangle,\pi_1\rangle \boldsymbol{D}[\boldsymbol{f}] = \overline{\boldsymbol{h}}\,\langle \boldsymbol{g},\boldsymbol{k}\rangle \boldsymbol{D}[\boldsymbol{f}].$$

DR.7 Consider,

$$\langle\langle \boldsymbol{0}, \boldsymbol{g}\rangle, \langle \boldsymbol{h}, \boldsymbol{k}\rangle\rangle \boldsymbol{D}[\boldsymbol{D}[\boldsymbol{f}]] = \langle\langle\langle 0,g\rangle,\langle h,k\rangle\rangle,\pi_1\rangle a_\times ND[a_\times ND[f]]$$

$$= \langle\langle 0,g\rangle,\langle\langle h,k\rangle,\pi_1\rangle\rangle ND[a_\times ND[f]] = \langle\langle\langle 0,g\rangle,0\rangle,\langle\langle h,k\rangle,\pi_1\rangle\rangle D[a_\times ND[f]]$$

$$= \langle\langle\langle 0,g\rangle,0\rangle,\langle\langle h,k\rangle,\pi_1\rangle\rangle (a_\times N \times a_\times N) D[D[f]]$$

$$= \langle\langle 0,\langle g,0\rangle\rangle,\langle\langle h,0\rangle,\langle k,\pi_1\rangle\rangle\rangle D[D[f]] = \langle\langle 0,\langle h,0\rangle\rangle,\langle\langle g,0\rangle,\langle k,\pi_1\rangle\rangle\rangle D[D[f]]$$

$$= \cdots$$

$$= \langle\langle \boldsymbol{0}, \boldsymbol{h}\rangle, \langle \boldsymbol{g}, \boldsymbol{k}\rangle\rangle \boldsymbol{D}[\boldsymbol{D}[\boldsymbol{f}]].$$

DR.8 Recall that $\overline{\boldsymbol{f}} = \overline{f}\,\pi_0$. Then,

$$\boldsymbol{D}[\overline{\boldsymbol{f}}\,] = a_\times D_{\times,0}[\overline{f}\,\pi_0] = a_\times(\langle 1,0\rangle \times 1)(1 \times \overline{f}\,)\pi_0\pi_0$$

$$= a_\times(1 \times \overline{f}\,)(\langle 1,0\rangle \times 1)\pi_0\pi_0 = a_\times(1 \times \overline{f}\,)\pi_0$$

$$= \langle \pi_0\pi_0, \langle \pi_0\pi_1,\pi_1\rangle \overline{f}\,\rangle\pi_0 = \overline{\langle \pi_0\pi_1,\pi_1\rangle f}\,\pi_0\pi_0 = \overline{\boldsymbol{\pi_1 f}}\,\pi_0\pi_0$$

$$= \overline{\boldsymbol{\pi_1 f}}\,\pi_0.$$

DR.9 Consider,

$$\begin{aligned}\overline{D[f]} &= \overline{a_x D_{\times,0}[f]}\,\pi_0 = \overline{a_\times(\langle 1,0\rangle \times 1)(1\times \overline{f})}\,\pi_0\\ &= \overline{\langle \pi_0\pi_0\langle 1,0\rangle, \langle \pi_0\pi_1,\pi_1\rangle\overline{f}\rangle}\,\pi_0\\ &= \overline{\langle \pi_0\pi_1,\pi_1\rangle f}\,\pi_0 = \overline{\pi_1 f}\,.\end{aligned}$$

Therefore, $\mathbb{X}[A]$ is a differential restriction category. □

Denote $\mathbb{X}[A]$ with the above differential restriction structure by $\mathbb{X}[A]_{\mathsf{D}}$.

Note that a strongly linear map in $\mathbb{X}[A]_{\mathsf{D}}$ is f such that

$$a_\times D_{\times,0}[f] = \boldsymbol{D}[f] \leq \pi_0 f = (\pi_0 \times 1)f.$$

Proposition 4.2.1. *A map f is strongly linear if and only if f is strongly linear in its first variable.*

Proof. Assume f is strongly linear. Then,

$$D_{\times,0}[f] = a_\times^{-1} a_\times D_{\times,0}[f] \leq a_\times^{-1}(\pi_0 \times 1)f = (\pi_1 \times 1)f,$$

so f is strongly linear in its first argument.

Conversely, assume that f is strongly linear in its first argument. Then

$$a_\times D_{\times,0}[f] \leq a_\times(\pi_1 \times 1)f = (\pi_0 \times 1)f,$$

so f is strongly linear. □

4.3 Idempotent Splitting

In [36], Manzonetto showed that a source of models for the untyped differential lambda calculus are *linear reflexive objects.* These are objects A in a Cartesian closed differential category for which $(A \Rightarrow A)$ is a retract of A and such that the s, r giving the retraction are both linear.

In this section, we will show that a more general class of idempotents can be split in a differential restriction category to again obtain a differential restriction category. The key observation made in this section is that when these idempotents split, they have the property that the retraction is linear; however, the section need not be.

Suppose that $e = ee$ splits as rs, and consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & E \arrow[r, equal] \arrow[d, "s"'] & E \arrow[r, "g"] \arrow[d, "s"] & Y \\ & A \arrow[r, "e"'] \arrow[ur, "r"] & A & \end{tikzcd}$$

It is desirable that the splitting interacts well with the derivative. Denote $f^{(0)} := f$ and $f^{(n)} := f^{(n-1)} \times f^{(n-1)}$. Also, we will use $E \overset{(r,s)}{\lhd} A$ to denote the splitting of an idempotent e as rs where $E \xrightarrow{s} A$ and $A \xrightarrow{r} E$.

Definition 4.3.1. *An* **n-differential splitting** *is* $E \overset{(r,s)}{\lhd} A$ *such that* r *is strongly additive and linear, and for all* $k \leq n$ *and all* $X \xrightarrow{f} E$, $E \xrightarrow{g} Y$*:*

$$D^k[f] = D^k[fs]r \qquad \textit{and} \qquad D^k[g] = s^{(k)} D^k[rg].$$

Given an n-differential splitting, call the idempotent, rs, **n-differentially split.**

In the above, the conditions on each n are, as far as we can see, independent. In this thesis we only consider, in detail, 2-differential splittings which is sufficient for theorem 4.3.1. For arbitrary n , the proofs in this chapter are more complicated because they rely on higher order chain rules. We believe the results hold for arbitrary n, and further, we believe that proofs may be obtained by using Cockett and Seely's Faa di Bruno construction [12] which provides the formulae for higher order chain rules.

It is clear that if $E \overset{(r,s)}{\lhd} A, E' \overset{(r',s')}{\lhd} A$ are n-differential splittings, and $E \xrightarrow{f} E'$, then for $k \leq n$ the derivative can be characterized as $D^k[f] = s^{(k)} D^k[rfs']r'$, or equivalently as $e^{(k)} D^k[rfs']e'$.

When the retraction of an n-differential splitting is linear, we observe:

Proposition 4.3.1 ($n \leq 2$). *Suppose* $E \overset{(r,s)}{\lhd} A$ *is such that* r *is strongly additive and linear. Then* $E \overset{(r,s)}{\lhd} A$ *an* n*-differential splitting if and only if for all* $k \leq n$, $D^k[s]r = \pi_0 \cdots \pi_0$ *(*k *times).*

Proof. Suppose $E \overset{(r,s)}{\lhd} A$ is such that r is strongly additive and linear.
($\Rightarrow$): First, assume $E \overset{(r,s)}{\lhd} A$ is an n-differential splitting. Then,

$$\pi_0 = D[1] = D[1s]r = D[s]r$$

$$\pi_0\pi_0 = D[\pi_0] = D[D[1]] = D[D[s]]r$$

($\Leftarrow$): Conversely assume that $D[s]r = \pi_0$ and $D^2[s]r = \pi_0\pi_0$.

$$\begin{aligned}
D[g] &= (s \times s)(r \times s)D[g] = (s \times s)(\overline{r} \times 1)\langle D[g], \pi_1 r\rangle D[g] \\
&= (s \times s)D[rg] \\
D^2[g] &= s^{(2)}((r \times r) \times (r \times r))D^2[g] \\
&= s^{(2)}(1 \times 1 \times \overline{r} \times 1)((r \times r) \times (r \times r))D^2[g] \\
&= s^{(2)}\langle \pi_0(r \times r), \langle \pi_1 D[r], \pi_1\pi_1 r\rangle\rangle D^2[g] \\
&= s^{(2)}\langle\langle \pi_0\pi_0 r, \pi_0\pi_1 r\rangle, \langle \pi_1 D[r], \pi_1\pi_1 r\rangle\rangle D^2[g] \\
&= s^{(2)}(1 \times \overline{r} \times 1 \times 1)\langle\langle \pi_0\pi_0 r, (\pi_1 \times \pi_1)\pi_0 r\rangle, \langle \pi_1 D[r], \pi_1\pi_1 r\rangle\rangle D^2[g] \\
&= s^{(2)}\langle\langle \pi_0\pi_0 r, (\pi_1 \times \pi_1)D[r]\rangle, \langle \pi_1 D[r], \pi_1\pi_1 r\rangle\rangle D^2[g] \\
&= s^{(2)}(\overline{r} \times 1 \times 1 \times 1)\langle\langle \pi_0\pi_0 r, (\pi_1 \times \pi_1)D[r]\rangle, \langle \pi_1 D[r], \pi_1\pi_1 r\rangle\rangle D^2[g] \\
&= s^{(2)}\langle\langle D^2[r], (\pi_1 \times \pi_1)D[r]\rangle, \langle \pi_1 D[r], \pi_1\pi_1 r\rangle\rangle D^2[g] \\
&= s^{(2)}\langle D[\langle D[r], \pi_1\rangle], \pi_1\langle D[r], \pi_1 r\rangle\rangle D^2[g] \\
&= s^{(2)}D[\langle D[r], \pi_1 r\rangle D[g]] \\
&= s^{(2)}D^2[rg]
\end{aligned}$$

$$D[f] = D[f] = \langle D[f], \pi_1 \rangle \pi_0 = \langle D[f, \pi_1 f \rangle D[s]r = D[fs]r$$

$$\begin{aligned} D^2[f] &= \langle \langle D^2[f], (\pi_1 \times \pi_1) D[f] \rangle, \pi_1 \langle D[f], \pi_1 f \rangle \rangle \pi_0 \pi_0 \\ &= \langle \langle D^2[f], (\pi_1 \times \pi_1) D[g] \rangle, \pi_1 \langle D[f], \pi_1 f \rangle \rangle D^2[s]r \\ &= \langle D[\langle D[f], \pi_1 f \rangle], \pi_1 \langle D[f], \pi_1 f \rangle \rangle D^2[s]r \\ &= D[\langle D[f], \pi_1 f \rangle D[s]]r \\ &= D^2[fs]r; \end{aligned}$$

Thus, $E \overset{(r,s)}{\lhd} A$ is a 2-differential splitting. $\square$

Note that the above calculations involving g used only linearity.

The following lemma will be useful in calculations:

Lemma 4.3.1. *Let r be and linear and $sr = 1$, then:*

(i) *(a)* $D[rs]\overline{r} = D[rs]$ $\Leftrightarrow$ *(b)* $D[s]\overline{r} = D[s]$ $\Leftrightarrow$ *(c)* $D[s]r = \pi_0$,

(ii) *(a)* $D^2[rs]\overline{r} = D[rs]$ $\Leftrightarrow$ *(b)* $D^2[s]\overline{r} = D^2[s]$ $\Leftrightarrow$ *(c)* $D^2[s]r = \pi_0\pi_0$.

Proof. For the first set of equivalences:

(a) $\Rightarrow$ *(b)*: $D[s]\overline{r} = (s \times s)D[rs]\overline{r} = (s \times s)D[rs] = D[s]$,

(b) $\Rightarrow$ *(c)*:

$$\begin{aligned} D[s]r &= \langle D[s]\pi_1 s \rangle \pi_0 r = \langle D[s], \pi_1 s \rangle (1 \times \overline{r}) \pi_0 r = \langle D[s], \pi_1 s \rangle (\overline{r} \times 1) D[r] \\ &= \langle D[s]\overline{r}, \pi_1 s \rangle D[r] = D[sr] = \pi_0, \end{aligned}$$

(c) $\Rightarrow$ *(a)*: Note that $D[rs]\overline{r} \leq D[rs]$, thus they are equal if they have the same restriction. Consider,

$$\overline{D[rs]\overline{r}} = \overline{\langle D[r], \pi_1 r \rangle D[s] r} = \overline{\langle D[r], \pi_1 r \rangle \pi_0} = \overline{D[r]}.$$

For the next set of equivalences, first note that $D^2[s] = (s \times s \times s \times s)D^2[rs]$. Then:

$(a) \Rightarrow (b)$:

$$D^2[s]\overline{r} = (s \times s \times s \times s)D^2[rs]\overline{r} = (s \times s \times s \times s)D^2[rs] = D^2[s].$$

$(b) \Rightarrow (c)$:

$$\begin{aligned} D^2[s]r &= \langle D^2[s], \pi_1 D[s]\rangle \pi_0 r = \langle D^2[s]\overline{r}, \pi_1 D[s]\rangle \pi_0 r \\ &= \langle D^2[s], \pi_1 D[s]\rangle D[r] = D[D[s]r] = D[\pi_0] = \pi_0\pi_0, \end{aligned}$$

$(c) \Rightarrow (a)$: $D[rs]\overline{r} \leq D[rs]$, and they will be shown equal by showing they have the same restriction.

$$\begin{aligned} \overline{D^2[rs]r} &= \overline{\langle\langle D^2[r], (\pi_1 \times \pi_1)D[r]\rangle, \pi_1\langle D[r], \pi_1 r\rangle\rangle D^2[s]r} \\ &= \overline{\langle\langle D^2[r], (\pi_1 \times \pi_1)D[r]\rangle, \pi_1\langle D[r], \pi_1 r\rangle\rangle \pi_0\pi_0} = \overline{D^2[r]} \\ &= \overline{\pi_1\pi_1 r} = \overline{\pi_1\pi_1 rs} = \overline{D^2[rs]}\,. \end{aligned}$$

□

n-Differential splittings are unique up to a linear isomorphism. Further, an idempotent may split with (just) a linear retraction in many ways, and if any one of these splittings is, in addition, an n-differential splitting, then all of the splittings are n-differential splittings.

Proposition 4.3.2. *If*

$$\begin{tikzcd} & E \arrow[dr, "s"] & \\ A \arrow[ur, "r"] \arrow[rr, "e"] \arrow[dr, "r'"'] & & A \\ & E' \arrow[ur, "s'"'] & \end{tikzcd}$$

commutes, and r, r' are linear. Then *(i) There is a unique isomorphism $E \xrightarrow{\alpha} E'$ such that $r\alpha = r'$ and $\alpha s' = s$. Moreover, α is linear.*

(ii) ($n \leq 2$) If, in addition, r, r' are strongly additive and $E \overset{(r,s)}{\lhd} A$ is an n-differential splitting then so is $E' \overset{(r',s')}{\lhd} A$.

Proof. Suppose r, r' are linear. Then,

(i) The proof that $\alpha := sr'$ is the unique isomorphism such that $r\alpha = r'$ and $\alpha s's$ is the same as for proposition 3.3.2. To see that α is linear, consider:

$$D[\alpha] = (s \times s)D[r\alpha] = (s \times s)D[r'] \smile (s \times s)\pi_r' = \pi_0 sr'.$$

(ii) Suppose that r, r' are strongly additive and $E \overset{(r,s)}{\lhd} A$ is an n-differential splitting. Since r' is linear, e will show that $E' \overset{(r',s')}{\lhd} A$ is an n-differential splitting using proposition 4.3.1. To show that $D[s']r' = \pi_0$ use that since α^{-1} is total and linear, then for any g, $D[\alpha^{-1}g] = (\alpha^{-1} \times \alpha^{-1})D[g]$.

$$D[s']r' = D[\alpha^{-1}s]r\alpha = \langle D[\alpha^{-1}], \pi_1\alpha^{-1}\rangle D[s]r\alpha = (\alpha^{-1} \times \alpha^{-1})\pi_0\alpha = \pi_0,$$

To show that $D^2[s']r' = \pi_0\pi_0$, use that since α^{-1} is total and linear , then for any g, $D^2[\alpha^{-1}g] = (\alpha^{-1})^{(2)}D[g]$.

$$D^2[s']r' = D^2[\alpha^{-1}s]r\alpha = (\alpha^{-1})^{(2)}D^2[s']r' = (\alpha^{-1})^{(2)}\pi_0\pi_0\alpha = \pi_0\pi_0$$

□

Next, we characterize the idempotents which split as n-differential splittings:

Definition 4.3.2. *Let $\mathbb{X}$ be a differential restriction category. An idempotent e is an* **n-differential idempotent** *when $\overline{e}$ is strongly additive, $(f+g)e \geq (fe+ge)e$, $D[e]e \smile \pi_0 e$, $(e \times 1)D[e] = (\overline{e} \times 1)D[e]$, and for all $k \leq n$, $D^k[e]\overline{e} = D^k[e]$.*

n-Differential idempotents have the following properties:

Lemma 4.3.2 ($n \leq 2$). *The following hold in any differential restriction category:*

(i) If e is n-differentially split, then e is a n-differential idempotent.

(ii) If e, e' are n-differential idempotents, then $e \times e'$ is an n-differential idempotent.

(iii) If $e = \overline{e}$ then e is an n-differential idempotent if and only if e is total (i.e. the identity).

Proof. (i) Suppose $e = rs$ is n-differentially split. The additive properties follow because of lemma (3.3.3). The first thing to show is $D[rs]rs \smile \pi_0 rs$. Consider,

$$D[rs]rs = \langle D[r], \pi_1 r\rangle D[s]rs = \langle D[r], \pi_1 r\rangle \pi_0 s = D[r]s \smile \pi_0 rs.$$

Next we show $(rs \times 1)D[rs] = (\overline{rs} \times 1)D[rs]$. Consider,

$$(rs \times 1)D[rs] = (rs \times 1)\langle D[r], \pi_1 r\rangle D[s] = (rsr \times r)D[s] = (r \times r)D[s]$$
$$= (\overline{r} \times 1)\langle D[r], \pi_1 r\rangle D[s] = (\overline{rs} \times 1)D[rs].$$

The other two properties are from lemma (4.3.1). Thus, rs is an n-differential idempotent.

(ii) Suppose e, e' are n-differential idempotents. The additive properties hold by lemma (3.3.2). Next, we show $D[e \times e'](e \times e') \smile \pi_0(e \times e')$. First, note that $\overline{D[e \times e'](e \times e')} = \overline{\pi_1\pi_0 e}\,\overline{\pi_1\pi_1 e'}$.

$$\overline{\pi_0(e \times e')}\, D[e \times e'](e \times e') = \overline{\pi_0\pi_0 e}\,\overline{\pi_0\pi_1 e'}\,\langle(\pi_0 \times \pi_0)D[e], (\pi_1 \times \pi_1)D[e']\rangle(e \times e')$$
$$= \langle(\pi_0 \times \pi_0)(\overline{e} \times 1)D[e]e, (\pi_1 \times \pi_1)(\overline{e'} \times 1)D[e']e'\rangle$$
$$= \langle(\pi_0 \times \pi_0)\overline{D[e]}\,\pi_0 e, (\pi_1 \times \pi_1)\overline{D[e']}\,\pi_0 e'\rangle = \overline{\pi_1\pi_0 e}\,\overline{\pi_1\pi_1 e'}\,\langle\pi_0\pi_0 e, \pi_0\pi_1 e'\rangle$$
$$= \overline{D[e \times e'](e \times e')}\,\pi_0(e \times e'),$$

as required. Next, we show $((e \times e') \times 1)D[e \times e'] = ((\overline{e} \times \overline{e'}) \times 1)D[e \times e']$. Use lemma (4.3.3) in the first and last steps, and consider,

$$((e \times e') \times 1)D[e \times e'] = \langle (e \times 1)D[e], (e' \times 1)D[e'] \rangle = \langle (\overline{e} \times 1)D[e], (\overline{e'} \times 1)D[e'] \rangle$$

$$= ((\overline{e} \times \overline{e'}) \times 1)D[e \times e'],$$

as required. Next, we will show $D[e \times e'](\overline{e} \times \overline{e'}) = D[e \times e']$. Consider,

$$D[e \times e'](\overline{e} \times \overline{e'}) = \langle (\pi_0 \times \pi_0)D[e], (\pi_1 \times \pi_1)D[e'] \rangle (\overline{e} \times \overline{e'})$$

$$= \langle (\pi_0 \times \pi_0)D[e]\overline{e}, (\pi_1 \times \pi_1)D[e']\overline{e'} \rangle = \langle (\pi_0 \times \pi_0)D[e], (\pi_1 \times \pi_1)D[e'] \rangle = D[e \times e'],$$

as required. Finally, we will show $D^2[e \times e'](\overline{e} \times \overline{e'})$. Consider,

$$D[D[e \times e']](\overline{e} \times \overline{e'}) = \langle (\pi_0 \times \pi_0 \times \pi_0 \times \pi_0)D^2[e], (\pi_1 \times \pi_1 \times \pi_1 \times \pi_1)D^2[e'] \rangle (\overline{e} \times \overline{e'})$$

$$= \langle (\pi_0 \times \pi_0 \times \pi_0 \times \pi_0)D^2[e], (\pi_1 \times \pi_1 \times \pi_1 \times \pi_1)D^2[e'] \rangle = D^2[e \times e'].$$

Thus, $e \times e'$ is an n-differential idempotent.

(iii) Assume $e = \overline{e}$ and that e is an n-differential idempotent. In particular, from $D[e]\overline{e} = D[e]$ we have

$$(1 \times \overline{e})\pi_0 = D[\overline{e}] = D[e] = D[e]\overline{e} = (\overline{e} \times \overline{e})\pi_0.$$

Together with $0\overline{e} = 0$, we have,

$$\overline{e} = \langle \overline{e}, 0 \rangle \pi_0 = \langle 1, 0 \rangle (\overline{e} \times \overline{e})\pi_0 = \langle 1, 0 \rangle (1 \times \overline{e})\pi_0 = \langle 1, 0\overline{e} \rangle \pi_0 = 1,$$

thus e is total.

□

Every n-differentially split idempotent is an n-differential idempotent; however, n-differential idempotents need not split. A differential restriction category is an **n-split differential restriction category** when every n-differential idempotent is n-differentially split.

Proposition (4.3.1) says that for $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ to be an n-split differential restriction category, if $e \xrightarrow{f} e'$, the derivative of f must be:

$$D[f] := (e \times e)D[efe']e' = (e \times e)D[f]e'.$$

Denote $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ with this differential structure by $\mathsf{Split}_{\mathcal{E}_{\mathcal{D}}}(\mathbb{X})$.

The goal in the remainder of this section is to prove the following theorem.

Theorem 4.3.1. *Let $\mathbb{X}$ be a differential restriction category. $\mathsf{Split}_{\mathcal{E}_{\mathcal{D}}}(\mathbb{X})$ is a 2-split differential restriction category if and only if $\mathcal{E}_{\mathcal{D}}$ is the class of 2-differential idempotents.*

The following lemma will be useful in calculations.

Lemma 4.3.3. *In a differential restriction category, the following hold:*

(i) If $\overline{e}$ is strongly additive then, $(f+g)e \geq (fe+ge)e$ if and only if $(f\overline{e}+g\overline{e})e = (fe+ge)e$.

(ii) If e is an n-differential idempotent, then $(fD[e]+gD[e])e = (fD[e]e+gD[e]e)e$.

(iii) For all a,b,c,d,e,f,g:

(a) $\langle\langle a,b\rangle,\langle c,d\rangle\rangle D[f\times g] = \langle\langle a,c\rangle D[f],\langle b,d\rangle D[g]\rangle$;

(b) $(1\times 1\times \overline{e}\times 1)D[g] = D[(\overline{e}\times 1)g]$;

(c) $\langle\langle a,b\rangle,\langle c,d\rangle\rangle D[(e\times 1)g] = \langle\langle\langle a,c\rangle D[e],b\rangle,\langle ce,d\rangle\rangle D[g]$.

(iv) If $f = ef$ and $f = fe'$ where e,e' are n-differential idempotents, then:

(a) $(\overline{e}\times 1)D[f] = (e\times 1)D[f]$;

(b) $\langle 0e,k\rangle D[f] = \overline{kf}\,0$;

(c) $\langle\langle a,b\rangle,\langle c,d\rangle\rangle(1\times 1\times\overline{e}\times 1)D^2[f] = \langle\langle\langle a,c\rangle D[e],b\rangle,\langle ce,d\rangle\rangle D^2[f]$;

(d) $D[f] = D[f]\overline{e'}$;

(e) $D^2[f] = D^2[f]\overline{e'}$;

(f) $D[D[f]e']e' = D^2[f]e'$.

Proof. (i) Suppose $\overline{e}$ is strongly additive. Consider,

$$
\begin{aligned}
&(f+g)e \geq (fe+ge)e \\
&\Leftrightarrow \overline{(fe+ge)e}\,(f+g)e = (fe+ge)e \\
&\Leftrightarrow \overline{fe}\,\overline{ge}\,(f+g)e = (fe+ge)e \qquad \text{lemma 3.3.1} \\
&\Leftrightarrow (f\overline{e} + g\overline{e})e = (fe+ge)e
\end{aligned}
$$

(ii) Suppose e is an n-differential idempotent. Then,

$$
\begin{aligned}
(fD[e]+gD[e])e &= (fD[e]\overline{e} + gD[e]\overline{e})e \\
&= (fD[e]e + gD[e]e)e \qquad \text{part (i)}
\end{aligned}
$$

(iii) Consider the following:

(a)

$$
\langle\langle a,b\rangle,\langle c,d\rangle\rangle D[f\times g] = \langle\langle a,b\rangle,\langle c,d\rangle\rangle\langle(\pi_0\times\pi_0)D[f],(\pi_1\times\pi_1)D[g]\rangle
$$
$$
= \langle\langle a,c\rangle D[f],\langle b,d\rangle D[g]\rangle;
$$

(b)

$$
D[(\overline{e}\times 1)g] = \langle D[\overline{e}\times 1],\pi_1(\overline{e}\times 1)\rangle D[g] = \langle\langle(\pi_0\times\pi_0)D[\overline{e}],\pi_0\pi_1\rangle,\pi_1(\overline{e}\times 1)\rangle D[g]
$$
$$
= \langle\langle\overline{\pi_1\pi_0 e}\,\pi_0(1\times 1),\pi_1(\overline{e}\times 1)\rangle D[g] = (1\times 1\times\overline{e}\times 1)D[g];
$$

(c)

$$
\langle\langle a,b\rangle,\langle c,d\rangle\rangle D[(e\times 1)g] = \langle\langle a,b\rangle,\langle c,d\rangle\rangle\langle D[e\times 1],\pi_1(e\times 1)\rangle D[g]
$$
$$
= \langle\langle\langle a,c\rangle D[e],b\rangle,\langle ce,d\rangle\rangle D[g] \qquad \text{part (a);}
$$

(iv) Assume $f = ef$, $f = fe'$, and e, e' are n-differential idempotents. Consider:

(a) Note that $(\overline{e} \times 1)\pi_1 = (e \times 1)\pi_1$. Then

$$(\overline{e} \times 1)D[f] = (\overline{e} \times 1)D[ef] = (\overline{e} \times 1)\langle D[e], \pi_1 e\rangle D[f]$$

$$= \langle(\overline{e} \times 1)D[e], (\overline{e} \times 1)\pi_1 e\rangle D[f] = \langle(e \times 1)D[e], (e \times 1)\pi_1 D[e]$$

$$= (e \times 1)\langle D[e], \pi_1 e\rangle D[f] = (e \times 1)D[f].$$

(b) Use that $\overline{e}$ is strongly additive; in particular, $0\overline{e} = 0$.

$$\langle 0e, k\rangle D[f] = \langle 0, k\rangle(e \times 1)D[f] = \langle 0\overline{e}, k\rangle D[f] = \langle 0, k\rangle D[f] = \overline{kf}\, 0.$$

(c) Use part (iii.b), (iv.a), then (iii.c):

$$\langle\langle a, b\rangle, \langle c, d\rangle\rangle(1 \times 1 \times \overline{e} \times 1)D^2[f] = \langle\langle a, b\rangle, \langle c, d\rangle\rangle D[(\overline{e} \times 1)D[f]]$$

$$= \langle\langle a, b\rangle, \langle c, d\rangle\rangle D[(e \times 1)D[f]] = \langle\langle\langle a, c\rangle D[e], b\rangle, \langle ce, d\rangle\rangle D^2[f].$$

(d) Consider,

$$D[f] = \langle D[f], \pi_1 f\rangle D[e'] = \langle D[f], \pi_1 f] D[e']\overline{e'} = D[f]\overline{e'}.$$

(e) Consider,

$$D^2[f] = D^2[fe'] = \langle\langle D^2[f], (\pi_1 \times \pi_1)D[f]\rangle, \pi_1\langle D[f], \pi_1 f\rangle\rangle D^2[e']$$

$$= \langle\langle D^2[f], (\pi_1 \times \pi_1)D[f]\rangle, \pi_1\langle D[f], \pi_1 f\rangle\rangle D^2[e']\overline{e'} = D^2[f]\overline{e'}$$

(f) Consider,

$$D[D[f]e']e' = \langle D^2[f], \pi_1 D[f]\rangle D[e']e' = \langle D^2[f]\overline{e'}, \pi_1 D[f]\rangle D[e']e'$$

$$= \langle D^2[f], \pi_1 D[f]\rangle(\overline{e'} \times 1)D[e']e' = \langle D^2[f], \pi_1 D[f]\rangle\overline{D[e']e'}\,\pi_0 e'$$

$$= \langle D^2[f], \pi_1 D[f]\overline{e'}\rangle\pi_0 e' = D^2[f]e'$$

$\square$

Next, we show that $\mathsf{Split}_{\mathcal{E}_\mathcal{D}}(\mathbb{X})$ is an n-split differential restriction category. $\mathcal{E}$ consists of only n-differential idempotents hence they are retractively additive; thus, theorem (3.3.1) says that $\mathsf{Split}_{\mathcal{E}_\mathcal{D}}(\mathbb{X})$ is a Cartesian left additive restriction category. Next we must check the differential restriction axioms:

DR.1 For the zero case:

$$(e \times e)D[e0e']e' = (e \times e)D[0e']e' = (e \times e)(0 \times 0)(\overline{e} \times 1)D[e']e'$$
$$= (e0 \times e0)\overline{\pi_1 e'}\,\pi_0 e' = (e \times e)(0 \times 0)e' = (e \times e)0e'.$$

For the sum:

$$(e \times e)D[(f+g)e')]e' = (e \times e)\langle D[f+g], \pi_1(f+g)\rangle D[e']e'$$
$$= (e \times e)\langle D[f] + D[g], \pi_1(f+g)\rangle D[e']e'$$
$$= (e \times e)(\langle D[f], \pi_1(f+g)\rangle D[e'] + \langle D[g], \pi_1(f+g)\rangle D[e'])e'$$
$$= (e \times e)(\langle D[f], \pi_1(f+g)\rangle D[e']e' + \langle D[g], \pi_1(f+g)\rangle D[e']e')e' \qquad \text{lemma 4.3.3}$$
$$= (e \times e)(\langle D[f], \pi_1 f(f+g)\rangle \overline{\pi_1 e'}\,\pi_0 e' + \langle D[g], \pi_1(f+g)\rangle \overline{\pi_1 e'}\,\pi_0 e')e'$$
$$= (e \times e)(\overline{\pi_1(fe' + ge')e'}\,D[f]e' + \overline{\pi_1(fe' + ge')e'}\,D[g]e')e'$$
$$= (e \times e)\overline{\pi_1 fe'}\,\overline{\pi_1 ge'}\,(D[f]e' + D[g]e')e'$$
$$= (e \times e)D[f]e' + (e \times e)D[g]e'.$$

DR.2 For the zero case:

$$\langle e0e', g\rangle(e' \times e')D[f]e'' = (e \times e)\langle 0e', g\rangle D[f]e'' = (e \times e)\overline{gf}\,0e''$$

For the sum:

$$\langle (h+k)e, g\rangle(e \times e)D[f]e' = \langle h+k, g\rangle(\overline{e} \times 1)D[f]e' = \langle h+k, g\rangle D[f]e'$$
$$= (\langle h, g\rangle D[f] + \langle k, h\rangle D[f])e'$$
$$= (\langle h, k\rangle(e \times e)D[f]e' + \langle g, k\rangle(e \times e)D[f]e')e' \qquad \text{lemma 4.3.3}$$

DR.3 Let $f : e \longrightarrow e_1$ and $g : e \longrightarrow e_2$.

$$(e \times e)D[\langle f, g\rangle](e_1 \times e_2) = \langle (e \times e)D[f]e_1, (e \times e)D[g]e_2\rangle.$$

DR.4 Consider the following,

$$((e \times e') \times (e \times e'))D[(e \times e')\pi_0]e = ((e \times e') \times (e \times e'))\langle D[e \times e'], \pi_1(e \times e')\rangle\pi_0\pi_0 e$$

$$= ((e \times e') \times (e \times e'))D[e \times e]\pi_0 e = ((e \times e') \times (e \times e'))\overline{\pi_1\pi_1 e'}\,(\pi_0 \times \pi_0)D[e]e$$

$$= ((e \times e') \times (e \times e'))(\pi_0 \times \pi_0)D[e]e = (\overline{\pi_1 e'}\,\pi_0 e \times \overline{\pi_1 e'}\,\pi_0 e)(\overline{e} \times 1)D[e]e$$

$$= (\overline{\pi_1 e'}\,\pi_0 e \times \overline{\pi_1 e'}\,\pi_0 e)\overline{\pi_1 e}\,\pi_0 e = ((e \times e') \times (e \times e'))\pi_0\pi_0 e$$

$$= ((e \times e') \times (e \times e'))\pi_0\pi_0$$

DR.5 Consider,

$$\langle (e_1 \times e_1)D[f]e_2, (e_1 \times e_1)\pi_1 f\rangle(e_2 \times e_2)D[g]e_3 = (e_1 \times e_1)\langle D[f], \pi_1 f\rangle(e_2 \times 1)D[g]e_3$$

$$(e_1 \times e_1)\langle D[f], \pi_1 f\rangle(\overline{e_2} \times 1)D[g]e_3 = (e_1 \times e_1)\langle D[f], \pi_1 f\rangle D[g]e_3$$

$$(e_1 \times e_1)D[fg]e_3.$$

DR.6 Consider,

$$\langle\langle g, d0e\rangle, \langle h, k\rangle\rangle(e \times e \times e \times e)D[(e \times e)D[f]e']e'$$

$$= d\langle\langle g, 0e\rangle, \langle h, k\rangle\rangle\langle D[e \times e], \pi_1(e \times e)\rangle D[D[f]e']e'$$

$$= d\langle\langle\langle g, h\rangle D[e], \langle 0e, k\rangle D[e]\rangle, \langle h, k\rangle\rangle D^2[f]e'$$

$$= d\langle\langle\langle g, h\rangle D[e], 0\rangle, \langle h, k\rangle\rangle D^2[f]e' = d\overline{h}\,\langle\langle g, h\rangle D[e], k\rangle D[fe]'$$

$$= d\langle\langle g, h\rangle D[e]\overline{e}\,, k\rangle D[f]e' = d\langle g, h\rangle D[e]e, k\rangle D[f]e'$$

$$= d\langle\langle g, h\rangle\pi_0 e, k\rangle D[f]e' = d\overline{h}\,\langle g, k\rangle D[f]e'.$$

DR.7 Consider:

$$
\begin{aligned}
&\langle\langle d0e, g\rangle, \langle h, k\rangle\rangle(e \times e \times e \times e)D[(e \times e)D[f]e']e' \\
&= d\langle\langle 0e, g\rangle, \langle h, k\rangle\rangle\langle D[e \times e], \pi_1(e \times e)\rangle D[D[f]e']e' \\
&= d\langle\langle\langle 0e, h\rangle D[e], \langle g, k\rangle D[e]\rangle, \langle h, k\rangle\rangle D^2[f]e' \\
&= d\langle\langle 0, \langle g, k\rangle D[e]\rangle, \langle h, k\rangle\rangle D^2[f]e' \\
&= d\langle\langle 0, h\rangle, \langle\langle g, k\rangle D[e], k\rangle\rangle D^2[f]e' \qquad \textbf{DR.7} \\
&= d\langle\langle 0, h\rangle, \langle\langle g, k\rangle D[e], k\rangle\rangle(1 \times 1 \times \overline{e} \times 1)D^2[f]e \\
&= d\langle\langle\langle 0, \langle g, k\rangle D[e]\rangle D[e], h\rangle, \langle\langle g, k\rangle D[e]e, k\rangle\rangle D^2[f]e' \qquad \text{lemma 4.3.3} \\
&= d\langle\langle 0, h\rangle, \langle g, k\rangle\rangle D^2[f]e' \\
&= \cdots \\
&= \langle\langle d0e, h\rangle, \langle g, k\rangle\rangle(e \times e \times e \times e)D[(e \times e)D[f]e']e'.
\end{aligned}
$$

DR.8 Consider:

$$
(e \times e)D[\overline{f}\,]e = (e \times e)\overline{\pi_1 f}\,\pi_0 e = (e \times e)\overline{(e \times e)\pi_1 f}\,(e \times e)\pi_0.
$$

DR.9 Consider:

$$
\begin{aligned}
&(e \times e)\overline{(e \times e)D[f]e'} = (e \times e)\overline{(e \times e)\langle D[f], \pi_1 f\rangle D[e']e'} \\
&= (e \times e)\overline{(e \times e)\langle D[f], \pi_1 f\rangle D[e']\overline{e'}} = (e \times e)\overline{(e \times e)\langle D[f], \pi_1 f\rangle D[e']} \\
&= (e \times e)\overline{(e \times e)D[f]} = (e \times e)\overline{(e \times e)\pi_1 f}\,.
\end{aligned}
$$

Next, we will show that each n-differential idempotent is n-differentially split. First, every idempotent $e \in \mathcal{E}$ splits in $\mathsf{Split}_{\mathcal{E}_D}(\mathbb{X})$ as:

$$
\begin{tikzcd}
d \arrow[rr, "e"] \arrow[dr, "e"'] & & d \\
& e \arrow[ur, "e"'] &
\end{tikzcd}
$$

Theorem 3.3.1 proves that the retraction, $e : d \longrightarrow e$, is strongly additive. To show that the retraction is linear, we must show that

$$(d \times d)D[e]e \smile (d \times d)\pi_0 e$$

which is true by definition. Lemma 4.3.1 proves that d is a linear 2-differential splitting. Thus $\mathsf{Split}_{\mathcal{E}_\mathcal{D}}(\mathbb{X})$ is a differential restriction category in which every retractively linear idempotent is linear differentially splits. $\square$

4.4 Term Logic

We extend the term logic for Cartesian left additive restriction categories to differential restriction categories, and and then prove this extension is sound and complete with respect to a translation into differential restriction categories. This term logic generalizes the term logic described by Blute *et al* [2] to the restriction setting. It is worth noting that the syntax of this logic is related to by Ehrhard and Regnier's differential λ-calculus, the syntax here is an uncurried form of Ehrhard and Regnier's syntax. Using the derivative in uncurried form makes using the term logic more intuitive, and also makes the completeness theorem possible without requiring closed structure.

The syntax is extended with a formal derivative operation.

$$\begin{aligned} T &:= V \mid () \mid (T, \ldots, T) \mid \{T.T\}T \mid 0 \mid T + T \mid \frac{\partial T}{\partial T}(T) \cdot T \mid T_{|T} \\ Ty &:= \mathbb{T} \mid \mathbf{1} \mid Ty \ \times \cdots \times \ Ty \end{aligned}$$

The type judgments are extended in table (4.1).

Of the six classes of equalities we only need to update the cut elimination rules and equations for the logic.

Cut-elimination The following equalities are written as directed equalities because they

Basic terms:

$$\frac{}{\Gamma, x : A \vdash x : A}\ \textsc{Proj} \qquad \frac{\Gamma \vdash t : A \quad \Gamma, p : A \vdash s : B}{\Gamma \vdash \{p.s\}T : B}\ \textsc{Cut}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \cdots \quad \Gamma \vdash t_n : A_n \quad \sigma(f) = ([A_1, \ldots, A_n], B)}{\Gamma \vdash f(t_1, \ldots, t_n) : B}\ \textsc{Fun}$$

Cartesian terms:

$$\frac{}{\Gamma \vdash () : \mathbf{1}}\ \textsc{Unit} \qquad \frac{\Gamma \vdash t_1 : A_1 \quad \cdots \Gamma \vdash t_n : A_n}{\Gamma \vdash (t_1, \ldots, t_n) : A_1 \times \cdots \times A_n}\ \textsc{Tuple}$$

$$\frac{\Gamma, t_1 : A_1, \ldots, t_n : A_n \vdash t : A}{\Gamma, (t_1, \ldots, t_n) : A_1 \times \cdots A_n \vdash t : A}\ \textsc{Pat}$$

Additive Terms:

$$\frac{}{\Gamma \vdash 0 : A}\ \textsc{Zero} \qquad \frac{\Gamma \vdash t_1 : A \quad \Gamma \vdash t_2 : A}{\Gamma t_1 + t_2 : A}\ \textsc{Add}$$

Differential Terms:

$$\frac{\Gamma, p : A \vdash t : B \quad \Gamma \vdash s : A \quad \Gamma \vdash U : A}{\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u : B}\ \mathrm{D}$$

Partial Terms:

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash s : B}{\Gamma \vdash t_{|s} : A}\ \textsc{Rest}$$

Table 4.1: Differential Restriction Term Formation Rules

Table 3.2 and
ESB.12 $\{x.\frac{\partial f}{\partial y}(s) \cdot u\}t \Rrightarrow \frac{\partial \{x.f\}t}{\partial y}(\{x.s\}t) \cdot \{x.u\}t$

Table 4.2: Differential restriction cut elimiation

can be viewed as a rewriting system on terms which removes occurences of cut. We extend the cut elimination rules in table 4.2.

Equations The equalities for differential restriction term logic are presented in table 4.3.

In table 4.3, the variable contexts are not explicitly mentioned. However, one should mind these variable contexts. For example, **DRL.18** does not make sense unless p' does not occur in either s or t.

Note that cut elimination holds in this logic as well. Lemmas (2.5.1 and 2.5.2) and thus corollary (2.5.1) have analogous counterparts. The proofs by induction in lemma (2.5.1) simply require three extra cases: one for the zero, one for the sum , and one for the derivative. The case for the derivative is a bit tricky.

To extend lemma (2.5.1), consider the following calculation used in the inner induction: proving the statement when the pattern is a variable. In the following calculation,

Table 3.3 and

DRL.11 $\frac{\partial t_{|s}}{\partial p}(v) \cdot u = \left(\frac{\partial t}{\partial p}(v) \cdot u\right)_{|\{p.s\}v}$ and $\frac{\partial t}{\partial p}(s_{|r}) \cdot u = \frac{\partial t}{\partial p}(s) \cdot (u_{|r}) = \left(\frac{\partial t}{\partial p}(s) \cdot u\right)_{|r}$

DRL.12 $t_{|\frac{\partial f}{\partial p}(s) \cdot u} = t_{|s,u,\{p.f\}s}$

DRL.13 $\frac{\partial t_1 + t_2}{\partial p}(s) \cdot u = \frac{\partial t_1}{\partial p}(s) \cdot u + \frac{\partial t_2}{\partial p}(s) \cdot u$ and $\frac{\partial 0}{\partial p}(s) \cdot u = 0_{|u,s}$;

DRL.14 $\frac{\partial t}{\partial p}(s) \cdot (u_1 + u_2) = \frac{\partial t}{\partial p}(s) \cdot u_1 + \frac{\partial t}{\partial p}(s) \cdot u_2$ and $\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot 0 = \Gamma \vdash 0_{|\{p.t\}s}$;

DRL.15

$$\begin{aligned} &\tfrac{\partial x}{\partial x}(s) \cdot u = u_{|s}, \\ &\tfrac{\partial t}{\partial (p,p')}((s,s')) \cdot (u,0) = \tfrac{\partial \{p'.t\}s'}{\partial p}(s) \cdot u, \text{ and} \\ &\tfrac{\partial t}{\partial (p,p')}((s,s')) \cdot (0,u') = \tfrac{\partial \{p.t\}s}{\partial p'}(s') \cdot u'; \end{aligned}$$

DRL.16 $\frac{\partial (t_1,t_2)}{\partial p}(s) \cdot u = \left(\frac{\partial t_1}{\partial p}(s) \cdot u, \frac{\partial t_2}{\partial p}(s) \cdot u\right)$;

DRL.17 $\frac{\partial \{p'.t\}t'}{\partial p}(s) \cdot u = \frac{\partial t}{\partial p'}(\{p.t'\}s) \cdot \frac{\partial t'}{\partial p}(s) \cdot u$
(chain rule; we require that no free variable of p occur in t);

DRL.18 $\frac{\partial \frac{\partial t}{\partial p}(s) \cdot p'}{\partial p'}(r) \cdot u = \left(\frac{\partial t}{\partial p}(s) \cdot u\right)_{|r}$;

DRL.19 $\frac{\partial \frac{\partial t}{\partial p_1}(s_1) \cdot u_1}{\partial p_2}(s_2) \cdot u_2 = \frac{\partial \frac{\partial t}{\partial p_2}(s_2) \cdot u_2}{\partial p_1}(s_1) \cdot u_1$;

Table 4.3: Differential term logic equations

no variables in p are in s'.

$$\begin{aligned}
\left(\{x.\frac{\partial t'}{\partial p}(s)\cdot u\}t\right)_{|s'} &= \left(\frac{\partial\{x.t'\}t}{\partial p}(\{x.s\}t)\cdot\{x.u\}t\right)_{|s'} \\
&= \left(\frac{\partial\{x.t'\}t}{\partial p}(\{x.s\}t)\cdot\{x.u\}t\right)_{|s',\{x.s\}t} \\
&= \left(\frac{\partial\{x.t'\}t}{\partial p}(\{x.s\}t)\cdot\{x.u\}t\right)_{|s'|\{x.s\}t} \\
&= \left(\frac{\partial\{x.t'\}t}{\partial p}(\{x.s\}t)\cdot\{x.u\}t\right)_{|\{p.s'\}(\{x.s\}t)} \\
&= \frac{\partial(\{x.t'\}t)_{|s'}}{\partial p}(\{x.s\}t)\cdot\{x.u\}t \qquad \textbf{DRL.11} \\
&= \frac{\partial\{x.t'_{|s'}\}t}{\partial p}(\{x.s\}t)\cdot\{x.u\}t \\
&= \{x.\frac{\partial t'_{|s'}}{\partial p}(s)\cdot u\}t \\
&= \{x.\left(\frac{\partial t'}{\partial p}(s)\cdot u\right)_{|\{p.s'\}s}\}t \qquad \textbf{DRL.11} \\
&= \{x.\left(\frac{\partial t'}{\partial p}(s)\cdot u\right)_{|s',s}\}t \\
&= \{x.\left(\frac{\partial t'}{\partial p}(s)\cdot u\right)_{|s'}\}t
\end{aligned}$$

To extend lemma (2.5.2), note that no variable in p occurs in t'.

$$\begin{aligned}
\{x.\frac{\partial t}{\partial p}(s)\cdot u\}t' &= \frac{\partial\{x.t\}t'}{\partial p}(\{x.s\}t')\cdot\{x.u\}t' \\
&= \frac{\partial([t'/x]\,t)_{|t'}}{\partial p}(([t'/x]\,s)_{|t'})\cdot\left(([t'/x]\,u)_{|t'}\right) \\
&= \left(\frac{\partial[t'/x]\,t}{\partial p}([t'/x]\,s)\cdot[t'/x]\,u\right)_{|t',t',\{p.t'\}[t'/x]s} \\
&= \left(\frac{\partial[t'/x]\,t}{\partial p}([t'/x]\,s)\cdot[t'/x]\,u\right)_{|t',[t'/x]s} \\
&= \left(\frac{\partial[t'/x]\,t}{\partial p}([t'/x]\,s)\cdot[t'/x]\,u\right)_{|t'} \\
&= \left([t'/x]\left(\frac{\partial t}{\partial p}(s)\cdot u\right)\right)_{|t'}
\end{aligned}$$

Now, we prove a few equations that are useful for handling the derivative of terms.

Lemma 4.4.1. *The following equalities hold in differential restriction term logic.*

(i) $\frac{\partial t}{\partial ()}(()) \cdot () = 0_{|t}$

(ii) If no variable in p *occurs in* t *then* $\frac{\partial t}{\partial p}(s) \cdot u = 0_{|t,u,s}$

(iii) If no variable in p' *occurs in* t *then* $\frac{\partial t}{\partial (p,p')}(s,s') \cdot (u,u') = \left(\frac{\partial t}{\partial p}(s) \cdot u\right)_{|\{p.t\}s,s',u'}$*. Similarly, if no variable in* p *occurs in* t *then* $\frac{\partial t}{\partial (p,p')}(s,s') \cdot (u,u') = \left(\frac{\partial t}{\partial p'}(s') \cdot u'\right)_{|\{p'.t\}s',s,u}$*.*

(iv) $\frac{\partial t}{\partial (p,p')}(s,s') \cdot (u,u') = \frac{\partial \{p'.t\}s'}{\partial p}(s) \cdot u + \frac{\partial \{p.t\}s}{\partial p'}(s') \cdot u'$*.*

(v) $\frac{\partial t}{\partial (p,p')}(s,s') \cdot (u,u') = \frac{\partial t}{\partial (p',p)}(s',s) \cdot (u',u)$*.*

(vi) If f *is linear in its first argument; i.e.* $\frac{\partial f(x,u)}{\partial x}(a) \cdot v \smile f(v,u)$ *(when* $x \notin u$*) then,*

$$\frac{\partial f(s,u)}{\partial x}(a) \cdot v = f\left(\frac{\partial s}{\partial x}(a) \cdot v, \{x.u\}a\right) + \frac{\partial f(\{x.s\}a, z_1)}{\partial z_1}(\{x.u\}a) \cdot \frac{\partial u}{\partial x}(a) \cdot v.$$

Proof. First note that $() = 0$. Then,

(i)

$$\frac{\partial t}{\partial ()}(()) \cdot () = \frac{\partial t}{\partial ()}(0) \cdot 0 = 0_{|\{().t\}0} = 0_{|t_{|0}} = 0_{|t}$$

(ii) That no variable in p occurs in t makes the use of **DRL.17** valid. Consider

$$\begin{aligned}
\frac{\partial t}{\partial p}(s) \cdot u &= \frac{\partial \{().t\}()}{\partial p}(s) \cdot u = \frac{\partial \{().t\}0}{\partial p}(s) \cdot u \\
&= \frac{\partial t}{\partial ()}(\{p.0\}s) \cdot \left(\frac{\partial 0}{\partial p}(s) \cdot u\right) \\
&= \left(\frac{\partial t}{\partial ()}(0) \cdot 0\right)_{|u,s} = 0_{|t,u,s}
\end{aligned}$$

(iii) Assume that no variable in p occurs in t. Then

$$\begin{aligned}
\frac{\partial t}{\partial(p,p')}(s,s')\cdot(u,u') &= \frac{\partial t}{\partial(p,p')}(s,s')\cdot(u,0) + \frac{\partial t}{\partial(p,p')}(s,s')\cdot(0,u') \\
&= \frac{\partial\{p'.t\}s'}{\partial p}(s)\cdot u + \frac{\partial\{p.t\}s}{\partial p'}(s')\cdot u' \\
&= 0_{|\{p'.t\}s',s,u} + \frac{\partial\{p.t\}s}{\partial p'}(s')\cdot u' \\
&= \left(0 + \frac{\partial\{p.t\}s}{\partial p'}(s')\cdot u'\right)_{|\{p'.t\}s',s,u} \\
&= \left(\frac{\partial t_{|s}}{\partial p'}(s')\cdot u'\right)_{|\{p'.t\}s',s,u} \\
&= \left(\frac{\partial t}{\partial p'}(s')\cdot u'\right)_{|\{p'.t\}s',s,u,\{p'.s\}s'} \\
&= \left(\frac{\partial t}{\partial p'}(s')\cdot u'\right)_{|\{p'.t\}s',s,u,s,s'} \\
&= \left(\frac{\partial t}{\partial p'}(s')\cdot u'\right)_{|\{p'.t\}s',s,u}
\end{aligned}$$

The other statement holds by a symmetric argument.

(iv) This is immediate from the first two steps of the previous calculation.

(v) The sum is commutative, so this follows from the previous identity.

Table 3.4 and
TCR.10 $[\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u]\!] = \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t]\!]]$

Table 4.4: Translation of Differential Terms

(vi) Consider the following calculation,

$$
\begin{aligned}
&\frac{\partial f(s,u)}{\partial x}(a) \cdot v = \frac{\partial \{(z_0,z_1).f(z_0,z_1)\}(s,u)}{\partial x}(a) \cdot v \\
&= \frac{\partial f(z_0,z_1)}{\partial(z_0,z_1)}(\{x.(s,u)\}a) \cdot \frac{\partial(s,u)}{\partial x}(a) \cdot v \\
&= \frac{\partial f(z_0,z_1)}{\partial(z_0,z_1)}(\{x.s\}a, \{x.u\}a) \cdot \left(\frac{\partial s}{\partial x}(a) \cdot v, \frac{\partial u}{\partial x}(a) \cdot v\right) \\
&= \frac{\partial\{z_0.f(z_0,z_1)\}\{x.s\}a}{\partial z_0}(\{x.s\}a) \cdot \frac{\partial s}{\partial x}(a) \cdot v + \frac{\partial\{z_1.f(z_0,z_1)\}xsa}{\partial z_1}(\{x.u\}a) \cdot \frac{\partial u}{\partial x}(a) \cdot v \\
&= \frac{\partial f(z_0,\{x.u\}a)}{\partial z_0}(\{x.s\}a) \cdot \frac{\partial s}{\partial x}(a) \cdot v + \frac{\partial f(\{x.s\}a,z_1)}{\partial z_1}(\{x.u\}a) \cdot \frac{\partial u}{\partial x}(a) \cdot v \\
&= f\left(\frac{\partial s}{\partial x}(a) \cdot v, \{x.u\}a\right)_{|\{x.s\}a} + \frac{\partial f(\{x.s\}a,z_1)}{\partial z_1}(\{x.u\}a) \cdot \frac{\partial u}{\partial x}(a) \cdot v \\
&= f\left(\frac{\partial s}{\partial x}(a) \cdot v_{|\{x.s\}a}, \{x.u\}a\right) + \frac{\partial f(\{x.s\}a,z_1)}{\partial z_1}(\{x.u\}a) \cdot \frac{\partial u}{\partial x}(a) \cdot v \\
&= f\left(\frac{\partial s}{\partial x}(a) \cdot v, \{x.u\}a\right) + \frac{\partial f(\{x.s\}a,z_1)}{\partial z_1}(\{x.u\}a) \cdot \frac{\partial u}{\partial x}(a) \cdot v
\end{aligned}
$$

□

The soundness theorem generalizes Blute *et al* [2] proposition 4.3.1 to the partial term setting.

4.4.1 Soundness

First, the extended translation of terms $[\![\_]\!]$ into a differential restriction category is given in table 4.4

To prove the soundness of this translation, it suffices to show [**DRL**.11-19] hold.

Theorem 4.4.1. *The translation defined in table (4.4) is sound.*

*Proof.***DRL.11** Consider the following calculation for the first part:

$$
\begin{aligned}
[\![\Gamma \vdash \frac{\partial t_{|s}}{\partial p}(v) \cdot u]\!] &= \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash v]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t_{|s}]\!]] \\
&= \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash v]\!], 1\rangle\rangle D[\overline{[\![(p,\Gamma) \vdash s]\!]}\, [\![(p,\Gamma) \vdash t]\!]] \\
&= \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash v]\!], 1\rangle\rangle \langle D[[\![(p,\Gamma) \vdash s]\!]], \pi_1 [\![(p,\Gamma) \vdash s]\!]\rangle D[[\![(p,\Gamma) \vdash t]\!]] \\
&= \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash v]\!], 1\rangle\rangle (1 \times [\![(p,\Gamma) \vdash s]\!]) D[[\![(p,\Gamma) \vdash t]\!]] \\
&= \overline{\langle[\![\Gamma \vdash v]\!], 1\rangle [\![(p,\Gamma) \vdash s]\!]}\, \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash v]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t]\!]] \\
&= \overline{[\![\Gamma \vdash \{p.s\}v]\!]}\, [\![\Gamma \vdash \frac{\partial t}{\partial p}(v) \cdot u]\!] \\
&= [\![\Gamma \vdash \left(\frac{\partial t}{\partial p}(v) \cdot u\right)_{\{p.s\}v}]\!]
\end{aligned}
$$

For the second part, consider that

$$
[\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u_{|r}]\!] = \overline{[\![\Gamma \vdash r]\!]}\, \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t]\!]]
$$

Then,

$$
\begin{aligned}
&\overline{[\![\Gamma \vdash r]\!]}\, \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t]\!]] \\
&= \langle\langle\overline{[\![\Gamma \vdash r]\!]}\, [\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t]\!]] = [\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot (u_{|r})]\!]
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{[\![\Gamma \vdash r]\!]}\, \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle[\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t]\!]] \\
&= \langle\langle[\![\Gamma \vdash u]\!], 0\rangle, \langle\overline{[\![\Gamma \vdash r]\!]}\, [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p,\Gamma) \vdash t]\!]] = [\![\Gamma \vdash \frac{\partial t}{\partial p}(s_{|r}) \cdot u]\!]
\end{aligned}
$$

So that

$$
[\![\Gamma \vdash \left(\frac{\partial t}{\partial p}(s) \cdot u\right)_{|r}]\!] = [\![\Gamma \vdash \frac{\partial t}{\partial p}(s_{|r}) \cdot u]\!] = [\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot (u_{|r})]\!]
$$

as required.

DRL.12 Consider the following calculation

$$
\begin{aligned}
\llbracket \Gamma \vdash t_{|\frac{\partial f}{\partial p}(s)\cdot u} \rrbracket &= \overline{\llbracket \Gamma \vdash \frac{\partial f}{\partial p}(s) \cdot u \rrbracket} \, \llbracket \Gamma \vdash t \rrbracket \\
&= \overline{\langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle D[\llbracket (p,\Gamma) \vdash f \rrbracket]} \, \llbracket \Gamma \vdash t \rrbracket \\
&= \overline{\langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle (1 \times \overline{\llbracket (p,\Gamma) \vdash f \rrbracket})} \, \llbracket \Gamma \vdash t \rrbracket \\
&= \overline{\overline{\langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle \llbracket (p,\Gamma) \vdash f \rrbracket} \, \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle} \, \llbracket \Gamma \vdash t \rrbracket \\
&= \overline{\langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle \llbracket (p,\Gamma) \vdash f \rrbracket} \, \overline{\langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle} \, \llbracket \Gamma \vdash t \rrbracket \\
&= \overline{\llbracket \Gamma \vdash \{p.f\}s \rrbracket} \, \overline{\llbracket \Gamma \vdash u \rrbracket} \, \overline{\llbracket \Gamma \vdash s \rrbracket} \, \llbracket \Gamma \vdash t \rrbracket \\
&= \llbracket \Gamma \vdash t_{|s,u,\{p.f\}s} \rrbracket
\end{aligned}
$$

DRL.13 For the zero case:

$$
\begin{aligned}
\llbracket \Gamma \vdash \frac{\partial 0}{\partial p}(s) \cdot u \rrbracket &= \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle D[\llbracket (p,\Gamma) \vdash 0 \rrbracket] \\
&= \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle D[0] = \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle 0 \\
&= \overline{\langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle} \, 0 = \overline{\llbracket \Gamma \vdash u \rrbracket} \, \overline{\llbracket \Gamma \vdash s \rrbracket} \, 0 \\
&= \llbracket \Gamma \vdash 0_{|u,s} \rrbracket
\end{aligned}
$$

For the nonzero case:

$$
\begin{aligned}
\llbracket \Gamma \vdash \frac{\partial t_1 + t_2}{\partial p}(s) \cdot u \rrbracket &= \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle D[\llbracket (p,\Gamma) \vdash t_1 + t_2 \rrbracket] \\
&= \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle \, (D[\llbracket (p,\Gamma) \vdash t_1 \rrbracket] + D[\llbracket (p,\Gamma) \vdash t_2 \rrbracket]) \\
&= \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle D[\llbracket (p,\Gamma) \vdash t_1 \rrbracket] + \langle\langle \llbracket \Gamma \vdash u \rrbracket, 0\rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1\rangle\rangle D[\llbracket (p,\Gamma) \vdash t_2 \rrbracket] \\
&= \llbracket \Gamma \vdash \frac{\partial t_1}{\partial p}(s) \cdot u \rrbracket + \llbracket \Gamma \vdash \frac{\partial t_2}{\partial p}(s) \cdot u \rrbracket \\
&= \llbracket \Gamma \vdash \frac{\partial t_1}{\partial p}(s) \cdot u + \frac{\partial t_2}{\partial p}(s) \cdot u \rrbracket
\end{aligned}
$$

DRL.14 For the zero case:

$$[\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot 0]\!] = \langle\langle 0, 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]]$$

$$= \langle 0, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]] = \overline{\langle [\![\Gamma \vdash s]\!], 1\rangle [\![(p, \Gamma) \vdash t]\!]}\, 0$$

$$= \overline{[\![\Gamma \vdash \{p.t\}s]\!]}\, 0 = [\![\Gamma \vdash 0_{|\{p.t\}s}]\!]$$

For the nonzero case:

$$[\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u_1 + u_2]\!] = \langle\langle [\![\Gamma \vdash u_1 + u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]]$$

$$= \langle\langle [\![\Gamma \vdash u_1]\!] + [\![\Gamma \vdash u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]]$$

$$= \langle\langle [\![\Gamma \vdash u_1]\!], 0\rangle + \langle [\![\Gamma \vdash u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]]$$

$$= \langle\langle [\![\Gamma \vdash u_1]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]] + \langle\langle [\![\Gamma \vdash u_2]\!], 0\rangle, [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]]$$

$$= [\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u_1]\!] + [\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u_2]\!]$$

$$= [\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u_1 + \frac{\partial t}{\partial p}(s) \cdot u_2]\!]$$

DRL.15 For the first part:

$$[\![\Gamma \vdash \frac{\partial x}{\partial x}(s) \cdot u]\!] = \langle\langle [\![\Gamma \vdash u]\!], 1\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![x, \Gamma \vdash x]\!]]$$

$$= \langle\langle [\![\Gamma \vdash u]\!], 1\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[\pi_0 [\![x \vdash x]\!]]$$

$$= \langle\langle [\![\Gamma \vdash u]\!], 1\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[\pi_0]$$

$$= \langle\langle [\![\Gamma \vdash u]\!], 1\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle \pi_0 \pi_0$$

$$= \overline{[\![\Gamma \vdash s]\!]}\, [\![\Gamma \vdash u]\!] = [\![\Gamma \vdash u_{|s}]\!]$$

For the second part, **DR.2** is used; in particular, that $\langle 0, 1\rangle D[h] = \overline{h}\, 0$. We will also use the fact the reassociation maps and commutativity maps for the product are linear; for example, $\langle\langle a, b\rangle, \langle e, d\rangle\rangle D[c_\times f] = \langle\langle a, b\rangle, \langle e, d\rangle\rangle (c_\times \times c_\times) D[f] = \langle\langle b, a\rangle, \langle d, e\rangle\rangle D[f]$. The preceding fact can be combined with the fact that $c_\times [\![(a, b) \vdash t]\!] = [\![(b, a) \vdash t]\!]$ to reorder the context under the derivative. An analogous fact may be used to reassociate a context under the derivative. Now, consider the following:

$$
\begin{aligned}
&[\![\Gamma \vdash \frac{\partial t}{\partial(p,p')}(s,s')\cdot(u,0)]\!]\\
&= \langle\langle\langle[\![\Gamma \vdash u]\!],0\rangle,0\rangle,\langle\langle[\![\Gamma \vdash s]\!],[\![\Gamma \vdash s']\!]\rangle,1\rangle\rangle D[[\![((p,p'),\Gamma)\vdash t]\!]]\\
&= \langle\langle\langle 0,[\![\Gamma \vdash u]\!]\rangle,0\rangle,\langle\langle[\![\Gamma \vdash s']\!],[\![\Gamma \vdash s]\!]\rangle,1\rangle\rangle D[[\![((p',p),\Gamma)\vdash t]\!]]\\
&= \langle\langle 0,\langle[\![\Gamma \vdash u]\!],0\rangle\rangle,\langle[\![\Gamma \vdash s']\!],\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\rangle D[[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle\langle 0,1\rangle D[[\![\Gamma \vdash s']\!]],\langle[\![\Gamma \vdash u]\!],0\rangle\rangle,\langle[\![\Gamma \vdash s']\!],\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\rangle D[[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle\langle(\pi_1\times\pi_1)D[[\![\Gamma \vdash s']\!]],\pi_0\rangle,\pi_1\langle\pi_1[\![\Gamma \vdash s']\!],1\rangle\rangle\\
&\quad D[[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle\langle\langle D[\pi_1],\pi_1\pi_1\rangle D[[\![\Gamma \vdash s']\!]],\pi_0\rangle,\pi_1\langle\pi_1[\![\Gamma \vdash s']\!],1\rangle\rangle\\
&\quad D[[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle\langle D[\pi_1[\![\Gamma \vdash s']\!]],\pi_0\rangle,\pi_1\langle\pi_1[\![\Gamma \vdash s']\!],1\rangle\rangle\\
&\quad D[[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle\langle D[\pi_1[\![\Gamma \vdash s']\!]],D[1]\rangle,\pi_1\langle\pi_1[\![\Gamma \vdash s']\!],1\rangle\rangle\\
&\quad D[[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle D[\langle\pi_1[\![\Gamma \vdash s']\!],1\rangle],\pi_1\langle\pi_1[\![\Gamma \vdash s']\!],1\rangle\rangle\\
&\quad D[[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle D[\langle\pi_1[\![\Gamma \vdash s']\!],1\rangle[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle D[\langle[\![(p,\Gamma)\vdash s']\!],1\rangle[\![(p',(p,\Gamma))\vdash t]\!]]\\
&= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle D[[\![(p,\Gamma)\vdash \{p'.t\}s']\!]]\\
&= [\![\Gamma \vdash \frac{\partial\{p'.t\}s'}{\partial p}(s)\cdot u]\!]
\end{aligned}
$$

A similar calculation works for the third part.

DRL.16 Consider the following:

$$\llbracket \Gamma \vdash \frac{\partial (t_1, t_2)}{\partial p}(s) \cdot u \rrbracket = \langle \langle \llbracket \Gamma \vdash u \rrbracket, 0 \rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1 \rangle \rangle D[\llbracket (p, \Gamma) \vdash (t_1, t_2) \rrbracket]$$

$$= \langle \langle \llbracket \Gamma \vdash u \rrbracket, 0 \rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1 \rangle \rangle D[\langle \llbracket (p, \Gamma) \vdash t_1 \rrbracket, \llbracket (p, \Gamma) \vdash t_2 \rrbracket \rangle]$$

$$= \langle \langle \llbracket \Gamma \vdash u \rrbracket, 0 \rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1 \rangle \rangle \langle D[\llbracket (p, \Gamma) \vdash t_1 \rrbracket], D[\llbracket (p, \Gamma) \vdash t_2 \rrbracket] \rangle$$

$$= \langle \langle \langle \llbracket \Gamma \vdash u \rrbracket, 0 \rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1 \rangle \rangle D[\llbracket (p, \Gamma) \vdash t_1 \rrbracket],$$

$$\langle \langle \llbracket \Gamma \vdash u \rrbracket, 0 \rangle, \langle \llbracket \Gamma \vdash s \rrbracket, 1 \rangle \rangle D[\llbracket (p, \Gamma) \vdash t_2 \rrbracket] \rangle$$

$$= \langle \llbracket \Gamma \vdash \frac{\partial t_1}{\partial p}(s) \cdot u \rrbracket, \llbracket \Gamma \vdash \frac{\partial t_2}{\partial p}(s) \cdot u \rrbracket \rangle = \llbracket \Gamma \vdash \left(\frac{\partial t_1}{\partial p}(s) \cdot u, \frac{\partial t_2}{\partial p}(s) \cdot u \right) \rrbracket$$

DRL.17 Assume no variable in p is in t. Recall that $a_\times^{-1}(c_\times \times 1)a_\times \pi_1 = 1 \times \pi_1$. Consider the following

$$[\![\Gamma \vdash \frac{\partial\{p'.t\}t'}{\partial p}(s)\cdot u]\!] = \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle D[[\![(p,\Gamma)\vdash \{p'.t\}t']\!]]$$

$$= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle D[\langle[\![(p,\Gamma)\vdash t']\!],1\rangle[\![(p',(p,\Gamma))\vdash t]\!]]$$

$$= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle D[\langle[\![(p,\Gamma)\vdash t']\!],1\rangle],\pi_1\langle[\![(p,\Gamma)\vdash t']\!],1\rangle\rangle$$
$$D[[\![(p',(p,\Gamma))\vdash t]\!]]$$

$$= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle D[\langle[\![(p,\Gamma)\vdash t']\!],1\rangle],\pi_1\langle[\![(p,\Gamma)\vdash t']\!],1\rangle\rangle$$
$$D[a_\times^{-1}(c_\times \times 1)a_\times[\![(p,(p',\Gamma))\vdash t]\!]]$$

$$= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle D[\langle[\![(p,\Gamma)\vdash t']\!],1\rangle],\pi_1\langle[\![(p,\Gamma)\vdash t']\!],1\rangle\rangle$$
$$(a_\times^{-1}(c_\times \times 1)a_\times \times a_\times^{-1}(c_\times \times 1)a_\times)D[[\![(p,(p',\Gamma))\vdash t]\!]]$$

$$= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle D[\langle[\![(p,\Gamma)\vdash t']\!],1\rangle],\pi_1\langle[\![(p,\Gamma)\vdash t']\!],1\rangle\rangle$$
$$(a_\times^{-1}(c_\times \times 1)a_\times \times a_\times^{-1}(c_\times \times 1)a_\times)D[\pi_1[\![((p',\Gamma))\vdash t]\!]]$$

$$= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle\langle D[[\![(p,\Gamma)\vdash t']\!]],\pi_0\rangle,\langle\pi_1[\![(p,\Gamma)\vdash t']\!],\pi_1\rangle\rangle$$
$$((1\times\pi_1)\times(1\times\pi_1))D[[\![((p',\Gamma))\vdash t]\!]]$$

$$= \langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle\langle\langle D[[\![(p,\Gamma)\vdash t']\!]],\pi_0\pi_1\rangle,\langle\pi_1[\![(p,\Gamma)\vdash t']\!],\pi_1\pi_1\rangle\rangle$$
$$D[[\![((p',\Gamma))\vdash t]\!]]$$

$$= \langle\langle\langle\langle[\![\Gamma \vdash u]\!],0\rangle,\langle[\![\Gamma \vdash s]\!],1\rangle\rangle D[[\![(p,\Gamma)\vdash t']\!]],0\rangle,\langle\langle[\![\Gamma \vdash s]\!],1\rangle[\![(p,\Gamma)\vdash t']\!],1\rangle\rangle$$
$$D[[\![((p',\Gamma))\vdash t]\!]]$$

$$= \langle\langle[\![\Gamma \vdash \frac{\partial t'}{\partial p}(s)\cdot u]\!],0\rangle,\langle[\![\Gamma \vdash \{p.t'\}s]\!],1\rangle\rangle D[[\![(p',\Gamma)\vdash t]\!]]$$

$$= [\![\Gamma \vdash \frac{\partial t}{\partial p'}(\{p.t'\}s)\cdot\frac{\partial t'}{\partial p}(s)\cdot u]\!]$$

DRL.18 Assume p' does not occur in s or t. Consider the following

$$
\begin{aligned}
&[\![\Gamma \vdash \frac{\partial \frac{\partial t}{\partial p}(s) \cdot p'}{\partial p'}(r) \cdot u]\!] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash r]\!], 1\rangle\rangle D[[\![p', \Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot p']\!]] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash r]\!], 1\rangle\rangle D[\langle\langle [\![p', \Gamma \vdash p']\!], 0\rangle, \langle [\![p', \Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, (p', \Gamma)) \vdash t]\!]]] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash r]\!], 1\rangle\rangle D[\langle\langle \pi_0, 0\rangle, \langle [\![p', \Gamma \vdash s]\!], 1\rangle\rangle((1 \times \pi_1) \times (1 \times \pi_1)) \\
&\quad D[[\![(p, \Gamma) \vdash t]\!]]] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash r]\!], 1\rangle\rangle D[\langle\langle \pi_0, 0\rangle, \langle \pi_1 [\![\Gamma \vdash s]\!], \pi_1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]]] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash r]\!], 1\rangle\rangle \langle D[\langle\langle \pi_0, 0\rangle, \langle \pi_1 [\![\Gamma \vdash s]\!], \pi_1\rangle\rangle], \\
&\quad \pi_1 \langle\langle \pi_0, 0\rangle, \langle \pi_1 [\![\Gamma \vdash s]\!], \pi_1\rangle\rangle\rangle D^2[[\![(p, \Gamma) \vdash t]\!]] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash r]\!], 1\rangle\rangle \langle\langle\langle \pi_0\pi_0, 0\rangle, \langle (\pi_1 \times \pi_1) D[[\![\Gamma \vdash s]\!]], \pi_0\pi_1\rangle\rangle, \\
&\quad \pi_1 \langle\langle \pi_0, 0\rangle, \langle \pi_1 [\![\Gamma \vdash s]\!], \pi_1\rangle\rangle\rangle D^2[[\![(p, \Gamma) \vdash t]\!]] \\
&= \langle\langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle\langle 0, 1\rangle D[[\![\Gamma \vdash s]\!]], 0\rangle\rangle, \\
&\quad \langle\langle [\![\Gamma \vdash r]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D^2[[\![(p, \Gamma) \vdash t]\!]] \\
&= \langle\langle\langle [\![\Gamma \vdash u]\!], 0\rangle, 0\rangle, \langle\langle [\![\Gamma \vdash r]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle\rangle D^2[[\![(p, \Gamma) \vdash t]\!]] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle\langle [\![\Gamma \vdash r]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle\rangle\ (\langle 1, 0\rangle \times 1)\ D^2[[\![(p, \Gamma) \vdash t]\!]] \\
&= \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle\langle [\![\Gamma \vdash r]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle\rangle (1 \times \pi_1) D[[\![(p, \Gamma) \vdash t]\!]] \\
&= \overline{[\![\Gamma \vdash r]\!]}\, \langle\langle [\![\Gamma \vdash u]\!], 0\rangle, \langle [\![\Gamma \vdash s]\!], 1\rangle\rangle D[[\![(p, \Gamma) \vdash t]\!]] \\
&= \overline{[\![\Gamma \vdash r]\!]}\, [\![\Gamma \vdash \frac{\partial t}{\partial p}(s) \cdot u]\!] = [\![\Gamma \vdash \left(\frac{\partial t}{\partial p}(s) \cdot u\right)_{|r}]\!]
\end{aligned}
$$

DRL.19 Assume that p_1, p_2 do not occur in any of s_1, s_2, u_1, u_2. Denote the map $a_\times^{-1}(c_\times \times 1)a_x$ by tw. Consider the following,

$$
\begin{aligned}
&[\![\Gamma \vdash \frac{\partial \frac{\partial t}{\partial p_1}(s_1) \cdot u_1}{\partial p_2}(s_2) \cdot u_2]\!] \\
&= \langle\langle [\![\Gamma \vdash u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle D[\langle\langle [\![p_2, \Gamma \vdash u_1]\!], 0\rangle, \langle [\![p_2, \Gamma \vdash s_1]\!], 1\rangle\rangle \\
&\quad D[[\![(p_1, (p_2, \Gamma)) \vdash t]\!]]] \\
&= \langle\langle [\![\Gamma \vdash u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle D[\langle\langle \pi_1 [\![\Gamma \vdash u_1]\!], 0\rangle, \langle \pi_1 [\![\Gamma \vdash s_1]\!], 1\rangle\rangle \\
&\quad D[[\![(p_1, (p_2, \Gamma)) \vdash t]\!]]] \\
&= \langle\langle [\![\Gamma \vdash u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle \langle D[\langle\langle \pi_1 [\![\Gamma \vdash u_1]\!], 0\rangle, \langle \pi_1 [\![\Gamma \vdash s_1]\!], 1\rangle\rangle], \\
&\quad \pi_1 \langle\langle \pi_1 [\![\Gamma \vdash u_1]\!], 0\rangle, \langle \pi_1 [\![\Gamma \vdash s_1]\!], 1\rangle\rangle\rangle D^2[[\![(p_1, (p_2, \Gamma)) \vdash t]\!]] \\
&= \langle\langle [\![\Gamma \vdash u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle \langle\langle\langle (\pi_1 \times \pi_1) D[[\![\Gamma \vdash u_1]\!]], 0\rangle, \langle (\pi_1 \times \pi_1) D[[\![\Gamma \vdash s_1]\!]], \pi_0\rangle\rangle, \\
&\quad \langle\langle \pi_1 \pi_1 [\![\Gamma \vdash u_1]\!], 0\rangle, \langle \pi_1 \pi_1 [\![\Gamma \vdash s_1]\!], \pi_1\rangle\rangle\rangle D^2[[\![(p_1, (p_2, \Gamma)) \vdash t]\!]] \\
&= \langle\langle\langle\langle 0, 1\rangle D[[\![\Gamma \vdash u_1]\!]], 0\rangle, \langle\langle 0, 1\rangle D[[\![\Gamma \vdash s_1]\!]], \langle [\![\Gamma \vdash u_2]\!], 0\rangle\rangle\rangle, \\
&\quad \langle\langle [\![\Gamma \vdash u_1]\!], 0\rangle, \langle [\![\Gamma \vdash s_1]\!], \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle\rangle\rangle D^2[[\![(p_1, (p_2, \Gamma)) \vdash t]\!]] \\
&= \langle\langle 0, \langle 0, \langle [\![\Gamma \vdash u_2]\!], 0\rangle\rangle\rangle, \langle\langle [\![\Gamma \vdash u_1]\!], 0\rangle, \langle [\![\Gamma \vdash s_1]\!], \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle\rangle\rangle \\
&\quad D^2[[\![(p_1, (p_2, \Gamma)) \vdash t]\!]] \\
&= \langle\langle 0, \langle [\![\Gamma \vdash u_1]\!], \langle 0, 0\rangle\rangle\rangle, \langle\langle 0, \langle [\![\Gamma \vdash u_2]\!], 0\rangle\rangle, \langle [\![\Gamma \vdash s_1]\!], \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle\rangle\rangle \\
&\quad D^2[tw [\![(p_2, (p_1, \Gamma)) \vdash t]\!]] \\
&= \langle\langle 0, \langle [\![\Gamma \vdash u_1]\!], \langle 0, 0\rangle\rangle\rangle, \langle\langle 0, \langle [\![\Gamma \vdash u_2]\!], 0\rangle\rangle, \langle [\![\Gamma \vdash s_1]\!], \langle [\![\Gamma \vdash s_2]\!], 1\rangle\rangle\rangle\rangle \\
&\quad (tw \times tw \times tw \times tw) D^2[[\![(p_2, (p_1, \Gamma)) \vdash t]\!]] \\
&= \langle\langle 0, \langle 0, \langle [\![\Gamma \vdash u_1]\!], 0\rangle\rangle\rangle, \langle\langle [\![\Gamma \vdash u_2]\!], 0\rangle, \langle [\![\Gamma \vdash s_2]\!], \langle [\![\Gamma \vdash s_1]\!], 1\rangle\rangle\rangle\rangle \\
&\quad D^2[[\![(p_2, (p_1, \Gamma)) \vdash t]\!]] \\
&= \cdots \\
&= [\![\Gamma \vdash \frac{\partial \frac{\partial t}{\partial p_2}(s_2) \cdot u_2}{\partial p_1}(s_1) \cdot u_1]\!]
\end{aligned}
$$

Therefore the interpretation is sound. $\square$

4.4.2 Completeness

We form the classifying category, now of a differential restriction theory, so derivatives must be added. The derivative of a map is given by:

$$\frac{f = p \mapsto t}{D[f] = (p',p) \mapsto \frac{\partial t}{\partial p}(p) \cdot p'}$$

(where $p, p' : A$). It remains to show that there is an h such that $D[h]\langle 1, !\rangle(1 \times s) \bullet r = \pi_0$.

As a remark on the differential structure in $\mathbb{C}[\mathcal{T}]$, consider the derivative $(p',p) \mapsto \frac{\partial f}{\partial p}(p) \cdot p'$. The derivative binds, in f, the variables of p. This has the effect that under composition $q \mapsto \{(p',p).\frac{\partial f}{\partial p}(p) \cdot p'\}(m_1, m_2)$, an alpha conversion is required of f, and the result, after moving some of the restrictions around, is $q \mapsto \frac{\partial f_{|m_1,m_2}}{\partial p}(m_2) \cdot m_1$. Using **DRL.11**, this is equal to

$$\left(\frac{\partial f}{\partial p}(m_2) \cdot m_1\right)_{|\{p.m_2\}m_2, \{p.m_1\}m_2} = \left(\frac{\partial f}{\partial p}(m_2) \cdot m_1\right)_{|m_2,m_1} = \frac{\partial f}{\partial p}(m_2) \cdot m_1.$$

The completeness theorem generalizes the completeness theorem of Blute *et al* [2] to the setting of partial terms.

Theorem 4.4.2. *For every differential restriction theory $\mathcal{T}$, $\mathbb{C}[\mathcal{T}]$ is a differential restriction category.*

Proof. **DR.1**

$$D[p \mapsto 0] = (p',p) \mapsto \frac{\partial 0}{\partial p}(p) \cdot p' = (p',p) \mapsto 0_{|p,p'} = (p',p) \mapsto 0$$

where the last step is justified because the restriction of a variable pattern disappears.

$$\begin{aligned} D[p \mapsto t_1 + t_2] &= (p',p) \mapsto \frac{\partial t_1 + t_2}{\partial p}(p) \cdot p' = (p',p) \mapsto \frac{\partial t_1}{\partial p}(p) \cdot p' + \frac{\partial t_2}{\partial p}(p) \cdot p' \\ &= \left((p',p) \mapsto \frac{\partial t_1}{\partial p}(p) \cdot p'\right) + \left((p,p') \mapsto \frac{\partial t_2}{\partial p}(p) \cdot p'\right) \\ &= D[p \mapsto t_1] + D[p \mapsto t_2] \end{aligned}$$

DR.2 For the zero case,

$$
\begin{aligned}
\langle 0, g\rangle D[f] &= (p \mapsto (0, g))((q', q) \mapsto \frac{\partial f}{\partial q}(q) \cdot q') \\
&= p \mapsto \frac{\partial \{q.f\}g}{\partial q}(g) \cdot 0 \\
&= p \mapsto 0_{|\{q.\{q.f\}g\}g} = p \mapsto 0_{|(\{q.f\}g)_{|g}} \\
&= p \mapsto 0_{|\{q.f\}(g_{|g})} = p \mapsto 0_{|\{q.f\}g} \\
&= p \mapsto (\{p.0\}p)_{|\{q.f\}g} = p \mapsto \{p.0\}\big(p_{|\{q.f\}g}\big) \\
&= (p \mapsto p_{|\{q.f\}g})(p \mapsto 0) = \overline{p \mapsto \{q.f\}g}\,(p \mapsto 0) \\
&= \overline{(p \mapsto g)(q \mapsto f)}\,(p \mapsto 0) = \overline{gf}\,0
\end{aligned}
$$

For the sum case,

$$
\begin{aligned}
&\langle g + h, k\rangle D[f] \\
&= (p \mapsto (g + h, k))((q', q) \mapsto \frac{\partial f}{\partial q}(q) \cdot q') \\
&= p \mapsto \frac{\partial(\{q.f\}k)_{|g,h}}{\partial q}(k) \cdot (g + h) \\
&= p \mapsto \left(\frac{\partial \{q.f\}k}{\partial q}(k) \cdot (g + h)\right)_{|\{q.g\}k, \{q.h\}k} \\
&= p \mapsto \left(\frac{\partial \{q.f\}k}{\partial q}(k) \cdot g + \frac{\partial \{q.f\}k}{\partial q}(k) \cdot h\right)_{|\{q.g\}k, \{q.h\}k} \\
&= \left(p \mapsto \frac{\partial(\{q.f\}k)_{|g}}{\partial q}(k) \cdot g\right) + \left(p \mapsto \frac{\partial(\{q.f\}k)_{|h}}{\partial q}(k) \cdot h\right) \\
&= (p \mapsto (g, k))((q', q) \mapsto \frac{\partial f}{\partial q}(q) \cdot q') + (p \mapsto (h, k))((q', q) \mapsto \frac{\partial f}{\partial q}(q) \cdot q') \\
&= \langle g, k\rangle D[f] + \langle h, k\rangle D[f]
\end{aligned}
$$

DR.3 For π_0, use lemma (4.4.1), then **DRL.15**.

$$\begin{aligned}
D[\pi_0] = D[(p,q) \mapsto p] &= ((p',q'),(p,q)) \mapsto \frac{\partial p}{\partial(p,q)}(p,q) \cdot (p',q') \\
&= ((p',q'),(p,q)) \mapsto \left(\frac{\partial p}{\partial p}(p) \cdot p'\right)_{|\{p.p\}p,q',q} \\
&= ((p',q'),(p,q)) \mapsto p'_{|p,q',q} \\
&= ((p',q'),(p,q)) \mapsto p' = \pi_0\pi_0
\end{aligned}$$

The last step is justified because restrictions of variable patterns vanish. A similar argument works for π_1.

DR.4 Use **DRL.16**,

$$\begin{aligned}
D[\langle f,g\rangle] = D[p \mapsto (f,g)] &= (p',p) \mapsto \frac{\partial(f,g)}{\partial p}(p) \cdot p' \\
&= (p',p) \mapsto \left(\frac{\partial f}{\partial p}(p) \cdot p', \frac{\partial g}{\partial p}(p) \cdot p'\right) \\
&= \langle D[f], D[g]\rangle
\end{aligned}$$

DR.5 In the following, note that p does occur in g, and so the use of **DRL.17** is valid. We also must use the above remark on the differential structure in $\mathbb{C}[\mathcal{T}]$ on the third to last step.

$$\begin{aligned}
D[fg] = D[(p \mapsto f)(q \mapsto g)] &= D[p \mapsto \{q.g\}f] \\
&= (p',p) \mapsto \frac{\partial\{q.g\}f}{\partial p}(p) \cdot p' \\
&= (p',p) \mapsto \frac{\partial g}{\partial q}(\{p.f\}p) \cdot \left(\frac{\partial f}{\partial p}(p) \cdot p'\right) \\
&= (p',p) \mapsto \frac{\partial g}{\partial q}(f) \cdot \left(\frac{\partial f}{\partial p}(p) \cdot p'\right) \\
&= (p',p) \mapsto \{(q',q).\left(\frac{\partial g}{\partial q}(q) \cdot q'\right)\}\left(\frac{\partial f}{\partial p}(p) \cdot p', f\right) \\
&= \left((p',p)\left(\mapsto \frac{\partial f}{\partial p}(p) \cdot p', f\right)\right)\left((q',q) \mapsto \frac{\partial g}{\partial q}(q) \cdot q'\right) \\
&= \langle D[f], \pi_1 f\rangle D[g]
\end{aligned}$$

DR.6 For this use **DRL.15** then **DRL.18**.

$$
\begin{aligned}
&(\langle 1,0\rangle \times 1)D^2[f]\\
&= ((q',(p',p)) \mapsto ((q',0),(p',p)))(((q',q),(p',p)) \mapsto \frac{\partial \frac{\partial f}{\partial p}(p)\cdot p'}{\partial p',p}(p',p)\cdot(q',q)\\
&= (q',(p',p)) \mapsto \frac{\partial \frac{\partial f}{\partial p}(p)\cdot p'}{\partial (p',p)}(p',p)\cdot(q',0)\\
&= (q',(p',p)) \mapsto \frac{\partial\{p.\frac{\partial f}{\partial p}(p)\cdot p'\}p}{\partial p'}(p')\cdot q'\\
&= (q',(p',p)) \mapsto \frac{\partial \frac{\partial f}{\partial p}(p)\cdot p'}{\partial p'}(p')\cdot q'\\
&= (q',(p',p)) \mapsto \left(\frac{\partial f}{\partial p}(p)\cdot q'\right)_{|p'} = (q',(p',p)) \mapsto \frac{\partial f}{\partial p}(p)\cdot q'\\
&= ((q',(p',p)) \mapsto (q',p))((q',p) \mapsto \frac{\partial f}{\partial p}(p)\cdot q') = (1\times \pi_1)D[f]
\end{aligned}
$$

DR.7 For this, if one tries to directly prove **DR.7**, one quickly runs into bound variable issues, and importantly, **DRL.19** cannot be used. However, note that

$$
\begin{aligned}
&\langle\langle\langle 0,0\rangle,\langle h,0\rangle\rangle,\langle\langle 0,g\rangle,\langle k_1,k_2\rangle\rangle\rangle D^2[f] = \langle\langle\langle 0,0\rangle,\langle 0,g\rangle\rangle,\langle\langle h,0\rangle,\langle k_1,k_2\rangle\rangle\rangle D^2[f]\\
&\implies \langle\langle 0,\langle h,0\rangle\rangle,\langle\langle 0,g\rangle,\langle 0,k\rangle\rangle\rangle D^2[(\pi_0+\pi_1)f]\\
&\qquad = \langle\langle 0,\langle 0,g\rangle\rangle,\langle\langle h,0\rangle,\langle 0,k\rangle\rangle\rangle D^2[(\pi_0+\pi_1)f]\\
&\implies \langle\langle 0,\langle h,0\rangle\rangle,\langle\langle 0,g\rangle,\langle 0,k\rangle\rangle\rangle (\pi_0+\pi_1)^4 D^2[f]\\
&\qquad = \langle\langle 0,\langle 0,g\rangle\rangle,\langle\langle h,0\rangle,\langle 0,k\rangle\rangle\rangle (\pi_0+\pi_1)^4 D^2[f]\\
&\implies \langle\langle 0,h\rangle,\langle g,k\rangle\rangle D^2[f] = \langle\langle 0,g\rangle,\langle h,k\rangle\rangle D^2[f]
\end{aligned}
$$

In other words, the first equation implies **DR.7**, and crucially, the first equation can be shown in $\mathbb{C}[\mathcal{T}]$ without the same troubles. Consider,

$$
\begin{aligned}
&\langle\langle\langle 0,0\rangle,\langle h,0\rangle\rangle,\langle\langle 0,g\rangle,\langle k_1,k_2\rangle\rangle\rangle D^2[f] \\
&= (a \mapsto (((0,0),(h,0)),((0,g),(k_1,k_2))))D^2[(x_1,x_2)\mapsto f] \\
&= (a \mapsto (((0,0),(h,0)),((0,g),(k_1,k_2)))) \\
&\qquad D[((u_1,u_2),(s_1,s_2)) \mapsto \frac{\partial f}{\partial(x_1,x_2)}(s_1,s_2)\cdot(u_1,u_2)] \\
&= (a \mapsto (((0,0),(h,0)),((0,g),(k_1,k_2)))) \\
&\qquad ((((v_1,v_2),(v_3,v_4)),((r_1,r_2),(r_3,r_4))) \mapsto \\
&\qquad \frac{\partial \frac{\partial f}{\partial(x_1,x_2)}(s_1,s_2)\cdot(u_1,u_2)}{\partial((u_1,u_2),(s_1,s_2))}((r_1,r_2),(r_3,r_4))\cdot((v_1,v_2),(v_3,v_4))) \\
&= a \mapsto \frac{\partial \frac{\partial f}{\partial(x_1,x_2)}(s_1,s_2)\cdot(u_1,u_2)}{\partial((u_1,u_2),(s_1,s_2))}((0,g),(k_1,k_2))\cdot((0,0),(h,0)) \\
&= a \mapsto \frac{\partial \frac{\partial f}{\partial(x_1,x_2)}(s_1,s_2)\cdot(0,g)}{\partial(s_1,s_2)}(k_1,k_2)\cdot(h,0) \qquad \textbf{DRL.15} \\
&= a \mapsto \frac{\partial \frac{\partial f}{\partial(x_1,x_2)}(s_1,k_2)\cdot(0,g)}{\partial s_1}(k_1)\cdot h \qquad \textbf{DRL.15} \\
&= a \mapsto \frac{\partial \frac{\partial\{x_1.f\}s_1}{\partial x_2}(k_2)\cdot g}{\partial s_1}(k_1)\cdot h \qquad \textbf{DRL.15} \\
&= a \mapsto \frac{\partial \frac{\partial f}{\partial x_2}(k_2)\cdot g}{\partial x_1}(k_1)\cdot h \qquad \alpha\text{-rename} \\
&= a \mapsto \frac{\partial \frac{\partial f}{\partial x_1}(k_1)\cdot h}{\partial x_2}(k_2)\cdot g \qquad \textbf{DRL.19} \\
&= \cdots \\
&= \langle\langle\langle 0,0\rangle,\langle 0,g\rangle\rangle,\langle\langle h,0\rangle,\langle k_1,k_2\rangle\rangle\rangle D^2[f]
\end{aligned}
$$

DR.8 Consider,

$$
\begin{aligned}
D[\overline{f}\,] = D[p \mapsto p_{|f}] &= (p',p) \mapsto \frac{\partial p_{|f}}{\partial p}(p) \cdot p' \\
&= (p',p) \mapsto \left(\frac{\partial p}{\partial p}(p) \cdot p'\right)_{|f} = (p',p) \mapsto p'_{|f} \\
&= \left((p',p) \mapsto (p',p)_{|f}\right)((p',p) \mapsto p') = \overline{(p',p) \mapsto f}\,\pi_0 \\
&= \overline{(p',p) \mapsto \{p.p\} f}\,\pi_0 = \overline{\pi_1 f}\,\pi_0
\end{aligned}
$$

DR.9 Consider,

$$
\begin{aligned}
\overline{D[f]} &= \overline{(p',p) \mapsto \frac{\partial f}{\partial p}(p) \cdot p'} \\
&= (p',p) \mapsto (p',p)_{|\frac{\partial f}{\partial p}(p) \cdot p'} \\
&= (p',p) \mapsto (p',p)_{|f} \\
&= \overline{\pi_1 f}
\end{aligned}
$$

Thus, $\mathbb{C}[\mathcal{T}]$ is a differential restriction category as proposed.

□

Theorem (4.4.2) establishes the completeness of differential restriction categories as models of differential restriction term logic.

4.5 Turing Categories

4.6 Differential Turing Categories and DPCAs

We are now ready to speculate on what a differential Turing category must be. Clearly it must be an additive Turing category but also the key idempotent splittings which make every object retractions of the Turing object must become at least 2-differential splittings. In fact, we shall propose that they should be ∞-differential splittings.

The issue is how to arrange this in a uniform manner. Our proposed definition is:

Definition 4.6.1. *Let $\mathbb{X}$ be a differential restriction category:*

(i) A **differential index** *for $f : A \times B \longrightarrow C$ in $\bullet_{BC} : T \times B \longrightarrow C$ is a map $s_f : A \longrightarrow T$ such that (using the term logic):*

- *$s(x) \bullet_{BC} y = f(x, y)$ (so it is an index in the usual sense);*
- *$(u, v) \mapsto \frac{\partial s(x)}{\partial x}(u) \cdot v$ is a differential index for $((u, v), y) \mapsto \frac{\partial f(x,y)}{\partial x}(u) \cdot v$.*

(ii) A map $\bullet_{BC} : T \times B \longrightarrow C$ is said to be **differentially universal** *in case it is both linear and strongly additive in its first argument and each map $f : A \times B \longrightarrow C$ has a differential index in $\bullet_{BC}$.*

(iii) $\mathbb{X}$ is a **differential Turing category** *if it has* **differential Turing structure,** *that is an object T with for each pair of objects B and C a differentially universal map $\bullet_{BC} : T \times B \longrightarrow C$.*

Note the derivative of a differential index for f is required to be not only an ordinary index for the derivative of f with respect to its first argument, but also a differential index - so its derivative is in turn a differential index.

Recall that there is a recognition theorem for Turing categories: it is reasonable to see whether the analogue holds for differential Turing categories. For this we must show that $A \triangleleft_{(\langle 1,! \rangle_{T_{1,A}}, s)} T$ is an ∞-differential splitting for each n. For this we need, for example, $D[s]\langle 1, ! \rangle_{T_{1,A}} = \pi_0$. In the term logic this calculation is:

$$\frac{\partial s(x)}{\partial x}(u) \cdot v \bullet () = \frac{\partial s(x) \bullet ()}{\partial x}(u) \cdot v = \frac{\partial x}{\partial x}(u) \cdot v = v$$

as required. The calculation for the higher derivatives is similar.

A similar calculation makes the standard encoding for pairs $T \triangleleft T \times T$ an ∞-differential splitting.

These observations then guide the definition of a *differential partial combinatory algebra*, $\mathbb{A}$, because $\mathsf{Comp}(\mathbb{A})$ must be a differential Turing category. Therefore, there must be a combinator d such that $\mathsf{d}\,x\,u\,v = \frac{\partial x\,z}{\partial z}(u) \cdot v$ which satisfies the required equations.

While there remains considerable details to fill (which are beyond the scope of this thesis), what has been achieved is a better understanding of which idempotents can be split to maintain additive and differential structure. Manzonetto's [36], had conjectured only strongly additive linear idempotent could be split: this led him proposing a complicated encoding of pairing via a linear idempotent. This encoding led him to further conjecture that the addition would have to be idempotent. What has been shown in this thesis is that one can split more general idempotents, and avoid Manzonetto's dilemma.

What this thesis has not done is to provide an example of a differential Turing category which is either not total or does not have an idempotent addition. Rather it has shown that such examples have not actually been precluded. Providing such examples (as was done for left additive Turing categories) would be critical in determining how reasonable our proposed definition, above, really is.

4.7 Conclusion

In Ehrhard and Regnier's differential λ-calculus, application is assumed to be linear in the first argument [23]. A natural next step is to consider differential applicative systems, and to assume that application should be linear in its first argument as well.

Manzonetto [36] realized that it was necessary to encode pairing as an idempotent which when split provided both additive and differential structure. He conjectured [37] that for an idempotent to do this, it is necessary to require that the idempotent itself is linear.

A significant contribution of this thesis has been to analyze this assumption in detail (and in more generality for the partial case). It turns out that the situation is more delicate: the idempotents themselves do not need to be linear. Thus Manzonetto's conjecture was too strong.

An immediate consequence of allowing more general idempotents is that the standard pairing combinator can be used. Thus, Manzonetto's complex pairing combinator, which in turn led him to suggest that the idempotent must be linear, is unnecessary. Therefore, we have opened the possibility that there are models of the differential lambda calculus which do not have an idempotent addition.

This thesis has not provided any (non-trivial) models of differential Turing categories. It is known that total models exist (see [36]) although the addition in these models is idempotent. The next step in this work is to develop models which illustrate both that addition need not be idempotent and that application can be partial.

Bibliography

[1] R. Blute, J. Cockett, and R. Seely. ! and ? storage as tensorial strength. *Mathematical Structures in Computer Science*, 6:313–351, 1996.

[2] R.F Blute, J.R.B. Cockett, and R.A.G. Seely. Cartesian differential categories. *Theory and Application of Categories*, 22:622–672, 2009.

[3] Richard Blute, Robin Cockett, and Robert Seely. Differential categories. *Mathematical Structures in Computer Science*, 16:1049–1083, December 2006.

[4] F. Borceux and D. Dejean. Cauchy completion in category theory. *Cahiers de Topologie et Geometrie Differentielle Categoriques*, 27:133–146, 1986.

[5] G. Boudol. The lambda-calculus with multiplicities. In *CONCUR '93 Proceedings of the 4th International Conference on Concurrency Theory*, 1993.

[6] G. Boudol, P. Curien, and C. Lavatelli. A semantics for lambda calculi with resources. *Mathematical Structures in Computer Science*, 9:437–482, 1999.

[7] Antonio Bucciarelli, Thomas Ehrhard, and Giulio Manzonetto. Categorical models for simply typed resource calculi. *Electronic Notes in Theoretical Computer Science*, 265:213–230, September 2010.

[8] A. Carraro, T. Ehrhard, and A. Salibra. Resource combinatory algebras. *Mathematical Foundations of Computer Science*, 6281:233–245, 2010.

[9] J. Cockett and P. Hofstra. Notes on partial term logic.

[10] J. Cockett and P. Hofstra. Categorical simulations. *Journal of Pure and Applied Algebra*, 214:1835–1853, 2010.

[11] J. Cockett and P. Hofstra. Unitary theories, unitary categories. In *Mathematical Foundations of Programming Semantics*, 2010.

[12] J. Cockett and R. Seely. The faa di bruno construction. *Theory and Application of Categories*, 25:393–425, 2011.

[13] J.R.B Cockett and P. Hofstra. Introduction to turing categories. *Annals of Pure and Applied Logic.*, 156.:183–209., 2008.

[14] J.R.B. Cockett and S. Lack. Restriction categories iii: colimits, partial limits and extensivity. *Mathematical Structures in Computer Science*, 17(4):775–817, 2007.

[15] Robin Cockett, Geoff Crutwell, and Jonathan Gallagher. Differential restriction categories. *Theory and Application of Categories*, 25:537–613, 2011.

[16] Robin Cockett and Stepen Lack. Restriction categories i: Categories of partial maps. *Theoretical Computer Science*, 270:223–259, 2002.

[17] G. Cousineau, P. Curien, and M. Mauny. The categorical abstract machine. *The Science of Computer Programming*, 8:173–202, 1987.

[18] Roy Crole. *Categories for Types.* Cambridge University Press, 1994.

[19] Ugo de Liguoro and Piperno Adolfo. Nondeterministic extensions of untyped lambda calculus. *Information and Computation*, 122:149–177, 1995.

[20] Klaus Denecke and Shelly Wismath. *Universal Algebra and Coalgebra.* World Scientific Pub. Co. Inc., 2009.

[21] Nachum Dershowitz and Zohar Manna. Proving termination with multiset orderings. *Communications of the ACM*, 22(8), August 1979.

[22] Thomas Ehrhard. On kothe sequence spaces and linear logic. In *Mathematical Structures in Computer Science*, 2002.

[23] Thomas Ehrhard and Laurent Regnier. The differential lambda calculus. *Theoretical Computer Science*, 309:1–41, 2003.

[24] S. Feferman. A language and axioms for explicit mathematics. *Lecture Notes in Mathematics*, 450:87–139, 1975.

[25] J.-Y. Girard. Coherent banach spaces: a continuous denotational semantics. *Theoretical Computer Science*, 227:297–321, 1999.

[26] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–101, January 1987.

[27] Claudio Hermida and Bart. Jacobs. Fibrations with indeterminates: contextual and functional completeness for polymorphic lambda calculi. *Mathematical Structures in Computer Science*, 5:501–531, 1995.

[28] M. Hyland and A. Schalk. Glueing and orthogonaligy for models of linear logic. *Theoretical Computer Science*, 294:183–231, 2001.

[29] B. Jacobs. *Categorical logic and type theory.* Elsevier, 1999.

[30] M. Karoubi. *K Theory, an Introduction.* Springer, 1978.

[31] J. Lambek and P. Scott. *An introduction to higher order categorical logic.* Cambridge University Press, 1986.

[32] J. Longley. *Realizability Toposes and Language Semantics.* PhD thesis, University of Edinburgh, 1994.

[33] J. Longley. Computability structures, simulations, and realizability. *Mathematical Structures in Computer Science*, 2011 (submitted).

[34] Saunders Maclane. *Categories for the working mathematician.* Springer, 1998.

[35] E. Manes. *Algebraic Theories.* Springer, 1976.

[36] G. Manzonetto. What is a categorical model of the differential and resource lambda calculi? *Mathematical Structures in Computer Science*, 2010 (submitted).

[37] G. Manzonetto. A differential model theory for resource lambda calculi. Talk slides at FMCS 2011., May 2011. http://pages.cpsc.ucalgary.ca/ robin/FMCS/FMCS2011/ManzonettoPartI.pdf http://pages.cpsc.ucalgary.ca/ robin/FMCS/FMCS2011/ManzonettoPartII.pdf.

[38] Robin Milner. *A Calculus of Communicating Systems.* Lecture Notes in Computer Science. Springer Verlag, 1980.

[39] Eugenio Moggi. *The Partial Lambda Calculus.* PhD thesis, University of Edinburgh, 1988.

[40] G. Plotkin. Dijkstras predicate transformers and smyth's power domain. In *Abstract Software Specificatoins*, 1979.

[41] E. Robinson and G. Rosolini. Categories of partial maps. *Information and Computation*, 79:95–130, 1988.

[42] Sixth ASIAN Symposium on Programming Languages and Systems. *Parallel Reduction in Resource Lambda-Calculus*, 2009.

[43] M. Smyth. Power domains. *Journal of Computer and System Sciences*, 16:23–36, 1978.